Classification of simple linearly compact n-Lie superalgebras

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Abstract

We classify simple linearly compact n-Lie superalgebras with $n > 2$ over a field $\mathbb{F}$ of characteristic 0. The classification is based on a bijective correspondence between non-abelian n-Lie superalgebras and transitive $\mathbb{Z}$-graded Lie superalgebras of the form $L = \oplus_{j=-1}^{n-1} L_j$, where $\dim L_{n-1} = 1$, L_{-1} and L_{n-1} generate L, and $[L_j, L_{n-j-1}] = 0$ for all j, thereby reducing it to the known classification of simple linearly compact Lie superalgebras and their $\mathbb{Z}$-gradings. The list consists of four examples, one of them being the $n+1$-dimensional vector product n-Lie algebra, and the remaining three infinite-dimensional n-Lie algebras.

Introduction

Given an integer $n \geq 2$, an n-Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$, endowed with an n-ary anti-commutative product

$$\Lambda^n \mathfrak{g} \to \mathfrak{g}, \quad a_1 \wedge \ldots \wedge a_n \mapsto [a_1, \ldots, a_n],$$

subject to the following Filippov-Jacobi identity:

$$(0.1) \quad \begin{aligned} [a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]] = [[a_1, \ldots, a_{n-1}, b_1], b_2, \ldots, b_n] + [b_1, [a_1, \ldots, a_{n-1}, b_2], b_3, \ldots, b_n] \\ + \ldots + [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, b_n]]. \end{aligned}$$

The meaning of this identity is similar to that of the usual Jacobi identity for a Lie algebra (which is a 2-Lie algebra), namely, given $a_1, \ldots, a_{n-1} \in \mathfrak{g}$, the map $D_{a_1,\ldots,a_{n-1}} : \mathfrak{g} \mapsto \mathfrak{g}$, given by $D_{a_1,\ldots a_{n-1}}(a) = [a_1, \ldots, a_{n-1}, a]$, is a derivation of the n-ary bracket. These derivations are called inner.

The notion of an n-Lie algebra was introduced by V.T. Filippov in 1985 [11]. In this and several subsequent papers, [12], [18], [19], [20], a structure theory of finite-dimensional n-Lie algebras over a field $\mathbb{F}$ of characteristic 0 was developed. In particular, W.X. Ling in [20] discovered the following disappointing feature of n-Lie algebras for $n > 2$: there exists only one simple finite-dimensional n-Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. It is given by the vector product of n vectors in the $n+1$-dimensional vector space V, endowed with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Recall that, choosing dual bases $\{a_i\}$ and $\{a^i\}$ of V, i.e., $(a_i, a^j) = \delta_{ij}, i, j = 1, ..., n+1$, the vector product of n vectors from the basis $\{a_i\}$ is defined as the following n-ary bracket:

$$[a_{i_1}, \ldots, a_{i_n}] = \epsilon_{i_1,\ldots,i_{n+1}} a^{i_{n+1}},$$

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where $\epsilon_{i_i,\ldots,i_{n+1}}$ is a non-zero totally antisymmetric tensor with values in $\mathbb{F}$, and extended by n-linearity. This is a simple n-Lie algebra, which is called the vector product n-Lie algebra; we denote it by O^n.

Another example of an n-Lie algebra appeared earlier in Nambu's generalization of Hamiltonian dynamics [23]. It is the space $\mathcal{C}^\infty(M)$ of $\mathcal{C}^\infty$-functions on a finite-dimensional manifold M, endowed with the following n-ary bracket, associated to n commuting vector fields $D_1, \ldots, D_n$ on M:

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} D_1(f_1) & \ldots & D_1(f_n) \\ \ldots & \ldots & \ldots \\ D_n(f_1) & \ldots & D_n(f_n) \end{pmatrix}. \tag{0.2}$$

The fact that this n-ary bracket satisfies the Filippov-Jacobi identity was noticed later by Filippov (who was unaware of Nambu's work), and by Takhtajan [25], who introduced the notion of an n-Poisson algebra (and was unaware of Filippov's work).

A more recent important example of an n-Lie algebra structure on $\mathcal{C}^\infty(M)$, given by Dzhumadildaev [6], is associated to $n-1$ commuting vector fields $D_1, \ldots, D_{n-1}$ on M:

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} f_1 & \ldots & f_n \\ D_1(f_1) & \ldots & D_1(f_n) \\ \ldots & \ldots & \ldots \\ D_{n-1}(f_1) & \ldots & D_{n-1}(f_n) \end{pmatrix}. \tag{0.3}$$

In fact, Dzhumadildaev considered examples (0.2) and (0.3) in a more general context, where $\mathcal{C}^\infty(M)$ is replaced by an arbitrary commutative associative algebra A over $\mathbb{F}$ and the D_i by derivations of A. He showed in [9] that (0.2) and (0.3) satisfy the Filippov-Jacobi identity if and only if the vector space $\sum_i \mathbb{F}D_i$ is closed under the Lie bracket.

In the past few years there has been some interest in n-Lie algebras in the physics community, related to M-branes in string theory. We shall quote here two sources — a survey paper [13], containing a rather extensive list of references, and a paper by Friedmann [14], where simple finite-dimensional 3-Lie algebras over $\mathbb{C}$ were classified (she was unaware of the earlier work). At the same time we (the authors of the present paper) have been completing our work [5] on simple rigid linearly compact superalgebras, and it occurred to us that the method of this work also applies to the classification of simple linearly compact n-Lie superalgebras!

Our main result can be stated as follows.

Theorem 0.1 *(a) Any simple linearly compact n-Lie algebra with $n > 2$, over an algebraically closed field $\mathbb{F}$ of characteristic 0, is isomorphic to one of the following four examples:*

(i) the $n+1$-dimensional vector product n-Lie algebra O^n;

(ii) the n-Lie algebra, denoted by S^n, which is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \ldots, x_n]]$, endowed with the n-ary bracket (0.2), where $D_i = \frac{\partial}{\partial x_i}$;

(iii) the n-Lie algebra, denoted by W^n, which is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \ldots, x_{n-1}]]$, endowed with the n-ary bracket (0.3), where $D_i = \frac{\partial}{\partial x_i}$;

(iv) the n-Lie algebra, denoted by SW^n, which is the direct sum of $n-1$ copies of $\mathbb{F}[[x]]$, endowed

with the following n-ary bracket, where $f^{\langle j\rangle}$ is an element of the j^{th} copy and $f' = \frac{df}{dx}$:

$$[f_1^{\langle j_1\rangle}, \dots f_n^{\langle j_n\rangle}] = 0, \quad \textit{unless} \ \ \{j_1, \dots, j_n\} \supset \{1, \dots, n-1\},$$
$$[f_1^{\langle 1\rangle}, \dots, f_{k-1}^{\langle k-1\rangle}, f_k^{\langle k\rangle}, f_{k+1}^{\langle k\rangle}, f_{k+2}^{\langle k+1\rangle}, \dots, f_n^{\langle n-1\rangle}]$$
$$= (-1)^{k+n}(f_1 \dots f_{k-1}(f'_k f_{k+1} - f'_{k+1} f_k) f_{k+2} \dots f_n)^{\langle k\rangle}.$$

(b) There are no simple linearly compact n-Lie superalgebras over $\mathbb{F}$, which are not n-Lie algebras, if $n > 2$.

Recall that a linearly compact algebra is a topological algebra, whose underlying vector space is linearly compact, namely is a topological product of finite-dimensional vector spaces, endowed with discrete topology (and it is assumed that the algebra product is continuous in this topology). In particular, any finite-dimensional algebra is automatically linearly compact. The basic example of an infinite-dimensional linearly compact space is the space of formal power series $\mathbb{F}[[x_1, \dots, x_k]]$, endowed with the formal topology, or a direct sum of a finite number of such spaces.

The proof of Theorem 0.1 is based on a construction, which associates to an n-Lie (super)algebra $\mathfrak{g}$ a pair (Lie $\mathfrak{g}$, μ), where Lie $\mathfrak{g} = \prod_{j\geq -1}$ Lie ${}_j\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie superalgebra of depth 1 and $\mu \in$ Lie ${}_{n-1}\mathfrak{g}$, such that the following properties hold:

(L1) Lie $\mathfrak{g}$ is transitive, i.e., if $a \in$ Lie ${}_j\mathfrak{g}$ with $j \geq 0$ and $[a,$ Lie ${}_{-1}\mathfrak{g}] = 0$, then $a = 0$;

(L2) Lie $\mathfrak{g}$ is generated by Lie ${}_{-1}\mathfrak{g}$ and μ;

(L3) $[\mu,$ Lie ${}_0\mathfrak{g}] = 0$.

A pair (L, μ), where $L = \prod_{j\geq -1} L_j$ is a transitive $\mathbb{Z}$-graded Lie superalgebra and $\mu \in L_{n-1}$, such that (L2) and (L3) hold, is called *admissible*.

The construction of the admissible pair (Lie $\mathfrak{g}, \mu$), associated to an n-Lie (super)algebra $\mathfrak{g}$, uses the universal $\mathbb{Z}$-graded Lie superalgebra $W(V) = \prod_{j\geq -1} W_j(V)$, associated to a vector superspace V (see Section 1 for details). One has $W_j(V) =$ Hom $(S^{j+1}V, V)$, so that an element $\mu \in W_{n-1}(V)$ defines a commutative n-superalgebra structure on V and vice versa. *Universality* means that any transitive $\mathbb{Z}$-graded Lie superalgebra $L = \prod_{j\geq -1} L_j$ with $L_{-1} = V$ canonically embeds in $W(V)$ (the embedding being given by $L_j \ni a \mapsto \varphi_a \in W_j(V)$, where $\varphi_a(a_1, \dots, a_{j+1}) = [\dots[a, a_1], \dots, a_{j+1}]$).

So, given a commutative n-ary product on a superspace V, we get an element $\mu \in W_{n-1}(V)$, and we denote by Lie V the $\mathbb{Z}$-graded subalgebra of $W(V)$, generated by $W_{-1}(V)$ and μ. The pair (Lie V, μ) obviously satisfies properties (L1) and (L2).

How do we pass from commutative to anti-commutative n-superalgebras? Given a commutative n-superalgebra V with n-ary product $(a_1, \dots, a_n)$, the vector superspace ΠV (Π stands, as usual, for reversing the parity) becomes an anti-commutative n-superalgebra with n-ary product

$$[a_1, \dots, a_n] = p(a_1, \dots, a_n)(a_1, \dots, a_n), \tag{0.4}$$

where

$$p(a_1, \dots, a_n) = \begin{cases} (-1)^{p(a_1)+p(a_3)+\dots+p(a_{n-1})} & \text{if } \ n \text{ is even} \\ (-1)^{p(a_2)+p(a_4)+\dots+p(a_{n-1})} & \text{if } \ n \text{ is odd}, \end{cases} \tag{0.5}$$

and vice versa.

