Explicit inverse of a tridiagonal (p, r)–Toeplitz matrix

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Abstract

Tridiagonal matrices appears in many contexts in pure and applied mathematics, so the study of the inverse of these matrices becomes of specific interest. In recent years the invertibility of nonsingular tridiagonal matrices has been quite investigated in different fields, not only from the theoretical point of view (either in the framework of linear algebra or in the ambit of numerical analysis), but also due to applications, for instance in the study of sound propagation problems or certain quantum oscillators. However, explicit inverses are known only in a few cases, in particular when the tridiagonal matrix has constant diagonals or the coefficients of these diagonals are subjected to some restrictions like the tridiagonal k–Toeplitz matrices, such that their three diagonals are formed by k-periodic sequences.

The recent formulae for the inversion of tridiagonal k–Toeplitz matrices are based, more o less directly, on the solution of difference equations with periodic coefficients, though all of them use complex formulation that in fact don't take into account the periodicity of the coefficients.

This contribution presents the explicit inverse of a tridiagonal matrix (p, r)–Toeplitz, which diagonal coefficients are in a more general class of sequences than periodic ones, that we have called quasi-periodic sequences. A tridiagonal matrix $A = (a_{ij})$ of order $n + 2$ is (p, r)–Toeplitz if there exists $m \in \mathbb{N} \setminus \{0\}$ such that $n + 2 = mp$ and

$$a_{i+p,j+p} = r\, a_{ij}, \quad i, j = 0, \cdots, (m-1)p$$

More generally, if A is a (p, r)-âĂŞToeplitz, then

$$a_{i+kp,j+kp} = r^k\, a_{ij}, \quad i, j = 0, \cdots, (m-1)p, \quad k = 0, \cdots, (m-1)$$

we develop a procedure that reduces any linear second order difference equation with periodic coefficients to a difference equation of the same kind but with constant coefficients. Therefore, the solutions of the former equations can be expressed in terms of Chebyshev polynomials This fact explain why Chebyshev polynomials are ubiquitous in

the above mentioned papers In addition, we show that this results are true when the coefficients are in a more general class of sequences, that we have called quasi-periodic sequences As a by-product, the inversion of these class of tridiagonal matrices could be explicitly obtained through the resolution of boundary value problems on a path

Making use of the theory of orthogonal polynomials, we will give the explicit inverse of tridiagonal 2-Toeplitz and 3-Toeplitz matrices, based on recent results from which enables us to state some conditions for the existence of A^-1. We obtain explicit formulas for the entries of the inverse of a based on results from the theory of orthogonal polynomials and it is shown that the entries of the inverse of such a matrix are given in terms of Chebyshev polynomials of the second kind.

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It is well–known that any second order linear difference equation with constant coefficients is equivalent to a Chebyshev equation. The aim of this work is to extend this result to any second order linear difference equation with quasi–periodic coefficients.

We say that a sequence z, $z\colon \mathbb{Z} \longrightarrow \mathbb{C}$, is *quasi–periodic with period* $p \in \mathbb{N} \setminus \{0\}$ if there exists $r \in \mathbb{C} \setminus \{0\}$ such that

$$z(k+p) = r\, z(k), \quad k \in \mathbb{Z},$$

Given a and b two quasi–periodic sequences with period p, we can find a function $Q_p(a;b)$ such that if the sequence z is a solution of the symmetric second order linear difference equation

$$a(k)z(k+1) - b(k)z(k) + a(k-1)z(k-1) = 0, \quad k \in \mathbb{Z},$$

then for any $m = 0, \dots, p-1$, $z_{p,m}$ the subsequence of z defined as

$$z_{p,m}(k) = z(kp+m), \quad k \in \mathbb{Z}.$$

is a solution of the Chebyshev equation

$$2Q_p(a;b)z_{p,m}(k) = z_{p,m}(k+1) + z_{p,m}(k-1), \quad k \in \mathbb{Z}.$$

The function Q_p only depends on the first p values, counting from 0, of the sequences a and b and it can be determined by a non–linear recurrence. Moreover, we obtain a closed expression for the function Q_p for any period.

