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Neutrosophic Set Approach for Characterizations of Left Almost Semigroups

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Abstract. In this paper we have defined neutrosophic ideals, neutrosophic interior ideals, netrosophic quasi-ideals and neutrosophic bi-ideals (neutrosophic generalized bi-ideals) and proved some results related to them. Furthermore, we have done some characterization of a neutrosophic LA-semigroup by the properties of its neutrosophic ideals. It has been proved that in a neutrosophic intra-regular LA-semigroup neutrosophic left, right, two-sided, interior, bi-ideal, generalized bi-ideal and quasi-ideals coincide and we have also proved that the set of neutrosophic ideals of a neutrosophic intra-regular LA-semigroup forms a semilattice structure.

Keywords: Neutrosophic LA-semigroup; neutrosophic intra-regular LA-semigroup; neutrosophic left invertive law; neutrosophic ideal.

Introduction

It is well known fact that common models with their limited and restricted boundaries of truth and falsehood are insufficient to detect the reality so there is a need to discover and introduce some other phenomenon that address the daily life problems in a more appropriate way. In different fields of life many problems arise which are full of uncertainties and classical methods are not enough to deal and solve them. In fact, reality of real life problems cannot be represented by models with just crisp assumptions with only yes or no because of such certain assumptions may lead us to completely wrong solutions. To overcome this problem, Lotfi A.Zadeh in 1965 introduced the idea of a fuzzy set which help to describe the behaviour of systems that are too complex or are ill-defined to admit precise mathematical analysis by classical methods. He discovered the relationships of probability and fuzzy set theory which has appropriate approach to deal with uncertainties. According to him every set is not crisp and fuzzy set is one of the example that is not crisp. This fuzzy set help us to reduce the chances of failures in modelling.. Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. Mordeson et al. has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy finite state mechanics and fuzzy languages etc.

Zadeh introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. Atanassov introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set. He has coined the words neutrosophy and neutrosophic. In 2013 he refined the neutrosophic set to n components: $t_1, t_2, \ldots$; $i_1, i_2, \ldots$; $f_1, f_2, \ldots$. The words neutrosophy and neutrosophic were coined/invented by F. Smarandache in his 1998 book. Etymologically, neutro-sophy (noun) [French neutre <Latin neuter, neutral, and Greek sophia, skill/wisdom] means knowledge of neutral thought. While neutrosophic (adjective), means having the nature of, or having the characteristic of Neutrosophy.

Recently, several theories have been presented to dispute with uncertainty, vagueness and imprecision. Theory of probability, fuzzy set theory, intutionistic fuzzy sets, rough set theory etc., are consistently being used as actively operative tools to deal with multiform uncertainties and imprecision enclosed in a system. But all these above theories failed to deal with indeterminate and inconsistent infomation. Therefore, due to the existance of indeterminancy in various world problems, neutrosophy founds its way into the modern research. Neutrosophy was developed in attempt to generalize fuzzy logic. Neutrosophy is a Latin world "neuter" - neutral, Greek "sophia" - skill/wisdom). Neutrosophy is a branch of philosophy, introduced by Florentin Smarandache which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is

the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.
Inpiring from the realities of real life phenomenons like sport games (winning/ tie/ defeating), votes (yes/ NA/ no) and decision making (making a decision/ hesitating/ not making), F. Smrandache introduced a new concept of a neutrosophic set (NS in short) in 1995, which is the generalization of a fuzzy sets and intutionistic fuzzy set. NS is described by membership degree, indeterminate degree and non-membership degree. The idea of NS generates the theory of neutrosophic sets by giving representation to indeterminates. This theory is considered as complete representation of almost every model of all real-world problems. Therefore, if uncertainty is involved in a problem we use fuzzy theory while dealing indeterminacy, we need neutrosophic theory. In fact this theory has several applications in many different fields like control theory, databases, medical diagnosis problem and decision making problems.
Using Neutrosophic theory, Vasantha Kandasmy and Florentin Smarandache introduced the concept of neutrosophic algebraic structures in 2003. Some of the neutrosophic algebraic structures introduced and studied including neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids. Madad Khan et al., for the first time introduced the idea of a neutrosophic AG-groupoid in [13].

1 Preliminaries

Abel-Grassmann's Groupoid (abbreviated as an AG-groupoid or LA-semigroup) was first introduced by Naseeruddin and Kazim in 1972. LA-semigroup is a groupoid S whose elements satisfy the left invertive law $(ab)c=(cb)a$ for all a, b, $c \in S$. LA-semigroup generalizes the concept of commutative semigroups and have an important application within the theory of flocks. In addition to applications, a variety of properties have been studied for AG-groupoids and related structures. An LA-semigroup is a non-associative algebraic structure that is generally considered as a midway between a groupoid and a commutative semigroup but is very close to commutative semigroup because most of their properties are similar to commutative semigroup. Every commutative semigroup is an AG-groupoid but not vice versa. Thus AG-groupoids can also be non-associative, however, they do not necessarily have the Latin square property. An LA-semigroup S can have left identity e (unique) i.e $ea=a$ for all $a \in S$ but it cannot have a right identity because if it has, then S becomes a commutative semigroup. An element s of LA-semigroup S is called idempotent if $s^2=s$ and if holds for all elements of S then S is called idempotent LA-semigroup.
Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. In 2003 W.B Vasantha Kandasamy and Florentin Smarandache introduced algebraic structures (such as neutrosophic semigroup, neutrosophic ring, etc.). Madad Khan et al., for the first time introduced the idea of a neutrosophic LA-semigroup in [Madad Saima]. Moreover $SUI=\{a+bI :$ where a, $b \in S$ and I is literal indeterminacy such that $I^2=I\}$ becomes neutrosophic LA-semigroup under the operation defined as:
$(a+bI)*(c+dI)=ac+bdI$ for all $(a+bI)$, $(c+dI) \in SUI$. That is $(SUI,*)$ becomes neutrosophic LA-semigroup. They represented it by $N(S)$.
$[(a_1+a_2I)(b_1+b_2I)](c_1+c_2I)=[(c_1+c_2I)(b_1+b_2I)](a_1+a_2I),$
holds for all $(a_1+a_2I),$ $(b_1+b_2I),$ $(c_1+c_2I) \in N(S)$.
It is since then called the neutrosophic left invertive law. A neutrosophic groupoid satisfying the left invertive law is called a neutrosophic left almost semigroup and is abbreviated as neutrosophic LA-semigroup.
In a neutrosophic LA-semigroup $N(S)$ medial law holds i.e

$$[(a_1+a_2I)(b_1+b_2I)][(c_1+c_2I)(d_1+d_2I)] = [(a_1+a_2I)(c_1+c_2I)][(b_1+b_2I)(d_1+d_2I)],$$

holds for all (a_1+a_2I), (b_1+b_2I), (c_1+c_2I), $(d_1+d_2I) \in N(S)$.
There can be a unique left identity in a neutrosophic LA-semigroup. In a neutrosophic LA-semigroup $N(S)$ with left identity $(e+eI)$ the following laws hold for all (a_1+a_2I), (b_1+b_2I), (c_1+c_2I), $(d_1+d_2I) \in N(S)$.

$$[(a_1+a_2I)(b_1+b_2I)][(c_1+c_2I)(d_1+d_2I)] = [(d_1+d_2I)(b_1+b_2I)][(c_1+c_2I)(a_1+a_2I)],$$

$$[(a_1+a_2I)(b_1+b_2I)][(c_1+c_2I)(d_1+d_2I)]=[(d_1+d_2I)(c_1+c_2I)][(b_1+b_2I)(a_1+a_2I)],$$

and

$$(a_1+a_2I)[(b_1+b_2I)(c_1+c_2I)]=(b_1+b_2I)[(a_1+a_2I)(c_1+c_2I)].$$

(3) is called neutrosophic paramedial law and a neutrosophic LA semigroup satisfies (5) is called neutrosophic AG**-groupoid.

Now, $(a+bI)^2 = a+bI$ implies $a+bI$ is idempotent and if holds for all $a+bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup.

2 Neutrosophic LA-semigroups

Example 2.1 *Let* $S=\{1,2,3\}$ *with binary operation* $"\cdot"$ *is an LA-semigroup with left identity* 3 *and has the following Calley's table:*

$\cdot$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

then

$N(S)=\{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$

is an example of neutrosophic LA-semigroup under the operation $"*"$ and has the following Callay's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$1+2I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$1+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+1I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$2+2I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$
$2+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+1I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$3+2I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$

It is important to note that if $N(S)$ contains left identity $3+3I$ then $(N(S))^2 = N(S)$.

Lemma 2.1: *If a neutrosophic LA-semigroup* $N(S)$ *contains left identity* $e+Ie$ *then the following conditions hold.*

(i) $N(S)N(L)=N(L)$ *for every neutrosophic left ideal* $N(L)$ *of* $N(S)$.

(ii) $N(R)N(S)=N(R)$ *for every neutrosophic right ideal* $N(R)$ *of* $N(S)$.

Proof (i) Let $N(L)$ be the neutrosophic left ideal of $N(S)$ implies that $N(S)N(L)\subseteq N(L)$. Let $a+bI \in N(L)$ and since $a+bI=(e+eI)(a+bI)\in N(S)N(L)$ which implies that $N(L)\subseteq N(S)N(L)$. Thus $N(L)=N(S)N(L)$.

(ii) Let $N(R)$ be the neutrosophic right ideal of $N(S)$. Then $N(R)N(S)\subseteq N(R)$. Now,let $a+bI\in N(R)$. Then

$$\begin{aligned} a+bI &= (e+eI)(a+bI) \\ &= [(e+eI)(e+eI)](a+bI). \\ &= [(a+bI)(e+eI)](e+eI) \\ &\in (N(R)N(S))N(S) \\ &\subseteq N(R)N(S). \end{aligned}$$

Thus $N(R)\subseteq N(R)N(S)$. Hence $N(R)N(S)=N(R)$.