Thus, given an anti-commutative n-superalgebra $\mathfrak{g}$, with n-ary product $[a_1, \dots, a_n]$, we consider the vector superspace $\Pi\mathfrak{g}$ with commutative n-ary product $(a_1, \dots, a_n)$, given by (0.4), consider the element $\mu \in W_{n-1}(\Pi\mathfrak{g})$, corresponding to the latter n-ary product, and let $\mathrm{Lie}\ \mathfrak{g}$ be the graded subalgebra of $W(\Pi\mathfrak{g})$, generated by $W_{-1}(\Pi\mathfrak{g})$ and μ.

Note that properties (L1) and (L2) of the pair $(\mathrm{Lie}\ \mathfrak{g}, \mu)$ still hold, and it remains to note that property (L3) is equivalent to the (super analogue of the) Filippov-Jacobi identity. Finally, the simplicity of the n-Lie (super)algebra $\mathfrak{g}$ is equivalent to

(L4) the $\mathrm{Lie}\ _0\mathfrak{g}$-module $\mathrm{Lie}\ _{-1}\mathfrak{g}$ is irreducible.

An admissible pair, satisfying property (L4) is called irreducible. Thus, the proof of Theorem 0.1 reduces to the classification of all irreducible admissible pairs (L, μ), where L is a linearly compact Lie superalgebra. It is not difficult to show, as in [5], that $S \subseteq L \subseteq \mathrm{Der}\ S$, where S is a simple linearly compact Lie superalgebra and $\mathrm{Der}\ S$ is the Lie superalgebra of its continuous derivations (it is at this point that the condition $n > 2$ is essential).

Up to now the arguments worked over an arbitrary field $\mathbb{F}$. In the case $\mathbb{F}$ is algebraically closed of characteristic 0, there is a complete classification of simple linearly compact Lie superalgebras, their derivations and their $\mathbb{Z}$-gradings [16], [17], [2], [3]. Applying these classifications completes the proof of Theorem 0.1

For example, if $\mathfrak{g}$ is a finite-dimensional simple n-Lie superalgebra, then $\dim\ \mathrm{Lie}\ \mathfrak{g} < \infty$, and from [16] we see that the only possibility for $\mathrm{Lie}\ \mathfrak{g}$ is $L = P/\mathbb{F}1$, where P is the Lie superalgebra defined by the super Poisson bracket on the Grassmann algebra in the indeterminates $\xi_1, \dots, \xi_{n+1}$, given by

$$\{\xi_i, \xi_j\} = b_{ij}\ ,\ \ i, j = 1, \dots, n+1. \tag{0.6}$$

where (b_{ij}) is a non-degenerate symmetric matrix, the $\mathbb{Z}$-grading on L being given by $\deg(\xi_{i_1} \dots \xi_{i_s}) = s - 2$, and $\mu = \xi_1\xi_2 \dots \xi_{n+1}$. We conclude that $\mathfrak{g}$ is the vector product n-Lie algebra. (The proof of this result in the non-super case, obtained by Ling [20], is based on the study of the linear Lie algebra spanned by the derivations $D_{a_1,\dots,a_{n-1}}$, and is applicable neither in the super nor in the infinite-dimensional case.) We have no *a priori* proof of part (b) of Theorem 0.1 — it comes out only after the classification process.

If $\mathrm{char}\ \mathbb{F} = 0$, we have a more precise result on the structure of an admissible pair (L, μ).

Theorem 0.2 *If $\mathrm{char}\ \mathbb{F} = 0$ and (L, μ) is an admissible pair, then $L = \oplus_{j=-1}^{n-1} L_j$, where $L_{n-1} = \mathbb{F}\mu$, $S := \oplus_{j=-1}^{n-2} L_j$ is an ideal in L, and $L_j = (\mathrm{ad}\ L_{-1})^{n-j-1}\mu$ for $j = 0, \dots, n-1$.*

Of course, Theorem 0.2 reduces significantly the case wise inspection in the proof of Theorem 0.1. Moreover, Theorem 0.2 can also be used in the study of representations of n-Lie algebras. Namely a representation of an n-Lie algebra $\mathfrak{g}$ in a vector space M corresponds to an n-Lie algebra structure on the semidirect product $L_{-1} = \mathfrak{g} \ltimes M$, where M is an abelian ideal. Hence, by Theorem 0.2, we obtain a graded representation of the Lie superalgebra $S = \oplus_{i=-1}^{n-2} L_i$ in the graded supervector space $L(M) = \sum_{i=-1}^{n-2} M_j$, $M_j = (\mathrm{ad}\ M)^{n-j-1}\mu$ so that $L_i M_j \subset M_{i+j}$. Such "degenerate" representations of the Lie superalgebra S are not difficult to classify, and this corresponds to a classification of representations of the n-Lie algebra $\mathfrak{g}$. In particular, representations of the n-Lie algebra O^n correspond to "degenerate" representations of the simple Lie superalgebra $H(0, n)$

(finite-dimensional representations of O^n were classified in [7], using Ling's method, mentioned above).

Finally, note that, using our discussion on $\mathbb{F}$-forms of simple linearly compact Lie superalgebras in [3], we can extend Theorem 0.1 to the case of an arbitrary field $\mathbb{F}$ of characteristic 0. The result is almost the same, namely the $\mathbb{F}$-forms are as follows: O^n, depending on the equivalence class of the symmetric bilinear form up to a non-zero factor, and the n-Lie algebras S^n, W^n and SW^n over $\mathbb{F}$.

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1 Preliminaries on n-superalgebras

Let V be a vector superspace over a field $\mathbb{F}$, namely we have a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$ in a direct sum of subspaces, where $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is called the subspace of even (resp. odd) elements; if $v \in V_\alpha$, $\alpha \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ we write $p(v) = \alpha$. Given two vector superspaces U and V, the space $\mathrm{Hom}\,(U, V)$ is naturally a vector superspace, for which even (resp. odd) elements are parity preserving (resp. reversing) maps; also $U \otimes V$ is a vector superspace via letting $p(a \otimes b) = p(a) + p(b)$ for $a \in U$, $b \in V$.

In particular, the tensor algebra $T(V) = \oplus_{j \in \mathbb{Z}_+} T^j(V)$ is an associative superalgebra. The symmetric (resp. exterior) superalgebra over V is the quotient of the superalgebra $T(V)$ by the 2-sided ideal, generated by the elements $a \otimes b - (-1)^{p(a)p(b)} b \otimes a$ (resp. $a \otimes b + (-1)^{p(a)p(b)} b \otimes a$), where $a, b \in V$. They are denoted by $S(V)$ and $\Lambda(V)$ respectively. Both inherit a $\mathbb{Z}$-grading from $T(V)$: $S(V) = \oplus_{j \in \mathbb{Z}_+} S^j$, $\Lambda(V) = \oplus_{j \in \mathbb{Z}_+} \Lambda^j(V)$.

A well known trivial, but important, observation is that the reversal of parity of V, i.e., taking the superspace ΠV, where $(\Pi V)_\alpha = V_{\alpha + \bar{1}}$, establishes a canonical isomorphism:

$$S(\Pi V) \simeq \Lambda(V)\,. \tag{1.1}$$

Definition 1.1 *Let $n \in \mathbb{Z}_+$ and let V be a vector superspace. An n-superalgebra structure (or n-ary product) on V of parity $\alpha \in \mathbb{Z}/2\mathbb{Z}$ is a linear map $\mu : T^n(V) \to V$ of parity α. A commutative (resp. anti-commutative) n-superalgebra of parity α is a linear map $\mu : S^n(V) \to V$ (resp. $\Lambda^n V \to V$) of parity α, denoted by $\mu(a_1 \otimes \cdots \otimes a_n) = (a_1, \ldots, a_n)$ (resp. $= [a_1, \ldots, a_n]$).*

Lemma 1.2 *Let (V, μ) be an anti-commutative n-superalgebra. Then $(\Pi V, \bar{\mu})$ is a commutative n-superalgebra (of parity $p(\mu) + n - 1$ mod 2) with the n-ary product*

$$\bar{\mu}(a_1 \otimes \cdots \otimes a_n) = p(a_1, \ldots, a_n)\mu(a_1 \otimes \ldots \otimes a_n), \tag{1.2}$$

where

$$p(a_1, \ldots, a_n) = (-1)^{\sum_{k=0}^{[\frac{n-2}{2}]} p(a_{n-1-2k})}, \tag{1.3}$$

and vice versa.

Proof. Denote by p' the parity in ΠV. Then, for $a_1, \dots, a_n \in V$, $p'(\bar{\mu}(a_1 \otimes \cdots \otimes a_n)) = p(\mu) + 1 + \sum_{i=1}^n p(a_i) = \sum_{i=1}^n p'(a_i) + p(\mu) + n + 1 \mod 2$, i.e., $p'(\bar{\mu}) = p(\mu) + n - 1 \mod 2$. Besides, we have $p(a_1, \dots, a_i, a_{i+1}, \dots, a_n) p(a_1, \dots, a_{i+1}, a_i, \dots, a_n) = (-1)^{p(a_i)+p(a_{i+1})}$, hence:

$$\bar{\mu}(a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) = p(a_1, \dots, a_n) \mu(a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n)$$

$$= -(-1)^{p(a_i)p(a_{i+1})} p(a_1, \dots, a_n) \mu(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n)$$

$$= -(-1)^{p(a_i)p(a_{i+1})} p(a_1, \dots, a_n) p(a_1, \dots, a_{i+1}, a_i, \dots, a_n) \bar{\mu}(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n)$$

$$= (-1)^{p'(a_i)p'(a_{i+1})} \bar{\mu}(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n).$$

□

Definition 1.3 *A derivation D of parity $\alpha \in \mathbb{Z}/2\mathbb{Z}$ of an n-superalgebra (V, μ) is an endomorphism of the vector superspace V of parity α, such that:*

$$D(\mu(a_1 \otimes \cdots \otimes a_n)) = (-1)^{\alpha p(\mu)} (\mu(Da_1 \otimes a_2 \otimes \cdots \otimes a_n) + (-1)^{\alpha p(a_1)} \mu(a_1 \otimes Da_2 \otimes \cdots \otimes a_n) + \cdots$$
$$+ (-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}))} \mu(a_1 \otimes \cdots \otimes D(a_n)))\,.$$

It is clear that derivations of an n-superalgebra V form a Lie superalgebra, which is denoted by $\mathrm{Der}\, V$. It is not difficult to show that all inner derivations of an n-Lie algebra $\mathfrak{g}$ span an ideal of $\mathrm{Der}\, \mathfrak{g}$ (see e.g. [7]), denoted by $\mathrm{Inder}\, \mathfrak{g}$.

