On the Approximability of the Maximum Induced Matching Problem

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Abstract

In this paper we consider the approximability of the maximum induced matching problem (MIM). We give an approximation algorithm with asymptotic performance ratio $d-1$ for MIM in d-regular graphs, for each $d \geq 3$. We also prove that MIM is APX-complete in d-regular graphs, for each $d \geq 3$.

Keywords: induced matching; strong matching; regular graph; approximation algorithm; APX-completeness

1 Introduction

For a given graph G, an *induced matching* M is a set of non-intersecting edges in $E(G)$ such that no two edges in M are joined by an edge of G. In other words, the set of edges in the subgraph of G induced by $V(M)$ coincides with M. For all other relevant graph-theoretic definitions, the reader is referred to [3]. Let $\beta^*(G), \beta_0(G)$ and $\gamma(G)$ denote the size of a maximum induced matching, a maximum independent set and a minimum dominating set respectively, for a given graph G. Define MIM, MIS [13, problem GT20] and MDS [13, problem GT2] to be the problems of determining $\beta^*(G), \beta_0(G)$ and $\gamma(G)$ respectively, for a given graph G. Let MIMD, MISD and MDSD denote the decision versions of MIM, MIS and MDS respectively.

Stockmeyer and Vazirani [27] introduced MIM as a variant of the maximum matching problem and motivated MIM as the "risk-free" marriage problem: find the maximum number of married couples such that each married person is compatible with no married person other than his/her spouse.

Induced matchings have stimulated a great deal of interest in the discrete mathematics community, since finding large induced matchings is a subtask of finding a *strong edge colouring* (i.e. a proper colouring of the edges such that no edge is adjacent to two edges of the same colour) using a small number of colours. (See [24, 26] for algorithmic results relating to strong edge colourings and see [23] for a survey of previous work in this area.)

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MIMD is NP-complete – this was first shown in [27], where it was demonstrated that the result holds even for bipartite graphs of maximum degree 4. Also, Zito [28] showed that MIMD is NP-complete for $4k$-regular graphs, for $k \geq 1$. Cameron [5] independently established NP-completeness of MIMD in bipartite graphs. Her transformation begins from MISD in arbitrary graphs; by starting from the NP-complete restriction of MISD to graphs of maximum degree 3 [13, problem GT20], the bipartite graph constructed as an instance of MIMD has vertices of degree 7. Lozin [22] proved that MIMD is NP-complete for certain classes of bipartite graphs, including bipartite graphs of maximum degree 3. Additionally, Ko and Shepherd [19] asserted that there is a close relationship between the parameters β^* and γ: namely $\beta^*(S(G)) + \gamma(G) = n$ for any graph G, where $n = |V(G)|$ and $S(G)$ denotes the *subdivision graph* of G (i.e. the graph obtained from G by replacing each edge $e = \{u, w\}$ by two edges $\{u, v_e\}, \{v_e, w\}$, where v_e is a newly-introduced vertex). Thus, since MDSD is NP-complete for planar cubic graphs [17], an immediate corollary of Ko and Shepherd's observation is that MIMD is NP-complete for planar bipartite graphs, where each vertex in one partite set has degree 2 and each vertex in the other partite set has degree 3.[1] Kobler and Rotics [20] showed that MIMD is NP-complete in Hamiltonian graphs, claw-free graphs, chair-free graphs, line graphs and d-regular graphs, for $d \geq 5$.

On the other hand, MIM has been shown to be solvable in polynomial time for several graph classes, including chordal graphs [5], circular arc graphs [14], trapezoid graphs, interval-dimension graphs, cocomparability graphs, interval graphs [15], prime bipartite $(Star_{1,2,3}, Sun_4)$-free graphs [22], (P_5, D_n)-free graphs [20, 21], $(P_5, K_{1,n})$-free graphs [20], $(P_k, K_{1,n})$-free graphs [21], (bull,chair)-free graphs, line graphs of Hamiltonian graphs [20], graphs of bounded clique width, including (chair,co-P,gem)-free graphs [4], weakly chordal graphs [7], graphs of bounded asteroidal index, bipartite permutation graphs [8], polygon-circle graphs, interval-filament graphs and asteroidal triple-free graphs [6]. Lastly, Kobler and Rotics [20] gave a polynomial-time algorithm that either finds a maximum induced matching in a given graph G, or else reports that $\beta^*(G) < \beta_1(G)$, where $\beta_1(G)$ denotes the size of a maximum matching in G.

For MIM in trees, Fricke and Laskar [12] gave a linear-time algorithm. Independently, Zito [28] and Golumbic and Lewenstein [15], have constructed simpler linear-time algorithms. Note that in [12, 14], induced matchings are referred to as *strong matchings*.