A subset $N(Q)$ of an neutrosophic LA-semigroup is called neutrosophic quasi-ideal if $N(Q)N(S)\cap N(S)N(Q)\subseteq N(Q)$. A subset $N(I)$ of an LA-semigroup $N(S)$ is called idempotent if $(N(I))^2 = N(I)$.

Lemma 2.2: *The intersection of a neutrosophic left ideal* $N(L)$ *and a neutrosophic right ideal* $N(R)$ *of a neutrosophic LA-semigroup* $N(S)$ *is a neutrosophic quasi-ideal of* $N(S)$.

Proof Let $N(L)$ and $N(R)$ be the neutrosophic left and right ideals of neutrosophic LA-semigroup $N(S)$ resp.

Since $N(L)\cap N(R)\subseteq N(R)$ and $N(L)\cap N(R)\subseteq N(L)$ and $N(S)N(L)\subseteq N(L)$ and $N(R)N(S)\subseteq N(R)$. Thus

$$\begin{aligned} &(N(L)\cap N(R))N(S)\cap N(S)(N(L)\cap N(R)) \\ &\subseteq N(R)N(S)\cap N(S)N(L) \\ &\subseteq N(R)\cap N(L) \\ &= N(L)\cap N(R). \end{aligned}$$

Hence, $N(L)\cap N(R)$ is a neutrosophic quasi-ideal of $N(S)$.

A subset(neutrosophic LA-subsemigroup) $N(B)$ of a neutrosophic LA-semigroup $N(S)$ is called neutrosophic generalized bi-ideal(neutosophic bi-ideal) of $N(S)$ if $(N(B)N(S))N(B)\subseteq N(B)$.

Lemma 2.3: *If* $N(B)$ *is a neutrosophic bi-ideal of a neutrosophic LA-semigroup* $N(S)$ *with left identity* $e+eI$ *, then* $((x_1+Iy_1)N(B))(x_2+Iy_2)$ *is also a neutrosophic bi-ideal of* $N(S)$ *, for any* x_1+Iy_1 *and* x_2+Iy_2 *in* $N(S)$.

Proof Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$, now using (1), (2), (3) and (4), we get

$$
\begin{aligned}
&[\{\{(x_1+y_1I)N(B)\}(x_2+y_2I)\}N(S)][\{(x_1+y_1I)N(B)\}(x_2+y_2I)]\\
&=[\{N(S)(x_2+y_2I)\}\{(x_1+y_1I)N(B)\}][\{(x_1+y_1I)N(B)\}(x_2+y_2I)]\\
&=[\{\{(x_1+y_1I)N(B)\}(x_2+y_2I)\}\{(x_1+y_1I)N(B)\}][N(S)(x_2+y_2I)]\\
&=[\{\{(x_1+y_1I)N(B)\}(x_1+y_1I)\}\{(x_2+y_2I)N(B)\}][N(S)(x_2+y_2I)]\\
&=[\{\{(x_1+y_1I)N(B)\}(x_1+y_1I)\}N(S)][\{(x_2+y_2I)N(B)\}(x_2+y_2I)]\\
&=[\{N(S)(x_1+y_1I)\}\{(x_1+y_1I)N(B)\}][\{(x_2+y_2I)N(B)\}(x_2+y_2I)]\\
&=[\{N(B)(x_1+y_1I)\}\{(x_1+y_1I)N(S)\}][\{(x_2+y_2I)N(B)\}(x_2+y_2I)]\\
&=[\{N(B)(x_1+y_1I)\}\{(x_2+y_2I)N(B)\}][\{(x_1+y_1I)N(S)\}(x_2+y_2I)]\\
&\subseteq[\{N(B)(x_1+y_1I)\}\{x_2+y_2I)N(B)\}]N(S)\\
&=[\{N(B)(x_1+y_1I)\}\{(x_2+y_2I)N(B)\}][(e+eI)N(S)]\\
&=[\{N(B)(x_1+y_1I)\}(e+eI)][\{(x_2+y_2I)N(B)\}N(S)]\\
&=[\{(e+eI)(x_1+y_1I)\}N(B)][\{N(S)N(B)\}(x_2+y_2I)]\\
&=[(x_2+y_2I)\{N(S)N(B)\}][N(B)(x_1+y_1I)]\\
&=[\{(e+eI)(x_2+y_2I)\}(N(S)N(B))][N(B)(x_1+y_1I)]\\
&=[\{N(B)N(S)\}\{(x_2+y_2I)(e+eI)\}][N(B)(x_1+y_1I)]\\
&=[(N(B)N(S))N(B)][\{(x_2+y_2I)(e+eI)\}(x_1+y_1I)]\\
&\subseteq N(B)[\{(x_2+y_2I)(e+eI)\}(x_1+y_1I)]\\
&=[(x_2+y_2I)(e+eI)][N(B)(x_1+y_1I)]\\
&=[(x_1+y_1I)N(B)][(e+eI)(x_2+y_2I]\\
&=[(x_1+y_1I)N(B)](x_2+y_2I).
\end{aligned}
$$

A subset $N(I)$ of a neutrosophic LA-semigroup $N(S)$ is called a neutrosophic interior ideal if $(N(S)N(I))N(S)\subseteq N(I)$.

A subset $N(M)$ of a neutrosophic LA-semigroup $N(S)$ is called a neutrosophic minimal left (right, two sided, interior, quasi- or bi-) ideal if it does not contains any other neutrosophic left (right, two sided, interior, quasi- or bi-) ideal of $N(S)$ other than itself.

Lemma 2.4: *If $N(M)$ is a minimal bi-ideal of $N(S)$ with left identity and $N(B)$ is any arbitrary neutrosophic bi-ideal of $N(S)$, then $N(M)=((x_1+Iy_1)N(B))(x_2+Iy_2)$, for every (x_1+y_1I), $(x_1+y_2I)\in N(M)$.*

Proof Let $N(M)$ be a neutrosophic minimal bi-ideal and $N(B)$ be any neutrosophic bi-ideal of $N(S)$, then by Lemma 2.3, $[(x_1+y_1I)N(B)](x_2+y_2I)$ is a neutrosophic bi-ideal of $N(S)$ for every (x_1+y_1I), $(x_2+y_2I)\in N(S)$. Let (x_1+y_1I), $(x_2+y_2I)\in N(M)$, we have

$$
\begin{aligned}
[(x_1+y_1I)N(B)](x_2+y_2I)&\subseteq[N(M)N(B)]N(M)\\
&\subseteq[N(M)N(S)]N(M)\\
&\subseteq N(M).
\end{aligned}
$$

But $N(M)$ is a neutrosophic minimal bi-ideal, so $[(x_1+y_1I)N(B)](x_{2+}y_2I)=N(M)$.

Lemma 2.5: *In a neutrosophic LA-semigroup $N(S)$ with left identity, every idempotent neutrosophic quasi-ideal is a neutrosophic bi-ideal of $N(S)$.*

Proof *Let $N(Q)$ be an idempotent neutrosophic quasi-ideal of $N(S)$, then clearly $N(Q)$ is a neutrosophic LA-subsemigroup too.*

$$
\begin{aligned}
(N(Q)N(S))N(Q)&\subseteq(N(Q)N(S))N(S)\\
&=(N(S)N(S))N(Q)\\
&=N(S)N(Q),\text{ and}\\
(N(Q)N(S))N(Q)&\subseteq(N(S)N(S))N(Q)\\
&=(N(S)N(S))(N(Q)N(Q))\\
&=(N(Q)N(Q))(N(S)N(S))\\
&=N(Q)N(S).
\end{aligned}
$$

Thus

$$(N(Q)N(S))N(Q)\subseteq(N(Q)N(S))\cap(N(S)N(Q))\subseteq N(Q)$$

. Hence, $N(Q)$ is a neutrosophic bi-ideal of $N(S)$.

Lemma 2.6: *If $N(A)$ is an idempotent neutrosophic quasi-ideal of a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, then $N(A)N(B)$ is a neutrosophic bi-ideal of $N(S)$, where $N(B)$ is any neutrosophic subset of $N(S)$.*

Proof *Let $N(A)$ be the neutrosophic quasi-ideal of $N(S)$ and $N(B)$ be any subset of $N(S)$.*

$$
\begin{aligned}
&((N(A)N(B))N(S))(N(A)N(B))\\
&=((N(S)N(B))N(A))(N(A)N(B))\\
&\subseteq((N(S)N(S))N(A))(N(A)N(B))\\
&=(N(S)N(A))(N(A)N(B))\\
&=(N(B)N(A))(N(A)N(S))\\
&=((N(A)N(S))N(A))N(B)\\
&\subseteq N(A)N(B)
\end{aligned}
$$

Hence $N(A)N(B)$ is neutrosophic bi-ideal of $N(S)$.

