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ON EXISTENCE OF SOLUTIONS TO THE CAPUTO TYPE FRACTIONAL ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we establish the existence of solutions to the fractional order three-point boundary value problems by utilizing Banach contraction principle and Schaefer's fixed point theorem.

1. Introduction

Fractional calculus deals with integration and differentiation of an arbitrary real order. Fractional order derivatives provide a new approach for modeling complex phenomena in physics, mechanics, control systems, flow in porous media, signal and image processing, aerodynamics, electromagnetics, viscoelasticity and especially dealing with memory and hereditary properties of various real materials [5, 6, 9, 11]. Compared to integer order models, the fractional order models offer better description of underlying processes. In consequence, fractional order differential equations have achieved great deal of interest and attention of researchers [1, 2, 4, 7, 10, 12, 13, 14, 16, 17, 18].

Zhang [20] obtained the existence and uniqueness of solutions for two-point fractional order boundary value problem

$$D^{\alpha}_{0^+}u(t) = f\big(t, u(t)\big),\ t \in (0,1),$$
$$u(0) + u'(0) = 0,\ u(1) + u'(1) = 0,$$

where $D^{\alpha}_{0^+}$ is the Caputo fractional order derivative and $1 < \alpha \leq 2$.

Shi and Zhang [17] given sufficient conditions for the existence of at least one solution for fractional order boundary value problem

$$D^{\delta}u(t) + g(t, u) = 0,\ t \in (0,1),$$
$$u(0) = a,\ u(1) = b,$$

where $1 < \delta \leq 2$, $g : [0,1] \times R \to R$ and D^{δ} is the Caputo fractional order derivative, using upper and lower solutions method.

Benchohra, Hamani and Ntouyas [3] derived sufficient conditions for the existence of at least one solution to the fractional order boundary value problem

$${}^{C}D^{\alpha}u(t) = f\big(t, u(t)\big),\ 0 < t < T,$$
$$u(0) = g(u),\ u(T) = u_T,$$

where $1 < \alpha \leq 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional order derivative, by using Schaefer's fixed point theorem. Also they established criteria for the uniqueness of solutions by virtue of the Banach fixed point theorem.

In [8], the authors studied the existence and uniqueness of solutions to the boundary value problem

$$D^{\alpha}u(t) = f(t, u(t), u'(t)),\ t \in (0,1),$$
$$u(0) = 0,\ D^{p}u(1) = \delta D^{p}u(\eta),$$

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where D is the Caputo fractional order derivative, $1 < \alpha \leq 2$, $0 < \delta < p < 1$, $0 < \eta \leq 1$, by using some fixed point theorem.

In this paper we consider the fractional order three-point boundary value problem

$$ {}^{C}D^{q}y(t) = f\big(t, y(t)\big),\ t \in (0,1), \tag{1.1} $$

$$ \begin{cases} y(0) + y'(0) + y''(0) = 0, \\ y(\eta) + y'(\eta) + y''(\eta) = 0, \\ y(1) + y'(1) + y''(1) = 0, \end{cases} \tag{1.2} $$

where $q \in (2,3], \eta \in (0,1)$ and ${}^{C}D^{q}$ is the standard Caputo fractional order derivative.

We assume that the following conditions hold throughout the paper:

(A1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous,
(A2) there exists a positive constant $\mathcal{M}$ such that $\big|f\big(t, y\big)\big| \leq \mathcal{M}$, for each $t \in [0,1]$ and $y \in \mathbb{R}$,
(A3) there exists a positive real constant ξ such that $|f(t,y) - f(t,\overline{y})| \leq \xi |y - \overline{y}|$ for each $t \in [0,1]$, for all $y,\ \overline{y} \in \mathbb{R}$.

1.1. **Preliminaries.** In this section, we present some definitions and lemmas that are useful in the proof of our main results.

Definition 1.1. [11] *The Riemann–Liouville fractional integral of order $p > 0$ of a function $f : [0,+\infty) \to \mathbb{R}$ is given by*

$$ I^{p}_{0^+} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau, $$

provided the right-hand side is defined.

Definition 1.2. [11] *The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [a,+\infty) \to \mathbb{R}$ is given by*

$$ {}^{C}_{a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,\ \ (n-1 < \alpha < n), $$

provided the right-hand side is defined.

Lemma 1.1. [11] *Let $\alpha > 0$, then the fractional order differential equation*

$$ {}^{C}D^{\alpha}u(t) = 0 $$

has solution, $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},\ c_i \in \mathbb{R},\ i = 0,1,2,\cdots,n-1,\ n = [\alpha] + 1$.

