Collective decision making under qualitative possibilistic uncertainty : principles and characterization

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Collective Decision Making under Qualitative Possibilistic Uncertainty: Principles and Characterization

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Résumé

Cette Thèse pose la question de la décision collective sous incertitude possibiliste. On propose différents règles de décision collective qualitative et on montre que dans un contexte possibiliste, l'utilisation d'une fonction d'agrégation collective pessimiste égalitariste ne souffre pas du problème du Timing Effect. On étend ensuite les travaux de Dubois et Prade (1995, 1998) relatifs à l'axiomatisation des règles de décision qualitatives (l'utilité pessimiste) au cadre de décision collective et montre que si la décision collective comme les décisions individuelles satisfont les axiomes de Dubois et Prade ainsi que certains axiomes relatifs à la décision collective, particulièrement l'axiome de Pareto unanimité, alors l'agrégation collective égalitariste s'impose. Le tableau est ensuite complété par une axiomatisation d'un pendant optimiste de cette règle de décision collective.

Le système axiomatique que nous avons développé peut être vu comme un pendant ordinal du théorème de Harsanyi (1955). Ce résultat á été démontré selon un formalisme qui et basé sur le modèle de de Von NeuMann and Morgenstern (1948) et permet de comparer des loteries possibilistes. Par ailleurs, on propose une première tentative pour la carectérisation des règles de décision collectives qualitatives selon le formalisme de Savage (1972) qui offre une représentation des décisions par des actes au lieu des loteries.

De point de vue algorithmique, on considère l'optimisation des stratégies dans les arbres de décision possibilistes en utilisant les critères de décision caractérisés dans la première partie de ce travail. On offre une adaptation de l'algorithme de *Programmation Dynamique* pour les critères monotones et on propose un algorithme de *Programmation Multi-dynamique* et un algorithme de *Branch and Bound* pour les critères qui ne satisfont pas la monotonie.

Finalement, on établit une comparaison empirique des différents algorithmes proposés. On mesure les CPU temps d'exécution qui augmentent linéairement en fonction de la taille de l'arbre mais restent abordable même pour des grands arbres. Ensuite, nous étudions le pourcentage d'exactitude de l'approximation des algorithmes exacts par *Programmation Dynamique*: Il apparaît que pour le critère U_{ante}^{-max} l'approximation de l'algorithme de *Programmation Multi-dynamique* n'est pas bonne. Mais, ceci n'est pas si dramatique puisque cet algorithme est polynomial (et efficace dans la pratique). Cependant, pour la règle U_{ante}^{+min} l'approximation par *Programmation Dynamique* est bonne et on peut dire qu'il devrait être possible d'éviter une énumération complète par *Branch and Bound* pour obtenir les stratégies optimales .

Mots clefs: théorie de décision, théorie de possibilité, axiomatisation, choix collectif, arbres de décision

Abstract

This Thesis raises the question of collective decision making under possibilistic uncertainty. We propose several collective qualitative decision rules and show that in the context of a possibilistic representation of uncertainty, the use of an egalitarian pessimistic collective utility function allows us to get rid of the Timing Effect. Making a step further, we prove that if both the agents' preferences and the collective ranking of the decisions satisfy Dubois and Prade's axioms (1995, 1998) and some additional axioms relative to collective choice, in particular Pareto unanimity, then the egalitarian collective aggregation is compulsory. The picture is then completed by the proposition and the characterization of an optimistic counterpart of this pessimistic decision rule.

Our axiomatic system can be seen as an ordinal counterpart of Harsanyi's theorem (1955). We prove this result in a formalism that is based on Von NeuMann and Morgenstern framework (1948) and compares possibilisitc lotteries. Besides, we propose a first attempt to provide a characterization of collective qualitative decision rules in Savage's formalism; where decisions are represented by acts rather than by lotteries.

From an algorithmic standpoint, we consider strategy optimization in possibilistic decision trees using the decision rules characterized in the first part of this work. So, we provide an adaptation of the *Dynamic Programming* algorithm for criteria that satisfy the property of monotonicity and propose a *Multi-Dynamic programming* and a *Branch and Bound* algorithm for those that are not monotonic.

Finally, we provide an empirical comparison of the different algorithms proposed. We measure the execution CPU times that increases linearly according to the size of the tree and it remains affordable in average even for big trees. Then, we study the accuracy percentage of the approximation of the pertinent exact algorithms by *Dynamic Programming*: It appears that for U_{ante}^{-max} criterion the approximation of *Multi-dynamic programming* is not so good. Yet, this is not so dramatic since this algorithm is polynomial (and efficient in practice). However, for U_{ante}^{+min} decision rule the approximation by *Dynamic Programming* is good and we can say that it should be possible to avoid a full *Branch and Bound* enumeration to find optimal strategies.

Keywords: decision theory, possibility theory, axiomatization, collective choice, decision trees

Contents

INTRODUCTION

Decision theory is an important field in Artificial Intelligence (AI) that has emerged to support decision making, particularly decision making under uncertainty. Making a *decision* amounts to comparing possible alternatives and choose the best among them on the basis of a *criterion* or a *decision rule*. The use of one or another criterion depends essentially on the nature of the decision maker preferences and on the manner he or she expresses his knowledge. Besides, in order to solve any decision problem, one should clearly define the environment of decision. In the last decades, a significant progress has been noticed in this domain leading to several decision models [12,63,65]. Each of them is motivated by one or several concerns such as uncertainty, collectivity, multiple objectives, or multiple criteria handling.

In this Thesis, we are interested in decision making problems where the knowledge about states of nature is pervaded with uncertainty. The most well known approach for decision making under uncertainty is the *expected utility* model. This model relies on a probabilistic representation of uncertainty and it owes its success to an axiomatic justification for both *objective* (VNM's approach [72]) and *subjective* (Savage approach [82]) modeling of uncertainty.

In some cases, numerical information are unavailable and the decision maker is not able to express his preferences quantitatively. Therefore, probability theory becomes inappropriate, and gives a way to other uncertainty theories such as *possibility theory* where only an order between different consequences may be required. The latter offers a simple and natural framework to handle qualitative uncertainty. However, giving up the probabilistic quantification of uncertainty yields giving up the EU criterion as well. Fortunately, the growth of possibilistic decision theory has led to the proposal and the characterization of a panoply of possibilistic decision rules. In this work, we are, particularly, interested in optimistic and pessimistic criteria that are the qualitative counterparts of EU criterion [30, 33, 36].

In addition to uncertainty, another important aspect should be considered when making decisions, is "time". Generally, a decision maker does not make simple non related decisions but he follows a *strategy* i.e. a sequence of decisions that are executed successively. That is why graphical models are of special interest. Such models indeed, allow a clear description of the problem with a simple representation of different decision scenarios. Besides, they offer valuable computational methods; based on Dynamic Programming and variable elicitation; to compare them. In the literature, we can specify four main families of decision models which are: decision trees [78], influence diagrams [60], Markov decision processes [5] and Valuation Based Systems [86].

Simultaneously, in recent years there is a growing interest for *collective decision problems* in the Artificial Intelligence community. This field, is inherited from classical sciences in particular the branch of economic theory, social choice and welfare theory. Collective decision making aims to provide a social or collective utility function that can be used to evaluate different policies and to make decisions that satisfy not only one person but a group of persons or *agents*. Then, we can say that the main question to answer in the context of collective decision is how to aggregate individual utilities in order to obtain a collective one.

However, decision making process usually requires to deal with uncertainty. So, as for single decision making, collective decision making under uncertainty has been studied by many scholars. A seminal work has been proposed in 1955 by Harsanyi [59] to solve collective decision problems under risk. In fact, under the assumption that individual preference relations as well as the collective one satisfy VNM's axioms and considering the Pareto indifference axiom, Harsanyi proved that the collective utility is a weighted sum of the individual expected utilities. This representation theorem has always been interpreted as a justification of utilitarian aggregation.

Many contributions were inspired by this theorem, some of them have been developed to clarify and to confirm the use of the sum of expected utilities in context of collective decision making under uncertainty [94]. Others, like Myerson [74] proved that the use of Harsanyi's Theorem is the only way to avoid *timing effect*. More recently, several works [52, 70] have been proposed as an attempt to extend Harsanyi's theorem to the context of subjective expected utility model. However, Harsanyi's approach has been criticized by other researchers such that Diamonds [20] who suggested the use of egalitarianism to aggregate individual preferences.

Nowadays, collective decision problems under uncertainty form an attractive field that is always in vogue. To the best of our knowledge, all the contributions provided in this domain rely on probability theory. However, as we have mentioned before, this framework is inappropriate to handle partial ignorance or qualitative uncertainty. In this work, we attempt to tackle this problematic and to answer the following question: How to solve a collective decision problem under qualitative uncertainty?

In this thesis, we propose eight qualitative decision rules that differ with regard to the decision maker attitude toward uncertainty, the used aggregation function and the time where uncertainty is integrated. Hence, we consider uncertainty then a multi-agent aggregation (*ex-post* aggregation) or the contrary collectivity then uncertainty (*ex-ante* aggregation). Afterward, we have studied the relationship between those rules and proved that the coincidence between *ex-ante* and *ex-post* approach is not synonym of utilitarianism. This result has motivated us to provide a qualitative egalitarian counterpart of Harsanyi's theorem. So, we have performed an axiomatic system relative to the full min oriented and the full max oriented criteria in the style of VNM (objective uncertainty). Besides, we have proposed an attempt to extend Savage framework (subjective uncertainty) and its axiomatic system to handle collective decision problems.

In addition to this theoretical contribution, we have considered sequential decision making in possibilistic decision trees and provided new algorithms to find optimal strategies in such models using the different criteria proposed. Besides, we have performed an experimental study to compare the different algorithms and evaluate the quality of their solutions.

This thesis is decomposed into two main parts:

The first part presents an overview on decision making under uncertainty in various aspects:

1. Chapter 1 introduces the basic concepts relative to decision theory and recalls the most known decision model, the expected utility theory, and presents its axiomatic justification in both objective and subjective cases.

2. Chapter 2 presents possibility theory and the qualitative possibilistic decision rules. In this chapter we focus on optimistic and pessimistic utilities (the qualitative counter parts of expect utility) and we evoke their axiomatization in both VNM's and Savage's style.

3. Chapter 3 is devoted to collective decision making. We especially, detail Harsanyi's theorem and present a brief overview of related works.

4. Chapter 4 is dedicated to sequential decision problems using decision trees. These graphical models are presented in a probabilistic and a possibilistic (qualitative) version.

The second part of the thesis represents our main contributions. It is structured as follows:

1. Chapter 5 is the core of this thesis and it defines a new approach for collective decision making under qualitative uncertainty. In this chapter, we propose eight qualitative decision rules to solve underlined problems on the basis of optimistic and pessimistic utilities. Besides, we study the relations between these criteria and we propose an axiomatic system for criteria that are fully min oriented or max oriented.

2. Chapter 6 extends qualitative decision trees to collective sequential problems and provides algorithmic solutions for each of the qualitative decision rules proposed in Chapter 5.

3. Chapter 7 is dedicated to an experimental validation of the proposed algorithms.

4. Finally, Chapter 8 presents an extension of Savage's approach and its axiomatic system to solve collective decision problems and exposes the first steps to the characterization of the full min-oriented and max-oriented collective decision rules in the style of Savage.

Principle results of this Thesis are published in [6–8].

Part I

Background

CHAPITRE 1

Decision making under probabilistic uncertainty

1.1 Introduction

According to an old scientific tradition, decision making theory is a prevalent concern in cognitive psychology, economics and social sciences. In this domain, still in vogue, several studies have been carried out to justify the use of this or that decision criterion to compare alternatives. This justification is based on a set of axioms capturing the choice behavior of the decision maker.

The most known models for decision making under uncertainty are based on the expected utility (EU) criterion. It relies on a probabilistic representation of uncertainty. This classical decision rule has been provided with an axiomatic justification, in the middle of the last century.

In this chapter, we study the properties of this model (i.e. the expected utility model) and we recall its axiomatic foundations. First, we present the main definitions and notations relative to decision theory under uncertainty in Section 1.2. Then, Section 1.3 details the *expected utility* (EU) criterion and its axiomatization in the context of risk. Finally, Section 1.4 is devoted to *subjective expected utility* (SEU) and Savage's axiomatic system.

1.2 Basic concepts of decision theory

Decision making is primarily a matter of choosing between alternatives that most commonly are expressed implicitly. Thus, solving a decision problem amounts to providing appealing choice with respect to the specification of the decision problem at hand, i.e. the available knowledge about the environment and the decision maker preferences relative to possible results of different actions.

Decision problems under uncertainty

Formally, a decision making problem under uncertainty is defined by a set of possible states of the world S (states of nature) and a set of possible outcomes or consequences denoted by X. A decision also called action or act, assigns a consequence to each state of nature, it is a mapping from S to X. Besides, it is assumed that any decision maker is able to provide a well-defined preference relation denoted by $\succeq$ over the set of consequences.

Uncertainty

In order to make a decision, it is necessary beforehand to clearly define the knowledge over the set of states of nature and the associated consequences. According to the type of available information relative to states of nature, we can distinguish several forms of uncertainty, namely:

- **Probabilistic uncertainty:** where uncertainty can be modeled via a probability distribution. We can distinguish two types of probabilities:
  1. *Objective probabilities:* indicate the relative frequency of the realization of events. A situation of uncertainty characterized by objective probability is called a situation of *risk*. In this report, as in literature, we use the term risk to indicate objective probabilistic uncertainty.
  2. *Subjective probabilities:* are probabilistic measures derived from an individual's personal belief about whether an event is likely to occur when no frequencies are available.
- **Non probabilistic uncertainty:** Since probability theory presents some draw-backs to represent ignorance, in particular total ignorance, several non probabilistic theories (also named non classical theories of uncertainty) have been defined to cope with this issue, such as *imprecise probabilities* [90], *evidence theory* [85], *rough set theory* [75] and *possibility theory* [27, 31, 100, 101] that will be investigated in the next chapter.

Preferences

A preference relation can be defined by a complete pre-order order over a set of consequences. Formally, the preference between two consequences x and $y \in X$ is denoted by $x \succeq y$. It means that "x is at least as good as y" for the decision maker. $\succ$ denotes the asymmetric part of $\succeq$ and $\sim$ its symmetrical part such that:

- The strict preference relation $\succ$ is defined by $x \succ y$ if and only if $x \succeq y$ and $y \not\succeq x$ and it means that"x is strictly preferred to y".

- The indifference relation $\sim$ is defined by $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$ and it means that " x and y are indifferently preferred".

Generally, a preference relation $\succeq$ is assumed to verify a set of properties (axioms). The main axioms are defined as follows [41]:

Axiom. *(Completeness) $\forall x, y$ either $x \succeq y$ or $y \succeq x$.*

Axiom. *(Antisymmetry) If $x \succeq y$ and $y \succeq x$ then $x = y$.*

Axiom. *(Asymmetry) If x is strictly preferred to y, then y is not strictly preferred to x.*

$$x \succ y \text{ then } \neg(y \succ x).$$

Axiom. *(Transitivity) If x is at least as good as y and y is at least as good as z, then x is at least as good as z.*

$$\text{If } (x \succeq y \text{ and } y \succeq z) \text{ then } (x \succeq z).$$

Axiom. *(Reflexivity)The relation $\succeq$ over a set X is reflexive if every element of X is related to itself.*

$$\forall x \in X,\ x \succeq x.$$

In the remainder of this work, we are in particular interested by preference relations defining pre-order over consequences. The relation $\succeq$ is a pre-order on X iff it is transitive and reflexive. If the relation $\succeq$ satisfies also the completeness then we have a complete pre-order.

The preference relation of an agent over a set of consequences is usually represented by a utility function. This latter, denoted by u, is a mapping from the set of consequences X to a numerical or ordinal scale U ($u : X \mapsto U$). Formally, U = $\{u_1, \ldots u_n\}$ is a totally ordered set that belongs to $\mathbb{R}$ such that $u_1 \leq \cdots \leq u_n$. The worst (resp. best) utility is denoted by $u_\perp$ (resp. $u_\top$).

1.3 Expected Utility: Von Neumann and Morgenstern's theory

In this section, we present the *expected utility* model, the most known framework to deal with decision making problems under risk. *Expected utility (EU)* theory has emerged in 1738 with Bernoulli [10] in "Commentarii Academiae Scientiarum Imperialis Petropolitanae" (translated as "Exposition of a new theory on the measurement of risk"). It was proposed in order to solve the Saint Petersburg paradox [17].

This paradox is a theoretical game used in economics related to probability and decision theories. It presents the human beings behavior faced to random events. The value of the random variable is probably small with infinite expected payoff. Bernoulli has shown that in this situation, probability theory dictates a decision that no reasonable player would take. Thus to solve this problem, he proposed to use expected utility rather than expected value (outcome).

This proposition has motivated Von Neumann and Morgenstern [72] to provide an axiomatic system to justify the use of expected utility and to define necessary conditions under which a utility function exists.

Von Neumann and Morgenstern's approach

In a framework of decision making under risk, the decision maker's knowledge about the states of nature S is represented by a probability distribution. Besides, it is also assumed that the decision maker has a well defined weak preference relation $\succeq$ on the set of possible consequences X of its choices.

In Von Neumann and Morgenstern's (VNM) approach, an act or a decision is represented by a probability distribution over the set of possible outcomes. It is called a *simple probabilistic lottery* and it is denoted by $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$, where $\lambda_i = p(x_i)$ is the probability that the decision leads to an outcome x_i.

A probabilistic compound lottery denoted by $\langle\ \lambda_1/L_1, \ldots, \lambda_m/L_m\ \rangle$ is a normalized probability distribution over the set of lotteries where λ_i is the probability to obtain lottery L_i.

Therefore, a decision making problem under risk can be represented using:

- A set of consequences (outcomes) X,
- A set of probabilistic lotteries $\mathcal{L}$, where each lottery L_i is a probability distribution p over the set of consequences X,
- A utility function u: $X \mapsto U$.

Any compound probabilistic lottery can be reduced into a simple one, denoted by $Reduction$ ($\langle\ \lambda_1/L_1, \ldots, \lambda_m/L_m\ \rangle$). Formally, it is defined by:

$$Reduction(\langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle) = \langle \sum_{j=1..m} (\lambda_j \times \lambda_1^j)/x_1, \ldots, \sum_{j=1..m} (\lambda_j \times \lambda_n^j)/x_n \rangle \quad (1.1)$$

Example 1.1. *Let $L_1 = \langle 0.7/10, 0.3/20 \rangle$ and $L_2 = \langle 0.4/10, 0.6/20 \rangle$ be two probabilistic lotteries. Consider the compound lottery L' = $\langle$ 0.5/L, 0.5/L' $\rangle$. L' can be reduced into the single stage lottery L'' = $\langle$ 0.55/10, 0.45/20 $\rangle$ presented in Figure 1.1.*

Obviously, the reduction of a simple lottery gives a simple lottery as result. Since, the sum ($\sum$) and the product ($\times$) operations are polynomial, the reduction is polynomial in the size of the

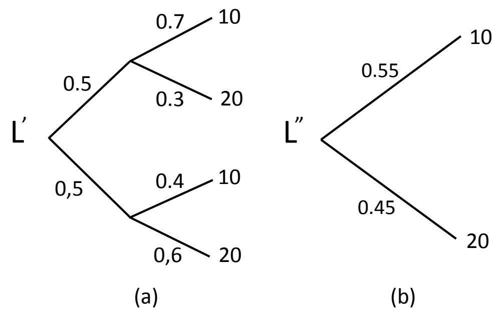

Figure 1.1: A compound probabilistic lottery (a) and its reduction (b)

compound lottery. Note that the size of a simple probabilistic lottery is simply the number of its outcomes; and the size of a compound lottery is the sum of the sizes of its sub-lotteries.

Solving a decision problem under risk amounts to evaluating risky alternatives and choosing among them. In this context, Von Neumann and Morgenstern's proposed to use the *expected utility* (EU) criterion to compare lotteries, i.e. probability distribution over consequences. Formally, the computation of the expected utility of a lottery L is performed as follows:

Definition 1.1. *Given a probabilistic lottery $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$ and a utility function u, the computation of the expected utility of L (denoted by $EU(L)$) is computed by:*

$$EU(L) = \sum_{x_i \in X} \lambda_i \times u(x_i). \tag{1.2}$$

Example 1.2. *Let $L = \langle 0.2/5, 0.2/15, 0.6/30 \rangle$ and $L' = \langle 0.7/0, 0.3/20 \rangle$ be two probabilistic lotteries. Using Equation 1.2, we have $EU(L) = (0.2 \times 5) + (0.2 \times 15) + (0.6 \times 30) = 22$ and $EU(L') = (0.7 \times 0) + (0.3 \times 20) = 6$.*
So, we can deduce that L is preferred to L'.

Formally, considering the expected utility criterion, any preference order $\succeq_{EU}$ defined on simple probabilistic lotteries can be extended to compare compound lotteries as follows:

$$L \succeq_{EU} L' \text{ iff } Reduction(L) \succeq_{EU} Reduction(L'). \tag{1.3}$$

Axiomatic system of Von Neumann and Morgenstern

At the middle of the last century, Von Neumann and Morgenstern have provided an axiomatic justification to the expected utility criterion [72]. They have shown that if the preference relation of any decision maker satisfies a set of axioms of rationality, then it can be represented via a utility function (EU) over lotteries. This axiomatic system is detailed in the following section.

Von Neumann and Morgenstern axiomatic system that characterize preference relation over lotteries is based on three fundamental axioms defined as follows:

Axiom. *($VNM1$: Weak order) The preference relation $\succeq$ is reflexive, complete and transitive.*

Axiom. *($VNM2$: Continuity) For any three probabilistic lotteries L, L' and $L^{''}$, if L is at least good as L' and L' is at least good as $L^{''}$ then there is a probability p for which the rational agent (DM) will be indifferent between lottery L' and the lottery in which L comes with probability p, $L^{''}$ with probability $(1-p)$.*

$$L \succeq L' \succeq L^{''} \Rightarrow \exists\, p \ \in]0,1[\ s.t.\ pL + (1-p)L^{''} \sim L'.$$

Axiom. *($VNM3$: Independence) For any probabilistic lotteries L, L' and $L^{''}$, if a DM prefers L to L', then his preferences over these lotteries isn't affected by mixing in a third one.*

$$L \succeq L' \Leftrightarrow \exists\, p \in]0,1]\ s.t.\ pL + (1-p)L^{''}\ \succeq\ pL' + (1-p)L^{''}.$$

The key axiom of VNM's formalism is the *independence* axiom ($VNM3$). It is the central axiom of the (objective) expected utility model and it can be interpreted as follows: If the decision maker prefers L to L' and he have to choose between $p \times L + (1-p) \times L''$ and $p \times L' + (1-p) \times L''$ then he will prefer the mixture $p \times L + (1-p) \times L''$ to $p \times L + (1-p) \times L''$ whatever the probability of the event that happens.

Von Neumann and Morgenstern [72] have claimed that a preference relation $\succeq$ satisfying the above axioms can be represented by an expected utility function. Their representation theorem is defined by:

Theorem 1.1. *If the preference relation $\succeq$ satisfies axioms $VNM1, \ldots, VNM3$ then it is exists a utility function u: $X \mapsto \mathbb{R}$ over the set of lotteries $\mathcal{L}$ such that:*

$$\forall L, L',\ L \succeq L' \quad \Leftrightarrow \quad EU(L) \geq EU(L'). \tag{1.4}$$

Theorem 1.1 can be described as follows: if the preference relation of any decision maker satisfies completeness, transitivity, continuity and independence axioms, then the DM behavior's obeys the maximization of the expected utility.

1.4 Subjective Expected Utility: Savage's approach

The VNM's model is a powerful and attractive tool to handle decision making under risk. However in real world problems, objective probabilities are not always available. To deal with such cases, Savage provided an extension of expected utility theory to a subjective context, namely *Subjective expected utility* [82].

Savage approach

In Savage's framework, a decision problem is represented by:

- A set of consequences X,
- A set of states of nature S (a subset E of S is an event),
- A decision (or "act") is a function f: $S \mapsto X$ and the set of acts is denoted by $\mathcal{F}$,
- $\succeq$ a preference relation on $\mathcal{F}$; $\succ$ denotes its asymmetric part, $\sim$ its symmetric part.

The subjective expected utility was developed by Savage in 1954 [82] as follows:

Definition 1.2. *Given a probabilistic distribution p over S and a utility function u on X, the subjective expected utility of an act f (denoted by $SEU(f)$) is defined by:*

$$SEU(f) = \sum_{s_i \in S} p(s_i) \times u(f(s_i)). \tag{1.5}$$

Example 1.3. *Let $S = \{s_1, s_2, s_3, s_4\}$ be a set of states of nature such that $p(s_1) = 0.2$, $p(s_2) = 0.1$, $p(s_3) = 0.4$ and $p(s_4) = 0.3$. Consider the two acts f and g given in Table 1.1.*
The utility value associated to each consequence is defined as follows: $u(f(s_1))$ = $u(g(s_2))$ = 5,

Acts/States	s_1	s_2	s_3	s_4
f	$f(s_1)$	$f(s_2)$	$f(s_3)$	$f(s_4)$
g	$g(s_1)$	$g(s_2)$	$g(s_3)$	$g(s_4)$

Table 1.1: Consequences of acts f and g

$u(f(s_2))$ = $u(g(s_3))$ = 10, $u(f(s_3))$ = $u(g(s_1))$ = 15, $u(f(s_4))$ = $u(g(s_4))$ = 20.

Using Equation 1.5, we have $SEU(f) = (0.2 \times 5) + (0.1 \times 10) + (0.4 \times 15) + (0.3 \times 20) = 14$ and $SEU(g) = (0.2 \times 15) + (0.1 \times 20) + (0.4 \times 5) + (0.3 \times 20) = 13$. So, f is preferred to g.

The axiomatic system of Savage

Like VNM's framework, the preference relation $\succeq$ between acts has to satisfy a certain number of axioms of rationality. Before addressing the axiomatic system relative to Savage's approach for subjective expected utility we need to define some basic notions [82]:

Definition 1.3. *(Constant act) A constant act $f_x \in \mathcal{F}$ provides the same consequence $x \in X$, whatever the state of the nature:*

$$\forall s \in S,\ f_x(s) = x.$$

Definition 1.4. *(Compound act) Giving two acts f and $g \in \mathcal{F}$ and an event $E \subseteq S$. The compound act fEg is defined by:*

$$fEg(s) = \begin{cases} f(s) & if \quad s \in E, \\ g(s) & if \quad otherwise. \end{cases}$$

This notion (of compound acts) is illustrated by Figure 1.2.

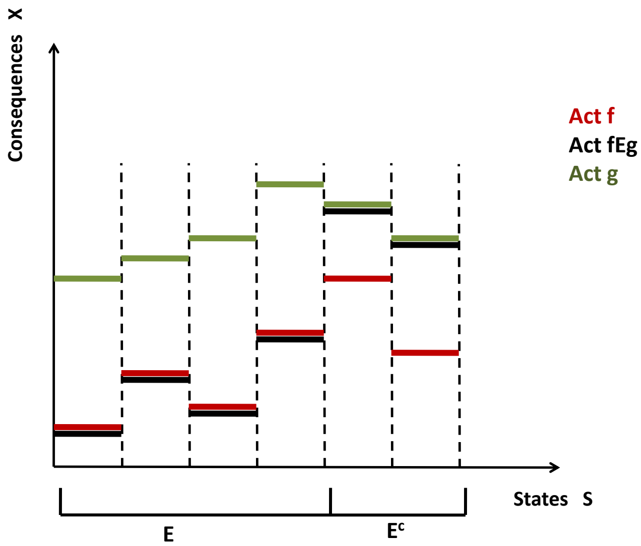

Figure 1.2: Compound act fEg

Definition 1.5. *(Null event) An event $E \subseteq S$ is said to be null iff:*

$$\forall f, g, h \in \mathcal{F},\ fEh \sim gEh.$$

Definition 1.6. *(Conditioning on E) For any event $E \subseteq S$, for any acts f and g, $f \succeq_E g$ iff $\forall h, fEh \succeq gEh$.*

Savage's axiomatic System is based on the following 6 axioms [82]:

Axiom. *($Sav1$: Complete Pre-order) The preference relation $\succeq$ is complete and transitive.*

Axiom. *($Sav2$: Sure Thing Principle) For all acts $f, g, h, h' \in \mathcal{F}$ and for every event $E \subseteq S$:*

$$fEh \succeq gEh \text{ iff } fEh' \succeq gEh'.$$

Axiom. *($Sav3$: Conditioning over constant acts) For any not null event $E \subseteq S$, and any constant acts f_x, $g_y \in \mathcal{F}$ it holds that:*

$$\forall E \subseteq S,\ f_x \succeq_E g_y \text{ iff } \forall h \in \mathcal{F}, f_xEh \succeq g_yEh.$$

Axiom. *($Sav4$: Projection from acts over events) For any consequences $x, y, x', y' \in X$, for any constant acts $f_x, g_y, f'_{x'}, g'_{y'} \in \mathcal{F}$. If $x \succ y$ and $x' \succ y'$, then $\forall E, D \subseteq S$ we have:*

$$f_xEg_y \succeq f_xDg_y \text{ iff } f'_{x'}Eg'_{y'} \succeq f'_{x'}Dg'_{y'}.$$

Axiom. *($Sav5$: Non triviality) $\exists\, f, g \in \mathcal{F}$, such that $f \succ g$.*

The use of Axiom $Sav5$ avoids the case of having only one consequence or the case where all consequences are equally preferred.

The basic axiom of Savage approach is the Sure Thing Principle ($Sav2$) that can be interpreted as follows: The preference order between two compound acts does not depend on their common consequences in a subset of S. Thus, Savage has showed that if the decision maker's preference among acts satisfies axioms $Sav1$ to $Sav5$ as well as two technical axioms of continuity and monotonicity, then this preference relation can be represented by an expected utility from the set of acts to the reals. So, for any act f, $SEU(f)$ is the expected utility of the consequence of f in the sense of a probability distribution on S. This implies the existence of a probability distribution p on S and a utility function u on X such that an act f is preferred to an another act g iff the subjective utility of f is higher than the subjective utility of g ($SEU(f) \succeq SEU(g)$).

1.5 Conclusion

The popularity of the expected utility models is essentially based on two strong axiomatic justifications: Von Neumann and Morgenstern's axiomatic system for decision making under risk and Savage's axiomatization for decision making under uncertainty. However, it have been proved that these formalisms cannot represent all decision makers' behaviors. Allais [2] and Ellsberg [37] questioned the axiomatic foundations for respectively VNM's and Savage approaches for expected utility.

Many research contributions have been proposed to overcome these limits, most of them are based on the *Choquet integral* [16] that is probably the most well known criterion. The latter is based on a generalized measure of uncertainty and has proven to be an important tool for decision making under risk and uncertainty. Besides, we can cite the *Rank-Dependent Utility* criterion [77] for decision under risk and its variants [83, 97] that have received an axiomatic justification.

These criteria involve the use of a quantitative representation of uncertainty. However, when the agent is unable to express his uncertainty and preferences numerically and he can only give an order among different alternatives, the previous criteria remain inappropriate. To solve such problems, qualitative decision models have been introduced with the emergence of other decision rules such as *Sugeno integral* [88] decision rule that is considered as an ordinal counterpart of *Choquet integral*. In particular, one may consider qualitative possibilistic decision rules that have emerged with the growth of Possibility theory. These alternatives are detailed in the next Chapter.

CHAPITRE 2

Decision Making under Possibilistic uncertainty

2.1 Introduction

Despite its success, it was proved that probability theory is appropriate only when numerical information are available and it presents some limits in regards to the representation of total ignorance and modeling qualitative uncertainty. Moreover, in decision theory, it has been proved that individuals reasoning does not necessarily conform to expected utility's assumptions.

Giving up the probabilistic quantification of uncertainty, yields to give up the EU criterion as well. Possibility theory presents an alternative model that offers a natural and simple framework to handle all kind of uncertainty, especially qualitative uncertainty and total ignorance. The development of possibilistic decision theory has lead to the proposition and the characterization of several possibilistic decision rules, namely, optimistic and pessimistic utilities that are qualitative counterparts of the EU criterion.

In this chapter, we will first give some basic elements of possibility theory in Section 2.2. Then in Section 2.3, we will present possibilistic decision rules, in particular pessimistic and optimistic utilities. Finally, in Section 2.4 we will focus on the axiomatization of pessimistic and optimistic utilities and we will detail their axiomatic systems in both VNM and Savage styles.