Lemma 2.7:*If* $N(L)$ *is a neutrosophic left ideal and* $N(R)$ *is a neutrosophic right ideal of a neutrosophic LA-semigroup* $N(S)$ *with left identity* $e+eI$ *then* $N(L)\cup N(L)N(S)$ *and* $N(R)\cup N(S)N(R)$ *are neutrosophic two sided ideals of* $N(S)$ *.*

Proof Let $N(R)$ be a neutrosophic right ideal of $N(S)$ then by using (3) and (4), we have

$$\begin{aligned}&[N(R)\cup N(S)N(R)]N(S)\\ &= N(R)N(S)\cup[N(S)N(R)]N(S)\\ &\subseteq N(R)\cup[N(S)N(R)][N(S)N(S)]\\ &= N(R)\cup[N(S)N(S)][N(R)N(S)]\\ &= N(R)\cup N(S)[N(R)N(S)]\\ &= N(R)\cup N(R)[N(S)N(S)]\\ &= N(R)\cup N(R)N(S)\\ &= N(R)\subseteq N(R)\cup N(S)N(R).\end{aligned}$$

and

$$\begin{aligned}&N(S)[N(R)\cup N(S)N(R)]\\ &= N(S)N(R)\cup N(S)[N(S)N(R)]\\ &= N(S)N(R)\cup[N(S)N(S)][N(S)N(R)]\\ &= N(S)N(R)\cup[N(R)N(S)][N(S)N(S)]\\ &\subseteq N(S)N(R)\cup N(R)[N(S)N(S)]\\ &= N(S)N(R)\cup N(R)N(S)\\ &\subseteq N(S)N(R)\cup N(R)\\ &= N(R)\cup N(S)N(R).\end{aligned}$$

Hence $[N(R)\cup N(S)N(R)]$ is a neutrosophic two sided ideal of $N(S)$. Similarly we can show that $[N(L)\cup N(S)N(L)]$ is a neutrosophic two-sided ideal of $N(S)$.

Lemma 2.8: *A subset* $N(I)$ *of a neutrosophic LA-semigroup* $N(S)$ *with left identity* $e+eI$ *is a neutrosophic right ideal of* $N(S)$ *if and only if it is a neutrosophic interior ideal of* $N(S)$.

Proof Let $N(I)$ be a neutrosophic right ideal of $N(S)$

$$\begin{aligned}N(S)N(I) &= [N(S)N(S)]N(I)\\ &= [N(I)N(S)]N(S)\\ &\subseteq N(I)N(S)\\ &\subseteq N(I).\end{aligned}$$

So $N(I)$ is a neutrosophic two-sided ideal of $N(S)$, so is a neutrosophic interior ideal of $N(S)$.

Conversely, assume that $N(I)$ is a neutrosophic interior ideal of $N(S)$, then by using (4) and (3), we have

$$\begin{aligned}N(I)N(S) &= N(I)[N(S)N(S)]\\ &= N(S)[N(I)N(S)]\\ &= [N(S)N(S)][N(I)N(S)]\\ &= [N(S)N(I)][N(S)N(S)]\\ &= [N(S)N(I)]N(S)\\ &\subseteq N(I).\end{aligned}$$

If $N(A)$ and $N(M)$ are neutrosophic two-sided ideals of a neutrosophic LA-semigroup $N(S)$, such that $(N(A))^2\subseteq N(M)$ implies $N(A)\subseteq N(M)$, then $N(M)$ is called neutrosophic semiprime.

Theorem 2.1: *In a neutrosophic LA-semigroup* $N(S)$ *with left identity* $e+eI$ *, the following conditions are equivalent.*

(i) *If* $N(A)$ *and* $N(M)$ *are neutrosophic two-sided ideals of* $N(S)$*, then* $(N(A))^2\subseteq N(M)$ *implies* $N(A)\subseteq N(M)$.

(ii) *If* $N(R)$ *is a neutrosophic right ideal of* $N(S)$ *and* $N(M)$ *is a neutrosophic two-sided ideal of* $N(S)$ *then* $(N(R))^2\subseteq N(M)$ *implies* $N(R)\subseteq N(M)$.

(iii) *If* $N(L)$ *is a neutrosophic left ideal of* $N(S)$ *and* $N(M)$ *is a neutrosophic two-sided ideal of* $N(S)$ *then* $(N(L))^2\subseteq N(M)$ *implies* $N(L)\subseteq N(M)$.

Proof $(i)\Rightarrow(iii)$

Let $N(L)$ be a left ideal of $N(S)$ and $[N(L)]^2\subseteq N(M)$, then by Lemma ref: slrs , $N(L)\cup N(L)N(S)$ is a neutrosophic two sided ideal of $N(S)$, therefore by assumption (i), we have $[N(L)\cup N(L)N(S)]^2\subseteq N(M)$ which implies $[N(L)\cup N(L)N(S)]\subseteq N(M)$ which further implies that $N(L)\subseteq N(M)$.

$(iii)\Rightarrow(ii)$ and $(ii)\Rightarrow(i)$ are obvious.

Theorem 2.2: *A neutrosophic left ideal* $N(M)$ *of a neutrosophic LA-semigroup* $N(S)$ *with left identity* $e+eI$ *is neutrosophic quasi semiprime if and only if* $(a_1+b_1I)^2\in N(M)$ *implies* $a_1+b_1I\in N(M)$.

Proof Let $N(M)$ be a neutrosophic semiprime left ideal of $N(S)$ and $(a_1+b_1I)^2 \in N(M)$. Since $N(S)(a_1+b_1I)^2$ is a neutrosophic left ideal of $N(S)$ containing $(a_1+Ib_1)^2$, also $(a_1+b_1I)^2 \in N(M)$, therefore we have $(a_1+b_1I)^2 \in N(S)(a_1+b_1I)^2 \subseteq N(M)$. But by using (2), we have

$$\begin{aligned} N(S)[a_1+b_1I]^2 &= N(S)[(a_1+b_1I)(a_1+b_1I)]. \\ &= [N(S)N(S)][(a_1+b_1I)(a_1+b_1I)] \\ &= [N(S)(a_1+b_1I)][N(S)(a_1+b_1I)] \\ &= [N(S)(a_1+b_1I)]^2. \end{aligned}$$

Therefore, $[N(S)(a_1+b_1I)]^2 \subseteq N(M)$, but $N(M)$ is neutrosophic semiprime ideal so $N(S)(a_1+b_1I) \subseteq N(M)$. Since $(a_1+b_1I) \in N(S)(a_1+b_1I),$ therefore $(a_1+b_1I) \in N(M)$.

Conversely, assume that $N(I)$ is an ideal of $N(S)$ and let $(N(I))^2 \subseteq N(M)$ and $(a_1+b_1I) \in N(I)$ implies that $(a_1+b_1I)^2 \in (N(I))^2$, which implies that $(a_1+b_1I)^2 \in N(M)$ which further implies that $(a_1+b_1I) \in N(M)$. Therefore, $(N(I))^2 \subseteq N(M)$ implies $N(I) \subseteq N(M)$. Hence $N(M)$ is a neutrosophic semiprime ideal.

A neutrosophic LA-semigroup $N(S)$ is called neutrosophic left (right) quasi-regular if every neutrosophic left (right) ideal of $N(S)$ is idempotent.

Theorem 2.3: *A neutrosophic LA-semigroup $N(S)$ with left identity is neutrosophic left quasi-regular if and only if $a+bI \in [N(S)(a+bI)][N(S)(a+bI)]$.*

Proof Let $N(L)$ be any left ideal of $N(S)$ and $a+bI \in [N(S)(a+bI)][N(S)(a+bI)]$. Now for each $l_1+l_2I \in N(L)$, we have

$$\begin{aligned} l_1+l_2I &\in [N(S)(l_1+l_2I)][N(S)(l_1+l_2I)] \\ &\subseteq [N(S)N(L)][N(S)N(L)] \\ &\subseteq N(L)N(L) = (N(L))^2. \end{aligned}$$

Therefore, $N(L) = (N(L))^2$.

Conversely, assume that $N(A) = (N(A))^2$ for every neutrosophic left ideal $N(A)$ of $N(S)$. Since $N(S)(a+bI)$ is a neutrosophic left ideal of $N(S)$. So, $a+bI \in N(S)(a+bI) = [N(S)(a+bI)][N(S)(a+bI)]$.

Theorem 2.4: *The subset $N(I)$ of a neutrosophic left quasi-regular LA-semigroup $N(S)$ is a neutrosophic left ideal of $N(S)$ if and only if it is a neutrosophic right ideal of $N(S)$.*

Proof Let $N(L)$ be a neutrosophic left ideal of $N(S)$ and $s_1+s_2I \in N(S)$ therefore, by Theorem 2.3 and (1), we have

$$\begin{aligned} &(l_1+l_2I)(s_1+s_2I) \\ &= [\{(x_1+x_2I)(l_1+l_2I)\}\{(y_1+y_2I)(l_1+l_2I)\}](s_1+s_2I) \\ &= [\{(s_1+s_2I)\{(y_1+y_2I)(l_1+l_2I)\}\}][(x_1+x_2I)(l_1+l_2I)] \\ &\in [\{N(S)\{N(S)N(L)\}\}][N(S)N(L)] \\ &= [N(S)N(L)][N(S)N(L)] \\ &\subseteq N(L)N(L) = N(L). \end{aligned}$$

Conversely, assume that $N(I)$ is a neutrosophic right ideal of $N(S)$, as $N(S)$ is itself a neytrosophic left ideal and by assumption $N(S)$ is idempotent, therefore by using (2), we have

$$\begin{aligned} N(S)N(I) &= [N(S)N(S)]N(I) \\ &= [N(I)N(S)]N(S) \\ &\subseteq N(I)N(S) \subseteq N(I). \end{aligned}$$

This implies $N(I)$ is neutrosophic left bideal too.

Lemma 2.9: *The intersection of any number of neutrosophic quasi-ideals of $N(S)$ is either empty or quasi-ideal of $N(S)$.*

Proof Let $N(Q_1)$ and $N(Q_2)$ be two netrosophic quasi ideals of neutrosophic LA-semigroup $N(S)$. If $N(Q_1)$ and $N(Q_2)$ are distinct then their intersection must be empty but if not then

$$\begin{aligned} &N(S)[N(Q_1)\cap N(Q_2)]\cap[N(Q_1)\cap N(Q_2)]N(S) \\ &= [N(S)N(Q_1)\cap N(S)N(Q_2)]\cap[N(Q_1)N(S)\cap N(Q_2)N(S)] \\ &= [N(S)N(Q_1)\cap N(Q_1)N(S)]\cap[N(S)N(Q_2)\cap N(Q_2)N(S)] \\ &\subseteq N(Q_1)\cap N(Q_2). \end{aligned}$$

Therefore, $N(Q_1)\cap N(Q_2)$ is a neutrosophic quasi-ideal.

