

# Degree in Mathematics

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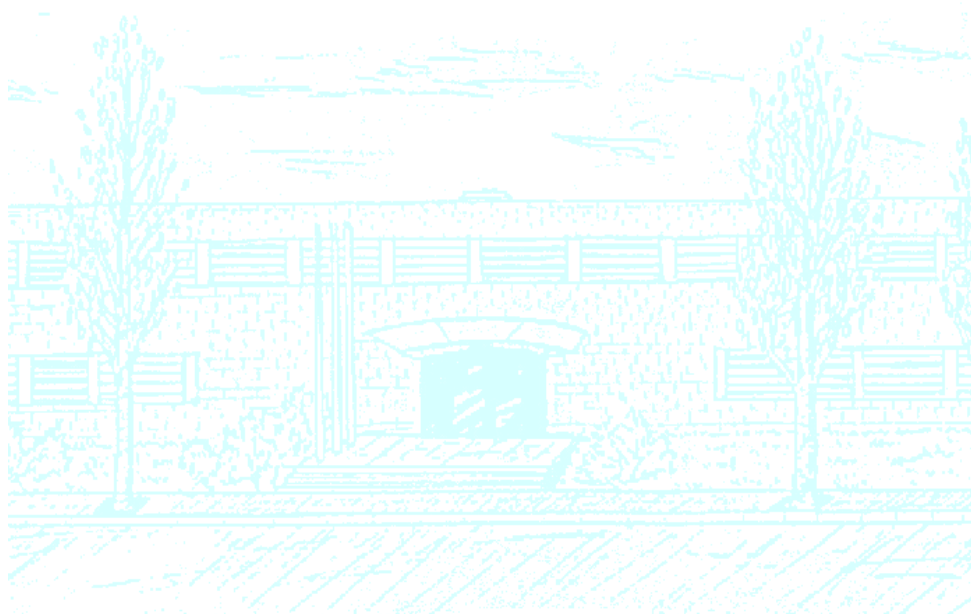
**Title:** The Riemann zeta function and its applications

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## Notation

- $s$  and  $z$  usually denote complex variables and they are used interchangeably.
- We denote the real part of  $s$  with  $\Re s$  or  $\sigma$ , and the imaginary part with  $\Im s$  or  $t$  depending on the context.
- In some places we use  $\tau$  instead of  $t$ , that is defined to be usually  $\tau = |t| + 1$  or  $\tau = |t| + 2$ . This is done so that the bounds or inequalities we are talking about remain valid for every  $t$ .
- When we describe contours by giving vertices or endpoints, the integration is supposed to be done with the orientation that matches the ordering of the endpoints or the vertices.
- In some places we write  $\int_{c-i\infty}^{c+i\infty}$  or similar expressions. This notation means by definition  $\lim_{N \rightarrow \infty} \int_{c-iN}^{c+iN} = \int_{c-i\infty}^{c+i\infty}$ .
- By  $f(x) = O(g(x))$  we mean that there exists  $C$  such that  $|f(x)| \leq C|g(x)|$
- $f \ll g$  means that  $f(x) = O(g(x))$ .
- By  $f(x) \approx g(x)$  we mean that  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$
- By  $f(x) \asymp g(x)$  we mean that there exist constants  $A, B > 0$  such that  $Ag(x) \leq f(x) \leq Bg(x)$
- $\{x\}$  and  $\lfloor x \rfloor$  denote the fractional part of  $x$  and the floor function of  $x$  respectively. Hence  $x = \lfloor x \rfloor + \{x\}$ .
- $\sum_p$  usually denotes that  $p$  runs over the prime numbers.
- $(a, b) = c$  means that the greatest common divisor of  $a$  and  $b$  is  $c$ .
- $a|b$  means that  $a$  divides  $b$ , and  $a \nmid b$  means that  $a$  does not divide  $b$ .

## 0 Introduction

The aim of this thesis is to expose two classical results in Analytic Number Theory: the Prime Number Theorem and Dirichlet's theorem on arithmetic progressions. We state them:

The **prime number theorem** gives an asymptotic formula for  $\pi(x)$ , which is defined as follows:

$$\pi(x) = \text{the number of primes between 1 and } x.$$

The asymptotic formula is the following

$$\pi(x) \approx \frac{x}{\log x}$$

**Dirichlet's theorem** tells us that given  $h$  and  $l$  coprime integers, the following sequence

$$\{l, h + l, 2h + l, \dots\}$$

contains infinitely many primes.

A powerful tool used to prove these theorems is the Riemann zeta function.

The Riemann zeta  $\zeta(s)$  function is a complex-valued function that was introduced to understand the behavior of prime numbers. For  $\Re s > 1$  its value can be defined through the following expression

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + 2^{-s} + 3^{-s} + \dots$$

An important property of the expression above was discovered by Euler (1707-1783): he noticed that  $\zeta(s)$  can be also expressed as the following infinite product over the prime numbers when  $\Re s > 1$

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}} = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \dots$$

He also computed some particular values of  $\zeta$ , although he only considered  $\zeta$  to be meaningful for integer values of  $s$ . This last expression is called an **Euler product** and it is what really relates  $\zeta(s)$  to prime numbers.

Riemann (1826-1866) found the connection between the prime numbers and the  $\zeta$  function. He gave an incomplete proof sketch of the prime number theorem, and also proved that  $\zeta(s)$  has nice properties:

1. It has a **analytic continuation** defined in the whole complex plane
2. It satisfies a **functional equation**: if  $s \neq 0, 1$

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s)$$

That is,  $\zeta$  is symmetric in some sense by the map  $s \mapsto 1 - s$ .

He also proved or almost proved other results in his memoir[10]. In fact, the notation he used there is still used today. But there is one region where  $\zeta$  cannot be yet controlled: the critical strip  $0 < \Re s < 1$ . Riemann conjectured that all the zeros in this region are in the critical line  $s = \frac{1}{2} + it$ . That is, all the non-trivial zeros have  $\Re s = \frac{1}{2}$ . This is known as the **Riemann hypothesis** and remains as an unsolved problem. Its proof would lead to simpler proofs and better error terms in this area and several results conditionally proven on Riemann hypothesis would become unconditionally proven. All known zeros in this region verify Riemann hypothesis.

The Riemann zeta function is an example of the more general concept of zeta function, which has applications in other areas of mathematics. There are lots of conjectures on the behavior of zeta functions, like the Birch and Swinerton-Dyer conjecture that relates the rank of an elliptic curve with the analytic properties of its associated zeta function.

We now give a brief summary of the contents:

- The first section introduces some necessary background in analysis, although we assume the reader has already some prior knowledge in complex analysis.
- In the second section we give a brief look at Dirichlet series: we define what is a Dirichlet series and prove some useful results on convergence of Dirichlet series. We also define the inverse Mellin transform and prove Perron's formula.
- The third section exposes the elementary theory of the Riemann zeta function: its definition, analytic continuation and properties. We also give a proof sketch for a formula that counts the number of non-trivial zeros in the critical strip.
- Dirichlet's theorem is proven in the fourth section. We define the group of characters of a finite abelian group and prove its finiteness, and then we prove the orthogonality relations and define what is a Dirichlet character and an L-function.
- The prime number theorem is proven in the fifth section. We begin by giving some historical results and context and then move on to the preliminaries and proof. An essential step is to find the zero-free regions of the Riemann zeta function.
- In the last section we prove Hardy's theorem, a result closely related to the Riemann hypothesis: it states that there are infinitely many zeros in the critical line.

I would like to thank Jordi Quer for his time and dedication, as well as for his valuable suggestions. I am sure this work would not be the same without his guidance. I would also like to express my gratitude to my family for his unconditional support.

## 1 Some results in analysis

Here we introduce concepts and results in mathematical analysis that will be used in the following sections. A good reference is [14].

### 1.1 A crash course in the basics

Most of the functions we will deal with are holomorphic. Holomorphicity is the natural generalization of differentiability for complex-valued functions.

