

Weighted Linear Dynamic Logic

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We introduce a weighted linear dynamic logic (weighted *LDL* for short) and show the expressive equivalence of its formulas to weighted rational expressions. This adds a new characterization for recognizable series to the fundamental Schützenberger theorem. Surprisingly, the equivalence does not require any restriction to our weighted *LDL*. Our results hold over arbitrary (resp. totally complete) semirings for finite (resp. infinite) words. As a consequence, the equivalence problem for weighted *LDL* formulas over fields is decidable in doubly exponential time. In contrast to classical logics, we show that our weighted *LDL* is expressively incomparable to weighted *LTL* for finite words. We determine a fragment of the weighted *LTL* such that series over finite and infinite words definable by *LTL* formulas in this fragment are definable also by weighted *LDL* formulas.

1 Introduction

Linear Temporal Logic (*LTL* for short) is widely used in several areas of Computer Science like, for instance in model checking where it plays the role of a specification language [3, 22], and in artificial intelligence [22]. Nevertheless, *LTL* formulas are expressively weaker than finite automata, namely the class of *LTL*-definable languages coincides with the class of First-Order (*FO* for short) logic definable languages (cf. [7] for an excellent survey on the topic). Therefore, it was greatly desirable, especially for applications, to have a logic which combines the complexity properties of reasoning on *LTL* and the expressive power of finite automata. This was recently achieved in [22], where the authors introduced a *Linear Dynamic Logic* (*LDL* for short) which is a combination of *Propositional Dynamic Logic* (cf. [23]) and *LTL*. The satisfiability, validity, and logical implication of *LDL* formulas interpreted over finite words were proved to be PSPACE-complete [22, 21], as for *LTL*. This was obtained by a translation of *LDL* formulas to finite automata. Similar results were stated for *LDL* formulas interpreted over infinite words in [38].

In the weighted setup, a Büchi type theorem stating the coincidence of recognizable series with the ones defined in a fragment of a weighted Monadic Second-Order (*MSO* for short) logic over semirings, was firstly proved in [8] (cf. also [9]). Then, weighted *MSO* logics have been investigated for several objects, including trees, pictures, nested words, graphs, and timed words. The weight structure of the semiring has been also replaced by more general ones incorporating average or discounting of weights. Most of the results work for finite as well as infinite objects. A weighted version of *LTL* over De Morgan algebras was firstly introduced in [25]. In [15] the authors proved several characterizations of *LTL*-definable and *LTL*- ω -definable series over arbitrary bounded lattices. Recently, a weighted *LTL* with averaging modalities was studied in [5], and a weighted *LTL* over idempotent and zero-divisor free semirings satisfying completeness axioms was investigated in [28, 30]. In [1, 2] the authors considered a discounted *LTL* with values in $[0, 1]$ and in [28, 29] in the max-plus semiring.

It is the goal of this paper to introduce and investigate a *weighted LDL* over arbitrary semirings. Our work is motivated as follows. In recent applications like verification of systems [6] and artificial intel-

ligence (cf. for instance [26]), classical automata have been replaced by quantitative ones. Therefore it is highly desirable to have a quantitative logic which is expressively equivalent to weighted automata. However, the class of series which are definable by all weighted *MSO* logic sentences exceeds that of recognizable series. Furthermore, the weighted *FO* logic over finite words is, in general, expressively incomparable to weighted finite automata [8], and this is shown here also for the weighted *LTL*. Therefore, in view of the results of [22, 21] for *LDL*, we investigate weighted *LDL*. We show that our weighted *LDL* is expressively equivalent to weighted finite automata over semirings. Surprisingly, there is no need to consider, as for the weighted *MSO* logic, any fragment of our logic to achieve the aforementioned equivalence. Our results hold for finite and infinite words and this shows the robustness of our theory and in turn the robustness of the *LDL* of [22, 21, 38]. Our main results are as follows.

- The class of *LDL*-definable series coincides with the class of generalized rational series over arbitrary semirings.
- The class of *LDL*-definable series coincides with the class of recognizable series over commutative semirings. This extends the fundamental Schützenberger theorem, for commutative semirings, with a logic directed characterization.
- The equivalence problem for weighted *LDL* formulas is decidable in doubly exponential time for a large class of weight structures including computable fields, as the realizability problem for *LDL* [21].
- The class of *LDL*- ω -definable series coincides with the class of generalized ω -rational series over totally complete semirings.
- The class of *LDL*- ω -definable series coincides with the class of ω -recognizable series over totally commutative complete semirings.

Our weighted *LDL* consists of the classical, unweighted *LDL* of [22] with the same interpretation and a copy of it which is interpreted quantitatively. Therefore, practitioners can use the classical *LDL* part as they are used to, and the copy of it in the same way to compute quantitative interpretation. A similar approach was followed for weighted *MSO* logic recently in [20]. While the translation of the restricted weighted *MSO* logic formulas of [9] to weighted automata as for *MSO* is non-elementary, the translation of the present weighted *LDL* into weighted automata can be done in doubly exponential time, as for *LDL*. We prove that our weighted *LDL* interpreted over finite words, is in general expressively incomparable to weighted *LTL* of [28, 30]. We define a fragment of that weighted *LTL* and prove that series over finite and infinite words definable by weighted *LTL* formulas in this fragment are definable as well by weighted *LDL* formulas. Furthermore, our weighted *LDL* is expressively equivalent to weighted conjunction-free μ -calculus [31] for a particular class of semirings.

2 Semirings and rational operations

Let A be an alphabet, i.e., a finite nonempty set. As usually, we denote by A^* (resp. A^ω) the set of all finite (resp. infinite) words over A and $A^+ = A^* \setminus \{\varepsilon\}$, where ε is the empty word. We write a finite (resp. infinite) word often as $w = w(0) \dots w(n-1)$ (resp. $w = w(0)w(1) \dots$) where $w(i) \in A$ for every $i \geq 0$. For every finite (resp. infinite) word $w = w(0) \dots w(n-1)$ (resp. $w = w(0)w(1) \dots$) and every $0 \leq i \leq n-1$ (resp. $i \geq 0$) we denote by $w_{\geq i}$ the suffix $w(i) \dots w(n-1)$ (resp. $w(i)w(i+1) \dots$) of w . In the sequel, we use the letter a with indices to denote the elements of an alphabet A .

A *semiring* $(K, +, \cdot, 0, 1)$ is denoted simply by K if the operations and the constant elements are understood. If no confusion is caused, we shall denote the operation \cdot simply by concatenation. The result of the empty product as usual equals to 1.

Throughout the paper A will denote an alphabet and K a semiring.