Now we recall the construction of the universal Lie superalgebra $W(V)$, associated to the vector superspace V. For an integer $k \geq -1$ let $W_k(V) = \mathrm{Hom}\, (S^{k+1}(V), V)$, in other words , $W_k(V)$ is the vector superspace of all commutative $k+1$-superalgebra structures on V, in particular, $W_{-1}(V) = V$, $W_0(V) = \mathrm{End}\, (V)$, $W_1(V)$ is the space of all commutative superalgebra structures on V, etc. We endow the vector superspace

$$W(V) = \prod_{k=-1}^{\infty} W_k(V)$$

with a product $f \Box g$, making $W(V)$ a $\mathbb{Z}$-graded superalgebra, given by the following formula for $f \in W_p(V)$, $g \in W_q(V)$:
(1.4)
$$f \Box g(x_0, \dots, x_{p+q}) = \sum_{\substack{i_0 < \cdots < i_q \\ i_{q+1} < \cdots < i_{p+q}}} \epsilon(i_0, \dots, i_q, i_{q+1}, \dots, i_{p+q}) f(g(x_{i_0}, \dots, x_{i_q}), x_{i_{q+1}}, \dots, x_{i_{p+q}}),$$

where $\epsilon = (-1)^N$, N being the number of interchanges of indices of odd x_i's in the permutation $\sigma(s) = i_s$, $s = 0, 1, \dots, p+q$. Then the bracket

$$[f, g] = f \Box g - (-1)^{p(f)p(q)} g \Box f \qquad (1.5)$$

defines a Lie superalgebra structure on $W(V)$.

Lemma 1.4 *Let V be a vector superspace and let $\mu \in W_{n-1}(V)$, $D \in W_0(V)$. Then*

(a) $[\mu, D] = 0$ *if and only if D is a derivation of the n-superalgebra (V, μ).*

(b) $\quad$ *D is a derivation of parity α of the commutative n-superalgebra (V, μ) if and only if D is a derivation of parity α of the anti-commutative n- superalgebra $(\Pi V, \bar{\mu})$, where*

$$\bar{\mu}(a_1 \otimes \cdots \otimes a_n) = p(a_1, \ldots, a_n)\mu(a_1 \otimes \cdots \otimes a_n)\,,$$

and $p(a_1, \ldots, a_n)$ is defined by (1.3).

Proof. By (1.5) and (1.4), we have:

$$[\mu, D](b_1 \otimes \cdots \otimes b_n) = (\mu \square D)(b_1 \otimes \cdots \otimes b_n) - (-1)^{\alpha p(\mu)}(D \square \mu)(b_1 \otimes \cdots \otimes b_n) =$$

$$\sum_{\substack{i_1 \\ i_2 < \cdots < i_n}} (\varepsilon(i_1, \ldots, i_n)\mu(D(b_{i_1}) \otimes b_{i_2} \otimes \cdots \otimes b_{i_n})) - (-1)^{\alpha p(\mu)} D(\mu(b_1 \otimes \cdots \otimes b_n)),$$

where α is the parity of D. Therefore $[\mu, D] = 0$ if and only if

$$D(\mu(b_1 \otimes \cdots \otimes b_n)) = (-1)^{\alpha p(\mu)} \sum_{\substack{i_1 \\ i_2 < \cdots < i_n}} (\varepsilon(i_1, \ldots, i_n)\mu(D(b_{i_1}) \otimes b_{i_2} \otimes \cdots \otimes b_{i_n}))$$

$$= (-1)^{\alpha p(\mu)}(\mu(D(b_1) \otimes b_2 \otimes \cdots \otimes b_{n-1}) + (-1)^{\alpha p(b_1)}\mu(b_1 \otimes D(b_2) \otimes \cdots \otimes b_{n-1}) + \ldots$$
$$+ (-1)^{\alpha(p(b_1) + \cdots + p(b_{n-1}))}\mu(b_1 \otimes \cdots \otimes b_{n-1} \otimes D(b_n))$$

i.e., if and only if D is a derivation of (V, μ) of parity α, proving (a).

In order to prove (b), note that D is a derivation of parity α of (V, μ) if and only if

$$D(\bar{\mu}(a_1 \otimes \cdots \otimes a_n) = p(a_1, \ldots, a_n)D(\mu(a_1 \otimes \cdots \otimes a_n)) = p(a_1, \ldots, a_n)((-1)^{\alpha p(\mu)}(\mu(D(a_1) \otimes a_2 \otimes \ldots a_n)$$
$$+ (-1)^{\alpha p(a_1)}\mu(a_1 \otimes D(a_2) \otimes \ldots a_n) + \cdots + (-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}))}\mu(a_1 \otimes \cdots \otimes D(a_n)))$$
$$= p(a_1, \ldots, a_n)(-1)^{\alpha p(\mu)} p(D(a_1), a_2 \ldots, a_n)(\bar{\mu}(D(a_1) \otimes a_2 \cdots \otimes a_n)$$
$$+ (-1)^{\alpha p(a_1)} p(D(a_1), a_2, \ldots, a_n) p(a_1, D(a_2), \ldots, a_n)\bar{\mu}(a_1 \otimes D(a_2) \cdots \otimes a_n) + \ldots$$
$$+ (-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}))} p(D(a_1), a_2, \ldots, a_n) p(a_1, \ldots, D(a_n))\bar{\mu}(a_1 \otimes \cdots \otimes D(a_n)))$$

If n is even, we have:

$$\begin{gathered} p(a_1, \ldots, a_n)(-1)^{\alpha p(\mu)} p(D(a_1), a_2 \ldots, a_n) = (-1)^{\alpha(p(\mu)+1)}, \\ (-1)^{\alpha p(a_1)} p(D(a_1), a_2 \ldots, a_n) p(a_1, D(a_2), \ldots, a_n) = (-1)^{\alpha(p(a_1)+1)}, \\ \vdots \\ (-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}))} p(D(a_1), a_2 \ldots, a_n) p(a_1, \ldots, D(a_n)) = (-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}) + 1)} \end{gathered}$$

If n is odd, we have:

$$\begin{gathered} p(a_1, \ldots, a_n)(-1)^{\alpha p(\mu)} p(D(a_1), a_2 \ldots, a_n) = (-1)^{\alpha p(\mu)}, \\ (-1)^{\alpha p(a_1)} p(D(a_1), a_2 \ldots, a_n) p(a_1, D(a_2), \ldots, a_n) = (-1)^{\alpha(p(a_1)+1)}, \\ \vdots \\ (-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}))} p(D(a_1), a_2 \ldots, a_n) p(a_1, \ldots, D(a_n))(-1)^{\alpha(p(a_1) + \cdots + p(a_{n-1}))}. \end{gathered}$$

Since $\bar{\mu}$ has parity equal to $p(\mu) + n - 1 \bmod 2$, (b) is proved. $\square$

2 The main construction

Let $(\mathfrak{g}, \mu)$ be an anti-commutative n-superalgebra over a field $\mathbb{F}$ with n-ary product $[a_1, \ldots, a_n]$, and let $V = \Pi\mathfrak{g}$. Consider the universal Lie superalgebra $W(V) = \prod_{k=-1}^{\infty} W_k(V)$, and let $\bar{\mu} \in W_{n-1}(V)$ be the element defined by (1.2). Let $\text{Lie}\,\mathfrak{g} = \prod_{j=-1}^{\infty} \text{Lie}\,_j\mathfrak{g}$ be the $\mathbb{Z}$-graded subalgebra of the Lie superalgebra $W(V)$, generated by $W_{-1}(V) = V$ and $\bar{\mu}$.

Lemma 2.1 *(a) Lie $\mathfrak{g}$ is a transitive subalgebra of $W(V)$.*

(b) If $D \in$ Lie $_0\mathfrak{g}$, then the action of D on Lie $_{-1}\mathfrak{g} = V(= \Pi\mathfrak{g})$ is a derivation of the n-superalgebra $\mathfrak{g}$ if and only if $[D, \bar{\mu}] = 0$.

(c) Lie $_0\mathfrak{g}$ is generated by elements of the form

$$(\text{ad}\, a_1) \ldots (\text{ad}\, a_{n-1})\bar{\mu}\,, \text{ where } a_i \in \text{Lie}\,_{-1}\mathfrak{g} = V\,. \tag{2.1}$$

Proof. (a) is clear since $W(V)$ is transitive and the latter holds since, for $f \in W_k(V)$ and $a, a_1, \ldots, a_k \in W_{-1}(V) = V$ one has:

$$[f, a](a_1, \ldots, a_k) = f(a, a_1, \ldots, a_k)\,.$$

(b) follows from Lemma 1.4.

In order to prove (c) let $\tilde{L}_{-1} = V$ and let $\tilde{L}_0$ be the subalgebra of the Lie algebra $W_0(V)$, generated by elements (2.1). Let $\prod_{j\geq -1} \tilde{L}_j$ be the full prolongation of $\tilde{L}_{-1} \oplus \tilde{L}_0$, i.e., $\tilde{L}_j = \{a \in W_j(V) | [a, \tilde{L}_{-1}] \subset \tilde{L}_{j-1}\}$ for $j \geq 1$. This is a subalgebra of $W(V)$, containing V and $\bar{\mu}$, hence Lie $\mathfrak{g}$. This proves (c). □

Definition 2.2 *An n-Lie superalgebra is an anti-commutative n-superalgebra $\mathfrak{g}$ of parity α, such that all endomorphisms $D_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{g}$ ($a_1, \ldots a_{n-1} \in \mathfrak{g}$), defined by*

$$D_{a_1,\ldots a_{n-1}}(a) = [a_1, \ldots, a_{n-1}, a],$$

are derivations of $\mathfrak{g}$, i.e., the following Filippov-Jacobi identity holds:

$$\begin{aligned}[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]] = (-1)^{\alpha(p(a_1)+\cdots+p(a_{n-1}))}([[a_1, \ldots, a_{n-1}, b_1], b_2, \ldots, b_n]+\\ (-1)^{p(b_1)(p(a_1)+\cdots+p(a_{n-1}))}[b_1, [a_1, \ldots, a_{n-1}, b_2], b_3, \ldots, b_n]+\\ \cdots + (-1)^{(p(b_1)+\cdots+p(b_{n-1}))(p(a_1)+\cdots+p(a_{n-1}))}[b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, b_n]]).\end{aligned} \tag{2.2}$$

Recall from the introduction that the pair (L, μ), where $L = \prod_{j\geq -1} L_j$ is a $\mathbb{Z}$-graded Lie superalgebra and $\mu \in L_{n-1}$, is called admissible if properties (L1),(L2) and (L3) hold. Two admissible pairs (L, μ) and (L', μ') are called isomorphic if there exists a Lie superalgebra isomorphism $\phi : L \mapsto L'$, such that $\phi : L_j = L'_j$ for all j and $\phi(\mu) \in \mathbb{F}^\times \mu'$.

