Green's kernel for subdivision networks

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In this work we calculate the Green's kernel for subdivision networks in terms of the corresponding kernel of original networks. Our techniques are based on the study of discrete operators using discrete Potential Theory. As a by–product we also obtain the effective resistances and the Kirchhoff index in terms of the corresponding parameters of each factor network.

The subdivision network $\Gamma^S = (V^S, E^S, c)$ of a given network $\Gamma = (V, E, c)$ is obtained by adding a new vertex v_{xy} at every edge $\{x, y\} \in E$ and by defining new conductances $c(x, v_{xy})$ so as to satisfy the *electrical compatibility* condition

$$\frac{1}{c(x,y)} = \frac{1}{c(x,v_{xy})} + \frac{1}{c(y,v_{xy})}.$$

The role of Green's kernel is crucial in this work. We take advantage of its relation not only with the evaluation of effective resistances of a network but also with the Kirchhoff index.

1 The Green kernel of a subdivision network

Taking into account the relation between Poisson problems on Γ^S and Γ, we obtain the expression of the Green kernel of a subdivision network, G^S, in terms of Green's kernel of the base network. From now on we consider the function on $\mathcal{C}(V)$, $\pi^S(x) = \sum\limits_{y\sim x} \alpha(x,y)$ and the constant

$$\beta = \frac{1}{(n+m)^2} \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y) + \frac{1}{(n+m)^2} \sum_{x\sim y} \frac{1}{k(v_{xy})}.$$

Proposition 1.1. *Let Γ^S be the subdivison network of Γ, then for any $x, z \in V$ and $v_{xy}, v_{zt} \in V'$,* *the Green kernel of* Γ^S *is given by*

$$
\begin{aligned}
G^S(x,z) &= G(x,z) - \frac{1}{n+m}\sum_{\ell\in V}\Big[G(x,\ell)+G(z,\ell)\Big]\pi^S(\ell)+\beta,\\
G^S(v_{xy},z) &= \alpha(x,y)G(x,z)+\alpha(y,x)G(y,z)\\
&- \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)+G(z,\ell)\Big]\pi^S(\ell) - \frac{1}{(n+m)k(v_{xy})}+\beta,\\
G^S(v_{xy},v_{zt}) &= \alpha(z,t)\Big(\alpha(x,y)G(x,z)+\alpha(y,x)G(y,z)\Big)+\alpha(t,z)\Big(\alpha(x,y)G(x,t)+\alpha(y,x)G(y,t)\Big)\\
&- \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)+\alpha(z,t)G(z,\ell)+\alpha(t,z)G(t,\ell)\Big]\pi^S(\ell)\\
&+ \frac{\varepsilon_{v_{zt}}(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)k(v_{xy})} - \frac{1}{(n+m)k(v_{zt})}+\beta.
\end{aligned}
$$

Proof. Suppose $z\in V$, and let $h_z=\varepsilon_z-\dfrac{1}{n+m}$. Then, for every $x\in V$

$$\underline{h}_z(x)=\varepsilon_z(x)-\frac{1}{n+m}-\frac{1}{n+m}\sum_{y\sim x}\alpha(x,y)=\varepsilon_z(x)-\frac{1}{n+m}(1+\pi^S(x)).$$

Hence, the Poisson problem to solve is $\mathcal{L}(u_z)=\underline{h}_z$, and, using the Green kernel for Γ, we obtain

$$u_z(x)=G(\varepsilon_z)(x)-\frac{1}{n+m}\sum_{\ell\in V}G(x,\ell)\pi^S(\ell)=G(x,z)-\frac{1}{n+m}\sum_{\ell\in V}G(x,\ell)\pi^S(\ell).$$

Then,

$$
\begin{aligned}
G_z^S(x) &= u_z^{h_z}(x)-\frac{1}{(n+m)}\sum_{r\sim s}\frac{h_z(v_{rs})}{k(v_{rs})}-\frac{1}{(n+m)}\sum_{r\sim s}[\alpha(r,s)u_z(r)+\alpha(s,r)u_z(s)]\\
&= G(x,z)-\frac{1}{n+m}\sum_{\ell\in V}G(x,\ell)\pi^S(\ell)+\frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})}\\
&- \frac{1}{(n+m)}\sum_{r\sim s}\alpha(r,s)\left[G(r,z)-\frac{1}{n+m}\sum_{\ell\in V}G(r,\ell)\pi^S(\ell)\right]\\
&- \frac{1}{(n+m)}\sum_{r\sim s}\alpha(s,r)\left[G(s,z)-\frac{1}{n+m}\sum_{\ell\in V}G(s,\ell)\pi^S(\ell)\right]\\
&= G(x,z)-\frac{1}{n+m}\sum_{\ell\in V}\Big[G(x,\ell)+G(z,\ell)\Big]\pi^S(\ell)+\frac{1}{(n+m)^2}\sum_{r,s}G(s,r)\pi^S(r)\pi^S(s)\\
&+ \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})}.
\end{aligned}
$$

