

# Using discrete-time mathematical programming to optimise the extraction rate of a durable non-renewable resource with a single primary supplier

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## ABSTRACT

A non-linear discrete-time mathematical program model is proposed to determining the optimal extraction policy for a single primary supplier of a durable non-renewable resource, such as gemstones or some metals. Karush, Kuhn and Tucker conditions allow obtaining analytic solutions and general properties of them in some specific settings. Moreover, provided that the objective function (i.e., the discounted value of the incomes throughout the planning horizon) is concave, the model can be easily solved, even using standard commercial solver. However, the analysis of the solutions obtained for different assumptions of the values of the parameters show that the optimal extraction policies and the corresponding prices do not exhibit a general shape.

Keywords: durable non-renewable resources; single primary supplier; non-linear programming

## 1. Introduction

This paper deals with the optimal production policy for a single owner of the primary source of a durable non-renewable (therefore, exhaustible) resource, such as gold or diamonds. The problem is approached by means of non-linear discrete-time mathematical programming, what allows, under very general assumptions and by means of Karush, Kuhn and Tucker conditions, obtaining analytic solutions and general properties of them in some particular settings and computing easily the optimal policies.

Although most economic theory is not explicit about whether inputs into production are renewable or non-renewable, this distinction has significant implications of the optimal policies of producing and pricing the resource.

Natural resources can be renewable (e.g. fish stocks or forests) or non-renewable (all minerals). Among the latter, some (gemstones, precious metals and other metals like copper) are durable, whereas others (e.g. all kinds of fossil fuels, phosphates and other

mineral fertilizers, and fossil water) are not. Non-durable resources disappear as such when they are used (burnt or dispersed), while durable resources may be reused, perhaps after recycling.

Therefore, when a non-renewable resource is durable, at any time there is an inventory of the resource in the ground and an inventory of the already used amounts of the resource that are potentially reusable.

Since the seminal papers by Gray (1914) and Hotelling (1931), where the famous Hotelling's rule concerning the price evolution of an exhaustible resource in a competitive market is stated, a certain number of papers and books on the economics of non-renewable resources have been published. The great majority of these publications (many relevant references can be found in Corominas and Fossas, 2015), explicitly or not, deal exclusively with non-durable resources, while the literature on the economics of durable non-renewable resources is relatively scarce. From the ten references included in a recent paper on this subject that deals with the prices of durable exhaustible resources under stochastic investment opportunities (Atewamba and Gaudet, 2014) only one (Levhary and Pindyck, 1981) is specifically devoted to durable non-renewable resources. Hence, this topic remains largely unexplored.

In any case, as it will be shown below, research on this issue has revolved mainly around the conditions under which the Hotelling rule is valid or it is not. However, the objective of the present paper is to show the use of a non-linear discrete-time mathematical programming model to find the optimal extraction policies of the single owner of a durable non-renewable resource in a variety of scenarios.

The implications of the durability for a monopolist<sup>1</sup> were analysed in Coase (1972), where was stated what later has been known as the Coase Conjecture, namely, that if a durable-goods monopolist were unable to precommit to a future sales trajectory, market power would disappear "in the twinkling of an eye". Karp (1993, 1996) specifies several settings in which the Conjecture fails.

Some papers deal with durable renewable goods monopolies, considering the problems derived from the fact that the sale of their products creates a secondary market beyond the control of the monopolists, which lead to compare selling versus renting (Bulow, 1982; Suslow, 1986; Malueg and Solow, 1987, 1989). Although the

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<sup>1</sup> Note that to speak of monopoly in relation to a durable good can be considered to a certain extent as an abuse of language, since the supply can come from the possessor of the primary source of the good and from any of its holders. However, for the sake of simplicity, as many authors do, the terms monopoly and monopolistic are used in the present paper in this specific sense.

possibility of renting, which arises in these settings, are hardly applicable to durable exhaustive resources, these questions are considered in Malueg and Solow (1988, 1990) in spite of that those two papers deal with this kind of resources.

Other researches concerning the economics of durable non-renewable resources focus on the validity of Hotelling's rule for this kind of resources, as it is shown below.