The following corollary of Lemma 2.1 is immediate.

Corollary 2.3 *If $\mathfrak{g}$ is an n-Lie superalgebra, then the pair (Lie $\mathfrak{g}, \bar{\mu}$) is admissible.*

Now it is easy to prove the following key result.

Proposition 2.4 *The map $\mathfrak{g} \mapsto (\text{ Lie } \mathfrak{g}, \bar{\mu})$ induces a bijection between isomorphism classes of n-Lie superalgebras, considered up to rescaling the n-ary bracket, and isomorphism classes of admissible pairs. Under this bijection, simple n-Lie algebras correspond to irreducible admissible pairs. Moreover, $\mathfrak{g}$ is linearly compact if and only if Lie $\mathfrak{g}$ is.*

Proof. Given an admissible pair (L, μ), where $L = \prod_{j \geq -1} L_j$, $\mu \in L_{n-1}$, we let $\mathfrak{g} = \Pi L_{-1}$, and define an n-ary bracket on $\mathfrak{g}$ by the formula

$$[a_1, \ldots, a_n] = p(a_1, \ldots, a_n)[...[\mu, a_1] \ldots, a_n],$$

where $p(a_1, \ldots, a_n)$ is given by (1.3). Obviously, this n-ary bracket is anti-commutative. The Filippov-Jacobi identity follows from the property (L3) using the embedding of L in $W(L_{-1})$ and applying Lemma 2.1(b). Thus, $\mathfrak{g}$ is an n-Lie superalgebra. Due to properties (L1) and (L2), we obtain the bijection of the map in question. It is obvious that $\mathfrak{g}$ is simple if and only if the pair (L, μ) is irreducible. The fact that the linear compactness of $\mathfrak{g}$ implies that of Lie $\mathfrak{g}$ is proved in the same way as Proposition 7.2(c) from [5]. $\square$

Remark 2.5 If $\mathfrak{g}$ is a finite-dimensional n-Lie algebra, then $\text{Lie}_{-1}\mathfrak{g} = \Pi\mathfrak{g}$ is purely odd, hence $\dim W(\Pi\mathfrak{g}) < \infty$ and therefore $\dim \text{ Lie } \mathfrak{g} < \infty$. In the super case this follows from Theorem 0.2 if $\text{char } \mathbb{F} = 0$, and from the fact that any finite-dimensional subspace of $W(V)$ generates a finite-dimensional subalgebra if $\text{char } \mathbb{F} > 0$. Thus, an n-Lie superalgebra $\mathfrak{g}$ is finite-dimensional if and only if the Lie superalgebra Lie $\mathfrak{g}$ is finite-dimensional.

Remark 2.6 Let V be a vector superspace. Recall that a sequence of anti-commutative $(n+1)$-ary products d_n , $n = 0, 1, \ldots$, of parity $n + 1 \mod 2$ on V endow V with a structure of a homotopy Lie algebra if they satisfy a sequence of certain quadratic identities, which mean that $d_0^2 = 0$, d_1 is a Lie (super)algebra bracket modulo the image of d_0 and d_0 is the derivation of this bracket, etc [24]. (Usually one also requires a $\mathbb{Z}$-grading on V for which d_n has degree $n - 1$, but we ignore this requirement here.) On the other hand, recall that if μ_n is an $(n + 1)$-ary anti-commutative product on V of parity $n + 1 \mod 2$, then the $(n + 1)$-ary product $\bar{\mu}_n$, defined in Lemma 1.4, is a commutative odd product on the vector superspace ΠV. It is easy to see that the sequence of $(n + 1)$-ary products $\bar{\mu}_n$ define a homotopy Lie algebra structure on ΠV if and only if the odd element $\mu = \sum_n \bar{\mu}_n \in W(\Pi V)$ satisfies the identity $[\mu, \mu] = 0$. As above, we can associate to a given homotopy Lie algebra structure on V the subalgebra of $W(\Pi V)$, denoted by Lie (V, μ), which is generated by $W_{-1}(\Pi V)$ and all $\bar{\mu}_n$, $n = 0, 1, \ldots$. If the superspace V is linearly compact and the homotopy Lie algebra is simple with $\bar{\mu}_n \neq 0$ for some $n > 2$, then the derived algebra of Lie (V, μ) is simple, hence is of one of the types $X(m, n)$, according to the classification of [17]. Then the simple homotopy Lie algebra is called of type $X(m, n)$. (Of course, there are many homotopy Lie algebras of a given type.) Lemma 3.1 below shows, in particular, that in characteristic 0 any n-Lie superalgebra of parity $n \mod 2$ is a homotopy Lie algebra, for which $\bar{\mu}_j = 0$ if $j \neq n - 1$. This was proved earlier in [8] and [22].

3 Proof of Theorem 0.2

For the sake of simplicity we consider the n-Lie algebra case, i.e. we assume that L_{-1} is purely odd. The same proof works verbatim when L_{-1} is not purely odd, using identity (2.2). Alternatively, the use of the standard Grassmann envelope argument reduces the case of n-Lie superalgebras of even parity to the case of n-Lie algebras.

First, introduce some notation. Let S_{2n-1} be the group of permutations of the $2n-1$ element set $\{1,\dots,2n-1\}$ and, for $\sigma \in S_{2n-1}$, let $\varepsilon(\sigma)$ be the sign of σ. Denote by S the subset of S_{2n-1} consisting of permutations σ, such that $\sigma(1) < \dots < \sigma(n-1)$, $\sigma(n) < \dots < \sigma(2n-1)$. Consider the following subsets of S (l and s stand for "long" and "short" as in [8]):

$$S^{l_1} = \{\sigma \in S|\ \sigma(2n-1) = 2n-1\},$$

$$S^{s_1} = \{\sigma \in S|\ \sigma(n-1) = 2n-1\}.$$

It is immediate to see that $S = S^{l_1} \cup S^{s_1}$. Likewise, let

$$S^{l_1 l_2} = \{\sigma \in S^{l_1} \mid \sigma(2n-2) = 2n-2\},$$

$$S^{l_1 s_2} = \{\sigma \in S^{l_1} \mid \sigma(n-1) = 2n-2\},$$

$$S^{s_1 l_2} = \{\sigma \in S^{s_1} \mid \sigma(2n-1) = 2n-2\},$$

$$S^{s_1 s_2} = \{\sigma \in S^{s_1} \mid \sigma(n-2) = 2n-2\}.$$

Then $S^{l_1} = S^{l_1 l_2} \cup S^{l_1 s_2}$, and $S^{s_1} = S^{s_1 l_2} \cup S^{s_1 s_2}$. Likewise, we define the subsets $S^{a_1 \dots a_k}$, with $a = l$ or $a = s$, for $1 \le k \le 2n-1$, so that

$$S^{a_1 \dots a_{k-1}} = S^{a_1 \dots a_{k-1} s_k} \cup S^{a_1 \dots a_{k-1} l_k}. \tag{3.1}$$

Lemma 3.1 *If $(L = \prod_{j \ge -1} L_j, \mu)$ is an admissible pair, then $[(adL_{-1})^{n-j-1}\mu, \mu] = 0$ for every $j = 0, \dots, n-1$.*

Proof. If (L,μ) is an admissible pair, then, by Lemma 2.1(c), $L_0 = (adL_{-1})^{n-1}\mu$, hence $[(adL_{-1})^{n-1}$ $\mu, \mu] = 0$ by property (L3). Now we will show that $[(adL_{-1})^{n-j-1}\mu, \mu] = 0$ for every $j = 1, \dots, n-1$. By Lemma 2.1(b) and property (L3), the Filippov-Jacobi identity holds for elements in ΠL_{-1} with product (1.2). Let $x_1, \dots, x_{n-j-1} \in L_{-1}$. By definition, $[\mu, [x_1, \dots, [x_{n-j-1}, \mu]]] = \mu \Box [x_1, \dots, [x_{n-j-1}, \mu]] - (-1)^{j(n-1)}\, [x_1, \dots, [x_{n-j-1}, \mu]] \Box \mu$. One checks by a direct calculation, using the Filippov- Jacobi identity, that

$$\mu \Box Alt[x_1, \dots, [x_{n-j-1}, \mu]] - (-1)^{j(n-1)} [x_1, \dots, [x_{n-j-1}, \mu]] \Box \mu = 0$$

where, for $f \in Hom(V^{\otimes k}, V)$, $Altf \in Hom(V^{\otimes k}, V)$ denotes the alternator of f, i.e., $Altf(a_1, \dots, a_k)$ $= \sum_{\sigma \in S_k} f(a_{\sigma(1)}, \dots, a_{\sigma(k)})$. Hence, since $\mu \in W_{n-1}(L_{-1})$, we have

$$[\mu, [x_1, \dots, [x_{n-j-1}, \mu]]] = ((j+1)! + 1)\mu \Box [x_1, \dots, [x_{n-j-1}, \mu]].$$

We will prove a stronger statement than the lemma, namely, we will show that, for every $j = 1, \dots, n-1$ one has:

$$\mu \Box [x_1, \dots, [x_{n-j-1}, \mu]] = 0. \tag{3.2}$$

Note that, by definition, for $a_1, \dots, a_{n+j} \in L_{-1}$, we have:

$$(\mu \Box [x_1, \dots, [x_{n-j-1}, \mu]])(a_1, \dots, a_{n+j}) = \sum_{\substack{\sigma(1) < \dots < \sigma(j+1) \\ \sigma(j+2) < \dots < \sigma(n+j)}} \varepsilon(\sigma) \mu([x_1, \dots, [x_{n-j-1}, \mu]](a_{\sigma(1)}, \dots, a_{\sigma(j+1)}),$$

$$a_{\sigma(j+2)},\dots,a_{\sigma(n+j)}) = \sum_{\substack{\sigma(1)<\cdots<\sigma(j+1)\\ \sigma(j+2)<\cdots<\sigma(n+j)}} \varepsilon(\sigma)\mu(\mu(x_{n-j-1},\dots,x_1,a_{\sigma(1)},\dots,a_{\sigma(j+1)}),a_{\sigma(j+2)},\dots,a_{\sigma(n+j)}).$$