Lemma 1.2. [11] *Let $\alpha > 0$, then $I^{\alpha}\ {}^{C}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$ for $c_i \in \mathbb{R},\ i = 0,1,2,\cdots,n-1,\ n = [\alpha] + 1$.*

The rest of the paper is organized as follows. In Section 2, we establish the conditions for the existence of solutions to the fractional order boundary value problem (1.1)-(1.2) by using different fixed point theorems such as Banach contraction principle and Schaefer's fixed point theorem. In Section 3, we present examples to illustrate the applicability of the conditions.

2. Main results

In this section, by stating some lemmas, we establish sufficient conditions for the existence of solutions to the fractional order boundary value problem (1.1)-(1.2) using certain fixed point theorems.

Let $\mathcal{B} = C^{q}\big([0,1], \mathbb{R}\big)$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ equipped with the norm

$$ \|y\| = \max_{t \in [0,1]} \big|y(t)\big|. $$

Lemma 2.1. *Let $\Delta = (\eta^2 - \eta)\Gamma(q) \neq 0$. Assume that condition $(A1)$ is satisfied. A function $y \in C^q\big([0,1], \mathbb{R}\big)$ is a solution of the fractional integral equation*

$$\tag{2.1} \begin{cases} y(t) = \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, y(s)\big)ds + \left[\frac{t^2 - 3t + 1}{\Delta}\right] \times \int_0^\eta \Phi(s) f\big(s, y(s)\big)ds \\ \qquad + \left[\dfrac{\eta^2(t-1) + \eta t(2-t)}{\Delta}\right] \times \displaystyle\int_0^1 \Psi^*(s) f\big(s, y(s)\big)ds, \end{cases}$$

where

$$\tag{2.2} \begin{cases} \Phi(s) =(\eta - s)^{q-1} + (q-1)(\eta - s)^{q-2} + \big(q^2 - 3q + 2\big)(\eta - s)^{q-3}, \\ \Psi^*(s) =(1 - s)^{q-1} + (q-1)(1 - s)^{q-2} + \big(q^2 - 3q + 2\big)(1 - s)^{q-3}, \end{cases}$$

if and only if y is a solution of the fractional order boundary value problem (1.1)-(1.2).

Proof. Let $y(t) \in C^q[0,1]$ be the solution of fractional order boundary value problem (1.1)-(1.2). An equivalent integral equation for (1.1) and is given by

$$\tag{2.3} y(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y(s))ds + k_1 + k_2 t + k_3 t^2.$$

Using the conditions (1.2) to $y(t)$ in (2.3) and by algebraic calculations, we can get

$$\begin{aligned} k_1 &= \frac{1}{\Delta}\Big[\int_0^\eta \Phi(s) f\big(s, y(s)\big)ds - \eta^2 \int_0^1 \Psi^*(s) f\big(s, y(s)\big)ds\Big] \\ k_2 &= \frac{1}{\Delta}\Big[(\eta^2 + 2\eta)\int_0^1 \Psi^*(s) f\big(s, y(s)\big)ds - 3\int_0^1 \Phi(s) f\big(s, y(s)\big)ds\Big] \\ k_3 &= \frac{1}{\Delta}\Big[\int_0^\eta \Phi(s) f\big(s, y(s)\big)ds - \eta \int_0^1 \Psi^*(s) f\big(s, y(s)\big)ds\Big]. \end{aligned}$$

By substituting the values of k_1, k_2, k_3 in (2.3), we obtain (2.1). This completes the proof. □

Now we establish the existence of solution to the fractional order boundary value problem (1.1)-(1.2) by an application of Banach contraction principle [15].

Lemma 2.2. (**Banach contraction principle**) *Let (X, d) be a complete metric space. Let $T : X \to X$ be a contraction. That is, there exists $\lambda \in [0,1)$ such that $d(T_x, T_y) \leq \lambda d(x,y)$, for all $x, y \in X$. Then there exists a unique fixed point of T.*

Theorem 2.3. *Assume that the conditions $(A1)$ and $(A3)$ are satisfied. If*

$$\tag{2.4} \zeta = \left|\frac{\xi(\eta + 1)(\eta^{q-2} + 1)}{\Gamma(q+1)(\eta - 1)}\right| < 1,$$

then the fractional order boundary value problem (1.1)-(1.2) *has a solution on* $[0,1]$.

Proof. We transform the fractional order boundary value problem (1.1)-(1.2) in to a fixed point problem by defining an operator T.