Let  $\Omega \subset \mathbb{C}$  be a region (a non-empty, connected open subset).

**Definition 1.1.** We say  $f : \Omega \rightarrow \mathbb{C}$  is **holomorphic** if

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} := f'(z)$$

exists for all  $z \in \Omega$ .

$f$  is **entire** if  $\Omega = \mathbb{C}$ .

However, being holomorphic is a strong property and has interesting consequences:

**Theorem 1.2** (Morera's theorem). *Let  $\Omega \subset \mathbb{C}$  be an open subset and let  $f : \Omega \rightarrow \mathbb{C}$  be a continuous function. Then  $f$  is holomorphic if for any closed triangle  $T \subset \Omega$  one has*

$$\int_{\partial T} f = 0$$

**Theorem 1.3.** *Let  $f_n$  be a sequence of holomorphic functions defined on  $\Omega$  such that  $f_n \rightarrow f$  uniformly over each compact subset of  $\Omega$ .*

*Then  $f$  is holomorphic.*

The following theorem is crucial for our purposes, because it lets us to use the calculus of residues.

**Theorem 1.4** (Cauchy's theorem). *Let  $\Omega \subset \mathbb{C}$  be a simply connected open subset and let  $f : \Omega \rightarrow \mathbb{C}$  be an holomorphic function. Then*

$$\int_C f = 0$$

*for any closed rectifiable path (curve) in  $\Omega$ .*

**Theorem 1.5.** *Let  $\Omega \subset \mathbb{C}$  be an open subset and let  $F(z, s) : \Omega \times [a, b] \rightarrow \mathbb{C}$  be a complex-valued function such that*

- For each fixed  $s \in [a, b]$ ,  $F(z, s)$  is holomorphic with respect to  $z$
- $F(z, s)$  is continuous in  $\Omega \times [a, b]$

then

$$f(z) = \int_a^b F(z, s) ds$$

is holomorphic in  $\Omega$ .

**Theorem 1.6** (Vanishing theorem). *Let  $z_n$  be a sequence of points in a connected open subset  $\Omega \subset \mathbb{C}$  and let  $f : \Omega \rightarrow \mathbb{C}$  be an holomorphic function such that  $f(z_n) = 0$  for all  $n$ . If  $\lim z_n = z \in \Omega$  then  $f \equiv 0$  ( $f$  is identically zero).*

**Definition 1.7** (Analytic continuation). Let  $\Omega \subset \mathbb{C}$  be an open subset and  $f : \Omega \rightarrow \mathbb{C}$  be an holomorphic function. We say  $F : \Sigma \rightarrow \mathbb{C}$  is an **analytic continuation** of  $f$  if  $\Omega \subset \Sigma$  and  $F = f$  in  $\Omega$  and  $F$  is holomorphic on  $\Sigma$ .

From the definition of analytic continuation and the vanishing theorem one can see that the analytic continuation is unique: if  $\Sigma$  and  $\Sigma'$  are both open connected subsets of  $\mathbb{C}$  and  $F : \Sigma \rightarrow \mathbb{C}$  and  $F' : \Sigma' \rightarrow \mathbb{C}$  are both analytic continuations of  $f$  then  $F = F'$  in  $\Sigma \cap \Sigma'$  because by hypothesis  $F = f = F'$  in  $\Omega$ .

For example, the function

$$f(z) = 1 + z + z^2 + \dots$$

is only defined when  $z \in \Omega = \{|z| < 1\}$  and  $f(z) = \frac{1}{1-z}$  in  $\Omega$ , so  $\frac{1}{1-z}$  is an analytic continuation of  $f$  to  $\mathbb{C} - \{1\}$ .

**Theorem 1.8** (Weierstrass M-test). *Let  $f_n(z)$  be a family of complex-valued functions all defined in  $\Omega \subset \mathbb{C}$  such that for every  $n$  there exists  $M_n$  verifying*

$$|f_n(z)| \leq M_n$$

for all  $z \in \Omega$ , and such that the series

$$\sum_{n=1}^{\infty} M_n = C < \infty$$

converges. Then the sequence of complex-valued functions  $S_N(z) = \sum_{n=1}^N f_n(z)$  converges uniformly in  $\Omega$  to  $f$  where

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

*Proof.* Fix  $z \in \Omega$ :

1. If  $B_N = \sum_{n=1}^N M_n \rightarrow C$  converges (in  $\mathbb{R}$ ) then in particular  $B_N$  is a Cauchy sequence.
2. We have that

$$|S_N(z) - S_M(z)| \leq \sum_{n=N}^M M_n$$

This last quantity can be made arbitrarily small ( $< \epsilon$ ) choosing  $N, M$  sufficiently large independently of the chosen  $z$ , because of step 1.

□



## 1.2 Riemann-Stieltjes integral

**Definition 1.9** (Riemann-Stieltjes integral). Let  $f$  and  $g$  be two real-valued functions defined in  $[a, b] \subset \mathbb{R}$ . Let  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$  be a partition of  $[a, b]$  and define  $S(x_n, \chi_n)$  as follows:

$$S(x_n, \chi_n) = \sum_{n=1}^n f(\chi_n)(g(x_n) - g(x_{n-1}))$$

where  $x_{n-1} \leq \chi_n \leq x_n$  are arbitrary.

The expression above is called a **Riemann-Stieltjes sum**. We say that the Riemann-Stieltjes integral  $\int f dg = I$  exists if all Riemann-Stieltjes sums are arbitrarily close to  $I$  when the mesh size is small, that is:

Given  $\epsilon > 0$  there exists  $\delta > 0$  such that for every partition  $P = \{x_0, \dots, x_n\}$  with  $\max_i |x_i - x_{i-1}| < \delta$  and every subset  $\{\chi_n\}_n \subset [a, b]$  such that  $\chi_n \in [x_{n-1}, x_n]$  one has  $|S(x_n, \chi_n) - I| < \epsilon$ .

Definition 1.9 can be thought as a generalization of the Riemann integral, because this latter is a particular case of definition 1.9 setting  $g(x) = x$ . Several known results about integration still hold like integration by parts. Under mild regularity conditions on  $f, g$  it can be proven that the Riemann-Stieltjes integral exists:

**Theorem 1.10.** *If  $f$  is continuous and  $g$  is of bounded variation then  $\int f dg$  exists.*

We recall that  $f : [a, b] \rightarrow \mathbb{R}$  is of **bounded variation** if

$$\text{Var}_{[a,b]}(f) = \sup_{\mathcal{P}} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \infty$$

where  $\mathcal{P} = \{a = x_0 \leq \dots \leq x_n = b\}$  is a partition of  $[a, b]$ . This is the case for instance if  $f \in C^1([a, b])$  because then  $|f'| \leq M$  so by the mean value theorem

$$\begin{aligned} \text{Var}_{[a,b]}(f) &= \sup_{\mathcal{P}} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \frac{|x_i - x_{n-1}|}{|x_i - x_{i-1}|} \\ &\leq \sum_{i=1}^n |f'(x'_i)| |x_i - x_{i-1}| \leq M(b - a) < \infty \end{aligned}$$

where  $x'_i \in [x_{i-1}, x_i]$  are given by the mean value theorem.

It turns out summation by parts is a particular case of integration by parts for Riemann-Stieltjes integrals, because one important application is that we can express sums as integrals.

**Corollary 1.11.** *Let  $\{a_n\}_{n \geq 1} \subset \mathbb{R}$  be a sequence of real numbers. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and define  $A(x) = \sum_{n \leq x} a_n$ . Then*

$$\sum_{n=1}^N a_n f(n) = \int_{1-\epsilon}^N f(x) dA(x) \quad (1.1)$$

for all  $\epsilon > 0$ .