A *formal series* (or simply *series*) over A^* and K is a mapping $s : A^* \rightarrow K$. We denote by $K \langle\langle A^* \rangle\rangle$ the class of all series over A^* and K . The *constant series* \tilde{k} ($k \in K$) is defined, for every $w \in A^*$, by $\tilde{k}(w) = k$. The *characteristic series* 1_L of a language $L \subseteq A^*$ is given by $1_L(w) = 1$ if $w \in L$ and $1_L(w) = 0$ otherwise. If $L = \{w\}$ is a singleton, then we write w in place of $1_{\{w\}}$. Let $s, r \in K \langle\langle A^* \rangle\rangle$ and $k \in K$. The *sum* $s + r$, the *products with scalars* ks and sk as well as the *Hadamard product* $s \odot r$ are defined elementwise by $(s + r)(w) = s(w) + r(w)$, $(ks)(w) = ks(w)$, $(sk)(w) = s(w)k$, $(s \odot r)(w) = s(w)r(w)$ for every $w \in A^*$. Trivially, the structure $(K \langle\langle A^* \rangle\rangle, +, \odot, \tilde{0}, \tilde{1})$ is a semiring. The *Cauchy product* $s \cdot r \in K \langle\langle A^* \rangle\rangle$ is determined by $(s \cdot r)(w) = \sum_{w=uv} s(u)r(v)$ for every $w \in A^*$. The *n th-iteration* $s^n \in K \langle\langle A^* \rangle\rangle$ ($n \geq 0$) is defined inductively by $s^0 = \varepsilon$ and $s^{n+1} = s \cdot s^n$ for every $n \geq 0$. The series s is called *proper* if $s(\varepsilon) = 0$. If s is proper, then for every $w \in A^*$ and $n > |w|$ we have $s^n(w) = 0$. The *iteration* $s^+ \in K \langle\langle A^* \rangle\rangle$ of a *proper series* s is defined by $s^+ = \sum_{n>0} s^n$.

The class of *weighted rational expressions over A and K* [11] is given by the grammar $E ::= ka \mid E + E \mid E \cdot E \mid E^+$ where $k \in K$ and $a \in A \cup \{\varepsilon\}$. We denote by $RE(K, A)$ the class of all such weighted rational expressions over A and K . For the relationship with weighted logics, we will need to consider the Hadamard product as a rational operation. Therefore, we introduce the class of *generalized weighted rational expressions over A and K* which is given by the grammar $E ::= ka \mid E + E \mid E \cdot E \mid E^+ \mid E \odot E$, where $k \in K$ and $a \in A \cup \{\varepsilon\}$. We shall denote by $GRE(K, A)$ the class of generalized weighted rational expressions over A and K . The *semantics* of a (generalized) weighted rational expression E is a series $\|E\| \in K \langle\langle A^* \rangle\rangle$ which is defined inductively by $\|ka\| = ka$, $\|E + E'\| = \|E\| + \|E'\|$, $\|E \cdot E'\| = \|E\| \cdot \|E'\|$, $\|E^+\| = \|E\|^+$ (if $\|E\|$ is proper; otherwise undefined), $\|E \odot E'\| = \|E\| \odot \|E'\|$. A series $s \in K \langle\langle A^* \rangle\rangle$ is called *rational* (resp. *g-rational*) if there is a weighted (resp. generalized weighted) rational expression E such that $s = \|E\|$. The following result is the fundamental Schützenberger theorem stating the coincidence of rational and recognizable series, i.e., series accepted by weighted automata. For the theory on weighted automata we refer the reader to [18, 34, 12].

Theorem 1 [35, 18, 34] *Let K be a semiring and A an alphabet. Then a series $s \in K \langle\langle A^* \rangle\rangle$ is rational iff it is recognizable.*

It is well-known (cf. [36, 4, 11]) that if the semiring K is commutative, then the class of recognizable series over A and K is closed under Hadamard product. Consequently, if K is commutative, then a series $s \in K \langle\langle A^* \rangle\rangle$ is g-rational iff it is recognizable.

3 Weighted linear dynamic logic on finite words

In this section, we introduce the weighted linear dynamic logic (weighted *LDL* for short). Our main result states the coincidence of the classes of g-rational series and series definable by weighted *LDL* formulas. First, we recall the *LDL* from [22]. For the definition of our weighted *LDL* below, we need to modify the notations used for the semantics of *LDL* formulas in [22]. For every letter $a \in A$ we consider an atomic proposition p_a and we let $P = \{p_a \mid a \in A\}$. For every $p \in P$ we identify $\neg \neg p$ with p .

Definition 2 The syntax of LDL formulas ψ over A is given by the grammar

$$\begin{aligned}\psi &::= true \mid p_a \mid \neg\psi \mid \psi \wedge \psi \mid \langle \theta \rangle \psi \\ \theta &::= \phi \mid \psi? \mid \theta + \theta \mid \theta; \theta \mid \theta^+\end{aligned}$$

where $p_a \in P$ and ϕ denotes a propositional formula over the atomic propositions in P .

Next, for every LDL formula ψ and $w \in A^*$ we define the satisfaction relation $w \models \psi$, inductively on the structure of ψ , as follows:

- $w \models true$,
- $w \models p_a$ iff $w(0) = a$,
- $w \models \neg\psi$ iff $w \not\models \psi$,
- $w \models \psi_1 \wedge \psi_2$ iff $w \models \psi_1$ and $w \models \psi_2$,
- $w \models \langle \phi \rangle \psi$ iff $w \models \phi$ and $w_{\geq 1} \models \psi$,
- $w \models \langle \psi_1? \rangle \psi_2$ iff $w \models \psi_1$ and $w \models \psi_2$,
- $w \models \langle \theta_1 + \theta_2 \rangle \psi$ iff $w \models \langle \theta_1 \rangle \psi$ or $w \models \langle \theta_2 \rangle \psi$,
- $w \models \langle \theta_1; \theta_2 \rangle \psi$ iff $w = uv$, $u \models \langle \theta_1 \rangle true$, and $v \models \langle \theta_2 \rangle \psi$,
- $w \models \langle \theta^+ \rangle \psi$ iff there exists n with $1 \leq n \leq |w|$ such that $w \models \langle \theta^n \rangle \psi$,

where θ^n , $n \geq 1$ is defined inductively by $\theta^1 = \theta$ and $\theta^n = \theta^{n-1}; \theta$ for $n > 1$.

We let $false = \neg true$. For an LDL formula ψ , we let $L(\psi) = \{w \in A^* \mid w \models \psi\}$, the language defined by ψ . A language $L \subseteq A^*$ is called *LDL-definable* if there is an LDL formula ψ such that $L = L(\psi)$.

Theorem 3 [22] A language $L \subseteq A^*$ is LDL-definable iff L is rational.

Definition 4 The syntax of formulas ϕ of the weighted LDL over A and K is given by the grammar

$$\begin{aligned}\phi &::= k \mid \psi \mid \phi \oplus \phi \mid \phi \otimes \phi \mid \langle \rho \rangle \phi \\ \rho &::= \phi \mid \phi? \mid \rho \oplus \rho \mid \rho \cdot \rho \mid \rho^\oplus\end{aligned}$$

where $k \in K$, ϕ denotes a propositional formula over the atomic propositions in P , and ψ denotes an LDL formula as in Definition 2.

