Homotopy Analysis Method to determine Magneto Hydrodynamics flow of compressible fluid in a channel with porous walls

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ABSTRACT: In this article magnetohydrodynamics (MHD) boundary layer flow of compressible fluid in a channel with porous walls is researched. In this study it is shown that the nonlinear Navier-Stokes equations can be reduced to an ordinary differential equation, using the similarity transformations and boundary layer approximations. Analytical solution of the developed nonlinear equation is carried out by the Homotopy Analysis Method (HAM). In addition to applying HAM for solving obtained equation, the result of the mentioned method is compared with a type of numerical analysis as Boundary Value Problem method (BVP) and a good agreement is seen. The effects of the Reynolds number and Hartman number are investigated.

Key Words: MHD Flow, Compressible Fluid, Boundary layer, HAM, BVP.

Contents

Nomenclature

BVP boundary value problem method
B_0 uniform static magnetic field
HAM Homotopy Analysis Method
$\hbar$ Auxiliary parameter
H Auxiliary function
L Non-linear operator
f similarity function
H channel width (m)
M Hartman number
p Pressure(Pa)
Re Reynolds number

u   x velocity(m/s)
v   y velocity(m/s)

Greek symbols
μ dynamic viscosity $(N.s/m^2)$
ρ density (kg/m^3)
σ electrical conductivity (Siemens/m, where Siemens=$1/\Omega$)

1. Introduction

Magnetohydrodynamics is essential in plasma physics and astrophysics and studies the motion of electrically conducting media in the presence of a magnetic field. In natural systems include the Earth's core and solar flares, and in the engineering world, the electromagnetic casting of metals and the confinement of plasmas MHD effects are important [1]. Recently reactor designs commonly involve the use of electrically conducting liquid metals, in fusion engineering, are much of the interest [2].

In recent decades many attempts have been made to develop analytical methods for solving such nonlinear equations. One of them is the perturbation method [3], which is strongly dependent on a so called small parameter to be defined according to the physics of the problem. Thus, it is worth developing some new analytical techniques,which are independent of defining a small parameter such as Homotopy Perturbation Method (HPM) [4,5,6,7], Variational Iteration Method (VIM) [8,9]. In fact the perturbation method cannot provide a simple way to adjust and control the region and rate of convergence of a particular approximated series. Liao introduced the basic idea of Homotopy in topology to propose a general analytical method for nonlinear problems, namely the Homotopy Analysis Method [10,11], that does not need any small parameter. This method has been successfully applied to solve many types of nonlinear problems [12,13,14].

In order to determine the velocity components, HAM is applied to solve the resulting nonlinear differential equation. Then the solution is compared with Boundary Value Problem Method. An ordinary non-linear differential equation can be derived from the governing differential equations by using similarity transformation.

2. Description of the problem

The two-dimensional MHD flow of a compressible fluid in a porous channel with suction and injection is investigated. The geometry of the problem is shown in figure (1-a) and (1-b). The x-axis is taken along the centerline of the channel and the y-axis transverse to these. The flow is symmetric about both axes. The porous walls of the channel are at $y = H/2$ and $y = \Gamma H/2$. The fluid injection or suction takes place through the porous walls with velocity $V_0/2$. Here $V_0 > 0$ corresponds to suction and $V_0 < 0$ for injection. Let u and v be the velocity components along the x- and y-axes respectively, and B_0 is a uniform static magnetic field in y-direction. The compressible electrically conducting fluid that flows though the axial direction in the channel will induce a magnetic field in the medium in an applied magnetic
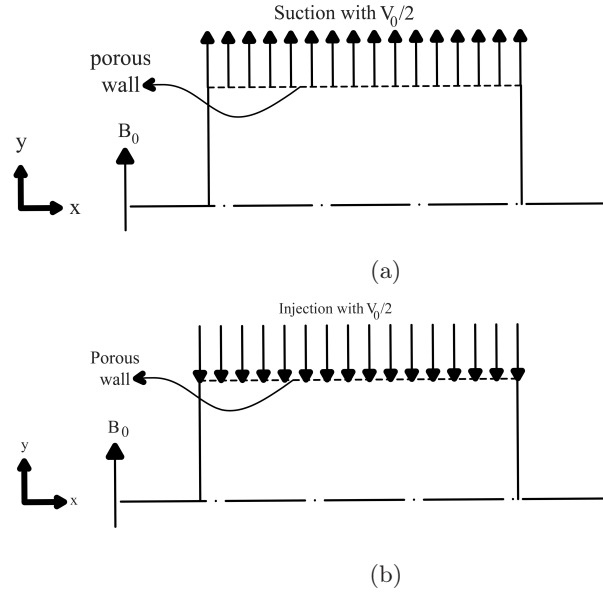
Figure 1: Axial section of the channel in case of (a) suction (b) injection