Given the NP-hardness of MIM in graphs of bounded degree, it is of interest to consider approximation algorithms in this setting. (Terminology relating to approximability used in this section is defined in Section 2.) Zito [28] has shown that for d-regular graphs, MIM is approximable within $d-(d-1)/(2d-1)$, for each $d \geq 3$. In Section 3 we improve this bound by presenting an approximation algorithm for MIM in d-regular graphs which has asymptotic performance ratio $d-1$, for each $d \geq 3$. We also show that MIM admits a polynomial-time approximation scheme for planar graphs of maximum degree 3.

On the other hand, Zito [28] also showed that, for each $k \geq 1$, MIM is APX-complete for $4k$-regular graphs. In Section 4 we extend this result to show that MIM is APX-complete for d-regular graphs, for each $d \geq 3$. We also establish the APX-completeness of MIM for bipartite graphs of maximum degree 3.

2 Preliminaries

In this section we define the notation and terminology related to approximability that will be used in this paper. The reader may find a more detailed description of the topic in [2].

[1] Ko and Shepherd [18] also asserted that MIMD is NP-complete for planar cubic graphs; however their reasoning contains an error. In Section 4 we prove this result as a corollary of establishing the APX-completeness of MIM in cubic graphs.

Let P be an optimisation problem and let A_P be an *approximation algorithm* for P, i.e. an algorithm that returns a feasible solution for a given instance of P. For every instance x of P and for every feasible solution y of x, let $c_P(x, y)$ denote the *cost* of y. For a given instance x of P, let $opt_P(x)$ denote the optimal cost of a feasible solution and let $A_P(x)$ denote the cost of the feasible solution constructed by A_P. If P is a maximisation (respectively minimisation) problem, we define

$$R_{A_P}(x) = \frac{opt_P(x)}{A_P(x)} \quad \left(R_{A_P}(x) = \frac{A_P(x)}{opt_P(x)} \right).$$

The *(absolute) performance ratio* R_{A_P} of A_P is defined as follows:

$$R_{A_P} = \inf\{c \geq 1 \ : \ R_{A_P}(x) \leq c \text{ for all instances } x \text{ of } P\}.$$

If P admits an approximation algorithm with performance ratio c, then P is *approximable within c*. The *asymptotic performance ratio* $R^{\infty}_{A_P}$ of A_P is defined as follows:

$$R^{\infty}_{A_P} = \inf\{c \geq 1 \ : \ \exists N \bullet R_{A_P}(x) \leq c \text{ for all instances } x \text{ of } P \text{ with } opt_P(x) \geq N\}.$$

We say that P admits a *polynomial time approximation scheme* (PTAS) if, given any $\varepsilon > 1$ there exists an approximation algorithm $A_{P,\varepsilon}$ such that $R_{A_{P,\varepsilon}} \leq \varepsilon$ and, for every instance x of P, $A_{P,\varepsilon}$ runs in time polynomial in $|x|$.

Some optimisation problems do not admit a PTAS unless P=NP. One way to prove such a result is to use an approximation preserving reduction. Although several notions of such reductions have been proposed (see for example [10]), the L-reduction defined in [25] is perhaps the easiest one to use. We now give a definition of this reduction.

Definition 2.1. *Let P and Q be two optimisation problems. An* L-reduction *from P to Q is a four-tuple $(t_1, t_2, \alpha, \beta)$ where t_1, t_2 are polynomial time computable functions and α, β are positive constants with the following properties:*

1. *t_1 maps instances of P to instances of Q such that, for every instance x of P, $opt_Q(t_1(x)) \leq \alpha \cdot opt_P(x)$.*
2. *for every instance x of P, t_2 maps pairs $(t_1(x), y')$ (where y' is a feasible solution of $t_1(x)$) to a feasible solution y of x such that*

$$|opt_P(x) - c_P(x, t_2(t_1(x), y'))| \leq \beta |opt_Q(t_1(x)) - c_Q(t_1(x), y')|.$$

If there is an L-reduction from P to Q, we say that P is L-reducible *to Q and denote this by $P \leq_{\mathrm{L}} Q$.*

Let APX denote the class of optimisation problems that are approximable within c, for some constant c. Suppose that Q is a problem in APX. By [2, Lemma 8.2], we may define Q to be *APX-complete* if $P \leq_{\mathrm{L}} Q$ for every problem P in APX. By transitivity of the L-reduction [25], to show that Q is APX-complete it is sufficient to show that $P \leq_{\mathrm{L}} Q$ for some APX-complete problem P.

If Q is APX-complete, then Q does not admit a PTAS unless P=NP [2, p.261]. In particular, there is some constant c such that the problem of approximating Q within c is NP-hard. The following result, proved in [28], can assist with computing such a constant c.