2.2 Basics on possibility theory

Possibility theory is a framework to handle uncertainty issued from Fuzzy Sets theory. It has been introduced by Zadeh [100] and further developed by Dubois and Prade [27, 28].

Possibility distribution

The concept of possibility distribution is the core of possibility theory and allows to represent qualitative knowledge about the real world. A possibility distribution π maps each state s in the universe of discourse S to a degree in a linearly ordered scale V, exemplified by $[0, 1]$. The values of this measure is interpreted as in table 2.1.

$\pi(s) = 0$	s is impossible
$\pi(s) = 1$	s is totally possible
$\pi(s) > \pi(s')$	s is preferred to s' (or is more plausible)

Table 2.1: Possibility distribution π

The possibilistic scale V can be interpreted in two manners:
- In the ***numerical*** or ***quantitative*** interpretation, the values of the possibility distribution make sense.
- In the ***ordinal*** or ***qualitative*** interpretation, the possibility degrees reflect only an order between the possible values.

Independently of the used scale, the extreme cases of knowledge are ***complete knowledge*** and ***total ignorance***. In the first case, we assign 1 to a state s_0 (totally possible) and 0 otherwise, i.e. $\exists\ s_0,\ \pi(s_0) = 1\ and\ \forall s \neq s_0,\ \pi(s) = 0$. In the second one, we assign 1 to all situations i.e. $\pi(s) = 1,\ \forall s \in S$.

A possibility distribution π is said to be *normalized* if there exists at least one element of S which is totally possible.

$$\max_{s \in S} \pi(s)\ =\ 1. \tag{2.1}$$

Possibility and necessity measures

In probability theory, uncertain knowledge about any event E is represented by a single probability measure P. This measure is auto-dual i.e. we can deduce the probability degree assigned to E from the probability P of its complement $(\neg E)$: $P(E) = 1 - P(\neg E)$.

In the contrary, possibility theory is characterized by the use of two dual measures: the possibility measure Π and necessity measure N.

- **Possibility measure** Π: Given a possibility distribution π, the possibility of any event $E \subseteq S$ is defined by:

$$\Pi(E) = \max_{s \in E} \pi(s). \tag{2.2}$$

$\Pi(E)$ denotes the possibility degree evaluating at which level E is **consistent** with the knowledge represented by π.

- **Necessity measure** N: Given a possibility distribution π, the necessity measure of an event $E \subseteq S$ is defined by:

$$N(E) = 1 - \Pi(\neg E) = \min_{s \in E} (1 - \pi(s)). \tag{2.3}$$

$N(E)$ denotes the necessity degree evaluating at which level E is **certainly** implied by the knowledge.

Example 2.1. *Let us consider the possibility distribution π describing the opinion of a doctor concerning the diagnosis of a patient. The universe of discourse related to this problem is the set of three diseases (d_1, d_2, d_3) and a healthy case (h): $S = \{d_1, d_2, d_3, h\}$.*

$$\pi(d_1) = 0.5, \quad \pi(d_2) = 1, \quad \pi(d_3) = 0.7, \quad \pi(h) = 0.$$

Note that π is normalized since $\max(0.5, 1, 0.7, 0) = 1$. For instance, the possibility degree that the patient has disease d_1 is 0.5.

Consider an event $E = \{d_1, d_3\}$ i.e. the patient suffers from d_1 or d_3. Then, the possibility and the necessity measures of this event are:

- $\Pi(E) = \max(0.5, 0.7) = 0.7$.
- $N(E) = 1 - \max(1, 0) = 0$.

Possibilistic lotteries

Following Dubois and Prade's possibilistic approach to decision making under uncertainty, a decision can be seen as a possibility distribution over a finite set of outcomes [30, 36]. A decision can be represented by a possibility distribution on a set of consequences X, called a (simple) *possibilistic lottery*, denoted by $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$; where $\lambda_j = \pi(x_j)$ is the possibility that the decision leads to consequence x_j. This degree is also denoted by $L[x_j]$.The outcome relative to each consequence x_j can be represented by a utility function u_j. Thus, we can rewrite the possibilistic lottery by $L = \langle \lambda_1/u_1, \ldots, \lambda_n/u_n \rangle$.

Similar to the notion of probabilistic lottery (detailed in Chapter 1 Section 3), Dubois and Prade have defined the notion of *possibilistic compound lottery*. Such a lottery is a normalized possibility distribution over a set of (simple or compound) lotteries and it is denoted by $L = \langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle$, where λ_i being the possibility of getting lottery L_i according to L. The possibility $\pi_{i,j}$ of getting consequence x_j from one of sub lotteries L_i depends on the possibility

λ_i of getting L_i and on the conditional possibility $\lambda_i^j = \pi(x_j \mid L_i)$ of getting x_j from L_i. So, we shall say that $\pi_{i,j} = \min(\lambda_i, \lambda_i^j)$. Hence, the possibility of getting x_j from a a lottery L = $\langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle$ is simply the *max*, over all L_i's, of $\pi_{i,j}$.

In Dubois and Prade's possibilistic decision theory [30,36], a compound lottery is assumed to be indifferent according to the decision maker's preference to the simple lottery defined by:

$$Reduction\ (L) = \langle \max_{i=1..m} \min(\lambda_i, \lambda_i^1)/x_1, \ldots, \max_{i=1..m} \min(\lambda_i, \lambda_i^n)/x_n \rangle \tag{2.4}$$

Example 2.2. *Let* $L_1 = \langle 1/0.5, 0.7/0.3 \rangle$ *and* $L_2 = \langle 0.6/0.5, 1/0.3 \rangle$ *be two simple possibilistic lotteries, and let* $L' = \langle 1/L_1, 0.8/L_2 \rangle$ *be a compound lottery represented in Figure 2.1 (a). The reduction of lottery* L' *into a simple lottery* L''*; presented by (b) in Figure 2.1; can be calculated using Equation 2.4 as follows:*

- $L''(x_1) = \max(\min(1, 1), \min(0.8, 0.6)) = 1$ *and*
- $L''(x_2) = \max(\min(1, 0.7), \min(0.8, 1)) = 0.8$*,*

So, L'' *= Reduction* $(L') = \langle 1/0.5, 0.8/0.3 \rangle$.

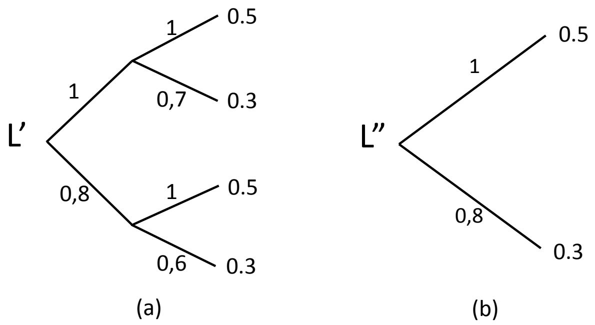

Figure 2.1: A possibilistic compound lottery (a) and its reduction (b).

The reduction of a compound lottery is polynomial in the size of the lottery since the $\min$ and the $\max$ operators are polynomial.

2.3 Possibilistic decision criteria

The development of possibility theory has led to the emergence of several possibilistic decision rules depending on the (quantitative or qualitative) nature of information describing the decision problem.

These decision criteria can be gathered into four classes that differ according to the interpretation of the used scale to encode uncertainty and utility (ordinal or numerical) as well as the commensurability between these scales i.e. if we are able or not to measure possibility and utility using a common standard:

- **Utility and possibility scales are commensurate and purely ordinal:** In this class, we have possibilistic qualitative criteria namely optimistic (denoted by U_{opt}) and pessimistic (denoted by U_{pes}) [30,36] integrals as well as possibilistic binary utilities (denoted by PU) [50].
- **Utility and possibility scales are commensurate:** This category concerns the Order of Magnitude Utility (denoted by $OMEU$) [76,96].
- **Utility and possibility scales are commensurate or not:** We can find *possibility-based likely dominance* (denoted by $L\Pi$) and *necessity-based likely dominance* (denoted by LN) [23,39].
- **Utility and possibility scales are commensurate and utilities are quantitative:** In this class we have *possibility-based Choquet integrals* (denoted by Ch_{Π}) and *necessity-based Choquet integrals* (denoted by Ch_N) [79].

In the sequel of this Chapter, we present the definition of each of the above possibilistic decision rules. Especially we detail the pessimistic and optimistic utilities and we focus on their axiomatization, that constitutes the basis of our contribution (Part 2).

Pessimistic utility

In a possibilistic framework, the evaluation of a lottery consists of the combination of possibility degrees π (encoding uncertainty) and utilities $u(x)$ (relative to possible consequences). To ensure this combination, Dubois and Prade [30, 34] have proposed pessimistic and optimistic criteria under the assumption that the uncertainty and utility scales are commensurate and purely ordinal. These decision rules constitute a qualitative counterpart to the expected utility criterion and depend on the decision maker behavior (an optimistic or pessimistic person).

The pessimistic criterion was initially defined by Whalen [95] and it generalizes the Wald criterion [89]. This criterion supposes that the decision maker considers the bad and plausible consequences. It estimates to what extent it is certain (i.e. necessary according to measure N) that L reaches a good utility.

Definition 2.1. *Let $L = \langle \lambda_1/x_1, \dots, \lambda_n/x_n \rangle$ be a possibilistic lottery, the pessimistic utility of L, denoted by U_{pes} is computed as follows:*

$$U_{pes}(L) = \max_{i=1..n} \min(u(x_i), n(L[x_i])). \tag{2.5}$$

where n is an order reversing function (e.g. n(L[x]) = 1 - L[x]).

Particular values for U_{pes} are as follows:

- If L assigns 1 to the worst utility ($u_\perp$) and 0 to all other utilities then $U_{pes}(L) = 0$.
- If L assigns 1 to the best utility $u_\top$ and 0 to all other utilities then $U_{pes}(L) = 1$.

Example 2.3. *Let $L = \langle 1/0.2, 0.7/0.5, 0.4/0.6\rangle$ and $L' = \langle 1/0.8, 0.3/0.7, 0.5/0.9\rangle$ be two possibilistic lotteries.*

Using Equation 2.5 we have:

- $U_{pes}(L)$ *= min(max(0.2,0),max(0.5,0.3),max(0.6,0.6)) = 0.2.*
- $U_{pes}(L')$ *= min(max(0.8,0),max(0.7,0.7),max(0.9,0.5)) = 0.7.*

$\Rightarrow U_{pes}(L') > U_{pes}(L)$ *so* $L' \succ_{U_{pes}} L$.

Dubois and Prade have proposed an adaptation of the pessimistic utility to evaluate acts in style of Savage. The definition of this decision rule is as follows:

Definition 2.2. *Given a possibilistic distribution π over S and a utility function u on the set of consequences X, the pessimistic utility of an act f is defined by:*

$$U_{pes}(f) = \min_{s_i \in S} \max(u(f(s_i)), 1 - \pi(s_i)). \tag{2.6}$$

Optimistic utility

The optimistic criterion was initially proposed by Yager [98, 99] and it can be seen as a mild version of the *maximax* criterion. This criterion estimates to what extend it is possible that L reaches a good utility. It captures the optimistic behavior of the decision maker that makes at least one of the good consequences highly plausible.

Definition 2.3. *Let $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n\rangle$ be a possibilistic lottery, the optimistic utility of L, denoted by U_{opt} is computed as follows:*

$$U_{opt}(L) = \max_{i=1..n} \min(u(x_i), L[x_i]). \tag{2.7}$$

Example 2.4. *Let $L = \langle 1/0.2, 0.7/0.5, 0.4/0.6\rangle$ and $L' = \langle 1/0.8, 0.3/0.7, 0.5/0.9\rangle$ be two possibilistic lotteries.*

Using Equation 2.7 we have:

- $U_{opt}(L)$ = *max(min(0.2,1),min(0.5,0.7),min(0.4,0.6)) = 0.5.*
- $U_{opt}(L')$ = *max(min(1,0.8),min(0.3,0.7),min(0.5,0.9)) = 0.8.*

$\Rightarrow U_{opt}(L') > U_{opt}(L)$ *so* $L' \succ_{U_{opt}} L$.

Particular values for U_{opt} are as follows:

- If L assigns 1 to the worst utility ($u_\bot$) and 0 to all other utilities then $U_{opt}(L) = 1$.
- If L assigns 1 to the best utility ($u_\top$) and 0 to all other utilities then $U_{opt}(L) = 0$.

Similar to its pessimistic counterpart, this criterion was also defined in Savage framework to choose among acts rather than lotteries. Its definition is as follows:

Definition 2.4. *Given a possibilistic distribution π over a set S and a utility function u on a set of consequences X, the optimistic utility of an act f is defined by:*

$$U_{opt}(f) = \max_{s_i \in S} \min(u(f(s_i)), \pi(s_i)). \tag{2.8}$$

It is important to note that the pessimistic and optimistic utilities represent particular cases of a more general criterion based on Sugeno integrals [25, 67, 88]:

$$S_{\gamma,u}(L) = \max_{\lambda \in [0,1]} \min(\lambda, \gamma(F_\lambda)). \tag{2.9}$$

where $F_\lambda = \{s_i \in S, u(f(s_i)) \geq \lambda\}$, is a set of preferred states for act f. γ is a monotonic set-function that reflects the decision maker attitude toward uncertainty. U_{opt} is recovered when γ is the *possibility measure* Π and U_{pes} is recovered when γ corresponds to *necessity measure* N.

Other possibilistic decision rules

Binary utilities (PU)

Binary possibilistic utility is a qualitative decision criterion that unifies the pessimistic and optimistic utilities. It has been proposed and axiomatized by Giang and Shenoy [50, 51] considering the importance of the best and the worst lotteries when making decision. In this bipolar model, the utility of an outcome is represented by a pair $u = \langle \overline{u}, \underline{u} \rangle$ where $max(\overline{u}, \underline{u}) = 1$. $\overline{u}$ is interpreted as the possibility of getting the ideal, good reward (denoted $\top$) and $\underline{u}$ is interpreted as the possibility of getting the anti ideal, bad reward (denoted $\bot$).

The lottery $\langle \lambda_1/u_1, \ldots, \lambda_n/u_n \rangle$ can be seen as a compound lottery, where each $u_i = \langle \overline{u_i}, \underline{u_i} \rangle$ is considered as a basic lottery $\langle \overline{u_i}/\top, \underline{u_i}/\bot \rangle$. Hence, the evaluation of this lottery comes down to the evaluation of its reduction such that:

Definition 2.5. *Let $L = \langle \lambda_1/u_1, \ldots, \lambda_n/u_n \rangle$ be a possibilistic lottery, the binary utility of L denoted by PU is computed as follows:*

$$\begin{aligned} &PU(\langle \lambda_1/u_1, \ldots, \lambda_n/u_n \rangle) = \\ &Reduction(\lambda_1/\langle \overline{u_1}/\top, \underline{u_1}/\bot \rangle, \ldots, \lambda_m \wedge \langle \overline{u_n}/\top, \underline{u_n}/\bot \rangle) \\ &= \langle \max_{j=1..m} (\min(\lambda_j, \overline{u_j}))/\top, \max_{j=1..m} (\min(\lambda_j, \underline{u_j}))/\bot \rangle \end{aligned} \tag{2.10}$$

As qualitative utilities and many other qualitative decision rules, binary utility suffers from a lack of decisiveness. However, a refinement has been proposed to obtain a more discriminating criterion [92, 93].

The possibilistic likely dominance LN and $L\Pi$

This criterion was proposed to make decisions when the utility and possibility scales are not commensurate [23, 39]. It does not assign global utilities to decisions but compare them using a pair-wise comparison between alternatives, hence the particularity of this decision rule. Possibilistic likely dominance can be defined in the ordinal setting by LN or in the numerical setting by $L\Pi$.

Giving two possibilistic lotteries $L_1 = \langle \lambda_1^1/x_1^1, \ldots, \lambda_n^1/x_n^1 \rangle$ and $L_2 = \langle \lambda_1^2/x_1^2, \ldots, \lambda_n^2/x_n^2 \rangle$, we say that "$L_1$ is as least as good as L_2" as soon as the possibility (in the case of $L\Pi$) or the necessity (in the case of LN) of the event *"the utility of L_1 is as least as good as the utility of L_2"* is greater or equal to the likelihood (possibility or necessity) of the event *"the utility of L_2 is as least as good as the utility of L_1"*. Formally:

Definition 2.6. *Let $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$ be a possibilistic lottery, the possibility likely dominance relations $\succeq_{LN}$ and $\succeq_{L\Pi}$ are defined as follows:*

$$L_1 \succeq_{LN} L_2 \text{ iff } N(L_1 \geq L_2) \geq N(L_2 \geq L_1). \tag{2.11}$$

where $N(L_1 \geq L_2) = 1 - \max_{u_i^1, u_j^2 \text{ s.t } u_i^1 < u_j^2} (\min(\lambda_i^1, \lambda_j^2))$.

$$L_1 \succeq_{L\Pi} L_2 \text{ iff } \Pi(L_1 \geq L_2) \geq \Pi(L_2 \geq L_1). \tag{2.12}$$

where $\Pi(L_1 \geq L_2) = \max_{u_i^1, u_j^2 \text{ s.t } u_i^1 \geq u_j^2} (\min(\lambda_i^1, \lambda_j^2))$.

Order of Magnitude Expected Utility (OMEU)

Order of magnitude expected utility is based on "disbelief functions" or "Kappa-rankings" measures initially proposed by [87]. A measure $\kappa : 2^S \to Z^+ \cup \{+\infty\}$ is a kappa-ranking iff:

- $\min_{s \in S} \kappa(s\}) = 0$

- $\kappa(E) = \min_{s \in E} \kappa(\{s\})$ if $\emptyset \neq E \subseteq S$, $\kappa(\emptyset) = +\infty$

This criterion have received an interpretation in term of order of magnitude of "small" probabilities. Hence, "$\kappa(E) = i$" is equivalent to $P(E)$ is of the same order of ε^i, for a given fixed infinitesimal ε. In this context, an event E is more likely than event D if and only if $\kappa(E) < \kappa(D)$. In [29], authors have pointed out that there exists a close link between kappa-rankings and possibility measures, insofar as any kappa-ranking can be represented by a possibility measure, and vice versa. Thus, order of magnitude utilities have been defined in the same way to rank outcomes in term of dissatisfaction degrees [76, 96].

Definition 2.7. *Let $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$ an order of magnitude lottery, the order of magnitude of the expected utility of L is computed as follows:*

$$OMEU(L) = \min_{i=1,n} \{\lambda_i + u(x_i)\} \tag{2.13}$$

Possibilistic Choquet integrals

When the decision problem is defined using heterogeneous information and the knowledge about states of nature is possibilistic and the utility degrees are cardinal, then Choquet integrals seem to be an appropriate decision rule to consider. This criterion allows the use of any monotonic set function μ, also called a capacity or fuzzy measure that may be a probability measure, necessity measure, possibility measure, or belief functions measure etc.

In the case of possibilistic decision problem and considering the decision maker behavior, we use the *necessity-based Choquet integrals* (μ = N) for cautious decision makers and the *possibility-based Choquet integrals* (μ = Π) for adventurous ones. Hence the definition of possibilistic Choquet integrals is as follows:

Definition 2.8. *Let $L = \langle \lambda_1/u_1, \ldots, \lambda_n/u_n \rangle$ a possibilistic lottery, the necessity-based Choquet integrals of L denoted by $Ch_N(L)$ is defined by:*

$$Ch_N(L) = u_1 + \sum_{i=2..p} (u_i - u_{i-1}) \times N(L \geq u_i). \tag{2.14}$$

The possibility-based Choquet integrals of L denoted by $Ch_\Pi(L)$ is defined by:

$$Ch_\Pi(L) = u_1 + \sum_{i=2..p} (u_i - u_{i-1}) \times \Pi(L \geq u_i). \tag{2.15}$$

2.4 Axiomatization of pessimistic and optimistic utilities

Under the assumption that the utility and the possibility scales are commensurate and purely ordinal, Dubois and Prade [30, 36] have proposed pessimistic and optimistic utilities to evaluate decisions. Similar to their probabilistic counterpart, these criteria have received a theorical characterization in both VNM and Savage frameworks. These axiomatic systems are detailed in the remaining of this Section.

Axiomatization in the style of VNM

Qualitative (pessimistic and optimistic) utilities have been axiomatized in the style of VNM [30, 36] to characterize preference relations between possibilistic lotteries.

Axiomatization of U_{pes} in the style of VNM

Let $\succeq$ be a preference relation on a set of possibilistic lotteries $\mathcal{L}$. The axiomatic system relative to the pessimistic utility U_{pes}, proposed by Dubois and Prade [30] relies on the following axioms:

Axiom. *($A1$: Total pre-order) $\succeq$ is complete and transitive.*

Axiom. *($A2$: Certainty equivalence) $\forall Y \subseteq X$, $\exists x \in Y$ s.t. x (L[x] =1; L[y] = 0 $\forall y \neq x$) and Y (L[y] =1 $\forall y \in Y$; L[y] = 0 otherwise) are equivalent for $\succeq$.*

Axiom. *($A3^-$:Risk aversion) $\forall x \in X$, if $L[x] \leq L'[x]$ ($L[x]$ is more specific than $L'[x]$), then $L \succeq L'$.*

Axiom. *($A4$: Independence) If L and L' are equivalents, then $\langle \lambda/L, \mu/L'' \rangle$ and $\langle \lambda/L', \mu/L'' \rangle$ are also equivalents, for any λ, μ s.t. $\max(\lambda, \mu) = 1$.*

Axiom. *($A5$: Reduction of lotteries) For any (compound possibilistic lottery) L, $L \sim Reduction(L)$.*

Axiom. *($A6^-$: Continuity) $\forall x \in X$, if $L'[x] \leq L[x]$ then $\exists \lambda$ s.t. $L' \sim \langle 1/L, \lambda/X \rangle$.*

In 1998, Dubois et al. [36] have performed a deeper study of this characterization and they have provided an axiomatizatic system with a lower number of axioms. Just four axioms are necessary and sufficient for the representation theorem of the pessimistic utility.

In fact, authors have shown that it is possible to ignore the use of Axiom $A5$ relative to reduction of compound lotteries since it is implicitly obtained by the definition of possibilistic lotteries. Besides, they have proved that Axiom $A2$ is also useless and redundant since it is a consequence of the use of Axiom $A1$, Axiom $A4$ and Axiom $A6^-$.

Thus, a new axiomatic system was proposed for pessimistic utility. It encompasses Axiom $A1$, Axiom $A3^-$, Axiom $A4$ and a new form of the continuity axiom (Axiom $A6^-$') that is defined by:

Axiom. *($A6^-$': Continuity) $\forall x \in X$, $\forall L[x]$, $\exists \lambda \in [0, 1]$ s.t. $L[x] \sim \langle 1/u_\top, \lambda/u_\perp \rangle$. Where $u_\top$ and $u_\perp$ are respectively the best and the worst utility.*

The detailed axiomatic system has allowed Dubois and Prade to provide a representation theorem relative to the pessimistic utility such that:

Theorem 2.1. *If a preference relation $\succeq$ on $\mathcal{L}$ satisfies axioms $A1 \dots A6^-$ (or axioms $A1$, $A3^-$, $A4$ and $A6^-$') then there exists a possibility distribution π: $S \mapsto [0,1]$, an order reversing and a utility function u: $X \mapsto [0,1]$ such that:*

$$L \succeq L' \quad \textit{iff} \quad U_{pes}(L) \geq U_{pes}(L'). \tag{2.16}$$

Axiomatization of U_{opt} in the style of VNM

In their work [30], Dubois et al. have defined the characterization of the optimistic utility U_{opt} from the axiomatization relative to the pessimistic utility by replacing axioms $A3^-$ and $A6^-$ by their optimistic counterparts i.e. axioms $A3^+$ and $A6^+$ defined by:

Axiom. *($A3^+$: Uncertainty attraction) $\forall x \in X$, if $L[x] \geq L'[x]$ ($L'[x]$ is more specific than $L[x]$), then $L' \succeq L$.*

Axiom. *($A6^+$: Continuity) $\forall x \in X$, if $L'[x] \leq L[x]$ then $\exists \lambda$ s.t. $L' \sim \langle \lambda/L, 1/X \rangle$.*

Furthermore, in [36] they have provided an improved axiomatic system for the optimistic utility by resuming axioms $A1$ and $A4$ that are the same as the pessimistic case, replacing Axiom $A3^-$ by Axiom $A3^+$ and finally providing an optimistic counterpart of Axiom $A6^-$' such as:

Axiom. *($A6^+$': Continuity) $\forall x \in X$, $\forall L[x]$, $\exists \lambda \in [0,1]$ s.t. $L[x] \sim \langle \lambda/u_\top, 1/u_\bot \rangle$. Where $u_\top$ and $u_\bot$are respectively the best and the worst utility.*

Considering those axioms, the representation theorem relative to the optimistic utility is defined as the following [30, 36]:

Theorem 2.2. *If a preference relation $\succeq$ on $\mathcal{L}$ satisfies axioms $A1 \dots A6^+$ (or axioms $A1$, $A3^+$, $A4$ and $A6^+$') then there exists a possibility distribution π: $S \mapsto [0,1]$ and a utility function u: $L \mapsto [0,1]$ such that:*

$$L \succeq L' \quad \textit{iff} \quad U_{opt}(L) \geq U_{opt}(L'). \tag{2.17}$$

Axiomatization in the style of Savage

Axiomatization of U_{pes} in the style of Savage

As mentioned before, in Savage framework, preference relations are defined over acts rather than lotteries. In this context, Dubois et al. have provided an axiomatic system for U_{pes} [34, 35]. Considering a set of acts $\mathcal{F}$ and a set of consequences X, this axiomatic system is defined by the following axioms:

Axiom. *($Sav1$: Complete pre-order) The preference relation $\succeq$ is reflexive, complete and transitive.*

Axiom. *($Sav5$: Non triviality) $\exists\, f, g \in \mathcal{F}$, such that $f \succ g$.*

These axioms are the same axioms proposed by Savage. However, authors have proposed a weak version of Axiom $Sav3$ defined by:

Axiom. *($WSav3$: Weak coherence with constant acts) For any not null event $E \subseteq S$, f_x and g_y two constant acts, then $f_x \succ_E g_y \Rightarrow f_xEh \succeq g_yEh$, $\forall h \in \mathcal{F}$.*

Besides, authors have proposed two additional axioms reflecting the decision maker behavior.

Axiom. *(RDD: Restricted disjunctive dominance) Let f and g be any two acts and a constant act f_x of value x: $f \succ g$ and $f \succ f_x \Rightarrow f \succ g \vee f_x$ (where $g \vee f_x$ gives the best of the results of g(s) and f_x(s) in each state s).*

Axiom. *(Pes: Pessimism) $\forall f, g \in \mathcal{F}$ and any event $E \subseteq S$. If $fEg \succ f$ then $f \succeq gEf$.*

Axiom RDD is interpreted as follows: if an act f is preferred to an act g and also preferred to a constant act f_x, then f still preferred to g even if the worst consequences of g are improved to the value x. Axiom Pes implies that if changing f into g may improve the expectation of act f when $\neg E$ occurs then there is no way to improve the act by changing f into g when E occurs. On the basis of this axiomatic system, [34, 35] have proposed a representation theorem of the pessimistic utility as follows:

Theorem 2.3. *If a preference relation $\succeq$ on acts satisfies the axioms $Sav1$, $Sav5$, $WSav3$, RDD and Pes then there exists a utility function u: $X \mapsto [0, 1]$ and a possibility distribution π: $S \mapsto [0, 1]$ such that $\forall f, g \in \mathcal{F}$:*

$$f \succeq g \quad \text{iff} \quad U_{pes}(f) \geq U_{pes}(g). \tag{2.18}$$

Axiomatization of U_{opt} in the style of Savage

The axiomatic system of U_{opt} in the context of Savage shares some axioms with its pessimistic counterpart namely: axioms $Sav1$, $Sav5$ and $WSav3$. The remaining axioms are:

Axiom. *(RCD: Restricted conjunctive dominance) Let f and g be any two acts and $[x]$ be a constant act of value x: $g \succ f$ and $f_x \succ f \Rightarrow g \wedge f_x \succ f$ (where $g \wedge f_x$ gives the worst of the results of g(s) and f_x(s) in each state s).*

Axiom. *(Opt: Optimism) $\forall f, g \in \mathcal{F}$ and $E \subseteq S$. If $f \succ fEg$ then $gEf \succeq f$.*

Axiom RCD is the dual property of the restricted disjunctive dominance. It allows a partial decomposability of qualitative utility with respect to the conjunction of acts in the case where one of them is constant. Axiom Opt implies that changing f into g does not improve the expectation of act f when $\neg E$ occurs. However, we can form a better act by changing f into g when E occurs.

Using the axioms presented above, Dubois and Prade [34, 35] have proposed the following representation theorem of optimistic utility:

Theorem 2.4. *If a preference relation $\succeq$ on acts satisfies axioms $Sav1$, $Sav5$, $WSav3$, RCD and Opt then there exists a utility function u: $X \mapsto [0,1]$ and a possibility distribution π: $S \mapsto [0,1]$ such that $\forall f, g \in \mathcal{F}$:*

$$f \succeq g \quad \textit{iff} \quad U_{opt}(f) \geq U_{opt}(g). \tag{2.19}$$

Note that in [12], Dubois et al. have proposed to replace axioms RCD and RDD by a non-compensation assumption defined as follows:

Axiom. *(NC: Non Compensation) Whatever $E \subseteq S$, $f_x, g_y \in \mathcal{F}$:*

$$\begin{cases} h_\top E g_y \sim g_y & \textit{or} \quad h_\top E g_y \sim h_\top E h_\bot \\ f_x E h_\bot \sim f_x & \textit{or} \quad f_x E h_\bot \sim h_\top E h_\bot \end{cases}$$

where $h_\top$ (resp. $h_\bot$) is the constant act that gives always the best (resp. worst) consequence $\top$ (resp. $\bot$).

2.5 Conclusion

In this chapter, we have presented an overview of possibility theory that offers a flexible and simple framework to represent uncertainty as well as the main possibilistic decision rules. We especially investigated qualitative utilities (U_{opt} and U_{pes}) and their axiomatic systems in both VNM and Savage frameworks.

As mentioned before, these criteria present the qualitative counter part of the expected utility model and constitute the groundwork of our contribution (Part 2). We have chosen these decision rules among others because we are interested to make decisions in a purely ordinal context and assuming a certain form of commensurability between the plausibility and the utility scales. Besides, these criteria have received a solid axiomatic justification that guarantee the rationality of the decision making process considering the decision maker attitude.

CHAPITRE 3

Collective Decision Making

3.1 Introduction

A Multi-Agent System (MAS) includes several entities (called agents) interacting within a common environment. An agent can be a virtual or physical autonomous entity such as software programs, robots, or human beings. Each interacts with others in order to meet preset goals or objectives according to available resources and skills. This paradigm describes many situations in real world problems, especially, collective decision making where decisions involves a group of agents instead of a single one.

In this context, the main issue is how to make a decision that satisfies everybody. However, decision theory as presented in the previous chapters appears inappropriate to answer this question. So, it was necessary to provide an extension of this discipline to be able to handle not only the preferences of one decision maker but all concerned ones. Hence, the development of collective decision theory.

This chapter focuses on the basics of collective decision theory and especially on Harsanyi's theorem and works relative to social choice theory. The remaining of this chapter is structured as follows: Section 3.2 defines utilitarian and egalitarian collective utility functions. Section 3.3 details Harsanyi's theorem regarding collective decision making under risk, in the style of VNM's expected utility framework. Finally, Section 3.4 is devoted to study the main proposed extensions and criticisms relative to Harsanyi's theory.

3.2 Collective utility functions

In collective decision problems, we suppose that agents are able to express their satisfaction over alternatives by a cardinal or ordinal preference relation. Then, the collective utility function (CUF) is the result of aggregating individual preferences (see [73] for more details).

Formally, we define a collective decision problem by a set $A = \{1, \ldots, p\}$ of agents, where each agent $i \in A$ is supposed to express his preferences on a set of consequences X, by a ranking function or a utility function u_i that associates to each element of X a value in a subset of $\mathbb{R}^+$ (typically in the interval $[0, 1]$). Thus to solve this problem, we need to define for each $x \in X$ a collective utility degree that reflects the collective preference.