Now, generalizing the result and let

$N(Q_1), N(Q_2), \ldots, N(Q_n)$ be the n-number of neutrosophic quasi ideals of neutrosophic quasi-ideals of $N(S)$ and assume that their intersection is not empty then

$$N(S)[N(Q_1)\cap N(Q_2)\cap\ldots\cap N(Q_n)]\cap[N(Q_1)\cap N(Q_2)\cap\ldots\cap N(Q_n)]N(S)$$
$$=[N(S)N(Q_1)\cap N(S)N(Q_2)\cap\ldots\cap N(S)N(Q_n)]\cap$$
$$[N(Q_1)N(S)\cap N(Q_2)N(S)\cap\ldots\cap N(Q_n)N(S)]$$
$$=[N(S)N(Q_1)\cap N(Q_1)N(S)]\cap[N(S)N(Q_2)\cap$$
$$N(Q_2)N(S)]\ldots[N(S)N(Q_n)\cap N(Q_n)N(S)]$$
$$\subseteq N(Q_1)\cap N(Q_2)\cap\ldots\cap N(Q_n).$$

Hence $N(Q_1)\cap N(Q_2)\cap\ldots\cap N(Q_n)$ is a neuteosophic quasi-ideal.

Therefore, the intersection of any number of neutrosophic quasi-ideals of $N(S)$ is either empty or quasi-ideal of $N(S)$.

3 Neutrosophic Regular LA-semigroups

An element $a+bI$ of a neutrosophic LA-semigroup $N(S)$ is called regular if there exists $x+yI \in N(S)$ such that $a+bI=[(a+bI)(x+yI)](a+bI)$, and $N(S)$ is called neutrosophic regular LA-semigroup if every element of $N(S)$ is regular.

Example *Let* $S=\{1,2,3\}$ *with binary operation* "$\cdot$" *given in the following Callay's table, is a regular LA-semigroup with left identity* 4

$\cdot$	1	2	3	4
1	3	4	1	2
2	2	1	4	3
3	4	3	2	1
4	1	2	3	4

then

$$N(S)=\{1+1I,1+2I,1+3I,2+1I,2+2I,2+3I,3+1I,3+2I,3+3I,4+1I,4+2I,4+3I,4+4I\}$$

is an example of neutrosophic regular LA-semigroup under the operation "$*$" and has the following Callay's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$1+1I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$
$1+2I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$
$1+3I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$
$1+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$
$3+1I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$
$2+2I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$
$2+3I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$
$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$
$3+1I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$
$3+2I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$
$3+3I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$
$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$
$4+1I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$
$4+2I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$
$4+3I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$
$4+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$

Clearly $N(S)$ is a neutrosophic LA-semigroup also $[(1+1I)(4+4I)](2+3I)\neq(1+1I)[(4+4I)(2+3I)]$, so $N(S)$ is non-associative and is regular because $(1+1I)=[(1+1I)(2+2I)](1+1I)$, $(2+2I)=[(2+2I)(3+3I)](2+2I)$, $(3+2I)=[(3+2I)(1+3I)](3+2I)$, $(4+1I)=[(4+1I)(4+2I)](4+1I)$, $(4+4I)=[(4+4I)(4+4I)](4+4I)$ etc.

Note that in a neutrosophic regular LA-semigroup, $[N(S)]^2=N(S)$.

Lemma 3.1: *If* $N(A)$ *is a neutrosophic bi-ideal(generalized bi-ideal) of a regular neutrosophic LA-semigroup* $N(S)$ *then* $[N(A)N(S)]N(A)=N(A)$.

Proof Let $N(A)$ be a bi-ideal(generalized bi-ideal) of $N(S)$, then $[N(A)N(S)]N(A)\subseteq N(A)$.

Let $a+bI\in N(A)$, since $N(S)$ is neutrosophic regular LA-semigroup so there exists an element $x+yI\in N(S)$ such that $a+bI=[(a+bI)(x+yI)](a+bI)$, therefore, $a+bI=[(a+bI)(x+bI)](a+bI)\in[N(A)N(S)]N(A)$.

This implies that $N(A)\subseteq[N(A)N(S)]N(A)$. Hence $[N(A)N(S)]N(A)=N(A)$.

Lemma 3.2: *If* $N(A)$ *and* $N(B)$ *are any neutrosophic ideals of a neutrosophic regular LA-semigroup* $N(S)$, *then* $N(A)\cap N(B)=N(A)N(B)$.

Proof Assume that $N(A)$ and $N(B)$ are any neutrosophic ideals of $N(S)$ so $N(A)N(B)\subseteq N(A)N(S)\subseteq N(A)$ and $N(A)N(B)\subseteq N(S)N(B)\subseteq N(B)$. This implies that $N(A)N(B)\subseteq N(A)\cap N(B)$. Let $a+bI\in N(A)\cap N(B)$, then $a+bI\in N(A)$ and

$a+bI \in N(B)$. Since $N(S)$ is a neutrosophic regular AG-groupoid, so there exist $x+yI$ such that $a+bI=[(a+bI)(x+yI)](a+bI) \in [N(A)N(S)]N(B) \subseteq N(A)N(B)$, which implies that $N(A)\cap N(B) \subseteq N(A)N(B)$. Hence $N(A)N(B)=N(A)\cap N(B)$.

Lemma 3.3: *If $N(A)$ and $N(B)$ are any neutrosophic ideals of a neutrosophic regular LA-semigroup $N(S)$, then $N(A)N(B)=N(B)N(A)$.*

Proof Let $N(A)$ and $N(B)$ be any neutrosophic ideals of a neutrosophic regular LA-semigroup $N(S)$. Now, let $a_1+a_2I \in N(A)$ and $b_1+b_2I \in N(B)$. Since, $N(A) \subseteq N(S)$ and $N(B) \subseteq N(S)$ and $N(S)$ is a neutrosophic regular LA-semigroup so there exist x_1+x_2I, $y_1+y_2I \in N(S)$ such that $a_1+a_2I=[(a_1+a_2I)(x_1+x_2I)](a_1+a_2I)$ and $b_1+b_2I=[(b_1+b_2I)(y_1+y_2I)](b_1+b_2I)$.

Now, let $(a_1+a_2I)(b_1+b_2I) \in N(A)N(B)$ but

$$\begin{aligned}
&(a_1+a_2I)(b_1+b_2I)\\
&=[\{(a_1+a_2I)(x_1+x_2I)\}(a_1+a_2I)]\\
&\quad[\{(b_1+b_2I)(y_1+y_2I)\}(b_1+b_2I)]\\
&\in[\{N(A)N(S)\}N(A)][\{N(B)N(S)\}N(B)]\\
&\subseteq[N(A)N(A)][N(B)N(B)]\\
&=[N(B)N(B)][N(A)N(A)]\\
&\subseteq N(B)N(A)
\end{aligned}$$

$N(A)N(B) \subseteq N(B)N(A)$.

Now, let $(b_1+b_2I)(a_1+a_2I) \in N(B)N(A)$ but

$$\begin{aligned}
(b_1+b_2I)(a_1+a_2I)&=[\{(b_1+b_2I)(y_1+y_2I)\}(b_1+b_2I)]\\
&\quad[\{(a_1+a_2I)(x_1+x_2I)\}(a_1+a_2I)]\\
&\in[\{N(B)N(S)\}N(B)][\{N(A)N(S)\}N(A)]\\
&\subseteq[N(B)N(B)][N(A)N(A)]\\
&=[N(A)N(A)][N(B)N(B)]\\
&\subseteq N(A)N(B).
\end{aligned}$$

Since $N(B)N(A) \subseteq N(A)N(B)$. Hence $N(A)N(B)=N(B)N(A)$.

Lemma 3.4; *Every neutrosophic bi-ideal of a regular neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$ is a neutrosophic quasi-ideal of $N(S)$.*

Proof Let $N(B)$ be a bi-ideal of $N(S)$ and $(s_1+s_2I)(b_1+b_2I) \in N(S)N(B)$, for $s_1+s_2I \in N(S)$ and $b_1+b_2I \in N(B)$. Since $N(S)$ is a neutrosophic regular LA-semigroup, so there exists x_1+x_2I in $N(S)$ such that $b_1+b_2I=[(b_1+b_2I)(x_1+x_2I)](b_1+b_2I)$, then by using (4) and (1), we have

$$\begin{aligned}
&(s_1+s_2I)(b_1+b_2I)\\
&=(s_1+s_2I)[\{(b_1+b_2I)(x_1+x_2I)\}(b_1+b_2I)]\\
&=[(b_1+b_2I)(x_1+x_2I)][(s_1+s_2I)(b_1+b_2I)]\\
&=[\{(s_1+s_2I)(b_1+b_2I)\}(x_1+x_2I)](b_1+b_2I)\\
&=[(s_1+s_2I)\{\{(b_1+b_2I)(x_1+x_2I)\}(b_1+b_2I)\}(x_1+x_2I)](b_1+bI)\\
&=[[\{(b_1+b_2I)(x_1+x_2I)\}\{(s_1+s_2I)(b_1+b_2I)\}](x_1+x_2I)](b_1+b_2I)\\
&=[\{(x_1+x_2I)((s_1+s_2I)(b_1+b_2I))\}\{(b_1+b_2I)(x_1+x_2I)\}](b_1+b_2I)\\
&=[(b_1+b_2I)[\{(x_1+x_2I)\{(s_1+s_2I)(b_1+b_2I)\}\}](x_1+x_2I)\}](b_1+b_2I)\\
&\in[N(B)N(S)]N(B)\\
&\subseteq N(B).
\end{aligned}$$

Therefore, $N(B)N(S)\cap N(S)N(B) \subseteq N(S)N(B) \subseteq N(B)$.