Proposition 2.2. *Let P and Q be two optimisation problems such that there is an L-reduction from P to Q with parameters α and β. Suppose that it is NP-hard to approximate P within c. Then it is NP-hard to approximate Q within $\frac{\alpha\beta c}{(\alpha\beta - 1)c + 1}$.*

select $e = \{u, v\}$ from $E(G)$
$M \leftarrow e$
$E(G) \leftarrow E(G) \setminus (adj(N(u)) \cup adj(N(v)))$
$V(G) \leftarrow V(G) \setminus (N(u) \cup N(v))$
while $(E(G) \neq \emptyset)$
do
  $j \leftarrow \min_{u \in V(G)} \{deg(u)\}$
  select u from V_j
  $k \leftarrow \min_{v \in N(u)} \{deg(v)\}$
  select v from $V_k \cap N(u)$
  $M \leftarrow M \cup \{u, v\}$
  $E(G) \leftarrow E(G) \setminus (adj(N(u)) \cup adj(N(v)))$
  $V(G) \leftarrow V(G) \setminus (N(u) \cup N(v))$
od

Figure 1: The algorithm MinGreedy.

3 Approximation algorithms

We begin this section by presenting a greedy algorithm for approximating MIM in regular graphs. In what follows, for a graph $G = (V, E)$, V_i denotes the set of vertices of current degree i in G and we let Y_i represent $|V_i|$. The set of neighbours of a vertex $v \in V$ is represented by $N(v)$ and for $S \subseteq V$, $adj(S)$ denotes the set of edges incident with the vertices in S.

The algorithm MinGreedy shown in Figure 1 takes a d-regular graph G as input ($d \geq 3$) and returns an induced matching $M \subseteq E(G)$. We assume the input graph to be connected, otherwise the algorithm may be applied to each connected component. Then, for each step of the algorithm, after the first and before its completion, $\sum_{i=1}^{d-1} Y_i > 0$. The first step of the algorithm involves adding an arbitrary edge to M and deleting the appropriate edges and vertices. At each subsequent step, a vertex u is chosen from those of current minimum degree and a vertex v is chosen from the vertices of current minimum degree in $N(u)$. The edge $\{u, v\}$ is added to M and the appropriate edges and vertices are deleted.

We now establish a lower bound on the size of the induced matching returned by the algorithm MinGreedy.

Theorem 3.1. *For every integer $d \geq 3$, given a connected d-regular graph on n vertices, the algorithm MinGreedy returns an induced matching of size at least*

$$\frac{d(n-2)}{2(2d-1)(d-1)}.$$

Proof. Let $\alpha = (2d-1)(d-1)$. In the first step, the total number of edges deleted is at most

$$1 + 2(d-1) + 2(d-1)^2 = 2d(d-1) + 1 = \alpha + d.$$

Subsequently, at each step, the total number of edges deleted is at most

$$1 + (d-1) + (d-2) + (d-1)^2 + (d-2)(d-1) = (2d-1)(d-1) = \alpha.$$

Assuming the worst case in each step, we have

$$|M| \geq 1 + \frac{\frac{dn}{2} - \alpha - d}{\alpha} = \frac{d(n-2)}{2\alpha} = \frac{d(n-2)}{2(2d-1)(d-1)}. \tag{1}$$

□

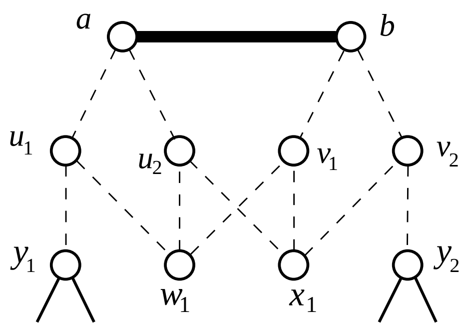

Figure 2: An example first step operation for a cubic graph.

Corollary 3.2. *For every integer $d \geq 3$, the algorithm MinGreedy approximates MIM for d-regular graphs with asymptotic performance ratio $d-1$.*

Proof. Zito [28] showed that the size of an optimum induced matching M^* of a d-regular graph on n vertices satisfies the inequality

$$|M^*| \leq \frac{dn}{2(2d-1)}. \tag{2}$$

Using the bounds given in (1) and (2), we have

$$\frac{|M^*|}{|M|} \leq \frac{\frac{dn}{2(2d-1)}}{\frac{d(n-2)}{2(2d-1)(d-1)}} = \frac{n}{n-2}(d-1).$$

□

It may be shown that for each $d \geq 3$, there exists an infinite family of d-regular graphs for which the algorithm MinGreedy only realises the lower bound given in (1). In order to demonstrate this, we consider the *operations* performed by the algorithm (an operation being the process of selecting an edge for inclusion into the induced matching and the subsequent deletion of the necessary vertices and edges). An operation may be described in terms of a subgraph that indicates the selected edge and those vertices and edges that are subsequently deleted.