We denote by $LDL(K, A)$ the set of all weighted LDL formulas ϕ over A and K . We represent the semantics $\|\phi\|$ of formulas $\phi \in LDL(K, A)$ as series in $K \langle\langle A^* \rangle\rangle$. For the semantics of LDL formulas ψ we use the satisfaction relation as defined above.

Definition 5 Let $\phi \in LDL(K, A)$. The semantics of ϕ is a series $\|\phi\| \in K \langle\langle A^* \rangle\rangle$. For every $w \in A^*$ the value $\|\phi\|(w)$ is defined inductively as follows:

$$\begin{aligned}\|k\|(w) &= k, & \|\phi_1 \oplus \phi_2\|(w) &= \|\phi_1\|(w) + \|\phi_2\|(w), \\ \|\psi\|(w) &= \begin{cases} 1 & \text{if } w \models \psi \\ 0 & \text{otherwise} \end{cases}, & \|\phi_1 \otimes \phi_2\|(w) &= \|\phi_1\|(w) \cdot \|\phi_2\|(w), \\ \|\langle \phi \rangle \phi\|(w) &= \|\phi\|(w) \cdot \|\phi\|(w_{\geq 1}), & \|\langle \phi_1? \rangle \phi_2\|(w) &= \|\phi_1\|(w) \cdot \|\phi_2\|(w), \\ \|\langle \rho_1 \oplus \rho_2 \rangle \phi\|(w) &= \|\langle \rho_1 \rangle \phi\|(w) + \|\langle \rho_2 \rangle \phi\|(w), & \|\langle \rho^\oplus \rangle \phi\|(w) &= \sum_{n \geq 1} \|\langle \rho^n \rangle \phi\|(w), \\ \|\langle \rho_1 \cdot \rho_2 \rangle \phi\|(w) &= \sum_{w=uv} (\|\langle \rho_1 \rangle true\|(u) \cdot \|\langle \rho_2 \rangle \phi\|(v)),\end{aligned}$$

where for the definition of $\|\langle \rho^\oplus \rangle \phi\|(w)$ we assume that $\|\langle \rho \rangle true\|$ is proper, and ρ^n , $n \geq 1$ is defined inductively by $\rho^1 = \rho$ and $\rho^n = \rho^{n-1} \cdot \rho$ for $n > 1$.

A series $s \in K \langle\langle A^* \rangle\rangle$ is called *LDL-definable* if there is a formula $\varphi \in \text{LDL}(K, A)$ such that $s = \|\varphi\|$. For $K = \mathbb{B}$ (the Boolean semiring) and any $L \subseteq A^*$, clearly L is *LDL-definable* iff $1_L \in \mathbb{B} \langle\langle A^* \rangle\rangle$ is *LDL-definable*, and therefore our weighted *LDL* generalizes *LDL*.

Example 6 We consider the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers, $a \in A$, $k \in \mathbb{N} \setminus \{0\}$, and the weighted *LDL* formula

$$\varphi = \langle \langle \langle \langle (k \otimes p_a) ? \rangle \text{Last} \rangle \cdot \langle \langle (k \otimes p_a) ? \rangle \text{Last} \rangle \rangle^\oplus \text{true} \oplus \bigwedge_{a' \in A} \neg p_{a'},$$

where *Last* denotes the *LDL* formula $\text{Last} ::= \langle \text{true} \rangle \bigwedge_{a' \in A} \neg p_{a'}$. For every $w = a_0 \dots a_{n-1} \in A^*$ and $0 \leq i \leq n-1$ we get

$$w_{\geq i} \models \text{Last} \text{ iff } w_{\geq i+1} \not\models p_{a'} \text{ for every } a' \in A \text{ iff } i = n-1,$$

and we can easily see that $\|\varphi\|(w) = k^{2^n}$ whenever $w = a^{2^n}$ for some $n \geq 0$, and $\|\varphi\|(w) = 0$ otherwise. Furthermore, the series $\|\varphi\|$ is not definable by any weighted *FO* logic sentence (cf. [8]) or weighted *LTL* formula (cf. Section 5). Indeed, let us assume that there is a weighted *FO* logic sentence (resp. *LTL* formula) φ' such that $\|\varphi'\| = \|\varphi\|$. Then, by replacing the non zero weights in φ' with *true* we get an *FO* logic sentence (resp. *LTL* formula) φ'' whose language is $(aa)^*$, which is impossible (cf. [7]).

Next we show that generalized weighted rational expressions can be translated to weighted *LDL* formulas in linear time.

Theorem 7 For every generalized weighted rational expression $E \in \text{GRE}(K, A)$ we can construct, in linear time, a weighted *LDL* formula $\varphi_E \in \text{LDL}(K, A)$ with $\|\varphi_E\| = \|E\|$.

Proof. [Sketch] We proceed by induction on the structure of generalized weighted rational expressions in $\text{GRE}(K, A)$. For this, we define for every $E \in \text{GRE}(K, A)$ the weighted *LDL* formula $\varphi_E \in \text{LDL}(K, A)$ as follows.

- If $E = k\varepsilon$ with $k \in K$, then $\varphi_E = k \otimes \bigwedge_{a \in A} \neg p_a$.
- If $E = ka$ with $k \in K, a \in A$, then $\varphi_E = \langle (k \otimes p_a) ? \rangle \text{Last}$.
- If $E = E_1 + E_2$, then $\varphi_E = \varphi_{E_1} \oplus \varphi_{E_2}$.
- If $E = E_1 \cdot E_2$, then $\varphi_E = \langle \varphi_{E_1} ? \cdot \varphi_{E_2} ? \rangle \text{true}$.
- If $E = E_1^+$, then $\varphi_E = \langle (\varphi_{E_1} ?)^\oplus \rangle \text{true}$.
- If $E = E_1 \odot E_2$, then $\varphi_E = \langle \varphi_{E_1} ? \rangle \varphi_{E_2}$. □

The next theorem shows that also the converse result holds. More precisely, we show that for every $\varphi \in \text{LDL}(K, A)$ we can construct a generalized weighted rational expression $E_\varphi \in \text{GRE}(K, A)$ such that $\|E_\varphi\| = \|\varphi\|$. For this, we first translate every *LDL* formula into a rational expression using Theorem 3. The complexity of an inductive translation would be non-elementary since for every occurrence of a negation symbol we need an exponential complementation construction. However, one can follow the translation of [22, 21] with a doubly exponential construction. We shall need the following lemma.

Lemma 8 Let E be a rational expression over A and $L(E)$ the language defined by E . Then, there is an $E' \in \text{RE}(K, A)$ such that $\|E'\|(w) = 1$ if $w \in L(E)$ and $\|E'\|(w) = 0$ otherwise, for every $w \in A^*$.

Proof. [Sketch] We consider a deterministic automaton for the rational expression E and construct a weighted automaton over A and K , with weights 0 and 1. □

Theorem 9 For every weighted LDL formula $\varphi \in LDL(K, A)$ we can construct a generalized weighted rational expression $E_\varphi \in GRE(K, A)$ such that $\|E_\varphi\| = \|\varphi\|$.