When the collective preference depends only on individual utilities, it can be obtained by a collective utility function of the form $Agg(x) = f(u_1(x), \ldots, u_p(x))$.

Utilitarian collective utility functions

Classical utilitarian utility theory prescribes that the best decisions are those which maximize the sum of individual utilities [45,73]. This collective utility function that is also known as Benthamite social welfare function can be defined as follows:

Definition 3.1. *Given a set of agents A, a consequence $x \in X$ and a set of utility functions $u_i(x)$, where $u_i(x)$ is the utility of x according to agent i, the additive collective utility function Agg^{sum} is defined by:*

$$Agg^{sum}(x) = \sum_{i \in A} u_i(x) \tag{3.1}$$

In the case where agents don't have the same importance (e.g. in an administration board or direction committee), an importance degree or a weight w_i is associated to each i where the agent where the highest weight is the most important. This leads to the use of a weighted sum:

$$Agg^{sum}(x) = \sum_{i \in A} w_i \times u_i(x) \tag{3.2}$$

Example 3.1. *Consider a consequence $x \in X$ and two agents 1 and 2 where the utility of x relative to each agent is respectively $u_1(x) = 0.4$ and $u_2(x) = 0.3$. When agents have the same importance, their collective utilitarian utility can be computed using Equation 3.1 as follows:*
$Agg^{sum}(x)$ = 0.4 + 0.3 = 0.7.
However in the case where agents are not equally important, example agent 1 is more important than agent 2 with $w_1 = 0.5$ and $w_2 = 0.2$. The collective weighted utilitarian utility is performed using Equation 3.2 as follows: $Agg^{sum}(x) = 0.4 \times 0.5 + 0.3 \times 0.2 = 0.26$.

This function has several good properties [49, 55] but fails to ensure equity between agents.

Egalitarian collective utility functions

Contrary to the utilitarian approach, the egalitarian approach proposes to maximize the satisfaction of the least satisfied agent. This collective utility function is defined by:

Definition 3.2. *Given a set of agents A, a consequence $x \in X$ and a set of utility functions $u_i(x)$, where $u_i(x)$ is the utility of x according to agent i, the egalitarian utility function is defined by:*

$$Agg^{min}(x) = \min_{i \in A} u_i(x) \tag{3.3}$$

When the agents are not equally important, a weight w_i can be associated to each agent i. The collective utility is then defined in a conjunctive (cautious) way using a weighted minimum:

$$Agg^{min}(x) = \min_{i \in A} \max((1 - w_i), u_i(x)) \tag{3.4}$$

Example 3.2. *Consider a consequence $x \in X$ and two agents 1 and 2 where the utility of x relative to each agent is respectively $u_1(x) = 0.4$ and $u_2(x) = 0.3$. When these agents have the same importance, the egalitarian utility function can be computed using Equation 3.3 as follows:*
$Agg^{min}(x)$ = min *(0.4 , 0.3) = 0.3.*
In the case where agents are not equally important, example agent 1 is more important than agent 2 such that $w_1 = 0.5$ and $w_2 = 0.2$, the weighted egalitarian utility can be computed using Equation 3.4 such that:
$Agg^{min}(x)$ = $\min(\max((1 - 0.5), 0.4), \max((1 - 0.2), 0.3))$ = *0.5.*

Notice that a non-egalitarian (max oriented) counterpart of the egalitarian utility can be used to aggregate agents' preferences using a disjunctive aggregation. This collective utility function can be defined for the case when agents have same importance (Equation 3.5) as well as the case when they are not equally important (Equation 3.6) as follows[1]:

$$Agg^{max}(x) = \max_{i \in A} u_i(x) \tag{3.5}$$

$$Agg^{max}(x) = \max_{i \in A} \min(w_i, u_i(x)) \tag{3.6}$$

3.3 Harsanyi's Theorem and related concepts

In the presence of risk, i.e. when the information about states of nature is probabilistic, the most popular criterion to compare decisions is the expected utility. In this context, an elementary decision is modeled by a probabilistic lottery over a set X of its possible outcomes. The preferences

[1]The min oriented and max oriented decision rules have received an interpretation in context of Multi-criteria decision making to aggregate criteria rather than agents preferences [26].

of the decision maker are supposed to be captured by a utility function assigning a numerical value to each consequence. The evaluation of a lottery is performed through the computation of its expected utility (the higher is the better).

When several agents are involved, the aggregation of individual preferences under risk raises a particular problem depending on when the utility of the agents is to be evaluated, before or after the consideration of uncertainty. This yields two different approaches that are the *ex-ante* and *ex-post* aggregations.

ex-ante and ex-post aggregations

In collective decision making problem under risk, we have to specify the operator used to aggregate the individual utilities: the **sum** operator for the utilitarian approach or the **min** in the case of egalitarianism. However, it is necessary also to define the moment to consider uncertainty. Hence, the definition of the *ex-ante* and *ex-post* approaches:

- The *ex-ante* approach consists in computing separately the (EU_i) expected utility for each agent i considering the uncertainty and then perform the aggregation of these expected utilities.
- The *ex-post* approach combines the decision makers utilities and determines the collective utility function (CUF) for each consequence x by defining the collective utility $Agg(x)$ and then considers the uncertainty. It comes down to a problem of mono-agent decision making under uncertainty (this agent being “the collectivity”).

This terminology has been introduced in economics and social welfare theory and then it has been extended to several fields related to collective decision making such as cooperative multi-agent system and especially Bayesian coalitional games [62].

Timing effect

In the probabilistic context, utilitarianism comes down to calculating either the collective expected utility of each consequence (*ex-ante*), or the aggregation of the individual expected utilities (*ex-post*) however both give exactly the same result. Egalitarianism prescribes to maximize either the expectation of the minimum of the satisfaction degrees or to compute the minimum of the mathematical expectations. The two approaches do not always coincide: this phenomenon have been identified by Myerson [74] as “Timing Effect" and it can be observed in the following counter-example.

Counter-example 3.1. *Consider two agents* 1 *and* 2 *and two lotteries* L_1 *and* L_2 *on* $X = \{x_1, x_2\}$ *depicted in Figure 3.1. The probability degrees for consequence* x_1 *(resp.* x_2*) are respectively 0.7*

(resp 0.3) for L_1 and 0.2 (resp 0.8) for L_2. The utility functions relative to consequences x_1 and x_2 for agent 1 (resp. agent 2) are defined such that: $u_1(x_1) = 0.3$ and $u_1(x_2) = 1$ (resp. $u_2(x_1) = 0.5$ and $u_2(x_2) = 0.4$).
The expected value of the minimum of the utilities are:
$0.7 \times \min(0.3, 0.5) + 0.3 \times \min(1, 0.4) = 0.33 \quad for\ L_1$
$0.2 \times \min(0.3, 0.5) + 0.8 \times \min(1, 0.4) = 0.38 \quad for\ L_2$
So ex-post, $L_2 \succ L_1$.
On the contrary, using ex-ante, computing the minimum of the expected utilities leads to $L_1 \succ L_2$, since:
$\min(0.7 \times 0.3 + 0.3 \times 1, 0.7 \times 0.5 + 0.3 \times 0.4) = 0.47 \quad for\ L_1$
$\min(0.2 \times 0.3 + 0.8 \times 1, 0.2 \times 0.5 + 0.8 \times 0.4) = 0.42 \quad for\ L_2.$

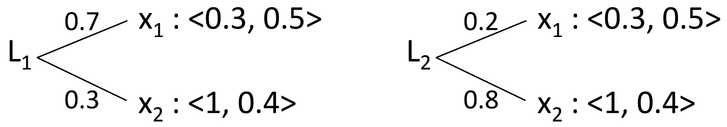

Figure 3.1: Two probabilistic lotteries in a bi-agent context.

It has been proved that in the case of egalitarian decision making under risk, the size of the problem may considerably affect the coincidence between *ex-ante* and *ex-post* aggregation [66]. In other words, the greater the number of uncertain states of the world is, more the possibility of equity between the two approaches is rare.

Moreover, the author has proved in the same work that in an egalitarian context, the only way to obtain equity between *ex-ante* and *ex-post* collective expected utilities is to consider that the less satisfied agent is always the same and this for any consequence $x \in X$.

The timing effect is not present in the probabilistic utilitarian approach, it has been proved that utilitarianism is the only way to avoid this problem in presence of risk. This conclusion is due to Harsanyi's contribution to social welfare theory [59], that has pervaded the literature until now. This result and related work are the subject of the remaining of this chapter.

Harsanyi's representation theorem

Following Fleming [43], Harsanyi provided a representation theorem for collective decision making in the context of Von Neumann and Morgenstern's expected utility theory [59]. This theorem is often interpreted as a justification of utilitarianism and it is based on three fundamental axioms, detailed in the following.

Axiom. *(Hars1) The preference relation of each agent i over lotteries satisfies Von Neumann and Morgenstern's axioms.*

This axiom summarizes VNM's assumptions relative to expected utility for each individual utility $u_i(x)$ and enforces the representation of individual preferences by expected utilities.

Axiom. *(Hars2) The collective preference relation satisfies also Von Neumann and Morgenstern's axioms.*

Axiom Hars2 requires, again, the use of expected utility to represent the preference relation of the collectivity.

In addition to these two axioms, Harsanyi imposes another condition relating collective preferences to individual ones, which is Pareto indifference axiom.

Axiom. *(PI: Pareto Indifference) If two decisions are indifferent for each agent they are considered as collectively indifferent.*

In [59], Harsanyi has shown that if the three assumptions (axioms Hars1, Hars2 and PI) are satisfied by individual as well as collective preferences i.e. (i) the collective preference satisfies Von Neumann and Morgenstern's axioms, (ii) the preferences of each agent also satisfy these axioms, and (iii) if two lotteries are indifferent for each agent they are considered as collectively indifferent (Pareto indifference axiom), then the only appropriate collective utility function (CUF) is the utilitarian one.

Theorem 3.1. *If axioms Hars1, Hars2 and PI are satisfied by the collective preference relation as well as individual ones and considering anonymity, there exists a set of individual utilities $u_i(x)$ relative to a consequence $x \in X$ and weights $\lambda_i \in \mathbb{R}$ such that:*

$$Agg(x) = \sum_{i \in A} \lambda_i \times u_i(x) \tag{3.7}$$

Under the assumption of anonymity, Harsanyi proved that Agg(x) can be written as the sum of individual utilities ($Agg(x) = \sum_{i \in A} u_i(x)$). On the basis of Harsanyi's results, Myerson [74] has proved that in presence of risk only the use of an affine collective aggregation function allows to overcome the Timing Effect problem. Conversely; in the probabilistic case; any attempt to introduce equity causes a divergence between the *ex-post* and *ex-ante* approaches.

3.4 Beyond Harsanyi's theorem

Many researches have surrounded Harsanyi's aggregation theorem by proposing several variants and criticisms. The main related works are detailed in the following.

Extension of Harsanyi's theorem

Harsanyi's aggregation theorem has been questioned w.r.t. weights λ_i defined in Equation 3.7. In the initial contribution, these weights are not necessary positive. This problem was naturally solved by strengthening Pareto Indifference [19] into Strong Pareto condition that is defined by:

Axiom. *(SP: Strong Pareto) If each agent weakly prefers L_1 to L_2 ($\forall j \neq i, L_1 \succeq_j L_2$) and at least one of them strictly prefers L_1 to L_2 ($L_1 \succ_i L_2$) then the collectivity strictly prefers lottery L_1 to L_2 ($L_1 \succ L_2$).*

Moreover, the problem of negative weights can also be solved using a weak version of Pareto (i.e. Weak Pareto axiom) defined as follows:

Axiom. *(WP: Weak Pareto) If each agent weakly prefers L_1 to L_2 ($\forall i, L_1 \succeq_i L_2$) then the collectivity weakly prefers lottery L_1 to L_2 ($L_1 \succeq L_2$).*

Further, Fishburn [42] and Coulhon and Mongin [18] pointed out that weights may be not uniquely defined. In order to solve this problem, Mongin et al. [69] have proposed the following *Independent Prospects* assumption as an additional condition to ensure uniqueness.

Axiom. *(Independent Prospects) For every agent i there exists a pair of lotteries for which he is not indifferent.*

Although, Domotor [22] and Border [11] have considered that this condition is unnecessary and that it is implicitly used as structural assumption by Harsanyi.

Since its emergence a panoply of works have been proposed to extend Harsanyi's Theorem, more recent ones are [15, 44, 47] that have been developed to confirm Harsanyi's approach and to extend it to deal with the case of an infinite population or agents with incomplete information.

Criticisms and impossibility results

Diamond [20] was the first to challenge Harsanyi's result, and especially the use of the VNM expected utility to express the collective preference. He proposed to abandon VNM's expected utility axioms in favor of the *egalitarianism*. He illustrated his argumentation on the basis of this classical example:

Example 3.3. *Consider two agents 1 and 2, and two equally probable situations; say tossing a fair coin. Let L and L' two probabilistic lotteries (presented in Figure 3.2) describing two cases: In the first case (lottery L), whatever the result of tossing, agent 1 gets everything and agent 2 gets nothing. In the other case, agent 1 gets everything if head and agent 2 gets everything if tails.*

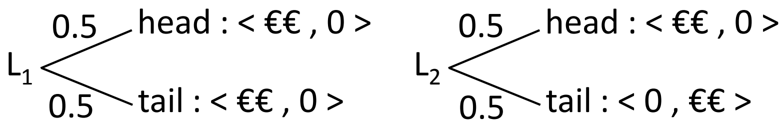

Figure 3.2: Example of Diamond [20]

Considering Harsanyi's axioms these two decisions are equivalent, and the ex-ante *and* ex-post *aggregations give the same results. In fact, for the* ex-ante *aggregation we have:*
$U(L_1) = [0.5 \times (\text{€€} + 0)] + [0.5 \times (\text{€€} + 0)] = \text{€€}$
$U(L_2) = [0.5 \times (\text{€€} + 0)] + [0.5 \times (0 + \text{€€})] = \text{€€}$
The same result is also obtained using the ex-post *aggregation such that:*
$U(L_1) = [(0.5 \times \text{€€}) + (0.5 \times 0)] + [(0.5 \times \text{€€}) + (0.5 \times 0)] = \text{€€}$
$U(L_2) = [(0.5 \times \text{€€}) + (0.5 \times 0)] + [(0.5 \times 0) + (0.5 \times \text{€€})] = \text{€€}$
However, from an ethical point of view lottery L_2 is better than L_1 because in L_1, for any situation (head or tail) agent 1 gets every thing $U_1(L_1) = \text{€€}$ and $U_2(L_1)$ = 0. While, in L_2 each agent can have half reward $U_1(L_2) = U_2(L_2) = \text{€}$.

Diamond pointed out this problem and deduced from this example that social preference may violate axiom $Hars2$, and more precisely the independence axiom of VNM's approach.

The Harsanyi-Diamond debate about the compelling nature of expected utility theory for collective decision making spawned a large literature. Besides, researchers were also motivated by the notion of subjective beliefs, that is more realistic. So, they gave up the VNM's expected utility in favor of subjective expected theory.

Hylland and Zeckhauser [61], Hammond [58], Broome [13], Seidenfeld et al. [84], Mongin [69], many others and recently [1] have followed this line. All these works have provided the same conclusion i.e. the aggregation of individual preferences into a collective subjective expected utility is not so easy to perform. In fact, replacing VNM's formalism with a subjective one and assuming that individuals as well as the collectivity obey subjective expected utility axioms while considering the Pareto condition, may lead to a negative result. In other words, when individuals are simultaneously heterogeneous in terms of tastes and beliefs, imposing the satisfaction of subjective utility axioms for collective as well as individual preference relations leads to an impossibility result [71].

To meet this challenge, Weymark [94] and Mongin [69] have provided variants of Harsanyi's theorem using subjective probabilities instead of VNM lotteries. But, these works are relevant only under the assumption that all the agents have the same beliefs, i.e. same probabilities which is a strong hypothesis.

The negative results are generally obtained due to technical assumptions considering subjective uncertainty. More recent works, have founded their arguments on the incompatibility between Pareto properties and axioms related to the subjective expected uncertainty models. Authors like

Broome [13], Gilboa, Samet and Schmeidler [53] condemn the Pareto property while others like Chambers and Hayashi [14] have proved, following Savage's axiomatization, that $Sav3$ and $Sav4$ are not compelling at a social level.

Despite their differences, the common conclusion of all these works is that it is impossible to perform the aggregation of social preferences under probabilistic uncertainty rather than risk, except for some special cases where all agents have the same probabilities and/or same utility functions [3, 69]. Making a step further, Gajdos, Tallon and Vergnaud [46] have shown that even by considering agents sharing homogenous beliefs, the collective aggregation cannot be proceeded unless an additional assumption defining individuals as well as society as uncertainty neutral.

In addition to this impossibility result [13,14,53], it has been proved that the use of the subjective framework leads to a gap between results obtained using *ex-ante* or *ex-post* aggregations. The choice of the former or the latter is until now controversial and it constitutes an attractive subject for future research.

3.5 Conclusion

In this chapter we have presented the problem of collective decision making, especially aggregating decision makers' preferences, in presence of probabilistic uncertainty. We have presented the Harsanyi's Theorem which proves that the weighted sum of individual expected utilities is the only way to avoid the difference between *ex-ante* and *ex-post* aggregation in presence of risk.

We have also considered collective aggregation in subjective frameworks of expected utilities: It appears that the majority of these studies have concluded to the existence of an impossibility result considering agents with subjective beliefs. However, it has been shown that it is possible to define a representation theorem by relaxing some constraints i.e. supposing that agents have same knowledge or same utilities.

CHAPITRE 4

Sequential decision making and decision trees

4.1 Introduction

In the previous chapters, we have been interested only in situations where just one decision has to be taken (one-stage decision making problems). However, in real world problems the decision maker is often facing a succession of decisions to be taken over time, i.e. *sequential decision problems*.

Graphical decision models such as decision trees [78], influence diagrams [60], Valuation Based Systems [86] and Markov decision process [5], offer a clear description of sequential decision problems that allow an easy definition of optimal strategies. Within existing models, we are particularly interested to the most known models namely decision trees ($\mathcal{DT}$s). This graphical representation is intuitive and constitutes a simple tool to represent decision problems at hand.

This formalism is the core of this chapter that is organized as follows: In Section 4.2, we define the graphical and numerical (probabilistic and possibilistic) components of decision trees. Then, we detail the optimization of these graphical models with regards to a probabilistic quantification of uncertainty in Section 4.3 and considering the possibilistic representation in Section 4.4.

4.2 Definition of decision trees ($\mathcal{DT}$ and $\Pi\mathcal{DT}$)

The most popular graphical decision model is the decision tree ($\mathcal{DT}$) proposed by Raiffa in 1968 [78]. These models encode the structure of the problem and represent all possible scenarios by several paths from the root to leaves of the tree. $\mathcal{DT}$s are defined using a graphical component and a numerical one as detailed in what follows.

Graphical component

A graphical component of a decision tree $\mathcal{T}$ is composed of a set of nodes $\mathcal{N}$ and a set of edges $\mathcal{E}$. The set of nodes is partitioned into three subsets C, D and LN:

- $C = \{C_1...C_n\}$ is a set of chance nodes which represent states of the world. Chance nodes are depicted by circles.
- $D = \{D_1...D_m\}$ is the set of decision nodes which represent decision options, or alternatives. They are depicted by rectangles and the label of each decision node is coherent with its temporal order i.e. if D_i is a descendant of D_j , then $i > j$.
- $LN = \{LN_1...LN_k\}$ is a set of leaves in the tree, they are called also utility leaves since they represent utilities: $\forall LN_i \in LN, u(LN_i)$ is the utility of being eventually in the node LN_i. In decision trees, leaves have no particular representation.

The root node is generally a decision node, denoted by D_0. Let N_i be a node that belongs to $\mathcal{N}$:

- Ω_{N_i} denotes the set of possible values of the node N_i.
- $Par(N_i)$ denotes the set of parents of N_i, its predecessors.
- $Succ(N_i)$ denotes the set of direct successors of N_i, its children. If N_i is a decision node $N_i = D_i \in D$, $Succ(D_i) \subset C$ i.e. a decision node is followed by chance nodes. If N_i is a chance node $N_i = C_i \in C$, $Succ(C_i) \subset D \cup LN$, i.e. $Succ(C_i)$ is the set of outcomes of C_i: either a leaf node is observed, or a decision node is reached (and then a new action should be executed). Leave nodes do not have successors.

A *scenario* (or “trajectory") is complete sequence of actions and observations that represents a path from the root to a leaf. The size $|\mathcal{T}|$ of a decision tree is the number of its edges which is equal to the number of its nodes minus 1.

Numerical component

The numerical component of decision trees consists on assigning utility values to leave nodes and labeling the edges outgoing from chance nodes. The quantification of the decision tree, i.e. the numerical component, depends essentially on the nature of uncertainty pertaining the problem and the theory used to represent it. In its classical version, decision trees are probabilistic. Hence, the uncertainty degrees following each $C_i \in C$ are represented by a conditional probability distribution p_i on $Succ(C_i)$, such that $\forall N_i \in Succ(C_i)$, $p_i(N_i) = P(C_i|Par(C_i))$. The value of p_i

depends in the values assigned to decision and chance nodes that depict the path from the root to C_i.

With the development of uncertainty theories, decision trees have been adapted to deal with other uncertainty models than the probabilistic one. Among these variants, we cite the possibilistic decision trees ($\Pi\mathcal{DT}$) that propose to model uncertainty according to possibility theory [48].

Possibilistic decision models share the same graphical component as the classical probabilistic ones; namely a set of Chance nodes C, a set of decision nodes D, a set of leave nodes LN and a set of edges connecting these nodes. However, the numerical component is different. In fact, in $\Pi\mathcal{DT}$s the uncertainty pertaining to the possible outcome of each $C_i \in C$, is defined by a *conditional possibility distribution* π_i on $Succ(C_i)$, such that $\forall N \in Succ(C_i)$, $\pi_i(C_i) = \Pi(C_i|path(C_i))$ where $path(C_i)$ denotes all the value assignments to chance and decision nodes on the path from the root to C_i. The utility value assigned to each leaf can be numerical (e.g. a currency gain) or ordinal (e.g. a satisfaction degree).

Example 4.1. *Figure 4.1 depicts an example of a probabilistic (a) and a possibilistic (b) decision tree. The two models share the same graphical component with three decision nodes* $D = \{D_0, D_1, D_2\}$*, six chance nodes* $C = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ *and ten leave nodes* $C = \{LN_1, LN_2, LN_3, LN_4, LN_5, LN_6, LN_7, LN_8, LN_9, LN_{10}\}$ *with* $U = \{0.1, 0.2, 0.3, 0.4, 0.5\}$. *However, the uncertainty degrees pertaining to chance nodes are defined with probability distributions in (a) and possibility distributions in (b).*

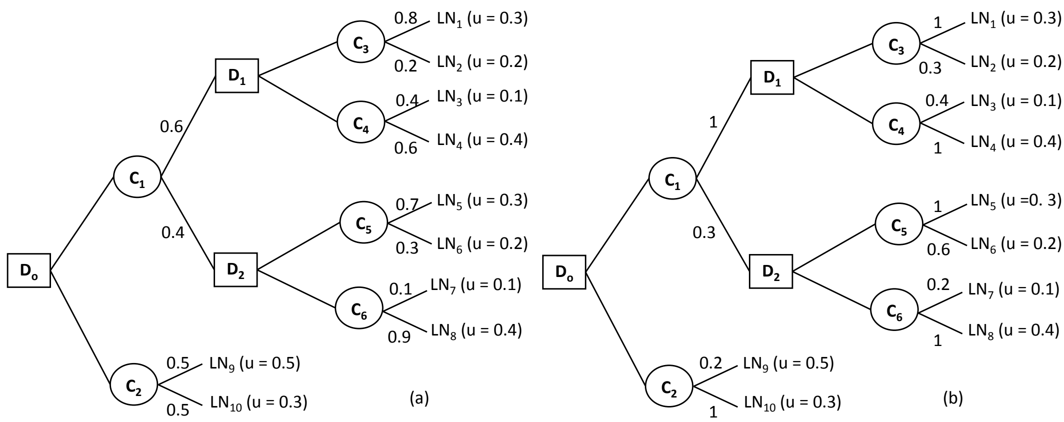

Figure 4.1: Example of a probabilistic decision tree (a) and a possibilistic decision tree (b).

4.3 Strategy optimization problem

Solving a decision tree amounts to building a *strategy* that selects an action (i.e. a chance node) for each reachable decision node. Formally, we define a strategy as a function δ from D to $C \cup \{\perp\}$. $\delta(D_i)$ is the action to be executed when a decision node D_i is reached. $\delta(D_i) = \perp$ means that no action has been selected for D_i (because either D_i cannot be reached or the strategy is partially defined). Admissible strategies should be:

- *sound*: $\forall D_i \in D, \delta(D_i) \in Succ(D_i) \cup \{\perp\}$.
- *complete*: (i)$\delta(D_0) \neq \perp$ and (ii)$\forall D_i$ s.t. $\delta(D_i) \neq \perp, \forall N \in Succ(\delta(D_i))$, either $\delta(N) \neq \perp$ or $N \in LN$.

Let Δ be the set of sound and complete strategies relative to a decision tree. Any strategy δ in Δ can be seen as a connected sub-tree of the decision tree whose arcs are of the form $(D_i; \delta(D_i))$. The optimization of the decision tree amounts to finding the optimal strategy δ^* within Δ w.r.t a decision criterion O. Formally, δ^* is optimal iff $\forall \delta_i \in \Delta$ we have $\delta^* \succeq_O \delta_i$ (i.e. δ^* is preferred to any strategy $\delta_i \in \Delta$ w.r.t. a decision criterion O).

Since leave nodes LN are labeled with utility degrees, chance node can be seen as a simple probabilistic lottery (for the most right chance nodes) or as a compound lottery (for the inner chance nodes). Then, each strategy δ_i can be seen as a compound lottery L_i. So, for evaluating and comparing different strategies, we can use the principle of lottery reduction and reduce each compound lottery to an equivalent simple one.

In order to define the optimal strategy, we should be able to compare different strategies w.r.t a decision criterion (O). Formally, a strategy $\delta^* \in \Delta$, is said to be optimal w.r.t. $\succeq_O$ iff:

$$\forall \delta' \in \Delta, Reduction(\delta^*) \succeq_O Reduction(\delta'). \tag{4.1}$$

Example 4.2. *The decision trees presented in Figure 4.1 contains 5 different strategies i.e. Δ={$\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$} where:*

- $\delta_1 : \delta_1(D_0) = C_1, \delta_1(D_1) = C_3, \delta_1(D_2) = C_5$;
- $\delta_2 : \delta_2(D_0) = C_1, \delta_2(D_1) = C_3, \delta_2(D_2) = C_6$;
- $\delta_3 : \delta_3(D_0) = C_1, \delta_3(D_1) = C_4, \delta_3(D_2) = C_5$;
- $\delta_4 : \delta_4(D_0) = C_1, \delta_4(D_1) = C_4, \delta_1(D_2) = C_6$;
- $\delta_5 : \delta_5(D_0) = C_2$.

For the probabilistic decision tree (Figure 4.1 (a)), the strategy δ_1 corresponds to the lottery $\langle 0.6/L_{C_3}, 0.4/L_{C_5}\rangle$ where $L_{C_3} = \langle 0.8/1, 0.2/5\rangle$ and $L_{C_5} = \langle 0.7/1, 0.3/4\rangle$. Then we have:

$Reduction(\delta_1) = \langle 0.76/1, 0.12/4, 0.12/5\rangle$.
For the possibilistic decision tree (Figure 4.1 (b)), the strategy δ_1 *corresponds to the lottery* $\langle 1/L_{C_3}, 0.5/L_{C_5}\rangle$ *where* $L_{C_3} = \langle 1/0.1, 0.2/0.5\rangle$ *and* $L_{C_5} = \langle 1/0.1, 0.3/0.4\rangle$*. The reduction is performed as follows:* $Reduction(\delta_1) = \langle 1/0.1, 0.3/0.4, 0.2/0.5\rangle$.

The size $|\delta|$ of a strategy δ is the sum of its number of nodes and edges; it is obviously lower than the size of the decision tree. Besides, the set of potential strategies is combinatorial (i.e., its size increases exponentially with the size of the tree). In a decision tree with m decision nodes and a branching factor equal to 2, the number of potential strategies is in $O(2^{\sqrt{m}})$. Then, the determination of an optimal strategy for a given representation and a given decision criterion is an algorithmic issue.

Furthermore, the difficulty of the optimization problem depends on some properties of the decision criterion O, especially the satisfaction of monotonicity (or weak monotonicity) and the transitivity properties that are respectively defined as follows:

Definition 4.1. *A preference order is said to be transitive iff whatever* L, L', L''*:*

$$L \succeq_O L' \text{ and } L' \succeq_O L'' \;\Rightarrow\; L \succeq_O L''; \tag{4.2}$$

$$L \succ_O L' \text{ and } L' \succ_O L'' \;\Rightarrow\; L \succ_O L''; \tag{4.3}$$

$$L \sim_O L' \text{ and } L' \sim_O L'' \;\Rightarrow\; L \sim_O L''. \tag{4.4}$$

Definition 4.2. *A preference order is said to be weakly monotonic iff whatever* L, L', L''*, whatever* (α, β) *such that* $max(\alpha, \beta) = 1$*:*

$$L \succeq_O L' \;\Rightarrow\; \langle \alpha/L, \beta/L''\rangle \succeq_O \langle \alpha/L', \beta/L''\rangle \tag{4.5}$$

The transitivity is interpreted as follows: if exists a preference relation between two lotteries L and L' w.r.t a decision criterion and the same relation exists between L' and L'' then this relation is also preserved between L and L''. Weak Monotonicity states that the combination of L (resp. L') with L'', does not change the initial order induced by O between L and L'. The satisfaction of these properties, is a necessary condition the use of *dynamic programming* Algorithm.

4.4 Optimization of probabilistic decision trees: Dynamic programming

Finding the optimal strategy in a decision tree may be performed via an exhaustive enumeration of Δ, that is a highly computational task. As an alternative method, Bellman proposed a recursive method; the so called Dynamic Programming algorithm; that builds the best strategy backwards, from leaves of the tree to its root [4]. However, the use of this algorithm is appealing only when the criterion to maximize is transitive and monotonic.

The optimization of standard probabilistic decision trees [78] amounts to maximizing the expected utility criterion (detailed in Section 1.2). Since the latter satisfies the monotonicity and transitivity properties, an optimal strategy can be computed in polytime with respect to the size of the tree using Dynamic programming algorithm (denoted by DynProg).

The principle of the backward reasoning procedure can be described as follows: when a chance node C is reached, an optimal sub-strategy is built for each of its children. These sub-strategies are combined w.r.t. their uncertainty degrees. Then, the resulting compound strategy is reduced to an equivalent simple lottery representing the current optimal sub-strategy. When a decision node D is reached, we select a decision D^* among all the possible ones $D \in Succ(X)$ leading to an optimal sub-strategy w.r.t. $\succeq_O$. The choice is performed by comparing the simple lotteries equivalent to each sub-strategy.

This principle is implemented by Algorithm (Line 2). Line (2) performs the comparison of two simple (probabilistic) lotteries according to the criterion O to optimize that is the expected utility (EU). These simple lotteries are the result of reduction performed in Line (1) where $Succ(N)$ is the set of successors of N and γ is the probability of the current node. $L[u_i]$ defines the probability degree to have the utility u_i in the lottery L, $\otimes$ is the product operator and $\oplus$ is the sum operator. Note that in the case of probabilistic decision tree (for the optimization of the expected utility criterion) this algorithm performs in polytime and its complexity is polynomial w.r.t. the size of the tree.

4.5 Optimization of possibilistic decision trees

The evaluation of a possibilistic decision tree, like the probabilistic one consists in finding the optimal strategy δ^* within Δ. As mentioned before, the optimization depends on the properties of the decision criterion O. If this criterion verifies the crucial properties of weak monotonicity and transitivity, an optimal strategy can be computed in polytime with respect to the size of the tree using the recursive method of Dynamic programming.