Therefore (3.2) is equivalent to the following:

$$\sum_{\sigma\in S^{l_1,\dots,l_{n-j-1}}} \varepsilon(\sigma)\mu(x_{\sigma(1)},\dots,x_{\sigma(n-1)},\mu(x_{\sigma(n)},\dots,x_{\sigma(2n-1)}))=0. \tag{3.3}$$

Set $A_\sigma=\varepsilon(\sigma)\mu(x_{\sigma(1)},\dots,x_{\sigma(n-1)},\mu(x_{\sigma(n)},\dots,x_{\sigma(2n-1)}))$, $Q_{l_1\dots l_t}=\sum_{\sigma\in S^{l_1\dots l_t}} A_\sigma$, and similarly define $Q_{a_1\dots a_t}$, where $a=s$ or $a=l$. Then (3.3) is equivalent to $Q_{l_1\dots l_{n-j-1}}=0$. In fact, we shall prove more:

$$Q_{l_1\dots l_t}=Q_{l_1\dots l_{t-1}s_t}=Q_{s_1\dots s_{t-1}l_t}=Q_{s_1\dots s_t}=0 \quad \text{for } t=0,\dots,n-2. \tag{3.4}$$

For $t=0$ and $t=1$, equality (3.4) can be proved as in [8, Proposition 2.1]. Namely, by the Filippov-Jacobi identity, for any $\sigma\in S^{s_1}$, A_σ can be written as a sum of n elements A_τ, where $\tau\in S^{l_1}$ is such that $\{\tau(1),\dots,\tau(n-1)\}\subset\{\sigma(n),\dots,\sigma(2n-1)\}$. Since the sets $\{\tau(1),\dots,\tau(n-1)\}$ and $\{\sigma(n),\dots,\sigma(2n-1)\}$ have $n-1$ and n elements, respectively, there exists only one i such that $\{\tau(1),\dots,\tau(n-1)\}\cup\{i\}=\{\sigma(n),\dots,\sigma(2n-1)\}$. Then $i\le 2n-2$ and $i\ne\tau(1),\dots,\tau(n-1)$. Therefore there are $n-1$ possibilities to choose i. It follows that

$$Q_{s_1}=(n-1)Q_{l_1}. \tag{3.5}$$

Likewise, by the Filippov-Jacobi identity, for any $\sigma\in S^{l_1}$, A_σ can be written as a sum of one element A_ρ with $\rho\in S^{l_1}$, and $n-1$ elements A_τ, with $\tau\in S^{s_1}$ such that $\{\tau(1),\dots,\tau(n-2)\}\subset\{\sigma(n),\dots,\sigma(2n-2)\}$. As above, there exists only one i such that $\{\tau(1),\dots,\tau(n-2)\}\cup\{i\}=\{\sigma(n),\dots,\sigma(2n-2)\}$. Such an i is different from $\tau(1),\dots,\tau(n-2)$ and $i\le 2n-2$. Hence there are n possibilities to choose i. Notice that

$$\rho=\begin{pmatrix} 1 & \dots & n-1 & n & \dots & 2n-2 & 2n-1\\ \sigma(n) & \dots & \sigma(2n-2) & \sigma(1) & \dots & \sigma(n-1) & 2n-1\end{pmatrix}, \tag{3.6}$$

hence $\varepsilon(\rho)=(-1)^{n-1}\varepsilon(\sigma)$. Therefore

$$Q_{l_1}=nQ_{s_1}+(-1)^{n-1}Q_{l_1}. \tag{3.7}$$

Equations (3.5) and (3.7) form a system of two linear equations in the two indeterminates Q_{s_1} and Q_{l_1}, whose determinant is equal to $n^2-n-1-(-1)^n$, which is different from zero for every $n>2$. It follows that $Q_{s_1}=0=Q_{l_1}$, i.e., (3.4) is proved for $t=1$. Since, as we have already noticed, $S=S^{l_1}+S^{s_1}$, (3.4) for $t=0$ also follows.

Now we argue by induction on t. We already proved (3.4) for $t=0$ and $t=1$. Assume that

$$Q_{l_1\dots l_{t-1}}=Q_{l_1\dots l_{t-2}s_{t-1}}=Q_{s_1\dots s_{t-2}l_{t-1}}=Q_{s_1\dots s_{t-1}}=0$$

for some $1\le t<n-2$. Similarly as above, by the Filippov-Jacobi identity, for any $\sigma\in S^{s_1\dots s_t}$, A_σ can be written as a sum of n elements A_τ with $\tau\in S^{l_1\dots l_t}$, such that $\{\tau(1),\dots,\tau(n-1)\}\subset\{\sigma(n),\dots,\sigma(2n-1)\}$, i.e., $\{\tau(1),\dots,\tau(n-1)\}\cup\{i\}=\{\sigma(n),\dots,\sigma(2n-1)\}$, for some $i\le 2n-t-1$ and $i\ne\tau(1),\dots,\tau(n-1)$. It follows that there are $n-t$ choices for i, hence

$$Q_{s_1\dots s_t}=(n-t)Q_{l_1\dots l_t}. \tag{3.8}$$

Likewise, if $\sigma \in S^{s_1\dots s_{t-1}l_t}$, then, by the Filippov-Jacobi identity, A_σ can be written as a sum of one element A_ρ with $\rho \in S^{l_1\dots l_t}$ as in (3.6), and $n-1$ elements A_τ with $\tau \in S^{l_1\dots l_{t-1}s_t}$, such that $\{\tau(1),\dots,\tau(n-2)\} \subset \{\sigma(n),\dots,\sigma(2n-2)\}$. As above, there exists only one i such that $\{\tau(1),\dots,\tau(n-2)\} \cup \{i\} = \{\sigma(n),\dots,\sigma(2n-2)\}$. Then $i \le 2n-t-1$, $i \ne \tau(1),\dots,\tau(n-2)$. It follows that

$$Q_{s_1\dots s_{t-1}l_t} = (n-t+1)Q_{l_1\dots l_{t-1}s_t} + (-1)^{n-1}Q_{l_1\dots l_t}. \tag{3.9}$$

Then, using (3.1) and the inductive hypotheses $Q_{s_1\dots s_{t-1}} = 0 = Q_{l_1\dots l_{t-1}}$, we get the following system of linear equations:

$$\begin{cases} Q_{s_1\dots s_t} = (n-t)Q_{l_1\dots l_t} \\ Q_{s_1\dots s_{t-1}l_t} = (n-t+1)Q_{l_1\dots l_{t-1}s_t} + (-1)^{n-1}Q_{l_1\dots l_t} \\ Q_{s_1\dots s_t} + Q_{s_1\dots s_{t-1}l_t} = 0 \\ Q_{l_1\dots l_t} + Q_{l_1\dots l_{t-1}s_t} = 0, \end{cases} \tag{3.10}$$

whose determinant is equal to $(-1)^n + 1$. It follows that if n is even then $Q_{l_1\dots l_t} = 0 = Q_{s_1\dots s_t} = Q_{s_1\dots s_{t-1}l_t} = Q_{l_1\dots l_{t-1}s_t}$, hence (3.4) is proved.

Now assume that n is odd. Then (3.10) reduces to

$$\begin{cases} Q_{s_1\dots s_t} = (n-t)Q_{l_1\dots l_t} \\ Q_{s_1\dots s_{t-1}l_t} = (n-t+1)Q_{l_1\dots l_{t-1}s_t} + Q_{l_1\dots l_t} \\ Q_{s_1\dots s_t} + Q_{s_1\dots s_{t-1}l_t} = 0. \end{cases} \tag{3.11}$$

Using the Filippov-Jacobi identity as above, one gets the following system of linear equations:

$$\begin{cases} Q_{s_1\dots s_{t-2}l_{t-1}l_t} = (n-t+2)Q_{l_1\dots l_{t-2}s_{t-1}s_t} - Q_{l_1\dots l_{t-2}s_{t-1}l_t} + Q_{l_1\dots l_{t-2}l_{t-1}s_t} \\ Q_{s_1\dots s_{t-2}l_{t-1}s_t} = (n-t+1)Q_{l_1\dots l_{t-2}s_{t-1}l_t} + Q_{l_1\dots l_{t-2}l_{t-1}l_t}. \end{cases} \tag{3.12}$$

Besides, using the inductive hypotheses, we get:

$$\begin{cases} Q_{s_1\dots s_{t-2}l_{t-1}l_t} + Q_{s_1\dots s_{t-2}l_{t-1}s_t} = Q_{s_1\dots s_{t-2}l_{t-1}} = 0 \\ Q_{l_1\dots l_{t-2}s_{t-1}s_t} + Q_{l_1\dots l_{t-2}s_{t-1}l_t} = Q_{l_1\dots l_{t-2}s_{t-1}} = 0 \\ Q_{l_1\dots l_{t-2}l_{t-1}s_t} + Q_{l_1\dots l_{t-2}l_{t-1}l_t} = Q_{l_1\dots l_{t-2}l_{t-1}} = 0. \end{cases} \tag{3.13}$$

Taking the sum of the two equations in (3.12), and using the three equations in (3.13), we get: $Q_{l_1\dots l_{t-2}s_{t-1}s_t} = Q_{l_1\dots l_{t-2}s_{t-1}l_t} = 0$. By arguing in the same way, one shows that

$$Q_{l_1\dots l_k s_{k+1} l_{k+2}\dots l_t} = 0 \quad \text{for every} \quad k = 0,\dots t-2. \tag{3.14}$$

Finally, using the Filippov-Jacobi identity as above, one gets the following system of linear equations:

$$\begin{cases} Q_{l_1\dots l_t} = nQ_{s_1\dots s_t} + (-1)^{n+t-2}Q_{s_1\dots s_{t-1}l_t} + \dots - Q_{s_1 l_2 s_3\dots s_t} + Q_{l_1 s_2\dots s_t} \\ Q_{s_1\dots s_{t-2}l_{t-1}s_t} = (n-t+1)Q_{l_1\dots l_{t-2}s_{t-1}l_t} + Q_{l_1\dots l_t} \\ \vdots \\ Q_{l_1 s_2\dots s_t} = (n-t+1)Q_{s_1 l_2\dots l_t} + Q_{l_1\dots l_t} \end{cases}$$

which reduces, by (3.14), to the following

$$\begin{cases} Q_{l_1\dots l_t} = nQ_{s_1\dots s_t} + (-1)^{n+t-2}Q_{s_1\dots s_{t-1}l_t} + \dots - Q_{s_1 l_2 s_3\dots s_t} + Q_{l_1 s_2\dots s_t} \\ Q_{s_1\dots s_{t-2}l_{t-1}s_t} = Q_{l_1\dots l_t} \\ \vdots \\ Q_{l_1 s_2\dots s_t} = Q_{l_1\dots l_t}. \end{cases} \tag{3.15}$$