Proof. We proceed by induction on the structure of $LDL(K, A)$ formulas φ . If $\varphi = \psi$ is an LDL formula, then by Theorem 3 it is expressively equivalent to a rational expression E_ψ . Then, by Lemma 8, we can assume that E_ψ is a weighted rational expression in $RE(K, A)$, hence in $GRE(K, A)$, whose semantics gets values 0 and 1 and we get $\|E_\psi\| = \|\psi\|$. Next, assume that $\varphi = k \in K$. It is straightforward that the generalized weighted rational expression $E_\varphi = k\varepsilon + k\varepsilon \cdot (1A)^+$, where $1A = \sum_{a \in A} a$, satisfies our claim. If $\varphi = \varphi_1 \oplus \varphi_2$ or $\varphi = \varphi_1 \otimes \varphi_2$, then we get our result by the induction hypothesis and the closure of generalized weighted rational expressions under sum and Hadamard product, respectively. Now assume that $\varphi = \langle \phi \rangle \varphi'$. By the induction hypothesis there are $E_\phi, E_{\varphi'} \in GRE(K, A)$ such that $\|E_\phi\| = \|\phi\|$ and $\|E_{\varphi'}\| = \|\varphi'\|$. We let $E_\varphi = E_\phi \odot (1A \cdot E_{\varphi'})$ and we get

$$\begin{aligned} \|E_\varphi\|(w) &= \|E_\phi\|(w) \cdot \|1A \cdot E_{\varphi'}\|(w) \\ &= \|E_\phi\|(w) \cdot \|1A\|(w(0)) \cdot \|E_{\varphi'}\|(w_{\geq 1}) \\ &= \|E_\phi\|(w) \cdot \|E_{\varphi'}\|(w_{\geq 1}) \\ &= \|\phi\|(w) \cdot \|\varphi'\|(w_{\geq 1}) \\ &= \|\langle \phi \rangle \varphi'\|(w) \end{aligned}$$

for every $w \in A^*$, hence $\|E_\varphi\| = \|\varphi\|$.

If $\varphi = \langle \varphi_1 ? \rangle \varphi_2$ or $\varphi = \langle \rho_1 \oplus \rho_2 \rangle \varphi'$ or $\varphi = \langle \rho_1 \cdot \rho_2 \rangle \varphi'$, then our claim holds true by the induction hypothesis and the closure of the class $GRE(K, A)$ under Hadamard product, sum, and Cauchy product, respectively. Finally, let $\varphi = \langle \rho^\oplus \rangle \varphi'$ and assume that $\|\varphi\|$ is defined and there are generalized weighted rational expressions E_1, E_2 such that $\|E_1\| = \|\langle \rho \rangle true\|$, which is proper, and $\|E_2\| = \|\langle \rho \rangle \varphi'\|$. Then, we let $E_\varphi = E_1^+ \cdot E_2 + E_2$ and for every $w \in A^*$ we get

$$\begin{aligned} \|E_\varphi\|(w) &= \|E_1^+ \cdot E_2 + E_2\|(w) \\ &= \sum_{w=uv, u \neq \varepsilon} (\|E_1^+\|(u) \cdot \|E_2\|(v)) + \|E_2\|(w) \\ &= \sum_{w=uv, u \neq \varepsilon} (\|E_1\|^{++}(u) \cdot \|E_2\|(v)) + \|E_2\|(w) \\ &= \sum_{w=uv, u \neq \varepsilon} (\|\langle \rho \rangle true\|^{++}(u) \cdot \|\langle \rho \rangle \varphi'\|(v)) + \|\langle \rho \rangle \varphi'\|(w) \\ &= \sum_{w=uv, u \neq \varepsilon} \sum_{m \geq 1} (\|\langle \rho \rangle true\|^m(u) \cdot \|\langle \rho \rangle \varphi'\|(v)) + \|\langle \rho \rangle \varphi'\|(w) \\ &= \sum_{n \geq 2} \|\langle \rho^n \rangle \varphi'\|(w) + \|\langle \rho \rangle \varphi'\|(w) \\ &= \sum_{n \geq 1} \|\langle \rho^n \rangle \varphi'\|(w) \\ &= \|\langle \rho^\oplus \rangle \varphi'\|(w), \end{aligned}$$

i.e., $\|E_\varphi\| = \|\langle \rho^\oplus \rangle \varphi'\|$ which concludes our proof. \square

By Theorems 7 and 9 we get our first main result.

Theorem 10 Let K be a semiring and A an alphabet. Then a series $s \in K \langle\langle A^* \rangle\rangle$ is LDL-definable iff it is g -rational.

By Theorem 10 and the discussion following Theorem 1, we immediately obtain the following consequence.

Corollary 11 *Let K be a commutative semiring and A an alphabet. A series $s \in K\langle\langle A^* \rangle\rangle$ is LDL-definable iff it is recognizable.*

The next proposition describes a doubly exponential translation of a weighted *LDL* formula to an expressively equivalent weighted automaton.

Proposition 12 *Let K be a commutative semiring and A an alphabet. For every weighted *LDL* formula φ we can construct, in doubly exponential time, a weighted automaton \mathcal{A}_φ such that $\|\mathcal{A}_\varphi\| = \|\varphi\|$.*

Proof. If φ is an *LDL* formula, then by [22, 21] we get a deterministic finite automaton accepting the language of φ which trivially can be considered as a weighted automaton with weights 0 and 1. Then, by applying structural induction on φ we prove our claim by well-known constructions on weighted automata (cf. [12]). More precisely, for the closure under sum we take the disjoint union of two weighted automata and for Hadamard product the product automaton. For the closure under Cauchy product we firstly construct the corresponding normalized weighted automata with one initial and final state respectively, and then identify the final state of the first automaton with the initial state of the second automaton. Finally for the plus-iteration, we get firstly the normalized weighted automaton and extend it with a copy of it. Then, we identify the final state of the original automaton with the copy states corresponding to the initial state and final state. The new automaton has the same initial state and the merging one as its final state. Since the translation of an *LDL* formula to a deterministic finite automaton is doubly exponential [22, 21] and the aforementioned constructions on weighted automata are polynomial, we obtain a doubly exponential translation of weighted *LDL* formulas to weighted automata. \square

The construction of the weighted automaton, as described in the above proposition, is not possible for any semiring, since, as is known [4], there are non-commutative semirings K and g -rational series $s \in K\langle\langle A^* \rangle\rangle$ which are not recognizable. On the other hand, it is well-known [11] that the equivalence of weighted automata is decidable whenever the weight structure is a computable field. More interestingly the complexity of checking the equivalence is cubic. Therefore, we get the third main result of our paper.