Let $T : \mathcal{B} \to \mathcal{B}$ be the operator defined by

$$\tag{2.5} \begin{cases} Ty(t) = \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, y(s)\big)ds - \Big[\frac{t^2 - 3t + 1}{\Delta}\Big] \times \int_0^\eta \Phi(s) f\big(s, y(s)\big)ds \\ \qquad - \Big[\dfrac{\eta^2(t-1) + \eta(2-t)t}{\Delta}\Big] \times \displaystyle\int_0^1 \Psi^*(s) f\big(s, y(s)\big)ds. \end{cases}$$

Clearly the fixed points of the operator T are the solutions of the fractional order boundary value problem (1.1)-(1.2). Now we show that the operator T is a contraction mapping. Let $y,\ \overline{y} \in \mathcal{B}$ and using (2.2), then for each $t \in [0,1]$, we have

$$\begin{aligned}
\big|Ty(t) - T\overline{y}(t)\big| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big| f\big(s, y(s)\big) - f\big(s, \overline{y}(s)\big)\Big| ds \\
& + \left|\frac{t^2 - 3t + 1}{\Delta}\right| \times \int_0^\eta \Phi(s) \Big| f\big(s, y(s)\big) - f\big(s, \overline{y}(s)\big)\Big| ds \\
& + \left|\frac{\eta^2(1-t) + \eta(t-2)t}{\Delta}\right| \times \int_0^1 \Psi^*(s) \Big| f\big(s, y(s)\big) - f\big(s, \overline{y}(s)\big)\Big| ds \\
\leq & \left\{ \frac{(t-s)^q}{q\Gamma(q)} + \frac{1}{|\Delta|} \times \int_0^\eta \Phi(s) ds + \left|\frac{\eta}{\Delta}\right| \times \int_0^1 \Psi^*(s) ds \right\} \Big[\xi \|y - \overline{y}\|_\infty\Big] \\
\leq & \left| \frac{\xi(\eta^{q-2} + 1)(\eta + 1)}{(\eta - 1)\Gamma(q+1)} \right| \big\|y - \overline{y}\big\|_\infty .
\end{aligned}$$

Hence $\|Ty(t) - T\overline{y}(t)\|_\infty \leq \zeta \|y - \overline{y}\|$. Therefore T is a contraction mapping. By the contraction mapping principle, the operator T has a fixed point and is the solution of the fractional order boundary value problem (1.1)-(1.2). □

Next we establish the result by applying Schaefer's fixed point theorem [19].

Lemma 2.4. (**Schaefer's fixed point theorem**) *Let $F : X \to X$ be a completely continuous operator. If the set $E(F) = \big\{x \in X : x = \lambda^* F x$, for some $\lambda^* \in [0,1]\big\}$ is bounded. Then F has fixed points.*

Theorem 2.5. *Assume that the conditions (A1)-(A2) are satisfied. Then the fractional order boundary value problem (1.1)-(1.2) has at least one solution on $[0,1]$.*

Proof. We prove that the operator T is defined by (2.5) has a fixed point by utilizing Schaefer's fixed point theorem. Now we establish the result in **4** steps.

Step 1. The operator T given by (2.5) is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\mathcal{B}$. Using (2.2) and for each $t \in [0,1]$, we have

$$\begin{aligned}
\big|Ty_n(t) - Ty(t)\big| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \Big| f\big(s, y_n(s)\big) - f\big(s, y(s)\big)\Big| ds + \left|\frac{t^2 - 3t + 1}{\Delta}\right| \times \\
& \int_0^\eta \Phi(s) \Big| f\big(s, y_n(s)\big) - f\big(s, y(s)\big)\Big| ds + \left|\frac{\eta^2(1-t) + \eta t(t-2)}{\Delta}\right| \times \\
& \int_0^1 \Psi^*(s) \Big| f\big(s, y_n(s)\big) - f\big(s, y(s)\big)\Big| ds \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \max_{s\in[0,1]} \Big| f\big(s, y_n(s)\big) - f\big(s, y(s)\big)\Big| ds + \\
& \frac{1}{|\Delta|} \times \int_0^\eta \Phi(s) \max_{s\in[0,1]} \Big| f\big(s, y_n(s)\big) - f\big(s, y(s)\big)\Big| ds + \\
& \frac{|\eta|}{|\Delta|} \times \int_0^1 \Psi^*(s) \max_{s\in[0,1]} \Big| f\big(s, y_n(s)\big) - f\big(s, y(s)\big)\Big| ds \\
\leq & \frac{\|f\big(\cdot, y_n(\cdot)\big) - f\big(\cdot, y(\cdot)\big)\|}{\Gamma(q)} \times \int_0^t (t-s)^{q-1} ds + \\
& \frac{1}{|\eta - \eta^2|} \times \int_0^\eta \Phi(s) ds + \left|\frac{\eta}{\eta - \eta^2}\right| \times \int_0^1 \Psi^*(s) ds \\
\leq & \left|\frac{(\eta+1)(\eta^{q-2}+1)}{\Gamma(q+1)}\right| \Big\| f\big(\cdot, y_n(\cdot)\big) - f\big(\cdot, y(\cdot)\big)\Big\| .
\end{aligned}$$