Such properties are satisfied by the qualitative possibilistic decision rules U_{opt}, U_{pes} and PU (they are monotonic and transitive). Thus, just like the case of the optimization of the expected utility, Algorithm (Line 2) may be used to obtain an optimal strategy in a possibilistic decision tree. In Line 1, the algorithm computes the reduction of the possibilistic lottery according to scale of evaluation: γ is the possibility measure for U_{opt} and PU and the necessity measure for U_{pes}, $\oplus$ is the max operator and $\otimes$ is the min operator. Line (2) allows the comparison of simple (probabilistic) lotteries according to the criterion O to optimize. Then, for these criteria the optimal strategy can be built in polytime in a recursive manner.

Algorithm 1: DynProg(N:Node, δ: Strategy)

Data: O is the criterion to optimize.
Result: A lottery L
begin
    // Initialization
    for $i \in \{1, \ldots, n\}$ **do** $L[u_i] \leftarrow 0$;
    // Leaves
    if $N \in LN$ **then** $L[u(N)] \leftarrow 1$;
    // Chance nodes
    if $N \in C$ **then**
        // Reduce the compound lottery
        foreach $Y \in Succ(N)$ **do**
            $L_Y \leftarrow DynProg(Y, \delta)$;
            for $i \in \{1, \ldots, n\}$ **do**
(Line 1) $L[u_i] \leftarrow \oplus(L[u_i], (\otimes(\gamma_N(Y), L_Y[u_i])))$;
                // Probabilistic lottery: $\otimes$ = product, $\oplus$ = sum
                // Possibilistic lottery: $\otimes$ = min, $\oplus$ = max
    // Decision nodes
    if $N \in D$ **then**
        // Choose the best decision
        $Y^* \leftarrow Succ(N).first$;
        foreach $Y \in Succ(N)$ **do**
            $L_Y \leftarrow DynProg(Y, \delta)$;
(Line 2)   **if** $L_Y \succ_O L_{Y^*}$ **then** $Y^* \leftarrow Y$;
        $\delta(N) \leftarrow Y^*$;
        $L \leftarrow L_{Y^*}$;
    return L;

For possibilistic likely dominance criteria which are monotonic but not fully transitive, an Extended Dynamic Programming algorithm (ExtDynProg) can be efficiently applied [64]. Finally, concerning possibilistic Choquet integrals that are not monotonic, the optimization problem is NP-hard and a Branch and bound algorithm (BB) has been proposed for binary possibilistic decision trees [9]. A detailed theoretical study on the problem of finding an optimal strategy in possibilistic decision trees has been proposed in [9, 38]. The main results of this study are summarized in table 4.1.

Example 4.3. *Let us consider the decision tree in Figure 4.1 (b). The optimal strategy relative to this possibilistic decision w.r.t. U_{opt} returned by Dynamic Programing (Algorithm (Line 2)) is highlighted by bold lines in Figure 4.2, where:*
$\delta^* : \delta^*(D_0) = C_1, \delta^*(D_1) = C_4, \delta^*(D_2) = C_6$.

Criterion O	Weak monotonicity	Transitivity	Algorithm	Complexity
U_{pes}	Yes	Yes	DynProg	P
U_{opt}	Yes	Yes	DynProg	P
PU	Yes	Yes	DynProg	P
OMEU	Yes	Yes	DynProg	P
LN	Yes	No	ExtDynProg	P
LΠ	Yes	No	ExtDynProg	P
Ch_N	No	Yes	BB	NP-hard
Ch_Π	No	Yes	BB	NP-hard

Table 4.1: Optimization of possibilistic criteria

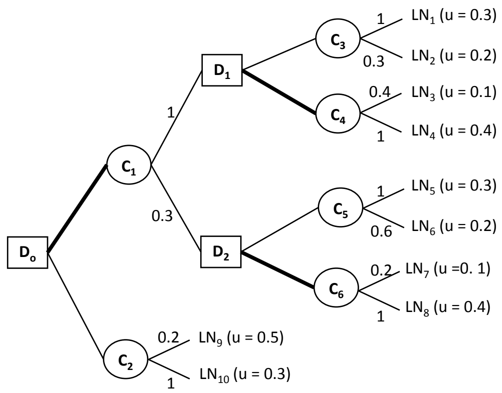

Figure 4.2: The optimal strategy $\delta^*(D_0) = C_1, \delta^*(D_1) = C_4, \delta^*(D_2) = C_6$.

4.6 Conclusion

In this chapter we have recalled the foundations of sequential decision problems i.e. evaluating strategies in probabilistic as well as possibilistic decision trees. After defining standard decision trees, we have exposed a more recent version of this model that are possibilistic decision trees. This graphical model, allows a natural and explicit model to handle sequential decision problems where uncertainty and utility degrees are possibilistic. Then, we focused on finding an optimal strategy w.r.t. the decision criterion and its properties.

The most naive solving method is exhaustive enumeration of all possible strategies to find the best one. However, given a large number of strategies in big decision trees, this method becomes intractable. As alternative, we can use a Dynamic Programming algorithm for probabilistic decision trees (EU criterion) as well as possibilistic decision trees where the criterion to optimize is transitive and monotonic i.e. for U_{pes}, U_{opt} and PU. For these criteria the optimal strategy can be built in polytime w.r.t. the size of the tree. For other decision rules that are not monotonic and/or not transitive more complex procedures have to be performed.

Part II

Contributions

CHAPITRE 5

Collective possibilistic decision making approach

5.1 Introduction

This chapter raises the question of collective decision making under possibilistic uncertainty. We propose a new qualitative decision rules to solve such problems depending on the decision makers' attitude with respect to uncertainty (i.e. optimistic or pessimistic) as well as the method performed to aggregate the decision makers' preferences relative to the different consequences; four *ex-ante* and four *ex-post* decision rules are defined and investigated.

We show that in context of a possibilistic representation of uncertainty, the use of an egalitarian collective utility function allows to get rid of the Timing Effect. Making a step further, we prove that if both the agents' preferences and the collective ranking of the decisions satisfy Pareto Unanimity and Dubois and Prade's axioms [30], particularly risk aversion, then the egalitarian collective aggregation is compulsory. This result can be seen as an ordinal counterpart of Harsanyi's theorem [59]. The picture is then completed by the proposition and the characterization of an optimistic counterpart of this pessimistic decision rule.

The next Section develops our proposition, defining four *ex-ante* and four *ex-post* aggregations, and shows that when the decision rules are either fully min-oriented or fully max-oriented, the *ex-ante* and the *ex-post* possibilistic aggregations provide the same result. Section 5.3 exposes our axiomatization system relative to the egalitarian pessimistic decision rule and propose our representation theorem. Finally, Section 5.4 presents a variant of this theorem adapted to the optimistic case.

5.2 Definition and properties of collective possibilistic decision rules

Harsanyi's theorem previously presented in Section 3.3 is strongly related to uncertainty modeling and is efficient only in a probabilistic view of uncertainty. In a possibilistic context, given a set X of consequences, a set A of agents we define a collective decision problem under possibilistic uncertainty as triplet $\langle \mathcal{L}, \vec{w}, \vec{u} \rangle$[1] where:

- $\mathcal{L}$ is a set of possibilistic lotteries;
- $\vec{w} \in [0,1]^p$ is a weighting vector: w_i denotes the weight of agent i;
- $\vec{u} = \langle u_1, \ldots, u_p \rangle$ is a vector of p utility functions on X: $u_i(x_j) \in [0,1]$ is the utility of x_j according to agent i;

Our aim is to compare lotteries according to decision maker's preferences relative to their different consequences (captured by the utility functions) and the importance of each agent (captured by the weighting vector). To do this, we can proceed in two different ways namely *ex-ante* or *ex-post*:

- The *ex-ante* aggregation consists in computing the (optimistic or pessimistic) utilities relative to each agent i, and then perform the aggregation (U^+ or U^-) using the agent's weights.
- The *ex-post* aggregation consists in first determining the aggregated utilities (Agg^{max} or Agg^{min}) relative to each possible consequence x_j of L and then combine them with the possibility degrees.

In the possibilistic framework, the decision maker's attitude with respect to uncertainty can be either optimistic (U^+) or pessimistic (U^-) and the aggregation of the agents preferences can be either conjunctive, egalitarian (Agg^{min}) or disjunctive, non egalitarian (Agg^{max}), hence the definition of four approaches of CDM under uncertainty, namely U^{+max}, U^{+min}, U^{-max} and U^{-min}; the first (resp. the second) sign denoting the attitude of the decision maker w.r.t. uncertainty (resp. agents preferences aggregation) [6].

Each of these utility functions can be computed either *ex-ante* or *ex-post*. Hence the definition of eight utilities:

[1]Classical problems of decision under possibilistic uncertainty are recovered when $|A| = 1$; Classical collective decision making problems are recovered when all the lotteries in $\mathcal{L}$ associate possibility 1 to some x_i and possibility 0 to all the other elements of X: $\mathcal{L}$ is identified to X, i.e. is a set of “alternatives" for the CDM problem.

Definition 5.1. *Given a possibilistic lottery L on X, a set of agents A, a vector of utility functions $\vec{u}$ and a weighting vector $\vec{w}$, let:*

$$U_{ante}^{+max}(L) = \max_{i \in A} \min(w_i, \max_{x_j \in X} \min(u_i(x_j), L[x_j])). \tag{5.1}$$

$$U_{ante}^{-min}(L) = \min_{i \in A} \max((1 - w_i), \min_{x_j \in X} \max(u_i(x_j), (1 - L[x_j]))). \tag{5.2}$$

$$U_{ante}^{+min}(L) = \min_{i \in A} \max((1 - w_i), \max_{x_j \in X} \min(u_i(x_j), L[x_j])). \tag{5.3}$$

$$U_{ante}^{-max}(L) = \max_{i \in A} \min(w_i, \min_{x_j \in X} \max(u_i(x_j), (1 - L[x_j]))). \tag{5.4}$$

$$U_{post}^{+max}(L) = \max_{x_j \in X} \min(L[x_j], \max_{i \in A} \min(u_i(x_j), w_i)). \tag{5.5}$$

$$U_{post}^{-min}(L) = \min_{x_j \in X} \max((1 - L[x_j]), \min_{i \in A} \max(u_i(x_j), (1 - w_i))). \tag{5.6}$$

$$U_{post}^{+min}(L) = \max_{x_j \in X} \min(L[x_j], \min_{i \in A} \max(u_i(x_j), (1 - w_i))). \tag{5.7}$$

$$U_{post}^{-max}(L) = \min_{x_j \in X} \max((1 - L[x_j]), \max_{i \in A} \min(u_i(x_j), w_i)). \tag{5.8}$$

It is obviously possible to define in the same way a series of utilitarian possibilistic utilities (U_{ante}^{-sum}, U_{ante}^{+sum}, etc.) and a series of max-oriented ones ($U_{post}^{-\max}$, $U_{post}^{+\max}$, etc.).

Example 5.1. *As a matter of fact, consider two agents 1 and 2, the first one being less important than the second ($w_1 = 0.6$, $w_2 = 1$), and the simple lotteries L_1, L_2 on $X = \{x_1, x_2, x_3\}$ depicted in Figure 5.1. Using the egalitarian aggregation we have:*

Figure 5.1: Two bi-agent possibilistic lotteries.

- $U_{post}^{-\min}(L_1) = \min(\max(1 - 0.6, \min\max(1 - 1, 0.9), \max(1 - 0.9, 0.1)),$
  $\max(1 - 1, \min\max(1 - 1, 0.1), \max(1 - 0.9, 0.8)))$
  = 0.1
- $U_{ante}^{-\min}(L_1) = \min(\max(1 - 1, \min(\max(1 - 0.6, 0.9), \max(1 - 1, 0.1))),$
  $\max(1 - 0.9, \min(\max(1 - 0.6, 0.1), \max(1 - 1, 0.8))))$
  $= 0.1$

- $U^{+\min}_{post}(L_1) = \min(\max(1-0.6, \max(\min(1, 0.9), \min(0.9, 0.1))),$
  $\max(1-1, \max(\min(1, 0.1), \min(0.9, 0.8))))$
  $= 0.8$

- $U^{+\min}_{ante}(L_1) = \max(\min(1, \min(\max(1-0.6, 0.9), \max(1-1, 0.1))),$
  $\min(0.9, \min(\max(1-0.6, 0.1), \max(1-1, 0.8))))$
  $= 0.4$

- $U^{-\min}_{ante}(L_2) = U^{-\min}_{post}(L_2) = U^{+\min}_{ante}(L_2) = U^{+\min}_{post}(L_2) = 0.8.$

The pessimistic aggregations are related to their optimistic counterparts by duality as stated by the following proposition.

Proposition 5.1. *Let $P = \langle \mathcal{L}, \vec{w}, \vec{u} \rangle$ be a qualitative decision problem, let $P^\tau = \langle \mathcal{L}, \vec{w}, \vec{u}^\tau \rangle$ be the inverse problem, i.e. the problem such that for any $x_j \in X, i \in A$, $u^\tau_i(x_j) = 1 - u_i(x_j)$. Then, for any $L \in \mathcal{L}$:*

$U^{+max}_{ante}(L) = 1 - U^{\tau-min}_{ante}(L)$ $\quad$ $U^{+max}_{post}(L) = 1 - U^{\tau-min}_{post}(L)$
$U^{-min}_{ante}(L) = 1 - U^{\tau+max}_{ante}(L)$ $\quad$ $U^{-min}_{post}(L) = 1 - U^{\tau+max}_{post}(L)$
$U^{+min}_{ante}(L) = 1 - U^{\tau-max}_{ante}(L)$ $\quad$ $U^{+min}_{post}(L) = 1 - U^{\tau-max}_{post}(L)$
$U^{-max}_{ante}(L) = 1 - U^{\tau+min}_{ante}(L)$ $\quad$ $U^{-max}_{post}(L) = 1 - U^{\tau+min}_{post}(L)$

Proof of Proposition 5.1.
Let us show that $U^{+\min}_{ante}(L) = 1 - U^{\tau-\max}_{ante}(L)$, (i.e. $1 - U^{+\min}_{post}(L) = U^{\tau-\max}_{post}(L)$), other utilities can be proved in the same way.

$1 - U^{+\min}_{ante}(L) = 1 - [\min_{i\in A} \max((1-w_i), \max_{x\in X} \min(u_i(x), L[x]))]$
$= \max_{i\in A} \min(w_i, \min \min_{x\in X} \max(1 - u_i(x), (1 - L[x])))$
$= \max_{i\in A} \min(w_i, \min_{x\in X} \max(u^\tau_i(x), (1 - L[x])))$
$= U^{\tau-\max}_{ante}(L).$ □

Interestingly it appears that the coincidence between *ex-post* and *ex-ante* approaches does not imply utilitarianism. Indeed it is easy to show that:

Proposition 5.2.

$$U^{-\min}_{ante}(L) = U^{-\min}_{post}(L)$$
$$U^{+\max}_{ante}(L) = U^{+\max}_{post}(L)$$

Proof of Proposition 5.2.
In the following we provide the proof relative to decision rule $U^{-\min}$. The proof relative to $U^{+\max}$ can be deduced by replacing $\min$ by $\max$, $\max$ by $\min$, $(1 - L[x)$ by $L[x]$ and $(1 - w_i)$ by w_i.

$U^{-\min}_{post}(L) = \min_{x\in X} \max(1 - L[x], \min_{i\in A} \max(1 - w_i, u_i(x)))$
$= \min_{x\in X} \min_{i\in A} \max(1 - L[x], \max(1 - w_i, u_i(x)))$
$= \min_{i\in A} \min_{x\in X} \max(1 - L[x], \max(1 - w_i, u_i(x)))$
$= \min_{i\in A} \min_{x\in X} \max(1 - w_i, \max(1 - L[x], u_i(x)))$
$= \min_{i\in A} \max(1 - w_i, \min_{x\in X} \max(1 - L[x], u_i(x)))$
$= U^{-\min}_{ante}(L)$ □

We shall thus simply use the notation $U^{-\min}$ (resp. $U^{+\max}$) for the fully min-oriented (resp. max-oriented) aggregation. Such a coincidence happens neither in the case of $U^{+\min}$ nor in the case of $U^{-\max}$; with $U^{+\min}_{post}$ a lottery is good as soon as there exists a possible outcome satisfying all the agents; with $U^{+\min}_{ante}$ (resp. $U^{-\max}_{post}$), a lottery is good when each agent forecasts an outcome that is good for her (but it is not necessarily the same one for all): it may happen that $U^{+\min}_{post}(L) < U^{+\min}_{ante}(L)$ (resp. $U^{-\max}_{post}(L) < U^{-\max}_{post}(L)$), as shown by Counter-example 5.1.

Counter-example 5.1. *Consider two equally important agents* 1 *and* 2 *(*$w_1 = w_2 = 1$*) and two lotteries* L *and* L' *(cf. Figure 5.2) relative to three totally possible consequences* x_1*,* x_2 *and* x_3 *such that* x_1 *is good for* 1 *and bad for* 2*,* x_2 *is bad for* 1 *and good for* 2 *and* x_3 *is average for both agents i.e.* $L[x_1] = L[x_2] = L[x_3] = 1$*,* $u_1(x_1) = u_2(x_2) = 1$*,* $u_2(x_1) = u_1(x_2) = 0$ *and* $u_1(x_3) = u_2(x_3) = 0.5$*. We can check that* $U^{+min}_{ante}(L) = 1 \neq U^{+min}_{post}(L) = 0$*:*

Figure 5.2: Lotteries L and L' relative to counter-example 5.1

$$\begin{aligned} U^{+min}_{ante}(L) &= \min(\max((1 - w_1), \max(\min(u_1(x_1), L[x_1]), \min(u_1(x_2), L[x_2]))),\\ &\quad \max((1 - w_2), \max(\min(u_2(x_1), L[x_1]), \min(u_2(x_2), L[x_2])))).\\ &= \min(\max((1 - 1), \max(\min(1, 1), \min(0, 1))),\\ &\quad \max((1 - 1), \max(\min(0, 1), \min(1, 1))))\\ &= 1.\\ U^{+min}_{post}(L) &= \max(\min(L[x_1], \min(\max(u_1(x_1), (1 - w_1)), \max(u_2(x_1), (1 - w_2)))),\\ &\quad \min(L[x_2], \min(\max(u_1(x_2), (1 - w_1)), \max(u_2(x_2), (1 - w_2))))).\\ &= \max(\min(1, \min(\max(1, (1 - 1)), \max(0, (1 - 1)))),\\ &\quad \min(1, \min(\max(0, (1 - 1)), \max(1, (1 - 1)))))\\ &= 0. \end{aligned}$$

The ex-ante and ex-post approaches may lead to different rankings of lotteries. Besides while L' *presents a constant act, we can check that* $U^{+min}_{ante}(L') = U^{+min}_{post}(L') = 0.5$*.*
So we can verify that $U^{+min}_{ante}(L) > U^{+min}_{ante}(L')$ *while* $U^{+min}_{post}(L) < U^{+min}_{post}(L')$*.*

Using these same lotteries L and L', we can show that: $U^{-max}_{ante}(L) = 0 \neq U^{-max}_{post}(L) = 1$ *and that* $U^{-max}_{ante}(L') = U^{-max}_{post}(L') = 0.5$*; then* $U^{-max}_{post}(L') < U^{-max}_{post}(L)$ *while* $U^{-max}_{ante}(L') > U^{-max}_{ante}(L)$*: like* U^{+min}*,* U^{-max} *are subject to the timing effect.*

In summary, U^{-max} and U^{+min} suffer from the timing effect, but U^{-min} and U^{+max} do not. It holds that:

Proposition 5.3.

$$U^{+\min}_{ante}(L) \geq U^{+\min}_{post}(L)$$
$$U^{-\max}_{ante}(L) \leq U^{-\max}_{post}(L)$$

Proof of Proposition 5.3.
Let us show that $U^{+\min}_{ante}(L) \leq U^{+\min}_{post}(L)$, the one relative to $U^{-\max}$ can be obtained in the same way.

Let $u'_i(x) = \max(u_i(x), 1 - w_i)$
$U^{+\min}_{post}(L) = \max_{x \in X} \min(L[x], \min_{i \in A} \max(1 - w_i, u_i(x)))$
$= \max_{x \in X} \min(L[x], \min_{i \in A} u'_i(x))$
$= \max_{x \in X} \min_{i \in A} \min(L[x], u'_i(x))$

$U^{+\min}_{ante}(L) = \min_{i \in A} \max(1 - w_i, \max_{x \in X} \min(u_i(x), L[x]))$
$= \min_{i \in A} \max_{x \in X} \max(1 - w_i, \min(u_i(x), L[x]))$
$= \min_{i \in A} \max_{x \in X} \min(\max(1 - w_i, u_i(x)), \max(1 - w_i, L[x]))$
$= \min_{i \in A} \max_{x \in X} \min(u'_i(x), \max(1 - w_i, L[x]))$
Besides since, $\forall x \in X, \forall i \in A$, $\max(1 - w_i, L[x]) \geq L[x]$; we have:
(i) $\min_{i \in A} \max_{x \in X} \min(u'_i(x), \max(1 - w_i, L[x])) \geq \min_{i \in A} \max_{x \in X} \min(u'_i(x), L[x])$

Let f(x,i) = $\min(u'_i(x), L[x])$, then we have:
$\forall x \in X, \forall i \in A$, $\max_{x \in X} f(x, i) \geq f(x, i)$
$\min_{i \in A} \max_{x \in X} f(x, i) \geq \min_{i \in A} f(x, i)$; $\forall x \in X$
$\min_{i \in A} \max_{x \in X} f(x, i) \geq \max_{x \in X} \min_{i \in A} f(x, i)$
Then we obtain $\min_{i \in A} \max_{x \in X} \min(u'_i(x), L[x]) \geq \max_{x \in X} \min_{i \in A} \min(u'_i(x), L[x])$
From (i) and (ii) we can deduce that $U^{+\min}_{ante}(L) \geq U^{+\min}_{post}(L)$ □

5.3 Axiomatization of egalitarian pessimistic decision rule in style of VNM axiomatic system

In this section, we propose an axiomatization of the pessimistic egalitarian utility. Consider a set A of p agents, a finite set of consequences X, a possibilistic scale V, the set of possibilistic lotteries $\mathcal{L}$ obtained from V and X. The preference profile $\langle \succeq_1, \ldots, \succeq_p \rangle$ gathers the preference relations $\succeq_i$ of each agent i on $\mathcal{L}$. $\succeq$ denotes the collective preference on $\mathcal{L}$.

Let us denote by x the "constant" lottery leading to consequence x for sure (s.t. $L[x] = 1$ and $L[y] = 0$ for each $y \neq x$; e.g. L_2 in Figure 5.1): constant lotteries and elements of X are identified. In the same way, let Y be the lottery that represents a subset Y of X (it provides the possibility degree 1 to each $y \in Y$, and 0 otherwise). As we compute a globbal criterion, the values of this criterion are consequences in X So, we formulate first of all a completeness axiom on the set of consequences X:

Axiom (C, *Completeness of* X)**.** *$\forall x, y \in X, \forall B \subseteq A, \exists z \in X$ such that: $z \sim_i x$ if $i \in B$ and $z \sim_j y$ if $j \notin B$.*

This axiom requires that there exists a z in X that is indifferent to x for agents in B and indifferent to y for the others. When two agents are involved, axiom C says that if x and y are two elements of X, then X contains an element z corresponding to the vector of satisfaction $\langle x_i, y_j \rangle$. This axiom ensure the existence of a sort of “collective equivalence” that can be seen as a counterpart in the sens of social choice, of the certainty equivalence axiom (A2) defined by Dubois and Prade for the characterization of qualitative utility.

More generally, this axiom requires the set of lotteries to be rich enough to contain all the constant acts corresponding to all the vectors of satisfaction (in a sense, C deals more with $\mathcal{L}$ than with $\succeq$). This implies in particular that X contains a consequence x^* that is ideal for all the agents, and a consequence x_* anti-ideal for all the agents. When the set of consequences X is too small, it is harmless to extend and enrich it in order to obtain all the z that we need: in the following, axiom C is supposed by construction (in Harsanyi's paper it is implicit: X is identified with the set of utility vectors).

We now introduce the axiom of Pareto unanimity, that is essential for collective choice:

Axiom (P, *Pareto Unanimity*)**.** *If $\forall i \in A$, $L \succeq_i L'$, then $L \succeq L'$.*

A representation theorem for agents with different weights

Let us study the pessimistic collective decision rules in the light of the axioms and show that they are consistent, namely obeyed by the pessimistic egalitarian utility ($U^{-\min}$). Consider a set $\mathcal{L}$ of possibilistic lotteries built from a set X and a scale V; a set u_i, $i \in A$ of utility functions on X taking their values in $[0,1]$; and a weight vector $\vec{w} \in [0,1]^p$ (where w_i is the weight of agent i).

Let us resume Dubois and Prade's axioms relative to qualitative utilities (see Section 2.4). It holds that:

Proposition 5.4.
The relations $\succeq$ and $\succeq_i$ defined by:

$$L \succeq L' \text{ iff } U^{-\min}(L) \geq U^{-\min}(L'),$$

$$L \succeq_i L' \text{ iff } U_i^{-}(L) \geq U_i^{-}(L')$$

satisfy Dubois and Prade's axioms ($A1^-1 \ldots A6^-$ of page 24) relative to the pessimistic utility, as well as the Pareto Unanimity axiom.

Proof of Proposition 5.4.

- The relation $\succeq_i$ defined by U_i^- satisfies axioms $A1^- \ldots A6^-$:
  By definition, U_i^- is the pessimistic utility relative to each agent, then it is obvious that $\succeq_i$ satisfies axioms $A1^- \ldots A6^-$.
- The relation $\succeq$ defined by $U^{-\min}$ satisfies axioms $A1^- \ldots A6^-$:
  By definition we have $\forall i \in A$, $\succeq_i$ satisfies axioms $A1^- \ldots A6^-$.
  Besides, we have: $U_{post}^{-\min} = \min_{x \in X} \max(1 - L[x], \min_{\in A} \max(u_i(x), 1 - w_i))$
  Let $u(x) = \min_{i \in A} \max(u_i(x), 1 - w_i)$
  Then, $U_{post}^{-\min} = \min_{x \in X} \max(1 - L[x], u(x))$
  Hence, $U_{post}^{-\min}$ can be seen as a pessimistic utility based on the utility function u(x). Since, $U_{post}^{-\min} = U_{ante}^{-\min} = U^{-\min}$, so $U^{-\min}$ satisfies Dubois and Prade's axioms.
- The relation $\succeq$ defined by $U^{-\min}$ satisfies the Pareto unanimity axiom:
  Let: $\forall i \in A$, $L \succeq_i L'$
  By the definition of pessimistic utility we have: $L \succeq_i L'$ iff $U_i^-(L) \geq U_i^-(L')$
  Hence, $\forall i \in A, \max(1 - w_i, U_i^-(L)) \geq \max(1 - w_i, U_i^-(L))$
  Then, $\min_{i \in A} \max(1 - w_i, U_i^-(L)) \geq \min_{i \in A} \max(1 - w_i, U_i^-(L'))$
  We obtain, $U_{ante}^{-\min}(L) \geq U_{ante}^{-\min}(L')$; which implies that $L \succeq L'$
  Since $U_{ante}^{-\min} = U_{post}^{-\min} = U^{-\min}$, then $U^{-\min}$ satisfies axiom P (Pareto Unanimity axiom).

□

Regarding the other (non egalitarian) pessimistic utility $U^{-\max}$ the news are unfortunately very bad: not only they suffer from timing effect problem; but also due to the drowning effect we can show by the following counter-examples that Pareto unanimity axiom is violated by $U_{post}^{-\max}$ (counter-example 5.2) and that $U_{ante}^{-\max}$ may fail to satisfy weak independence (counter-example 5.3).

Counter-example 5.2. *Consider two agents* 1 *and* 2 *with different importance such that* $w_1 = 1$ *and* $w_2 = 0.8$ *and consider the two lotteries presented in Figure 5.3 on* $X = \{x_1, x_2, x_3\}$. *We get:*

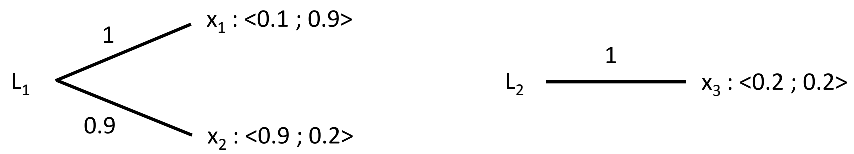

Figure 5.3: A counter-example to pareto unanimity for $U_{post}^{-\max}$ criterion.

- *For Agent* 1*:*

$$\begin{aligned} U^-(L_1) = &\ \min(\max((1-1), 0.1), \max((1-0.9), 0.9)) &= 0.1 \\ U^-(L_2) = &\ \max((1-1), 0.2) &= 0.2 \end{aligned}$$

- *For Agent* 2*:*

$$\begin{aligned} U^-(L_1) = &\ \min(\max((1-1), 0.9), \max((1-0.9), 0.2)) &= 0.2 \\ U^-(L_2) = &\ \max((1-1), 0.2) &= 0.2 \end{aligned}$$

While:

$$\begin{aligned} U_{post}^{-\max}(L_1) = &\ \min(\max((1-1), \max(\min(1, 0.1), \min(0.8, 0.9))), \\ &\ \max((1-0.9), \max(\min(1, 0.9), \min(0.8, 0.2)))) = 0.8 \\ U_{post}^{-\max}(L_2) = &\ \max((1-1), \max(\min(1, 0.2), \min(0.8, 0.2))) = 0.2 \end{aligned}$$

Hence, we can verify that Agent 1 *prefers* L_2 *to* L_1 *whereas lotteries* L_1 *and* L_2 *are equivalent for Agent* 2. *However, we can show that* $U_{post}^{-\max}(L_1) > U_{post}^{-\max}(L_2)$, *which contradicts Pareto Unanimity.*

Counter-example 5.3. *Consider two agents* 1 *and* 2*, where Agent* 2 *is more important than Agent* 1 *such that* $w_1 = 0.5$ *and* $w_2 = 1$*, and consider the three lotteries on* $X = \{x_1, x_2, x_3\}$ *depicted in Figure 5.4.*
Let L *and* L' *be the lotteries defined by:*
$L = \langle 1/L_1, 0.9/L_3 \rangle$ *and* $L' = \langle 1/L_2, 0.9/L_3 \rangle$.

We can verify that L_1 *and* L_2 *are equivalent:* $U_{ante}^{-\max}(L_1) = U_{ante}^{-\max}(L_2) = 0.5$ *whereas* $U_{ante}^{-\max}(L) = 0.5 \neq U_{ante}^{-\max}(L') = 0.4$*, which proves that* $U_{ante}^{-\max}$ *does not satisfy the weak independence axiom.*

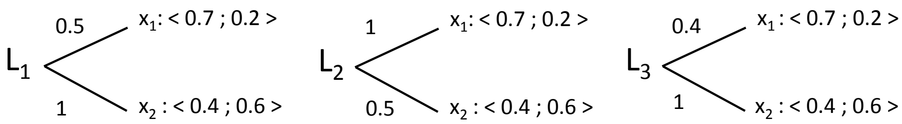

Figure 5.4: A counter-example to weak independence for $U_{ante}^{-\max}$ criterion.

Let us now go back to our egalitarian pessimistic utility and show that the relation that satisfies Completeness, Pareto Unanimity and the axioms of pessimistic utility can be captured by $U^{-\min}$ - thus providing a counterpart of Harsanyi's theorem that allows (weighted) equity. First, we sketch the proof in the general case, allowing more or less important agents: this provides a characterization of $U^{-\min}$ in its full generality. Then, we impose equity between agents in a second step - and rule out the weights.

We have seen that the set of axioms is consistent since satisfied by $U^{-\min}$. Let us now go the reverse way. Consider a relation $\succeq$ on $\mathcal{L}$ (built on X and V) that satisfies axioms $A1^- \dots A6^-$, a set of relations $\succeq_i$ on the same $\mathcal{L}$ that also satisfy these axioms, and suppose that the axioms of Pareto Unanimity and Completeness of X hold.

First of all, since axioms $A1^- \dots A6^-$ are satisfied by $\succeq$ and $\succeq_i$, these relations can be represented by pessimistic utilities -this is Dubois and Prade's theorem of representation. Let us now consider for any agent $i \in A$, the set $\top_i = \{x \in X\colon \forall y, x \succeq_i y\}$ of the best consequences according to i (this set cannot be empty because the $\succeq_i$'s are pre-orders). Thanks to Axiom C, there exists a consequence x^* that belongs to all the $\top_i$'s. By Pareto Unanimity, $x^* \succeq y, \forall y \in X$. In the same way, there exists a x_* such that $y \succeq x_*, \forall y \in X$.