System (3.15) implies the following equation:

$$Q_{l_1\ldots l_t} = nQ_{s_1\ldots s_t} + (-1)^{n+t-2}Q_{s_1\ldots s_{t-1}l_t} + \frac{(-1)^t+1}{2}Q_{l_1\ldots l_t}.$$

This equation together with (3.11) form a system of four linear equations in four indeterminates, whose determinant is equal to $(n-t+1)((-1)^{n-t}(n-t)+\frac{1-(-1)^t}{2}-n(n-t))$. It is different from 0 for every $t=1,\ldots,n-2$. Hence $Q_{l_1\ldots l_t}=0=Q_{l_1\ldots l_{t-1}s_t}=Q_{s_1\ldots s_{t-1}l_t}=Q_{s_1\ldots s_t}$, and (3.4) is proved. □

Proof of Theorem 0.2. Any element of the subalgebra generated by L_{-1} and μ is a linear combination of elements of the form:

$$[\ldots[[\ldots[[[\ldots[\mu,a_1],\ldots,a_s],\mu],b_1],\ldots,b_k],\mu],\ldots] \tag{3.16}$$

with $a_1,\ldots,a_s,b_1,\ldots,b_k,\ldots$ in L_{-1}. By Lemma 3.1, every element of the form $[[[\ldots[\mu,a_1],\ldots,a_s],\mu]$ is either 0 or an element in L_{-1}, therefore we can assume that μ appears only once in (3.16), i.e., any element of L lies in $[\ldots[\mu,L_{-1}],\ldots,L_{-1}]$. □

Remark 3.2 We conjecture that Theorem 0.2 holds also in non-zero characteristic if $\operatorname{char}\mathbb{F}\geq n$. Our argument works for $\operatorname{char}\mathbb{F}>(n-1)^2$. The following example shows that Theorem 0.2 (and Theorem 0.1) fails if $0<\operatorname{char}\mathbb{F}<n$. Let $\mathfrak{g}=\mathbb{F}a$ be a 1-dimensional odd space, which we endow by the following n-bracket: $[a,a,\ldots,a]=a$. The Filippov-Jacobi identity holds if $n=sp+1$, where $p=\operatorname{char}\mathbb{F}$ and s is a positive integer. However, Lie $\mathfrak{g}$ is not of the form described by Theorem 0.2.

4 Classification of irreducible admissible pairs

First, we briefly recall some examples of $\mathbb{Z}$-graded linearly compact Lie superalgebras over a field $\mathbb{F}$ of characteristic 0, and some of their properties. For more details, see [17] and [4].

Given a finite-dimensional vector superspace V of dimension $(m|n)$ (i.e. $\dim V_{\bar 0}=m$, $\dim V_{\bar 1}=n$), the universal Lie superalgebra $W(V)$ is isomorphic to the Lie superalgebra $W(m,n)$ of continuous derivations of the tensor product $\mathbb{F}(m,n)$ of the algebra of formal power series in m commuting variables $x_1,\ldots,x_m$ and the Grassmann algebra in n anti-commuting variables $\xi_1,\ldots,\xi_n$. Elements of $W(m,n)$ can be viewed as linear differential operators of the form

$$X=\sum_{i=1}^m P_i(x,\xi)\frac{\partial}{\partial x_i}+\sum_{j=1}^n Q_j(x,\xi)\frac{\partial}{\partial \xi_j},\ \ P_i,Q_j\in\mathbb{F}(m,n).$$

The Lie superalgebra $W(m,n)$ is simple linearly compact (and it is finite-dimensional if and only if $m=0$).

Letting $\deg x_i=-\deg\frac{\partial}{\partial x_i}=k_i$, $\deg\xi_i=-\deg\frac{\partial}{\partial\xi_i}=s_i$, where $k_i,s_i\in\mathbb{Z}$, defines a $\mathbb{Z}$-grading on $W(m,n)$, called the $\mathbb{Z}$-grading of type $(k_1,\ldots,k_m|s_1,\ldots,s_n)$. Any $\mathbb{Z}$-grading of $W(m,n)$ is conjugate (i.e. can be mapped by an automorphism of $W(m,n)$) to one of these. Clearly, such a grading has finite depth d (meaning that $W(m,n)_j\neq 0$ if and only if $j\geq -d$) if and only if $k_i\geq 0$ for all i. It is easy to show that the depth $d=1$ if all k_i's and s_i's are 0 or 1, or if all k_i's are 0, $s_j=-1$ for some j, and $s_i=0$ for every $i\neq j$.

Now we shall describe some closed (hence linearly compact) subalgebras of $W(m,n)$.

First, given a subalgebra L of $W(m,n)$, a continuous linear map $Div : L \to \mathbb{F}(m,n)$ is called a divergence if the action π_λ of L on $\mathbb{F}(m,n)$, given by

$$\pi_\lambda(X)f = Xf + (-1)^{p(X)p(f)}\lambda f Div X,\ \ X \in L,$$

is a representation of L in $\mathbb{F}(m,n)$ for any $\lambda \in \mathbb{F}$. Note that

$$S'_{Div}(L) := \{X \in L \mid Div X = 0\}$$

is a closed subalgebra of L. We denote by $S_{Div}(L)$ its derived subalgebra (recall that the derived subalgebra of $\mathfrak{g}$ is $[\mathfrak{g},\mathfrak{g}]$). An example of a divergence on $L = W(m,n)$ is the following, denoted by div:

$$div(\sum_{i=1}^m P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n Q_j \frac{\partial}{\partial \xi_j}) = \sum_{i=1}^m \frac{\partial P_i}{\partial x_i} + \sum_{j=1}^n (-1)^{p(Q_j)} \frac{\partial Q_j}{\partial \xi_j}.$$

Hence for any $\lambda \in \mathbb{F}$ we get the representation π_λ of $W(m,n)$ in $\mathbb{F}(m,n)$. Also, we get closed subalgebras $S'_{div}(W(m,n)) \supset S_{div}(W(m,n))$ denoted by $S'(m,n) \supset S(m,n)$. Recall that $S'(m,n) = S(m,n)$ is simple if $m > 1$, and

$$S'(1,n) = S(1,n) \oplus \mathbb{F}\xi_1 \ldots \xi_n \frac{\partial}{\partial x_1}, \tag{4.1}$$

where $S(1,n)$ is a simple ideal.

The $\mathbb{Z}$-gradings of type $(k_1,\ldots,k_m|s_1,\ldots,s_n)$ of $W(m,n)$ induce ones on $S'(m,n)$ and $S(m,n)$ and any $\mathbb{Z}$-grading is conjugate to those. The description of $\mathbb{Z}$-gradings of depth 1 for $S'(m,n)$ and $S(m,n)$ is the same as for $W(m,n)$.

Next examples of subalgebras of $W(m,n)$, needed in this paper, are of the form

$$L(\omega) = \{X \in W(m,n) \mid X\omega = 0\},$$

where ω is a differential form.

In the case $m = 2k$ is even, consider the symplectic differential form

$$\omega_s = 2\sum_{i=1}^k dx_i \wedge dx_{k+i} + \sum_{i=1}^n d\xi_i d\xi_{k-i+1}.$$

The corresponding subalgebra $L(\omega_s)$ is denoted by $H'(m,n)$ and is called a Hamiltonian superalgebra. This superalgebra is simple, hence coincides with its derived subalgebra $H(m,n)$, unless $m = 0$, when the Hamiltonian superalgebra is finite-dimensional.

It is convenient to consider the "Poisson" realization of $H(m,n)$. For that let $p_i = x_i$, $q_i = x_{k+i}$, $i = 1,\ldots,k$, and introduce on $\mathbb{F}(m,n)$ the structure of a Poisson superalgebra $P(m,n)$ by letting the non-zero brackets between generators to be as follows:

$$\{p_i, q_i\} = 1 = \{\xi_i, \xi_{n-i+1}\},$$

and extend by the Leibniz rule. Then the map $P(m,n) \to H'(m,n)$, given by $f \mapsto \sum_{i=1}^k (\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}) - (-1)^{p(f)}\sum_{i=1}^k \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_{k-i+1}}$, defines a surjective Lie superalgebra homomorphism with kernel

$\mathbb{F}1$. Thus, $H'(m,n) = P(m,n)/\mathbb{F}1$. In this realization $H(0,n)$ is spanned by all monomials in ξ_i mod $\mathbb{F}1$ except for the one of top degree, and we have:

$$H'(0,n) = H(0,n) \oplus \mathbb{F}\xi_1 \ldots \xi_n. \tag{4.2}$$

Note that $H(0,n)$ is simple if and only if $n \geq 4$.

All $\mathbb{Z}$-gradings of depth 1 of $H'(0,n)$ are, up to conjugacy, those of type $(|1,\ldots,1)$, $(|1,0,\ldots,0,-1)$, and $(|\underbrace{1,\ldots,1}_{n/2},0,\ldots,0)$, if n is even [5].

Another example is $HO(n,n) = L(\omega_{os}) \subset W(n,n)$, where $\omega_{os} = \sum_{i=1}^n dx_i d\xi_i$ is an odd symplectic form. This Lie superalgebra is simple if and only if $n \geq 2$. It contains the important for this paper subalgebra $SHO'(n,n) = HO(n,n) \cap S'(n,n)$. Its derived subalgebra $SHO(n,n)$ is simple if and only if $n \geq 3$.

Again, it is convenient to consider a "Poisson" realization of $HO(n,n)$. For this consider the Buttin bracket on $\Pi\mathbb{F}(n,n)$:

$$\{f,g\}_B = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} - (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i}\right).$$

This is a Lie superalgebra, which we denote by $PO(n,n)$, and the map $PO(n,n) \to HO(n,n)$, given by

$$f \mapsto \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial \xi_i} - (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x_i}\right)$$

is a surjective Lie superalgebra homomorphism, whose kernel is $\mathbb{F}1$. Thus, $HO(n,n) = P(n,n)/\mathbb{F}1$. In this realization we have:

$$SHO'(n,n) = \{f \in P(n,n)/\mathbb{F}1 \mid \Delta f = 0\},$$

where $\Delta = \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial \xi_i}$ is the odd Laplace operator. Then $SHO(n,n)$ is an ideal of codimension 1 in $SHO'(n,n)$, and we have:

$$SHO'(n,n) = SHO(n,n) \oplus \mathbb{F}\xi_1 \ldots \xi_n. \tag{4.3}$$

All $\mathbb{Z}$-gradings of depth 1 of $SHO'(n,n)$ are, up to conjugacy, those of type $(1,\ldots,1|1,\ldots,1)$, $(0,\ldots,0,1|0,\ldots,0,-1)$, and $(\underbrace{1,\ldots,1}_{k},0,\ldots,0|\underbrace{0,\ldots,0}_{k},1,\ldots,1)$, where $k = 0,\ldots,n$ [5].