Stewart (1980) uses an optimisation discrete-time model and the Lagrange multiplier technique (what implies the assumption that the resource has to be depleted unless it is unlimitedly available) to compare, regarding the production throughout a finite time horizon of a durable exhaustible resource, the strategy of a competitive extractive industry with that of a monopolistic one. The author considers a general demand function that may vary from period to period and the notion of quasi-durability, which is quantified by means of a coefficient corresponding to the fraction of the stock of the extracted resource that remains from one period to the next. Stewart concludes that Hotelling's rule applies to competitive and monopolistic markets, although in these latter, contrarily to that happens in the former, the optimal strategy may lead to falling prices.

Levhari and Pindyck (1981), a fundamental contribution on the subject, using a continuous time infinite horizon formulation with growing demand and the Maximum Principle, criticise Stewart's conclusions and argue that, although in a competitive market the price minus the marginal cost will rise at the rate of interest, this does not imply that price is steadily rising. The authors also discuss briefly the case of monopolistic markets and conclude that this rule does not hold in them. Besides, they point out that the evolution of the prices of durable resources "have shown long secular declines during at least part of their history, and in many cases have indeed been U-shaped over the long term (50-100 years)" and show that, under specific assumptions, their models can explain these behaviours.

Chilton (1984), however, show that, if a convenient definition of marginal revenue is used, Hotelling's rule extends to the case of monopolistic extraction of a durable good.

Malueg and Solow (1988) analyse in detail the two-periods case under the assumptions of monopoly, static linear demand function, and perfect durability. They adapt a model from Bulow (1982), with the additional assumption that the resource is exhaustible. Their analysis focusses on the differences that exhaustibility induces in the monopoly equilibrium of durable resources.

The same authors (Malueg and Solow, 1990) analyse if monopoly leads or not to overconservation in the case of durable exhaustive resources. They use two models

with static linear demand functions and an infinite horizon (a discrete-time model with perfect durability and a continuous-time one in which costs are an increasing function of cumulative production) and obtain from them similar results, with the general conclusions that monopoly is overconservative and prices fall monotonically during the production period..

In the present paper, a discrete-time non-linear mathematical programming model is proposed for determining the optimal policies of the single primary supplier of a durable exhaustible resource, under a variety of assumptions. This approach allows dealing with any evolution of the demand function throughout the time, any number of periods of the planning horizon and either with perfect durability or any degree of partial durability. Moreover, it makes easier the computation of the optimal policies and permits also analysing the properties of these policies in diverse settings.

The structure of the rest of the paper is as follows. The adopted assumptions and the mathematical programming model are stated in section 2. The properties of the optimal solutions in several particular settings are discussed in section 3, which also contains numerical examples. Section 4 closes the paper with some concluding remarks and future research lines.

## 2. Assumptions and model formulation

We consider a finite planning horizon divided into  $T$  periods. The equilibrium price,  $p_t$ , for each period,  $t$ , is a function,  $\varphi_t$ , of the stock of resource in circulation,  $s_t$ ; i. e.  $p_t = \varphi_t(s_t)$ <sup>2</sup>. At the beginning of the planning horizon, the single primary supplier possesses an amount  $R$  of the resource and the stock in circulation is  $s_0$ .

We assume that the costs of production and distribution are negligible, although they could be easily incorporated if they are constant or depending on time and not on the amount of resource in the hands of the monopolist.

The stock in circulation in any period,  $s_t$ , is assumed to be equal to  $\rho \cdot s_{t-1} + x_t$ , where  $x_t (\geq 0)$  is the amount of resource extracted and introduced into the market by the monopolist in period  $t$  and  $\rho (\in (0,1])$  is the proportion of the stock available in  $t-1$

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<sup>2</sup> Although some authors (e. g. Stewart, 1980) refer the equivalents of  $\varphi_t$  function as the inverse demand functions, others (Levhari and Pindyck, 1981) avoid the use of this denomination, as we do in the present paper (except when describing the work of authors that use it). Note that, strictly speaking,  $\varphi_t$  is the relation between the stock of resource in circulation and the price and that the stock in circulation does not necessarily coincides with the supply of the resource, in the sense of the amount put to sale.

that is still available in  $t$  (a part of the available stock can deteriorate, be dispersed or lost or not considered marketable by its owners). The value zero is excluded, because in this case the resource would be non-durable;  $\rho = 1$  corresponds to a perfectly durable resource and  $0 < \rho < 1$  to the infinitely many degrees of partial durability. Of course, the value of this parameter may depend on time; however, we assume, for the sake of simplicity of the formulations, that it does not (relaxing this assumption, on the other hand, is straightforward).

