Discrete Inverse Problems on Grids

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1 Recovering the conductances on a 3–dimensional grid

A three dimensional grid is the discretization of any cuboid in $\mathbb{R}^3$. Let us take three integers $\ell_i \in \mathbb{N}$, $i = 1, 2, 3$. We define the *three dimensional grid with boundary* as the network $\Gamma = (V, c)$ with vertex set

$$V = \{x_{ijk} : i = 0, \dots, \ell_1 + 1,\ j = 0, \dots, \ell_2 + 1,\ k = 0, \dots, \ell_3 + 1\}$$

and conductivity function c given by

$$c(x_{ijk}, x_{pqr}) > 0 \ \text{ when } \begin{cases} p = i \pm 1, & q = j \text{ and } r = k, \\ p = i, & q = j \pm 1 \text{ and } r = k, \\ p = i, & q = j \text{ and } r = k \pm 1 \end{cases}$$

for all $i = 1, \dots, \ell_1$, $j = 1, \dots, \ell_2$ and $k = 1, \dots, \ell_3$, $c(x_{ij0}, x_{ij1}) > 0$, $c(x_{ij\ell_3}, x_{ij\ell_3+1}) > 0$, $c(x_{0jk}, x_{1jk}) > 0$, $c(x_{\ell_1 jk}, x_{\ell_1+1jk}) > 0$, $c(x_{i0k}, x_{i1k}) > 0$ and $c(x_{i\ell_2 k}, x_{i\ell_2+1k}) > 0$ for all $i = 1, \dots, \ell_1$, $j = 1, \dots, \ell_2$ and $k = 1, \dots, \ell_3$ and $c(x, y) = 0$ otherwise. We say that i, j and k are the *first*, the *second* and the *third component* of the vertex $x_{ijk} \in V$, respectively.

We define the following sets of vertices

$$\begin{aligned} A_j &= \{x_{ijk} : i = 1, \dots, \ell_1,\ k = 1, \dots, \ell_3\}, & j &= 0, \dots, \ell_2 + 1, \\ R_j &= \left\{x_{ijk} : i = 0, \ell_1 + 1,\ k = 1, \dots, \ell_3\right\} \cup \left\{x_{ijk} : i = 1, \dots, \ell_1,\ k = 0, \ell_3 + 1\right\}, & j &= 1, \dots, \ell_2. \end{aligned}$$

If we consider the set $F = \bigcup\limits_{j=1}^{\ell_2} A_j$, then its boundary is given by $\delta(F) = A \cup R \cup B$, where

$$A = A_0, \quad R = \bigcup_{i=1}^{\ell_2} R_i \quad \text{and} \quad B = A_{\ell_2+1}.$$

See Figure 1 for an illustration of a three dimensional grid with $\ell_1 = \ell_3 = 2$ and $\ell_2 = 3$, and the associated boundary sets A, B and $R_1, \dots, R_{\ell_2}$.

Given an index $i \in \{1, \dots, a\}$, we consider the *partial layers* of vertices

$$D_i = \left\{x_{ijk} \in \bar{F} : k = a + 1 - i, \dots, \ell,\ j = 1, \dots, p\right\},$$

$D_{a+1} = \left\{x_{a+1\,jk} \in \bar{F} : k = 1, \dots, \ell, j = 1, \dots, p\right\}$ and $D_0 = \left\{x_{0jk} \in \bar{F} : k = a + 1, \dots, \ell, j = 1, \dots, p\right\}$. In particular, $D_{a+1}, D_0 \subset R$.

The recovery of conductances on a 3 dimensional grid is an iterative process, for we are not able to give explicit formulae for all the conductances at the same time but we can give a recovery algorithm instead. Hence, we describe the algorithm in steps, each of them requiring the information obtained in the last one.

To start with, let $\mathsf{N_q}$ be an irreducible and symmetric M–matrix of order $n = 2(\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3)$ satisfying that is a response matrix. Let $\lambda \geq 0$ be the lowest eigenvalue of $\mathsf{N_q}$ and $\sigma \in \Omega(\delta(F))$ the eigenvector associated with λ. In addition, we choose $\omega \in \Omega(\bar{F})$ such that $\omega = k\sigma$ on $\delta(F)$ with $0 < k < 1$.