*Proof.*

- This follows from theorem 1.10 and the definition 1.9. Consider the following term in a Riemann-Stieltjes sum

$$f(\chi_n)(A(x_{n+1}) - A(x_n))$$

If  $x_i$  and  $x_{i+1}$  both lie in the same interval  $[n, n+1)$  for some  $n$ , then  $A(x_i) = A(x_{i+1})$  so this term will vanish in the Riemann-Stieltjes sum, otherwise  $x_i \in [n-1, n)$  and  $x_{i+1} \in [n, n+1)$  without loss of generality and  $A(x_{i+1}) - A(x_i) = a_i$ .

- It is clear that  $A(x)$  has a finite number of discontinuities in  $1 - \epsilon \leq x \leq n$ . In particular is piecewise- $C^1$  and of bounded variation.
- Observe that we integrate from  $1 - \epsilon$  to capture the first jump of  $A$  at  $x = 1$ , corresponding to the first term of the sum in (1.1).

□

Here  $\{a_j\}_{j \geq 1}, \{b_j\}_{j \geq 1} \subset \mathbb{R}$  are sequences of real numbers.

**Lemma 1.12** (Abel's lemma). *Let  $A_{n,m} = \sum_{j=n}^m a_j$ ,  $S_{n,m} = \sum_{j=n}^m a_j b_j$  then*

$$S_{n,m} = b_m A_{n,m} + \sum_{j=n}^{m-1} A_{n,j} (b_j - b_{j+1})$$

*Proof.* This can be proven directly, although it can be thought of as integration by parts in the context of Riemann-Stieltjes integral (definition 1.9). □

The following proposition is used in the proof of proposition 4.10 in the section dedicated to Dirichlet's theorem on arithmetic progressions (theorem 4.20).

**Proposition 1.13** (Abel's criterion). *Suppose  $|A_{n,m}| \leq C$  is bounded for all  $n, m$  and  $\{b_n\}_{n \geq 1} \subset \mathbb{R}$  is a strictly decreasing sequence of real numbers and  $b_n \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Then the series  $\sum_{n=1}^{\infty} a_n b_n$  converges.*

*Proof.*

- Using last lemma 1.12 we have

$$|S_{n,m}| \leq C|b_m| + C \sum_{j=n}^{m-1} |b_j - b_{j+1}| = C|b_n|$$

- We conclude that the partial sums of the series  $S_{n,m}$  form a Cauchy sequence because  $|b_n| \rightarrow 0$ .

□

### 1.3 The argument principle

Let  $\beta$  be a non-zero meromorphic function in the region  $\Omega$  and let  $\rho$  be fixed. Then  $\beta$  can be written as

$$\beta(s) = (s - \rho)^{k_\rho} H_\rho(s)$$

where  $k_\rho$  is the **order** of  $\beta$  at  $\rho$  and  $H_\rho(s)$  is a non-vanishing holomorphic function in a neighborhood of  $\rho$ . One says  $\rho$  is a **zero** if  $k_\rho > 0$  and is a **pole** if  $k_\rho < 0$ .

Let  $C$  be a simple closed contour in  $\mathbb{C}$  and let  $\Omega$  be an open subset of  $\mathbb{C}$  containing  $C$  and its interior. If  $C$  does not contain any zero or pole of  $\beta$  then the **argument principle** states the following:

$$S(C) = Z(C) - P(C) = \frac{1}{2\pi i} \int_C \frac{\beta'(s)}{\beta(s)} ds \quad (1.2)$$

where  $Z(C), P(C)$  denotes the number of zeros and poles inside  $C$  (respectively) with multiplicity.

This can be seen by observing that for every zero or pole  $\rho$  ( $k_\rho \neq 0$ ) one has

$$\beta'(s) = k_\rho (s - \rho)^{k_\rho - 1} H_\rho(s) + (s - \rho)^{k_\rho} H_\rho'(s)$$

Therefore, if we denote by  $h_\rho(s) = \frac{H_\rho'(s)}{H_\rho(s)}$  we have that in a neighborhood of  $\rho$

$$\frac{\beta'(s)}{\beta(s)} = \frac{k_\rho (s - \rho)^{k_\rho - 1} H_\rho(s) + (s - \rho)^{k_\rho} H_\rho'(s)}{(s - \rho)^{k_\rho} H_\rho(s)} = \frac{k_\rho}{s - \rho} + h_\rho(s) \quad (1.3)$$

Observe  $h_\rho(s)$  is holomorphic. If we apply this argument for each zero or pole of  $\beta$  and integrate over a sufficiently small circular contour  $C_\rho$  around  $\rho$ , such that the hypotheses on  $H_\rho(s)$  hold (that is, it does not vanish) then Cauchy's theorem will ignore the holomorphic part  $h_\rho(s)$ . It is not hard to prove that  $C$  and the small contours  $C_\rho$  around each  $\rho$  verify

$$\int_C \frac{\beta'(s)}{\beta(s)} ds = \sum_{\rho \text{ is a zero or a pole}} \int_{C_\rho} \frac{\beta'(s)}{\beta(s)} ds$$

In fact, this follows from the fact that the cycles  $C$  and  $\sum_\rho C_\rho$  are homologous in  $H_1(X)$  where  $X = \Omega - \{\text{poles and zeros of } \beta\}$ . Hence by splitting the sum in positive and negative terms we have (1.2).

### 1.4 Schwarz reflection principle

This principle lets us extend an holomorphic function under certain circumstances to a larger domain.

Let  $\Omega$  be an open subset of  $\mathbb{C}$  invariant by complex conjugation.

Then we can define  $\Omega^+ = \Omega \cap \{z \in \mathbb{C} : \Im z > 0\}$  and  $\Omega^- = \Omega \cap \{z \in \mathbb{C} : \Im z < 0\}$  the subsets of  $\Omega$  with positive and negative imaginary parts, respectively.

**Theorem 1.14** (Schwarz reflection principle). *Let  $f : \Omega^+ \rightarrow \mathbb{C}$  be an holomorphic function that extends continuously to  $I = \Omega \cap \{\Im z = 0\}$ . Then the complex-valued function  $F : \Omega \rightarrow \mathbb{C}$  defined by*

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup I \\ f(\bar{z}) & \text{if } z \in \Omega^- \end{cases}$$

*is holomorphic in  $\Omega$ .*

This follows from a more general principle

**Lemma 1.15.** *Let  $f^+ : \Omega^+ \rightarrow \mathbb{C}$  and  $f^- : \Omega^- \rightarrow \mathbb{C}$  be holomorphic and  $f^+ = f^-$  on  $I$ . Then the function  $f : \Omega \rightarrow \mathbb{C}$  defined by*

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}$$

*is an holomorphic function in  $\Omega$ .*

*Proof.* This is a consequence of Morera's theorem (theorem 1.2), which states that  $f$  is holomorphic in a given region if for any given triangle-shaped contour  $T \subset \Omega$  then

$$\int_{\partial T} f = 0$$

As noted in [14], we would integrate over any triangle  $T$  inside  $\Omega$  and check if the continuous extension  $f$  of  $f^+$  and  $f^-$  verifies  $\int_T f = 0$ , because this condition implies holomorphicity. If  $T$  is completely contained in  $\Omega^+$  or  $\Omega^-$  then it is clear because each  $f^+$ ,  $f^-$  was already holomorphic. In the other case, we can split the triangle along  $I$  in other triangulable polygons contained in  $\Omega^+$  and  $\Omega^-$ . The contributions along  $I$  cancel because of the orientation and the continuity of  $f$ .  $\square$

We will also use the Schwarz reflection principle to simplify some arguments dealing with integration along contours which are invariant by complex conjugation.

## 1.5 The Jacobi theta function

In section 3.3 we find the analytic continuation of the Riemann zeta function in the same way Riemann did. In his method he used the Jacobi theta function and some its properties, and we will prove them now using the Fourier transform and the Poisson summation formula.