Let  $\alpha_t (t = 1, \dots, T)$  be the discount factor corresponding to period  $t$ .

Then, the policy that maximises the present value of the single supplier can be determined by means of solving the following mathematical program:

Model MODER

(Monopolistic Optimisation for a Durable Exhaustible Resource)

$$\text{maximise } z = \sum_{t=1}^T \alpha_t \cdot \varphi_t(s_t) \cdot x_t = \sum_{t=1}^T \alpha_t \cdot \varphi_t \left( \rho^t \cdot s_0 + \sum_{\tau=1}^t \rho^{t-\tau} \cdot x_\tau \right) \cdot x_t$$

s.t.

$$\sum_{t=1}^T x_t \leq R \quad (1)$$

$$-x_t \leq 0 \quad t = 1, \dots, T \quad (2)$$

Given that the constraints are linear and define a feasible solution set with interior points, if the objective function is concave the Karush, Kuhn and Tucker (KKT) conditions are necessary and sufficient for optimality.

As it is known, a necessary and sufficient condition for the concavity of the function is that the Hessian matrix be negative semi-definite. That this condition holds or does not depends on the specific properties of the  $\varphi_t$  functions.

Then, in some specific settings the use of KKT conditions allows deducting analytic expressions for the optimum values of the variables and, hence, general properties of the optimal solutions for the corresponding setting.

Moreover, if concavity holds, the model, having only the non-negativity constraints and the linear constraint concerning the availability amount of the resource in the hands of the monopolist, is easy to solve using any commercial mathematical programming (even Excel can be used, provided that the value of  $T$  is not too high).

Calling  $u (\geq 0)$  the multiplier associated with constraint (1) and  $v_t (\geq 0; t = 1, \dots, T)$  those associated with constraints (2), the KKT conditions for MODER can be written as follows:

$$\begin{aligned}
-\frac{\partial z}{\partial x_t} + u - v_t &= 0 \quad (t = 1, \dots, T) \\
-\alpha_t \cdot \varphi'_t(s_t) \cdot \frac{\partial s_t}{\partial x_t} \cdot x_t - \alpha_t \cdot \varphi_t(s_t) - \sum_{\tau=t+1}^T \alpha_\tau \cdot \varphi'_\tau(s_\tau) \cdot \frac{\partial s_\tau}{\partial x_t} \cdot x_\tau + u - v_t &= \\
= -\alpha_t \cdot \varphi'_t(s_t) \cdot x_t - \alpha_t \cdot \varphi_t(s_t) - \sum_{\tau=t+1}^T \alpha_\tau \cdot \varphi'_\tau(s_\tau) \cdot \rho^{\tau-t} \cdot x_\tau + u - v_t &= 0 \quad (t = 1, \dots, T)
\end{aligned}$$

### 3. Optimal policies for specific settings, with linear $\varphi_t$ functions

In this section, we will use linear  $\varphi_t$  functions:

$$p_t = \varphi_t(s_t) = \frac{P_t}{Q_t} \cdot (Q_t - s_t)$$

where  $P_t$  is the choke price (i. e., the limit price when  $s_t$  goes to 0) and  $Q_t$  is the maximum amount of the resource in circulation.

Then, the objective function of MODER becomes:

$$\begin{aligned}
z &= \sum_{t=1}^T \alpha_t \cdot \frac{P_t}{Q_t} \left( Q_t - \rho^t \cdot s_0 - \sum_{\tau=1}^t \rho^{t-\tau} \cdot x_\tau \right) \cdot x_t = \\
&= \sum_{t=1}^T \beta_t \cdot \Delta_t \cdot x_t - \sum_{t=1}^T \beta_t \cdot x_t^2 - \sum_{t=1}^{T-1} \sum_{\tau=1}^t \beta_t \cdot \rho^{t-\tau} \cdot x_\tau \cdot x_t
\end{aligned}$$

where  $\beta_t = \frac{\alpha_t \cdot P_t}{Q_t}$  and  $\Delta_t = Q_t - \rho^t \cdot s_0$ .