The next important for us example is

$$KO(n,n+1) = \{X \in W(n,n+1) \mid X\omega_{oc} = f\omega_{oc} \text{ for some } f \in \mathbb{F}(n,n+1)\},$$

where $\omega_{oc} = d\xi_{n+1} + \sum_{i=1}^n (\xi_i dx_i + x_i d\xi_i)$ is an odd contact form. This superalgebra is simple for all $n \geq 1$. Another realization of this Lie superalgebra is $PO(n,n+1) = \Pi\mathbb{F}(n,n+1)$ with the bracket $\{f,g\}_{BO} = (2-E)f\frac{\partial g}{\partial \xi_{n+1}} - (-1)^{p(f)}\frac{\partial f}{\partial \xi_{n+1}}(2-E)g - \sum_{i=1}^n (\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} - (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i})$, where $E = \sum_{i=1}^n (x_i\frac{\partial}{\partial x_i} + \xi_i\frac{\partial}{\partial \xi_i})$. The isomorphism $PO(n,n+1) \to KO(n,n+1)$ is given by $f \mapsto (2-E)f\frac{\partial}{\partial \xi_{n+1}} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_{n+1}}E - \sum_{i=1}^n (\frac{\partial f}{\partial x_i}\frac{\partial}{\partial \xi_i} - (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x_i})$. It turns out that for each $\beta \in \mathbb{F}$ the Lie superalgebra $KO(n,n+1)$ admits a divergence

$$div_\beta f = \Delta f + (E - n\beta)\frac{\partial f}{\partial \xi_{n+1}}, \ f \in PO(n,n+1).$$

We let

$$SKO'(n,n+1;\beta)=\{f\in PO(n,n+1)\mid div_\beta f=0\}.$$

This Lie superalgebra is not always simple, but its derived algebra, denoted by $SKO(n,n+1;\beta)$, is simple if and only if $n\geq 2$. In fact, $SKO'(n,n+1;\beta)=SKO(n,n+1;\beta)$, unless $\beta=1$ or $\beta=\frac{n-2}{n}$. In the latter cases $SKO(n,n+1;\beta)$ is an ideal of codimension 1 in $SKO'(n,n+1;\beta)$, and we have:

$$SKO'(n,n+1;1)=SKO(n,n+1;1)+\mathbb{F}\xi_1\dots\xi_{n+1}, \tag{4.4}$$

$$SKO'(n,n+1;\frac{n-2}{n})=SKO(n,n+1;\frac{n-2}{n})+\mathbb{F}\xi_1\dots\xi_n. \tag{4.5}$$

All $\mathbb{Z}$-gradings of depth 1 of $SKO'(n,n+1;\beta)$ are, up to conjugacy, of type $(0,\dots,0,1|0,\dots,0,-1,0)$, and $(\underbrace{1,\dots,1}_{k},0,\dots,0|\underbrace{0,\dots,0}_{k},1,\dots,1)$, where $k=0,\dots,n$ [5].

Theorem 4.1 *Let $(L=\oplus_{j=-1}^{n-1}L_j,\mu)$ be an irreducible admissible pair over an algebraically closed field $\mathbb{F}$ of characteristic 0, where L is a linearly compact Lie superalgebra, and $n>2$. Then*

(a) $L=\oplus_{j=-1}^{n-1}L_j$ is a semidirect product of the simple ideal $S=\oplus_{j=-1}^{n-2}L_j$ and the 1-dimensional subalgebra $L_{n-1}=\mathbb{F}\mu$, where μ is an outer derivation of L, such that $[\mu,L_0]=0$.

(b) The pair (L,μ) is isomorphic to one of the following four irreducible admissible pairs:

- *(i) $(H'(0,n+1),\xi_1\dots\xi_{n+1})$, $n\geq 3$, with the grading of type $(|1,\dots,1)$;*
- *(ii) $(SHO'(n,n),\xi_1\dots\xi_n)$, $n\geq 3$, with the grading of type $(0,\dots,0|1,\dots,1)$;*
- *(iii) $(SKO'(n-1,n;1),\xi_1\dots\xi_{n-1}\xi_n)$, $n\geq 3$, with the grading of type $(0,\dots,0|1,\dots,1)$;*
- *(iv) $(S(1,n-1),\xi_1\dots\xi_{n-1}\frac{\partial}{\partial x})$, $n\geq 3$, with the grading of type $(0|1,\dots,1)$.*

Proof. The decomposition $L=S\rtimes\mathbb{F}\mu$ in (a) follows from Theorem 0.2. The fact that S is simple is proved in the same way as in [5, Theorem 7.3]. Indeed, S is the minimal among non-zero closed ideals of L, since if I is a non-zero closed ideal of L, then $I\cap L_{-1}\neq\emptyset$ by transitivity, hence, by irreducibility, $I\cap L_{-1}=L_{-1}$, from which it follows that I contains S. Next, by the super-analogue of Cartan-Guillemin's theorem [1, 15], established in [10], $S=S'\hat{\otimes}\Lambda(m,h)$, for some simple linearly compact Lie superalgebra S' and some $m,h\in\mathbb{Z}_{\geq 0}$, and μ lies in $Der(S'\hat{\otimes}\mathcal{O}(m,h))$. Since $Der(S'\hat{\otimes}\mathcal{O}(m,h))=DerS'\hat{\otimes}\mathcal{O}(m,h)+1\otimes W(m,h)$ [10], we have: $\mu=\sum_i(d_i\otimes a_i)+1\otimes\mu'$ for some $d_i\in DerS'$, $a_i\in\mathcal{O}(m,h)$ and $\mu'\in W(m,h)$.

First consider the case when μ is even. Then μ' is an even element of $W(m,h)$ hence, by the minimality of the ideal $S'\hat{\otimes}\mathcal{O}(m,h)$, $h=0$. Now suppose $m\geq 1$. If μ' lies in the non-negative part of $W(m,0)$ with the grading of type $(1,\dots,1|)$, then the ideal generated by $S'x_1$ is a proper μ-invariant ideal of $S'\hat{\otimes}\mathcal{O}(m,0)$, contradicting its minimality. Therefore we may assume, up to a linear change of indeterminates, that $\mu'=\frac{\partial}{\partial x_1}+D$, for some derivation D lying in the non-negative part of $W(m,0)$. Since μ lies in L_{n-1}, we have $\deg(x_1)=-n+1$, but this is a contradiction since the $\mathbb{Z}$-grading of L has depth 1. It follows that $m=0$.

Now consider the case when μ is odd. Consider the grading of $W(m,h)$ of type $(1,\dots,1|1,\dots,1)$, and denote by $W(m,h)_{\geq 0}$ its non-negative part. If $\mu'\in W(m,h)_{\geq 0}$, then the minimality of the ideal

$S'\hat{\otimes}\mathcal{O}(m,h)$ implies $m=h=0$. Now suppose that $h\geq 1$ and that μ' has a non-zero projection on $W(m,h)_{-1}$. Then, up to a linear change of indeterminates, $\mu'=\frac{\partial}{\partial\xi_1}+D$ for some odd derivation $D\in W(m,h)_{\geq 0}$. Since μ lies in L_{n-1}, we have $\deg(\xi_1)=-n+1<-1$. Since $L\supset S'\xi_1$ and the grading of L has depth 1, it follows that every element in S' has positive degree, but this is a contradiction since S' is simple. This concludes the proof of the simplicity of S.

In order to prove (b), note that the grading operator D of the simple $\mathbb{Z}$-graded Lie superalgebra S is an outer derivation (since $[\mu,L_0]=0, D\notin L_0$). Another outer derivation of S is μ. From the classification of simple linearly compact Lie superalgebras [16], [17] and their derivations in [16], [17], [2, Proposition 1.8] (see also Lemma 4.3 below), we see that the only possibilities for L are $H'(0,n+1)$, $SHO'(n,n)$, $SKO'(n-1,n;1)$, $SKO'(n-1,n;\frac{n-2}{n})$, and $S'(1,n)$ for $n\geq 3$. From the description of $\mathbb{Z}$-gradings of depth 1 of these Lie superalgebras, given above, it follows that $L=SKO'(n-1,n;\frac{n-1}{n})$ is ruled out, whereas for the remaining four possibilities for L only the grading of type $(0,\dots,0|1,\dots,1)$ is possible, and for them indeed $L_{n-1}=\mathbb{F}\mu$, where μ is as described above. It is immediate to check that in these four cases the pair (L,μ) is admissible. Irreducibility of the L_0-module L_{-1} follows automatically from the simplicity of S since its depth is 1. □

Remark 4.2 In cases (i)–(iv) of Theorem 4.1(b) the subalgebra L_0 and the L_0-module ΠL_{-1} are as follows:

- (i) $L_0\cong so_{n+1}(\mathbb{F})$, $\Pi L_{-1}=\mathbb{F}^{n+1}$ with the standard action of $so_{n+1}(\mathbb{F})$;
- (ii) $L_0\cong S(n,0)$, $\Pi L_{-1}=\mathbb{F}[[x_1,\dots,x_n]]/\mathbb{F}1$, where $\mathbb{F}[[x_1,\dots,x_n]]$ is the standard module over $S(n,0)$;
- (iii) $L_0\cong W(n-1,0)$, $\Pi L_{-1}=\mathbb{F}[[x_1,\dots,x_{n-1}]]$, which carries the representation $\pi_{\lambda=-1}$ of $W(n-1,0)$;
- (iv) $L_0\cong W(1,0)\ltimes sl_{n-1}(\mathbb{F}[[x]])$, $\Pi L_{-1}=\mathbb{F}^{n-1}\otimes\mathbb{F}[[x]]$ with the standard action of $sl_{n-1}(\mathbb{F}[[x]])$ and the representation $\pi_{\lambda=-1/(n-1)}$ of $W(1,0)$ on $\mathbb{F}[[x]]$.

As we have seen, an important part of the classification of irreducible admissible pairs is the description of derivations of simple linearly compact Lie algebras. This description is based on the following simple lemma.