Finally, we extend our results to solve general second order linear difference equations with quasi–periodic coefficients a, b and c

$$a(k)z(k+1) - b(k)z(k) + c(k-1)z(k-1) = 0, \quad k \in \mathbb{Z}.$$

1 Introduction

Tridiagonal matrices are commonly named *Jacobi matrices*, and the computation of its inverse is in relation with discrete Schrödinger operators on a finite path. If we consider $n \in \mathbb{N} \setminus \{0\}$, the set $\mathcal{M}_n(\mathbb{R})$ of matrices with order $n \in \mathbb{N}$ and real coefficients, and the sequences $\{a(k)\}_{k=0}^n$, $\{b(k)\}_{k=0}^{n+1}$, $\{c(k)\}_{k=0}^n \subset \mathbb{R}$, a Jacobi matrix $\mathsf{J}(a,b,c) \in \mathcal{M}_{n+2}(\mathbb{R})$ has the following structure:

$$\mathsf{J}(a,b,c) = \begin{bmatrix} b(0) & -a(0) & 0 & \cdots & 0 & 0 \\ -c(0) & b(1) & -a(1) & \cdots & 0 & 0 \\ 0 & -c(1) & b(2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b(n) & -a(n) \\ 0 & 0 & 0 & \cdots & -c(n) & b(n+1) \end{bmatrix}$$

As in [?], we have chosen to write down the coefficients outside the main diagonal with negative sign. This is only a convenience convention, motivated by the mentioned relationship between Jacobi matrices and Schrödinger operators on a path. We will also require $a(k) \neq 0$ and $c(k) \neq 0$, $k = 0, \dots, n$, so in other case $\mathsf{J}(a,b,c)$ is reducible and the inversion problem leads to the invertibility of a matrix of lower order.

The matrix $\mathsf{J}(a,b,c)$ is invertible iff for each $\mathsf{f} \in \mathbb{R}^{n+2}$ exists $\mathsf{u} \in \mathbb{R}^{n+2}$ such that

$$\left\{ \begin{array}{rl} b(0)u(0) - a(0)u(1) = f(0), & \\ -a(k)u(k+1) + b(k)u(k) - c(k-1)u(k-1) = f(k), & k = 1, \dots, n, \\ -c(n)u(n) + b(n+1)u(n+1) = f(n+1). & \end{array} \right. \tag{1}$$

We can recognize in previous identities the structure of a boundary value problem associated to a second order linear difference equation or, equivalently, a Schrödinger operator on the path $\mathrm{I} = \{0, \dots, n+1\}$ with boundary $\delta(\mathrm{I}) = \{0, n+1\}$ and, hence $\overset{\circ}{\mathrm{I}} = \{1, \dots, n\}$.

Let $\mathcal{C}(\mathrm{I})$ be the vector space of real functions whose domain is the set I, and the functions $u, f \in \mathcal{C}(\mathrm{I})$. Using this functional notation, Equations (1) are equivalent to the Sturm–Liouville value problem

$$\mathcal{L}_q(u) = f \ \text{ en } \overset{\circ}{\mathrm{I}}, \quad \mathcal{L}_q(u)(0) = f(0) \ \text{ y } \ \mathcal{L}_q(u)(n+1) = f(n+1). \tag{2}$$

Therefore, $\mathsf{J}(a,b,c)$ is invertible iff $\mathcal{L}_q$ is invertible. In terms of the boundary value problem, the invertibility conditions for $\mathsf{J}(a,b,c)$ are exactly the same conditions to ensure that the boundary value problem is *regular*, that is with a unique solution, and the computation of its inverse can be reduced to the calculus of this solution. Implicitly or explicitly, to determine the solutions for initial or final value problems is the strategy followed to achieve the inversion of tridiagonal matrices, see i.e. [8, 9], but either it is not analyzed the general case, or the explicit expressions of these solutions are not obtained, or the expressions obtained are excessively cumbersome.

2 Regular boundary value problems

It is well-known that every initial value problem for the Schrödinger equation on I has a unique solution. Specifically, the set $\mathcal{S}$ of solutions of the homogeneous Schrödinger equation is a vector space whose $\dim \mathcal{S} = 2$, while for any $f \in \mathcal{C}(\mathrm{I})$, the set $\mathcal{S}(f)$ of solutions of the Schrödinger equation with data f satisfies $\mathcal{S}(f) \neq \emptyset$ and given $u \in \mathcal{S}(f)$, it is verified $\mathcal{S}(f) = u + \mathcal{S}$.