Using this notation, the elements of the Hessian are:

$$h_{tt} = -2 \cdot \beta_t \quad (t = 1, \dots, T); \quad h_{t\tau} = -\rho^{|t-\tau|} \cdot \beta_\tau \quad (t = 1, \dots, T; \tau \neq t)$$

And the KKT conditions:

$$-\beta_t \cdot \Delta_t + 2 \cdot \beta_t \cdot x_t + \beta_t \cdot \sum_{\tau=1}^{t-1} \rho^{t-\tau} \cdot x_\tau + \sum_{\tau=t+1}^T \beta_\tau \cdot \rho^{\tau-t} \cdot x_\tau + u - v_t = 0 \quad (t = 1, \dots, T)$$

In every particular case it is easy to check whether the Hessian is positive-semidefinite and, if it is, solve the mathematical program using an appropriate solver.

Additionally, in some specific settings, analytical expressions can be found for the optimal values of the variables. In the rest of this section, three of these settings are

analysed: (i)  $T = 2$ ; (ii)  $T = 3, s_0 = 0; Q_t = Q, P_t = P, \alpha_t = 1 (t = 1, 2, 3)$ ; (iii)  $s_0 = 0; Q_t = Q, P_t = P, \alpha_t = 1 \forall t, \rho = 1$ . Finally, numerical results for two examples of more general settings are presented.

### 3.1. Linear $\varphi_t$ functions with $T = 2$

This case is similar to that dealt with in Malueg and Solow (1988). However, here the model is more general, because the  $\varphi_t$  functions corresponding to the two periods may be different and the degree of durability may have any value in  $(0, 1]$ .

The objective function in this case is:

$$z = \beta_1 \cdot \Delta_1 \cdot x_1 + \beta_2 \cdot \Delta_2 \cdot x_2 - \beta_1 \cdot x_1^2 - \beta_2 \cdot x_2^2 - \rho \cdot \beta_2 \cdot x_1 \cdot x_2$$

Dividing this expression by  $\beta_1$  and replacing  $\beta_2 / \beta_1$  with  $\theta$ , the following objective function results:

$$\hat{z} = \Delta_1 \cdot x_1 + \theta \cdot \Delta_2 \cdot x_2 - x_1^2 - \theta \cdot x_2^2 - \rho \cdot \theta \cdot x_1 \cdot x_2$$

s.t.  $x_1 + x_2 \leq R, -x_1 \leq 0, -x_2 \leq 0$

Then  $H(\hat{z}) = \begin{pmatrix} -2 & -\rho \cdot \theta \\ -\rho \cdot \theta & -2 \cdot \theta \end{pmatrix}$  and the condition for  $\hat{z}$  to be concave is:

$$\det H(\hat{z}) = \begin{vmatrix} -2 & -\rho \cdot \theta \\ -\rho \cdot \theta & -2 \cdot \theta \end{vmatrix} = 4 \cdot \theta - \rho^2 \cdot \theta^2 \geq 0 \quad (\rho^2 \cdot \theta \leq 4)$$

Therefore, assuming that this condition holds, KKT conditions are necessary and sufficient for optimality:

$$\begin{aligned} -\Delta_1 + 2 \cdot x_1 + \rho \cdot \theta \cdot x_2 + u - v_1 &= 0 \\ -\theta \cdot \Delta_2 + \rho \cdot \theta \cdot x_1 + 2 \cdot \theta \cdot x_2 + u - v_2 &= 0 \end{aligned}$$

In this case, the mathematical program has only three constraints, each one of them can be active or not in an optimal solution. Taken into account that the case in which the two non-negative constraints are active is trivial (this can happen if and only if  $R = 0$ ), there are six subcases:

Subcase 1.1:  $(0 < x_1 < R, x_2 = 0) \Rightarrow u = 0, v_1 = 0, v_2 \geq 0$

Then,  $x_1 = \Delta_1 / 2, v_2 = \theta \cdot (\rho \cdot \Delta_1 / 2 - \Delta_2)$ , and the conditions for this solution to be valid are:  $\Delta_1 / 2 < R, \Delta_2 \leq \rho \cdot \Delta_1 / 2$ .