Now, if $z \in V$ for every $v_{xy} \in V'$

$$
\begin{aligned}
G_z^S(v_{xy}) &= \frac{h_z(v_{xy})}{k(v_{xy})} + \alpha(x,y)u_z(x) + \alpha(y,x)u_z(y) \\
&- \frac{1}{(n+m)} \sum_{r\sim s} \frac{h_z(v_{rs})}{k(v_{rs})} - \frac{1}{(n+m)} \sum_{r\sim s} [\alpha(r,s)u_z(r) + \alpha(s,r)u_z(s)] \\
&= -\frac{1}{(n+m)k(v_{xy})} + \alpha(x,y)G(x,z) + \alpha(y,x)G(y,z) \\
&- \frac{1}{n+m} \sum_{\ell\in V} \big[\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell)\big]\pi^S(\ell) \\
&+ \frac{1}{(n+m)^2} \sum_{r\sim s} \frac{1}{k(v_{rs})} - \frac{1}{(n+m)} \sum_{r\sim s} \alpha(r,s) \left[G(r,z) - \frac{1}{n+m} \sum_{\ell\in V} G(r,\ell)\pi^S(\ell)\right] \\
&- \frac{1}{(n+m)} \sum_{r\sim s} \alpha(s,r) \left[G(s,z) - \frac{1}{n+m} \sum_{\ell\in V} G(s,\ell)\pi^S(\ell)\right] \\
&= -\frac{1}{(n+m)k(v_{xy})} + \alpha(x,y)G(x,z) + \alpha(y,x)G(y,z) \\
&- \frac{1}{n+m} \sum_{\ell\in V} \big[\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell) + G(z,\ell)\big]\pi^S(\ell) \\
&+ \frac{1}{(n+m)^2} \sum_{r\sim s} \frac{1}{k(v_{rs})} + \frac{1}{(n+m)^2} \sum_{r,s} G(s,r)\pi^S(r)\pi^S(s).
\end{aligned}
$$

Suppose now $v_{zt} \in V$, and let $h_{v_{zt}} = \varepsilon_{v_{zt}} - \dfrac{1}{n+m}$. Then, for every $x \in V$

$$
\begin{aligned}
\underline{h}_{v_{zt}}(x) &= \varepsilon_{v_{zt}}(x) - \frac{1}{n+m} + \sum_{y\in V} \alpha(x,y) \left(\varepsilon_{v_{zt}}(v_{xy}) - \frac{1}{n+m}\right) \\
&= -\frac{1}{n+m}(1+\pi^S(x)) + \alpha(z,t)\varepsilon_z(x) + \alpha(t,z)\varepsilon_t(x).
\end{aligned}
$$

Hence, the Poisson problem to solve is $\mathcal{L}(u_{v_{zt}}) = \underline{h}_{v_{zt}}$, and, using Green's kernel for Γ, we obtain

$$
u_{v_{zt}}(x) = -\frac{1}{n+m} \sum_{\ell\in V} G(x,\ell)\pi^S(\ell) + \alpha(z,t)G(x,z) + \alpha(t,z)G(x,t).
$$

Then,

$$\begin{aligned}
G^S_{v_{zt}}(v_{xy}) &= \frac{h_{v_{zt}}(v_{xy})}{k(v_{xy})} + \alpha(x,y)u_{v_{zt}}(x) + \alpha(y,x)u_{v_{zt}}(y) \\
&- \frac{1}{(n+m)}\sum_{r\sim s}\frac{h_{v_{zt}}(v_{rs})}{k(v_{rs})} - \frac{1}{(n+m)}\sum_{r\sim s}[\alpha(r,s)u_{v_{zt}}(r) + \alpha(s,r)u_{v_{zt}}(s)] \\
&= \frac{\varepsilon_{v_{zt}}(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)k(v_{xy})} \\
&- \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell)\Big)\pi^S(\ell) \\
&+ \alpha(z,t)\Big(\alpha(x,y)G(x,z) + \alpha(y,x)G(y,z)\Big) + \alpha(t,z)\Big(\alpha(x,y)G(x,t) + \alpha(y,x)G(y,t)\Big) \\
&- \frac{1}{(n+m)k(v_{zt})} + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} \\
&- \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(z,t)G(z,\ell) + \alpha(t,z)G(t,\ell)\Big)\pi^S(\ell) \\
&+ \quad \frac{1}{(n+m)^2}\sum_{r,s\in V} G(r,s)\pi^S(r)\pi^S(s) \\
&= \alpha(z,t)\Big(\alpha(x,y)G(x,z) + \alpha(y,x)G(y,z)\Big) + \alpha(t,z)\Big(\alpha(x,y)G(x,t) + \alpha(y,x)G(y,t)\Big) \\
&- \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell) + \alpha(z,t)G(z,\ell) + \alpha(t,z)G(t,\ell)\Big)\pi^S(\ell) \\
&+ \frac{\varepsilon_{v_{zt}}(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)k(v_{xy})} - \frac{1}{(n+m)k(v_{zt})} + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} \\
&+ \quad \frac{1}{(n+m)^2}\sum_{r,s\in V} G(r,s)\pi^S(r)\pi^S(s). \quad \square
\end{aligned}$$

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