Given two solutions $u, v \in \mathcal{C}(\mathrm{I})$ of the homogeneous Schrödinger equation, their *wronskian* or *casoratian*, see [1], is $w[u,v] \in \mathcal{C}(\mathrm{I})$ defined as

$$w[u,v](k) = u(k)v(k+1) - v(k)u(k+1), \quad 0 \leq k \leq n,$$

and as $w[u,v](n+1) = w[u,v](n)$. Either $w[u,v] = 0$ or $w[u,v]$ is always non null. Moreover, u and v are linearly dependent iff their wronskian is null.

The wronskian allow us to obtain a basis of $\mathcal{S}$ and, hence, to express the solution of all initial value problem associated to the homogeneous equation in terms of that basis. The solution of the complete equation with data f, we just have to add a particular solution. The *Green's function* of the homogeneous Schrödinger equation on I can be obtained from a basis of $\mathcal{S}$ and it allow us to obtain a particular solution, see [4, Lemma 1.2 y Proposition 1.3]. Therefore, to solve the boundary value problem (2) it will be enough to choose the appropriated basis of solutions. Moreover, it will be very useful to introduce the *companion function* defined as $\rho(k) = \prod\limits_{s=0}^{k-1} \dfrac{a(s)}{c(s)}$, $k = 0, \ldots, n+1$.

Theorem 2.1. *Consider Φ and Ψ the unique solutions of the homogeneous Schrödinger equation on $\overset{\circ}{\mathrm{I}}$ that verify*

$$\Phi(0) = a(0), \; \Phi(1) = b(0), \; \Psi(n) = -b(n+1), \; \Psi(n+1) = -c(n).$$

Then, $a(0)\left(b(0)\Psi(0) - a(0)\Psi(1)\right) = a(n)\rho(n)\left(c(n)\Phi(n) - b(n+1)\Phi(n+1)\right)$, *the Schrödinger operator $\mathcal{L}_q$ is invertible iff $b(0)\Psi(0) \neq a(0)\Psi(1)$ and, moreover, given $f \in \mathcal{C}(\mathrm{I})$, for any $k = 0, \ldots, n+1$,*

$$(\mathcal{L}_q)^{-1}(f)(k) = \sum_{s \in \mathrm{I}} \frac{\Phi(\min\{k,s\})\Psi(\max\{k,s\})}{a(0)\Big[b(0)\Psi(0) - a(0)\Psi(1)\Big]}\rho(s)f(s).$$

Proof. The value $b(0)\Psi(0) - a(0)\Psi(1)$ corresponds to the wronskian $w[\Psi,\Phi](0)$. For the rest of the demonstration, consult [4, Section 3]. □

3 The inverse of a tridiagonal (p,r)–Toeplitz matrix

A *Toeplitz matrix* is a square matrix with constant diagonals. Therefore, a tridiagonal matrix (or Jacobi matrix) which is also a Toeplitz matrix has the three main diagonals constant and the rest are null.

Definition 3.1. *Consider $p \in \mathbb{N} \setminus \{0\}$ and $r \in \mathbb{R} \setminus \{0\}$. The Jacobi matrix $\mathsf{J}(a,b,c)$, where $a, b, c \in \mathcal{C}(\mathrm{I})$, $a(k) \neq 0$ and $c(k) \neq 0$ for any $k = 0, \ldots, n$, $a(n+1) = c(n)$ and $c(n+1) = a(n)$, is a (p,r)–*Toeplitz matrix *if it exists $m \in \mathbb{N} \setminus \{0\}$ such that $n+2 = mp$, and a, b and c are quasi–periodic coefficients; that is*

$$a(p+j) = ra(j), \quad b(p+j) = rb(j) \quad \textit{and} \quad c(p+j) = rc(j), \quad j = 0, \ldots, (m-1)p.$$

If $r = 1$, the Jacobi $(p, 1)$–Toeplitz matrices are the ones so–called *tridiagonal p–Toeplitz matrices*, see for instance [7], whose coefficients are periodic with period p. When $p = 1$ too, the Jacobi $(1, 1)$–Toeplitz matrices are the matrices referenced at the beginning of this section, the *tridiagonal and Toeplitz matrices.* Note also that Jacobi $(1, r)$–Toeplitz matrices are those whose diagonals are *geometrical sequences with ratio* r.