Since f is continuous, we have

$$\Big\|Ty_n(t)-Ty(t)\Big\| \leq \left|\frac{(\eta+1)(\eta^{q-2}+1)}{\Gamma(q+1)(n-1)}\right| \Big\|f\big(\cdot,y_n(\cdot)\big)-f\big(\cdot,y(\cdot)\big)\Big\| \to 0 \quad \text{as } n\to\infty.$$

Therefore the operator T is continuous.

Step 2. The operator T maps bounded sets in to bounded sets in $\mathcal{B}$. Now we will show that for any $\vartheta>0$, there exits a positive constant m^* such that for each $y\in B_\vartheta=\big\{y\in\mathcal{B}:\|y\|_\infty\leq\vartheta\big\}$, we have $\|Ty\|_\infty\leq m^*$. By the condition $(A2)$ and (2.2), for each $t\in[0,1]$, we have

$$\begin{aligned}
|Ty(t)| \leq& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\Big|f\big(s,y(s)\big)\Big|ds+\left|\frac{t^2-3t+1}{\Delta}\right|\times\int_0^\eta \Phi(s)\Big|f\big(s,y(s)\big)\Big|ds\\
&+\left|\frac{\eta(1-t)+\eta t(t-2)}{\Delta}\right|\times\int_0^1\Psi^*(s)f\big(s,y(s)\big)ds\\
\leq& \mathcal{M}\left\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds+\frac{1}{|\Delta|}\times\int_0^\eta\Phi(s)ds+\left|\frac{\eta}{\Delta}\right|\times\int_0^1\Psi^*(s)ds\right\}\\
\leq& \mathcal{M}\left|\frac{(\eta+q)(1+\eta^{q-2})}{\Gamma(q+1)(\eta-1)}\right|.
\end{aligned}$$

Thus

$$\|Ty\|_\infty\leq\mathcal{M}\left|\frac{(\eta+q)(1+\eta^{q-2})}{\Gamma(q+1)(\eta-1)}\right|=m^*.$$

Step 3. The operator T maps bounded sets into equicontinuous sets $\mathcal{B}$. Let $t_1,t_2\in(0,1],\ t_1<t_2,\ B_\vartheta$ be a bounded set of $\mathcal{B}$ as in **Step 2** and $y\in B_\vartheta$. Then

$$\begin{aligned}
|Ty(t_2)-Ty(t_1)| \leq& \int_0^{t_1}\frac{[(t_2-s)^{q-1}-(t_1-s)^{q-1}]}{\Gamma(q)}f\big(s,y(s)\big)ds-\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f\big(s,y(s)\big)ds\\
&-\left[\frac{(t_2^2-3t_2+1)-(t_1^2-3t_1+1)}{\Delta}\right]\times\int_0^\eta f\big(s,y(s)\big)\Phi(s)ds\\
&-\left[\frac{\eta^2(t_2^2-t_1^2)+\eta(t_1^2-t_2^2+2t_2-2t_1)}{\Delta}\right]\times\int_0^1 f\big(s,y(s)\big)ds\\
\leq& \mathcal{M}\left\{\Big[\frac{(t_2-t_1)^q-(t_2-t_1)^q}{\Gamma(q)}\Big]-\frac{(t_2^2-t_1^2)-3(t_2-t_1)}{\Delta\big[\eta^q+q\eta^{q-1}+q(q-1)\eta^{q-2}\big]}\right.\\
&\left.-\frac{(1+q^2)\eta^2\big[(\eta-1)(t_2^2-t_1^2)+(2t_2-2t_1)\big]}{\Delta}\right\}.
\end{aligned}$$

As $t_2\to t_1$, the right hand side of the above inequality tends to zero. From **Steps 1** to **3,** we can conclude that $T:\mathcal{B}\to\mathcal{B}$ is completely continuous.