For any $k \geq 1$, we construct a d-regular graph $G_{d,k}$ for which the algorithm MinGreedy has its poorest worst-case performance – this graph consists of two parts. Firstly, we have a subgraph that will be processed by the initial step of the algorithm. Secondly, the remaining part of the graph will consist of a chain of repeating subgraphs that are processed by the main body of the algorithm.

It may be verified that there exists an operation that may be performed as the first step of the algorithm that destroys vertices of degree d and generates two vertices of degree $d-1$. We call this a *first step operation.* As an example, Figure 2 shows one such operation for $d = 3$. Here the bold edge is added to the matching, the dashed edges are deleted and the vertices adjacent only to bold or dashed edges are deleted.

For larger d, the construction is as follows. Start with an edge $\{a, b\}$ and introduce two new sets of vertices $U = \{u_1, u_2, \ldots, u_{d-1}\}$ and $V = \{v_1, v_2, \ldots, v_{d-1}\}$. Connect vertex a to all vertices in U and connect vertex b to all vertices in V. Now introduce two more sets of new vertices $W = \{w_1, w_2, \ldots, w_{d-2}\}$ and $X = \{x_1, x_2, \ldots, x_{d-2}\}$. Connect each vertex in W to every vertex in U and connect each vertex in X to every vertex in V. Every vertex in $U \cup V$ now has degree $d-1$. For each i ($1 \leq i \leq d-2$), connect vertex w_i to vertex v_i and connect vertex x_i to vertex u_{i+1}. Finally introduce two more vertices y_1, y_2, connect

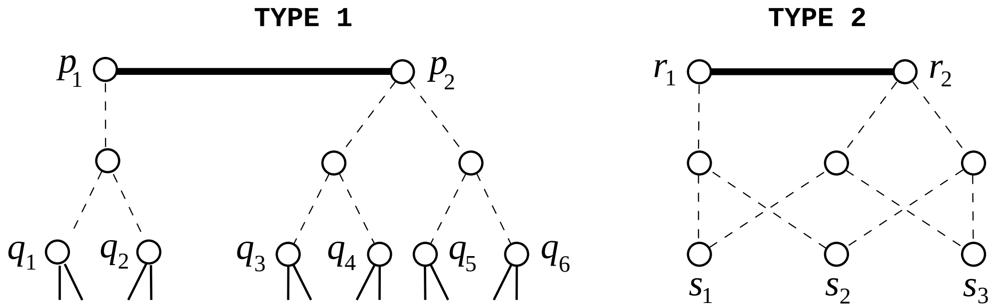

Figure 3: Operations for the algorithm MinGreedy on a cubic graph.

vertex y_1 to vertex u_1 and connect vertex y_2 to vertex v_{d-1}. Each of the vertices y_1 and y_2 has $d-1$ additional neighbours to be defined.

We now define two additional operations which we will call Type 1 and Type 2. The subgraph for Type 1 consists of an edge with endpoints p_1 and p_2 having degrees $d-1$ and d respectively. Let N_1 denote the neighbours of these endpoints other than p_1 and p_2 themselves. The size of N_1 is $2d-3$ and all vertices in N_1 have degree d. All neighbours of vertices in N_1, apart from p_1 and p_2, are also of degree d and are distinct; we label these vertices as $q_1, q_2, \ldots, q_D$, where $D = (2d-3)(d-1)$. The subgraph for Type 2 consists of an edge with endpoints r_1 and r_2 having degrees $d-1$ and d respectively. Let N_2 denote the neighbours of these endpoints other than r_1 and r_2 themselves. The size of N_2 is $2d-3$ and all vertices in N_2 have degree d. The vertices in N_2 form one half of the vertices of a $(d-1)$-regular bipartite graph on two sets of $2d-3$ vertices. Label the other half of the vertices in this bipartite subgraph with $s_1, s_2, \ldots, s_{2d-3}$. As an example, Type 1 and Type 2 operations for $d=3$ are given in Figure 3. In each instance the bold edges in Figure 3 are added by the algorithm MinGreedy to the induced matching, the dashed edges are deleted and the vertices adjacent only to bold or dashed edges are deleted.

We form a *repeating component* by *merging* operations together. By this we mean, for a pair of successive operations, we identify vertices in the first operation with vertices in the second operation. The repeating component is constructed from a number of Type 1 and Type 2 operations, together with a single Type 1a operation. A Type 1a operation is almost identical to a Type 1 operation except that vertex p_2 has $d-2$ neighbours of degree d and one neighbour of degree $d-1$ (other than p_1).