Subcase 1.2:  $(x_1 = R, x_2 = 0) \Rightarrow u \geq 0, v_1 = 0, v_2 \geq 0$

The solution corresponding to this subcase is optimal iff  $u = \Delta_1 - 2 \cdot R \geq 0, v_2 = u + \rho \cdot \theta \cdot R - \theta \cdot \Delta_2 = \Delta_1 - 2 \cdot R + \rho \cdot \theta \cdot R - \theta \cdot \Delta_2 \geq 0$ , i.e.,  $\Delta_1 \geq 2 \cdot R$ ,  $\Delta_2 \leq \rho \cdot R + (\Delta_1 - 2 \cdot R) / \theta$ .

Subcase 1.3:  $(x_1 = 0, 0 < x_2 < R) \Rightarrow u \geq 0, v_1 \geq 0, v_2 = 0$

$x_2 = \Delta_2 / 2, v_1 = \rho \cdot \theta \cdot \Delta_2 / 2 - \Delta_1$ ; valid iff  $\Delta_2 / 2 < R, \Delta_1 \leq \rho \cdot \theta \cdot \Delta_2 / 2$ .

Subcase 1.4:  $(x_1 = 0, x_2 = R) \Rightarrow u \geq 0, v_1 \geq 0, v_2 = 0$

The solution corresponding to this subcase is optimal iff  $u = \theta \cdot (\Delta_2 - 2 \cdot R) \geq 0, v_1 = u + \rho \cdot \theta \cdot R - \Delta_1 = \theta \cdot (\Delta_2 - 2 \cdot R) + \rho \cdot \theta \cdot R - \Delta_1 \geq 0$ , i.e.,  $\Delta_2 \geq 2 \cdot R$ ,  $\Delta_1 \leq \theta \cdot (\Delta_2 - 2 \cdot R + \rho \cdot R)$ .

Subcase 1.5:  $(x_1, x_2 > 0, x_1 + x_2 < R) \Rightarrow u = 0, v_1 = 0, v_2 = 0$

$x_1 = (2 \cdot \Delta_1 - \rho \cdot \theta \cdot \Delta_2) / (4 - \rho^2 \cdot \theta), x_2 = (2 \cdot \Delta_2 - \rho \cdot \Delta_1) / (4 - \rho^2 \cdot \theta)$ . This solution is optimal provided that the expressions defining  $x_1, x_2$  are  $> 0$  and  $x_1 + x_2 = ((2 - \rho) \cdot \Delta_1 + (2 - \rho \cdot \theta) \cdot \Delta_2) / (4 - \rho^2 \cdot \theta) < R$  (note that the denominators of these expressions cannot be negative when the objective function is concave). In this subcase, the condition  $\rho_1 > \rho_2$  holds without exception.

Subcase 1.6:  $(x_1, x_2 > 0, x_1 + x_2 = R) \Rightarrow u \geq 0, v_1 = 0, v_2 = 0$

$$x_1 = (\Delta_1 - \theta \cdot \Delta_2 + (2 - \rho) \cdot \theta \cdot R) / (2 \cdot (1 - \rho \cdot \theta + \theta)),$$

$$x_2 = (-\Delta_1 + \theta \cdot \Delta_2 + (2 - \rho \cdot \theta) \cdot R) / (2 \cdot (1 - \rho \cdot \theta + \theta))$$

This solution is optimal provided that the expressions defining  $x_1, x_2$  are  $> 0$  (note that the denominator is  $\geq 2$ ) and

$$u = \theta \cdot ((2 - \rho) \cdot \Delta_1 + (2 - \rho \cdot \theta) \cdot \Delta_2 + (\rho^2 \cdot \theta - 4) \cdot R) / (2 \cdot (1 - \rho \cdot \theta + \theta)) \geq 0,$$

or, equivalently,  $(2 - \rho) \cdot \Delta_1 + (2 - \rho \cdot \theta) \cdot \Delta_2 \geq (4 - \rho^2 \cdot \theta) \cdot R$ . In this subcase, prices may be decreasing, stable or increasing, depending on the values of the parameters for each specific instance.