We define the Jacobi theta function as follows:

**Definition 1.16.** The Jacobi theta function  $\vartheta$ : For  $t > 0$  define

$$\vartheta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

We define the Fourier transform as follows:

**Definition 1.17.** The Fourier transform of  $f$  is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx$$

if the integral converges absolutely.

The Fourier transform can be thought as an continuous analogue of the Fourier series, where the Fourier coefficients are now  $\hat{f}$ .

We now state without proof an important result we will use. One can find a proof of this in [13].

**Theorem 1.18** (Inversion theorem). *Let  $f$  be a continuous function and  $\int_{\mathbb{R}} |f| < \infty$ . Then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x \xi} d\xi$$

The Poisson summation formula holds for functions of **moderate decrease** which are holomorphic in an infinite strip, but it also holds for a more general class of functions which need not be holomorphic. We are interested in the case of moderate decrease:

**Definition 1.19** (Functions of moderate decrease). We say  $f \in \mathfrak{F}_t$  if

1.  $f$  is holomorphic in the strip  $\{z \in \mathbb{C} \text{ such that } |\Im z| < t\}$

2.  $|f(x + iy)| \leq \frac{A}{1+x^2}$  when  $x \in \mathbb{R}$  and  $|y| \leq t$

**Theorem 1.20** (Poisson summation formula). *Let  $f$  and  $\hat{f} \in \mathfrak{F}_t$ . Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

*Proof.*

- We use a rectangular contour depending on  $N$  (see figure 1). By Cauchy theorem we have

$$I_N = \frac{1}{2\pi i} \int_{\Delta_N} \frac{f(z)}{e^{2\pi i z} - 1} dz = \sum_{-N < n < N} f(n)$$

because  $(e^{2\pi i z} - 1)^{-1}$  has a simple pole with residue 1 at each integer satisfying  $-N < n < N$ , because they are inside the chosen contour  $\Delta_N$ .

- We deal with the integral along the horizontal segments by substituting  $(e^{2\pi i z} - 1)^{-1}$  with a geometric series converging along each segment. Denote by  $V(t, n)$  the integral along the vertical segments. We have

$$I_N = V(t, N) - \int_{-N-\frac{1}{2}+it}^{N+\frac{1}{2}+it} \sum_{n=0}^{\infty} f(z)e^{2\pi i z n} dz + \int_{-N-\frac{1}{2}-it}^{N+\frac{1}{2}-it} \sum_{n=1}^{\infty} f(z)e^{-2\pi i z n} dz$$

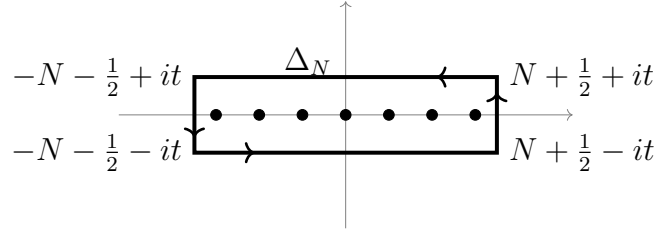


Figure 1: The rectangular contour used in theorem 1.20

- If we let  $N \rightarrow \infty$  the integral along the vertical segments  $V(t, N)$  tends to zero as  $N \rightarrow \infty$ , because of the moderate decrease of  $f$ . This results in

$$I_\infty = - \int_{-\infty+it}^{\infty+it} \sum_{n=0}^{\infty} f(z) e^{2\pi izn} dz + \int_{-\infty-it}^{\infty-it} \sum_{n=1}^{\infty} f(z) e^{-2\pi izn} dz$$

- It is valid to exchange sum with integrals. The integrands converge uniformly to a holomorphic function, and we could restrict the integral over a compact segment to exchange sum with integral in the compact segment. The integral outside the compact segment can be made arbitrarily small because of the moderate decrease of  $f$ . Then

$$I_\infty = - \sum_{n=0}^{\infty} \int_{-\infty+it}^{\infty+it} f(z) e^{2\pi izn} dz + \sum_{n=1}^{\infty} \int_{-\infty-it}^{\infty-it} f(z) e^{-2\pi izn} dz$$

- But we can shift the contours  $[-\infty \pm it, \infty \pm it]$  back to the real line. We consider only the upper horizontal segment, and the other is treated similarly: let  $n \in \mathbb{Z}$  and consider the infinite rectangular contour  $R^+$  with endpoints at  $-\infty, \infty, \infty + it, -\infty + it$ . By Cauchy theorem and the holomorphicity of  $f$  in the infinite strip we have

$$\begin{aligned} \int_{R^+} f(z) e^{-2\pi izn} dz &= 0 = \int_{-\infty}^{\infty} f(z) e^{-2\pi izn} dz - \int_{-\infty+it}^{\infty+it} f(z) e^{-2\pi izn} dz \\ &= \hat{f}(n) - \int_{-\infty+it}^{\infty+it} f(z) e^{-2\pi izn} dz \end{aligned}$$

- Hence

$$I_\infty = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

□

In the following proposition we prove an important property of the Jacobi theta function:

**Proposition 1.21.** *Functional equation for  $\vartheta$ : For  $t > 0$  we have*

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \vartheta(t) = \sqrt{\frac{1}{t}} \vartheta\left(\frac{1}{t}\right) = \sqrt{\frac{1}{t}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi n^2}{t}}$$

*Proof.*

1. We will calculate the Fourier transform  $\hat{f}$  when  $f = e^{-\pi x^2 t}$  and  $t > 0$ . In this case it is useful to complete the square:

$$\pi x^2 t + 2\pi i x \xi = \pi(x\sqrt{t} + i\xi\sqrt{1/t})^2 + \pi\xi^2/t$$

because all the terms in the integrand are exponential terms. Then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2 t} e^{-2\pi i x \xi} dx = e^{-\pi \frac{\xi^2}{t}} \int_{-\infty}^{\infty} e^{-\pi(x\sqrt{t} + \frac{i\xi}{\sqrt{t}})^2} dx$$

2. We deal now with this last integral. This is done by shifting down the contour to the real line, so that we are left with a Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\pi(x\sqrt{t} + \frac{i\xi}{\sqrt{t}})^2} dx = \int_{-\infty}^{\infty} e^{-\pi x^2 \sqrt{t}} dx = \sqrt{\frac{1}{t}}$$

The shift is done by using a rectangular contour with endpoints at  $-N, N, N + \frac{i\xi}{\sqrt{t}}, -N + \frac{i\xi}{\sqrt{t}}$  and Cauchy theorem: observe that  $e^{-\pi x^2 t} e^{-2\pi i x \xi}$  is entire for all  $t, \xi$  and the integrals along the vertical segments tend to zero as  $N \rightarrow \infty$  because  $t > 0$ . Hence

$$\hat{f}(\xi) = e^{-\pi \frac{\xi^2}{t}} \sqrt{\frac{1}{t}} \tag{1.4}$$

3. Now apply theorem 1.20 to  $f = e^{-\pi x^2 t}$  and its Fourier transform given by (1.4). □

## 1.6 The $\Gamma$ function: definition and properties

The aim of this section is to expose the  $\Gamma$  function and its properties. Although they are interesting on their own, they are essential to understand the proof of the functional equation of the Riemann zeta function (corollary 3.3) and Hardy's theorem (section 6), a result closely related to the Riemann hypothesis (conjecture 3.17).

**Definition 1.22** (Definition of  $\Gamma$  in the halfplane  $\Re s > 0$ ). If  $\Re s > 0$  we define  $\Gamma(s)$  as

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$$

It can be shown that the expression above is well-defined and defines an holomorphic function in the half-plane  $\Re s > 0$ .

$\Gamma$  was originally discovered by Bernoulli, Goldbach and Euler as an extension of the factorial of integer numbers  $n! = 1 \cdot 2 \cdot \dots \cdot n$ , because it is not hard to prove by integration by parts the following relation:

$$s\Gamma(s) = \Gamma(s + 1)$$

It is natural to ask if it can be extended to the whole complex plane.

