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Some Fixed Point Results in Dislocated Probability Menger Metric Spaces *

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abstract: In this work, we shall give some new results about generalized common fixed point theorems for two mappings $f : X \to X$ and $T : X^k \to X$, where X is dislocated probability quasi Menger metric space (briefly, DP_qM-Space) or dislocated probability Menger metric space (briefly, DPM-Space). Our result extends and generalizes many well known results.

Key Words: Dislocated probabilistic Menger metric space, Dislocated probabilistic quasi Menger metric space, Coincidence and common fixed points, Weakly compatible mappings, Occasionally weakly compatible mappings.

Contents

1. Introduction

An interesting and important generalization of the notion of metric space was introduced by, Karl Menger [13] in 1942 under the name of statistical metric space, which is now called probabilistic metric space (PM-space). The idea of K. Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In fact the study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [17,18].

In an interesting paper [8], Hicks observed that fixed point theorems for certain contraction mappings on a Menger space endowed with a triangular t-norm may be obtained from corresponding results in metric spaces. In 1989, Kent and Richardson [11] introduced the class of probabilistic quasi-metric spaces (briefly, PQM-spaces) and proved common fixed point theorems. The study of fixed points of mappings in probabilistic quasi metric spaces is in nascent stage.

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Before we proceed we must state some definitions, known facts, and, technical results to be used in the sequel. The concepts used are that of [6,17].

Definition 1.1. *A distribution function is a function $F : [-\infty, \infty] \to [0, 1]$, that is non-decreasing and left continuous on $\mathbb{R}$, moreover, $F(-\infty) = 0$ and $F(\infty) = 1$.*

The set of all the distribution functions ($d.f.$) is denoted by Δ, and the set of those distribution functions such that $F(0) = 0$ is denoted by Δ^+. In particular for every $x_0 \geq 0$, ε_{x_0} is the d.f. defined by

$$\varepsilon_{x_0} = \begin{cases} 1 & \text{if } x > x_0, \\ 0 & \text{if } x \leq x_0. \end{cases}$$

The space Δ^+ is partially ordered by the usual pointwise ordering of functions, the maximal element for Δ^+ in this order is ε_0.

Definition 1.2. *A probabilistic metric space (abbreviated, PM-space) is an ordered pair (X, F), where X is a non-empty set and $F : X \times X \to \Delta^+$ ($F(p, q)$ is denoted by $F_{p,q}$) satisfies the following conditions:*

(PM1) $F_{p,q}(t) = 1$ *for all* $t > 0$, *iff* $p = q$,

(PM2) $F_{p,q}(t) = F_{q,p}(t)$,

(PM3) *If* $F_{p,q}(t) = 1$ *and* $F_{q,r}(s) = 1$, *then* $F_{p,r}(t + s) = 1$,

for every $p, q, r \in X$ *and* $t, s \geq 0$.

Definition 1.3. *A mapping $\tau : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:*

(i) $\tau(a, b) = \tau(b, a)$,

(ii) $\tau(a, \tau(b, c)) = \tau(\tau(a, b), c)$,

(iii) $\tau(a, b) \geq \tau(c, d)$ whenever $a \geq c$ and $b \geq d$,

(iv) $\tau(a, 1) = a$,

for every $a, b, c, d \in [0, 1]$.

Definition 1.4. *A Menger space is a triplet (X, F, τ), where (X, F) is PM-space and τ is a t-norm such that for all $p, q, r \in X$ and for all $t, s \geq 0$,*

$$F_{p,r}(t + s) \geq \tau(F_{p,q}(t), F_{q,r}(s)).$$

Definition 1.5. *A dislocated probabilistic quasi Menger space (abbreviated, DP_qM-space) is a triplet (X, F, τ), where X is a non empty set, τ is a t-norm and $F : X \times X \to \Delta^+$ ($F(p, q)$ is denoted by $F_{p,q}$) satisfies the following conditions:*

(i) $F_{p,q}(t) = 1$ *and* $F_{q,p}(t) = 1 \Rightarrow p = q$,

(ii) $F_{p,r}(t+s) \geq \tau(F_{p,q}(t), F_{q,r}(s))$,

for every $p, q, r \in X$ *and* $t, s \geq 0$.

Example 1.6. *Let* $X = \mathbb{R}$, *define* $\tau(a,b) = a \cdot b$ *and*

$$F_{x,y}(t) = \frac{1}{e^{\frac{|x-y|+2|x|+|y|}{t}}},$$

for all $(x,y) \in X \times X, t \in (0,\infty)$. *Then* (X, F, τ) *is a* DP_qM*-space.*

Definition 1.7. *A dislocated probabilistic Menger space (abbreviated,* DPM*-space) is a* DP_qM*-space such that for all* $p, q \in X$, $F_{p,q} = F_{q,p}$.