Since a Jacobi (p, r)–Toeplitz matrix is in fact a Jacobi matrix, to determine its inverse, $\mathsf{J}^{-1} = \mathsf{R} = (r_{ks})$, is equivalent to obtain the inverse of the Schrödinger operator on a path described in Theorem 2.1. Our goal is to compute explicitly the functions $\Phi_{a,b,c}$ and $\Psi_{a,b,c}$.

The first result correspond to the easiest case, the Jacobi $(1, 1)$–Toeplitz matrices. In this case, the Schrödinger operator corresponds to a second order linear difference equation with constant coefficients, so its solution can be expressed in terms of Chebyshev polynomials. The expression obtained coincides with that published by Fonseca and Petronilho in [6, Corollary 4.1] and [7, Equation 4.26].

Proposition 3.2. *If* $\alpha\gamma \neq 0$, *the Jacobi* $(1, 1)$*–Toeplitz matrix*

$$\mathsf{J}(\alpha,\beta,\gamma) = \begin{bmatrix} \beta & -\alpha & 0 & \cdots & 0 & 0 \\ -\gamma & \beta & -\alpha & \cdots & 0 & 0 \\ 0 & -\gamma & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & -\alpha \\ 0 & 0 & 0 & \cdots & -\gamma & \beta \end{bmatrix}$$

is invertible iff

$$\beta \neq 2\sqrt{\alpha\gamma}\cos\left(\frac{k\pi}{n+3}\right), \qquad k = 1, \dots, n+2,$$

and then,

$$r_{ks} = \frac{1}{U_{n+2}(q)} \begin{cases} \alpha^{s-k}(\sqrt{\alpha\gamma})^{k-s-1}U_k(q)U_{n-s+1}(q), & \text{if } 0 \leq k \leq s \leq n+1, \\ \gamma^{k-s}(\sqrt{\alpha\gamma})^{s-k-1}U_s(q)U_{n-k+1}(q), & \text{if } 0 \leq s \leq k \leq n+1. \end{cases}$$

where $q = \dfrac{\beta}{2\sqrt{\alpha\gamma}}$.

Proof. The firs part is consequence of Theorem 2.1 taking into account that the functions $\Phi_{a,b,c}$ and $\Psi_{a,b,c}$ are the solutions of a second order linear homogeneous difference equation with constant coefficients, hence can be expressed as a linear combination of Chebyshev polynomials of second kind, applying [4, Theorem 2.4]. □

At the general case, the Jacobi (p, r)–Toeplitz matrix, the Schrödinger equation has quasi–periodic coefficients, so we can apply the results presented in [4], a previous work of the authors devoted to the study of these kind of equations. Our main result appears now as a consequence of [4, Theorem 3.3].

Theorem 3.3. *Consider $p, m \in \mathbb{N}^*$, such that $pm = n+2$, $r \in \mathbb{R}\setminus\{0\}$,* $\mathsf{J}(a,b,c)$ *a* Jacobi (p,r)–Toeplitz matrix, *the Floquet function $q_{p,r}$ and the functions*

$$\begin{aligned} u(k,\ell) &= \theta\Phi_{a,b,c}(p+\ell)U_{k-1}(q_{p,r}) - \Phi_{a,b,c}(\ell)U_{k-2}(q_{p,r}),\\ v(k,\ell) &= \Psi_{a,b,c}\big(n+2-2p+\ell\big)U_{m-k-2}(q_{p,r})\\ &\quad - \theta\Psi_{a,b,c}\big(n+2-p+\ell\big)U_{m-k-3}(q_{p,r}), \end{aligned}$$

defined for any $k = 0,\dots,m-1$ and for any $\ell = 0,\dots,p-1$, where

$$\Phi_{a,b,c}(k) = \Big(\prod_{j=1}^{k-1} a(j)\Big)^{-1}\Big[b(0)P_{k-1}(b,ac) - a(0)c(0)P_{k-2}(b_1,a_1c_1)\Big] \quad \textit{and}$$

$$\begin{aligned}\Psi_{a,b,c}(n+1-k) = \Big(\prod_{j=n+1-k}^{n-1} c(j)\Big)^{-1}\Big[&a(n)c(n)P_{k-2}(b_{n+1-k}, a_{n+1-k}c_{n+1-k})\\ &-b(n+1)P_{k-1}(b_{n+1-k}, a_{n+1-k}c_{n+1-k})\Big].\end{aligned}$$