**Proposition 1.23** (Meromorphic continuation for  $\Gamma$ ).  $\Gamma$  extends to a meromorphic function, with a simple pole at  $s = n$  for all  $n \leq 0$ .

*Proof.*

1. We split the integral expression for  $\Gamma$  into two other integrals, one over a compact segment:

$$\Gamma(s) = \int_0^1 x^{s-1} e^{-x} dx + \int_1^\infty x^{s-1} e^{-x} dx$$

- (a) The second term defines an entire function: consider the sequence of holomorphic functions  $S = \{\phi_N(s)\}_{N \geq 1}$ , where  $\phi_N(s)$  is defined as follows:

$$\phi_N(s) = \int_1^N x^{s-1} e^{-x} dx$$

If  $s \in K$  where  $K$  is a compact subset of  $\mathbb{C}$  we have for  $M > N$

$$|\phi_N(s) - \phi_M(s)| \leq \int_N^M x^{s-1} e^{-x} dx$$

This last quantity can be made  $< \epsilon$  by choosing  $N$  and  $M$  sufficiently large, independently of the chosen  $s \in K$ .

- (b) Hence  $S$  is a uniformly converging sequence of holomorphic functions, for each compact subset  $K$  of  $\mathbb{C}$ . So  $\phi_\infty(s)$  is holomorphic by theorem 1.3.
2. We rewrite the first integral and exchange summatory and integral (by the absolute convergence of the series for  $e^{-x}$ )

$$\int_0^1 x^{s-1} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+s)}$$

3. Therefore, the following expression defines a meromorphic function and extends the previous definition of  $\Gamma$  by construction:

$$\Gamma(s) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+s)} + \int_1^\infty x^{s-1} e^{-x} dx \quad (1.5)$$



4. The expression in (1.5) tells us  $\Gamma$  is a meromorphic function with a simple pole at  $s = -n$  and residue  $(-1)^n/n!$ , where  $n \geq 0$  is an integer. □

The following proposition implies  $\Gamma$  does not vanish.

**Proposition 1.24** (Euler reflection formula). *If  $0 < \Re s < 1$  then*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

*Proof.*

- We multiply and change variables in the integral expression for  $\Gamma$  by  $w = u/v$

$$\begin{aligned} \Gamma(s)\Gamma(1-s) &= \int_0^\infty \int_0^\infty e^{-u-v} u^{s-1} v^{-s} du dv \\ &= \int_0^\infty \int_0^\infty e^{-wv-v} w^{s-1} dw dv = \int_0^\infty \frac{w^{s-1}}{1+w} dw \end{aligned}$$

- This last integral converges absolutely because of the condition  $0 < \Re s < 1$ .
- The function  $F = w^{s-1}(1+w)^{-1}$  has a simple pole at  $w = -1$ . We apply the calculus of residues to  $F$  with  $C$  being the keyhole contour in figure 2, parametrized by  $(\epsilon, \delta, R)$ .
- Now let  $\epsilon, \delta \rightarrow 0$  and  $R \rightarrow \infty$  so that

$$\begin{aligned} \int_C F &= 2\pi i \operatorname{Res}\left(\frac{w^{s-1}}{1+w}, w = -1\right) = 2\pi i e^{\pi i(s-1)} \\ &= \int_0^\infty \frac{w^{s-1}}{1+w} dw - \int_0^\infty \frac{w^{s-1} e^{2\pi i(s-1)}}{1+w} dw = (1 - e^{2\pi i(s-1)})\Gamma(s)\Gamma(1-s) \end{aligned}$$

(because the integrals over  $C_\epsilon$  and  $C_R$  tend to 0)

- Using  $\sin(\pi s) = (e^{\pi i s} - e^{-\pi i s})/(2i)$  and dividing by  $(1 - e^{2\pi i(s-1)})$  gives the result. □

**Corollary 1.25** ( $\Gamma$  does not vanish). *For all  $s \in \mathbb{C}$  one has  $\Gamma(s) \neq 0$*

*Proof.*

- If we had  $\Gamma(z) = 0$  for some  $z$  we could translate this zero in a fixed region, say  $0 < \Re s < 1$ . That is,  $0 < \Re(z+n) < 1$  for some integer  $n$  and the functional equation  $z\Gamma(z) = \Gamma(z+1)$  implies  $z(z+1)(z+2)\dots(z+n-1)\Gamma(z) = 0 = \Gamma(z+n)$ .
- This contradicts proposition 1.24 because  $\pi/(\sin(\pi(z+n))) \neq 0$ . □

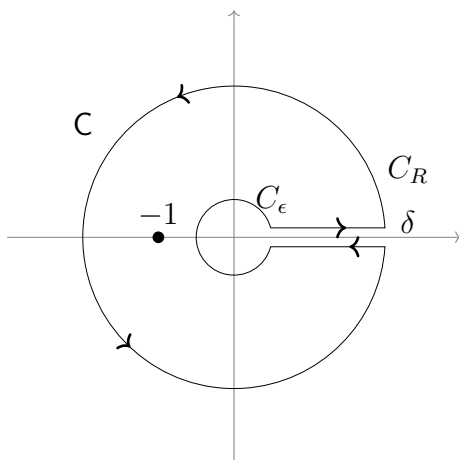


Figure 2: The contour used in proposition 1.24

### 1.6.1 Approximation and bounds for $\Gamma$

We state without proof Stirling's formula and a useful bound we apply in lemma 6.3. One can find a proof of these facts in [9].

**Theorem 1.26** (Stirling's formula). *Let  $\delta > 0$  and  $D(\delta) = \{z \in \mathbb{C} : |z| \geq \delta \text{ and } |\text{Arg } z| \leq \pi - \delta\}$ . Then*

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} \left(1 + O\left(\frac{1}{|s|}\right)\right)$$

*uniformly in  $D(\delta)$ . (That is, the implicit constant only depends on  $\delta$ .)*

This makes sense, because it is clear we are not taking into account the poles of  $\Gamma$ , i.e.  $D(\delta)$  does not contain any pole.

**Corollary 1.27** (A bound for  $|\Gamma|$ ). *Fix  $K$ . Then*

$$|\Gamma(x + iy)| = \sqrt{2\pi} e^{-\pi|y|/2} |y|^{x-1/2} \left(1 + r(x, y)\right)$$

*where  $|r| \rightarrow 0$  uniformly for  $x < K$  as  $|y| \rightarrow \infty$*

## 1.7 Jensen's inequality

The next theorem is well-known and it allows us to bound the number of zeros of analytic functions. It plays an important role in the proof of theorem 3.19, that gives an estimate on the number of zeros of  $\zeta$  in the critical strip (see section 3). It is used to derive corollary 1.30, essential to understand the proof of theorem 5.15, that gives a zero-free region for  $\zeta$  that we will use in the proof of the Prime Number Theorem (theorem 5.17).

Given  $R > 0$  we use the notation  $D_R$  to describe the following closed disk:

$$D_R = \{z \in \mathbb{C} : |z| \leq R\}$$

and  $\partial D_R$  denotes the boundary of  $D_R$ .

**Theorem 1.28.** *Let  $f$  be analytic in the region  $\Omega \supset D_R$  such that  $|f| \leq M$  inside  $D_R$  and  $f(0) \neq 0$ .*

*If we denote by  $Z_{f,r}$  the number of zeros of  $f$  inside  $D_r$  then*

$$Z_{f,r} \leq \frac{\log(M/|f(0)|)}{\log(R/r)}$$

for all  $r < R$

*Proof.*

- Let  $k = Z_{f,r}$  and let  $z_1, \dots, z_k$  be the zeros of  $f$  in  $D_r$ , not necessarily different. Observe there must be a finite number of them because  $D_R$  is compact so if we had an infinite number of zeros there should be a converging partial inside  $D_R$  and  $f$  would vanish identically by theorem 1.6, contradicting  $f(0) \neq 0$ .
- Consider

$$g(z) = f(z) \prod_{m=1}^k \frac{R^2 - z\bar{z}_m}{R(z - z_m)} \quad (1.6)$$

The  $m$ -th factor of (1.6) has a simple pole in  $z_m$  and has absolute value 1 on  $|z| = R$ , because for  $|z| = R$  one has

$$\left| \frac{R^2 - z\bar{z}_m}{R(z - z_m)} \right| = \left| \frac{z}{R} \right| \left| \frac{\bar{z} - \bar{z}_m}{z - z_m} \right| = 1$$

because  $z\bar{z} = R^2$ , and  $|(\bar{z} - \bar{z}_m)(z - z_m)^{-1}| = 1$ .