Take $2d-3$ copies, $C_1, C_2, \ldots, C_{2d-3}$, of Type 1 and merge them together by identifying vertex q_D in C_i, $1 \le i \le 2d-4$, with vertex p_1 in C_{i+1}. Then merge a copy of Type 1a by identifying vertex q_D in C_{2d-3} with vertex p_1 in the Type 1a. Now merge $(2d-1)(d-2)$ copies of Type 2 by identifying vertices $q_1, \ldots, q_{D-1}$ in the copies of the Type 1 and Type 1a subgraphs with vertices r_1, $s_1, \ldots, s_{2d-3}$ in the copies of the Type 2 subgraphs. The repeating component has a p_1 vertex of degree $d-1$ in one copy of a Type 1 subgraph, together with a vertex of degree $d-1$ in the Type 1a subgraph (the neighbour of p_2 of degree $d-1$ that is not labelled p_1). Again, an example for $d=3$ is given in Figure 4 where the two vertices of degree $d-1$ are labelled p_1 and z.

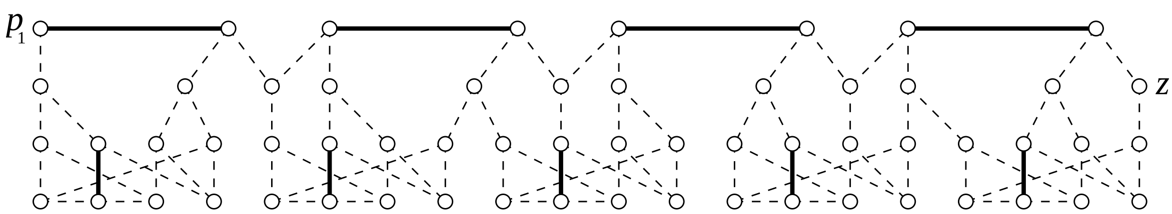

Figure 4: Repeating cubic component.

Next we form a *chain* of k repeating components, $D_1, D_2, \ldots D_k$, by connecting with an edge the vertex z of degree $d-1$ in D_i, $1 \le i \le k-1$, with the vertex p_1 of degree $d-1$ in D_{i+1}. (This leaves only one Type 1a operation, since the additional edges turn $k-1$ Type 1a operations into Type 1 operations.) Finally we obtain a d-regular graph $G_{d,k}$ by merging this graph with the subgraph for the first step operation. We do this by identifying vertices p_1 and z at either end of the chain of repeating components with vertices y_1 and y_2 from the first step operation.

In the worst case the algorithm MinGreedy would initially choose the edge $\{a, b\}$ in the first step operation, thus exposing the chain of repeating components. It would then perform $2k(d-1)-1$ Type 1 operations, then one Type 1a operation, followed by $k(2d-1)(d-2)$ operations of Type 2. Hence, in the worst case, the matching M satisfies $|M| = kd(2d-3)+1$. Let $\alpha = (2d-1)(d-1)$. The initial operation deletes $\alpha+d$ edges and each operation performed thereafter deletes α edges, except the Type 1a operation which deletes $\alpha - 1$ edges. Hence it may be verified that $G_{d,k}$ has $kd\alpha(2d-3) + \alpha + d - 1$ edges. Thus the lower bound of Theorem 3.1 gives $|M| \ge kd(2d-3)+1-\frac{1}{\alpha}$, i.e. $|M| \ge kd(2d-3)+1$ since $d \ge 3$. Therefore we have proved the following result.

Theorem 3.3. *For every $d \ge 3$ and $k \ge 1$, there exists a d-regular graph $G_{d,k}$ such that, in the worst case, the size of the induced matching returned by the algorithm MinGreedy on $G_{d,k}$ is equal to the lower bound given by Theorem 3.1.*

We conclude this section by considering the approximability of MIM in planar graphs. It turns out that MIM has a PTAS for planar graphs of maximum degree 3. To demonstrate this we introduce an additional graph problem. Given a graph $G = (V, E)$, define a set $S \subseteq V$ to be *2-independent* if, for any two vertices $u, v \in S$, the distance between them in G is at least 3. Let M2IS denote the problem of finding a maximum 2-independent set in a given graph G. Also let $L(G)$ denote the line graph of a given graph G. Clearly S is an induced matching in G if and only if S is a 2-independent set in $L(G)$. Duckworth et al. [11] showed that M2IS admits a PTAS for planar graphs. Now suppose that G is any planar graph of maximum degree 3. Then $L(G)$ is planar (c.f. [3, Theorem 10.4]), so the PTAS for M2IS in planar graphs, together with the simple reduction from MIM to M2IS, gives the following result.

Theorem 3.4. *MIM admits a PTAS for planar graphs of maximum degree 3.*

4 Non-approximability results

In this section we establish the APX-completeness of MIM in d-regular graphs, for each $d \ge 3$. Throughout this section we use the fact that MIM belongs to APX for bounded degree graphs [28]. We begin by proving APX-completeness for MIM in bipartite graphs of maximum degree 3.