Theorem 13 *Let K be a computable field and A an alphabet. Then, for every $\varphi, \varphi' \in \text{LDL}(K, A)$ the equality $\|\varphi\| = \|\varphi'\|$ is decidable in doubly exponential time.*

Corollary 14 *Let K be a computable field, A an alphabet, and $k \in K$. Then, for every $\varphi \in \text{LDL}(K, A)$ the equality $\|\varphi\| = \tilde{k}$ is decidable in doubly exponential time.*

Remark 15 *If K is an idempotent commutative semiring, then for every weighted *LDL* formula φ we can construct a weighted automaton \mathcal{A}_φ such that $\|\mathcal{A}_\varphi\| = \|\varphi\|$ in exponential time. Indeed, if φ is an *LDL* formula, then by [22, 21] in exponential time we get a nondeterministic finite automaton accepting the language of φ , which, since K is idempotent, can be considered as a weighted automaton with weights 0 and 1. Then proceed as before. In particular, if K is a bounded distributive lattice, the equivalence of two weighted automata over A and K and hence of two weighted *LDL*(K, A) formulas is again decidable [33].*

4 Weighted linear dynamic logic on infinite words

In this section we interpret weighted *LDL* formulas over infinite words. For this, we need our semiring to be equipped with infinite sums and products. More precisely, we assume that the semiring K is equipped, for every index set I , with an infinitary sum operation $\sum_I : K^I \rightarrow K$ such that for every family $(k_i \mid i \in I)$ of elements of K and $k \in K$ we have

$$\begin{aligned} \sum_{i \in \emptyset} k_i &= 0, & \sum_{i \in \{j\}} k_i &= k_j, & \sum_{i \in \{j,l\}} k_i &= k_j + k_l \text{ for } j \neq l, \\ \sum_{j \in J} \left(\sum_{i \in I_j} k_i \right) &= \sum_{i \in I} k_i, & \text{if } \bigcup_{j \in J} I_j &= I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j', \\ \sum_{i \in I} (k \cdot k_i) &= k \cdot \left(\sum_{i \in I} k_i \right), & \sum_{i \in I} (k_i \cdot k) &= \left(\sum_{i \in I} k_i \right) \cdot k. \end{aligned}$$

Then the semiring K together with the operations \sum_I is called *complete* [16, 24].

A complete semiring is said to be *totally complete* [17], if it is endowed with a countably infinite product operation satisfying for every sequence $(k_i \mid i \geq 0)$ of elements of K the subsequent conditions:

$$\prod_{i \geq 0} 1 = 1, \quad \prod_{i \geq 0} k_i = \prod_{i \geq 0} k'_i, \quad k_0 \cdot \prod_{i \geq 0} k_{i+1} = \prod_{i \geq 0} k_i, \quad \prod_{j \geq 1} \sum_{i \in I_j} k_i = \sum_{(i_1, i_2, \dots) \in I_1 \times I_2 \times \dots} \prod_{j \geq 1} k_{i_j},$$

where in the second equation $k'_0 = k_0 \cdot \dots \cdot k_{n_1}$, $k'_1 = k_{n_1+1} \cdot \dots \cdot k_{n_2}$, ... for any increasing sequence $0 < n_1 < n_2 < \dots$, and in the last equation I_1, I_2, \dots are arbitrary index sets.

Furthermore, we will call a totally complete semiring K *totally commutative complete* if it satisfies the equation:

$$\prod_{i \geq 0} (k_i \cdot k'_i) = \left(\prod_{i \geq 0} k_i \right) \cdot \left(\prod_{i \geq 0} k'_i \right).$$

Obviously a totally commutative complete semiring is commutative. We refer the reader to [11, 16, 24] for examples of complete semirings. Throughout this section we assume K to be a totally complete semiring. An *infinitary series* (or simply *series*) over A^ω and K is a mapping $s : A^\omega \rightarrow K$. We denote by $K \langle\langle A^\omega \rangle\rangle$ the class of all series over A^ω and K . The sum, the products with scalars, and the Hadamard product of series in $K \langle\langle A^\omega \rangle\rangle$ are defined elementwise as for series on finite words. The structure $(K \langle\langle A^\omega \rangle\rangle, +, \odot, \tilde{0}, \tilde{1})$ of infinitary series over A and K is a totally complete semiring. Next let $s \in K \langle\langle A^* \rangle\rangle$ and $r \in K \langle\langle A^\omega \rangle\rangle$. The *Cauchy product* $s \cdot r \in K \langle\langle A^\omega \rangle\rangle$ is determined by $(s \cdot r)(w) = \sum_{w=uv, u \in A^*} s(u)r(v)$ for every $w \in A^\omega$. Finally, the ω -iteration $s^\omega \in K \langle\langle A^\omega \rangle\rangle$ of a proper series $s \in K \langle\langle A^* \rangle\rangle$ is defined by $s^\omega(w) = \sum_{w=w_0 w_1 \dots} \prod_{i \geq 0} s(w_i)$.

Next, we recall *weighted ω -rational expressions over A and K* which are defined by the grammar $E ::= E + E \mid F \cdot E \mid F^\omega$ where F is any weighted rational expression. We denote by $\omega\text{-RE}(K, A)$ the class of all such weighted ω -rational expressions over A and K . Similarly we define the class of *generalized weighted ω -rational expressions over A and K* which is given by the grammar $E ::= E + E \mid F \cdot E \mid F^\omega \mid E \odot E$, where F is any generalized weighted rational expression. We shall denote by $\omega\text{-GRE}(K, A)$ the class of generalized weighted ω -rational expressions over A and K . The *semantics* of a (generalized) weighted ω -rational expression E is a series $\|E\| \in K \langle\langle A^\omega \rangle\rangle$ which is defined inductively by $\|E + E'\| = \|E\| + \|E'\|$, $\|F \cdot E\| = \|F\| \cdot \|E\|$, $\|F^\omega\| = \|F\|^\omega$ (if $\|F\|$ is proper; otherwise undefined), $\|E \odot E'\| = \|E\| \odot \|E'\|$. A series $s \in K \langle\langle A^\omega \rangle\rangle$ is called ω -rational (resp. *g- ω -rational*) if there is a weighted (resp. generalized weighted) ω -rational expression E such that $s = \|E\|$. The subsequent result states the

coincidence of ω -rational and ω -recognizable series, i.e., infinitary series accepted by weighted automata over infinite words. For the theory on weighted automata over infinite words we refer the reader to [18, 9].

Theorem 16 [18] *Let K be a totally complete semiring and A an alphabet. Then a series $s \in K \langle\langle A^\omega \rangle\rangle$ is ω -rational iff it is ω -recognizable.*

It is well-known (cf. [9]) that if the semiring K is totally commutative complete, then the class of ω -recognizable series over A and K is closed under Hadamard product. Consequently, if K is totally commutative complete, then a series $s \in K \langle\langle A^\omega \rangle\rangle$ is g - ω -rational iff it is ω -recognizable.

We shall need to extend the syntax of *LDL* formulas and weighted *LDL* formulas as follows.

Definition 17 [38] *The syntax of formulas ξ of the LDL over A , interpreted over infinite words, is given by the grammar*

$$\begin{aligned} \xi &::= \text{true} \mid p_a \mid \neg \xi \mid \xi \wedge \xi \mid \langle \eta \rangle \xi \\ \eta &::= \phi \mid \xi? \mid \eta + \eta \mid \theta; \eta \mid \theta^\omega \end{aligned}$$

where $p_a \in P$, ϕ denotes a propositional formula over the atomic propositions in P , and θ denotes an expression as in Definition 2.