field. The magnetic Reynolds number ($Re_m = \sigma\mu_m UL$) represents the relative strength of the induced field. In the above relation the characteristics such as U and L are the scale length and velocity and μ_m is magnetic permeability. If the magnetic Reynolds number is small, the induced magnetic field will be neglected [17]. It can be assumed that the electric field is zero as no external electric field is applied and the effect of polarization of the ionized fluid is negligible. The equations for the MHD boundary layer flow of a compressible fluid with are:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2.1}$$

$$\rho\left(\frac{u\partial(\rho u)}{\partial x} + v\frac{\partial(\rho v)}{\partial y}\right) = -\sigma B_0^2 u - \frac{\partial p}{\partial x} + \frac{4}{3}\mu\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\mu\frac{\partial^2 v}{\partial x\partial y} + \mu\frac{\partial^2 u}{\partial y^2} \tag{2.2}$$

Assuming the symmetry about the x-axis and no-slip conditions at $y = H/2$, we have:

$$\frac{\partial u}{\partial y} = 0, v = 0, \qquad at\, y = 0$$

$$u = 0, v = \frac{V_0}{2} \qquad y = \frac{H}{2} \tag{2.3}$$

The Equation (4) represents the non-dimensional parameters to rewrite the Equation (2) in the non-dimensional form, in which $f(y^*)$ is assumed as a similarity

function.

$$x^* = \frac{x}{H}, y^* = \frac{y}{H}, u = -V_0 x^* f'(y^*), v = V_0 f(y^*) \tag{2.4}$$

Applying the above equation, Equations (2) and (3) may be written as:

$$f''' Re(f'^2 - ff'') - M^2 f' = 0 \tag{2.5}$$

$$\begin{aligned} f = 0, \quad & f'' = 0 \quad & at\, y^* = 0 \\ f = \frac{1}{2}, \quad & f' = 0 \quad & at\, y^* = \frac{1}{2} \end{aligned} \tag{2.6}$$

Where $M^2 = \sigma B_0^2 H^2 / \mu$ and $Re = \rho H V_0 / \mu$ are known as Hartman number and Reynolds number respectively. To solve Equations (5) and (6), the DTM method is employed.

3. Implementation of the Homotopy Analysis Method

For HAM solutions, we choose the initial guess and auxiliary linear operator in the following form:

$$f_0(y^*) = -\frac{1}{2}y^{*3} + \frac{3}{4}y^*, \tag{3.1}$$

$$L(f) = f'''', \tag{3.2}$$

$$L\left(\frac{1}{6}c_1 y^{*3} + \frac{1}{2}c_2 y^{*2} + c_3 y^* + c_4 = 0\right), \tag{3.3}$$