Let (X, F, τ) be a $DPqM$-space and $F^{\ddagger}_{p,q}(t) = \min\{F_{p,q}(t), F_{q,p}(t)\}$ ($p, q \in X$ and $t \in [0,\infty]$), then it is easy to see that, $(X, F^{\ddagger}, \tau)$ is a DPM-space. In addition, if (X, F, τ) is a DPM-space, then $F^{\ddagger} = F$.

Definition 1.8. *Let* (X, F, τ) *be a* DP_qM*-space (or* DPM*-space),* $T : X^k \to X$ *and* $f : X \to X$ *be mappings. A point* $z \in X$ *is said to be a coincidence point of* f *and* T *if* $T(z, z, \ldots, z) = fz$. *The set of all coincidence points of the mappings* f *and* T *denoted by* $C(T, f)$. *A point* $z \in X$ *is said to be a common fixed point of* f *and* T *if* $T(z, z, \ldots, z) = fz = z$.

Definition 1.9. *The mappings* $T : X^k \to X$ *and* $f : X \to X$, *where* (X, F, τ) *is a* DP_qM*-space, are said to be weakly compatible (wc) if the mappings commuting at their coincidence points, i.e.* $T(fz, fz, \ldots, fz) = f(T(z, z, \ldots, z))$, *for all* $z \in C(T, f)$.

Very recently many authors proved some new fixed point theorems in dislocated quasi metric spaces, see [7,12,16]. In 1996, Jungck [9] introduced the notion of weakly compatible mappings which is more general than compatibility and proved fixed point theorems in absence of continuity of the involved mappings. In recent years, many mathematicians established a number of common fixed point theorems satisfying contractive type conditions and involving conditions on commutativity, completeness and suitable containment of ranges of the mappings. Al-Thagafi and Shahzad [2] introduced the notion of occasionally weakly compatible mappings in metric space, which is more general than weakly compatible mappings. Recently, Jungck and Rhoades [10] extensively studied the notion of occasionally weakly compatible mappings in semi-metric spaces. The notions of improving commutativity of self mappings have been extended to PM-spaces by many authors. For example, Singh and Jain [19] extended the notion of weak compatibility and Chauhan et al. [4] extended the notion of occasionally weak compatibility to PM-spaces. The fixed point theorems for occasionally weakly compatible mappings in different settings investigated by many researchers (e.g. [1,3,15,14]).

Definition 1.10. *The mappings* $T : X^k \to X$ *and* $f : X \to X$, *where* (X, F, τ) *is a* DP_qM*-space, are said to be occasionally weakly compatible (owc) if the mappings commuting at least one coincidence point, whenever* $C(T, f) \neq \phi$.

Definition 1.11. *Let (X, F, τ) be a $DPqM$-space. A left (right) open ball (abbreviated, L-open (R-open) ball) with center x and radius r $(0 < r < 1)$ in X is the set $B_L(x, r, t) = \{y \in X : F_{x,y}(t) > 1 - r\}$ $(B_R(x, r, t) = \{y \in X : F_{y,x}(t) > 1 - r\})$, for all $t \in (0, 1)$. Moreover, a open ball with center x and radius r $(0 < r < 1)$ in X is the set $B(x, r, t) = \{y \in X : F^{\ddagger}_{x,y}(t) > 1 - r\}$, for all $t \in (0, 1)$.*

Definition 1.12. *A sequence (x_n) in a DP_qM-space (X, F, τ) is said to be bi-convergent to a point $x \in X$ if and only if $\lim_{n\to\infty} F^{\ddagger}_{x_n,x}(t) = 1$ for all $t > 0$, in this case we say that limit of the sequence (x_n) is x. A sequence (x_n) is said to be left (right) Cauchy sequence if and only if*

$$\lim_{n\to\infty} F_{x_n,x_{n+p}}(t) = 1 \qquad (\lim_{n\to\infty} F_{x_{n+p},x_n}(t) = 1)$$

for all $t > 0$, $p \in \mathbb{N}$. Also, a sequence (x_n) is said to be bi-Cauchy if and only if $\lim_{n\to\infty} F^{\ddagger}_{x_n,x_{n+p}}(t) = 1$ for all $t > 0$, $p \in \mathbb{N}$.