Therefore, even for the simple case  $T = 2$  there is no single shape for the optimal policies, which depend on the specific values of the data. Some optimal policies imply the depletion of the resource, while others do not. For instance, for a perfectly durable resource ( $\rho = 1$ ), stable demand and discount factors equal to 1 (therefore, with  $\theta = 1$ ) and initial stock in the market equal to 0 (what implies, taking into account the preceding assumptions,  $\Delta_1 = \Delta_2 = \Delta$ ), the applicable subcases would be subcase 1.5 or



subcase 1.6, according if the value of the ratio  $\Delta / R$  is  $< 3/2$  (in this subcase,  $x_1 = x_2 = \Delta / 3$ ; therefore, at the end of the planning horizon an amount of the resource equal to  $R - 2\Delta / 3$  would remain in the ground) or  $\geq 3/2$  (in this subcase 6,  $x_1 = x_2 = R / 2$  and the resource would be depleted).

3.2. Linear  $\varphi_t$  function with  $T = 3, s_0 = 0; Q_t = Q, P_t = P, \alpha_t = 1 (t = 1, 2, 3)$

The assumptions that define this setting imply  $\beta_t = \beta (t = 1, 2, 3)$  (therefore, we can leave them aside)  $\Delta_t = \Delta (t = 1, 2, 3)$ . Therefore, the Hessian is:

$$H(z) = \begin{pmatrix} 2 & \rho & \rho^2 \\ \rho & 2 & \rho \\ \rho^2 & \rho & 2 \end{pmatrix},$$

which is positive definite for all possible values of  $\rho (0 < \rho \leq 1)$ .

Moreover, the KKT conditions:

$$\begin{aligned} 2 \cdot x_1 + \rho \cdot x_2 + \rho^2 \cdot x_3 + u &= \Delta \\ \rho \cdot x_1 + 2 \cdot x_2 + \rho \cdot x_3 + u &= \Delta \\ \rho^2 \cdot x_1 + \rho \cdot x_2 + 2 \cdot x_3 + u &= \Delta \end{aligned}$$

Subcase 2.1:  $x_1 + x_2 + x_3 < R$

The condition that defines this subcase implies  $u = 0$ . Therefore, KKT conditions read as follows:

$$\begin{aligned} 2 \cdot x_1 + \rho \cdot x_2 + \rho^2 \cdot x_3 &= \Delta \\ \rho \cdot x_1 + 2 \cdot x_2 + \rho \cdot x_3 &= \Delta \\ \rho^2 \cdot x_1 + \rho \cdot x_2 + 2 \cdot x_3 &= \Delta \end{aligned}$$

whose solution is  $x_1 = x_3 = \frac{2 - \rho}{4} \cdot \Delta, x_2 = \frac{2 - 2 \cdot \rho + \rho^2}{4} \cdot \Delta$ . These values have to fulfil

$$x_1 + x_2 + x_3 < R, \text{ i. e., } \frac{\Delta}{R} < \frac{4}{6 - 4 \cdot \rho + \rho^2}.$$

Note that  $x_2 \leq x_1 = x_3$ , since  $2 - 2 \cdot \rho + \rho^2 \leq 2 - \rho (0 < \rho \leq 1)$ . Therefore, in this case the optimal policy shows a "bowl effect" (i.e., it is U-shaped), which is maximally apparent when  $\rho = 2 - \sqrt{2}$  (then,  $x_2 / x_1 = 2 \cdot (\sqrt{2} - 1) = 0,828$ ).