Step 0

In this step we do not recover any conductance. However, we set the necessary tools to obtain them in future steps. Having fixed two indices $\mathfrak{r} \in \{1, \dots, p\}$ and $\mathfrak{s} \in \{1, \dots, \ell\}$, we consider the overdetermined partial Dirichlet–Neumann boundary value problem that consists in finding $u_{\mathfrak{rs}} \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u_{\mathfrak{rs}}) = 0 \text{ on } F, \quad u_{\mathfrak{rs}} = \varepsilon_{x_{a+1\,\mathfrak{rs}}} \text{ on } A \cup R \quad \text{and} \quad \frac{\partial u_{\mathfrak{rs}}}{\partial \mathsf{n}_F} = 0 \text{ on } A. \tag{1}$$

There exists a large set of vertices of the 3 dimensional grid Γ where $u_{\mathfrak{rs}} = 0$. We denote this set by

$$Z(u_{\mathfrak{rs}}) = \left\{x \in \bar{F} \,:\, u_{\mathfrak{rs}}(x) = 0\right\} = \mathsf{supp}(u_{\mathfrak{rs}})^c \subseteq \bar{F}.$$

Clearly, $A \subseteq Z(u_{\mathfrak{rs}})$. The size of $Z(u_{\mathfrak{rs}})$, however, is much bigger than the size of A.

Proposition 1.1. *Fixed $j = 1, \dots, \ell_2$ and $x_{ijk} \in R_j$, it is verified that for any $x \in A_s$, $0 \leq s \leq j$, and for any $x \in A_{j+r} \setminus N_r(x_{ij+rk})$, $r = 1, \dots, \ell_2 - j + 1$*

$$\widetilde{P}_q(x, x_{ijk}) = 0.$$

Moreover, fixed $x_{i0k} \in A$, it is verified that for any s and $x \in A_s \setminus N_{s-1}(x_{isk})$, $s = 1, \dots, \ell_2 + 1$

$$\widetilde{P}_q(x, x_{i0k}) = 0.$$

In particular, $\widetilde{\mathsf{P}}_{\mathsf{q}}(A_s; R_j) = 0$ for any $j = 1, \dots, \ell_2$ and $0 \leq s \leq j$ and $\widetilde{\mathsf{P}}_{\mathsf{q}}(A; A) = \widetilde{\mathsf{P}}_{\mathsf{q}}(A_1; A) = \mathsf{I}$.

Proposition 1.2. *The following equality is satisfied:*

$$Z(u_{\mathfrak{rs}}) = \left\{x_{ijk} \in \bar{F} \,:\, i = 1, \dots, a,\ k = 1, \dots, \ell,\ j = 1, \dots, r - \max\{0, i - a + k - \mathfrak{s}\}, r + \max\{0, i - a + k - \mathfrak{s}\}, \dots, p\right\}.$$

Step 1

Let us fix the indices $\mathfrak{r} \in \{1, \dots, p\}$ and $\mathfrak{s} \in \{1, \dots, \ell\}$ for this step and let us consider the unique solution $u_{\mathfrak{rs}} \in \mathcal{C}(\bar{F})$ of problem (2). We already know that $u_{\mathfrak{rs}} = 0$ on $A \cup (R \smallsetminus \{x_{a+1\,\mathfrak{rs}}\})$ and $u(x_{a+1\,\mathfrak{rs}}) = 1$. Moreover, the values of $u_{\mathfrak{rs}}$ on B are given by the matricial equation

$$\mathsf{u}_{\mathfrak{rs}_\mathsf{B}} = -\mathsf{N_q}(\mathsf{A}; \mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A}; \mathsf{x}_{\mathsf{a}+1\,\mathfrak{rs}}).$$

Notice that this means that all the values of $u_{\mathfrak{rs}}$ on B are known, for the Dirichlet–to–Robin map is known. In consequence, $u_{\mathfrak{rs}}$ is known on all the boundary $\delta(F)$. Let us define the vector $\mathsf{u_B} = \mathsf{u}_{\mathfrak{rs}_\mathsf{B}}$ for the sake of the simplicity of the notation. In Figure 1(b) we show all the information obtained at the end of this step.

Step 2

In this step we recover the conductances of all the boundary spikes by means of the boundary spike formula. However, we first need to determine some values of the modified Green matrix of a 3 dimensional grid.

Lemma 1.3. *For all $j = 1, \dots, p$ and $k = 1, \dots, \ell$, the value $\widetilde{G}_q(x_{ajk}, x_{ajk})$ is always zero.*

Now we are ready to determine the conductances $c(x_{0j}, x_{1j})$ for all $j = 1, \ldots, \ell$, using the boundary spike formula.