Also, Axiom C allows us to define the constant act x^i for any agent i, such as:

Definition 5.2. *$\forall x \in X$ and any agent i, let x^i be the constant lottery s.t. $x^i \sim_i x$ and $x^i \sim_j x^*$ for each $j \neq i$.*

x^i will be identified with the utility of x according to agent i: the influence of the other agents is neutralized (they get their best outcome, which behaves as a neutral element in the pessimistic approach).

Let $\Sigma_i = \{x^i, x \in X\}$. $(x^*)^i$ (resp. $(x_*)^i$) is the best (resp. worst) consequence according to agent i belong to Σ_i by definition. The union of the Σ_i's, that is to say $\Sigma = \{x^i\colon x \in X, i \in A\}$, plays an important role in our proof - it allows the construction of a common evaluation scale. Σ is naturally ordered by $\succeq$ and each Σ_i is ordered by $\succeq_i$. By construction, we have:

Proposition 5.5. $\forall x \in X, (x^*)^i \succeq_i x \succeq_i (x_*)^i$.

Proof of Proposition 5.5.
By definition of x^* according to Axiom C we have: $\forall i \in A, \forall x \in X; x^* \succeq_i x$.

Besides, by definition of $(x^*)^i$ we have: $\forall i \in A$; $(x^*)^i \sim_i x^*$.
Then, by transitivity of $\succeq_i$ we can obtain: (i) $(x^*)^i \succeq_i x$.
In the same way, by definition of x_* we have: $\forall i \in A, \forall x \in X; x \succeq_i x_*$.
and by definition of $(x_*)^i$ we have: $\forall i \in A$; $(x_*)^i \sim_i x_*$.
Also, by transitivity we can obtain: (ii) $x \succeq_i (x_*)^i$.
From (i) and (ii) we can deduce that: $\forall x \in X, (x^*)^i \succeq_i x \succeq_i (x_*)^i$. □

Moreover, we can show that:

Proposition 5.6. $\forall x^i \in \Sigma_i, (x^*)^i \succeq x^i \succeq (x_*)^i$

Proof of Proposition 5.6.
By definition of x^i, we have: $\forall i \in A, \forall x \in X$; $x^i \sim_i x$.
According to Proposition 5.5, we have: $\forall i \in A, \forall x \in X$; $(x^*)^i \succeq_i x$.
Then, by transitivity we obtain: $(x^*)^i \succeq_i x^i$.
Besides, $(x^*)^i \succeq_k x^i$, for all k (because $(x^*)^i \sim_k x^*$, whether k is equal to i or not).
Then, $(x^*)^i \succeq_k x^i$ and $\succeq_i$ is transitive; by Pareto Unanimity we obtain (i) $(x^*)^i \succeq x^i$.
In the same way, by definition of x^i, we have: $\forall i \in A, \forall x \in X$; $x^i \sim_i x$.
According to Proposition 5.5, we have: $\forall i \in A, \forall x \in X$; $x \succeq_i (x_*)^i$.
Then, by transitivity we obtain: $x^i \succeq_i (x_*)^i$; besides by Pareto Unanimity we obtain (ii) $x^i \succeq (x_*)^i$.
From (i) and (ii) we can deduce that: $\forall x^i, (x^*)^i \succeq x^i \succeq (x_*)^i$. □

$(x^*)^i$ is one of the best consequences for i and $(x_*)^i$ is one of her worst ones. It may happen that one of the x^i be indifferent w.r.t. $\succeq$ to $(x_*)^i$: i prefers x^i to $(x_*)^i$, but the collectivity does not; this is due to the fact that agent i is not so important, so the elements of X that are bad for her (e.g. $(x_*)^i$) are considered as not so bad for the collectivity.

Let us denote by $B_i = \{x^i \in \Sigma_i$: $x^i \sim (x_*)^i\}$ the set of the elements of Σ_i that are indifferent to $(x_*)^i$according to the collectivity, and this even if agent i makes a difference; the elements of B_i form an equivalence class according to $\succeq$ - but, again, not necessarily according to $\succeq_i$.
Let denote m_i the best element of B_i (according to $\succeq_i$) [2]. It reflects the importance of the agent: the greater is m_i, the less is the importance of i.

Formally[3], it can be defined as follows:

Definition 5.3. $\forall i \in A$, *let* $m_i = argmax_{\succeq_i}\{x^i$: $x^i \sim (x_*)^i\}$ *be the* discount degree *of* i.

We can now formulate the following two lemmas:

[2]If $|B_i| > 1$, m_i can be any one of its elements.
[3]In the following, there are many relations (preorders). For the sake of clarity, we may indicate for each minimum or maximum operation the preorder it relies on.

Lemma 5.1. $\forall x \in X, i = 1, p$, $x^i \sim \max_{\succeq_i}(m_i, x^i)$.

Proof of Lemma 5.1.

- Case 1: $x^i \succ_i m_i$: then trivially $\max_{\succeq_i}(m_i, x^i) = x^i$.
  Since $\succeq$ is reflexive, then $x^i \sim \max_{\succeq_i}(m_i, x^i)$.

- Case 2: $m_i \succeq_i x^i$: by definition of m_i, we have:
  m_i = $argmax_{\succeq_i}\{x^i$: $x^i \sim (x_*)^i\}$, then m_i is a particular x^i, such as $x^i \sim (x_*)^i$. Then, we can say that (i) $m_i \sim (x_*)^i$.
  Furthermore, from Proposition 5.6 we have (ii) $x^i \succeq (x_*)^i$.
  From (i) and (ii) and since $\succeq$ is transitive and reflexive we can say that $x^i \succeq m_i$.
  Hence, we can deduce that $m_i \succeq_i x^i$ involve $x^i \sim \max_{\succeq_i}(m_i, x^i)$.

□

Lemma 5.2. $\forall x \in X$, $x \sim$ $argmin_{\succeq}$ $\{x^i$: $i \in A\}$.

Proof of Lemma 5.2.
We consider the lottery $L = \langle 1/x^1, \ldots, 1/x^p \rangle$.
Since $\succeq$ is a pessimistic utility (satisfies Dubois and Prade's axioms) and the x^is are constant acts; then we have: $L \sim argmin_{\succeq}\{x^i, i \in A\}$.
Let us show that $L \sim x$.
Since $\succeq_i$s relations are pessimistic utilities (satisfy Dubois and Prade's axioms), for each agent i have: $L \sim_i argmin_{\succeq_i}\{x^k, k = 1, p\}$.
Since $\forall k \neq i, x^k \sim_i x^*$, and $x^* \succeq_i x^i$, by transitivity we have $\forall k \neq i, x^k \succeq_i x^i$. Then $L \sim_i x^i$.
Besides, since $x \sim_i x^i$ and $\sim_i$ is transitive, we obtain $L \sim_i x$, for each agent i. Using Pareto Unanimity axiom, we can deduce that $L \sim x$.
From $L \sim x$ and $L \sim argmin_{\succeq}\{x^i, i \in A\}$, we can deduce by transitivity that: $x \sim argmin_{\succeq}\{x^i, i \in A\}$. □

From Lemmas 5.1 and 5.2 we get:

Corollary 5.1. $x \sim$ $argmin_{\succeq}\{\max_{\succeq_i}(m_i, x^i)$: $i \in A\}$.

Proof of Corollary 5.1.
From Lemmas 5.1 and 5.2, we have respectively:
(i) $\forall x \in X, i \in A, x^i \sim \max_{\succeq_i}(m_i, x^i)$.
(ii) $\forall x \in X, x \sim$ $argmin_{\succeq}$ $\{x^i$: $i \in A\}$.
Then the proof of this corollary is trivial by replacing x^i in (ii) by $\max_{\succeq_i}(m_i, x^i)$ from (i). □

In order to show that a relation satisfying axioms $A1^- \ldots A6^-$ is a pessimistic utility, Dubois and Prade [30] built the scale $U = \{[x]\colon x \in X\}$ where $[x]$ is the equivalence class of x according to $\succeq$. U is totally ordered by $\succeq$ and these authors set $u(x) = [x]$. Here, we use the set $\Sigma = \{x^i\colon x \in X, i \in A\}$, partially ordered by the relation $\trianglerighteq$ defined by:

Definition 5.4. $\forall x, y \in X, i, j \in A$*:*

$x^i \trianglerighteq y^i$ *iff* $x^i \succeq_i y^i$

$x^i \trianglerighteq y^j$ *iff* $x_i \succeq_i m_i$, $y_j \succeq_j m_j$ *and* $x^i \succeq y^j$, $\forall i \neq j$.

This relation is a partial pre-order (it is reflexive and transitive) but if $x^i \prec m_i$ or $x^j \prec m_j$ for all $i \neq j$, x_i and y_j are not comparable: neither $x^i \trianglerighteq y^j$ nor $y^j \trianglerighteq x^i$ hold, but this is harmless. What is important is that (i) the restriction of $\trianglerighteq$ to each Σ_i is a pre-order (on Σ_i, $\trianglerighteq = \succeq_i$) and (ii) that any x_i that is as least as good as m_i (according to i) is comparable to any x_j that is as least as good as m_j (according to j) . Properties (i) and (ii) ensure that $v(x) = \min_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i)\colon i \in A\}$ exists.

Proposition 5.7.
The relation $\trianglerighteq$ is transitive.

Proof of Proposition 5.7.

- Case 1: $\forall x, y, z \in X$, $\forall i \in A$, if $x^i \trianglerighteq y^i$ and $y^i \trianglerighteq z^i$ then by definition of $\trianglerighteq$ we have: $x^i \succeq_i y^i$ and $y^i \succeq_i z^i$. Since $\succeq_i$ is transitive then $x^i \succeq_i z^i$ so, $x^i \trianglerighteq z^i$.
- Case 2: $\forall x, y, z \in X$, if $x^i \trianglerighteq y^i$ and $y^i \trianglerighteq z^j$ (such as: $i \neq j$), by definition of $\trianglerighteq$we can say that $x^i \succeq_i y^i$, then by Pareto Unanimity we can deduce that: $x^i \succeq y^i$.
  Besides, by Definition 5.4 we have: (i) $y^i \succeq_i m_i$, $z^j \succeq_j m_j$ and $y^i \succeq z^j$.
  We have $x^i \succeq y^i$ and $y^i \succeq z^j$ then by transitivity of $\succeq$ we obtain: (ii) $x^i \succeq z^j$
  From (i) and (ii) we can deduce that $x^i \succeq_i m_i$, $z^j \succeq_j m_j$ and $x^i \succeq z^j$ besides by definition of $\trianglerighteq$ we obtain $x^i \trianglerighteq z^j$.
- Case 3: $\forall x, y, z \in X$, if $x^i \trianglerighteq y^j$ and $y^j \trianglerighteq z^k$ (such as: $i \neq j \neq k$), by definition of $\trianglerighteq$ we have $x^i \succ_i m_i$, $y^j \succ_j m_j$ and $x^i \succ y^j$ (i) besides $y^j \succ_j m_j$, $z^k \succ_k m_k$ and $y^j \succ z^k$(ii). Then, from (i) and (ii) and by transitivity of $\succ$ we can conclude that $x^i \succ_i m_i$, $z^k \succ_k m_k$ and $x^i \succ z^k$, besides by definition of $\trianglerighteq$ obtain $x^i \trianglerighteq z^k$.

Then, $\forall x, y \in X$ and for all$i \neq j$; we can deduce that $\trianglerighteq$ is transitive. □

Then from Corollary 5.1 and Definition 5.4 it follows that: $x \sim v(x)$. Let k be the agent for which the min is reached in the expression of $v(x)$: $v(x) = \max_{\trianglerighteq}(m_k\ x^k)$ belongs to Σ_k and is such that $v(x) \succeq_k m_k$. Hence $v(x)$ and $v(y)$ are comparable w.r.t. $\trianglerighteq$, whatever x, y. This allows us to write:

Lemma 5.3.
$x \succeq y$ *iff* $argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i)$: $i \in A\} \trianglerighteq argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, y^i)$: $i \in A\}$.

Proof of Lemma 5.3.
By Lemma 5.1 we have: $\forall x \in X, i \in A$, $x^i \sim \max_{\succeq_i}(m_i, x^i)$.
Then, by Definition 5.4 we can say that: (i) $x^i \sim \max_{\trianglerighteq}(m_i, x^i)$.
Besides, by Lemma 5.2 we have: $\forall x \in X$, $x \sim\ argmin_{\succeq}\ \{x^i$: $i \in A\}$.
In the same way, by Definition 5.4 we can say that:
(ii) $x \sim\ argmin_{\trianglerighteq}\ \{x^i, i \in A\}$.
Using Corollary 5.1 and from (i) and (ii) we can obtain:
$x \sim argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i), i \in A\}$.
Then we can deduce that:
$x \sim\ argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i), i \in A\}$ and $y \sim\ argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, y^i), i \in A\}$
So, by Definition 5.4 we obtain:
$x \succeq y$ iff $argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i), i \in A\} \trianglerighteq argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, y^i), i \in A\}$. □

Because working with a partial preorder is not so convenient, we shall use any complete preorder $\trianglerighteq'$ on Σ such that $x \trianglerighteq y \implies x \trianglerighteq' y$ (there always exists one). Then we get:

Lemma 5.4.
$x \succeq y$ *iff* $argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, x^i)$: $i \in A\} \trianglerighteq' argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, y^i)$: $i \in A\}$.

Proof of Lemma 5.4.
The proof of this Lemma is similar to the proof relative to Lemma 5.3 since they express the same thing; with a partial pre-order ($\trianglerighteq$) on Σ for Lemma 5.3 and a complete pre-order ($\trianglerighteq'$) for Lemma 5.4. □

Since $\succeq$ satisfies axioms $A1^- \dots A6^-$, Dubois and Prade's result applies: there exists an order reversing function n s.t.:

$$L \succeq L' \text{ iff } \min_{x \in X \succeq} \max_{\succeq}(n(L[x]), u(x)) \succeq \min_{x \in X \succeq} \max_{\succeq}(n(L'[x]), u(x)).$$

Let us denote $u(x) =\ argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, x^i)$: $i \in A\}$ and $n^{ext}(v) = u(n(v))$ ($n(v)$ is an element of Σ). By applying Lemma 5.4, we can write:

$$L \succeq L' \text{ iff } \min_{x \in X \trianglerighteq'} \max_{\trianglerighteq'}(n^{ext}(L[x]), u(x)) \trianglerighteq' \min_{x \in X \trianglerighteq'} \max_{\trianglerighteq'}(n^{ext}(L'[x]), u(x)).$$

$n^{ext}(v)$, m_i and x^i, $u(x)$ belong to Σ. In order to get a total order, we consider the equivalence classes of Σ, i.e. the set $U^{ext} = \{[x]$: $x \in X\}$ where $[x]$ is the equivalence class of x w.r.t. $\trianglerighteq'$. Because $x =\ argmin_{\succeq}\{x^i$: $i \in A\}$ (Lemma 5.2) U^{ext} contains the equivalence classe of each $x \in X$ to $\succeq$, in particular, the equivalence classe $[x^i]$ of each x^i; U^{ext} is ordered by $\trianglerighteq'$ and is equipped with a maximal and a minimal elements ($[x^*]$ and $[x_*]$, respectively).

Setting $u_i(x) = [x^i]$, $nw_i = [m_i]$ and $n(v) = [n^{ext}(v)]$, we get:

$$L \succeq L' \text{ iff } \min_{\substack{\trianglerighteq' \\ x\in X}} \max_{\trianglerighteq'}(n(L[x]), \min_{\substack{\trianglerighteq' \\ i\in A}} max_{\trianglerighteq'}(u_i(x), nw_i)) \quad \trianglerighteq' \quad \min_{\substack{\trianglerighteq' \\ x\in X}} \max_{\trianglerighteq'}(n(L[x]), \min_{\substack{\trianglerighteq' \\ i\in A}} max_{\trianglerighteq'}(u_i(x), nw_i)).$$

Hence the main result of this work:

Theorem 5.1. *If the collective preference and individual preference relations satisfy axioms $A1^-$ ... $A6^-$, Pareto Unanimity (axiom P) and the axiom of completeness of X (axiom C) then there exists a scale U^{ext} totally ordered by $\trianglerighteq'$, a distribution of weights nw: $A \mapsto U^{ext}$, a series of functions u_i: $X \mapsto U^{ext}, i = 1, n$ and an order reversing function n: $V \mapsto U^{ext}$ s.t. for each couple of lotteries L and L':*

$$L \succeq L' \textit{ iff } \min_{x\in X} \max(n(L[x]), \min_{i\in A} \max(nw_i, u_i(x))) \quad \trianglerighteq' \quad \min_{x\in X} \max(n(L'[x]), \min_{i\in A} \max(nw_i, u_i(x))).$$

This theorem can be interpreted as follows: If Dubois and Prade's axioms $A1^- \dots A6^-$ relative to the pessimistic utility, as well as axioms P and C are satisfied by the collective preference relation $\succeq$ and the individual preference relations $\succeq_i$, then there exists a scale U^{ext} totally ordered by $\trianglerighteq'$, a distribution of weights, a set of functions $u_i(x)$ and an order reversing function such that for any pair of lotteries L and L', L is preferred by the collectivity to L' iff the pessimistic egalitarian utility $U^{-min}(L)$ is better than $U^{-min}(L')$.

Proof of Theorem 5.1.
According to Dubois and Prade's result relative to pessimistic utility, since $\succeq$ satisfies axioms $A1^- \dots A6^-$ it exists an order reversing function n: $V \mapsto X$ such that:

$$L \succeq L' \text{ iff } \min_{\substack{\succeq \\ x\in X}} \max_{\succeq}(n(L[x]), x) \succeq \min_{\substack{\succeq \\ x\in X}} \max_{\succeq}(n(L'[x]), x).$$

Using Definition 5.4 and since $\trianglerighteq$ relative to each Σ_i is a complete pre-ordre (on Σ_i, $\trianglerighteq = \succeq_i$) and each x^i is at least as good as m_i (for i) is comparable to each x^j that is at least as good as m_j (for j) we can say that: $u(x) = argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i)\text{: } i \in A\}$. Using Lemma 5.4 we can say that: $u(x) = argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, x^i), i \in A\}$
By applying Lemma 5.4 and using Dubois and Prade's definition relative to the pessimistic utility, we obtain:

$$L \succeq L' \text{ ssi } \min_{\substack{\trianglerighteq' \\ x\in X}} \max_{\trianglerighteq'}(n^{ext}(L[x]), u(x)) \quad \trianglerighteq' \quad \min_{\substack{\trianglerighteq' \\ x\in X}} \max_{\trianglerighteq'}(n^{ext}(L'[x]), u(x)).$$

$n^{ext}(v)$, m_i and x^i, $u(x)$ belongs to Σ. In order to have a total order, we consider the equivalence classes of Σ, i.e. the set $U^{ext} = \{[x]\text{: } x \in X\}$ where $[x]$ is an equivalence class of x for $\trianglerighteq'$. Since $x = argmin_{\succeq}\{x^i\text{: } i \in A\}$ (Lemma 5.2) U^{ext} contain the equivalence class of each $x \in X$ for $\succeq$, in particular, the equivalence class of $[x^i]$ for each x^i ; U^{ext} is totally ordered by $\trianglerighteq'$, where $[x^*]$ and $[x_*]$ are respectively the maximal and minimal elements.

Let $u_i(x) = [x^i]$, $nw_i = [m_i]$ and $n(v) = [n^{ext}(v)]$, then we obtain:

$$L \succeq L' \text{ ssi } \min_{x \in X}{}_{\trianglerighteq} \max{}_{\trianglerighteq}(n(L[x]), \min_{i \in A}{}_{\trianglerighteq} \max{}_{\trianglerighteq}(u_i(x), nw_i)) \quad \trianglerighteq \quad \min_{x \in X}{}_{\trianglerighteq} \max{}_{\trianglerighteq}(n(L'[x]), \min_{i \in A}{}_{\trianglerighteq} \max{}_{\trianglerighteq}(u_i(x), nw_i)).$$

□

A representation theorem for agents with the same importance (pure egalitarianism)

We can now add two axioms that leads to pure egalitarianism.

Axiom (E). $\forall i, j, (x_*)^i \sim (x_*)^j$.

Axiom (PW). $\forall i$, *if* $x \succ_i y$ *then* $x^i \succ y^i$.

By **E**, the dissatisfaction of one agent has no more power than the one of another agent. A direct consequence is that the agents have the same discount degree. By **PW**, each agent has some power (he makes the decision at least when every other one is totally happy with both x and y). It implies $m_i \sim_i x_*$, for each i. Because all i share the same discount m_i, Pareto Unanimity implies that $m_i \sim x_*$.

This provides a characterization of the full egalitarian CUF:

Theorem 5.2. *If the collective preference and individual preference relations satisfy axioms $A1^-$ ... $A6^-$, P, C, E and PW then there exists a scale U^{ext} totally ordered by $\trianglerighteq'$, a series of functions u_i: $X \mapsto U^{ext}$ and an order reversing function n: $V \mapsto U^{ext}$ such that:*

$$L \succeq L' \text{ iff } \min_{x \in X} \max(n(L[x]), \min_{i \in A} u_i(x)) \quad \trianglerighteq' \quad \min_{x \in X} \max(n(L'[x]), \min_{i \in A} u_i(x)).$$

Proof of Theorem 5.2.
From Lemma 5.1 we have: $\forall x \in X$, $i \in A$, $x^i \sim \max_{\succ_i}(m_i, x^i)$.
Then we can consider that m_i is a particular x^i such as: $x^i \sim (x_*)^i$, i.e. $m_i \sim (x_*)^i, \forall i \in A$. Using axiom E ($(x_*)^i \sim (x_*)^j$) we can say that $m_i \sim (x_*)^j$ $(j \neq i)$. Then we can deduce that: $\forall i, j; m_i \sim (x_*)^i \sim m_j \sim (x_*)^j$.
Using Lemma 5.4 we have:
$x \succeq y$ iff $argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, x^i) : i \in A\} \trianglerighteq' argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, y^i) : i \in A\}$.
Besides, since each agent i have the possibility to decide for the collectivity when others are indifferent (axiom PW), Then we can rewrite Lemma 5.4 such as:
$x \succeq y$ iff $argmin_{\trianglerighteq'}\{x^i : i \in A\} \trianglerighteq' argmin_{\trianglerighteq'}\{y^i : i \in A\}$.
Summing up Dubois and Prade's result relative to the pessimistic utility and considering u(x) = $argmin_{\trianglerighteq'} x^i$, we can rewrite Theorem 5.1 such as:

$$L \succeq L' \text{ ssi } \min_{x \in X} \max(n(L[x]), \min_{i \in A} u_i(x)) \quad \trianglerighteq' \quad \min_{x \in X} \max(n(L'[x]), \min_{i \in A} u_i(x)).$$

□

This theorem shows that not only egalitarianism and decision under uncertainty are compatible and can escape the Timing Effect, but egalitarianism is compulsory when the decision is to be made on a possibilistic and cautious basis. This can be interpreted as a justification of egalitarianism just like Harsanyi's theorem that is interpreted as a justification of utilitarianism.

5.4 Axiomatization of non egalitarian optimistic decision rule in style of VNM axiomatic system

Let us now propose a representation theorem relative to homogeneous optimistic utility function $U^{+\max}$, which is an optimistic counterpart of the previous representation theorem.

We consider a preference relation $\succeq$ on $\mathcal{L}$ that satisfies Dubois and Prade's axioms relative to the optimistic utility (presented in Section 2.3), a set of preference relations $\succeq_i$ on the same $\mathcal{L}$ satisfying the same axioms, and we suppose that Pareto Unanimity and the Completeness axioms are satisfied. Since Dubois and Prade's axioms ($A1^+ \ldots A6^+$) are satisfied by $\succeq$ and by any $\succeq_i$, it is obvious that these relations may be represented by optimistic utility functions.

A representation theorem for agents with different weights

Consider a set $\mathcal{L}$ of possibilistic lotteries built from a set X and a scale V; a set u_i, $i \in A$ of utility functions on X taking their values in $[0, 1]$; and a weight vector $\vec{w} \in [0, 1]^p$ (where w_i is the weight of agent i). Similar to $U^{-\min}$, the homogenous utility, it holds that:

Proposition 5.8.
The relations $\succeq$ and $\succeq_i$ defined by:

$$L \succeq L' \text{ iff } U^{+\max}(L) \geq U^{+\max}(L'),$$

$$L \succeq_i L' \text{ iff } U_i^{+}(L) \geq U_i^{+}(L')$$

satisfy axioms $A1^+ \ldots A6^+$, as well as the Pareto Unanimity axiom.

Proof of Proposition 5.8.
This proof relative to Proposition 5.8 is very similar to the one one relative to 5.4 by replacing the $\min$ operator by $\max$, $\max$ by $\min$, $(1 - L[x])$ by $L[x]$, $(1 - w_i)$ by w_i and axioms $A1^- \ldots A6^-$ by axioms $A1^+ \ldots A6^+$. □

As shown by counter-example 5.4, $U_{post}^{+\min}$ fails to satisfy Pareto Unanimity. Likewise, $U_{ante}^{+\min}$ violates the weak independence axiom (counter-example 5.5).

Counter-example 5.4. *Consider two not equally important agents* 1 *and* 2 *such that* w_1 *= 1 and* w_2 *= 0.8 and consider the two lotteries resented in Figure 5.5 on* $X = \{x_1, x_2, x_3\}$*. We get:*

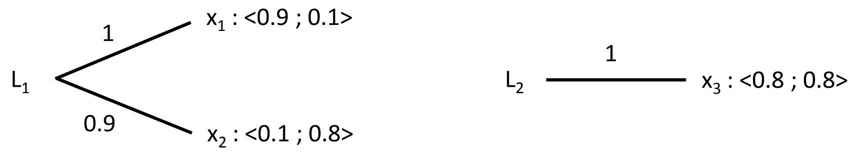

Figure 5.5: A counter-example to Pareto Unanimity for $U_{ante}^{+\min}$ criterion.

- *For Agent* 1*:*

$$\begin{array}{lll} U^+(L_1) = & \max(\min(1, 0.9), \min(0.9, 0.1)) & = 0.9 \\ U^+(L_2) = & \min(1, 0.8) & = 0.8 \end{array}$$

- *For Agent* 2*:*

$$\begin{array}{lll} U^+(L_1) = & \max(\min(1, 0.1), \min(0.9, 0.8)) & = 0.8 \\ U^+(L_2) = & \min(1, 0.8) & = 0.8 \end{array}$$

While:
$U_{post}^{+\min}(L_1) = \max(\min(1, \min(\max((1-1), 0.9), \max((1-0.8), 0.1))),$
$\min(0.9, \min(\max((1-1), 0.1), \max((1-1), 0.8)))) = 0.1$
$U_{post}^{+\min}(L_2) = \max(\min(1, \min(\max((1-1), 0.8), \max((1-0.8), 0.8)))) = 0.8$. *Hence, we can verify that Agent* 1 *prefers* L_1 *to* L_2 *whereas lotteries* L_1 *and* L_2 *are equivalent for Agent* 2*. So,* $U_{post}^{+\min}(L_2) > U_{post}^{+\min}(L_1)$*, which contradicts Pareto Unanimity.*

Counter-example 5.5. *Consider two agents* 1 *and* 2 *such that* w_1 *= 0.5 and* w_2 *= 1, and consider the three lotteries on* $X = \{x_1, x_2, x_3\}$ *depicted in Figure 5.6.*
Let L *and* L' *be the lotteries defined by:*
$L = \langle 1/L_1, 0.9/L_3 \rangle$ *and* $L' = \langle 1/L_2, 0.9/L_3 \rangle$*.*

We can verify that L_1 *and* L_2 *are equivalent:* $U_{ante}^{+\min}(L_1)$ = $U_{ante}^{+\min}(L_2)$ = *0.5 whereas* $U_{ante}^{+\min}(L)$ = *0.5* $\neq U_{ante}^{+\min}(L')$ = *0.6, which proves that* $U_{ante}^{+\min}$ *does not satisfy the weak independence axiom.*

In the previous section, we have seen that axiom C ensures the existence of a constant act x^* ideal for all the agents and a constant act x_* that is anti-ideal for all. We consider for each agent i, $\bot_i = \{x \in X, \forall y, y \succeq_i x\}$ the set of the least appreciated (worst) consequences to i, where x_* belongs obviously to all the $\bot_i$'s.

For each agent i, we define the constant act $x_i \in X$ as follows:

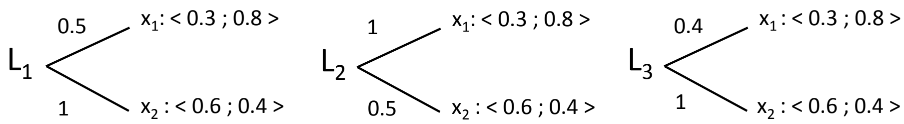

Figure 5.6: A counter-example to weak independence for $U_{ante}^{+\min}$ criterion.

Definition 5.5. *$\forall x \in X$, and any agent i, x_i is the constant lottery such that: $x_i \sim_i x$ and $x_i \sim_j x_*$; for each $j \neq i$*

x_i will be identified with the utility of x according to agent i: the influence of the other agents is neutralized (they get their worst outcome, which behaves as a neutral element in the optimistic approach). Let $\Sigma_i = \{x_i, x \in X\}$ be the set of constant acts relative to agent i. Σ_i is ordered by $\succeq_i$. We define Σ as the union of the Σ_i's, that is ordered by $\succeq$.

$(x^*)_i$ is one of the best consequences for i and $(x_*)_i$ is one of her worst ones. It may happen that one of the x_i be indifferent w.r.t. $\succeq$ to $(x^*)_i$: i prefers $(x^*)_i$ to x_i, but the collectivity does not; this is due to the fact that the agent i is not so important, so the consequences that are good for her, typically $(x^*)_i$, as considered as not so good for the collectivity. So i is not so important to impose her preferences to the group.

Let us denote $B^i = \{x_i \in \Sigma_i, x_i \sim (x^*)_i\}$, the set of elements of Σ_i that are indifferent to $(x^*)_i$ for the collectivity, and this even if agent i makes a difference; the elements of B^i form an equivalence class according to $\succeq$ but not necessarily according to $\succeq_i$.

Let m^i denotes the worst element of B_i according to $\succeq_i$ as follows:

Definition 5.6. $\forall i \in A$, $m^i = argmin_{\succeq_i}\{x_i$: $x_i \sim (x^*)_i\}$

We can now propose the two following lemmas:

Lemma 5.5.
$\forall x \in X, i = 1, p$, $x_i \sim \min_{\succeq_i}(m^i, x_i)$

Proof of Lemma 5.5.

- Case 1: $m^i \succ_i x_i$: then trivially $\min_{\succeq_i}(m^i, x_i) = x_i$.
  Since $\succeq$ is reflexive, $x_i \sim \min_{\succeq_i}(m^i, x_i)$.
- Case 2: $x_i \succeq_i m^i$: by definition of m^i, we have: $m^i = argmin_{\succeq_i}\{x_i$: $x_i \sim (x^*)_i\}$, then m^i is a particular x_i such as $x_i \sim (x^*)_i$. Then we can say that (i) $m^i \sim (x^*)_i$.
  Besides, from Proposition 5.6 we have (ii) $(x^*)_i \succeq x_i$.
  From (i) and (ii) and since $\succeq$ is transitive and reflexive we can say that $m^i \succeq x_i$.
  Hence, we can deduce that $x_i \succeq_i m^i$ involve $x_i \sim \min_{\succeq_i}(m^i, x_i)$.

□

Lemma 5.6.
$\forall x \in X$, $x \sim argmax_{\succeq} \{x_i, i \in A\}$.