Step 4. The set $A=\Big\{y\in\mathcal{B}:y=\lambda Ty \text{ for some } 0<\lambda<1\Big\}$ is bounded. Let $y\in A$, then $y=\lambda Ty$ for some $0<\lambda<1$. Therefore, for each $t\in[0,1]$, we have

$$\begin{aligned}
y(t)=&\lambda\left\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f\big(s,y(s)\big)ds-\left[\frac{t^2-3t+1}{\Delta}\right]\times\int_0^\eta\Phi(s)f\big(s,y(s)\big)ds\right.\\
&\left.-\left[\frac{\eta^2(t-1)+\eta t(2-t)}{\Delta}\right]\times\int_0^1\Psi^*(s)f\big(s,y(s)\big)ds\right\}
\end{aligned}$$

and

$$\begin{aligned}\Big|Ty(t)\Big| &\le \mathcal{M}\lambda\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \Big|\frac{t^2-3t+1}{\Delta}\Big| \times \int_0^\eta \Phi(s)ds \\ &\quad + \Big|\frac{\eta^2(t-1)+\eta t(2-t)}{\Delta}\Big| \times \int_0^1 \Psi^*(s)ds\Big\} \\ &\le \mathcal{M}\lambda\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \int_0^\eta \frac{\Phi(s)}{\Delta}ds + \eta\int_0^1 \frac{\Psi^*(s)}{\Delta}ds\Big\} \\ &\le \frac{\mathcal{M}(1+\eta)(1+\eta^{q-1})}{\Gamma(q+1)}.\end{aligned}$$

Thus for every $t \in [0,1]$, we have

$$\big\|Ty\big\|_\infty \le \mathcal{M}\left|\frac{(1+\eta)(1+\eta^{q-1})}{\Gamma(q+1)}\right|.$$

This shows that the set A is bounded. By Schaefer's fixed point theorem, the operator T has a fixed point which is a solution of the fractional order boundary value problem (1.1)-(1.2). □

3. Examples

In this section, we give two examples to illustrate the usefulness of our main results.

Example 3.1 Consider the fractional order three-point boundary value problem,

$$ {}^{C}D^{2.6}y(t) = \frac{3e^{-2t}y}{(19+5e^{2t})(3+y)},\ t \in (0,1), \tag{3.1}$$

$$\begin{cases} y(0)+y'(0)+y''(0) =0, \\ y\Big(\frac{2}{5}\Big)+y'\Big(\frac{2}{5}\Big)+y''\Big(\frac{2}{5}\Big) =0, \\ y(1)+y'(1)+y''(1) =0. \end{cases} \tag{3.2}$$

For $y, \overline{y} \in [0,\infty)$ and $t \in [0,1]$, we have

$$|f(t,y)-f(t,\overline{y})| = \frac{3e^{-2t}}{19+5e^{2t}}\left|\frac{y}{3+y} - \frac{\overline{y}}{3+\overline{y}}\right| \le \frac{9}{24}|y-\overline{y}|.$$

Now we check the condition (2.4) with $\xi = \frac{9}{24}$,

$$\zeta = \left|\frac{\frac{9}{24}\big(\frac{2}{5}+1\big)\big[\big(\frac{2}{5}\big)^{0.6}+1\big)\big]}{\Gamma(3.6)\big(\frac{-3}{5}\big)}\right| = 0.42 < 1.$$

Then all conditions of Theorem 2.3 are satisfied. Thus by Theorem 2.3 the fractional order boundary value problem (3.1)-(3.2) has unique solution on $[0,1]$.

Example 3.2 Consider the fractional order three-point boundary value problem,

$$ {}^{C}D^{2.5}y(t) = \frac{e^{-t}y}{(1+e^{t})(1+y)},\ t \in (0,1), \tag{3.3}$$

$$\begin{cases} y(0)+y'(0)+y''(0) =0, \\ y\Big(\frac{1}{2}\Big)+y'\Big(\frac{1}{2}\Big)+y''\Big(\frac{1}{2}\Big) =0, \\ y(1)+y'(1)+y''(1) =0. \end{cases} \tag{3.4}$$

By simple algebraic calculations, one can determine $m^* = \left|\frac{\frac{3}{2}\big(1+0.5^{0.5}\big)\big]}{\Gamma(3.5)\big(\frac{-1}{2}\big)}\right| = 1.6$ and $\mathcal{M} = \frac{1}{2}$. Then the fractional order boundary value problem (3.3)-(3.4) satisfies all conditions of Theorem 2.5. Hence it has unique solution on $[0,1]$.

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