Corollary 1.4. *The conductances of the edges joining the vertices of D_{a+1} with the vertices of D_a are given by*

$$c(x_{a+1\,jk}, x_{ajk}) = \frac{\omega(x_{a+1\,jk})}{\omega(x_{ajk})}\Big(\mathsf{N_q}(\mathsf{x_{a+1\,jk}};\mathsf{x_{a+1\,jk}}) - \mathsf{N_q}(\mathsf{x_{a+1\,jk}};\mathsf{B})\cdot\mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1}\cdot\mathsf{N_q}(\mathsf{A};\mathsf{x_{a+1\,jk}}) - \lambda\Big)$$

for all $j = 1, \ldots, p$ and $k = 1, \ldots, \ell$.

Step 3

Again, let us fix the indices $\mathfrak{r} \in \{1, \ldots, p\}$ and $\mathfrak{s} \in \{1, \ldots, \ell\}$ in this step and let us consider the unique solution $u_{\mathfrak{rs}} \in \mathcal{C}(\bar{F})$ of problem (2). Then, we know all the values of $u_{\mathfrak{rs}}$ on D_a, as the following result shows.

Lemma 1.5. *The values of $u_{\mathfrak{rs}}$ on D_a are given by*

$$u_{\mathfrak{rs}}(x_{ajk}) = \frac{1}{c(x_{a+1\,jk}, x_{ajk})}\Big(\lambda u(x_{a+1\,jk}) - \mathsf{N_q}(\mathsf{x_{a+1\,jk}};\mathsf{x_{a+1\,\mathfrak{rs}}}) - \mathsf{N_q}(\mathsf{x_{a+1\,jk}};\mathsf{B})\cdot\mathsf{u_B}\Big) + \frac{\omega(x_{ajk})}{\omega(x_{a+1\,jk})}u_{\mathfrak{rs}}(x_{a+1\,jk})$$

for all $j = 1, \ldots, p$ and $k = 1, \ldots, \ell$.

Step 4

Here we find the conductances of all the edges with both ends in D_a and such that the third compenent of their ends is different. However, we state a more general result.

Proposition 1.6. *Let $i \in \{0, \ldots, a-1\}$. For every $\mathfrak{r} = 1, \ldots, p$ and $\mathfrak{s} = 1, \ldots, \ell$, let us suppose that we know the values of $u_{\mathfrak{rs}}$ on D_{i+2} and D_{i+1}. Also, we suppose that the conductances of all the edges joining vertices from D_{i+2} and D_{i+1} are known. Now we fix the indices $\mathfrak{r} = 1, \ldots, p$ and $\mathfrak{s} = 1, \ldots, \ell$. Then, the conductances $c(x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i-1}, x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i})$ are also known. They are given by*

$$c(x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i-1}, x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i}) = -\frac{u_{\mathfrak{rs}}(x_{i+2\,\mathfrak{r}\,\mathfrak{s}+a-i-1})}{u_{\mathfrak{rs}}(x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i})}\,c(x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i-1}, x_{i+2\,\mathfrak{r}\,\mathfrak{s}+a-i-1}).$$

When $i = a-1$, Proposition 1.6 shows that $c(x_{a\mathfrak{rs}}, x_{a\mathfrak{r}\,\mathfrak{s}+1})$ is known for all $\mathfrak{r} = 1, \ldots, p$ and $\mathfrak{s} = 1, \ldots, \ell$. See Figure 1(e) in order to see all the known information at the end of this step.

Step 5

In this step we give the conductances of all the edges with both ends in D_a that are still unknown. Furthermore, we state a more general result.

Proposition 1.7. *Let $i \in \{0, \ldots, a-1\}$. For every $\mathfrak{r} = 1, \ldots, p$ and $\mathfrak{s} = 1, \ldots, \ell$, let us suppose that we know the values of $u_{\mathfrak{rs}}$ on D_{i+2} and D_{i+1}. Also, let us assume that we know the conductances of all the edges joining vertices from D_{i+2} and D_{i+1}, and the ones of the edges with both ends in D_{i+1} and such that the ends have different third component. Now we fix the indices $\mathfrak{r} = 1, \ldots, p$ and $\mathfrak{s} = 1, \ldots, \ell$. Then, the conductances $c(x_{i+1\,\mathfrak{r}+1\,\mathfrak{s}+a-i}, x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i})$ are also known. They are given by*

$$c(x_{i+1\,\mathfrak{r}+1\,\mathfrak{s}+a-i}, x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i}) = -\frac{u_{\mathfrak{rs}}(x_{i+1\,\mathfrak{r}+1\,\mathfrak{s}+a-i+1})}{u_{\mathfrak{rs}}(x_{i+1\,\mathfrak{r}\,\mathfrak{s}+a-i})}\,c(x_{i+1\,\mathfrak{r}+1\,\mathfrak{s}+a-i}, x_{i+1\,\mathfrak{r}+1\,\mathfrak{s}+a-i+1}).$$