The concept of left (right) Cauchy sequence is inspired from that of G-Cauchy sequence (it belongs to Grabiec [5]).

Definition 1.13. *A DP_qM-space (X, F, τ) is said to be left (or right) complete if and only if every left (or right) Cauchy sequence in X, is bi-convergent. Also, a DP_qM-space is said to be bi-complete if and only if every bi-Cauchy sequence in X, is bi-convergent.*

Clearly a sequence (x_n) in a DP_qM-space (X, F, τ) is *bi*-Cauchy sequence if and only if sequence (x_n) is a Cauchy sequence in the DPM-space $(X, F^{\ddagger}, \tau)$. Also, a DP_qM-space (X, F, τ) is *bi*-Complete if and only if the DPM-space $(X, F^{\ddagger}, \tau)$ is complete.

Proposition 1.1. *The limit of a bi-convergent sequence in a DP_qM-space (X, F, τ) is unique.*

Proof: Let (x_n) be a sequence in X and suppose that u and v are two limits of (x_n). By the hypothesis we have $F^{\ddagger}_{u,v}(t) \geq \tau(F^{\ddagger}_{u,x_n}(t/2), F^{\ddagger}_{x_n,v}(t/2))$ for all n. Now taking the limit as $n \to \infty$, so we have $F^{\ddagger}_{u,v}(t) \geq \tau(1, 1) = 1$. Hence $u = v$, the result follows. □

Proposition 1.2. *Let (X, F, τ) be a DP_qM-space (or DPM-space) and (x_n) be a sequence in X. If sequence (x_n) bi-converges (or converges) to $x \in X$, then $F_{x,x}(t) = 1$ for all $t > 0$.*

Proof: By using a similar argument as in the proof of the above proposition, the result follows. □

Proposition 1.3. *Let (X, F, τ) be a DP_qM-space (or DPM-space). If $f, g : X \to X$ is two mappings such that $fz = gz$ and $F^{\ddagger}_{fgz,gfz}(t) = 1$ (or $F_{fgz,gfz}(t) = 1$) for some $z \in X$ and $t \in [0, \infty)$, then $F_{ffz,ffz}(t) = 1$.*

Proof: Since $F^{\ddagger}_{fgz,gfz}(t) = 1$, so by the hypothesis we have $fgz = gfz$. Therefore $F_{ffz,ffz}(t) = F_{fgz,fgz}(t) = F_{fgz,gfz}(t) = 1$. □

The following results are immediate.

Lemma 1.14. *Let (X, F, τ) be a DP_qM-space (or DPM-space), $T : X^k \to X$ and $f : X \to X$ be occasionally weakly compatible mappings. If f and T have a unique point of coincidence then f and T are weakly compatible.*

Thus, if mappings f and T have a unique point of coincidence, then the pair (f, T) are weakly compatible if and only if they are occasionally weakly compatible.

Lemma 1.15. *Let (X, F, τ) be a DP_qM-space (or DPM-space), $T : X^k \to X$ and $f : X \to X$ be occasionally weakly compatible mappings. If pair (f, T) has a unique point of coincidence, then it has a unique common fixed point.*

The above lemma is also valid for weakly compatible mappings. The next example shows that if the point of coincidence is not unique, then occasionally weakly compatible mappings are more general than weakly compatible mappings.

Example 1.16. *Take $X = [0, 1]$, $fx = x^2$, $T(x_1, \cdots, x_k) = \frac{\sqrt[k]{x_1x_2\cdots x_k}}{2}$. It is obvious that $\{0, \frac{1}{2}\} \subseteq C(f, T)$, $fT0 = Tf0$ but $fT\frac{1}{2} \neq Tf\frac{1}{2}$ and so f and T are occasionally weakly compatible but not weakly compatible. Note that 0 and $\frac{1}{4}$ are two point of coincidence and 0 is the unique common fixed point.*

Definition 1.17. *The mapping f is said to be coincidentally idempotent (ci) with respect to T, if and only if f is idempotent at the coincidence points of f and T, i.e. $ffz = fz$, for all $z \in C(T, f)$.*

Definition 1.18. *The mapping f is said to be occasionally coincidentally idempotent (oci) with respect to T, if and only if f is idempotent at least one coincidence point, whenever $C(T, f) \neq \phi$.*

Clearly if f and T are coincidentally idempotent then they are oci. However, the Example (1.16) shows that the converse is not necessarily true.