Where $c_i(i = 1, 2, 3, 4)$ are constants. Let $P \in [0, 1]$ denotes the embedding parameter and $\hbar$ indicates non-zero auxiliary parameters. We then construct the following equations:
Zeroth-order deformation equations

$$(1 - P)L\left[F(y^*; p) - f_0(y^*)\right] = ph H(y^*) N\left[F(y^*; p)\right] \tag{3.4}$$

$$F(0; p) = 1; \quad F''(0, p) = 0 \quad F(\frac{1}{2}; p) = \frac{1}{2}, \quad F'(\frac{1}{2}; p) = 0 \tag{3.5}$$

$$N[F(y^*; p)] = \frac{d^4 F(y^*; p)}{dy^{*4}} + Re\left[\frac{dF(y^*; p)}{dy^*}\frac{d^2 F(y^*; p)}{dy^{*2}} - F(y^*; p)\frac{d^3 F(y^*; p)}{dy^{*3}}\right]$$
$$\tag{3.6}$$

$$- M^2 \frac{d^2 F(y^*; p)}{dy^{*2}}$$

For $p = 0$ and $p = 1$ we have

$$F(y^*; 0) = f_0(y^*) \quad F(y^*; 1) = f(y^*) \tag{3.7}$$

When p increases from 0 to 1 then $F(y^*; p)$ varies from $f_0(y^*)$ to $f(y^*)$. By Taylor's theorem and using equation (13), $F(y^*; p)$ can be expanded in a power series of p as follows:

$$F(y^*; p) = f_0(y^*) + \sum_{m-1}^{\infty} f_m(y^*)p^m \quad , \quad f_m(y^*) = \frac{1}{m!} \frac{\partial^m (F(y^*; p))}{\partial p^m} \bigg|_{p=0} \quad (3.8)$$

In which $\hbar$ is chosen in such a way that this series is convergent at $p = 1$, therefore we have through equation (14) that

$$f(y^*) = f_0(y^*) + \sum_{m-1}^{\infty} f_m(y^*), \quad (3.9)$$

mth -order deformation equations

$$L\left[f_m(y^*) - \chi_m f_{m-1}(y^*)\right] = hH(y^*)R_m(y^*) \quad (3.10)$$

$$F_m(0; p) = 0; \qquad F_m''(0; p) = 0, \qquad F_m(\tfrac{1}{2}; p) = 0, \qquad F_m'(\tfrac{1}{2}; p) = 0 \quad (3.11)$$

$$R_m(y^*) = f_{m-1}'''' + \sum_{k=0}^{m-1} \left[Re\left(f_{m-1-k}'f_k'' - f_{m-1-k}f_k'''\right)\right] - M^2 f_{m-1}'' \quad (3.12)$$

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \quad (3.13)$$

Now we determine the convergency of the result, the differential equation, and the auxiliary function according to the solution expression. So let us assume:

$$H(y^*) = 1 \quad (3.14)$$

We have found the answer by maple analytic solution device. The first deformation is presented below

$$f_1(y^*) = \frac{2}{35}hRey^{*7} + \frac{1}{10}hM^2y^{*5} + \left(-\frac{3}{280}hRe - \frac{1}{20}hM^2\right)y^{*3}$$
$$+ \left(\frac{1}{560}hRe + \frac{1}{160}hM^2\right)y^* \quad (3.15)$$

The solutions $f(y^*)$ were too long to be mentioned here, therefore, they are shown graphically