For every *LDL* formula ξ and $w \in A^\omega$ we define the satisfaction relation $w \models \xi$, inductively on the structure of ξ , as follows:

- $w \models \text{true}$,
- $w \models p_a$ iff $w(0) = a$,
- $w \models \neg \xi$ iff $w \not\models \xi$,
- $w \models \xi_1 \wedge \xi_2$ iff $w \models \xi_1$ and $w \models \xi_2$,
- $w \models \langle \phi \rangle \xi$ iff $w \models \phi$ and $w_{\geq 1} \models \xi$,
- $w \models \langle \xi_1? \rangle \xi_2$ iff $w \models \xi_1$ and $w \models \xi_2$,
- $w \models \langle \eta_1 + \eta_2 \rangle \xi$ iff $w \models \langle \eta_1 \rangle \xi$ or $w \models \langle \eta_2 \rangle \xi$,
- $w \models \langle \theta; \eta \rangle \xi$ iff $w = uv$ with $u \in A^*$, $u \models \langle \theta \rangle \text{true}$, and $v \models \langle \eta \rangle \xi$,
- $w \models \langle \theta^\omega \rangle \xi$ iff $\xi = \text{true}$, $w = w_0 w_1 \dots$, and $w_i \models \langle \theta \rangle \text{true}$ for every $i \geq 0$.

For an *LDL* formula ξ , we let $L_\omega(\xi) = \{w \in A^\omega \mid w \models \xi\}$, the infinitary language defined by ξ . An infinitary language $L \subseteq A^\omega$ is called *LDL- ω -definable* if there is an *LDL* formula ξ such that $L = L_\omega(\xi)$. The coincidence of ω -rational and *LDL- ω -definable* languages was stated in [38].

Theorem 18 [38] *A language $L \subseteq A^\omega$ is LDL- ω -definable iff L is ω -rational.*

Next we introduce the syntax of the weighted *LDL* formulas interpreted over infinite words.

Definition 19 *The syntax of formulas ζ of the weighted LDL over A and K , interpreted over infinite words, is given by the grammar*

$$\begin{aligned} \zeta &::= k \mid \xi \mid \zeta \oplus \zeta \mid \zeta \otimes \zeta \mid \langle \pi \rangle \zeta \\ \pi &::= \phi \mid \zeta? \mid \pi \oplus \pi \mid \rho \cdot \pi \mid \rho^\omega \end{aligned}$$

where $k \in K$, $p_a \in P$, ϕ denotes a propositional formula over the atomic propositions in P , ξ denotes an *LDL* formula as in Definition 17, and ρ an expression as in Definition 4.

We denote by $LDL_\omega(K, A)$ the set of all weighted LDL formulas ζ over A and K . We represent the semantics $\|\zeta\|_\omega$ of formulas $\zeta \in LDL_\omega(K, A)$ as series in $K \langle\langle A^\omega \rangle\rangle$. For the semantics of LDL formulas ξ interpreted over infinite words, we use the satisfaction relation \models as defined above.

Definition 20 Let $\zeta \in LDL_\omega(K, A)$. The semantics of ζ is a series $\|\zeta\|_\omega \in K \langle\langle A^\omega \rangle\rangle$. For every $w \in A^\omega$ the value $\|\zeta\|_\omega(w)$ is defined inductively as follows:

$$\begin{aligned} \|\mathbf{k}\|_\omega(w) &= k, & \|\zeta_1 \oplus \zeta_2\|_\omega(w) &= \|\zeta_1\|_\omega(w) + \|\zeta_2\|_\omega(w), \\ \|\xi\|_\omega(w) &= \begin{cases} 1 & \text{if } w \models \xi \\ 0 & \text{otherwise} \end{cases}, & \|\zeta_1 \otimes \zeta_2\|_\omega(w) &= \|\zeta_1\|_\omega(w) \cdot \|\zeta_2\|_\omega(w), \\ \|\langle \phi \rangle \zeta\|_\omega(w) &= \|\phi\|_\omega(w) \cdot \|\zeta\|_\omega(w_{\geq 1}), & \|\langle \zeta_1 ? \rangle \zeta_2\|_\omega(w) &= \|\zeta_1\|_\omega(w) \cdot \|\zeta_2\|_\omega(w), \\ \|\langle \pi_1 \oplus \pi_2 \rangle \zeta\|_\omega(w) &= \|\langle \pi_1 \rangle \zeta\|_\omega(w) + \|\langle \pi_2 \rangle \zeta\|_\omega(w), \\ \|\langle \rho \cdot \pi \rangle \zeta\|_\omega(w) &= \sum_{w=uv, u \in A^*} (\|\langle \rho \rangle \text{true}\|_\omega(u) \cdot \|\langle \pi \rangle \zeta\|_\omega(v)), \\ \|\langle \rho^\omega \rangle \zeta\|_\omega(w) &= \begin{cases} \sum_{w=w_0 w_1 \dots} \prod_{i \geq 0} \|\langle \rho \rangle \text{true}\|_\omega(w_i) & \text{if } \zeta = \text{true} \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where for the definition of $\|\langle \rho^\omega \rangle \zeta\|_\omega(w)$ we assume that $\|\langle \rho \rangle \text{true}\|_\omega$ is proper.

A series $s \in K \langle\langle A^\omega \rangle\rangle$ is called LDL - ω -definable if there is a formula $\zeta \in LDL_\omega(K, A)$ such that $s = \|\zeta\|_\omega$. For $K = \mathbb{B}$ and any $L \subseteq A^\omega$, clearly L is LDL - ω -definable iff $1_L \in \mathbb{B} \langle\langle A^\omega \rangle\rangle$ is LDL - ω -definable, and therefore our weighted LDL generalizes LDL over infinite words.

Example 21 Let $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ be the totally complete semiring of extended natural numbers, $A = \{a, b\}$, and $k \in \mathbb{N} \setminus \{0\}$. We consider the LDL formula $\psi_1 = \langle\langle (p_b ?) \text{Last} \rangle\rangle^\oplus \text{true} \vee (\neg p_a \wedge \neg p_b)$, the weighted LDL formula $\psi_2 = \langle\langle k \otimes p_a \rangle\rangle \text{Last}$, and we let

$$\zeta = \langle\langle (\psi_1 ? \cdot \psi_2 ? \cdot \psi_1 ? \cdot \psi_2 ?)^\oplus \oplus (\neg p_a \wedge \neg p_b) ? \cdot (\langle\langle p_b ? \rangle\rangle \text{Last} ?)^\omega \rangle\rangle \text{true}.$$

By a standard computation we can show that for every $w \in A^\omega$ we get $\|\zeta\|_\omega(w) = k^{|w|_a}$ whenever $|w|_a < \infty$ and it is even, and $\|\zeta\|_\omega(w) = 0$ otherwise. Furthermore, since the infinitary language $L = \{w \in A^\omega \mid w \text{ contains an even number of } a\text{'s}\}$ is not ω -star-free (cf. [32]), with a similar argument as in Example 6, we can show that the series $\|\zeta\|_\omega$ is not ω -definable by any weighted FO logic sentence (resp. LTL formula) (cf. Section 5 and [28, 30]) over the extended naturals.