When $i = a-1$, Proposition 1.7 shows that $c(x_{a\,\mathfrak{r}+1\,\mathfrak{s}+1}, x_{a\mathfrak{r}\,\mathfrak{s}+1})$ is known for all $\mathfrak{r} = 1, \ldots, p$ and $\mathfrak{s} = 1, \ldots, \ell$. See Figure 1(f) in order to see all the information gathered at the end of this step.

Step 6

Let us define the linear operator $\wp\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(F)$ given by the values

$$\wp(v)(x_{ijk}) = c(x_{ijk}, x_{ij\,k-1})v(x_{ij\,k-1}) + c(x_{ijk}, x_{ij\,k+1})v(x_{ij\,k+1}) + c(x_{ijk}, x_{i\,j-1\,k})v(x_{i\,j-1\,k})$$
$$+ c(x_{ijk}, x_{i\,j+1\,k})v(x_{i\,j+1\,k}) + c(x_{ijk}, x_{i+1\,jk})v(x_{i+1\,jk})$$

for all $v \in \mathcal{C}(\bar{F})$ and $x_{ijk} \in F$. This operator will be useful in this and also in the following steps.

In this step we give the conductances of all the edges joining the vertices from D_a and D_{a-1}. See Figure 1(g) in order to see all the information obtained at the end of this step.

Step 7

In this step we are able to obtain the unknown values of $u_{\mathfrak{rs}}$ on D_{a-1} for all $\mathfrak{r} = 1, \dots, p$ and $\mathfrak{s} = 1, \dots, \ell$. In fact, let us state a more general result.

Proposition 1.8. *Let $i \in \{0, \dots, a-1\}$. For every $\mathfrak{r} = 1, \dots, p$ and $\mathfrak{s} = 1, \dots, \ell$, let us suppose that we know the values of $u_{\mathfrak{rs}}$ on D_{i+2} and D_{i+1}. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i+2} and D_{i+1}, from D_{i+1} and D_i and the ones of the edges with both ends in D_{i+1}. Now fix the indices $\mathfrak{r} = 1, \dots, p$ and $\mathfrak{s} = 1, \dots, \ell$. Then, the values of $u_{\mathfrak{rs}}$ on D_i are also known. They are given by*

$$u_{\mathfrak{rs}}(x_{ijk}) = \frac{\wp(u_{\mathfrak{rs}})(x_{i+1\,jk})}{c(x_{i+1\,jk}, x_{ijk})} - \frac{\wp(\omega)(x_{i+1\,jk})}{\omega(x_{i+1\,jk})c(x_{i+1\,jk}, x_{ijk})}u_{\mathfrak{rs}}(x_{i+1\,jk}) - \frac{\omega(x_{ijk})}{\omega(x_{i+1\,jk})}u_{\mathfrak{rs}}(x_{i+1\,jk})$$

for all $j = 1, \dots, \ell$ and $k = \mathfrak{r} + a - i, \dots, \ell$.

Proof. Fixed three indices $i \in \{0, \dots, a-1\}$, $\mathfrak{r} = 1, \dots, p$ and $\mathfrak{s} = 1, \dots, \ell$, let $j \in \{1, \dots, p\}$ and $k \in \{\mathfrak{r} + a - i, \dots, \ell\}$. Observe that $\wp(\omega)(x_{i+1\,jk})$ and $\wp(u_{\mathfrak{rs}})(x_{i+1\,jk})$ are already known. Then,

$$0 = \mathcal{L}_q(u_{\mathfrak{rs}})(x_{i+1\,jk}) = \frac{u_{\mathfrak{rs}}(x_{i+1\,jk})}{\omega(x_{i+1\,jk})}\wp(\omega)(x_{i+1\,jk}) - \wp(u_{\mathfrak{rs}})(x_{i+1\,jk})$$
$$- c(x_{i+1\,jk}, x_{ijk})u_{\mathfrak{rs}}(x_{ijk}) + \frac{\omega(x_{ijk})}{\omega(x_{i+1\,jk})}c(x_{i+1\,jk}, x_{ijk})u_{\mathfrak{rs}}(x_{i+1\,jk})$$

and hence $u_{\mathfrak{rs}}(x_{ijk})$ is the unique unknown term of this equality. □