Lemma 3.5. *In a neutrosophic regular LA-semigroup $N(S)$, every neutrosophic ideal is idempotent.*

Proof. Let $N(I)$ be any neutrosophic ideal of neutrosophic regular LA-semigroup $N(S)$. As we know, $(N(I))^2 \subseteq N(I)$ and let $a+bI \in N(I)$, since $N(S)$ is regular so there exists an element $x+yI \in N(S)$ such that

$$\begin{aligned}
a+bI&=[(a+bI)(x+yI)](a+bI)\\
&\in[N(I)N(S)]N(I)\\
&\subseteq N(I)N(I)=(N(I))^2.
\end{aligned}$$

This implies $N(I) \subseteq (N(I))^2$. Hence, $(N(I))^2=N(I)$.

As $N(I)$ is the arbitrary neutrosophic ideal of $N(S)$. So every ideal of neutrosophic regular AG-groupoid is idempotent.

Corollary 3.1. *In a neutrosophic regular LA-semigroup $N(S)$, every neutrosophic right ideal is idempotent.*

Proof. Let $N(R)$ be any neutrosophic right ideal of neutrosophic regular LA-semigroup $N(S)$ then $N(R)N(S) \subseteq N(R)$ and $(N(R))^2 \subseteq N(R)$. Now,let

$a+bI \in N(R),$

as $N(S)$ is regular implies for $a+bI \in N(R)$,there exists $x+yI \in N(S)$ such that

$$\begin{aligned} a+bI &= [(a+bI)(x+yI)](a+bI) \\ &\in [N(R)N(S)]N(I) \\ &\subseteq N(R)N(R) \\ &= (N(R))^2. \end{aligned}$$

Thus $(N(R))^2 = N(R)$. Hence, $(N(R))^2 = N(R)$. So every neutrosophic right ideal of neutrosophic regular LA-semigroup $N(S)$ is idempotent.

Corollary 3.2: *In a neutrosophic regular LA-semigroup $N(S)$, every neutrosophic ideal is semiprime.*

Proof: Let $N(P)$ be any neutrosophic ideal of neutrosophic regular LA-semigroup $N(S)$ and let $N(I)$ be any other neutrosophic ideal such that $[N(I)]^2 \subseteq N(P).$

Now as every ideal of $N(S)$ is idempotent by lemma 3.5. So, $[N(I)]^2 = N(I)$ implies $N(I) \subseteq N(P)$. Hence, every neutrosophic ideal of $N(S)$ is semiprime.

4 Neutrosophic Intra-regular LA-semigroups

An LA-semigroup $N(S)$ is called neutrosophic intra-regular if for each element $a_1 + a_2 I \in N(S)$ there exist elements $(x_1 + x_2 I)$, $(y_1 + y_2 I) \in N(S)$ such that $a_1 + a_2 I = [(x_1 + x_2 I)(a_1 + a_2 I)^2](y_1 + y_2 I)$.

Example *Let $S = \{1,2,3\}$ with binary operation "$\cdot$" given in the following Callay's table, is an intra-regular LA-semigroup with left identity 2.*

$\cdot$	1	2	3
1	2	3	1
2	1	2	3
3	3	1	2

then

$N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$

is an example of neutrosophic intraregular LA-semigroup under the operation "$*$" and has the following Callay's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$
$1+2I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$1+3I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$2+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+2I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$2+3I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$3+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+2I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$3+3I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$

Clearly $N(S)$ is a neutrosophic LA-semigroup and is non-associative because $[(1+1I)*(2+2I)]*(2+3I) \neq (1+1I)*[(2+2I)*(2+3I)]$ and $N(S)$ is intra-regular as

$(1+1I) = [(1+3I)(1+1I)^2](2+31)$,

$(2+3I) = [(1+1I)(2+3I)^2](3+1I)$,

$(3+1I) = [(2+3I)(3+1I)^2](3+3I)$ etc.

Note that if $N(S)$ is a neutrosophic intra-regular LA-semigroup then $[N(S)]^2 = N(S)$.

Lemma 4.1: *In a neutrosophic intra-regular LA-semigroup $N(S)$ with left identity $e+eI$, every neutrosophic ideal is idempotent.*

Proof Let $N(I)$ be any neutrosophic ideal of a neutrosophic intraregular LA-semigroup $N(S)$ implies $[N(I)]^2 \subseteq N(I)$. Now, let $a_1 + a_2 I \in N(I)$ and since $N(I) \subseteq N(S)$ implies $a_1 + a_2 I \in N(S)$. Since $N(S)$ is a neutrosophic intra-regular LA-semigroup, so there exist $(x_1 + x_2 I)$, $(y_1 + y_2 I) \in N(S)$ such that

$$\begin{aligned} (a_1 + a_2 I) &= [(x_1 + x_2 I)(a_1 + a_2 I)^2](y_1 + y_2 I) \\ &\in [N(S)(N(I))^2]N(S) \\ &= [N(S)(N(I)N(I))]N(S) \\ &= (N(I)(N(S)N(I)))N(S) \\ &\subseteq (N(I)N(I))N(S) \\ &= (N(S)N(I))N(I) \\ &\subseteq N(I)N(I) \\ &= [N(I)]^2. \end{aligned}$$

Hence $[N(I)]^2 = N(I)$. As, $N(I)$ is arbitrary so every neutrosophic ideal of is idempotent in a neutrosophic intra-regular LA-semigroup $N(S)$ with left identity.

Lemma 4.2. *In a neutrosophic intra-regular LA-semigroup $N(S)$ with left identity $e+eI$* ,

$N(I)N(J)=N(I)\cap N(J)$ *, for every neutrosophic ideals* $N(I)$ *and* $N(J)$ *in* $N(S)$.

Proof: Let $N(I)$ and $N(J)$ be any neutrosophic ideals of $N(S)$, then obviously $N(I)N(J)\subseteq N(I)N(S)$ and $N(I)N(J)\subseteq N(S)N(J)$ implies $N(I)N(J)\subseteq N(I)\cap N(J)$. Since $N(I)\cap N(J)\subseteq N(I)$ and $N(I)\cap N(J)\subseteq N(J)$, then $[N(I)\cap N(J)]^2\subseteq N(I)N(J)$. Also $N(I)\cap N(J)$ is a neutrosophic ideal of $N(S)$, so using Lemma 4.1, we have $N(I)\cap N(J)=[N(I)\cap N(J)]^2\subseteq N(I)N(J)$. Hence $N(I)N(J)=N(I)\cap N(J)$.

Theorem 4.1. *For neutrosophic intra-regular AG-groupoid with left identity* $e+eI$ *, the following statements are equivalent.*

(i) $N(A)$ *is a neutrosophic left ideal of* $N(S)$.

(ii) $N(A)$ *is a neutrosophic right ideal of* $N(S)$.

(iii) $N(A)$ *is a neutrosophic ideal of* $N(S)$.

(iv) $N(A)$ *is a neutrosophic bi-ideal of* $N(S)$.

(v) $N(A)$ *is a neutrosophic generalized bi-ideal of* $N(S)$.

(vi) $N(A)$ *is a neutrosophic interior ideal of* $N(S)$.

(vii) $N(A)$ *is a neutrosophic quasi-ideal of* $N(S)$.

$(viii)$ $N(A)N(S)=N(A)$ *and* $N(S)N(A)=N(A)$.

Proof: $(i)\Rightarrow(viii)$

Let $N(A)$ be a neutrosophic left ideal of $N(S)$. By Lemma first, $N(S)N(A)=N(A)$. Now let $(a_1+a_2I)\in N(A)$ and $(s_1+s_2I)\in N(S)$, since $N(S)$ is a neutrosophic intra-regular LA-semigroup, so there exist (x_1+x_2I), $(y_1+y_2I)\in N(S)$ such that $(a_1+a_2I)=[(x_1+x_2I)(a_1+a_2I)^2](y_1+y_2I)$, therefore by (1), we have

$$\begin{aligned}(a_1+a_2I)(s_1+s_2I) &= [\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)](s_1+s_2I)\\ &= [\{(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}\}(y_1+y_2I)](s_1+s_2I)\\ &\in [\{N(S)\{N(A)N(A)\}\}N(S)]N(S)\\ &\subseteq [\{N(S)\{N(S)N(A)\}\}N(S)]N(S)\\ &\subseteq [\{N(S)N(A)\}N(S)]N(S)\\ &= [N(S)N(S)][N(S)N(A)]\\ &= N(S)[N(S)N(A)]\subseteq N(S)N(A)=N(A).\end{aligned}$$

which implies that $N(A)$ is a neutrosophic right ideal of $N(S)$, again by Lemma first, $N(A)N(S)=N(S)$.

$(viii)\Rightarrow(vii)$

Let $N(A)N(S)=N(A)$ and $N(S)N(A)=N(A)$ then $N(A)N(S)\cap N(S)N(A)=N(A)$, which clearly implies that $N(A)$ is a neutrosophic quasi-ideal of $N(S)$.

$(vii)\Rightarrow(vi)$

Let $N(A)$ be a quasi-ideal of $N(S)$. Now let $[(s_1+s_2I)(a_1+a_2I)](s_1+s_2I)\in[N(S)N(A)]N(S)$, since $N(S)$ is neutrosophic intra-regular LA-semigroup so there exist (x_1+x_2I), (y_1+y_2I), (p_1+p_2I), $(q_1+q_2I)\in N(S)$ such that $(s_1+s_2I)=[(x_1+x_2I)(s_1+s_2I)^2](y_1+y_2I)$ and $(a_1+a_2I)=[(p_1+p_2I)(a_1+a_2I)^2](q_1+q_2I)$.