The next theorem states that every generalized weighted ω -rational expression can be translated to a weighted LDL formula in linear time. The proof is done by induction on the structure of generalized weighted ω -rational expressions, as in the proof of Theorem 7.

Theorem 22 For every generalized weighted ω -rational expression $E \in \omega\text{-GRE}(K, A)$ we can construct, in linear time, a weighted LDL formula $\zeta_E \in LDL_\omega(K, A)$ with $\|\zeta_E\|_\omega = \|E\|_\omega$.

In the sequel, we show that also the converse result holds. For this, we need the subsequent lemma.

Lemma 23 Let E be an ω -rational expression over A and $L(E)$ the language defined by E . Then, there is an $E' \in \omega\text{-RE}(K, A)$ such that $\|E'\|_\omega(w) = 1$ if $w \in L(E)$ and $\|E'\|_\omega(w) = 0$ otherwise, for every $w \in A^\omega$.

Theorem 24 For every weighted LDL formula $\zeta \in LDL_\omega(K, A)$ we can construct a generalized weighted ω -rational expression $E_\zeta \in \omega\text{-GRE}(K, A)$ such that $\|E_\zeta\|_\omega = \|\zeta\|_\omega$.

Proof. [Sketch] By induction on the structure of $LDL_\omega(K, A)$ formulas ζ , using similar arguments as the ones in the proof of Theorem 9. More precisely, if $\zeta = \xi$ is an LDL formula, then we use Lemma 23. For the induction steps, we use the closure of generalized weighted ω -rational expressions under sum, Hadamard and Cauchy products, and ω -iteration. \square

By Theorems 22 and 24 we get the fourth main result of our paper.

Theorem 25 *Let K be a totally complete semiring and A an alphabet. Then a series $s \in K \langle\langle A^\omega \rangle\rangle$ is LDL - ω -definable iff it is g - ω -rational.*

By Theorem 25 and the discussion following Theorem 16 we get the subsequent corollary.

Corollary 26 *Let K be a totally commutative complete semiring and A an alphabet. A series $s \in K \langle\langle A^\omega \rangle\rangle$ is LDL - ω -definable iff it is ω -recognizable.*

Proposition 27 *Let K be an idempotent totally commutative complete semiring and A an alphabet. For every weighted LDL formula ζ we can construct, in exponential time, a weighted Büchi automaton \mathcal{A}_ζ such that $\|\mathcal{A}_\zeta\| = \|\zeta\|_\omega$.*

Proof. If ζ is an LDL formula, then it is an $PLDL$ (*parametric linear dynamic logic*) formula and, by [19] we get in exponential time a nondeterministic Büchi automaton accepting the language of ζ . This automaton can be considered as a weighted Büchi automaton with weights 0 and 1. Then, by applying structural induction on ζ we prove our claim by standard constructions on weighted Büchi automata. More precisely, for the closure under sum we take the disjoint union of two weighted Büchi automata. For Hadamard product we use the well-known product construction for Büchi automata, showing the closure of the class of ω -recognizable languages under intersection [37], reasonably translated to weighted setup. For the closure under Cauchy product we construct the corresponding normalized weighted automaton and initial weight normalized weighted Büchi automaton, and then identify the final state of the first automaton with the initial state of the second automaton. Finally, for the ω -iteration, we again get the normalized weighted automaton and identify its initial and final state. All the aforementioned constructions are polynomial, and our proof is completed. \square

In particular, if K is a bounded distributive lattice, the equivalence of two weighted automata over A and K on infinite words and hence of two $LDL(K, A)$ formulas is again decidable [10].

5 Comparison of weighted LDL to other weighted logics

In this last section we state the relation of our weighted LDL to weighted monadic second-order logic (weighted MSO logic for short), weighted linear temporal logic (weighted LTL for short) and weighted μ -calculus. The relation of LDL -definable series (resp. infinitary series) to weighted MSO logic definable series (resp. infinitary series) is immediately derived by [8, 9] and Corollary 11 (resp. by [9] and Corollary 26). We get the following consequences.

Corollary 28 *Let K be a commutative semiring and A an alphabet. A series $s \in K \langle\langle A^* \rangle\rangle$ is LDL -definable iff it is definable by a restricted weighted MSO logic sentence over A and K .*

Corollary 29 *Let K be a totally commutative complete semiring and A an alphabet. A series $s \in K \langle\langle A^\omega \rangle\rangle$ is LDL - ω -definable iff it is definable by a restricted weighted MSO logic sentence over A and K interpreted over infinite words.*

Weighted *LTL* has been investigated over De Morgan algebras [25], arbitrary bounded lattices [15], idempotent zero-divisor free totally commutative complete semirings [28, 30], with averaging modalities [5], with discounting over the interval $[0, 1]$ [1, 2], and with discounting over the max-plus semiring [28, 29]. Recently, a type of weighted *LTL* has been applied to robotics [26]. We need to recall first the classical *LTL* (cf. [3]). For every letter $a \in A$ we consider an atomic proposition p_a and we let $P = \{p_a \mid a \in A\}$. The syntax of *LTL* formulas over A is given by the grammar $\phi ::= \text{true} \mid p_a \mid \neg\phi \mid \phi \vee \psi \mid \bigcirc\phi \mid \phi U \psi$ where $p_a \in P$. Let ϕ be an *LTL* formula over A . For every $w = a_0 \dots a_{n-1} \in A^*$ and $0 \leq i \leq n-1$ (resp. $w = a_0 a_1 \dots \in A^\omega$ and $i \geq 0$) the satisfaction relation $w, i \models \phi$ is defined as usual (cf. for instance [3, 7]) by induction on the structure of ϕ .

The syntax of formulas φ of the *weighted LTL over A and K* is given by the grammar

$$\varphi ::= k \mid \phi \mid \varphi \oplus \psi \mid \varphi \otimes \psi \mid \bigcirc\varphi \mid \varphi \mathcal{U} \psi \mid \boxtimes\varphi$$

where $k \in K$, $p_a \in P$, and ϕ is an *LTL* formula over A .

We denote by $LTL(K, A)$ the class of all weighted *LTL* formulas φ over A and K . Firstly, we represent the semantics $\|\varphi\|$ of formulas $\varphi \in LTL(K, A)$ as series in $K \langle\langle A^* \rangle\rangle$. For the semantics of *LTL* formulas ϕ we use the satisfaction relation as defined above.