Subcase 2.2:  $x_1 + x_2 + x_3 = R$

Solving the four equations linear system yields:

$$x_1 = x_3 = \frac{2 - \rho}{6 - 4 \cdot \rho + \rho^2} \cdot R, x_2 = \frac{2 - 2 \cdot \rho + \rho^2}{6 - 4 \cdot \rho + \rho^2} \cdot R, u = \Delta - \frac{4}{6 - 4 \cdot \rho + \rho^2} \cdot R$$

That is optimal provided that  $u \geq 0$ , i.e.,  $\frac{\Delta}{R} \geq \frac{4}{6-4\rho+\rho^2}$ .

The optimal policy for this subcase shows the same bowl effect that the policy for subcase 2.1.

In both subcases, prices are strictly decreasing.

3.3. Linear  $\varphi_t$  function with  $\rho = 1$ ,  $s_0 = 0$ ;  $Q_t = Q$ ,  $P_t = P$ ,  $\alpha_t = 1$  ( $t = 1, \dots, T$ )

In this case, the KKT conditions read as follows:

$$-\Delta + 2 \cdot x_t + \sum_{\tau \neq t} x_\tau + u - v_t = -\Delta + x_t + \sum_{\tau=1}^T x_\tau + u - v_t = 0 \quad (t = 1, \dots, T)$$

Adding up these  $T$  equations gives:

$$-T \cdot \Delta + (T+1) \cdot \sum_{t=1}^T x_t + T \cdot u - \sum_{t=1}^T v_t = 0$$

Where either  $\sum_{t=1}^T x_t < R$  or  $\sum_{t=1}^T x_t = R$ .

Subcase 3.1:  $\sum_{t=1}^T x_t < R$

The condition defining this subcase implies  $u = 0$ . The solution  $x_t = \frac{1}{T+1} \cdot \Delta$  ( $t = 1, \dots, T$ )

fulfils the KKT conditions (since  $x_t > 0 \Rightarrow v_t = 0$ ) provided that  $\sum_{t=1}^T x_t = \frac{T}{T+1} \cdot \Delta < R$ , i.e.,

$$\Delta < \frac{T+1}{T} \cdot R.$$

Subcase 3.2:  $\sum_{t=1}^T x_t = R$

In this case,  $u = \Delta - \frac{T+1}{T} \cdot R$  ( $\geq 0$  iff  $\Delta \geq \frac{T+1}{T} \cdot R$ ) and  $x_t = \Delta - u - R = \frac{R}{T}$  ( $t = 1, \dots, T$ ).

In both subcases, prices decline regularly:

$$p_{t+1} = p_t - \frac{P}{\Delta} \cdot x_t \quad (t = 1, \dots, T-1), \text{ with } p_1 = P - \frac{P}{\Delta} \cdot x_1$$

In subcase 3.1,  $p_T = P / (T+1)$ ; therefore, the final price tends to zero as the number of periods tends to infinity). In 3.2,  $p_T = 0$ .

### 3.4. Numerical examples

#### 3.4.1. Linear $\varphi_t$ function with $s_0 = 0; Q_t = Q, P_t = P, \alpha_t = 1 (t=1, \dots, T)$

This example is an extension to any number of periods of the setting presented in 4.2. With  $T=10, Q=220, P=110, R=100, \rho = 0,9$ , the optimal productions, which deplete the resource, show an accentuated bowl effect (Fig. 1), while the prices decline, in the whole planning horizon, more than a 50% (Table 1).

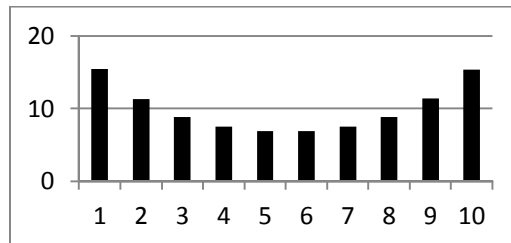


Fig. 1: Optimal productions in the 10 periods of the planning horizon for the example 4.4.1