Proposition 4.1. *MIM is APX-complete for bipartite graphs of maximum degree 3.*

Proof. Alimonti and Kann [1] proved that MDS is APX-complete for graphs of maximum degree 3. As mentioned in Section 1, Ko and Shepherd [18] proved that $\beta^*(S(G))+\gamma(G) = n$ for any graph G, where $n = |V(G)|$. It is known [16, p.50] that $\gamma(G) \ge \frac{n}{1+\Delta(G)}$, where $\Delta(G)$ denotes the maximum degree of G. Hence $\beta^*(S(G)) \le 3\gamma(G)$, so there is an L-reduction from MDS in graphs of maximum degree 3 to MIM in bipartite graphs of maximum degree 3, with constants $\alpha = 3$ and $\beta = 1$. □

Corollary 4.2. *It is NP-hard to approximate MIM in bipartite graphs of maximum degree 3 within $\frac{6600}{6659}$.*

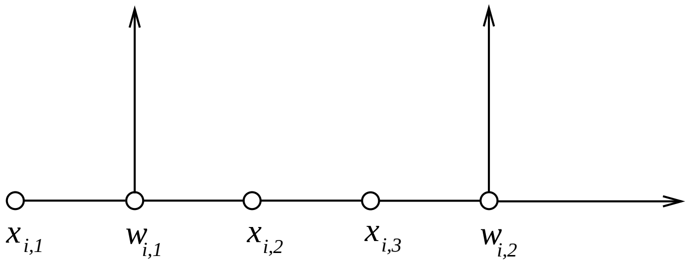

Figure 5: A typical subgraph C_i from the constructed instance of MIM (the arrows denote cross-edges).

Proof. Alimonti and Kann [1] proved that MDS is APX-complete for graphs of maximum degree 3. They gave an L-reduction from the problem of finding a minimum vertex cover in a given graph, which we denote by MVC [13, problem GT1], restricted to graphs of maximum degree 3, with constants $\alpha = 22$ and $\beta = 1$. As mentioned in the proof of Proposition 4.1, there is an L-reduction from MDS in graphs of maximum degree 3 to MIM in bipartite graphs of maximum degree 3 with constants $\alpha = 3$ and $\beta = 1$. The problem of approximating MVC in graphs of maximum degree 3 within $\frac{100}{99}$ is NP-hard [9]. Hence by combining these calculations, the result follows from Proposition 2.2. □

For arbitrary graphs of maximum degree 3, it is possible to substantially improve on the lower bound computed in Corollary 4.2 by considering the following alternative reduction, which uses techniques from [22].

Theorem 4.3. *MIM is APX-complete for graphs of maximum degree 3.*

Proof. We give a transformation from MIS in cubic graphs, which was shown to be APX-complete by Alimonti and Kann [1]. Let $G = (V, E)$ (a cubic graph) be an instance of MIS where $V = \{v_1, v_2, \ldots, v_n\}$. We form an instance $G' = (V', E')$ (graph of maximum degree 3) of MIM. For every vertex $v_i \in V$ $(1 \leq i \leq n)$, construct a subgraph C_i of G' as follows. Let the vertices in C_i be V_i, where $V_i = \{w_{i,1}, w_{i,2}, x_{i,1}, x_{i,2}, x_{i,3}\}$. Join all vertices in V_i to form a P_5 in the order $x_{i,1}, w_{i,1}, x_{i,2}, x_{i,3}, w_{i,2}$ (let F_i denote the four edges joining these vertices). Let $\{v_{i,1}, v_{i,2}, v_{i,3}\}$ be the set of vertices adjacent to v_i in G. Solely for ease of exposition, in what follows, we shall use $w_{i,3}$ to also denote the vertex $w_{i,2}$. For each j $(1 \leq j \leq 3)$, the vertex $w_{i,j}$ of C_i is joined in G' to exactly one of the $w_{r,s}$ vertices in the subgraph C_r corresponding to $v_{i,j}$, where $v_r = v_{i,j}$ and $1 \leq s \leq 3$ (call such an edge of G' a *cross-edge of* G'). (That is, there is a one-one correspondence between the edges of G and the cross-edges of G'.) There is obviously a degree of freedom involved in making such attachments, however the actual choice of assignment does not affect the remainder of the proof. We denote by E_i the union of F_i with the cross-edges of G' incident to vertices in V_i. Finally, let T_i denote the edges $\{x_{i,1}, w_{i,1}\}, \{x_{i,3}, w_{i,2}\}$.

Let $V' = \bigcup_{i=1}^{i=n} V_i$ and let $E' = \bigcup_{i=1}^{i=n} E_i$. It is clear that the graph G' constructed has maximum degree 3. A typical subgraph C_i of G' is illustrated in Figure 5.