- Hence  $g$  is analytic because each zero cancels a pole of the product if we count the zeros with multiplicity and  $|g| = |f| \leq M$  on  $|z| = R$ . Also,

$$|g(0)| = |f(0)| \prod_{m=1}^k \frac{R}{|z_m|} \quad (1.7)$$

and each term in (1.7) is  $\frac{R}{|z_m|} \geq \frac{R}{r} > 1$  since the zeros  $z_k$  are inside  $D_r$ . So  $\log(R/r) > 0$  and by the maximum modulus principle we have

$$M \geq |g(0)| \geq |f(0)| \left(\frac{R}{r}\right)^k \rightarrow Z_{f,r} = k \leq \frac{\log(M/|f(0)|)}{\log(R/r)}$$

□

## 1.8 Borel-Carathéodory lemma

This theorem bounds an analytic function in terms of its real part in a larger region.

**Theorem 1.29.** *Let  $h$  be analytic in the region  $\Omega \supset D_R$ ,  $h(0) = 0$  and  $|\Re h| \leq M$  on  $D_R$ . Then*

$$\begin{cases} |h(z)| \leq \frac{2Mr}{R-r} \\ |h'(z)| \leq \frac{2MR}{(R-r)^2} \end{cases} \quad (1.8)$$

hold for every  $z$  such that  $|z| \leq r < R$

*Proof.*

1. If we can prove that

$$\left| \frac{h^{(k)}(0)}{k!} \right| \leq \frac{2M}{R^k} \quad (1.9)$$

then the bounds in (1.8) will hold: if we expand  $h$  in power series at  $z = 0$  and recall that  $h(0) = 0$ , we can bound  $h$  and  $h'$  by a geometric series

$$|h(z)| \leq \sum_{k=1}^{\infty} \frac{2M}{R^k} r^k = \frac{2Mr}{R-r}$$

$$|h'(z)| \leq \sum_{k=1}^{\infty} \frac{2M}{R^k} k r^{k-1} = \frac{2M}{R} \frac{1}{(1 - \frac{r}{R})^2} = \frac{2MR}{(R-r)^2}$$

2. In order to prove the bound in (1.9) we will use Cauchy integral formula. Define  $I_k$  to be as follows

$$I_k = \frac{R^{-k}}{2\pi i} \int_{\partial D_R} h(z) z^{k-1} dz$$

This integral help us extract the coefficients of  $h$ .

3. By changing variables  $z = Re^{2\pi i\theta}$  we have that

$$I_k = \int_0^1 h(Re^{2\pi i\theta}) e^{2\pi ik\theta} d\theta = \frac{R^{-k}}{2\pi i} \int_{\partial D_R} h(z) z^{k-1} dz \quad (1.10)$$

4. In fact the number  $I_k$  in (1.10) is 0 for all  $k \geq 0$ :  $h(z)z^{k-1}$  is holomorphic because  $h(0) = 0$ .

If  $k = -m$  is negative, then we can expand  $h$  as an absolutely convergent power series around  $z = 0$  so

$$I_k = I_{-m} = R^m \frac{h^{(k)}(0)}{k!} \quad (1.11)$$

5. Let  $\phi \in [0, 2\pi]$  be a real number we will choose later and set  $k > 0$ . Denote by  $\eta_k$  the following linear combination of expressions in (1.10):

$$\eta_k = I_0 + \frac{1}{2}(I_k + I_{-k}) \cos(2\pi\phi) - \frac{1}{2i}(I_k - I_{-k}) \sin(2\pi\phi)$$

6. By step 4 the terms  $I_k = I_0 = 0$  vanish. If we apply (1.11) in the linear combination  $\eta_k$  we have

$$\begin{aligned} \eta_k &= I_0 + \frac{1}{2}(I_k + I_{-k}) \cos(2\pi\phi) - \frac{1}{2i}(I_k - I_{-k}) \sin(2\pi\phi) \\ &= 0 + \frac{1}{2}(0 + I_{-k}) \cos(2\pi\phi) - \frac{1}{2i}(0 - I_{-k}) \sin(2\pi\phi) \\ &= \frac{1}{2}I_{-k}(\cos 2\pi\phi - i \sin 2\pi\phi) = I_{-k} \frac{1}{2} e^{-2\pi i\phi} = R^k \frac{h^{(k)}(0)}{k!} e^{-2\pi i\phi} \end{aligned}$$

7. On the other hand, we express  $\eta_k$  in another way: by the identities  $\sin u = (e^{iu} - e^{-iu})(2i)^{-1}$  and  $\cos u = (e^{iu} + e^{-iu})2^{-1}$  and the expression for  $I_k$  in (1.10) we have

$$\begin{aligned} \eta_k &= \int_0^1 \left(1 + \frac{e^{2\pi ik\theta} + e^{-2\pi ik\theta}}{2} \cos(2\pi\phi) - \frac{e^{2\pi ik\theta} - e^{-2\pi ik\theta}}{2i} \sin(2\pi\phi)\right) h(Re^{2\pi i\theta}) d\theta \\ &= \int_0^1 (1 + \cos(2\pi k\theta) \cos(2\pi\phi) - \sin(2\pi k\theta) \sin(2\pi\phi)) h(Re^{2\pi i\theta}) d\theta \end{aligned}$$

8. By using the trigonometrical identity  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$  we rewrite the integral above

$$\eta_k = \int_0^1 h(Re^{2\pi i\theta}) \left(1 + \cos(2\pi(k\theta + \phi))\right) d\theta$$

9. Finally, if we take real parts we have by steps 6 and 8 the following expression

$$\begin{aligned} \Re(\eta_k) &= \Re\left(\frac{1}{2}R^k e^{-2\pi i\phi} \frac{h^{(k)}(0)}{k!}\right) = \Re \int_0^1 h(Re^{2\pi i\theta}) (1 + \cos(2\pi(k\theta + \phi))) d\theta \\ &\leq M \int_0^1 (1 + \cos(2\pi(k\theta + \phi))) d\theta = M \end{aligned}$$

for any real  $\phi$ , because  $|\Re h| \leq M$ . We choose  $\phi$  so that  $\eta_k = |\eta_k|$ . Hence

$$\Re \eta_k = \eta_k = |\eta_k| = \left| \frac{1}{2}R^k \frac{h^{(k)}(0)}{k!} \right| \leq M$$

This proves the bound in (1.9). □

**Corollary 1.30** (Combination of Jensen-Borel-Carathéodory). *Let  $f$  be analytic in  $\Omega \supset D_1 \supset D_R$  where  $\Omega$  is an open subset, with  $|f| \leq M$  in  $D_R$  and  $f(0) \neq 0$ .*

*Let  $r, R \in \mathbb{R}$  be fixed such that  $0 < r < R < 1$  and let  $z_1 \cdots z_k$  be the zeros of  $f$  inside  $D_R$  counted with multiplicity.*

*Then for  $|z| \leq r$  one has*

$$\frac{f'}{f}(z) = \sum_{m=1}^k \frac{1}{z - z_m} + O\left(\log \frac{M}{|f(0)|}\right)$$

where the implicit constant in  $O()$  only depends on  $r, R$ .