4. Convergence of the HAM solution

As pointed out by Liao [11], the convergence region and rate of solution series can be adjusted and controlled by means of the auxiliary parameter $\hbar$. To influence of $\hbar$ on the convergence of solution, we plot the so-called $\hbar$-curve of $f'''(0)$, as shown in Figures. 2 (a-c). The solutions converge for $\hbar$ values which are corresponding to the horizontal line segment in $\hbar$ curve. In order to investigate the range of
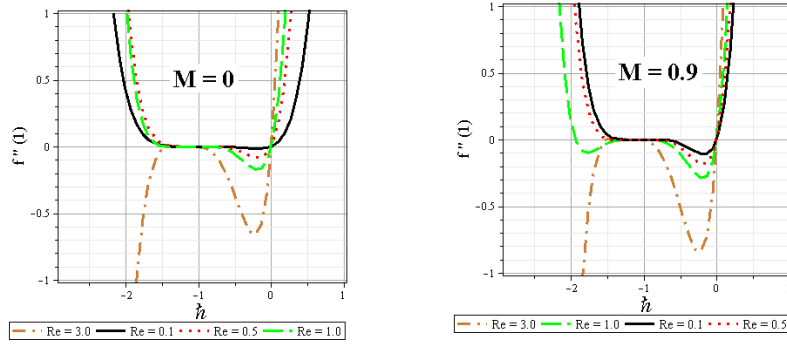
(a) The $\hbar$- validity for $M = 0$ and different value of Re

(b) The $\hbar$- validity for $M = 0.9$ and different value of Re
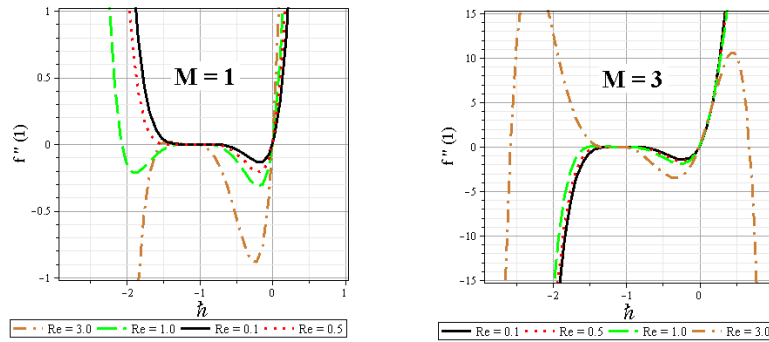
(c) The $\hbar$- validity for $M = 1$ and different value of Re

(d) The $\hbar$- validity for $M = 3$ and different value of Re

Figure 2: The $\hbar$- validity for different value of M and Re.

admissible values of the auxiliary parameter $\hbar$, for various quantities of Re and M, the curves of $\hbar$ were derived 9th-order approximations. Figures 2- 4 shows obtained admissible values for auxiliary parameter $\hbar$. In our case study, it is easy to discover that $h = -1.5$ is suitable value which is used for values of $0.1 < M < 0.9$ and $-5 < Re\omega < 5$.

5. Result and discussion

In the present study HAM method is applied to obtain an explicit analytic solution of compressible fluid in a channel under the presence of uniform magnetic field (Figure. 1). First, a comparison between the applied methods and numerical method for different values of active parameters is shown in Figures. 2. The numerical solution is performed using the algebra package Maple 16.0, to solve the
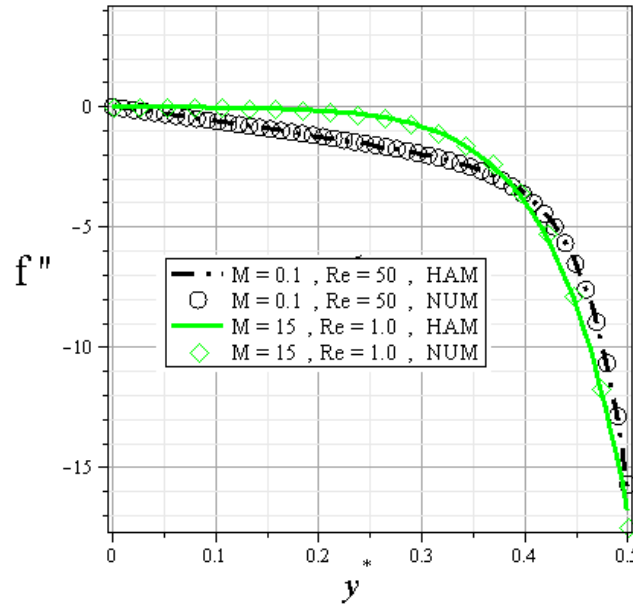
present case. The package uses a fourth order Runge-Kutta procedure for solving nonlinear boundary value (B-V) problem [18,19]. Validity of HAM is shown in Table 1 and Table 2. In these tables, the %Error is defined as:

$$\%Error = |f(y)_{NUM} - f(y)_{HAM}| \qquad (5.1)$$

The results are proved to be precise and accurate in solving a wide range of mathematical and engineering problems especially Fluid mechanic cases. This accuracy gives high confidence to us about validity of this problem and reveals an excellent agreement of engineering accuracy. This investigation is completed by depicting the effects of some important parameters to evaluate how these parameters influence on this fluid.

In figures (3) to (6) the effects of Hartman number and Reynolds number on the velocity components f and $f\prime$ are investigated. From figures (3) and 5), it is observed that as the Reynolds number and Hartman number increase, the similarity function (f) decreases. In the figures (4) and (6), toward the center point from $y* = 0$ to the suction side as the Hartman number and Reynolds number grow, f' decreases, but then this parameter increases. Hence the profile of the velocity component in x direction will have a common point that approximately takes place in $y* = 0.25$. So the stated point can be interpreted as a critical point in the formation of x direction flow.

Table 1.
The results of HAM and Numerical methods for $f(y^*)$ and $f'(y^*)$ for *Re=0.1* and M=*0.1*.

y^*	$f(y^*)$			$f'(y^*)$		
	HAM	NUM	Error	HAM	NUM	Error
0.00	0.000000000	0.000000000	0.00000E+00	1.499791227	1.499758471	3.27563E-05
0.05	0.074739731	0.074738119	1.61138E-06	1.484801403	1.484770242	3.11613E-05
0.10	0.147980474	0.147977410	3.06405E-06	1.439831575	1.439805143	2.64319E-05
0.15	0.218223202	0.218218996	4.20557E-06	1.364880477	1.364861712	1.87652E-05
0.20	0.283968789	0.283963891	4.89867E-06	1.259945355	1.259936764	8.59148E-06
0.25	0.343717918	0.343712882	5.03637E-06	1.125020985	1.125024259	3.27346E-06
0.30	0.395970919	0.395966353	4.56646E-06	0.960098314	0.960113727	1.54132E-05
0.35	0.439227538	0.439224010	3.52820E-06	0.765162700	0.765188233	2.55329E-05
0.40	0.471986606	0.471984502	2.10463E-06	0.540191761	0.540221878	3.01169E-05
0.45	0.492745589	0.492744895	6.93445E-07	0.285152828	0.285176850	2.40216E-05
0.50	0.500000000	0.500000000	9.99201E-16	0.000000000	0.000000000	0.00000E+00

Table 2.
The results of HAM and Numerical methods for $f(y^*)$ and $f'(y^*)$ for *Re=0.9* and M=*0.1*.

y^*	$f(y^*)$			$f'(y^*)$		
	HAM	NUM	Error	HAM	NUM	Error
0.00	0.000000000	0.000000000	0.00000E+00	1.498539189	1.498301370	0.000237819
0.05	0.074678066	0.074666353	1.17122E-05	1.483605578	1.483378489	0.000227090
0.10	0.147862762	0.147840409	2.23529E-05	1.438804132	1.438609137	0.000194995
0.15	0.218060627	0.218029762	3.08643E-05	1.364131265	1.363989177	0.000142088
0.20	0.283777844	0.283741600	3.62438E-05	1.259575168	1.259504976	7.01926E-05
0.25	0.343519625	0.343481986	3.76397E-05	1.125107010	1.125123200	1.61897E-05
0.30	0.395789064	0.395754531	3.45331E-05	0.960668501	0.960776347	0.000107846
0.35	0.439085274	0.439058235	2.70393E-05	0.766155734	0.766343831	0.000188097
0.40	0.471900624	0.471884257	1.63671E-05	0.541399169	0.541628393	0.000229224
0.45	0.492716873	0.492711394	5.47845E-06	0.286139603	0.286327577	0.000187974
0.50	0.500000000	0.500000000	0.00000E+00	0.000000000	0.000000000	0.00000E+00