Therefore using (2), (4), (3) and (1), we have

$$\begin{aligned}&[(s_1+s_2I)(a_1+a_2I)](s_1+s_2I)\\ &= [(s_1+s_2I)(a_1+a_2I)][\{(x_1+xI)(s_1+s_2I)^2\}(y_1+y_2I)]\\ &= [\{(s_1+s_2I)\{(x_1+x_2I)(s_1+s_2I)^2\}\}][(a_1+a_2I)(y_1+y_2I)]\\ &= (a_1+a_2I)[\{(s_1+s_2I)\{(x_1+x_2I)(s_1+s_2I)^2\}\}(y_1+y_2I)]\\ &\in N(A)N(S).\end{aligned}$$

and

$$\begin{aligned}&[(s_1+s_2I)(a_1+a_2I)](s_1+s_2I)\\ &= [(s_1+s_2I)\{\{(p_1+p_2I)(a_1+a_2I)^2\}(q_1+q_2I)\}](s_1+s_2I)\\ &= [\{(p_1+p_2I)(a_1+a_2I)^2\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I)\\ &= [\{(p_1+p_2I)\{(a_1+a_2I)(a_1+a_2I)\}\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I)\\ &= [\{(a_1+a_2I)\{(p_1+p_2I)(a_1+a_2I)\}\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I)\\ &= [\{(q_1+q_2I)(s_1+s_2I)\}\{\{(p_1+p_2I)(a_1+a_2I)\}(a_1+a_2I)\}](s_1+s_2I)\\ &= [\{(p_1+p_2I)(a_1+a_2I)\}\{\{(q_1+q_2I)(s_1+s_2I)\}(a_1+a_2I)\}](s_1+s_2I)\\ &= [\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}\{(a_1+a_2I)(p_1+p_2I)\}](s_1+s_2I)\\ &= [(a_1+a_2I)\{\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}(p_1+p_2I)\}](s_1+s_2I)\\ &= [(s_1+s_2I)\{\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}(p_1+p_2I)\}](a_1+a_2I)\\ &\in N(S)N(A)\subseteq N(A).\end{aligned}$$

which shows that $N(A)$ is a neutrosophic interior ideal of $N(S)$.

$(vi)\Rightarrow(v)$

Let $N(A)$ be a neutrosophic interior ideal of a neutrosophic intraregular LA-semigroup $N(S)$ and

$[(a_1+a_2I)(s_1+s_2I)](a_1+a_2I)\in[N(A)N(S)]N(A)$

. Now using (4) and (1), we get

$$[(a_1+a_2I)(s_1+s_2I)](a_1+a_2I)$$
$$=[(a_1+a_2I)(s_1+s_2I)][\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)]$$
$$=[(x_1+x_2I)(a_1+a_2I)^2][\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)]$$
$$=[(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}][\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)]$$
$$=[\{\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)\}\{(a_1+a_2I)(a_1+a_2I)\}](x_1+x_2I)$$
$$=[(a_1+a_2I)\{\{\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)\}(a_1+a_2I)\}](x_1+x_2I)$$
$$=[(a_1+a_2I)\{\{(a_1+a_2I)(y_1+y_2I)\}\{(a_1+a_2I)(s_1+s_2I)\}\}](x_1+x_2I)$$
$$=[\{(a_1+a_2I)(y_1+y_2I)\}\{(a_1+a_2I)\{(a_1+a_2I)(s_1+s_2I)\}\}](x_1+x_2I)$$
$$=[\{\{(a_1+a_2I)\{(a_1+a_2I)(s_1+s_2I)\}\}(y_1+y_2I)\}(a_1+a_2I)](x_1+x_2I)$$
$$\in[N(S)N(A)]N(S)\subseteq N(A).$$

$(v)\Rightarrow(iv)$

Let $N(A)$ be a neutrosophic generalized bi-ideal of $N(S)$. Let $a_1+a_2I\in N(A)$, and since $N(S)$ is neutrosophic intra-regular LA-semigroup so there exist (x_1+x_2I), (y_1+y_2I) in $N(S)$ such that $a_1+a_2I=[(x_1+x_2I)(a_1+a_2I)^2](y_1+y_2I),$ then using (3) and (4), we have

$$(a_1+a_2I)(a_1+a_2I)$$
$$=[\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)](a_1+a_2I)$$
$$=[\{(x_1+x_2I)(a_1+a_2I)^2\}\{(e_1+e_2I)(y_1+y_2I)\}](a_1+a_2I)$$
$$=[\{(y_1+y_2I)(e_1+e_2I)\}\{(a_1+a_2I)^2(x_1+x_2I)\}](a_1+a_2I)$$
$$=[(a_1+a_2I)^2\{\{(y_1+y_2I)(e_1+e_2I)\}(x_1+x_2I)\}](a_1+a_2I)$$
$$=[\{(a_1+a_2I)(a_1+a_2I)\}\{\{(y_1+y_2I)(e_1+e_2I)\}(x_1+x_2I)\}](a_1+a_2I)$$
$$=[\{(x_1+x_2I)\{(y_1+y_2I)(e_1+e_2I)\}\}\{(a_1+a_2I)(a_1+a_2I)\}](a_1+a_2I)$$
$$=[(a_1+a_2I)\{\{(x_1+x_2I)\{(y_1+y_2I)(e_1+e_2I)\}\}(a_1+a_2I)\}](a_1+a_2I)$$
$$\in[N(A)N(S)]N(A)\subseteq N(A).$$

Hence $N(A)$ is a neutrosophic bi-ideal of $N(S)$.

$(iv)\Rightarrow(iii)$

Let $N(A)$ be any neutrosophic bi-ideal of $N(S)$ and let $(a_1+a_2I)(s_1+s_2I)\in N(A)N(S)$. Since $N(S)$ is neutrosophic intra-regular LA-semigroup, so there exist (x_1+x_2I), $(y_1+y_2I)\in N(S)$ such that $(a_1+a_2I)=[(x_1+x_2I)(a_1+a_2I)^2](y_1+y_2I).$

Therefore, using (1), (3), (4) and (2), we have

$$(a_1+a_2I)(s_1+s_2I)$$
$$=[\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)](s_1+s_2I)$$
$$=[(s_1+s_2I)(y_1+y_2I)][(x_1+x_2I)(a_1+a_2I)^2]$$
$$=[(a_1+a_2I)^2(x_1+x_2I)][(y_1+y_2I)(s_1+s_2I)]$$
$$=[\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\}(a_1+a_2I)^2]$$
$$=[\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)][(a_1+a_2I)(a_1+a_2I)]$$
$$=[(a_1+a_2I)(a_1+a_2I)][(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}]$$
$$=[\{(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}\}(a_1+a_2I)](a_1+a_2I)$$
$$=\{(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}\}$$
$$\{\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)\}(a_1+a_2I)$$
$$=[\{(x_1+x_2I)(a_1+a_2I)^2\}\{\{(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}\}$$
$$(y_1+y_2I)\}](a_1+a_2I)$$
$$=[\{(y_1+y_2I)\{(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}\}\}$$
$$\{(a_1+a_2I)^2(x_1+x_2I)\}](a_1+a_2I)$$
$$=[(a_1+a_2I)^2\{\{(y_1+y_2I)\{(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}\}\}$$
$$(x_1+x_2I)\}](a_1+a_2I)$$
$$=[\{(a_1+a_2I)(a_1+a_2I)\}\{\{(y_1+y_2I)\{(x_1+x_2I)$$
$$\{(y_1+y_2I)(s_1+s_2I)\}\}\}(x_1+x_2I)\}](a_1+a_2I)$$
$$=[\{(x_1+x_2I)\{(y_1+y_2I)\{(x_1+x_2I)\{(y_1+y_2I)(s_1+s_2I)\}\}\}\}$$
$$\{(a_1+a_2I)(a_1+a_2I)\}](a_1+a_2I)$$
$$\in[N(A)N(S)]N(A)\subseteq N(A).$$

$$(s_1+s_2I)(a_1+a_2I)$$
$$=(s_1+s_2I)[\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)]$$
$$=[(x_1+x_2I)(a_1+a_2I)^2][(s_1+s_2I)(y_1+y_2I)]$$
$$=[(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}][(s_1+s_2I)(y_1+y_2I)]$$
$$=[(a_1+a_2I)\{(x_1+x_2I)(a_1+a_2I)\}][(s_1+s_2I)(y_1+y_2I)]$$
$$=[\{(s_1+s_2I)(y_1+y_2I)\}\{(x_1+x_2I)(a_1+a_2I)\}](a_1+a_2I)$$
$$=[\{(a_1+a_2I)(x_1+x_2I)\}\{(y_1+y_2I)(s_1+s_2I)\}](a_1+a_2I)$$
$$=[\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\}(a_1+a_2I)](a_1+a_2I)$$
$$=[\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\}$$
$$\{\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)\}](a_1+a_2I)$$
$$=[\{\{(y_1+y_2I)(s_1+s_2I)\}\{(x_1+x_2I)(a_1+a_2I)^2\}\}$$
$$\{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I)$$
$$=[\{\{(a_1+a_2I)^2(x_1+x_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\}$$
$$\{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I)$$
$$=[\{\{\{(a_1+a_2I)(a_1+a_2I)\}(x_1+x_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\}$$
$$\{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I)$$

$$
\begin{aligned}
&= [\{\{(x_1 + x_2I)(y_1 + y_2I)\}\{(s_1 + s_2I)(y_1 + y_2I)\}\}\\
&\quad \{\{(a_1 + a_2I)(a_1 + a_2I)\}(x_1 + x_2I)\}](a_1 + a_2I)\\
&= [(a_1 + a_2I)\{\{\{(s_1 + s_2I)(y_1 + y_2I)\}(x_1 + x_2I)\}\\
&\quad \{(y_1 + y_2I)(x_1 + x_2I)\}\}(a_1 + a_2I)\}](a_1 + a_2I)\\
&\in [N(A)N(S)]N(A)\\
&\subseteq N(A).
\end{aligned}
$$

Therefore, $N(A)$ is a neutrosophic ideal of $N(S)$.