We now demonstrate that $\beta^*(G') = n + \beta_0(G)$. Suppose that I is an independent set of G and that $k = |I|$. We construct a set of edges S as follows. For each i $(1 \leq i \leq n)$, if $v_i \in I$ then add the edges in T_i to S. If $v_i \notin I$ then add the edge $\{x_{i,2}, x_{i,3}\}$ to S. It may be verified that S is an induced matching in G' and $|S| = 2k + (n - k) = n + k$, so that $\beta^*(G') \geq n + \beta_0(G)$.

Conversely, let S be a maximum induced matching in G' and let i be given $(1 \leq i \leq n)$. Clearly $|S \cap E_i| \leq 2$ and it may be verified that $|S \cap E_i| = 2$ if and only if each of $w_{i,1}$ and $w_{i,2}$ is incident to some edge of S. Without loss of generality if $w_{i,1}$ is incident to some edge

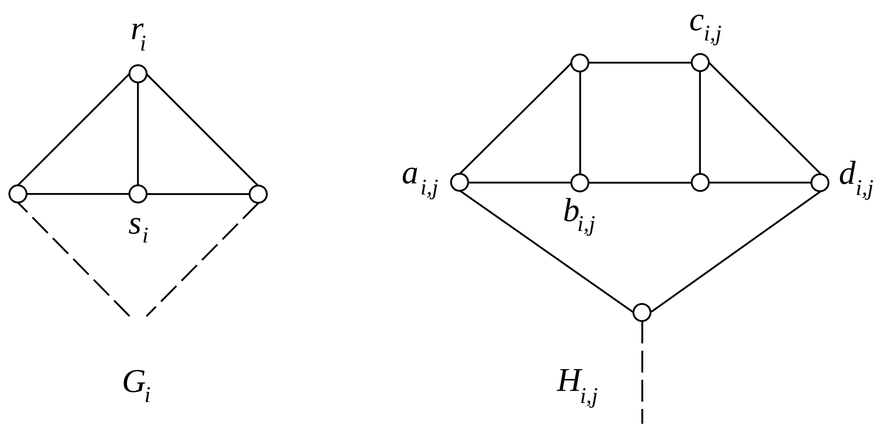

Figure 6: Typical gadgets in the constructed instance of MIM in cubic graphs.

e of S, we may assume that this edge is $e' = \{x_{i,1}, w_{i,1}\}$, for if not, we may replace e with e' in S. Similarly, if $w_{i,2}$ is incident to some edge e of S, we may assume that this edge is $e' = \{x_{i,3}, w_{i,2}\}$, for if not, we may replace e with e' in S (since $\{w_{i,1}, x_{i,2}\} \notin S$). Hence either $S \cap E_i = T_i$, or $|S \cap E_i| \leq 1$. Define $I = \{v_i \in V : S \cap E_i = T_i\}$. We firstly claim that I is independent in G. For if $\{v_i, v_j\} \in E$, then $\{w_{i,r}, w_{j,s}\} \in E'$, where $1 \leq r \leq 2$ and $1 \leq s \leq 2$. If $v_i \in I$ then $w_{i,r}$ is incident to some edge of $S \cap T_i$, but S is an induced matching in G', so $w_{j,s}$ cannot be incident to any edge of S. Thus $S \cap E_j \neq T_j$, so that $v_j \notin I$ as required. Let $k = |I|$ and suppose for sake of a contradiction that $k < \beta_0(G)$. Then $\beta^*(G') = |S| \leq 2k + (n-k) = n + k < n + \beta_0(G)$, a contradiction. Hence $k = \beta_0(G)$ so $\beta^*(G') = n + \beta_0(G)$ as required.

Note that any maximal independent set is also a dominating set, so $\beta_0(G) \geq \gamma(G)$. Also $\gamma(G) \geq \frac{n}{1+\Delta(G)}$ as mentioned in the proof of Proposition 4.1. Hence $\beta_0(G) \geq \frac{n}{4}$, so $\beta^*(G') \leq 5\beta_0(G)$. Thus our transformation is an L-reduction, from MIS in cubic graphs to MIS in graphs of maximum degree 3, with parameters $\alpha = 5$ and $\beta = 1$. □

Corollary 4.4. *It is NP-hard to approximate MIM in graphs of maximum degree 3 within* $\frac{475}{474}$.

Proof. It is NP-hard to approximate MIS in cubic graphs within $\frac{95}{94}$ [9]. The proof of Theorem 4.3 gives an L-reduction from MIS in cubic graphs to MIM in graphs of maximum degree 3 with constants $\alpha = 5$ and $\beta = 1$. The result follows by Proposition 2.2. □

We now extend the result of Theorem 4.3 to cubic graphs by adding suitable gadgets to vertices of degree 1 or 2.

Corollary 4.5. *MIM is APX-complete for cubic graphs.*

Proof. We use the same transformation as in the proof of Theorem 4.3, together with copies of the gadgets shown in Figure 6.