Proof of Lemma 5.6.
We consider lottery $L = \langle 1/x_1, \ldots, 1/x_p \rangle$.
Since $\succeq$ is an optimistic utility (it satisfies Dubois ans Prade's axioms) and x_is are constant acts; we have: $L \sim argmax_{\succeq}\{x_i, i \in A\}$.
Let us show that $L \sim x$.
Since $\succeq_i$s are optimistic utilities (satisfy Dubois and Prade's axioms), for each agent i we have: $L \sim_i argmax_{\succeq_i}\{x_k, k = 1, p\}$.
Since $\forall k \neq i, x_k \sim_i x_*$, and $x_i \succeq_i x_*$, by transitivity we have $\forall k \neq i, x_i \succeq_i x_k$. Then $L \sim_i x_i$.
Since $x \sim_i x_i$ and $\sim_i$ is transitive, we obtain $L \sim_i x$, and this for each agent i. Using Pareto Unanimity, we can deduce that $L \sim x$.
From $L \sim x$ and $L \sim argmax_{\succeq}\{x_i, i \in A\}$, we can conclude by transitivity that: $x \sim argmax_{\succeq}\{x_i, i \in A\}$. □

From the previous lemmas we can deduce that:

Corollary 5.2. $\forall x \in X$, $x \sim argmax_{\succeq}\{\min_{\succeq_i}(m^i, x_i), i \in A\}$.

Proof of Corollary 5.2.
From Lemma 5.5 and Lemma 5.6, we have respectively:
(i) $\forall x \in X, i \in A, x_i \sim \min_{\succeq_i}(m^i, x_i)$.
(ii) $\forall x \in X, x \sim argmax_{\succeq} \{x_i\colon i \in A\}$.
Then the proof of the corollary is trivial by replacing x_i from (ii) by $\min_{\succeq_i}(m^i, x_i)$ from (i). □

Let us now define the relation $\trianglerighteq_o$ on $\Sigma = \{x_i, x \in X, i \in A\}$ such as:

Definition 5.7. $\forall x, y \in X, i, j \in A$:

$x_i \trianglerighteq_o y_i$ *iff* $x_i \succeq_i y_i$

$\forall i \neq j$, $x_i \trianglerighteq_o y_j$ *iff* $m^i \succeq_i x_i$, $m^j \succeq_j y_j$ *and* $x_i \succeq y_j$

This relation is reflexive and transitive but it defines a partial preorder on Σ; it may happens that $m^i \prec x_i$ or $m^j \prec x_j$, for all $i \neq j$ so, x_i and y_j are not comparable: but this is harmless since (i) the restriction of $\trianglerighteq_o$ to each Σ_i is a pre-order (on Σ_i, $\trianglerighteq_o$ = $\succeq_i$) and (ii) that any x_i which is always equal or less than m^i according to i, is comparable to any x_j that is at most equal to m^j according to j. Properties (i) and (ii) ensure that $u(x) = \max_{\trianglerighteq_o}\{\min_{\trianglerighteq}(m^i, x_i)\colon i \in A\}$ exists.

Proposition 5.9. $\trianglerighteq_o$ *is transitive.*

Proof of Proposition 5.9.

- Case 1: $\forall x, y, z \in X$, $\forall i \in A$, if $x_i \trianglerighteq_o y_i$ and $y_i \trianglerighteq_o z_i$ then by definition of $\trianglerighteq_o$ we have: $x_i \succeq_i y_i$ and $y_i \succeq_i z_i$. Since $\succeq_i$ is transitive then $x_i \succeq_i z_i$ so, $x_i \trianglerighteq_o z_i$.

- Case 2: $\forall x, y, z \in X$, if $x_i \trianglerighteq_o y_i$ and $y_i \trianglerighteq_o z_j$ (such as: $i \neq j$), by definition of $\trianglerighteq_o$ we can deduce that $x_i \succeq_i y_i$, so by Pareto Unanimity we can write: $x_i \succeq y_i$.
  Besides, by Definition 5.7 we have: (i) $m^i \succeq_i y_i$, $m^j \succeq_j z_j$ and $y_i \succeq z_j$.
  We have $x_i \succeq y_i$ and $y_i \succeq z_j$ then by transitivity of $\succeq$ we obtain: (ii) $x_i \succeq z_j$
  From (i) and (ii) we can deduce that $m^i \succeq_i x_i$, $m^j \succeq_j z_j$ and $x_i \succeq z_j$ besides by definition of $\trianglerighteq_o$ we obtain $x_i \trianglerighteq_o z_j$.

- Case 3: $\forall x, y, z \in X$, if $x_i \trianglerighteq_o y_j$ and $y_j \trianglerighteq_o z_k$ (such as: $i \neq j \neq k$), by definition of $\trianglerighteq_o$ we have $m^i \succ_i x_i$, $m^j \succ_j y_j$ and $x_i \succ y_j$ (i) also $m^j \succ_j y_j$, $m^k \succ_k z_k$ and $y_j \succ z_k$(ii). From (i) and (ii), by transitivity of $\succ$ we can conclude that $m^i \succ_i x_i$, $m^k \succ_k z_k$ and $x_i \succ z_k$, also by definition of $\trianglerighteq_o$ we obtain $x_i \trianglerighteq_o z_k$.

So, $\forall x, y \in X$ and for all $i \neq j$; we can deduce that $\trianglerighteq_o$ is transitive. □

From Corollary 5.2 and Definition 5.7, it follows that:

Lemma 5.7.
$x \succeq y$ *iff* $argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, x_i), i \in A\} \trianglerighteq argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, y_i), i \in A\}$.

Proof of Lemma 5.7.
By Lemma 5.5 we have: $\forall x \in X, i \in A, x_i \sim \min_{\succeq_i}(m^i, x_i)$.
Then, by Definition 5.7 we can write: $x_i \sim \min_{\trianglerighteq_o}(m^i, x_i)$ (i).
Besides, by Lemma 5.6 we have: $\forall x \in X, x \sim\ argmax_{\succeq}\ \{x_i\colon i \in A\}$.
In the same way, by Definition 5.7 we can write:
(ii) $x \sim\ argmax_{\trianglerighteq_o}\ \{x_i, i \in A\}$.
Using Corollary 5.2 and from (i) and (ii) we obtain:
$x \sim argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, x_i), i \in A\}$.
Then we can deduce that:
$x \sim\ argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, x_i), i \in A\}$ and $y \sim\ argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, y_i), i \in A\}$
Hence, by Definition 5.7 we obtain:
$x \succeq y$ iff $argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, x_i), i \in A\} \trianglerighteq_o argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, y_i), i \in A\}$. □

Because working with a partial pre-order is not convenient, we shall use any complete pre-order $\trianglerighteq'_o$ on Σ, such that $x \trianglerighteq_o y \Rightarrow x \trianglerighteq'_o y$ (there always exists one). Then we get:

Lemma 5.8.
$x \succeq y$ *iff* $argmax_{\trianglerighteq'_o}\{\min_{\trianglerighteq'_o}(m^i, x_i), i \in A\} \trianglerighteq'_o argmax_{\trianglerighteq'_o}\{\min_{\trianglerighteq'_o}(m^i, y_i), i \in A\}$.

Proof of Lemma 5.8.
The proof of this Lemma is similar to the proof of Lemma 5.7 since they express the same thing; using a partial pre-order ($\trianglerighteq_o$) on Σ for Lemma 5.7 and a complete pre-order ($\trianglerighteq'_o$) for Lemma 5.8. □

Since $\succeq$ satisfies Dubois and Prade's axioms relative to the optimistic qualitative utility, we can resume their result: there exists a utility function u such that:
$L \succeq L'$ iff $\max_{\succeq, x\in X} \min_{\succeq}(L[x], u(x)) \succeq \max_{\succeq, x\in X} \min_{\succeq}(L'[x], u(x))$.

Let us denote $u(x) = argmax_{\trianglerighteq'_o}\{\min_{\trianglerighteq'_o}(m^i, x_i), i \in A\}$. By applying Lemma 5.8, we can write:
$L \succeq L'$ iff $\max_{\trianglerighteq'_o, x\in X} \min_{\trianglerighteq'_o}(L[x], u(x)) \trianglerighteq'_o \max_{\trianglerighteq'_o, x\in X} \min_{\trianglerighteq'_o}(L'[x], u(x))$.

Hence, an optimistic counter part of Theorem 5.1:

Theorem 5.3. *If the collective preference $\succeq$ and individual preference relations $\succeq_i$ satisfy axioms $A1^+ \dots A6^+$ relative to the optimistic utility; Pareto Unanimity and the axiom of completeness of X are satisfied, then there exists a scale U^{ext} totally ordered by $\trianglerighteq'_o$, a distribution of weights w: $A \mapsto U^{ext}$, a series of functions u_i: $X \mapsto U^{ext}$ such that for each couple of lotteries L and L':*
$L \succeq L'$ *iff* $\max_{x\in X} \min(L[x], \max_{i\in A} \min(u_i(x), w_i)) \trianglerighteq_o$
$\max_{x\in X} \min(L'[x], \max_{i\in A} \min(u_i(x), w_i))$.

Proof of Theorem 5.3.
Since $\succeq$ satisfies axioms $A1^+ \dots A6^+$ we can write:
$L \succeq L'$ iff $\max_{\succeq, x\in X} \min_{\succeq}(L[x], x) \succeq \max_{\succeq, x\in X} \min_{\succeq}(L'[x], x)$.
Using Definition 5.7 and since $\trianglerighteq_o$ for each Σ_i is a complete pre-ordre (on Σ_i, $\trianglerighteq_o = \succeq_i$) and each x_i that is equal or inferior m^i (for i) is comparable to each x_j that is inferior or equal to m^j (for j) we can say that: $u(x) = argmax_{\trianglerighteq_o}\{\min_{\trianglerighteq_o}(m^i, x_i)$: $i \in A\}$. Using Lemma 5.8 we can write:
$u(x) = argmax_{\trianglerighteq'_o}\{\min_{\trianglerighteq'_o}(m^i, x_i), i \in A\}$
By applying Lemma 5.8 and using Dubois and Prade's definition relative to the optimistic utility, we obtain:
$L \succeq L'$ ssi $\max_{\trianglerighteq'_o, x\in X} \min_{\trianglerighteq'_o}(L[x], u(x)) \trianglerighteq'_o \max_{\trianglerighteq'_o, x\in X} \min_{\trianglerighteq'_o}(L'[x], u(x))$.

m^i, x_i and $u(x)$ belongs to Σ. In order to have a total order, we consider the equivalence classes of Σ, i.e. the set $U^{ext} = \{[x]: x \in X\}$ where $[x]$ is the equivalence class of x for $\trianglerighteq'_o$. Since $x = argmax_{\succeq}\{x_i: i \in A\}$ (Lemma 5.6) U^{ext} contains the equivalence class for each $x \in X$ for $\succeq$, in particular, the equivalence class of $[x_i]$ for each x_i ; U^{ext} is totally ordered by $\trianglerighteq'_o$, where $[x^*]$ and $[x_*]$ are respectively the maximal and minimal elements.

Let $u_i(x) = [x_i]$ and $w_i = [m^i]$, then we obtain:

$$L \succeq L' \text{ ssi } \max_{x \in X}{}_{\trianglerighteq'_o} \min_{\trianglerighteq'_o}(L[x], \max_{i \in A}{}_{\trianglerighteq'_o} \min_{\trianglerighteq'_o}(u_i(x), w_i)) \quad \trianglerighteq'_o \quad \max_{x \in X}{}_{\trianglerighteq'_o} \min_{\trianglerighteq'_o}(L'[x], \max_{i \in A}{}_{\trianglerighteq'_o} \min_{\trianglerighteq'_o}(u_i(x), w_i)).$$

□

A representation theorem for agents with same importance

Now, to ensure the pure egalitarianism and to guarantee that all the agents have the same importance, we resume axiom PW presented previously, and we present an optimistic version of axiom E:

Axiom (E'). $\forall i, j, (x^*)_i \sim (x^*)_j$.

Thanks to axioms E' and PW, we can provide an optimistic version of Theorem 5.2:

Theorem 5.4. *If the collective preference $\succeq$ and individual preference relations $\succeq_i$ satisfy axioms $A1^+ \dots A6^+$, C, P, E and PW then there exists a scale U^{ext} totally ordered by $\trianglerighteq'_o$, a series of functions u_i: $X \mapsto U^{ext}$ such that for each couple of lotteries L and L':*

$$L \succeq L' \textit{ iff } \max_{x \in X} \min(L[x], \max_{i \in A} u_i(x)) \quad \trianglerighteq'_o \quad \max_{x \in X} \min(L'[x], \max_{i \in A} u_i(x)).$$

Proof of Theorem 5.4.
From Lemma 5.5 we have: $\forall x \in X, i \in A, x_i \sim \min_{\succ_i}(m^i, x_i)$.
Then we consider that m^i is a particular x_i such as: $x_i \sim (x^*)_i$. In other words: $m^i \sim (x^*)_i, \forall i \in A$.
Using axiom E' ($(x^*)_i \sim (x^*)_j$) we can say that $m^i \sim (x^*)_j$ ($j \neq i$). Hence we can deduce that: $\forall i, j m^i \sim (x^*)_i \sim m^j \sim (x^*)_j$.
From Lemma 5.8 we have:
$x \succeq y$ iff $argmax_{\trianglerighteq'_o}\{\min_{\trianglerighteq'_o}(m^i, x_i){:}i \in A\} \trianglerighteq'_o argmax_{\trianglerighteq'_o}\{\min_{\trianglerighteq'_o}(m^i, y_i){:}i \in A\}$.
In addition, since each agent i have the possibility to decide for the collectivity when others are indifferent (axiom PW), then we can rewrite Lemma 5.8 as the following:
$x \succeq y$ iff $argmax_{\trianglerighteq'_o}\{x_i{:}i \in A\} \trianglerighteq'_o argmax_{\trianglerighteq'_o}\{y_i{:}i \in A\}$.
Resuming Dubois and's results relative to the optimistic utility and considering u(x) = $argmax_{\trianglerighteq'_o} x_i$ then we can rewrite theorem 5.3 such as:

$$L \succeq L' \text{ iff } \max_{x \in X} \min(L[x], \max_{i \in A} u_i(x)) \quad \trianglerighteq'_o \quad \max_{x \in X} \min(L'[x], \max_{i \in A} u_i(x)).$$

□

5.5 Conclusion

This chapter has provided a decision theoretical approach for evaluating collective decision problems under possibilistic uncertainty. The combination of the collectivity dimension, namely egalitarian and not egalitarian approach to aggregate agents preferences; and the decision maker's attitude with respect to uncertainty (i.e. optimistic utility or pessimistic utility) leads to four approaches of collective decision making under possibilistic uncertainty. Considering that each of these utilities can be computed either *ex-ante* or *ex-post*, we have proposed the definition of eight aggregations, that eventually reduce to six: U^{-min}_{ante} (resp. U^{+max}_{ante}) has been shown to coincide with $U^{-min}post$ (resp. U^{+max}_{post}) such a coincidence does not happen for U^{+min} and U^{-max}, that suffer from timing effect.

Under the assumption, that all the agents have the same knowledge, we have provided an axiomatic system for U^{-min} and have shown that if both the collective preference and the individual preferences do satisfy Dubois and Prade's axioms, and in particular risk aversion, then an egalitarian CUF is mandatory. Finally, we have provided an optimistic counter part of this theorem relative to U^{+max}.These results can be considered as an ordinal counterpart to Harsanyi's theorem.

CHAPITRE 6

Algorithms for multi-agent optimization in possibilistic decision trees

6.1 Introduction

In Chapter 4, we have presented an algorithmic study relative to the optimization in decision trees, where decisions concern one decision maker. In this Chapter we deal with qualitative decision trees representing collective sequential decision problems. We focus on the optimization of the collective qualitative decision rules presented in Chapter 5 in such trees and we propose algorithms dedicated to find optimal strategies. A Dynamic Programming algorithm is performed to optimize criteria that satisfy the property of monotonicity and a Multi-Dynamic programming or a Branch and Bound algorithm is proposed for those that are not monotonic.

This Chapter is organized as follows: the next Section summarizes the adaptation of Dynamic Programming algorithm for the optimization of the *ex-post* decision rules. Section 6.2 details the adaptation of Dynamic Programming for the optimization of the U_{ante}^{-max} criterion and Section 6.3 is devoted to the optimization of the U_{ante}^{+min} using a Branch and Bound Algorithm.

6.2 Dynamic Programming algorithm for the optimization of "*ex-post*" decision rules

Let us now consider the strategy optimization in possibilistic (qualitative) decision trees represented in Section 4.4. As usual with decision trees, the set of potential strategies to compare, Δ, is exponential w.r.t. the size of the input, which means that an explicit evaluation of each strategy in Δ is not realistic. Such a problem can nevertheless be solved efficiently (i.e. in polytime, without an explicit evaluation of each strategy in the set) by a Dynamic Programming algorithm, as soon as the decision rule leads to transitive preferences and satisfies the principle of weak monotonicity. Formally, for any decision rule O (e.g. any of the decision rules proposed in the previous Chapter) over possibilistic lotteries, $\succeq_O$ is said to be weakly monotonic iff whatever L, L' and L'', whatever (α,β) such that $\max(\alpha, \beta) = 1$:

$$L \succeq_O L' \Rightarrow \langle \alpha/L, \beta/L'' \rangle \succeq_O \langle \alpha/L', \beta/L'' \rangle. \tag{6.1}$$

This property ensures that each sub strategy of an optimal strategy is optimal in its sub-tree.

Hopefully, each of the *ex-post* criteria satisfies the weak monotonicity property (because they collapse to a classical pessimistic U^- or optimistic U^+ utility, which are known to be monotonic [9, 48]). Like each of the utilities used in the previous Chapter, these criteria take their values on $\mathbb{R}^+$ and thus obviously define a transitive preference on strategies. This allows the use of a Dynamic Programming algorithm: the utility values pertaining to the leaves are aggregated; then an optimal strategy is built in an incremental way, from the last decision nodes to the root of the tree. The adaptation of this algorithm to the *ex-post* rules is detailed in Algorithm 2; which is not very different from its classical (probabilistic) version presented by Algorithm (Line 2) in Chapter 4, and polynomial in the number of edges and nodes in the tree, as usual.

Example 6.1. *Let us consider two equally important agents* 1 *and* 2 *(*$w_1 = w_2 = 1$*) and consider the decision tree* T *represented in Figure 6.1. The optimization of the egalitarian pessimistic utility* $U_{post}^{-\min}$ *decision rule in this decision tree using Algorithm 2 can be performed in two main steps:*

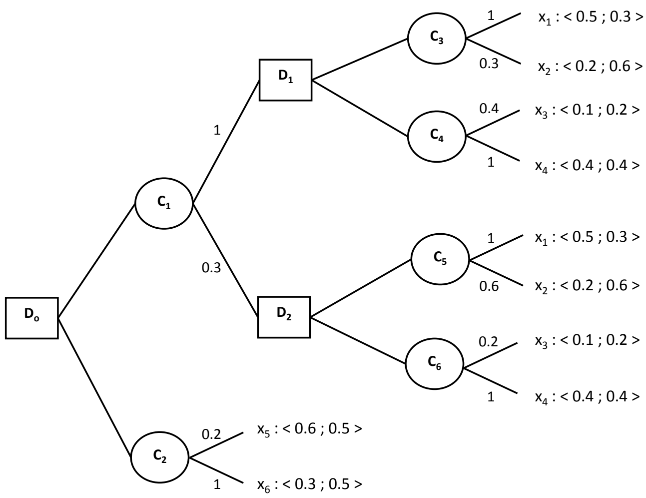

Figure 6.1: Possibilistic decision tree for Example 6.1.

- *First of all, we calculate the aggregated utility relative to each leave node* $LN \in N$ *and we transform each utility vector* $\vec{u_N}$ *to an aggregated utility* u_N*, as shown in Figure 6.2.*

- *Then, we perform the Dynamic Programing as for a possibilistic decision tree* Π *DT with a single agent.*

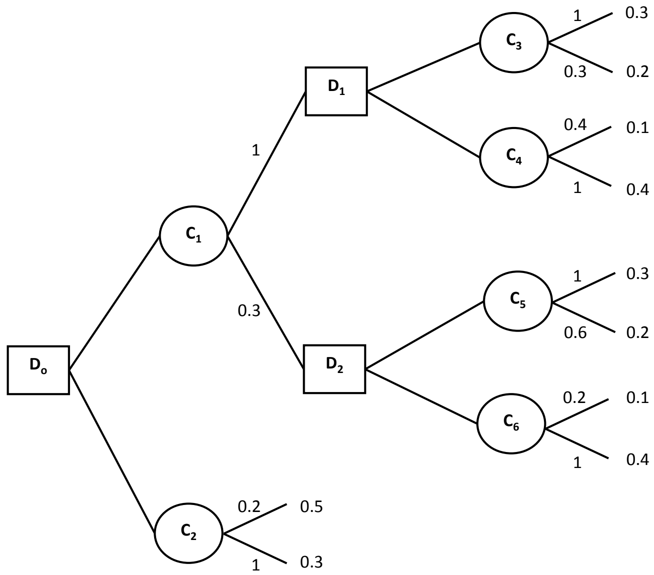

Figure 6.2: Possibilistic decision tree for Example 6.1 with aggregated utilities.

Concerning $U^{-\min}_{ante}$ and $U^{+\max}_{ante}$, recall that $U^{-\min}_{ante}$ (resp. $U^{+\max}_{ante}$) $= U^{-\min}_{post}$ (resp. $U^{+\max}_{post}$): the optimization can thus simply be performed by the *ex-post* Dynamic Programming algorithm (Algorithm 2).

6.3 Dynamic Programming algorithm for the optimization of "*ex-ante*" decision rules

It is possible to propose an *ex-ante* variant of Dynamic Programming for the optimization of the *ex-ante* criteria- see Algorithm 3. It keeps at each decision node a possibility distribution over the utility vectors corresponding to the decision made in the future. This algorithm provides the same results as Algorithm 2 for $U^{-\min}_{ante}$ and $U^{+\max}_{ante}$. However, Algorithm 3 may provide a good strategy for $U^{-\max}_{ante}$ and $U^{+\min}_{ante}$, but without any guarantee of optimality. This is because these decision rules do not satisfy the monotonicity principle stricto sensu [6].

Algorithm 2: DynProgPost: An adaptation of Dynamic Programming for the optimization of "*ex-post*" decision rules

```
Data: A node N
Result: An optimal strategy δ and its value u*
begin
    // Initialization
    for j ∈ {1,...,n} do
        L[u_j] ← 0;
    // Leaves
    if N ∈ LN then
        // (Disjunctive / Conjunctive) aggregation
        for i ∈ {1,...,p} do
            u'_N ← ((u_j)_i ⊗ ω_i) ;
            u_N ← (u_N ⊕ u'_N) ;
            L[u(N)] ← 1
    // where ⊗ = min (resp. max) , ω_i = w_i (resp. 1 - w_i) and ⊕ = max (resp. min) in the
    case of disjunctive (resp. conjunctive) aggregation
    // Chance nodes
    if N ∈ C then
        //Reduce the compound lottery
        foreach Y ∈ Succ(N) do
            L_Y ← DynProgPost(Y, δ);
            for i ∈ {1,...,n} do
                L[u_i] ← max(L[u_i], (min(π_N(Y), L_Y[u_i])));
    // Decision nodes
    if N ∈ D then
        // Choice of the best decision
        Y* ← Succ(N).first ;
        foreach Y ∈ Succ(N) do
            L_Y ← DynProgPost(Y, δ);
            if L_Y ≻_o L_Y* then
                Y* ← Y ;
        L ← L_Y* ;
        δ(N) ← Y* ;
    return δ;
```

Algorithm 3: DynProgAnte: An adaptation of Dynamic Programming algorithm for the optimization of "*ex-ante*" decision rules

```
Data: a node N
Result: An optimal strategy δ and its value u*
begin
    // Chance nodes
    if X ∈ C then
        // (Optimistic / Pessimistic) utility
        foreach Y ∈ Succ(N) do
            u_Y ← DynProgAnte(Y) ;
            for i ∈ {1, ..., p} do
                u'_N[i] = (u_i ⊗ λ_y) ;
                u_N[i] = (u_N[i] ⊕ u'_N[i]);
        // where ⊗ = min (resp. max), λ_Y = π(Y) (resp. 1 - π(Y)) and ⊕ = max (resp.
        min) in the case of optimistic (resp. pessimistic) utility
    // Decision nodes
    if X ∈ D then
        // Choice of the best decision
        Y* ← Succ(N).first ;
        foreach Y ∈ Succ(N) do
            for i ∈ {1, ..., p} do
                u_Y[i] ← (u_Y[i] ⊗ ω_i) ;
                u_Y ← (u_N ⊕ u'_N[i]) ;
        // where ⊗ = min (resp. max), ω_i = w_i (resp. 1 - w_i) and ⊕ = max (resp. min) in
        the case of disjunctive (resp. conjunctive) aggregation
        if u_Y ≻_o u_Y* then
            Y* ← Y ;
        u* ← u_Y* ;
        δ(N) ← Y* ;
    return u*;
```

6.4 Multi Dynamic Programming algorithm for the optimization of U_{ante}^{-max}

The lack of monotonicity is not as dramatic as it may seem, at least for $U_{ante}^{-\max}$. In this case indeed, we are looking for a strategy that is good w.r.t. U^- for at least one agent. This means that if it is possible to get for each agent i a strategy that optimizes U^- according to this agent (and this can be done by Dynamic Programming, since U^- does satisfy the principle of monotonicity), the one with the highest value for $U_{ante}^{-\max}(L)$ is globally optimal. Formally:

Proposition 6.1. $U_{ante}^{-\max}(L) = \max\limits_{i=1,p} \min(w_i, U_i^-(L))$
where $U_i^-(L) = \min_{x_j \in X} \max(1 - L[x_j], u_i(x_j))$ *is the pessimistic utility of* L *according to the sole agent* i.

It then follows that:

Proposition 6.2. *Let* $\mathcal{L}$ *be the set of lotteries that can be built on* X *and let:*

- $\Delta^* = \{L_1^*, \dots, L_p^*\}$ *s.t.* $\forall L \in \mathcal{L}$, $U_i^-(L_i^*) \geq U_i^-(L)$*;*
- $L^* \in \Delta^*$ *s.t.* $\forall L_j^* \in \Delta^*$*:*
  $\max\limits_{i=1,p} \min(w_i, U_i^-(L^*)) \geq \max\limits_{i=1,p} \min(w_i, U_i^-(L_j^*))$.

It holds that, for any $L \in \mathcal{L}, U_{ante}^{-\max}(L^*) \geq U_{ante}^{-\max}(L)$.

Hence, it is enough to optimize w.r.t. each agent separately: First, we keep each of the p strategies returned by the calls (**DynProg(T, i)**) to a classical Dynamic Programming algorithm (Algorithm (Line 2)); presented in Section 4.3. Then, we to select among them the strategy with the higher $U_{ante}^{-\max}$ value. This is the principle of the Multi Dynamic Programming approach detailed by Algorithm 4.

6.5 Exact optimization of U_{ante}^{+min}: a Branch and Bound algorithm

Let us finally study the case of $U_{ante}^{+\min}$. Since the preference order defined by this rule does not satisfy the principle of monotonicity, the use of the DynProgAnte algorithm (Algorithm 3) may be suboptimal . In this case like in the previous one, Dynamic Programming may provide a good strategy, but without any guarantee of optimality.

Algorithm 4: MultiDynProg: An adaptation of Dynamic Programming algorithm for the optimization of $U_{ante}^{-\max}$

```
Data: A tree T
Result: An optimal strategy δ* and its value u*
begin
    // Initialization
    u* = 0;
    for i ∈ {1,...,p} do
        // Leaves
        foreach N ∈ LN do
            u_N ← u⃗_N[i];
        δ_i = DynProg(T, i)^1;
        u_{δ_i} = Upes(δ_i);
        u*_i = max_{i=1...p} min(w_i, u*_i);
        if u*_i > u* then
            δ* ← δ_i;
            u* ← u*_i;
    return δ*;
```

As an alternative, we have chosen to proceed by an implicit enumeration via a Branch and Bound algorithm, following the approach proposed by Ben Amor et al. [9] for Possibilistic Choquet integrals and by Gildas and Spanjaard [54] for Rank Dependent Utility (both in the mono agent case). The Branch and Bound procedure (see Algorithm 5) takes as argument a partial strategy δ and an upper bound of the $U_{ante}^{+\min}$ value of its best extension. It returns the $U_{ante}^{+\min}$ value of the best strategy found so far, δ^{opt}. As initial value for δ we retain the empty strategy ($\delta(D_i) = \bot, \forall D_i$). We initialize δ^{opt} with the strategy provided by the Dynamic Programming: indeed, even not necessarily providing an optimal strategy, this algorithm generally provides a good one. At each step of the Branch and Bound algorithm, the current partial strategy, δ, is developed by the choice of an action for some unassigned decision node. When several decision nodes are candidate, the one with the minimal rank (i.e. the former one according to the temporal order) is developed. The recursive procedure backtracks when either the current strategy is complete (then $\delta^{opt}$and $U_{ante}^{+minopt}$ may be updated) or proves to be worse than the current δ^{opt} in any case.

In order to get an upper bound of the $U_{ante}^{+\min}$ utility of the best completion of δ, we call a function *UpperBound*(D_0, δ) that computes the optimistic U_i^+ utility w.r.t. each agent i separately and builds an optimal strategy δ_i relative to each i. It then returns the p strategies obtained using classical optimization of U_i^+ by Dynamic Programming, introduces the weights (computes $\min_{i \in A} \max((1 - w_i), U_i^+)$) and then selects, among them, the one with the highest $U_{ante}^{+\min}$.

Whenever the value $U^{+\min}$ returned by *UpperBound*(D_0, δ) is lower or equal to the $U^{+\min}$

value of the current strategy δ the algorithm backtracks, yielding the choice of another action for the last considered decision nodes. When δ is complete, $UpperBound(D_0, \delta) = U^{+\min}_{ante}(\delta)$.

Algorithm 5: BB: A Branch and Bound algorithm for the optimization of $U^{+\min}_{ante}$

Data: A (possibly partial) strategy δ, the evaluation of its utility ($U^{+\min}$)
Result: $U^{+minopt}$: the value of δ^{opt} i.e. the best strategy found so far
begin
  if $\delta = \emptyset$ **then** $D_{pend} \leftarrow \{D_0\}$;
  else
    $D_{pend} \leftarrow \{D_i \in D$ s.t. $\delta(D_i) = \perp$ and $\exists D_j, \delta(D_j) \neq \perp$ and $D_i \in Succ(\delta(D_j))\}$;
  // Is δ a complete strategy ?
  if $D_{pend} = \emptyset$ **then**
    if $U^{+\min} > U^{+-opt}$ **then**
      $\delta^{opt} \leftarrow \delta$;
      $U^{+minopt} \leftarrow U^{+\min}$;
  else
    $D_{next} \leftarrow arg\ min_{D_i \in D_{pend}}\ i$;
    foreach $C_i \in Succ(D_{next})$ **do**
      $\delta(D_{next}) \leftarrow C_i$;
      $U^{+\min} \leftarrow UpperBound(D_0, \delta)$;
      if $U^{+\min} > U^{+minopt}$ **then** $U^{+minopt} \leftarrow BB(U^{+\min}, \delta)$;
  return $U^{+minopt}$;

6.6 Conclusion

This Chapter focuses on the optimization of qualitative decision rules for collective decision making in context of possibilistic uncertainty. Since all the *ex-post* decision rules, as well as U^{+max}_{ante} and U^{-min}_{ante} satisfy the weak monotonicity property, we have proposed to use Dynamic Programming (Algorithm 2 for *ex-post* decision rules and Algorithm 3 for U^{+max}_{ante} and U^{-min}_{ante}). For the remaining utilities, the optimization was carried out using exact algorithms: Multi-Dynamic Programming for U^{+max}_{ante} (Algorithm 4) and Branch and Bound for U^{-min}_{ante} (Algorithm 5). The quality of solutions provided by these algorithms is studied in the next Chapter.

CHAPITRE 7

Experimental study

7.1 Introduction

This chapter is dedicated to the experimental study of the algorithms presented in Chapter 6. It is organized as follows: Section 7.2 explains the experimental protocol and specifies the structure of the implemented problems, then Section 7.3 provides an empirical comparison of different algorithms in term of CPU time and in term of “quality” of the solutions.

7.2 Experimental protocol

In order to explore the feasibility of our algorithms (proposed in the previous chapter), we have implemented them using Java; the computational experiments described in this chapter were carried out on a dedicated PC with an Intel(R) Core(TM)$i7$-2670 QMCPU, $2.2Ghz$ processor, 64 bits architecture, $6Gb$ RAM memory and under Windows 7 operating system. The aim of the experiments is twofold: First of all, we would like to explore the “quality” of solutions, for the problematic cases of $U_{ante}^{-\max}$ and $U_{ante}^{+\min}$, where Dynamical Programming can provide a suboptimal strategy and compare the performances of respectively the Multi-Dynamic Programming and the Branch and Bound approaches with those of the sole Dynamic Programming procedure. Secondly, we would like to compare the accuracy of results provided by the *ex-post* approach as an approximation of the *ex-ante* approach.