$(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious.

Lemma 4.4. *A neutrosophic LA-semigroup $N(S)$ with left identity $(e+eI)$ is intra-regular if and only if every neutrosophic bi-ideal of $N(S)$ is idempotent.*

Proof. Assume that $N(S)$ is a neutrosophic intra-regular LA-semigroup with left identity $(e+eI)$ and $N(B)$ is a neutrosophic bi-ideal of $N(S)$. Let $(b+bI) \in N(B)$, and since $N(S)$ is intra-regular so there exist (c_1+c_2I), (d_1+d_2I) in $N(S)$ such that $(b_1+b_2I) = [(c_1+c_2I)(b_1+b_2I)^2](d_1+d_2I)$, then by using (3), (4) and (1), we have

$$
\begin{aligned}
&(b_1 + b_2I)\\
&= [(c_1 + c_2I)(b_1 + b_2I)^2](d_1 + d_2I)\\
&= [\{(c_1 + c_2I)(b_1 + b_2I)^2\}\{(e + eI)(d_1 + d_2I)\}]\\
&= [\{(d_1 + d_2I)(e + eI)\}\{(b_1 + b_2I)^2(c_1 + c_2I)\}]\\
&= [(b_1 + b_2I)^2\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}]\\
&= [\{(b_1 + b_2I)(b_1 + b_2I)\}\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}]\\
&= [\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(b_1 + b_2I)](b_1 + b_2I)\\
&= [\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}\\
&\quad \{\{(c_1 + c_2I)(b_1 + b_2I)^2\}(d_1 + d_2I)\}](b_1 + b_2I)\\
&= [\{\{(c_1 + c_2I)(b_1 + b_2I)^2\}\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}\\
&\quad (d_1 + d_2I)\}](b_1 + b_2I)\\
&= [\{(c_1 + c_2I)\{(b_1 + b_2I)(b_1 + b_2I)\}\}\{\{(d_1 + d_2I)(e + eI)\}\\
&\quad (c_1 + c_2I)\}(d_1 + d_2I)](b_1 + b_2I)\\
&= [\{(b_1 + b_2I)\{(c_1 + c_2I)(b_1 + b_2I)\}\}\{\{\{(d_1 + d_2I)(e + eI)\}\\
&\quad (c_1 + c_2I)\}(d_1 + d_2I)\}](b_1 + b_2I)\\
&= [\{\{\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(d_1 + d_2I)\}\}\\
&\quad \{(c_1 + c_2I)(b_1 + b_2I)\}(b_1 + b_2I)](b_1 + b_2I)\\
&= [\{\{\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(d_1 + d_2I)\}\\
&\quad \{\{(c_1 + c_2I)\{\{(c_1 + c_2I)(b_1 + b_2I)^2\}\\
&\quad (d_1 + d_2I)\}\}(b_1 + b_2I)](b_1 + b_2I)\\
&= [\{\{\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(d_1 + d_2I)\}\}\{(c_1 + c_2I)\\
&\quad \{\{(c_1 + c_2I)\{(b_1 + b_2I)(b_1 + b_2I)\}\}(d_1 + d_2I))\}\}\\
&\quad (b_1 + b_2I)\}](b_1 + b_2I)\\
&= [\{\{\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(d_1 + d_2I)\}\}\{(c_1 + c_2I)\{\{(b_1 + b_2I)\\
&\quad \{(c_1 + c_2I)(b_1 + b_2I)\}\}(d_1 + d_2I)\}\}(b_1 + b_2I)](b_1 + b_2I)\\
&= [\{\{\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(d_1 + d_2I)\}\}\{(b_1 + b_2I)\\
&\quad \{\{(c_1 + c_2I)\{(c_1 + c_2I)(b_1 + b_2I)\}\}(d_1 + d_2I)\}\}(b_1 + b_2I)](b_1 + b_2I)\\
&= [(b_1 + b_2I)\{\{\{\{\{(d_1 + d_2I)(e + eI)\}(c_1 + c_2I)\}(d_1 + d_2I)\}\}\\
&\quad \{\{(c_1 + c_2I)\{(c_1 + c_2I)(b_1 + b_2I)\}\}(d_1 + d_2I)\}\}(b_1 + b_2I)](b_1 + b_2I)\\
&\in [\{N(B)N(S)\}N(B)]N(B) \subseteq N(B)N(B).
\end{aligned}
$$

Hence $[N(B)]^2 = N(B)$.

Conversely, since $N(S)(a+bI)$ is a neutrosophic bi-ideal of $N(S)$, and by assumption $N(S)(a+bI)$ is idempotent, so by using (2), we have

[illegible]

Hence $N(S)$ is neutrosophic intra-regular LA-semigroup.

Theorem 4.2. *In a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, the following statements are equivalent.*

(i) $N(S)$ *is intra-regular.*

(ii) *Every neutrosophic two sided ideal of $N(S)$ is semiprime.*

(iii) *Every neutrosophic right ideal of $N(S)$ is semiprime.*

(iv) *Every neutrosophic left ideal of $N(S)$ is semiprime.*

Proof: $(i) \Rightarrow (iv)$

Let $N(S)$ is intra-regular, then by Theorem equalient and Lemma 4.1, every neutrosophic left ideal of $N(S)$ is semiprime.

$(iv) \Rightarrow (iii)$

Let $N(R)$ be a neutrosophic right ideal and $N(I)$ be any neutrosophic ideal of $N(S)$ such that $[N(I)]^2 \subseteq N(R)$. Then clearly $[N(I)]^2 \subseteq N(R) \cup N(S)N(R)$. Now by Lemma 2.7, $N(R) \cup N(S)N(R)$ is a neutrosophic two-sided ideal of $N(S)$, so is neutrosophic left. Then by (iv) we have $N(I) \subseteq N(R) \cup N(S)N(R)$. Now using (1) we have

$$N(S)N(R) = [N(S)N(S)]N(R) = [N(R)N(S)]N(S) \subseteq N(R)N(S) \subseteq N(R).$$

This implies that $N(I) \subseteq N(R) \cup N(S)N(R) \subseteq N(R).$ Hence $N(R)$ is semiprime.

It is clear that $(iii) \Rightarrow (ii)$.

Now $(ii) \Rightarrow (i)$

Since $(a+bI)^2 N(S)$ is a neutrosophic right ideal of $N(S)$ containing $(a+bI)^2$ and clearly it is a neutrosophic two sided ideal so by assumption (ii), it is semiprime, therefore by Theorem 2.2, $(a+bI) \in (a+bI)^2 N(S).$ Thus using (4) and (3), we have

$$\begin{aligned} a+bI &\in (a+bI)^2 N(S) \\ &= (a+bI)^2[N(S)N(S)] \\ &= N(S)[(a+bI)^2 N(S)] \\ &= [N(S)N(S)][(a+bI)^2 N(S)] \\ &= [N(S)(a+bI)^2][N(S)N(S)] \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is intra-regular.

Theorem 4.3. *An LA-semigroup $N(S)$ with left identity $e+eI$ is intra-regular if and only if every neutrosophic left ideal of $N(S)$ is idempotent.*

Proof. Let $N(S)$ be a neutrosophic intra-regular LA-semigroup then by Theorem equalient and Lemma 4.1, every neutrosophic ideal of $N(S)$ is idempotent.

Conversely, assume that every neutrosophic left ideal of $N(S)$ is idempotent. Since $N(S)(a+bI)$ is a neutrosophic left ideal of $N(S)$, so by using (2), we have

$$\begin{aligned} a+bI &\in N(S)(a+bI) \\ &= [N(S)(a+bI)][N(S)(a+bI)] \\ &= [\{N(S)(a+bI)\}\{N(S)(a+bI)\}]\{N(S)(a+bI)\} \\ &= [\{N(S)N(S)\}\{(a+bI)(a+bI)\}]\{N(S)(a+bI)\} \\ &\subseteq [N(S)(a+bI)^2][N(S)N(S)] \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Theorem 4.4. *A neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$ is intra-regular if and only if $N(R) \cap N(L) \subseteq N(R)N(L)$, for every neutrosophic semiprime right ideal $N(R)$ and every neutrosophic left ideal $N(L)$ of $N(S)$.*

Proof. Let $N(S)$ be an intra-regular LA-semigroup, so by Theorem equalient $N(R)$ and $N(L)$ become neutrosophic ideals of $N(S)$, therefore by Lemma 4.2, $N(R) \cap N(L) \subseteq N(L)N(R),$ for every neutrosophic ideal $N(R)$ and $N(L)$ and by Theorem every ideal semiprime, $N(R)$ is semiprime.

Conversely, assume that $N(R) \cap N(L) \subseteq N(R)N(L)$ for every neutrosophic right ideal $N(R),$ which is semiprime and every neutrosophic left ideal $N(L)$ of $N(S)$. Since $(a+bI)^2 \in (a+bI)^2 N(S)$, which is a neutrosophic right ideal of $N(S)$ so is semiprime which implies that $(a+bI) \in (a+bI)^2 N(S)$. Now clearly $N(S)(a+bI)$ is a neutrosophic left ideal of $N(S)$ and $(a+bI) \in N(S)(a+bI)$. Therefore, using (3),we have

$$\begin{aligned} a+bI &\in [(a+bI)^2 N(S)] \cap [N(S)(a+bI)] \\ &\subseteq [(a+bI)^2 N(S)][N(S)(a+bI)] \\ &\subseteq [(a+bI)^2 N(S)][N(S)N(S)] \\ &= [(a+bI)^2 N(S)]N(S) \\ &= [\{(a+bI)(a+bI)\}N(S)]N(S) \\ &= [\{(a+bI)(a+bI)\}\{N(S)N(S)\}]N(S) \\ &= [\{N(S)N(S)\}\{(a+bI)(a+bI)\}]N(S) \\ &= [N(S)\{(a+bI)(a+bI)\}]N(S) \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Therefore, $N(S)$ is a neutrosophic intra-regular LA-semigroup.