6. Conclusion

In this research, an analytic method for solving of the two-dimensional magnetohydrodynamics (MHD) boundary layer flow of compressible fluid has been presented. Differential equations were transformed to algebraic equations, using Homotopy Analysis Method (HAM). Then HAM is compared with Boundary Value Problem (BVP) method as a numerical solution. The effects of different Reynolds number and Hartman number were investigated for the similarity functions f, f' used to determine the velocity components. It was found from the results, as the Hartman number and Reynolds number changed a common point appeared in the profile of the velocity component in x direction. When the velocity injection increased, it was clear that the suction force assisted the structural formation of y direction flow. This research has been also proved that HAM includes of high accuracy to solve different problems in the engineering field.
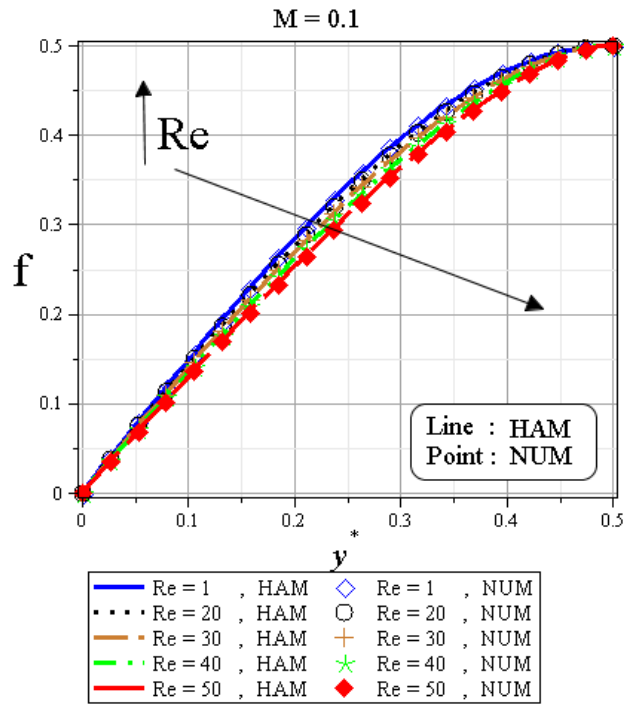
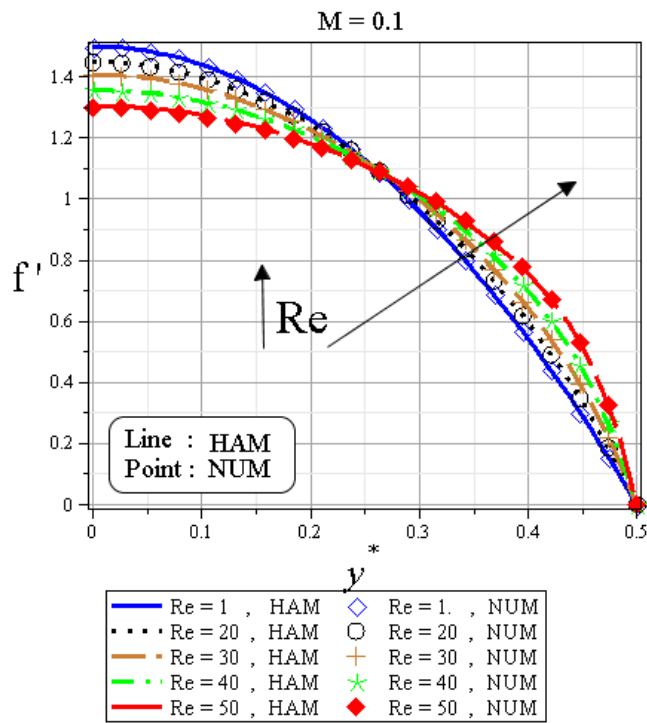Figure 3: Effects of the Reynolds number for $f(y*)$ on the 16th-order approximation $(M = 0.1)$