Then, $\mathsf{J}(a,b,c)$ *is invertible iff*

$$b(0)v(0,0) \neq a(0)v(0,1)$$

and, moreover,

$$r_{kp+\ell, sp+\hat\ell} = \frac{\rho(\hat\ell)}{a(0)r^s\theta^{k-s}d_{\mathsf{J}}}\begin{cases} u(k,\ell)v(s,\hat\ell), & si\ k<s,\\ u(s,\hat\ell)v(k,\ell), & si\ k<s,\\ u(s,\min\{\ell,\hat\ell\})v(s,\max\{\ell,\hat\ell\}), & si\ k=s,\end{cases}$$

where $d_{\mathsf{J}} = b(0)v(0,0) - a(0)v(0,1)$.

Proof. Theorem 3.3 in [4] establishes that

$$\Phi_{a,b,c}(kp+\ell) = \theta^{-k}u(k,\ell) \quad \text{and} \quad \Psi_{a,b,c}(kp+\ell) = \theta^{m-k-2}v(k,\ell),$$

for any $k = 0,\dots,m-1$ and for any $\ell = 0,\dots,p-1$. Taking into account $\rho(kp+\ell) = \rho(p)^k\rho(\ell)$ for any $k \in \mathbb{N}$ and for any $\ell = 0,\dots,p-1$ and considering the identities

$$\begin{aligned}\prod_{j=1}^{p-1+\ell} a(j) &= r^{\ell}\Big(\prod_{j=1}^{p-1} a(j)\Big)\Big(\prod_{j=0}^{\ell-1} a(j)\Big)\\ \prod_{j=n+1-p-\ell}^{n-1} c(j) &= r^{-\ell}\Big(\prod_{j=n+1-p}^{n-1} c(j)\Big)\Big(\prod_{j=n+1-\ell}^{n} c(j)\Big),\end{aligned}$$

to evaluate the required values of the functions $\Phi_{a,b,c}(k)$ and $\Psi_{a,b,c}(n+1-k)$, $k = 0,\dots,2p-1$, we just need to apply Theorem 2.1. □

The last result corresponds to applying the previous theorem to tridiagonal matrices whose diagonals are geometric sequences and, in our opinion, a new result in the literature.

Corollary 3.4. *If* $r \in \mathbb{R} \setminus \{0\}$*, the* Jacobi $(1,r)$–Toeplitz matrix

$$\mathsf{J}(\alpha,\beta,\gamma;r) = \begin{bmatrix} \beta & -\alpha & 0 & \cdots & 0 & 0 \\ -\gamma & \beta r & -\alpha r & \cdots & 0 & 0 \\ 0 & -\gamma r & \beta r^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta r^n & -\alpha r^n \\ 0 & 0 & 0 & \cdots & -\gamma r^n & \beta r^{n+1}. \end{bmatrix}$$

is invertible iff

$$\beta\sqrt{r} \neq 2\sqrt{\alpha\gamma}\cos\left(\frac{k\pi}{n+3}\right), \qquad k = 1,\dots,n+2,$$

and then,

$$r_{ks} = \frac{\left(\sqrt{\frac{\alpha r}{\gamma}}\right)^{s+1-k} U_{\min\{k,s\}}\left(\frac{\beta}{2}\sqrt{\frac{r}{\alpha\gamma}}\right) U_{n+1-\max\{k,s\}}\left(\frac{\beta}{2}\sqrt{\frac{r}{\alpha\gamma}}\right)}{\alpha r^s U_{n+2}\left(\frac{\beta}{2}\sqrt{\frac{r}{\alpha\gamma}}\right)}.$$

Supported by the Spanish Research Council under project MTM2014-60450-R.

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