Definition 1.19. *A function $\phi : [0, 1]^k = [0, 1] \times [0, 1] \times \cdots \times [0, 1] \to [0, 1]$ is said to be a Φ^k-function if it satisfies the following conditions:*

(i) ϕ is an increasing function, i.e, $x_1 \leq y_1, x_2 \leq y_2, \cdots, x_k \leq y_k$ implies

$$\phi(x_1, x_2, \cdots, x_k) \leq \phi(y_1, y_2, \cdots, y_k),$$

(ii) $\phi(t, t, \cdots, t) \geq t$, for all $t \in [0, \infty)$,

(iii) ϕ is continuous in all variables.

In this paper, we establish some coincidence point theorems for certain maps and common fixed point theorems for weakly compatible maps in DPM-space (DP_qM-space) under strict contractive conditions. Our results generalize many known results in DPM-space (DP_qM-space).

2. Coincidence and common fixed points in DP_qM-space and DPM-space

In this section we prove some coincidence and common fixed point results for two mappings $f : X \to X$ and $T : X^k \to X$ in a DP_qM-space or DPM-space. Now, we state our first main theorem.

Theorem 2.1. *Let (X, F, τ) be a DPM-space, $k \geq 2$ be an integer, $f : X \to X$ and $T : X^k \to X$ be mappings, such that $f(X)$ is complete and $T(X^k) \subseteq f(X)$. If*

$$F_{T(x_1,x_2,\ldots,x_k),T(x_2,x_3,\ldots,x_{k+1})}(qt) \geq \phi\left(F_{fx_1,fx_2}(t), F_{fx_2,fx_3}(t), \ldots, F_{fx_k,fx_{k+1}}(t)\right),$$

where $x_1, x_2, \ldots, x_{k+1}$ are arbitrary elements in X, $0 < q < 1$, $t \in [0,\infty)$ and ϕ is Φ^k-function. Then the sequence (y_n) defined by

$$y_n = f(x_{n+k}) = T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \tag{2.1}$$

for arbitrary elements $x_1, x_2, \ldots, x_k$ in X, converges to a point of coincidence of f and T.

Proof: If $\alpha_n = F_{y_n,y_{n+1}}(qt)$, then by the hypothesis, it is easy to see that

$$\alpha_{n+k} \geq \phi\left(\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k-1}\right).$$

Clearly, if $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$, then the sequence $(y_n)_{n=1}^{\infty}$ is constant and so it is a Cauchy sequence in $f(X)$. Otherwise, by induction on n, we will prove that

$$\alpha_n \geq \left(\frac{K-\theta^n}{K+\theta^n}\right)^2 \tag{2.2}$$

where $\theta = \frac{1}{q}$ and $K = \min\left\{\frac{\theta(1+\sqrt{\alpha_1})}{(1-\sqrt{\alpha_1})}, \frac{\theta^2(1+\sqrt{\alpha_2})}{(1-\sqrt{\alpha_2})}, \ldots, \frac{\theta^k(1+\sqrt{\alpha_k})}{(1-\sqrt{\alpha_k})}\right\}$. We can see that (2.2) is true for $n = 1, 2, \ldots, k$ by the definition of K. Assume that (2.2) is true for n, $n+1$ to $n+k-1$. Then by the hypothesis, we have

$$\begin{aligned}
\alpha_{n+k} &\geq \phi\left(\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k-1}\right)\\
&\geq \phi\left(\left(\frac{K-\theta^n}{K+\theta^n}\right)^2, \left(\frac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2, \ldots, \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\right)\\
&\geq \phi\left(\left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2, \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2, \ldots, \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\right)\\
&\geq \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\\
&\geq \left(\frac{K-\theta^{n+k}}{K+\theta^{n+k}}\right)^2.
\end{aligned}$$

Thus inductive proof of (2.2) is complete. Now for $p \in \mathbb{N}$ and $t \in [0, \infty)$, we have

$$\begin{aligned}
F_{y_{n+1},y_{n+p+1}}(t) \geq & \tau\Big(F_{y_{n+1},y_{n+2}}\Big(\frac{t}{2}\Big), \tau\Big(F_{y_{n+2},y_{n+3}}\Big(\frac{t}{2^2}\Big),\\
& \tau\Big(\cdots, \tau\Big(F_{y_{n+p-1},y_{n+p}}\Big(\frac{t}{2^{p-1}}\Big), F_{y_{n+p},y_{n+p+1}}\Big(\frac{t}{2^{p-1}}\Big)\Big)\cdots\Big)\Big)\Big)\\
\geq & \tau\Big(\Big(\frac{K-2^n}{K+2^n}\Big)^2, \tau\Big(\Big(\frac{K-2^{2(n+1)}}{K+2^{2(n+1)}}\Big)^2,\\
& \tau\Big(\cdots, \tau\Big(\Big(\frac{K-2^{(p-1)(n+p-2)}}{K+2^{(p-1)(n+p-2)}}\Big)^2,\\
& \Big(\frac{K-2^{(p-1)(n+p-1)}}{K+2^{(p-1)(n+p-1)}}\Big)^2\Big)\cdots\Big)\Big)\Big)\\
\geq & \tau(1, \tau(1, \tau(\cdots, \tau(1, 1)\cdots))) = 1, \quad as\, n \to \infty.
\end{aligned}$$