The experiments were performed on complete binary decision trees with different heights. We have considered four sets of problems, the number of decisions to be made in sequence (denoted seq) varying from 2 to 6, with an alternation of decision and chance nodes: at each decision level l (i.e. odd levels), the tree contains 2^{l-1} decision nodes followed by 2^l chance nodes. Thus, the number of chance nodes in each tree is equal to $|C| = |D| * 2$ and the number of leaves is equal to $|LN| = |D| + |C| + 1$. This means that for the set of problems with a sequence length $seq = 2$ (resp. 3, 4, 5, 6), the number of decision nodes is equal to $|D| = 5$ (resp. 21, 85, 341, 1365) and the number of nodes in the tree equal to $|N| = |D| + |C| + |LN| = 31$ (resp. 127, 511, 2047, 8191), see Figure 7.1 for an example of a structure of a generated decision tree with a sequence length $seq = 2$.

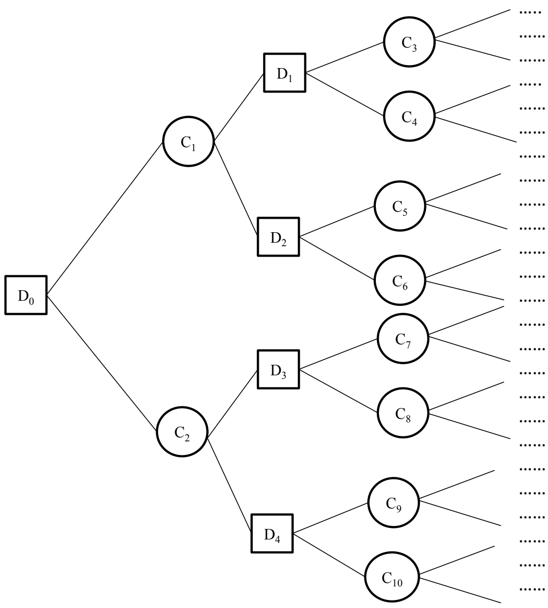

Figure 7.1: Structure of the generated decision trees for $seq = 2$.

The utility values as well as the weights degrees are randomly fired in the set $U = W = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ following an equiprobable distribution. Conditional possibilities are chosen randomly in $[0, 1]$ and normalized. Each of the four samples of problems contains 1000 randomly generated trees.

7.3 Results and interpretation

Feasibility analysis and temporal performances

Tables 7.1 and 7.2 present the execution CPU time of the proposed algorithms for the optimization of respectively the *ex-post* (Algorithm 2) and the *ex-ante* criteria (Algorithms 3, 4 and 5) in the case of 3 agents. Obviously, for any algorithm, the execution CPU time increases according to the size of the tree. But the times remains affordable in average even for big trees (341 and 1365 decision nodes). Unsurprisingly, we can check that the approximation performed using Dynamic Programming for $U_{ante}^{-\max}$ (resp. $U_{ante}^{+\min}$) are faster than the Multi-Dynamic Programming and the Branch and Bound algorithm, respectively, which are exact algorithms. Unsurprisingly also, the difference in CPU time between Branch and Bound and the approximation using Dynamic Programming is more important than the difference between Multi-Dynamic Programming and its approximation by Dynamic Programming.

	$U^{-\min}$	$U^{+\max}$	$U^{-\max}_{post}$	$U^{+\min}_{post}$
Decision nodes number\Algorithm	DynProg	DynProg	DynProg	DynProg
5	0.068	0.071	0.068	0.067
21	0.073	0.075	0.083	0.075
85	0.076	0.082	0.090	0.082
341	0.126	0.128	0.140	0.132
1365	0.215	0.207	0.235	0.211

Table 7.1: Average CPU time (in milliseconds) for *ex-post* utilities

	$U^{-\max}_{ante}$		$U^{+\min}_{ante}$	
Decision nodes number\Algorithm	DynProg	MultiDynProg	DynProg	BB
5	0.079	0.172	0.074	0.576
21	0.096	0.203	0.084	1.012
85	0.120	0.247	0.097	1.252
341	0.147	0.295	0.139	1.900
1365	0.254	1.068	0.231	5.054

Table 7.2: Average CPU time (in milliseconds) for *ex-ante* utilities

Furthermore, to study the effects of varying the number of agents, we consider the optimization of $U^{-\max}_{ante}$, for reasonable trees (85 decision nodes) with p agents from 4 to 7, using the more time-consuming algorithm (Branch and Bound). Clearly, as shown in Table 7.3, the average CPU time with 4 agents, is about 2 milliseconds and the maximal CPU time for decision trees with 7 agent is less than 4 milliseconds. Thus, we can say that the results are good enough to allow the handling of real-size problems.

	Number of agents			
	4	5	6	7
$U^{-\max}_{ante}$	2.190	3.237	3.782	3.863

Table 7.3: Average CPU time (in milliseconds) for $U^{-\max}_{ante}$ using Branch and Bound algorithms for trees with 85 decision nodes

Quality of the approximations of U^{-max}_{ante} and U^{+min}_{ante}

As mentioned above, the monotonicity principle does not apply for $U^{-\max}_{ante}$ and $U^{+\min}_{ante}$ and Dynamic Programming (Algorithm 3) can lead to suboptimal strategies. In order to get an optimal strategy for sure, we must use a complete algorithm, i.e. the Multi-Dynamic Programming (for $U_{ante}^{-\max}$) and Branch and Bound (for $U_{ante}^{+\min}$) which may be much slower.

On the other hand, Dynamic Programming may find good strategies, and even the optimal one. That is why we propose to estimate the quality of the approximations by comparing them to the values generated by the multi-Dynamic Programming (for $U_{ante}^{-\max}$) and Branch and Bound (for $U_{ante}^{+\min}$).

The comparisons are made as follows: we compute for different trees the number of cases for which the value provided by the approximation (Dynamical Programming) is the optimal one, and for the problems on which it fails, we report the closeness value equal to $\frac{V_{Approx}}{V_{Exact}}$ where V_{Approx} is relative to the strategy provided by Dynamic Programming (Algorithm 3) and V_{Exact} is the value provided by the exact Algorithms: Multi Dynamic Programming or Branch and Bound.

The results are presented in Tables 7.4 and 7.5: it appears that *ex-ante* Dynamic Programming provides a very good approximation for the $U_{ante}^{+\min}$ case: it provides the same optimal utility as the Branch and Bound Algorithm in about 70% of cases, with an average closeness value of 94%. This is not the case for the approximation of $U_{ante}^{-\max}$, where in most of the time, Dynamic Programming fails to provide the optimal strategy. However, this is not so bad news since Multi-Dynamic Programming is a polynomial algorithm and its experimental CPU time is very affordable.

	Number of Decision Nodes				
	5	21	85	341	1365
$U_{ante}^{-\max}$	17.3%	19%	22.1%	26.4%	31%
$U_{ante}^{+\min}$	87%	76.8%	68%	62.6%	59.6%

Table 7.4: Percentage of problems for which the Dynamical Programming value is correct

	Number of Decision Nodes				
	5	21	85	341	1365
$U_{ante}^{-\max}$	0.522	0.56	0.614	0.962	0.981
$U_{ante}^{+\min}$	0.97	0.95	0.94	0.93	0.91

Table 7.5: Average closeness value $\frac{U_{ante}(\delta_{approx})}{U_{ante}(\delta_{optimal})}$ of the strategy provided by *ex-ante* Dynamical Programming

Finally, we complete the picture by a comparison of the *ex-post* and *ex-ante* utilities; recall that they are strongly correlated: it holds that $U_{post}^{-\max}(L) \geq U_{ante}^{-\max}(L)$ and $U_{ante}^{+\min}(L) \geq U_{post}^{+\min}(L)$. It appears that in about 90% of the cases, $U_{post}^{+\min}(L) = U_{ante}^{+\min}(L)$ but $U_{post}^{-\max}(L) = U_{ante}^{-\max}(L)$ only in 15% of the cases. We have then repeated the experiment on decision tree (Tables 7.6 and 7.7).

It appears that in 90% of the cases, the optimal value of $U_{post}^{+\min}$ is also optimal for $U_{ante}^{+\min}$. Moreover, even when the *ex-post* approach fails to get a strategy optimal for the *ex-ante* one, the quality of the approximation is good, as shown in Table 7.7) . We can conclude that for $U^{+\min}$ the *ex-post* approach is to be a good approximation of the *ex-ante* one. Since *ex-post* Dynamical Programming is faster than the Branch and Bound, we can think about using it as an approximate algorithm for getting good *ex-ante* strategies. Unfortunately this is not the case for $U^{-\max}$. For this rule, the optimal value of $U^{-\max}_{post}$ is equal to $U^{-\max}_{ante}$ in only 20% of the cases, and the quality of the approximation is low.

	Number of Decision Nodes				
	5	21	85	341	1365
$U^{-\max}_{ante}$	15.4%	23.6%	30.7%	35.6%	40.4%
$U^{+\min}_{ante}$	91.7%	90.8%	88.2%	86.7%	76%

Table 7.6: Percentage of problems for which the *ex-post* optimal strategy is also *ex-ante* optimal

	Number of Decision Nodes				
	5	21	85	341	1365
$U^{-\max}_{ante}$	0.473	0.529	0.556	0.58	0.52
$U^{+\min}_{ante}$	0.989	0.975	0.946	0.928	0.98

Table 7.7: Average closeness value $\frac{U_{ante}(\delta)}{U_{ante}*}$ of the U_{ante} value provided by the best *ex-post* approximation compared to the value of the globally best *ex-ante* strategy

7.4 Conclusion

In this chapter, we have performed experiments on different decision trees built randomly in order to study the quality of solutions provided by the different algorithms provided in Chapter 6.

We have compared these algorithms w.r.t their execution CPU time, and then we have studied the percentage of accuracy between Dynamic Programming and the pertinent exact algorithms (Multi-dynamic programming and Branch and Bound), on one hand, and percentage of accuracy of the *ex-post* approach as an approximation of the *ex-ante* one, on the other hand.

For U^{+min}_{ante}, both approximations are good, and it should be possible to avoid a full Branch and Bound based enumeration. They are not so good for U^{+max}_{ante}, but this has a little impact, since we get a polynomial (and efficient in practice) algorithm for this rule.

CHAPITRE 8

Towards collective decision rules for agents with subjective knowledge

8.1 Introduction

In Chapter 5 we have proposed an axiomatic justification of the fully max-oriented and min-oriented utilities (U^{+max} and U^{-min}) in the style of Von Neuman and Morgenstern. In the present chapter, we are interested in a Savage like formalization.

In Section 8.2, we generalize Savage's framework to collective decision making by considering a new dimension relative to agents in addition to the two standard dimensions i.e. events and consequences. Then in Section 8.3, we focus our attention on the particular case of collective decision making under uncertainty by proposing axioms specific to agents and more precisely to their knowledge and their importance. Finally, we show in Section 8.4 that the qualitative collective decision rules U^{-min} and U^{+max} obey these axioms.

8.2 Extending Savage's framework for multi-agent decision making under uncertainty

In Savage's original setting [82], an act is a function that maps any state to a consequence. In a multi-agent setting, each agent gets a consequence. Formally, let us use the following notations:

- X is the set of consequences,
- S is a set of states,

- A is a set of p agents,
- An act is a function $f : S \times A \mapsto X$: $f(s,i)$ denotes the utility of $f(s)$ for agent i. We can also consider that f is a function from S to X^p and denote $f(s) \in X^p$ the vector of elements of X associated by f to s. $\mathcal{F}$ is the set of such functions.
- $\succeq$ a preference relation on $\mathcal{F}$; $\succ$ denotes its asymmetric part, $\sim$ its symmetric part.

First of all, we recall the following basic axioms of Savage that apply to our context without any change:

Axiom. *(Sav1: Weak order)* $\succeq$ *is complete and transitive.*

Axiom. *(Sav5: Non triviality)* $\exists\, f, g \in \mathcal{F}$ *such that* $f \succ g$.

The remaining axioms and definitions proposed by Savage need to be generalized.

Compound acts and the Sure Thing Principle

In Savage's framework, for any event $E \subseteq S$ and any pair of acts f, g, the *Compound act* fEg is the act that provides the same consequence as f for any state $s \in E$ and the same consequence than g for any state $s \notin E$. In a multi-agent framework, the composition of acts can be generalized as follows:

Definition 8.1 (Compound act)**.**

Given two acts f and g and any $P \subseteq S \times A$, the compound act fPg is defined by:

$$fPg(s,i) = \begin{cases} f(s,i) & \textit{if } (s,i) \in P, \\ g(s,i) & \textit{otherwise.} \end{cases}$$

Given two acts f and g and an agent $i \in A$, the $f\{i\}g$ denotes the compound act $f\{i\} \times Sg$ that provides the same consequences as f for agent i and the same consequences as g for the other agents:

$$f\{i\}g(s,j) = \begin{cases} f(s,j) & \textit{if } j = i, \\ g(s,j) & \textit{otherwise.} \end{cases}$$

Given two acts f and g and any event $E \subseteq S$, the compound act fEg denotes the compound act $fE \times Ag$ that provides the same consequences as f for the states in E and the same consequences as g for the other states:

$$fEg(s,i) = \begin{cases} f(s,i) & \textit{if } s \in E, \\ g(s,i) & \textit{otherwise .} \end{cases}$$

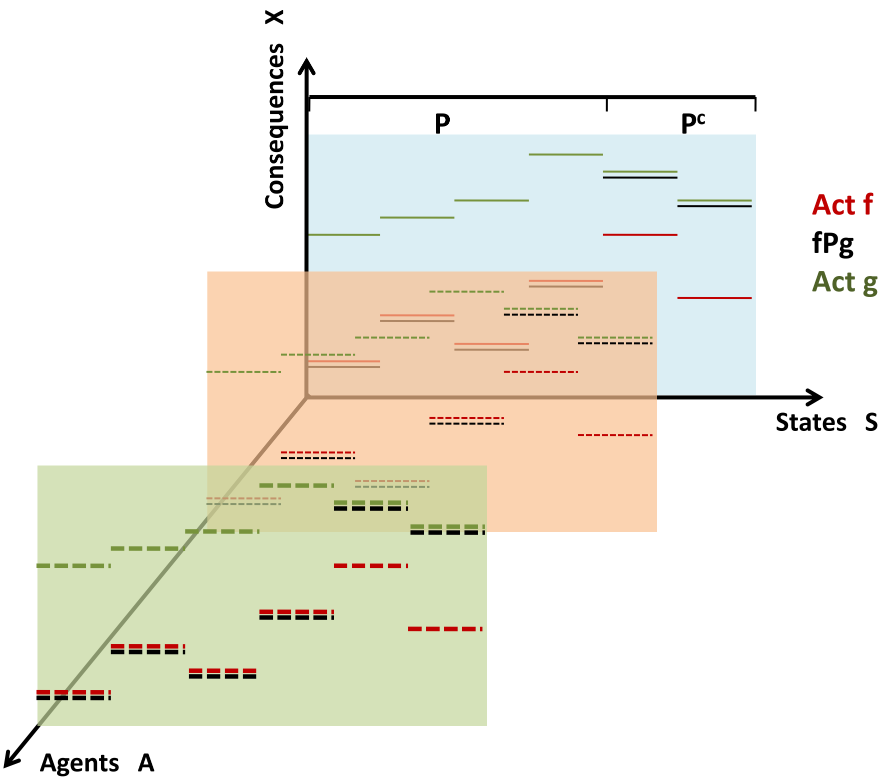

Figure 8.1: Multi-agent compound act

Figure 8.1 illustrates an example of a compound act for three agents.

This generalization of the notion of compound acts allows us to straightforwardly extend Savage's Sure Thing Principle to multi-agent decision making under uncertainty as follows:

Axiom (Sav2: "Sure Thing Principle"). *For any acts* $f, g, h, h' \in \mathcal{F}$, *for any* $P \subseteq S \times A$, $fPh \succeq gPh$ *iff* $fPh' \succeq gPh'$.

Grant et al. [56, 57] proposed a weak version of the Sure Thing Principle, which is mainly a principle of weak decomposability. Another weak form of the Sure Thing Principle has been defined by Dubois et al. [32, 34] as a weak independence axiom. They can be written as follows:

Axiom (wSTP: "Weak Decomposability"). *For any acts* $f, g \in \mathcal{F}$, *for any* $P \subseteq S \times A$, $fPg \succ g$ *and* $gPf \succ g \implies f \succ g$.

Axiom (wSav2: "Weak Independence"). *For any acts* $f, g, h, h' \in \mathcal{F}$, *for any* $P \subseteq S \times A$, $fPh \succ gPh \implies fPh' \succeq gPh'$.

The classical axioms are recovered when A contains only one agent.

After generalizing compound acts, we can now consider the following generalization of the conditioning by restricting $\succeq$ to a subset P of $A \times S$:

Definition 8.2. *(Conditioning on P)* $\forall P \subseteq A \times S$, $f \succeq_P g$ *iff* $\forall h, fPh \succeq gPh$.

Like in Savage's work, it follows from this definition that $\succeq_P$ is a weak order:

Proposition 8.1. *Consider any* $P \subseteq A \times S$*; we have:*

- $f \sim_P g$ *iff* $\forall h \in \mathcal{F}, fPh \sim gPh$.
- *if* $Sav1$ *holds, then* $f \succ_P g$ *iff* $\forall h \in \mathcal{F}, fPh \succeq gPh$ *and* $\exists\, h, fPh \succ gPh$.
- $\succeq_P$ *is transitive.*
- *if* $Sav1$ *and* $wSav2$ *holds, then* $\succeq_P$ *is complete.*

Proof of Proposition 8.1.

- $f \sim_P g$ writes $\forall h \in \mathcal{F}, fPh \succeq gPh$ and $\forall h \in \mathcal{F}, fPh \preceq gPh$ which is equivalent to $\forall h \in \mathcal{F}, fPh \sim gPh$
- $f \succ_P g$ writes $\forall h \in \mathcal{F}, fPh \succeq gPh$ and $\exists h \in \mathcal{F}, (gPh \not\succeq fPh)$; Since $\succeq$ is complete, we get $f \succ_P g$ iff $\forall h \in \mathcal{F}, fPh \succeq gPh$ and $\exists h \in \mathcal{F}, fPh \succ gPh$.
- The transitivity of $\succeq_P$ is obvious: $\forall h \in \mathcal{F}, fPh \succeq gPh$ and $\forall h \in \mathcal{F}, gPh \succeq kPh$ implies $\forall h \in \mathcal{F}, fPh \succeq kPh$.
- Consider any acts f,$g \in \mathcal{F}$. Three cases are considered (they are exclusive, and exhaustive thanks to $Sav1$):
  - If $fPh \succ gPh$ then by $wSav2$ we have $\forall h', fPh' \succeq gPh'$, i.e. $f \succeq_P g$;
  - If $gPh \succ fPh$ then by $wSav2$ we have $\forall h', gPh' \succeq fPh'$, i.e. $g \succeq_P f$;
  - If $\forall h \in \mathcal{F}, fPh \sim gPh$ we get $f \sim_P g$. Hence the completeness of $\succeq_P$.

□

We define the restriction of $\succeq$ to a subset of I agents, its restriction to a single agent i and its restriction to an event E in a similar way:

Definition 8.3.

- $\forall E \subseteq S, f \succeq_E g$ *iff* $f \succeq_{A \times E} g$.

- $\forall I \subseteq A, f \succeq_I g$ *iff* $f \succeq_{I \times S} g$.
- $\forall i \in A, f \succeq_i g$ *iff* $f \succeq_{\{i\}} g$.

That is to say: $f \succeq_E g$ iff for any h, $fEh \succeq gEh$, $f \succeq_I g$ iff for any h, $fIh \succeq gIh$ and $f \succeq_i g$ iff for any h, $f\{i\}h \succeq g\{i\}h$. As a corollary of the previous proposition, all these relations are complete and transitive.

In Savage's work, an event is null whenever it does not make any difference between the acts; the same definition can be used to characterize null agents:

Definition 8.4. *(Null agents, null events)*

- *A set of agents* $I \subseteq A$ *is null iff* $\forall f, g, h \in \mathcal{F}$, $fIh \sim gIh$.
- *An agent* $i \in A$ *is null whenever* $\{i\}$ *is null.*
- *An event* $E \subseteq S$ *is null iff* $\forall f, g, h \in \mathcal{F}$, $fEh \sim gEh$.

We will see in Section 8.2 how to derive the knowledge of the agents from the global preference relation $\succeq$. But to this extent, we need to develop the notion of constant act, which will allow us to derive from $\succeq$ the individual preferences on X.

Constant acts, individual and global preferences of X

Up to this point, the generalization of compound acts and conditioning notions was straightforward; it is slightly more complex for constant acts. Recall that constant acts in the sense of Savage are acts that are constant over the set of states. In our context, we get three notions: acts that gives the same vector of consequences for any state (constant acts), acts that do not make any difference between the agents i.e. in a given state, all receive the same consequence (we call then common acts) and acts that give the same consequence for all agents and all states (we call them representative acts).

Definition 8.5.

- *A* **constant act** f *is an act that maps the same vector of utilities to any state:* $\forall s, s' \in S$, $\forall i \in A$, $f(s,i) = f(s',i)$.
- *A* **common act** f *is an act such that whatever* f, *state* s *maps the same utility to any agent:* $\forall s \in S, \forall i, i' \in A$, $f(s,i) = f(s,i')$.
- $\forall x \in X$, *the* **representative act** *of* x *(denoted by [x]) is the constant (and common) act that provides* x *to any state and any agent:*
  $\forall s \in S$, $\forall i \in A$, $f(s,i) = x$.

A *constant act* maps the same vector to any state: what an agent gets does not depend on the state of the world. The set of constant acts is ordered by $\succeq$. By identification of X^p with the set of constant acts, any $\vec{x} \in X^p$, $\vec{x}$ also denotes the corresponding constant act. For any $\vec{x}, \vec{y} \in X^p$ we write $\vec{x} \succeq \vec{y}$ whenever the constant act providing $\vec{x}$ for any s is at least as good (according to $\succeq$) as the one providing $\vec{y}$ for any s: $\succeq$ is the common preference order on X^p.

A *representative act* $[x]$ is a constant act that maps the same consequence, x, to any state and any agent: everybody receives x, and this for any state. X is ordered by $\succeq$ (which is a weak order): we say that x is as least as good as y (and we denote $x \succeq y$) whenever the representative act $[x]$ is preferred to $[y]$.

In this work, $(X, \succeq)$ is interpreted as a scale of satisfaction common to all the agents. To this extend, we shall propose the following axiom:

Axiom (CS: Common Scale)**.** *For any non null agent* $i \in A$*,*
$[x] \succeq [y]$ *iff* $[x] \succeq_i [y]$.

For some decision rules, it may happen that an agent i is not important enough: his preference is drowned when x and y are not high in the scale, and thus not revealed by $\succeq_i$. In this case, the following weak version of CS may be preferred:

Axiom (wCS: Weak Common Scale)**.** *For any agent* $i \in A$*,*
$[x] \succeq [y]$ *implies* $[x] \succeq_i [y]$.

Let us now go back to Savage's axioms. The third axiom of Savage states that the preference on X is not sensitive to the conditioning on non null events; we write it here in its classical version and in its weak variant:

Axiom (Sav3)**.** *For any* $x, y \in X$*, for any non null event* $E \subseteq S$*,*
$[x] \succeq [y]$ *iff* $[x] \succeq_E [y]$.

Axiom (wSav3)**.** *For any* $x, y \in X$*, for any event* $E \subseteq S$*,*
$[x] \succeq [y] \implies [x] \succeq_E [y]$.

Axioms $Sav3$/$wSav3$ are very close to axioms CS/wCS; it can be possible to see them as two particular cases of a more general axiom ensuring the coherence on the conditioning of the preference over X on any $P \subseteq A \times S$ (and not only the subsets of the form $\{i\} \times S$ and the events $A \times E$).

Axiom (Cond)**.** *For any* $x, y \in X$*, for any non null* $P \subseteq A \times S$*,*
$[x] \succeq [y]$ *iff* $[x] \succeq_P [y]$.

Axiom (wCond)**.** *For any* $x, y \in X$*,for any* $P \subseteq A \times S$*,*
$[x] \succeq [y] \implies [x] \succeq_P [y]$.

Proposition 8.2.
If $Sav1$ and $wSav2$ hold then for any disjoint subsets P, P' of $A \times S$, if $f \succeq_P g$ and $f \succeq_{P'} g$ then $f \succeq_{P \cup P'} g$.

Proof of Proposition 8.2. Suppose that $f \succeq_P g$ and $f \succeq_{P'} g$, i.e. that $\forall h, fPh \succeq gPh$ and $\forall h, fP'h \succeq gP'h$. Then, thanks to $wSav2$ for any h we have:
$fPfP' \succeq gPfP'h$ and $\forall h, gPfP'h \succeq gPgP'h$; by transitivity we get $\forall h, fPfP'h \succeq gPgP'h$, i.e. $f \succeq_{P \cup P'} g$. □

Pareto Unanimity axiom; that is essential for collective decision making; is defined as follows:

Axiom (Pareto Unanimity)**.** *If $\forall i, s, f(s,i) \succeq g(s,i)$, then $f \succeq g$.*

It holds that:

Proposition 8.3. *If $Sav1$, $wSav2$ and $wSav3$ then Pareto Unanimity holds.*

Proof of Proposition 8.3. Suppose that $\forall(s,i), f(s,i) \succeq g(s,i)$; then by $wSav3$ we have:
$\forall(s,i), [f(s,i)] \succeq_{\{(s,i)\}} [g(s,i)]$; the repeated application of Proposition 8.2 leads to $f \succeq g$. □

Conversely, it is possible to show that $wCond$ (and thus $wSav3$ and wCS) is a consequence of Pareto.

Proposition 8.4. *If Pareto Unanimity is satisfied then $wCond$ holds.*

Proof of Proposition 8.4. Consider any $P \subseteq A \times S$ and let x and y be two consequences such that $[x] \succeq [y]$ (and thus $x \succeq y$) and consider the compound acts $[x]Ph$ and $[y]Ph$; it holds that $\forall i \in A$:

- if $(s,i) \in P$, then $[x]Ph(s,i) = x \succeq [y]Ph(s,i) = y$,
- if $(s,i) \notin P$, then $[x]Ph(s,i) = [y]Ph(s,i) = h(s,i)$.

Thus by Pareto Unanimity we have: $[x]Ph \succeq [y]Ph$. We did not put any restriction on h. Hence $\forall h$, $[x]Ph \succeq [y]Ph$, i.e. $[x] \succeq_P [y]$, which proves that $wCond$ is a consequence of Pareto Unanimity. □

Corollary 8.1. *If Pareto Unanimity is satisfied, then wCS holds.*

In other terms, Pareto unanimity also guarantees that construction is well founded, since it allows us to get back the weak form of Axiom CS (Coherence of the scale). Another consequence of Pareto unanimity is that the scale contains an ideal and anti ideal consequence, which we will denote $\top$ and $\bot$.

Proposition 8.5. *If Pareto Unanimity holds, then there exists* $\top$ *and* $\bot \in X$ *such as, whatever* $i \in A$*,* $h \in \mathcal{F}$*,*

- $[\top] \succeq f \succeq [\bot]$.
- $[\top] \succeq_i f \succeq_i [\bot]$.

Proof of Proposition 8.5. Consider two consequences $\top$ and $\bot \in X$ the best and worst consequences (Sav1 allows them to exist: the restriction of $\succeq$ on representative acts is a weak order).

- For any $f \in \mathcal{F}$, $s \in S$, $i \in A$, it holds that $[\top](s,i) = \top \succeq f(s,i)$; then Pareto implies that $[\top] \succeq f$. In a similar way, it holds that $f(s,i) \succeq [\bot](s,i) = \bot$; then Pareto implies that $f \succeq [\bot]$.
- For any $f, h \in \mathcal{F}$, $s \in S$, $i \in A$, consider the act $[\top]\{i\}h$ and $f\{i\}h$. It holds that $[\top]ih(s,i) = \top \succeq f\{i\}h(s,i) = f(s,i)$ for any s besides $[\top]ih(s,j) = h(s,j) \sim f\{i\}h(s,j)$ for any $s, j \neq i$. Then by Pareto unanimity we get $[\top]\{i\}h \succeq f\{i\}h$. Since we did not put any restriction on h, $[\top]\{i\}h \succeq f\{i\}h$ for any h, i.e. $[\top] \succeq_i f$.
- The proof of $f \succeq_i [\bot]$ is similar to the above proof of $[\top] \succeq_i f$.

□

Because Pareto is a consequence of $Sav1$, $wSav2$ and $wSav3$, we get:

Corollary 8.2. *If* $Sav1$*,* $wSav2$ *and* $wSav3$*, then there exists* $\top$ *and* $\bot \in X$ *such as, whatever* $i \in A$*,* $h \in \mathcal{F}$*,*

- $[\top] \succeq f \succeq [\bot]$.
- $[\top] \succeq_i f \succeq_i [\bot]$.

The proof of Corollary 8.2 is obvious and it is a direct consequence of Propositions 8.3 and 8.5.

Likelihood of the events

Classically, the relative likelihood of the events follows from the observation of binary acts, i.e. acts of the form $[x]E[y]$. This requires the satisfaction of the following axiom, which is a weak form of Savage's fourth axiom.

Axiom (Sav4)**.** *For any consequences $x, y, x', y' \in X$, if $[x] \succ [y]$ and $[x'] \succ [y']$, then $\forall E, D \subseteq S$ we have: $[x]E[y] \succeq [x]D[y] \Leftrightarrow [x']E[y'] \succeq [x']D[y']$.*

Axiom (wSav4)**.** *For any consequences $x, y, x', y' \in X$, if $[x] \succ [y], [x'] \succ [y']$ and $[x]E[y] \succ [x]D[y]$ then $[x']E[y'] \succeq [x']D[y']$.*

Following Savage, we can define the relative likelihood of events as the relation $\succeq$ on 2^S as:

Definition 8.6. *(Likelihood of events relative to collectivity)*
$E \succeq D$ iff $\forall x, y \in X$ s.t. $[x] \succ [y] : [x]E[y] \succeq [x]D[y]$.

We can also define the personal belief of agent i:

Definition 8.7. *(Likelihood of events relative to an agent)*
$E \succeq_i D$ iff $\forall x, y \in X$ s.t. $[x] \succ_i [y], [x]E[y] \succeq_i [x]D[y]$.

From $wSav4$, it follows that $E \succ D$ when $\forall x, y \in X$ s.t. $[x] \succ [y], [x]E[y] \succeq [x]D[y]$ and $\exists x, y \in X$ s.t. $[x] \succ [y], [x]E[y] \succ [x]D[y]$ (and similarly for the $\succeq_i$), and that:

Proposition 8.6. *If $Sav1$ and $wSav4$ are satisfied, then $\succeq$ is a weak order on 2^S.*

Proof of Proposition 8.6.

- Let D and E be any two events. We consider three cases; either there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]E[y] \succ [x]D[y]$, or there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]D[y] \succ [x]E[y]$, or for any $x, y \in X$ s.t. $[x] \succ [y]$, $[x]E[y] \sim [x]D[y]$. Notice that the three cases are exclusive, thanks to $wSav4$.
  - If there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]E[y] \succ [x]D[y]$, then by $wSav4$ $[x']E[y'] \succeq [x']D[y']$ for any x', y' such that $[x'] \succ [y']$ and thus $[x]E[y] \succeq [x]D[y]$.
  - If there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]D[y] \succ [x]E[y]$, we get $[x]D[y] \succeq [x]E[y]$ in the same way.
  - If for any $x, y \in X$ s.t. $[x] \succ [y]$, $[x]E[y] \sim [x]D[y]$ we get $[x]E[y] \succeq [x]D[y]$ and $[x]D[y] \succeq [x]E[y]$, i.e. $[x]E[y] \sim [x]D[y]$.

  $\succeq$ on 2^S is thus complete.