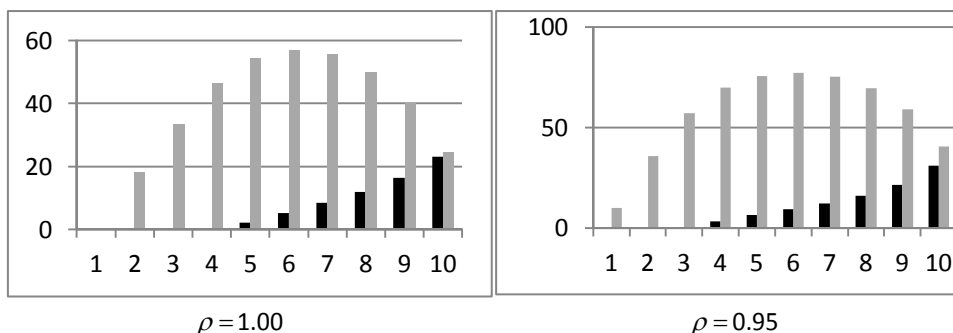
$t$	1	2	3	4	5	6	7	8	9	10
$x_t$	15,41	11,34	8,88	7,50	6,87	6,87	7,50	8,88	11,34	15,41
$P_t$	189,18	169,59	156,86	148,18	141,62	135,71	129,15	120,47	107,74	88,15

Table 1. Optimal productions and prices for the example 4.4.1

#### 3.4.2. Linear $\varphi_t$ functions with $T = 10, s_0 = 100, R = 100, Q_t = 100 + 10 \cdot (t - 1),$

$$P_t = 200, \alpha_t = 0.9^{t-1}$$

This example illustrates the significant effect of the parameter  $\rho$  on the shape of the optimal extraction policy and on the evolution of price. Note that when  $\rho = 1$  the price in the period 1 is 0, given that  $Q_1 = s_0$ .



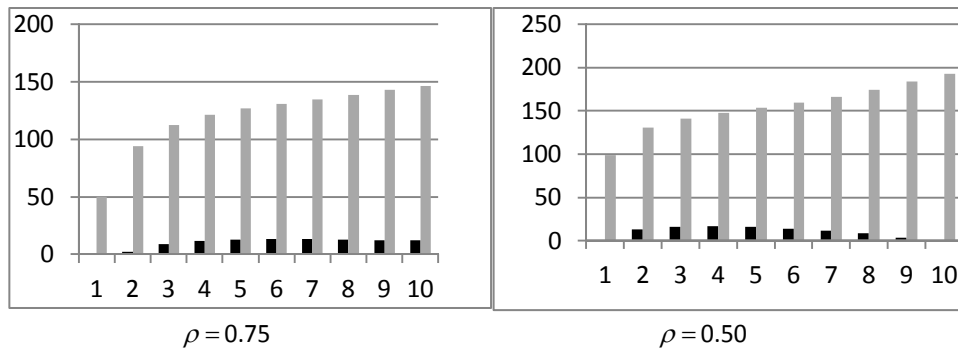


Fig. 2: Production (black bars) and prices (grey bars) for the optimal policies corresponding to the following data:

$$T = 10, s_0 = 100, R = 100, P_t = 200, Q_t = 100 + 10 \cdot (t - 1), \alpha_t = 0.9^{t-1}$$

In the case  $\rho = 1.00$  the total amount extracted is 66,968; in the other cases, the resource is depleted

#### 4. Conclusions and prospects

The problem of determining the optimal extraction policy of a durable non-renewable resource with a single primary supplier remains largely unexplored so far.

In this paper, a discrete-time mathematical programming model for the problem is proposed. The model allows dealing with  $\varphi_t$  functions (that give the equilibrium price as a function of the stock of resource in circulation) depending on time and with any degree of durability of the considered resource. Under some specific conditions on the shape of the  $\varphi_t$  functions and the values of the parameters, this formulation, using the Karush, Kuhn and Tucker conditions, allows studying the properties of the optimal solutions and computing easily the optimal extraction policy and the corresponding prices.

The applications of the model reveal that it is an efficient tool for determining the optimal extraction policies and the corresponding prices in a variety of settings.

Although in some particular settings the use of the KKT conditions yields analytic solutions and general properties of them, the analysis of the solutions obtained for different setting show that the optimal extraction policies and the corresponding prices do not exhibit a general shape.

Next future research on the problem will focus on incorporating the extraction costs in the model and extending the analysis to other types of  $\varphi_t$  functions, considering the possibility of diverse behaviours of the single primary suppliers and the other holders of the resource.

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