Hence (y_n) is a Cauchy sequence in $f(X)$ and so there exists v in $f(X)$ such that $\lim_{n\to\infty} y_n = v$. Let $v = f(u)$ for some $u \in X$. Then we have

$$\begin{aligned}
F_{T(u,u,\ldots,u),fu}(t) = & \lim_{n\to\infty} F_{T(u,u,\ldots,u),y_n}(t) = \lim_{n\to\infty} F_{T(u,u,\ldots,u),T(x_n,x_{n+1},\ldots,x_{n+k-1})}(t)\\
\geq & \lim_{n\to\infty} \tau\Big(F_{T(u,u,\ldots,u),T(u,u,\ldots,x_n)}\Big(\frac{t}{2}\Big),\\
& \tau\Big(F_{T(u,u,\ldots,x_n),T(u,u,\ldots,x_n,x_{n+1})}\Big(\frac{t}{2^2}\Big),\\
& \tau\Big(\cdots, \tau\Big(F_{T(u,u,x_n,\ldots,x_{n+k-3}),T(u,x_n,\ldots,x_{n+k-2})}\Big(\frac{t}{2^{k-1}}\Big),\\
& F_{T(u,x_n,\ldots,x_{n+k-2}),T(x_n,x_{n+1},\ldots,x_{n+k-1})}\Big(\frac{t}{2^{k-1}}\Big)\Big)\cdots\Big)\Big)\Big)\\
\geq & \lim_{n\to\infty} \tau\Big(\phi\Big(F_{fu,fu}(t), F_{fu,fu}(t), \ldots, F_{fu,fx_n}(t)\Big),\\
& \tau\Big(\phi\Big(F_{fu,fu}(t), \ldots, F_{fu,fx_n}(t), F_{fx_n,fx_{n+1}}(t)\Big),\\
& \tau\Big(\cdots, \tau\Big(\phi\Big(F_{fu,fu}(t), F_{fx_n,fx_{n+1}}(t), \ldots, F_{fx_{n+k-3},fx_{n+k-2}}(t)\Big),\\
& \phi\Big(F_{fu,fx_n}(t), F_{fx_n,fx_{n+1}}(t), \ldots, F_{fx_{n+k-2},fx_{n+k-1}}(t)\Big)\Big)\cdots\Big)\Big)\Big)\\
& = 1,
\end{aligned}$$

hence, $F_{T(u,u,\ldots,u),fu}(t) = 1$. Therefore $C(f,T) \neq \phi$ and v is a point of coincidence of f and T, as required. □

Corollary 2.2. *With the same hypotheses of the Theorem 2.1, if T and f are occasionally weakly compatible mappings, $0 < q < \frac{1}{2}$ and $\lim_{t\to\infty} F_{x,y}(t) = 1$ for all*

$x, y \in X$. Then the sequence (y_n) defined by (2.1) converges to a unique common fixed point of f and T.

Proof: By the Theorem 2.1, we see that the sequence (y_n) converges to $v = f(u)$ which is a point of coincidence of f and T. Then by the hypothesis, we have

$$\begin{aligned} F_{fu,fu}(qt) &= F_{T(u,u,\ldots,u),T(u,u,\ldots,u)}(qt) \\ &\geq \phi\Big(F_{fu,fu}(t), F_{fu,fu}(t), \ldots, F_{fu,fu}(t)\Big) \\ &\geq F_{fu,fu}(t) = F_{T(u,u,\ldots,u),T(u,u,\ldots,u)}(t) \\ &\geq \phi\Big(F_{fu,fu}(\frac{t}{q}), F_{fu,fu}(\frac{t}{q}), \ldots, F_{fu,fu}(\frac{t}{q})\Big) \\ &\geq F_{fu,fu}(\frac{t}{q}) \geq \ldots\ldots \geq F_{fu,fu}(\frac{t}{q^{n-1}}). \end{aligned}$$

As $n \to \infty$ we get $F_{fu,fu}(qt) = 1$. Suppose there exists $v^* \in X$ such that $f(u^*) = T(u^*, u^*, \ldots, u^*) = v^*$ for some u^* in $C(f, T)$. Similarly, we can show that $F_{fu^*,fu^*}(qt) = 1$.