Let i $(1 \leq i \leq n)$ be given and consider the subgraph C_i. Using the dashed edges, attach the graph G_i to $x_{i,1}$ and attach the graphs $H_{i,1}$ and $H_{i,2}$ to $x_{i,2}$ and $x_{i,3}$ respectively. Then the graph G' so obtained is cubic. It may be verified that any induced matching in G' has at most one edge from G_i and at most two edges from $H_{i,j}$, for any i $(1 \leq i \leq n)$ and j $(1 \leq j \leq 2)$. Furthermore these upper bounds can be attained by selecting, in particular, the edges $\{r_i, s_i\}, \{a_{i,j}, b_{i,j}\}$ and $\{c_{i,j}, d_{i,j}\}$ as shown in Figure 6. Each of these edges is at distance at least 2 from each of $x_{i,1}$, $x_{i,2}$ and $x_{i,3}$. Hence we have that $\beta^*(G') = 6n + \beta_0(G)$, so this revised transformation is an L-reduction with parameters $\alpha = 25$ and $\beta = 1$. □

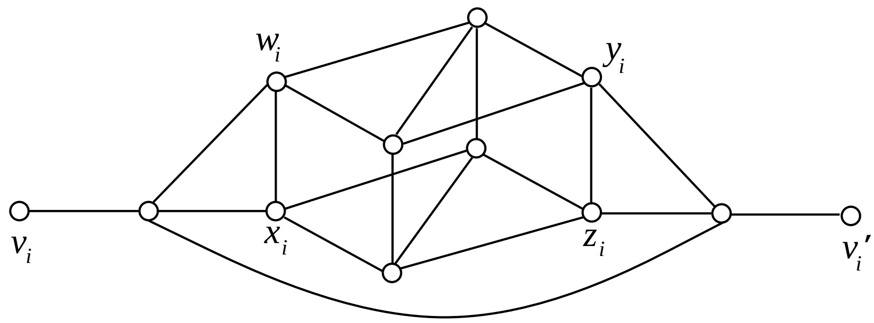

Figure 7: A typical gadget in the constructed instance of MIM in 4-regular graphs.

Corollary 4.6. *MIMD is NP-complete for planar cubic graphs.*

Proof. Clearly MIMD is in NP. The reduction described by Theorem 4.3 and Corollary 4.5 preserves the planarity of the input graph G. By reducing from the NP-complete restriction of MISD to planar cubic graphs [13, problem GT20], we obtain NP-completeness for MIMD in planar cubic graphs also. □

We continue this section by establishing the APX-completeness of MIM in d-regular graphs, for $d \geq 4$. We firstly consider the case $d = 4$.

Theorem 4.7. *MIM is APX-complete for 4-regular graphs.*

Proof. By Corollary 4.5, MIM is APX-complete for cubic graphs. Hence let $G = (V, E)$ (a cubic graph) be an instance of MIM and let $G' = (V', E')$ be a copy of G. Suppose that $V = \{v_1, v_2, \ldots, v_n\}$ and $V' = \{v_1', v_2', \ldots, v_n'\}$. We construct a 4-regular graph H as follows. Initially let $H = G \cup G'$. For each i ($1 \leq i \leq n$), connect v_i and v_i' using the subgraph H_i as shown in Figure 7.

It may be verified that any induced matching in H contains at most two edges from H_i ($1 \leq i \leq n$). Furthermore, this upper bound can be attained by selecting, in particular, the edges $\{w_i, x_i\}$ and $\{y_i, z_i\}$. Each of these edges is at distance at least 2 from each of v_i, v_i'. Hence we have that $\beta^*(H) = 2n + 2\beta^*(G)$. By Theorem 3.1, $\beta^*(G) \geq 3(n-2)/20$. Hence this transformation is an L-reduction with parameters $\alpha = \frac{58}{3}$ and $\beta = \frac{1}{2}$. □

We conclude this section by establishing APX-completeness for the case that $d \geq 5$. Note that Kobler and Rotics [20] showed that MIMD is NP-complete for d-regular graphs, for each $d \geq 5$. Their transformation takes a $(d-2)$-regular graph G as an instance of MISD and constructs a d-regular graph G' as an instance of MIMD, such that $\beta^*(G') = n + \beta_0(G)$, where $n = |V(G)|$. The same transformation is an L-reduction from MIS to MIM and, as in the proof of Theorem 4.3, the parameters are $\alpha = d$ and $\beta = 1$. Note that MIS is APX-complete for $(d-2)$-regular graphs, for each $d \geq 5$ [9]. Hence Kobler and Rotics' reduction immediately gives the following result.

Theorem 4.8. *MIM is APX-complete for d-regular graphs, for each $d \geq 5$.*

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