- Consider 3 events C, D, E such that $E \succeq D$ and $D \succeq C$. This writes for any $x, y \in X$ s.t. $[x] \succ [y] : [x]E[y] \succeq [x]D[y]$ and $[x]D[y] \succeq [x]C[y]$; the transitivity of $\succeq$ (Sav1) implies that for any $x, y \in X$ s.t. $[x] \succ [y] : [x]E[y] \succeq [x]C[y]$: we get $E \succeq C$, which proofs the transitivity of $\succeq$ on 2^S.

□

Proposition 8.7. *If $Sav1$, $wSav2$, $wSav3$ and $wSav4$ are satisfied, for any agent i, $\succeq_i$ is a weak order on 2^S.*

Proof of Proposition 8.7.

- Let D and E be any two events. By definition $[x]E[y] \succeq_i [x]D[y]$ iff $\forall h, [x]E \times \{i\}[y]\bar{E} \times \{i\}h \succeq [x]D \times \{i\}[y]\bar{D} \times \{i\}h$, and in particular $[x]E \times \{i\}[y]\bar{E} \times \{i\}[y] \succeq [x]D \times \{i\}[y]\bar{D} \times \{i\}[y]$, i.e. $[x]E \times \{i\}[y] \succeq [x]D \times \{i\}[y]$.
  We consider three cases; either there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]E \times \{i\}[y] \succ [x]D \times \{i\}[y]$, or there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]D \times \{i\}[y] \succ [x]E \times \{i\}[y]$, or for any $x, y \in X$ s.t. $[x]E \times \{i\}[y] \sim [x]D \times \{i\}[y]$. Notice that the three cases are exclusive.
  - If there exist $x, y \in X$ s.t. $[x]E \times \{i\}[y] \succ [x]D \times \{i\}[y]$, then by wSav4 $[x']E \times \{i\}[y'] \succeq [x']D \times \{i\}[y']$ for any x', y' such that $[x'] \succ [y']$; by wSav2 we get $\forall h, [x']E \times \{i\}[y']\bar{E} \times \{i\}h \succeq [x']D \times \{i\}[y']\bar{D} \times \{i\}h$, thus $[x]E[y] \succeq_i [x]D[y]$.
  - If there exist $x, y \in X$ s.t. $[x] \succ [y]$ and $[x]D[y] \succ [x]E[y]$, we get $[x]D[y] \succeq_i [x]E[y]$ in the same way.
  - If for any $x, y \in X$ s.t. $[x] \succ [y]$, $[x]E \times \{i\}[y] \sim [x]D \times \{i\}[y]$, by wSav2, its holds that $\forall h, [x]E \times \{i\}[y]\bar{E} \times \{i\}h \sim [x]D \times \{i\}[y]\bar{D} \times \{i\}h$. Thus, $x, y \in X$ s.t. $[x] \succ [y]$ we have $\forall h, [x]E \times \{i\}[y]\bar{E} \times \{i\}h \sim [x]D \times \{i\}[y]\bar{D} \times \{i\}h$ i.e. $[x]E[y] \sim_i [x]D[y]$.

  $\succeq_i$ on 2^S is thus complete.

- Consider 3 events C, D, E such that $E \succeq_i D$ and $D \succeq_i C$. This writes forall $x, y \in X$ s.t. $[x] \succ [y]$: $\forall h, [x]E \times \{i\}[y]\bar{E} \times \{i\}h \succeq [x]D \times \{i\}[y]\bar{D} \times \{i\}h$ and $\forall h, [x]D \times \{i\}[y]\bar{D} \times \{i\}h \succeq [x]C \times \{i\}[y]\bar{C} \times \{i\}h$. The transitivity of $\succeq$ ($Sav1$) implies that $x, y \in X$ s.t. $[x] \succ [y], \forall h, [x]E \times \{i\}[y]\bar{E} \times \{i\}h \succeq [x]C \times \{i\}[y]\bar{C} \times \{i\}h$, i.e. $\forall x, y \in X$ s.t. $[x] \succ [y][x]E[y] \succeq_i [x]C[y]$: we get $E \succeq C$, which proves the transitivity of $\succeq_i$ on 2^S.

□

8.3 Axioms for qualitative collective decision making under uncertainty

In the previous Section, we have simply generalized Savage's model so as to add a new dimension, the possibility of handling more than one agent. None of these axioms deals specifically with the influence of agents, but the axiom CS (Coherence of the scale) that was presented in the previous Section because strongly related to Pareto Unanimity. For the sake of brevity, we assume from this point, that $Sav1$, $wSav2$, $wSav4$, $Sav5$ are satisfied. Besides, we assume either the Pareto Unanimity axiom, that is a very weak and very natural assumption for collective decision making, or $wSav3$ that is equivalent to Pareto given $Sav1$ and $Sav2$. Let us now study additional axioms that are more related to the presence of several agents.

Additional axioms of collective decision making

The knowledge of an agent, i.e. the relative likelihood that each of them gives to events, may cohere or not. Of course, if $\forall i \in A, E \succeq_i D$, the principle of Pareto Unanimity implies that $E \succeq D$, i.e. if all the agents share the same knowledge then this knowledge is common (for the collectivity). However, the reciprocal is not necessarily true, since at the present point, nothing forbids that the agents disagree with the likelihood of events. To overcome this, we explicitly assume the following axiom of common knowledge:

Axiom ($wCok$: Weak Common Knowledge)**.** *For any events $E, D \subseteq S$, for any agent non null $i \in A$, $E \succeq D$ implies $E \succeq_i D$.*

It may happen that $E \succ D$ while $E \sim_i D$ because agent i is not important enough: her preference is drowned when the likelihood of E and D is not high in the scale, and thus not revealed by $\succeq_i$. When all the agents are decisive and share the same importance, a stronger axiom can be used:

Axiom (Cok: Common Knowledge)**.** *For any events $E, D \subseteq S$, for any agent non null $i \in A$, $E \succeq D$ iff $E \succeq_i D$.*

In some cases, all the agents are equally important; this corresponds to an axiom of anonymity, that guarantees the fairness of the decision: no agent is more important than another and the preference is always stable even by changing the role of any two agents. Formally, we write:

Axiom (Ano: Anonymity)**.** *For any agents $i, j \in A$, for any acts $f, f', g, g' \in \mathcal{F}$ such that $f(s,i) = f'(s,j)$, $f(s,j) = f'(s,i)$, $g(s,i) = g'(s,j)$, $g(s,j) = g'(s,i)$, $f \succeq g$ iff $f' \succeq g'$.*

Finally, we shall consider the following axiom that enables each agent to have some power to make decision when others are indifferent.

Axiom (Decision)**.** *For any agent $i \in A$, $\exists\, f, g, h \in \mathcal{F}$ such that $f_i h \succ g_i h$.*

Axioms of qualitative decision

We finally set Dubois et al's axioms relative to U_{opt} and U_{pes} [24,32,34] to the multi-agent context:

Axiom. *(RDD: Restricted disjunctive dominance) Let f and g be any two acts and $[x]$ be a representative act of value x: $f \succ g$ and $f \succ x \Rightarrow f \succ g \vee [x]$ (where $g \vee [x]$ gives the best of the results of $g(s,i)$ and x in each pair (s,i)).*

Axiom. *(RCD: Restricted conjunctive dominance) Let f and g be any two acts and $[x]$ be a representative act of value x: $g \succ f$ and $x \succ f \Rightarrow g \wedge [x] \succ f$ (where $g \wedge [x]$ gives the worst of the results of $g(s,i)$ and x in each pair (s,i)).*

Axiom. *(NC: Non compensation) Whatever $E \subseteq S$, $x, y \in X$:*

- $[\top]E[y] \sim y$ *or* $[\top]E[y] \sim [\top]E[\bot]$.
- $[x]E[\bot] \sim x$ *or* $[x]E[\bot] \sim [\top]E[\bot]$.

The meaning of the axioms does not change when extended to the multi-agent context: Axiom RDD says that if an act f is preferred to an act g and also preferred to a representative act x, then f still preferred to g even if the worst consequences of g are improved to the value x. Axiom RCD is the dual property of the restricted disjunctive dominance. It allows a partial decomposability of qualitative utility with respect to the conjunction of acts in the case where one of them is constant.

The other axioms can be written without any modification:

Axiom. *(DD: Disjunctive dominance) Let f,g,h be any three acts: $f \succ g$ and $f \succ h$ $\Rightarrow f \succ g \vee h$.*

Axiom. *(RCD: Conjunctive dominance) Let f,g,h be any three acts: $f \succ g$ and $f \succ h$ $\Rightarrow f \succ g \wedge h$.*

Axiom. *(Pes: Pessimism) $\forall f, g \in \mathcal{F}$ and $E \subseteq S$. If $fEg \succ f$ then $f \succeq gEf$.*

Axiom. *(Opt: Optimism) $\forall f, g \in \mathcal{F}$ and $E \subseteq S$. If $f \succ fEg$ then $gEf \succeq f$.*

8.4 Properties of collective qualitative decision rules in style of Savage

We now study the fully min-oriented and max-oriented decision rules in light of the axioms presented above. Hence, we show that these axioms are consistent and obeyed by the pessimistic egalitarian utility ($U^{-\min}$) and by its optimistic counterpart ($U^{+\max}$). We suppose here, without loss of generality, that X is a subset of $[0,1]$ and consider a set of acts $\mathcal{F}$ built from $S \times A$ to $X \subseteq [0,1]$ (with $\{0,1\} \subseteq X$); a possibility distribution of S and a set of weights w_i. $U^{-\min}$ and

$U^{+\max}$ write:

$$\begin{aligned}
U^{-\min}(f) &= \min_{i=1,p} \max(1-w_i, \min_{s\in S} \max(f(s,i), 1-\pi(s))). \\
&= \min_{s\in S} \max(1-\pi(s), \min_{i=1,p} \max(1-w_i, f(s,i))). \\
U^{+\max} &= \max_{i=1,p} \min(w_i, \max_{s\in S} \min(f(s,i), \pi(s))). \\
&= \max_{s\in S} \min(\pi(s), \max_{i=1,p} \min(w_i, f(s,i))).
\end{aligned}$$

It holds that:

Proposition 8.8.
The relation $\succeq$ defined by: $f \succeq g$ iff $U^{-\min}(f) \geq U^{-\min}(g)$, satisfies axioms $Sav1$, $Sav5$, $wSav2$, $wSav3$, $wSav4$, $ParetoUnanimity$, NC, RDD, PES, wCS and $wCoK$.

If we suppose that we are in a pure egalitarian context ($w_i = 1, i = 1, p$), we get:

$$U^{-\min}(f) = \min_{i=1,p} \min_{s\in S} \max(f(s,i), 1-\pi(s))).$$

In this case, the preference relation satisfies Ano and $Decision$. Since the agents' weights do not have a role anymore, CS and Cok are satisfied in their full strength:

Proposition 8.9.
The relation $\succeq$ defined by:

$$f \succeq g \text{ iff } \min_{i=1,p} \min_{s\in S} \max(f(s,i), 1-\pi(s))) \geq \min_{i=1,p} \min_{s\in S} \max(g(s,i), 1-\pi(s))),$$

satisfies axioms $Sav1$, $Sav5$, $wSav2$, $wSav3$, $wSav4$, $ParetoUnanimity$, NC, RDD, PES, CS, Cok, Ano as well as $Decision$.

Proof of Proposition 8.8.

- $\succeq$ trivially satisfies $Sav1$ (each act is given by a global utility on $[0,1]$).
- $\succeq$ satisfies $wSav2$:
  Consider $f, g, h \in \mathcal{F}$, $P \in S \times A$ such that $U^{-\min}(fPh) \succ U^{-\min}(gPh)$ and any fourth act h', Let:
  $a = \min_{s\in P} \max(1-\pi(s), \min_{i\in A} \max(1-w_i, f(s,i)))$,
  $b = \min_{s\in P} \max(1-\pi(s), \min_{i\in A} \max(1-w_i, g(s,i)))$,
  $c = \min_{s\in \bar{P}} \max(1-\pi(s), \min_{i\in A} \max(1-w_i, h(s,i)))$,
  $c' = \min_{s\in \bar{P}} \max(1-\pi(s), \min_{i\in A} \max(1-w_i, h'(s,i)))$.

It holds that: $U^{-\min}(fPh) = \min(a,c)$ $U^{-\min}(gPh) = \min(b,c)$
$U^{-\min}(fPh') = \min(a,c')$ $U^{-\min}(gPh) = \min(b,c')$

$U^{-\min}(fPh) \succ U^{-\min}(gPh)$ iff $\min(a,c) > \min(b,c)$. Then for any d, $\min(a,d) \geq \min(b,d)$; for $d = c'$ we get $\min(a,c') \geq \min(b,c')$, i.e. $U^{-\min}(fPh') \geq U^{-\min}(gPh')$. Hence $U^{-\min}(fPh) \succ U^{-\min}(gPh)$ implies $U^{-\min}(fPh') \succeq U^{-\min}(gPh')$ for any h': $wSav2$ is satisfied.

- Pareto Unanimity follows from the fact that the max and min operators are monotonic ($U^{-\min}$ do not decrease when one of the $f(s,i)$ increases).

- $wSav3$ follows from $Sav1$, $wSav2$ and Pareto Unanimity.

- $\succeq$ satisfies $wSav4$:
  When all the agents receive the same consequence, $U^{-\min}$ writes as a classical, mono agent pessimistic utility. Gambles of the form $[x]E[y]$ do assign the same consequence to all the agents (namely, whatever i, $[x]E[y](s,i) = x$ if $s \in E$ and $[x]E[y](s,i) = y$ is $s \notin E$) - we recover the mono agent case, with $U^{-\min}([x]E[y]) = median(x, N(E), y)$. The satisfaction of $wSav4$ follows from its satisfaction in the mono-agent case.

- $Sav5$ is satisfied by construction.

- $\succeq$ satisfies $wCoK$:
  Consider to events E, D such that $E \succeq D$, i.e. $\forall x, y$ such that $x > y$, $U^{-\min}(xEy) \geq U^{-\min}(xDy)$; as previously noticed, $U^{-\min}([x]E[y]) = median(x, N(E), y)$ and we have $U^{-\min}([x]D[y]) = median(x, N(D), y)$. Thus for any x, y such that $x > y$, we obtain: $median(x, N(E), y) \geq median(x, N(D), y)$.

  Suppose now that there is an agent i such that $D \succ_i E$, which means that

  - $\forall x > y, \forall h, U^{-\min}([x]E \times \{i\}[y]\bar{E} \times \{i\}h) \geq U^{-\min}([x]D \times \{i\}[y]\bar{D} \times \{i\}h)$,
  - $\exists x > y, \forall h, U^{-\min}([x]E \times \{i\}[y]\bar{E} \times \{i\}h) > U^{-\min}([x]D \times \{i\}[y]\bar{D} \times \{i\}h)$,

  The second point implies that $U^{-\min}([x]E \times \{i\}[y]\bar{E} \times \{i\}1) > U^{-\min}([x]D \times \{i\}[y]\bar{D} \times \{i\}1)$. Moreover, we can write: $U^{-\min}([x]E \times \{i\}[y]\bar{E} \times \{i\}1) = \min(\max(1 - w_i, median(x, N(E), y)), \max(1 - min_{j \neq i} w_j, 1)$ and $U^{-\min}([x]D \times \{i\}[y]\bar{D} \times \{i\}1) = \min(\max(1 - w_i, median(x, N(D), y)), \max(1 - min_{j \neq i} w_j, 1)$.
  We thus get $\max(1 - w_i, median(x, N(D), y)) > \max(1 - w_i, median(x, N(E), y))$, which implies $median(x, N(D), y)) > median(x, N(E), y))$.
  This contradicts $median(x, N(E), y) \geq median(x, N(D), y)$. Hence there cannot be a i such that $D \succ_i E$, i.e. $E \succeq_i D$. This shows that $E \succeq D$ implies $E \succeq_i D$: $wCoK$ is satisfied.

- wCS is a consequence of Pareto Unanimity and $Sav1$.

- $\succeq$ satisfies NC because whatever x, y, E, $U^{-\min}([x]E[y]) = median(x, N(E), y)$.

- $\succeq$ satisfies RDD because it satisfies NC, $Sav1$ and $wSav3$.

- $\succeq$ satisfies PES.
  This is due to the fact that $U^{-\min}$ is an ex-post utility; actually, setting $u(\vec{(f(s))}) = \min(\max(1-w_i, f(s,i)))$, we get $U^{-\min}(f) = \min_{s \in S} \max(1-\pi(s), u(\vec{(f(s))}))$: $U^{-\min}$ is a classical pessimistic utility (the sole agent being the collectivity) and thus satisfies PES.

□

Proof of Proposition 8.9. Using Proposition 8.8, we have proved that axioms $Sav1$, $Sav5$, $wSav2$, $wSav3$, $wSav4$, Pareto Unanimity, NC, RDD and PES are satisfied. For axioms CS , Cok, Ano and $Decision$ the proof is performed as follows:

- The relation $\succeq$ satisfies anonymity axiom (Ano):
  Consider acts f, g, f' and $g' \in \mathcal{F}$, and two agents i, $j \in A$ $f(s,i) = f'(s,j)$, $f(s,j) = f'(s,i)$, $g(s,i) = g'(s,j)$, $g(s,j) = g'(s,i)$. We have:
  $$\begin{array}{rcl} U^{-\min}(f) & = & \min \quad \begin{array}{l} \min_{s\in S} \max(f(s,i), 1-\pi(s)), \\ \min_{s\in S} \max(f(s,j), 1-\pi(s)), \\ \min_{k\neq i,j;s\in S} \max(f(s,k)), 1-\pi(s)). \end{array} \end{array}$$

  Besides, we have:
  $$\begin{array}{rcl} U^{-\min}(f') & = & \min \quad \begin{array}{l} \min_{s\in S} \max(f('s,i), 1-\pi(s)), \\ \min_{s\in S} \max(f'(s,j), 1-\pi(s)). \\ \min_{k\neq i,j;s\in S} \max(f'(s,k)), 1-\pi(s)). \end{array} \end{array}$$

  Since $f(s,i) = f'(s,j)$, $f(s,j) = f'(s,i)$ and let $f(s,k) = f'(s,k)$ for all $k \neq i,j$, we get $U^{-\min}(f) = U^{-\min}(f')$. We obtain in the same way $U^{-\min}(g) = U^{-\min}(g')$. Hence, $U^{-\min}(f) \geq U^{-\min}(g)$ iff $U^{-\min}(f') \geq U^{-\min}(g')$, which proves Ano.

- $\succeq$ satisfies Dec, since $U^{-\min}([1]\{i\}[1]) = 1 > U^{-\min}([0]\{i\}[1]) = 0$.

- $\succeq$ satisfies CS:
  Suppose that $[x] \succ [y]$, i.e. $U^{-\min}([x]) = x > U^{-\min}([y]) = y$. It holds that: $U^{-\min}([x]\{i\}1) = x > U^{-\min}([y]\{i\}1) = y$. Hence, $\forall h, U^{-\min}([x]\{i\}h) \geq U^{-\min}([y]\{i\}h)$ (by $wSav2$) and $\exists h, U^{-\min}([x]\{i\}h) > U^{-\min}([y]\{i\}h)$. Thus $[x] \succ_i [y]$.

- $\succeq$ satisfies CoK:

- We know that $wCok$ holds, thus $E \sim D$ implies $E \sim_i D$.
- Consider two events E, D such that $E \succ D$, i.e. $\forall x, y$ such that $x > y$, $U^{-\min}(xEy) > U^{-\min}(xDy)$. Since $U^{-\min}([x]E[y]) = median(x, N(E), y)$ and $U^{-\min}([x]D[y]) = median(x, N(D), y)$. Thus for any x, y such that $x > y$, $median(x, N(E), y) > median(x, N(D), y)$.
  Suppose now that there is a i such that $D \succeq_i E$, which means that
  * $\forall x > y, \forall h, U^{-\min}([x]E\times\{i\}[y]\bar{E}\times\{i\}h) \geq U^{-\min}(([x]D\times\{i\}[y]\bar{D}\times\{i\}h)$,

  This implies that $U^{-\min}([x]E\times\{i\}[y]\bar{E}\times\{i\}1) \geq U^{-\min}(([x]D\times\{i\}[y]\bar{D}\times\{i\}1)$.
  Moreover, $U^{-\min}([x]E\times\{i\}[y]\bar{E}\times\{i\}1) = median(x, N(E), y)$ and $U^{-\min}([x]D\times\{i\}[y]\bar{D}\times\{i\}1) = median(x, N(D), y)$.
  So we get $median(x, N(D), y) \geq median(x, N(E), y)$,
  which contradicts $median(x, N(E), y) > median(x, N(D), y)$. Hence there cannot be a i such that $D \succeq_i E$, i.e. $E \succeq_i D$.
  This shows that $E \succ D$ implies $E \succ_i D$:

We thus get $E \succeq D$ iff $E \succeq_i D$: CoK is satisfied.

□

The same kind of result can be obtained for the optimistic case.

Proposition 8.10.
The relation $\succeq$ defined by: $f \succeq g$ iff $U^{+\max}(f) \geq U^{+\max}(g)$, satisfies axioms $Sav1$, $Sav5$, $wSav2$, $wSav3$, $wSav4$, $ParetoUnanimity$, NC, RCD, OPT, wCS and $wCoK$.

If we suppose that we are in a pure egalitarian context ($w_i = 1, i = 1, p$), we get:

$$U^{+\max}(f) = \max_{i=1,p}\max_{s\in S} \min(f(s,i), \pi(s)))$$

.

and we recover Ano, $Decision$, CS and Cok:

Proposition 8.11.
The relation $\succeq$ defined by:

$$f \succeq g \textit{ iff } \max_{i=1,p}\max_{s\in S} \min(f(s,i), \pi(s))) \geq \max_{i=1,p}\max_{s\in S} \min(g(s,i), \pi(s))),$$

satisfies axioms $Sav1$, $Sav5$, $wSav2$, $wSav3$, $wSav4$, $ParetoUnanimity$, NC, RCD, OPT, CS, Cok, Ano as well as $Decision$.

This shows the coherence and the relevance of our set of axioms.

Proof of Proposition 8.10. The proofs are very similar to the ones presented for the pessimistic utility.

- $\succeq$ trivially satisfies Sav1 (each act is given by a global utility on $[0, 1]$).
- $\succeq$ satisfies wSav2:
  Consider f, g, h, P such that $U^{-\min}(fPh) \succ U^{-\min}(gPh)$ and any fourth act h'
  Let $a = \max_{s \in P} \min(\pi(s), \max_{i \in A} \min(w_i, f(s,i)))$, $b = \max_{s \in P} \min(\pi(s), \max_{i \in A} \min(w_i, g(s,i)))$, $c = \max_{s \in \bar{P}} \min(\pi(s), \max_{i \in A} \min(w_i, h(s,i)))$, $c' = \max_{s \in \bar{P}} \min(\pi(s), \max_{i \in A} \min(w_i, h'(s,i)))$,
  It holds that: $U^{+\max}(fPh) = \max(a, c)$ $U^{+\max}(gPh) = \max(b, c)$
  $U^{+\max}(fPh') = \max(a, c')$ $U^{+\max}(gPh) = \max(b, c')$

  $\max(a, c) > \max(b, c)$ implies that $\max(a, c') \geq \max(b, c')$ i.e.
  $U^{+\max}(fPh') \geq U^{+\max}(gPh')$.
- Pareto unanimity follows from the fact that the $\max$ and $\min$ operators are monotonic.
- $wSav3$ follows from $Sav1$, $wSav2$ and Pareto Unanimity.
- The proof of $wSav4$ is similar to the one given for the pessimistic case, considering that $U^{+\max}([x]E[y]) = median(x, \Pi(E), y)$.
- $Sav5$ is satisfied by construction.
- The proof of $wSav4$ are similar to the ones given for the pessimistic case, considering that $U^{+\max}([x]E[y])$ writes as a mono-agent optimistic utility.
- The proofs of $wCoK$ and NC are similar to the ones given for the pessimistic case, considering that $U^{+\max}([x]E[y]) = median(x, \Pi(E), y)\ \forall E \subseteq S$.
- wCS is a consequence of Pareto Unanimity and $Sav1$.
- $\succeq$ satisfies RCD because it satisfies NC, $Sav1$ and $wSav3$.
- $\succeq$ satisfies OPT since $U^{+\max}$ is an ex-post utility; setting $u((\vec{f}(s))) = \max(\min(w_i, f(s,i))$, we get $U^{+\max}(f) = \max_{s \in S} \min(\pi(s), u((\vec{f}(s)))$: $U^{+\max}$ is a classical optimistic utility and thus satisfies OPT.

□

Proof of Proposition 8.11. We only need to prove that axioms CS , Cok, Ano and $Decision$ are satisfied. The remaining proofs are easy and similar to the ones given for the pessimistic case.

- The relation $\succeq$ satisfies anonymity axiom (Ano):
  Consider acts f, g, f' and $g' \in \mathcal{F}$, and two agents i, $j \in A$ $f(s,i) = f'(s,j)$, $f(s,j) = f'(s,i)$, $g(s,i) = g'(s,j)$, $g(s,j) = g'(s,i)$. We have:
  $$U^{+\max}(f) \quad = \quad \max \quad \begin{array}{l} \max_{s \in S} \min(f(s,i)), \pi(s)), \\ \max_{s \in S} \min(f(s,j), \pi(s)). \\ \max_{k \neq i,j; s \in S} \min(f(s,k)), \pi(s)). \end{array}$$
  Besides, we have:
  $$U^{+\max}(f) \quad = \quad \max \quad \begin{array}{l} \max_{s \in S} \min(f'(s,i)), \pi(s)), \\ \max_{s \in S} \min(f'(s,j), \pi(s)). \\ \max_{k \neq i,j; s \in S} \min(f'(s,k)), \pi(s)). \end{array}$$
  Since $f(s,i) = f'(s,j)$, $f(s,j) = f'(s,i)$ and let $f(s,k) = f'(s,k)$ for all $k \neq i,j$, we get $U^{+\max}(f) = U^{+\max}(f')$.
  We obtain in the same way $U^{+\max}(g) = U^{+\max}(g')$
  Hence $U^{+\max}(f) \geq U^{+\max}(g)$ iff $U^{+\max}(f') \geq U^{+\max}(g')$, which proves Ano.

- $\succeq$ satisfies Dec, since $U^{+\max}([1]\{i\}[0]) = 1 > U^{+\max}([0]\{i\}[0]) = 0$

- $\succeq$ satisfies CS:
  Suppose that $[x] \succ [y]$ since $U^{+\max}([x]\{i\}0) = x > U^{+\max}([y]\{i\}0) = y$. Hence $\forall h, U^{+\max}([x]\{i\}h) \geq U^{+\max}([y]\{i\}h)$ (by wSav2) and $\exists\, h$,
  $U^{+\max}([x]\{i\}h) > U^{+\max}([y]\{i\}h)$. Thus $[x] \succ_i [y]$.

- The proofs of CoK is similar to the ones given for the pessimistic case, considering that $U^{+\max}([x]E[y]) = median(x, \Pi(E), y)$.

□

8.5 Conclusion

This Chapter makes a first attempt to propose an analysis of the possibilistic collective decision rules in the style of Savage. Some basic axioms ($Sav1$, $Sav5$, OPT, PES) stand without any change. Others like $wSav3$ and $wSav4$ are extended thanks to the notion of representative act of a consequence (that plays the same role as the notion of constant act in Savage's framework). We also extend or propose additional axioms specific to multi-agent context - e.g. Ano, Dec, Common Scale and Common Knowledge - . The last part of the Chapter shows that $U^{-\min}$ and

$U^{+\max}$ satisfy our extension of [24, 32, 34]'s axioms for respectively U_{pes} and U_{opt} as well as additional axioms. This axiomatic opens the way to a Savage-like representation theorem for $U^{-\min}$ and $U^{+\max}$ in the style of Savage that may be proposed.

The framework presented here imposes that all agents have the same knowledge (as in VNM's framework) and share the same preference order on X. However, it is also important to consider the case where agents have different view on the environment (different knowledge). It has been proved, in the case of probabilistic context [13, 14, 53], that subjective knowledge leads to an impossibility result but what about possibilistic uncertainty?

CONCLUSION

Our contribution proposes new decision criteria for collective decision making under possibilistic uncertainty for cautious as well as adventurous decision makers. This approach offers to decision makers the possibility to find the best decision which satisfies the collectivity taking into account the uncertainty aspect of the problem, the decision makers attitude towards this uncertainty, and the method of aggregating the agents' preferences. These considerations as well as the manner to combine them (either *ex-ante* or *ex-post* aggregation), give rise to different decision criteria namely, eight qualitative collective decision rules. We have proved the coincidence between the *ex-ante* and *ex-post* homogenous utilities for both the fully min-oriented and max-oriented cases: it is worth noticing that the use of qualitative decision criteria allows to get rid of the timing effect even while being egalitarian.

On the basis of a set of rational axioms, we have provided a representation theorem for the fully pessimistic and optimistic utilities in the style of VNM for agents with identical or different weights. This result can be seen as an ordinal counterpart of Harsanyi's theorem. Making a step further, we have provided a first attempt to propose an analysis of the multi-agent qualitative decision making in the style of Savage.

Furthermore, we have presented an algorithmic study relative to the optimization of the proposed collective qualitative decision rules in possibilistic decision trees. We have provided an adaptation of the Dynamic Programming algorithm for criteria that satisfy the monotonicity property, Multi-Dynamic Programing and Branch and Bound algorithms for those that are not monotonic. Finally, an experimental study has been performed to compare the different algorithms and to evaluate the quality of the solutions provided.

The seminal work of Harsanyi [59] is generally interpreted as a justification of utilitarianism. However, our work may be seen as an ordinal counter part of Harsanyi's theorem and as a justification of egalitarianism when pessimistic agents have to make decisions that satisfy the collectivity. These results have true been presented under the assumption that all agents share the same knowledge, which is seldom in real world problem. This consideration has been the topic of several works, always in the probabilistic context (in style of Savage), that led to an impossibility theorem when agents have heterogeneous knowledge about states of world [3, 69]. Thus, it is interesting to address this question in context of possibility theory. In this case, the definition of the decision rules must be revised especially for the *ex-post* aggregation when collectivity is first considered. Thus, we may consider only *ex-ante* decision rules or provide a new form of aggregation.

Moreover, in our formalism we suppose that all agents are either optimistic or pessimistic. But, it may happen that agents have different attitudes toward uncertainty: the group of decision makers may gather pessimistic as well as optimistic persons, then the study of such situations may be envisaged. Likewise, we shall extend our approach, considering that the importance of the decision makers is not absolute but may depend on the consequence of each decision.

Besides, the possibilistic aggregations used in this work are basically specializations of the Sugeno integral. So, we aim at generalizing the study of collective decision making under uncertainty through the development of double Sugeno Integrals. Also, on the basis of the pessimistic utility we have proved that egalitarianism is compulsory to avoid the timing effect problem. So, what about the use of other possibilistic decision rules (e.g. Binary possibilistic utility, Order of Magnitude Expected utility, Possibilistic likely dominance, etc.) or even the use of different uncertainty theories such evidence theory [85] or rough set theory [75].

Another line of research to consider is the refinement of the decision rules. In fact, since they are based on possibilistic utilities (U_{opt} and U_{pes}) our criteria as many qualitative decision rules suffer from a lack of decisiveness, called the “drowning effect", due to the use of the idempotent operations max and min. So, it is interesting to consider the refinement of these criteria to more discriminating ones on the basis of existing works in the literature [40, 91].

From an algorithmic point of view, the difference of attitude and/or knowledge of the agents may raise several difficulties. In these cases, the upcoming criteria may suffer from the lack of monotonicity and their optimization in possibilistic decision trees can not be performed using a Dynamic Programming algorithm anymore. An alternative idea is the deliberation of agents after each decision. In this case, the Dynamic Programing algorithm shall be used but it leads to an approach that is neither the *ex-ante* nor the *ex-post* aggregation.

Finally, we can notice that there is a growing interest in the literature to adapt graphical decision models (decision trees [78], influence diagrams [60], Markov decision process [5] or Valuation Based Systems [86]) to handle more complex decision problems. In [48] for example, the authors propose qualitative counterpart of influence diagrams. An adaptation of Markov decision process to the possibilistic context has also been proposed by [80, 81]. Other works have been provided to consider other aspects such as multiple objective decision making in influence diagrams [21, 68]. So, why not provide new graphical models for qualitative multi-criteria or multi-agent decision making problems.

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