Therefore, we have

$$\begin{aligned} F_{v,v^*}(q\frac{t}{2}) =& F_{T(u,u,u,\ldots,u),T(u^*,u^*,u^*,\ldots,u^*)}(q\frac{t}{2}) \geq \tau\Big(F_{T(u,u,u,\ldots,u),T(u,u,u,\ldots,u^*)}(\frac{qt}{2^2}), \\ & \tau\Big(F_{T(u,u,u,\ldots,u^*),T(u,u,u,\ldots,u^*,u^*)}(\frac{qt}{2^3}), \\ & \tau\Big(\cdots, \tau\Big(F_{T(u,u,u^*,\ldots,u^*),T(u,u^*,u^*,\ldots,u^*)}(\frac{qt}{2^k}), \\ & F_{T(u,u^*,u^*,\ldots,u^*),T(u^*,u^*,u^*,\ldots,u^*)}(\frac{qt}{2^k})\Big)\cdots\Big)\Big)\Big) \\ \geq& \tau\Big(\phi\Big(F_{fu,fu}(\frac{t}{2^2}), F_{fu,fu}(\frac{t}{2^2}), \ldots, F_{fu,fu^*}(\frac{t}{2^2}\Big), \\ & \tau\Big(\phi\Big(F_{fu,fu}(\frac{t}{2^3}), F_{fu,fu}(\frac{t}{2^3}), \ldots, F_{fu,fu^*}(\frac{t}{2^3}), F_{fu^*,fu^*}(\frac{t}{2^3})\Big), \\ & \tau\Big(\cdots, \tau\Big(\phi\Big(F_{fu,fu}(\frac{t}{2^k}), F_{fu,fu^*}(\frac{t}{2^k}), \ldots, F_{fu^*,fu^*}(\frac{t}{2^k})\Big), \\ & \phi\Big(F_{fu,fu^*}(\frac{t}{2^k}), F_{fu^*,fu^*}(\frac{t}{2^k}), \ldots, F_{fu^*,fu^*}(\frac{t}{2^k})\Big)\Big)\cdots\Big)\Big)\Big) \\ \geq& \tau\Big(F_{fu,fu^*}(\frac{t}{2^2}), \tau\Big(F_{fu,fu^*}(\frac{t}{2^3}), \tau\Big(\cdots, \\ & \tau\Big(F_{fu,fu^*}(\frac{t}{2^k}), F_{fu,fu^*}(\frac{t}{2^k})\Big)\cdots\Big)\Big)\Big) \end{aligned}$$

$$\begin{aligned}
=&\tau\Big(F_{T(u,u,\ldots,u),T(u^*,u^*,\ldots,u^*)}(\frac{t}{2^2}),\tau\Big(F_{T(u,u,\ldots,u),T(u^*,u^*,\ldots,u^*)}(\frac{t}{2^3}),\\
&\tau\Big(\cdots,\tau\Big(F_{T(u,u,\ldots,u),T(u^*,u^*,\ldots,u^*)}(\frac{t}{2^k}),\\
&F_{T(u,u,\ldots,u),T(u^*,u^*,\ldots,u^*)}(\frac{t}{2^k})\Big)\cdots\Big)\Big)\Big)\\
\geq&\tau\Big(F_{fu,fu^*}(\frac{t}{2^2q}),\tau\Big(F_{fu,fu^*}(\frac{t}{2^3q}),\\
&\tau\Big(\cdots,\tau\Big(F_{fu,fu^*}(\frac{t}{2^kq}),F_{fu,fu^*}(\frac{t}{2^kq})\Big)\cdots\Big)\Big)\Big).
\end{aligned}$$

Repeating the above process n times we get

$$\begin{aligned}
F_{v,v^*}(qt)\geq&\tau\Big(F_{fu,fu^*}(\frac{t}{2^{n+1}q^n}),\tau\Big(F_{fu,fu^*}(\frac{t}{2^{n+2}q^n}),\\
&\tau\Big(\cdots,\tau\Big(F_{fu,fu^*}(\frac{t}{2^{n+k-1}q^n}),F_{fu,fu^*}(\frac{t}{2^{n+k-1}q^n})\Big)\cdots\Big)\Big)\Big).
\end{aligned}$$

Taking the limit as $n\to\infty$ we get $F_{v,v^*}(qt)\geq 1$ and so $v=v^*$ i.e. v is the unique point of coincidence of f and T . Finally, by the Lemma 1.15, v is a unique common fixed point of f and T. □

Note that the condition $\lim_{t\to\infty}F_{x,y}(t)=1$ ensures the uniqueness of the point of coincidence. However in the next result we will remove the condition $\lim_{t\to\infty}F_{x,y}(t)=1$ and also increase the range of q.