In particular, when $i = a-1$, Proposition 1.8 shows that $u_{\mathfrak{rs}}$ is known on D_{a-1} for all $\mathfrak{r} = 1, \dots, p$ and $\mathfrak{s} = 1, \dots, \ell$. Observe that we already knew some of the values of $u_{\mathfrak{rs}}$ on D_{a-1}, which are those of the vertices in $Z(u_{\mathfrak{rs}})$. Figure 1(h) shows the information obtained until this step.

Step 8 and beyond

We keep repeating the same process to obtain more conductances, that is, we keep applying Proposition 1.6 from Step 4, then Proposition 1.7 from Step 5, **??** from Step 6 and then Proposition 1.8 from Step 7 for each $i = a-2, \dots, 1$. We stop when we have obtained all the conductances between and all the vertices in $\bigcup_{i=1}^{a} D_i$, see Figure 1(j).

The final step left is to rotate the grid (see Figure1(k)), that is, to consider the new boundary sets $A = \big\{x_{ij\,\ell+1} \in \bar{F} \,:\, i = 1, \dots, a,\, j = 1, \dots, p\big\}$ and $B = \big\{x_{ij0} \in \bar{F} \,:\, i = 1, \dots, a,\, j = 1, \dots, p\big\}$ instead of the previous ones, and consider now the overdetermined partial Dirichlet–Neumann boundary value problem

$$\mathcal{L}_q(u_{\mathfrak{rs}}) = 0 \;\text{ on }\; F, \quad u_{\mathfrak{rs}} = \varepsilon_{x_{0\mathfrak{rs}}} \;\text{ on }\; A \cup R \quad\text{and}\quad \frac{\partial u_{\mathfrak{rs}}}{\partial \mathsf{n}_{F}} = 0 \;\text{ on }\; A. \tag{2}$$

for $\mathfrak{r} \in \{1, \ldots, p\}$ and $\mathfrak{s} \in \{1, \ldots, \ell\}$. By proceeding analogously to the last steps, we obtain the lacking conductances of the 3 dimensional grid, see Figure 1(l).

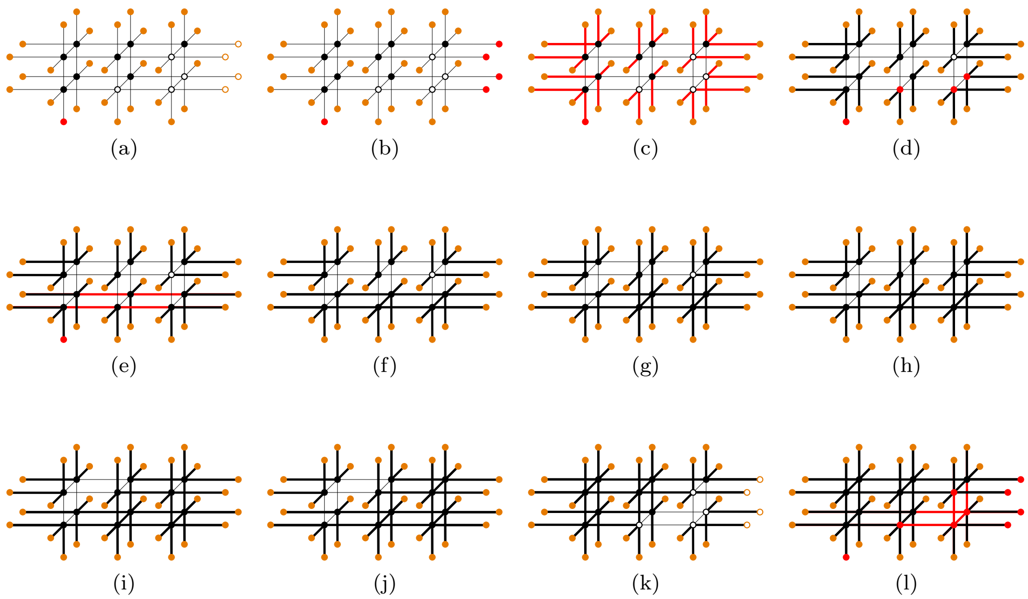

Figure 1: The bold items are the ones known at the end of each step for the case $\mathfrak{r} = \mathfrak{s} = 1$.

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