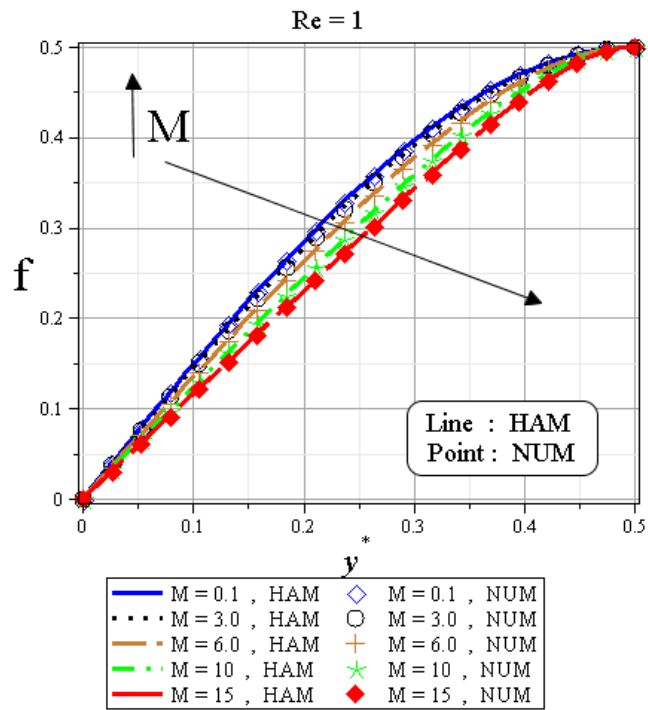Figure 4: Effects of the Reynolds number for $f'(y*)$ on the 16th-order approximation $(M = 2)$

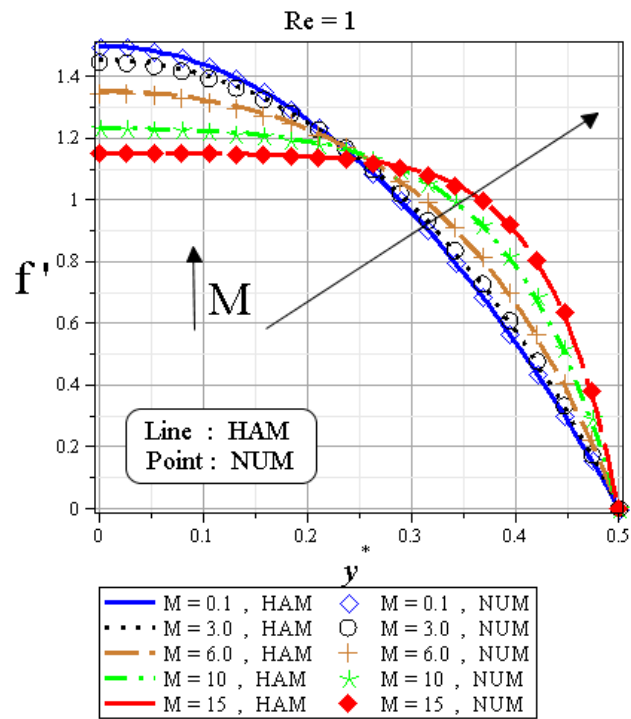Figure 5: Effects of the Hartman number for $f(y*)$ on the 15th-order approximation $(Re = 1)$

Figure 6: Effects of the Hartman number for $f'(y*)$ on the 15th-order approxima-
tion $(Re = 1)$

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