Corollary 2.3. *With the same hypotheses of the Theorem 2.1, if one of the following two conditions are satisfied:*

(i) f is oci with respect to T and the pair (f,T) is weakly compatible,

(ii) f is coincidentally idempotent with respect to T and the pair (f,T) is owc.

Then f and T have a common fixed point.

Proof: By the Theorem 2.1, it follows that $C(T,f)\neq\phi$. Now suppose that (i) is satisfied. So there will exist $z\in C(f,T)$ such that $ffz=fz$ and also $f(T(z,z,\ldots,z))=T(fz,fz,\ldots,fz)$. Thus we have $fz=ffz=f(T(z,z,\ldots,z))=T(fz,fz,\ldots,fz)$, i.e., fz is a common fixed point of f and T . The proof follows on the same lines in the other case also. □

Theorem 2.4. *Let (X,F,τ) be a DP_qM-space, $k\geq 2$ be an integer, $f:X\to X$ and $T:X^k\to X$ be mappings, such that $f(X)$ is R-complete (L-complete) and $T(X^k)\subseteq f(X)$. If*

$$F_{T(x_1,x_2,\ldots,x_k),T(x_{k+1},x_1,x_2,\ldots,x_{k-1})}(qt)\geq\phi\Big(F_{fx_1,fx_{k+1}}(t),F_{fx_2,fx_1}(t),\ldots,F_{fx_k,fx_{k-1}}(t)\Big),$$

where $x_1, x_2, \ldots, x_{k+1}$ *are arbitrary elements in* X, $0 < q < 1$, $t \in [0, \infty)$ *and* ϕ *is* Φ^k*-function. Then the sequence* (y_n) *defined by* (2.1) *converges to a point of coincidence of* f *and* T.

Proof: By modifying the proof of the Theorem 2.1, we can show that $C(f,T) \neq \phi$. □

Corollary 2.5. *With the same hypotheses of the Theorem 2.4, if* T *and* f *are weakly compatible mappings,* $0 < q < \frac{1}{2}$ *and* $\lim\limits_{t\to\infty} F_{x,y}(t) = 1$ *for all* $x, y \in X$. *Then the sequence* (y_n) *defined by* (2.1) *converges to a unique common fixed point of* f *and* T.

Proof: By modifying the proof of the Corollary 2.2, we can show that the sequence (y_n) defined by (2.1) converges to a unique common fixed point. □

Proof of the following corollary follows on the same lines as that of the Corollary 2.3.

Corollary 2.6. *With the same hypotheses of the Theorem 2.4, if one of the following two conditions are satisfied:*

(i) f *is oci with respect to* T *and the pair* (f,T) *is weakly compatible,*

(ii) f *is coincidentally idempotent with respect to* T *and the pair* (f,T) *is owc.*

Then f *and* T *have a common fixed.*

Taking $k = 1$ in the previous results, we get the following.

Theorem 2.7. *Let* (X, F, τ) *be a* DP_qM*-space,* $f : X \to X$ *and* $T : X \to X$ *be mappings, such that* $f(X)$ *is R-complete (L-complete) and* $T(X) \subseteq f(X)$. *If*

$$F_{Tx,Ty}(qt) \geq \phi\Big(F_{fx,fy}(t)\Big), \tag{2.3}$$

where x, y *are arbitrary elements in* X, $0 < q < \frac{1}{2}$, $t \in [0, \infty)$ *and* ϕ *is* Φ^1*-function. Then* f *and* T *have a coincidence point, i.e.,* $C(f,T) \neq \phi$. *Moreover, if* f *and* T *are weakly compatible and* $\lim\limits_{t\to\infty} F_{x,y}(t) = 1$ *for all* $x, y \in X$, *or one of the following two conditions are satisfied:*

(i) f *is oci with respect to* T *and the pair* (f,T) *is weakly compatible,*

(ii) f *is coincidentally idempotent with respect to* T *and the pair* (f,T) *is owc.*

Then the pair (f,T) *have a unique common fixed point.*

The above theorem is also valid for complete $f(X)$ in DPM-space. If we take f to be the identity mapping in the above corollaries, we get the following

Corollary 2.8. *Let (X, F, τ) be a R-complete (L-complete) DP_qM-space, $T : X \to X$ be a mapping, such that*

$$F_{Tx,Ty}(t) \geq F_{x,y}(t),$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$. Then T has a fixed point.