Lemma 4.3 *Let L be a linearly compact Lie superalgebra and let $\mathfrak{a}$ be a reductive subalgebra of L (i.e. the adjoint representation of $\mathfrak{a}$ on L decomposes in a direct product of finite-dimensional irreducible $\mathfrak{a}$-modules). Then any continuous derivation of L is a sum of an inner derivation and a derivation commuting with the adjoint action of $\mathfrak{a}$.*

Proof. [16] We have closed $\mathfrak{a}$-submodules:

$$\text{Inder } L\subset \text{ Der } L\subset EndL\,,$$

where Inder L and Der L denote the subspaces of all inner derivations and all continuous derivations of the Lie superalgebra L in the space of continuous endomorphisms of the linearly compact vector space L. Since $L=\prod_j V_j$, where V_j are finite-dimensional irreducible $\mathfrak{a}$-modules, we have: End $L=$

$\prod_{i,j}$ Hom (V_i, V_j), hence End L, and therefore Der L, decomposes into a direct product of irreducible $\mathfrak{a}$-submodules. Hence

$$\text{Der } L = \text{ Inder } L \oplus V,$$

where V is an $\mathfrak{a}$-submodule. But $\mathfrak{a}V \subset$ Inder L since Inder L is an ideal in Der L. Hence $\mathfrak{a}V = 0$, i.e., any derivation from V commutes with the adjoint action of $\mathfrak{a}$ on L. □

5 Classification of simple linearly compact n-Lie algebras over a field of characteristic 0, and their derivations

Proof of Theorem 0.1 By Proposition 2.4, the classification of simple linearly compact n-Lie algebras is equivalent to the classification of admissible pairs (L, μ), for which L is linearly compact. The list of the latter consists of the four examples (i)–(iv) given in Theorem 4.1(b). It is easy to see that the corresponding n-Lie algebras are O^n, S^n, W^n and SW^n. (By Lemma 1.4(a), we automatically get from $[\mu, L_0] = 0$ that the Filippov-Jacobi identity indeed holds.) □

The notation for the four simple n-Lie algebras comes from the following fact.

Proposition 5.1 *(a) The Lie algebra of continuous derivations of the n-Lie algebras O^n, S^n, W^n and SW^n is isomorphic to $so_{n+1}(\mathbb{F})$, $S(n,0)$, $W(n-1,0)$ and $W(1,0) \ltimes sl_{n-1}(\mathbb{F}[[x]])$, respectively. Its representation on the n-Lie algebra is described in Remark 4.2.*

(b) All continuous derivations of a simple linearly compact n-Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0 lie in the closure Inder $\mathfrak{g}$ of the span of the inner ones.

Proof. Let $\mathfrak{g}$ be one of the four simple n-Lie algebras and let Der $\mathfrak{g}$ be the Lie algebra of all continuous derivations of $\mathfrak{g}$. Then $L_0 :=$ Inder $\mathfrak{g}$ is an ideal of Der $\mathfrak{g}$. By Remark 4.2, L_0 is isomorphic to the Lie algebras listed in (a). But all derivations of the Lie algebras $L_0 = so_{n+1}(\mathbb{F})$, $W(n-1,0)$ and $W(1,0) \ltimes sl_{n-1}(\mathbb{F}[[x]])$ are inner, and Der $S(n,0) = S(n,0) \oplus \mathbb{F}E$, where $E = \sum_i x_i \frac{\partial}{\partial x_i}$. This is well known, except for the case $L = W(1,0) \ltimes sl_{n-1}(\mathbb{F}[[x]])$. We apply Lemma 4.3 to this case, taking $\mathfrak{a} = \mathbb{F}x\frac{d}{dx} \oplus sl_{n-1}(\mathbb{F})$. If D is an endomorphism of the vector space L, commuting with $\mathfrak{a}$, we have, by Schur's lemma :

$$D(x^k \frac{d}{dx}) = \alpha_k x^k \frac{d}{dx}, \quad D(x^k a) = \beta_k x^k a, \text{ for } a \in sl_{n-1}(\mathbb{F}), \text{where } \alpha_k, \beta_k \in \mathbb{F}.$$

Since D is also a derivation of L, we conclude that D is a multiple of ad $x\frac{d}{dx}$.

Let now $D \in$ Der $\mathfrak{g} \setminus$ (Inder $\mathfrak{g} = L_0$). Since $[D, L_0] \subset L_0$, D induces a derivation of L_0. Since all derivations of L_0 are inner, except for E in the case $\mathfrak{g} \cong S^n$, but E is not a derivation of $\mathfrak{g}$, we conclude that there exists $a \in L_0$, such that $D|_{L_0} =$ ad $a|_{L_0}$. Therefore $D' = D - a$ commutes with the action of L_0 on L_{-1}. But the latter representation is described in Remark 4.2, and, clearly, in all cases the only operators, commuting with the representation operators of L_0 on L_{-1}, are scalars. Since a non-zero scalar cannot be a derivation of $\mathfrak{g}$, we conclude that $D' = 0$, hence Der $\mathfrak{g} = L_0$. □

In conclusion we discuss $\mathbb{F}$-forms of the four simple n-Lie algebras, where $\mathbb{F}$ is a field of characteristic 0. Let $\overline{F} \supset \mathbb{F}$ be the algebraic closure of $\mathbb{F}$. Given a linearly compact n-Lie algebra $\mathfrak{g}$ over $\overline{\mathbb{F}}$, its $\mathbb{F}$-form is defined as an n-Lie algebra $\mathfrak{g}^{\mathbb{F}}$ over $\mathbb{F}$, such that $\overline{F} \otimes_{\mathbb{F}} \mathfrak{g}^{\mathbb{F}}$ is isomorphic to $\mathfrak{g}$.

Due to the bijection given by Proposition 2.4, the $\mathbb{F}$-forms of $\mathfrak{g}$ are in one-to-one correspondence with the $\mathbb{F}$-forms of the $\mathbb{Z}$-graded Lie superalgebras $S_{\mathfrak{g}} = [$ Lie $\mathfrak{g}$, Lie $\mathfrak{g}]$. But the latter are parameterized by the set $H^1(Gal, AutS_{\mathfrak{g}})$, where Gal is the Galois group of $\overline{\mathbb{F}}$ over $\mathbb{F}$, and $AutS_{\mathfrak{g}}$ is the group of continuous automorphisms of the Lie superalgebra $S_{\mathfrak{g}}$, preserving its $\mathbb{Z}$-grading (cf. [3]).

By the method of [3] it is easy to compute the group $AutS_{\mathfrak{g}}$, using Remark 4.2.

Proposition 5.2 *One has:*

$$AutS_{\mathfrak{g}} = G_{\mathfrak{g}} \ltimes \mathcal{U},$$

where $\mathcal{U}$ is a prounipotent group and $G_{\mathfrak{g}}$ is a reductive group, isomorphic to $O_{n+1}(\overline{\mathbb{F}})$, $GL_n(\overline{\mathbb{F}})$, $GL_{n-1}(\overline{\mathbb{F}})$ and $\overline{\mathbb{F}}^{\times} \times SL_{n-1}(\overline{F})$, if $\mathfrak{g}$ is isomorphic to O^n, S^n, W^n and SW^n over $\overline{\mathbb{F}}$, respectively.

We have $H^1(Gal, AutS_{\mathfrak{g}}) = H^1(G_{\mathfrak{g}}, Gal)$ (see, e.g., [3]). Furthermore, $H^1(G_{\mathfrak{g}}, Gal) = 1$ in the last three cases of Proposition 5.2, hence the only $\mathbb{F}$-forms of S^n, W^n and SW^n over $\overline{\mathbb{F}}$ are S^n, W^n and SW^n over $\mathbb{F}$. Finally, it follows from [3] that the $\mathbb{F}$-forms of the $\mathbb{Z}$-graded Lie superalgebra $H(0, n+1)$ are the derived algebras of the Lie superalgebras $P/\mathbb{F}1$, where P is a Poisson algebra, defined by (0.6). Hence $\mathbb{F}$-forms of O^n are vector product n-Lie algebras on $\mathbb{F}^{n+1}$, $n \geq 3$, with a non-degenerate symmetric bilinear form (up to isomorphism, these n-Lie algebras depend on the equivalence class of the bilinear form up to a non-zero factor).

Appendix A

Below we list all known examples of infinite-dimensional simple n-Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0 for $n \geq 3$.

Let A be a commutative associative algebra over $\mathbb{F}$ and let $\mathfrak{g}$ be a Lie algebra of derivations of A, such that A contains no non-trivial $\mathfrak{g}$-invariant ideals.

Example 1. $S(A, \mathfrak{g}) = A$, where $\mathfrak{g}$ is an n-dimensional Lie algebra with basis $D_1, \ldots, D_n$, the n-ary Lie bracket being

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} D_1(f_1) & \ldots & D_1(f_n) \\ \multicolumn{3}{c}{\dotfill} \\ D_n(f_1) & \ldots & D_1(f_n) \end{pmatrix}.$$

Example 2. $W(A, \mathfrak{g})$, where $\mathfrak{g}$ is an $n-1$-dimensional Lie algebra with basis $D_1, \ldots, D_{n-1}$, the n-ary Lie bracket being

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} f_1 & \ldots & f_n \\ D_1(f_1) & \ldots & D_1(f_n) \\ \multicolumn{3}{c}{\dotfill} \\ D_{n-1}(f_1) & \ldots & D_{n-1}(f_n) \end{pmatrix}.$$

Example 3. $SW(A, D) = A^{\langle 1 \rangle} \oplus \cdots \oplus A^{\langle n-1 \rangle}$ is the sum of $n-1$ copies of A and $\mathfrak{g} = \mathbb{F}D$, the n-ary Lie bracket being the following. For $h \in A$, denote by $h^{\langle k \rangle}$ the corresponding element in $A^{\langle k \rangle}$, then

$$[f_1^{\langle j_1 \rangle}, \ldots, f_n^{\langle j_n \rangle}] = 0, \quad \text{unless } \{j_1, \ldots, j_n\} \supset \{1, \ldots, n-1\};$$

$$[f_1^{\langle 1 \rangle}, \ldots, f_{k-1}^{\langle k-1 \rangle}, f_k^{\langle k \rangle}, f_{k+1}^{\langle k \rangle}, f_{k+2}^{\langle k+1 \rangle}, \ldots, f_n^{\langle n-1 \rangle}] =$$

$$(-1)^{k+n-1}(f_1 \ldots f_{k-1}(D(f_k)f_{k+1} - f_k D(f_{k+1}))f_{k+2} \ldots f_n)^{\langle k \rangle},$$

extended on $SW(A, D)$ by anticommutativity.

It is an open problem whether there exist any other simple infinite-dimensional n-Lie (super)algebras over an algebraically closed field of characteristic 0 if $n > 2$. In particular are there any examples of infinite-dimensional simple n-Lie superalgebras over a field of characteristic 0, which are not n-Lie algebras, if $n > 2$?

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