*Proof.*

1. It is clear we can suppose  $f \neq 0$  in  $\partial D_R$  by choosing some  $r < R < 1$ . Otherwise we could form a sequence of points  $p_i$  inside the compact set  $D_1$  with  $f(p_i) = 0$ , and  $f \equiv 0$  would vanish identically in  $\Omega$  by theorem 1.6, contradicting  $f(0) \neq 0$ .
2. Define  $g$  to be

$$g(z) = f(z) \prod_{m=1}^k \frac{R^2 - z\bar{z}_m}{R(z - z_m)}$$

which is the same function we used to prove Jensen's inequality (theorem 1.28).

- (a) We know by theorem 1.28 that  $k$ , the number of zeros inside the disk  $D_R$  verifies

$$k \leq \frac{\log(M/|f(0)|)}{\log(1/R)} \ll \log \frac{M}{|f(0)|}$$

where the implicit constant in  $\ll$  depends only on  $R$ . This bounds  $k$  in terms of  $M, f$  and  $R$ .

- (b)  $g$  does not vanish in  $D_R$ : each term  $(R(z - z_m))^{-1}$  cancels the zero in  $f$  as many times as the multiplicity of the zero  $z_m$  of  $f$ , because we are counting the zeros with multiplicity (so the same factor  $(R(z - z_m))^{-1}$  can appear in  $g$  many times).
- (c) For each term in the product

$$\left| \frac{R^2 - z\bar{z}_m}{R(z - z_m)} \right| = 1$$

by the same argument we used in theorem 1.28.

3. Hence  $|g| = |f|$  in  $\partial D_R$ . By the maximum modulus principle  $|g(z)| \leq M$  in  $D_R$ , and  $|g(0)| = |f(0)| \prod \frac{R}{|\bar{z}_m|} \geq |f(0)|$  because  $z_m \in D_R$ .
4. By step 2b one can define the analytic function  $h(z) = \log(g(z)/g(0))$  for  $z \in D_R$  and

$$\Re h(z) = \log |g(z)| - \log |g(0)| \leq \log M - \log |f(0)|$$

for  $z \in D_R$  by the observation in 3.

Observe that  $h(0) = 0$  by construction.

- (a)  $h$  verifies the hypothesis in Borel-Carathéodory lemma (theorem 1.29) so by (1.8) we have that for  $z \in D_r$

$$|h'(z)| \leq \frac{2R}{(R-r)^2} \log \frac{M}{|f(0)|} \ll \log \frac{M}{|f(0)|}$$

- (b) The product that defines  $g$  is finite, so it is valid to differentiate term by term. This results in

$$h' = \frac{g'}{g} = \frac{f'}{f} - \sum_{m=1}^k \frac{1}{z - z_m} + S$$

where

$$S = \sum_{m=1}^k \frac{1}{z - R^2/\bar{z}_m}$$

- (c) This last term  $S$  can be bounded, because by triangle inequality one has

$$\left| z - \frac{R^2}{\bar{z}_m} \right| \geq \left| |z| - \left| \frac{R^2}{\bar{z}_m} \right| \right| > R - r$$

because  $|R^2/\bar{z}_m| > R$ .

- (d) Hence by step 2a we have

$$|S| \leq \frac{k}{R-r} \ll \log \frac{M}{|f(0)|}$$

for  $z \in D_r$ .

The proof is complete. □

## 2 Dirichlet series

In the next sections many complex-valued functions will be defined in terms of Dirichlet series, so it is sensible to give a brief summary of their properties.

We begin by defining what is a Dirichlet series and some theorems dealing with convergence. Next we move on to the inverse Mellin transform and Perron's formula, because are essential to understand the proof of the prime number theorem (theorem 5.17).

The inverse Mellin transform will be used to encode in a Dirichlet series (the Riemann zeta function, section 3) the distribution of the prime numbers.

### 2.1 Definition

**Definition 2.1.** A **Dirichlet series** is a series of the form

$$\alpha(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where  $\{a_n\}_{n \geq 1}$  is a sequence of complex numbers.

This last definition can be seen as a particular case of  $\alpha(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$  where  $\lambda_n = \log n$ .

### 2.2 General convergence of Dirichlet series

We now prove that if a Dirichlet series converges at some point, it converges uniformly in a larger region. The region is a circular sector on the right of that point:

**Proposition 2.2.** *If the Dirichlet series  $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  converges for  $s_0$  then it converges uniformly in every domain verifying*

$$\Re s > \Re s_0, |\operatorname{Arg}(s - s_0)| \leq \alpha$$

for  $\alpha < \pi/2$ .

*Proof.*

- We put  $\lambda_n = \log n$  for convenience so that  $\alpha(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ .
- It is clear we can take  $s_0 = 0$ , by redefining the  $a_n \mapsto a_n e^{-\lambda_n s_0}$ .
- The hypothesis then is that  $\sum a_n$  is convergent, so in particular it is bounded  $|\sum a_n| \leq C$ .

With the same notation of proposition 1.13, we put  $b_j = e^{-\lambda_n s}$ . We could use Abel's criterion (proposition 1.13) to prove the convergence of the series if the  $b_j$  were real numbers.



- We solve this problem: if  $\sigma$  is the real part of  $s$ ,

$$\begin{aligned} |b_j - b_{j+1}| &= |e^{-\lambda_j s} - e^{-\lambda_{j+1} s}| = |s \int_{\lambda_j}^{\lambda_{j+1}} e^{-su} du| \\ &\leq |s| \int_{\lambda_j}^{\lambda_{j+1}} e^{-\sigma u} du = \frac{|s|}{\sigma} (e^{-\lambda_j \sigma} - e^{-\lambda_{j+1} \sigma}) \end{aligned}$$

- The condition  $|\text{Arg}(s - s_0)| \leq \alpha < \pi/2$  implies that  $\frac{|s|}{\sigma}$  is bounded.
- We can now apply the same argument we used in Abel's criterion (proposition 1.13), because  $\sigma > 0$  and  $\lambda_n \rightarrow \infty$  is strictly increasing. Then  $e^{-\lambda_j \sigma}$  is strictly decreasing.

□

**Definition 2.3** (Abscissa of convergence for Dirichlet series). Let  $\mathfrak{C}(\alpha)$  be the following set:

$$\mathfrak{C}(\alpha) = \inf\{\sigma' \in \mathbb{R} \text{ such that } \alpha(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges for any } s \text{ with } \sigma > \sigma'\}$$

The abscissa of convergence is defined as

$$\sigma_c = \inf \mathfrak{C}(\alpha)$$

**Definition 2.4** (Abscissa of absolute convergence for Dirichlet series). Let  $|\mathfrak{C}|(\alpha)$  be the following set

$$|\mathfrak{C}|(\alpha) = \{\sigma' \in \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} \left| \frac{a_n}{n^s} \right| \text{ converges for any } s \text{ with } \sigma > \sigma'\}$$

The abscissa of absolute convergence is defined as

$$\sigma_a = \inf |\mathfrak{C}|(\alpha)$$

In particular, one can reorder the terms of  $\alpha$  arbitrarily for any  $s$  with  $\sigma > \sigma_a$ .

We say  $\sigma_c = \infty$  when  $\mathfrak{C}(\alpha) = \emptyset$  and  $\sigma_c = -\infty$  when  $\mathfrak{C}(\alpha) = \mathbb{R}$ , that is, the series converge nowhere and everywhere respectively.

These abscissas tell us where does  $\alpha(s)$  converge, because for a given  $s$  with  $\Re s = \sigma > \sigma_c$  one can use proposition .

In fact, if we are dealing only with absolute convergence, instead of proposition 2.2 one has that for given  $s$  with  $\Re s = \sigma > \sigma_a$

$$|\alpha(s)| = \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_a}} < \infty$$

(because  $|n^s| = n^\sigma$ ) so the series for  $\alpha(s)$  converge absolutely too.