In what follows, we present some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results proved in this paper.

Example 2.9. *Let $X = [0, 2]$ and $d : X \times X \to X$ be given by $d(x, y) = |x-y|+x+y$ and define $F_{x,y} : [-\infty, \infty] \to [0, 1]$ by*

$$F_{x,y}(t) = \frac{t}{t + d(x, y)},$$

for every $x, y \in X$. Clearly, d is dislocated metric on X and (X, F, τ) is DPM-space with $\tau(a, b) = ab$ for all $a, b \in [0, \infty)$. Let $k \geq 2$ be an integer and $T : X^k \to X$ and $f : X \to X$ be defined by $T(x_1, x_2, \cdots, x_k) = \frac{(x_1^2+x_2^2+\cdots+x_k^2)}{2k}$ and $f(x) = x^2$. Then we have

$$\begin{aligned}
&F_{T(x_1,x_2,\cdots,x_k),T(x_2,x_3,\cdots,x_{k+1})}(qt)\\
&= \frac{qt}{qt + \frac{1}{2k}|x_1^2 - x_{k+1}^2| + \frac{1}{2k}(x_1^2 + 2x_2^2 + \cdots + 2x_k^2 + x_{k+1}^2)}\\
&\geq \frac{qt}{qt + \frac{1}{2k}|x_1^2 - x_2^2| + \frac{1}{2k}x_1^2 + \frac{1}{2k}x_2^2 + \cdots + \frac{1}{2k}|x_k^2 - x_{k+1}^2| + \frac{1}{2k}x_k^2 + \frac{1}{2k}x_{k+1}^2}(q = \frac{1}{2})\\
&\geq \frac{t}{t + \frac{1}{k}|x_1^2 - x_2^2| + \frac{1}{k}x_1^2 + \frac{1}{k}x_2^2 + \cdots + \frac{1}{k}|x_k^2 - x_{k+1}^2| + \frac{1}{k}x_k^2 + \frac{1}{k}x_{k+1}^2}\\
&\geq \min\left\{\frac{t}{t + |x_1^2 - x_2^2| + x_1^2 + x_2^2}, \cdots, \frac{t}{t + |x_k^2 - x_{k+1}^2| + x_k^2 + x_{k+1}^2}\right\}\\
&= \phi\left(F_{fx_1,fx_2}(t), F_{fx_2,fx_3}(t), \ldots, F_{fx_k,fx_{k+1}}(t)\right).
\end{aligned}$$

Therefore, f and T satisfy conditions of the Theorem 2.1 with $\phi(t_1, \cdots, t_k) = \min\{t_1, \cdots, t_k\}$ and $q = \frac{1}{2}$. So we see that $C(T, f) = \{0\}$, f and T commute at 0. Finally 0 is the unique common fixed point of f and T.

Example 2.10. *Let $X = \mathbb{R}$ and $d : X \times X \to X$ be given by*

$$d(x, y) = |x - y| + 2|x| + |y|$$

and define $F_{x,y} : [-\infty, \infty] \to [0, 1]$ by

$$F_{x,y}(t) = \left(\frac{1}{e^{\frac{d(x,y)}{t}}}\right)^{\frac{1}{2}},$$

for every $x, y \in X$*. Clearly,* d *is* R*-complete dislocated quasi metric on* X *and* (X, F, τ) *is* R*-complete* DP_qM*-space with* $\tau(a, b) = ab$ *for all* $a, b \in [0, \infty)$*. Let* $T : X^2 \to X$ *and* $f : X \to X$ *be defined by* $T(x, y) = \frac{-x+y}{4}$ *and* $f(x) = x$*. It is obvious* f *and* T *are weakly compatible mappings and* f *is coincidentally idempotent with respect to* T*. Also it is easy to see that* f *and* T *satisfy conditions of the Theorem 2.4 whit* $\phi(t_1, t_2) = \sqrt{t_1 \cdot t_2}$ *and* $q = \frac{1}{4}$*. So we see that* $C(T, f) = \{0\}$*. Finally* 0 *is the unique common fixed point of* f *and* T*.*

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