Theorem 4.5. *For a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, the following statements are equivalent.*

(i) $N(S)$ *is intra-regular.*

(ii) $N(L) \cap N(R) \subseteq N(L)N(R)$, *for every right ideal $N(R),$ which is neutrosophic semiprime and every neutrosophic left ideal $N(L)$ of $N(S)$.*

(iii) $N(L) \cap N(R) \subseteq [N(L)N(R)]N(L)$, *for every neutrosophic semiprime right ideal $N(R)$ and every neutrosophic left ideal $N(L)$.*

Proof $(i) \Rightarrow (iii)$

Let $N(S)$ be intra-regular and $N(L)$, $N(R)$ be any neutrosophic left and right ideals of $N(S)$ and let $a_1 + a_2I \in N(L) \cap N(R)$, which implies that $a_1 + a_2I \in N(L)$ and $a_1 + a_2I \in N(R)$. Since $N(S)$ is intra-regular so there exist $(x_1 + x_2I)$, $(y_1 + y_2I)$ in $N(S)$, such that $a_1 + a_2I = [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I)$, then by using (4), (1) and (3), we have

$$
\begin{aligned}
a_1 + a_2I &= [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I) \\
&= [(x_1 + x_2I)\{(a_1 + a_2I)(a_1 + a_2I)\}](y_1 + y_2I) \\
&= [(a_1 + a_2I)\{(x_1 + x_2I)(a_1 + a_2I)\}](y_1 + y_2I) \\
&= [(y_1 + y_2I)\{(x_1 + x_2I)(a_1 + a_2I)\}](a_1 + a_2I) \\
&= [(y_1 + y_2I)\{(x_1 + x_2I)\{\{(x_1 + x_2I)(a_1 + a_2I)^2\} \\
&\quad (y_1 + y_2I)\}\}](a_1 + a_2I) \\
&= [(y_1 + y_2I)\{\{(x_1 + x_2I)(a_1 + a_2I)^2\}\{(x_1 + x_2I)(y)\}\}](a_1 + a_2I) \\
&= [\{(x_1 + x_2I)(a_1 + a_2I)^2\}\{(y_1 + y_2I) \\
&\quad \{(x_1 + x_2I)(y_1 + y_2I)\}\}](a_1 + a_2I) \\
&= [\{(x_1 + x_2I)\{(a_1 + a_2I)(a_1 + a_2I)\}\}\{(y_1 + y_2I) \\
&\quad \{(x_1 + x_2I)(y_1 + y_2I)\}\}](a_1 + a_2I) \\
&= [\{(a_1 + a_2I)\{(x_1 + x_2I)(a_1 + a_2I)\}\}\{(y_1 + y_2I) \\
&\quad \{(x_1 + x_2I)(y_1 + y_2I)\}\}](a_1 + a_2I) \\
&\in [\{N(R)\{N(S)N(L)\}\}N(S)]N(L) \\
&\subseteq [\{N(R)N(L)\}N(S)]N(L) \\
&= [N(L)N(S)][N(R)N(L)] \\
&= [N(L)N(R)][N(S)N(L)] \\
&\subseteq [N(L)N(R)]N(L),
\end{aligned}
$$

which implies that $N(L) \cap N(R) \subseteq [N(L)N(R)]N(L)$. Also by Theorem every ideal semiprime, $N(L)$ is semiprime.

$(iii) \Rightarrow (ii)$

Let $N(R)$ and $N(L)$ be neutrosophic left and right ideals of $N(S)$ and $N(R)$ is semiprime, then by assumption (iii) and by (3), (4) and (1), we have

$$
\begin{aligned}
N(R) \cap N(L) &\subseteq [N(R)N(L)]N(R) \\
&\subseteq [N(R)N(L)]N(S) \\
&= [N(R)N(L)][N(S)N(S)] \\
&= [N(S)N(S)][N(L)N(R)] \\
&= N(L)[\{N(S)N(S)\}N(R)] \\
&= N(L)[\{N(R)N(S)\}N(S)] \\
&\subseteq N(L)[N(R)N(S)] \\
&\subseteq N(L)N(R).
\end{aligned}
$$

$(ii) \Rightarrow (i)$

Since $e + eI \in N(S)$ implies $a + bI \in N(S)(a + bI)$, which is a neutrosophic left ideal of $N(S)$, and $(a + bI)^2 \in (a + bI)^2 N(S)$, which is a semiprime neutrosophic right ideal of $N(S)$, therefore by Theorem 2.2 $a + bI \in (a + bI)^2 N(S)$. Now using (3) we have

$$
\begin{aligned}
a + bI &\in [N(S)(a + bI)] \cap [(a + bI)^2 N(S)] \\
&\subseteq [N(S)(a + bI)][(a + bI)^2 N(S)] \\
&\subseteq [N(S)N(S)][(a + bI)^2 N(S)] \\
&= [N(S)(a + bI)^2][N(S)N(S)] \\
&= [N(S)(a + bI)^2]N(S).
\end{aligned}
$$

Hence $N(S)$ is intra-regular

A neutrosophic LA-semigroup $N(S)$ is called totally ordered under inclusion if $N(P)$ and $N(Q)$ are any neutrosophic ideals of $N(S)$ such that either $N(P) \subseteq N(Q)$ or $N(Q) \subseteq N(P)$.

A neutrosophic ideal $N(P)$ of a neutrosophic LA-semigroup $N(S)$ is called strongly irreducible if $N(A) \cap N(B) \subseteq N(P)$ implies either $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$, for all neutrosophic ideals $N(A)$, $N(B)$ and $N(P)$ of $N(S)$.

Lemma 4.4. *Every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is prime if and only if it is strongly irreducible.*

Proof. Assume that every ideal of $N(S)$ is neutrosophic prime. Let $N(A)$ and $N(B)$ be any neutrosophic ideals of $N(S)$ so by Lemma 4.2, $N(A)N(B) = N(A) \cap N(B)$, where $N(A) \cap N(B)$ is neutrosophic ideal of $N(S)$. Now, let $N(A) \cap N(B) \subseteq N(P)$ where $N(P)$ is a neutrosophic ideal of $N(S)$ too. But by assumption every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is prime so is neutrosophic prime, therefore, $N(A)N(B) = N(A) \cap N(B) \subseteq N(P)$ implies $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$. Hence $N(S)$ is strongly irreducible.

Conversely, assume that $N(S)$ is strongly irreducible. Let

$N(A)$, $N(B)$ and $N(P)$ be any neutrosophic ideals of $N(S)$ such that $N(A)\cap N(B)\subseteq N(P)$ implies $N(A)\subseteq N(P)$ or $N(B)\subseteq N(P)$. Now, let $N(A)\cap N(B)\subseteq N(P)$ but $N(A)N(B)=N(A)\cap N(B)$ by lemma ij, $N(A)N(B)\subseteq N(P)$ implies $N(A)\subseteq N(P)$ or $N(B)\subseteq N(P)$. Since $N(P)$ is arbitrary neutrosophic ideal of $N(S)$ so very neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is prime.

Theorem 4.6. *Every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is neutrosophic prime if and only if $N(S)$ is totally ordered under inclusion.*

Proof. Assume that every ideal of $N(S)$ is neutrosophic prime. Let $N(P)$ and $N(Q)$ be any neutrosophic ideals of $N(S)$, so by Lemma 4.2, $N(P)N(Q)=N(P)\cap N(Q)$, where $N(P)\cap N(Q)$ is neutrosophic ideal of $N(S)$, so is neutrosophic prime, therefore, $N(P)N(Q)\subseteq N(P)\cap N(Q),$ which implies that $N(P)\subseteq N(P)\cap N(Q)$ or $N(Q)\subseteq N(P)\cap N(Q),$ which implies that $N(P)\subseteq N(Q)$ or $N(Q)\subseteq N(P)$. Hence $N(S)$ is totally ordered under inclusion.

Conversely, assume that $N(S)$ is totally ordered under inclusion. Let $N(I)$, $N(J)$ and $N(P)$ be any neutrosophic ideals of $N(S)$ such that $N(I)N(J)\subseteq N(P)$. Now without loss of generality assume that $N(I)\subseteq N(J)$ then

$$N(I)=[N(I)]^2=N(I)N(I)$$
$$\subseteq N(I)N(J)\subseteq N(P).$$

Therefore, either $N(I)\subseteq N(P)$ or $N(J)\subseteq N(P)$, which implies that $N(P)$ is neutrosophic prime.

Theorem 4.7. *The set of all neutrosophic ideals $N(I)_s$ of a neutrosophic intra-regular $N(S)$ with left identity $e+eI$ forms a semilattice structure.*

Proof. Let $N(A)$, $N(B)\in N(I)_s$, since $N(A)$ and $N(B)$ are neutrosophic ideals of $N(S)$ so we have

$$[N(A)N(B)]N(S)=[N(A)N(B)][N(S)N(S)]$$
$$=[N(A)N(S)][N(B)N(S)]$$
$$\subseteq N(A)N(B).$$

Also $N(S)[N(A)N(B)]=[N(S)N(S)][N(A)N(B)]$
$$=[N(S)N(A)][N(S)N(B)]$$
$$\subseteq N(A)N(B).$$

Thus $N(A)N(B)$ is a neutrosophic ideal of $N(S)$. Hence $N(I)_s$ is closed. Also using Lemma ij, we have,

$$N(A)N(B)=N(A)\cap N(B)=N(B)\cap N(A)=N(B)N(A)$$

which implies that $N(I)_s$ is commutative, so is associative. Now by using Lemma ii, $[N(A)]^2=N(A)$, for all $N(A)\in N(I)_s$. Hence $N(I)_s$ is semilattice.

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