Definition 30 Let $\varphi \in LTL(K, A)$. The semantics of φ is a series $\|\varphi\| \in K \langle\langle A^* \rangle\rangle$. For every $w \in A^*$, with $|w| = n$ ($n \geq 0$), the value $\|\varphi\|(w)$ is defined inductively as follows:

$$\begin{aligned} \|k\|(w) &= k, & \|\varphi \oplus \psi\|(w) &= \|\varphi\|(w) + \|\psi\|(w), \\ \|\phi\|(w) &= \begin{cases} 1 & \text{if } w \models \phi \\ 0 & \text{otherwise} \end{cases}, & \|\varphi \otimes \psi\|(w) &= \|\varphi\|(w) \cdot \|\psi\|(w), \\ \|\bigcirc\varphi\|(w) &= \|\varphi\|(w_{\geq 1}), & \|\boxtimes\varphi\|(w) &= \prod_{0 \leq i \leq n-1} \|\varphi\|(w_{\geq i}), \\ \|\varphi \mathcal{U} \psi\|(w) &= \sum_{0 \leq i \leq n-1} \left(\left(\prod_{0 \leq j < i} \|\varphi\|(w_{\geq j}) \right) \cdot \|\psi\|(w_{\geq i}) \right). \end{aligned}$$

A series $s \in K \langle\langle A^* \rangle\rangle$ is called *LTL-definable* if there is a formula $\varphi \in LTL(K, A)$ such that $s = \|\varphi\|$.

Example 31 We consider the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers and the *LTL* formulas $\varphi = \boxtimes 2$ and $\psi = \boxtimes \varphi$. Then, we can easily see that for every $w \in A^*$, we get $\|\varphi\|(w) = 2^{|w|}$ and $\|\psi\|(w) = 2^{2^{|w|}}$. It is well known (cf. Ex. 3.4 in [8]) that the series $\|\psi\|$ is not recognizable, and hence by Corollary 11 not *LDL-definable*.

By Examples 6 and 31 we immediately obtain the following proposition.

Proposition 32 The classes of *LDL-definable* and *LTL-definable* series over the semiring of natural numbers and an alphabet A are incomparable.

Next, we represent the semantics of formulas in $LTL(K, A)$ as infinitary series in $K \langle\langle A^\omega \rangle\rangle$.

Definition 33 Let K be a totally complete semiring and $\varphi \in LTL(K, A)$. The semantics of φ over infinite words is an infinitary series $\|\varphi\|_\omega \in K \langle\langle A^\omega \rangle\rangle$. For every $w \in A^\omega$ the value $\|\varphi\|_\omega(w)$ is defined inductively as in the case of finite words except for the operators \mathcal{U} and \boxtimes :

$$\begin{aligned} \|\varphi \mathcal{U} \psi\|_\omega(w) &= \sum_{i \geq 0} \left(\left(\prod_{0 \leq j < i} \|\varphi\|_\omega(w_{\geq j}) \right) \cdot \|\psi\|_\omega(w_{\geq i}) \right), \\ \|\boxtimes\varphi\|_\omega(w) &= \prod_{i \geq 0} \|\varphi\|_\omega(w_{\geq i}). \end{aligned}$$

A series $s \in K \langle\langle A^\omega \rangle\rangle$ is called *LTL- ω -definable* if there is a formula $\varphi \in LTL(K, A)$ such that $s = \|\varphi\|_\omega$. In view of Proposition 32, we define a fragment of our weighted *LTL*, and show that the class of series (resp. infinitary series) defined by *LTL* formulas in this fragment is in the class of *LDL*-definable (resp. *LDL- ω -definable*) ones. More precisely, an *LTL-step formula* is an *LTL*(K, A) formula of the form $\bigoplus_{1 \leq i \leq n} (k_i \otimes \varphi_i)$ where $k_i \in K$ and φ_i is an *LTL* formula for every $1 \leq i \leq n$. Then, we call a formula $\varphi \in LTL(K, A)$ *restricted* if whenever it contains a subformula of the form $\boxtimes \psi$ or $\psi \mathcal{U} \xi$, then ψ is an *LTL-step formula*. We shall denote by $rLTL(K, A)$ the set of all restricted *LTL*(K, A) formulas. A series $s \in K \langle\langle A^* \rangle\rangle$ (resp. $s \in K \langle\langle A^\omega \rangle\rangle$) is called *rLTL-definable* (resp. *rLTL- ω -definable*) if there is a formula $\varphi \in rLTL(K, A)$ such that $s = \|\varphi\|$ (resp. $s = \|\varphi\|_\omega$). By an inductive construction, we can show that every *rLTL*-definable (resp. *rLTL- ω -definable*) series is also definable (resp. ω -definable) by a restricted weighted *FO* logic sentence in the sense of [8]. Therefore, by Corollaries 28 and 29, we get respectively, the subsequent results.

Theorem 34 *Let K be a commutative semiring and A an alphabet. If a series $s \in K \langle\langle A^* \rangle\rangle$ is *rLTL-definable*, then it is *LDL-definable*.*

Theorem 35 *Let K be a totally commutative complete semiring and A an alphabet. If a series $s \in K \langle\langle A^\omega \rangle\rangle$ is *rLTL- ω -definable*, then it is *LDL- ω -definable*.*

A weighted μ -calculus over a particular class of semirings was investigated in [31] (cf. also [27]). More precisely, the author showed that the class of rational (resp. ω -rational) series over dc-semirings with the Arden fixed point property (resp. with infinite products and the Arden fixed point property) coincides with the class of series (resp. infinitary series) definable by the weighted conjunction-free μ -calculus. Therefore, by Corollaries 11, 26 and Theorem 4.5 in [31], we immediately obtain the following theorem.

Theorem 36 *Let K be a commutative (resp. totally commutative complete) dc-semiring with the Arden fixed point property and A an alphabet. Then a series $s \in K \langle\langle A^* \rangle\rangle$ (resp. $s \in K \langle\langle A^\omega \rangle\rangle$) is *LDL-definable* (resp. *LDL- ω -definable*) iff it is definable by a sentence of the weighted conjunction-free μ -calculus over A and K .*

6 Conclusion

We introduced a weighted linear dynamic logic for finite (resp. infinite) words over arbitrary (resp. totally complete) semirings and proved the expressive equivalence of formulas of this logic with generalized weighted rational (resp. ω -rational) expressions. In our proofs we used structural induction for both directions. We proved also that the translation of any weighted *LDL* formula to a weighted automaton can be done as well, by structural induction, using the corresponding translation of [22, 21] and well-known constructions on weighted automata. More interestingly, for the applications, the time complexity of the translation does not increase in the weighted setup. We recalled the weighted *LTL* and showed that the class of series defined by weighted *LTL* and weighted *LDL* formulas are, in general, incomparable, in contrast to the well known relation for classical logics. We defined a fragment of weighted *LTL*, which is larger than the one in recent works [28, 30], and showed that *LTL*-definable (resp. *LTL- ω -definable*) series in this fragment are also *LDL*-definable (resp. *LDL- ω -definable*). Recent applications require weighted automata (resp. weighted automata with input infinite words) over more general structures than semirings, for instance incorporating average or discounted computations of weights [6, 13, 14]. Therefore, it should be very interesting, especially for applications, to explore the expressive power of a weighted *LDL* over more general weight structures.

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