### 2.3 Landau's theorem

The proposition 2.2 states that if a Dirichlet series  $\alpha(s)$  converges for every  $\sigma > \sigma_c$ , then it converges in every circular sector shaped region to the right, but gives no information on convergence in the point  $s = \sigma_c$ . The next theorem states that if the coefficients  $a_n$  of  $\alpha$  are real and positive then  $\sigma_c$  is a singularity of  $\alpha$ .

It was proved in 1905 by Landau, but similar theorems for power series were already proven by Vivanti and Pringsheim.

This theorem is essential in the proof of Dirichlet's theorem on arithmetic progressions (theorem 4.20).

**Theorem 2.5.** *Let  $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  be a Dirichlet series and  $\sigma_c$  its abscissa of convergence. If the  $a_n$  are real positive numbers ( $a_n \geq 0$ ) then  $\sigma_c$  is a singularity of  $\alpha$ .*

*Proof.*

1. We note that if the  $a_n$  are positive for  $n$  sufficiently large the result also holds because we only have to ignore a finite number of terms, which clearly defines an holomorphic function.
2. It is clear that we can suppose that  $\sigma_c = 0$  by changing the coefficients in  $\alpha$  by putting  $a_n \mapsto a_n n^{-\sigma_c}$ .
3. Suppose 0 is not a singularity. Therefore  $\alpha$  is holomorphic there so it is also locally analytic, so we can find  $\delta > 0$  so that  $\alpha$  is given by a power series in  $D = \{\sigma > 0\} \cap \{|s| < \delta\}$ . Expand  $\alpha$  as a power series at  $s = 1$

$$\alpha(s) = \sum_{k=0}^{\infty} c_k (s-1)^k \quad (2.1)$$

where

$$c_k = \frac{\alpha^{(k)}(1)}{k!} = \frac{1}{k!} \sum_{n=1}^{\infty} \frac{a_n}{n} (-\log n)^k$$

The radius of convergence of the series in equation 2.1 is the distance from 1 to the nearest singularity of  $\alpha$ , but  $\alpha$  is analytic in  $D$  so this radius is at least  $\sqrt{1 + \delta^2} = 1 + \epsilon$  because  $\alpha$  does not have singularities on the right of  $\sigma = 0$  by proposition 2.2. Hence

$$\alpha(s) = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} \frac{a_n}{n} (\log n)^k \quad (2.2)$$

4. Now we restrict expression 2.2 for  $s$  real. If  $1-s > 0$  then all terms are positive, therefore expression 2.2 is absolutely converging series because it converges. Because of absolute convergence we can reorder the terms arbitrarily. So

$$\alpha(s) = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} \frac{a_n}{n} (\log n)^k = \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{(1-s)^k (\log n)^k}{k!} = \sum_{n=1}^{\infty} a_n n^{-s}$$

converges for  $-\epsilon < s < 1$ . But this is a contradiction because then  $\sigma_c \neq 0$ .  $\square$

**Corollary 2.6.** *If  $\alpha$  is a Dirichlet series with positive coefficients and  $\alpha(\sigma_0) = \sum_{n=1}^{\infty} a_n n^{-\sigma_0}$  converges then  $\sigma_c < \sigma_0$ .*

*Proof.* If  $\alpha$  converges in  $\sigma_0$  then  $\sigma_c \neq \sigma_0$  because of theorem 2.5. Therefore  $\sigma_c < \sigma_0$  by proposition 2.2.

In particular,  $\sigma_c = \Re \rho$  where  $\rho$  is the rightmost real-valued singularity of  $\alpha(s)$ .  $\square$

## 2.4 Euler products

We say the coefficients of a Dirichlet series  $a_n$  are **completely multiplicative** if  $a_{nm} = a_n a_m$  holds for arbitrary  $n, m$ . Then it is not hard to see that the partial products  $\alpha_N(s) = \prod_{p < N} \left(1 - \frac{a_p}{p^s}\right)^{-1}$  converge uniformly over compact sets for  $s$  with  $\sigma > \sigma_a$  because

$$\alpha_N(s) = \prod_{p < N} \left(1 - \frac{a_p}{p^s}\right)^{-1} = \prod_{p < N} \sum_{n=1}^{\infty} \frac{a_p^n}{p^{sn}} = \sum_{n \in S_N} \frac{a_n}{n^s}$$

where  $S_N$  is the set of positive integers divisible only by primes less than  $N$ . Hence

$$|\alpha_N(s) - \alpha(s)| = \left| \sum_{n \notin S_N} \frac{a_n}{n^s} \right| \leq \sum_{n \geq N} \left| \frac{a_n}{n^s} \right| = \epsilon_N$$

and it is clear  $\epsilon_N \rightarrow 0$  as  $N \rightarrow \infty$  for all  $s$  with  $\sigma \geq \sigma' > \sigma_a$  for arbitrarily fixed  $\sigma'$  because the partial sums of  $\sum \left| \frac{a_n}{n^s} \right|$  form a Cauchy sequence because  $\sigma > \sigma_a$ .

So we can write for  $s$  with  $\sigma > \sigma_a$

$$\alpha(s) = \prod_{p \text{ prime}} \left(1 - \frac{a_p}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

This last expression is an **Euler product** for  $\alpha$  because it resembles the expression for the Riemann zeta function as an infinite product Euler discovered (theorem 3.2).

## 2.5 The inverse Mellin transform

The inverse Mellin transform will be useful for our purposes. It will help us to prove the prime number theorem (theorem 5.17), and it is the idea that changes the discrete nature of the problem (counting prime numbers) to a problem in complex analysis and asymptotics.

**Theorem 2.7.** *Let  $\alpha$  be a Dirichlet series with  $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  convergent for  $\sigma > \sigma_c$ .*

*Let  $A(x) = \sum_{n \leq x} a_n$ . If  $\sigma > \max(0, \sigma_c)$  then*

$$\alpha(s) = s \int_1^{\infty} A(x) x^{-(s+1)} dx \quad (2.3)$$

This last expression is called the **inverse Mellin transform** of  $A(x)$ .

Moreover, if  $\sigma_c < 0$  then  $A(x)$  is bounded and if  $\sigma_c \geq 0$  then

$$\limsup_{x \rightarrow \infty} \frac{\log |A(x)|}{\log x} = \sigma_c \quad (2.4)$$

*Proof.*

- Denote the left hand side of (2.4) by

$$\phi = \limsup_{x \rightarrow \infty} \frac{\log |A(x)|}{\log x} \quad (2.5)$$

- Instead of integrating from 1 to  $\infty$  in (2.3), we integrate from 1 to  $N$ . Denote by  $\chi_n$  the characteristic function for the interval  $[n, \infty] \subset \mathbb{R}$ . Then  $A(x) = \sum_{n=1}^{\infty} a_n \chi_n(x)$ . Observe this latter series is a finite sum for each  $x$ . Therefore

$$\begin{aligned} \int_1^N A(x) x^{-(s+1)} dx &= \sum_{n=1}^{\infty} \int_1^N a_n \chi_n(x) x^{-(s+1)} dx = \sum_{n \leq N} \int_n^N a_n x^{-(s+1)} dx \\ &= - \sum_{n \leq N} a_n \frac{x^{-s}}{s} \Big|_n^N = \frac{1}{s} \sum_{n \leq N} \frac{a_n}{n^s} - \frac{1}{s} A(N) N^{-s} \end{aligned} \quad (2.6)$$

- If  $\sigma_c < 0$ , then by theorem 2.2 one has  $\alpha(0) = \sum_{n=1}^{\infty} a_n$  converges. Then  $A(x)$  is bounded and  $|A(N)N^{-s}| \rightarrow 0$  for  $\sigma > 0$  and the integral above converges absolutely. Therefore, by (2.6) the integral above converges to  $\alpha(s)/s$  as  $N \rightarrow \infty$ . This proves (2.3).
- Suppose  $\sigma_c \geq 0$  and let  $\theta > \phi$ . Then  $A(x) \ll x^\theta$  by (2.5). If  $\sigma > \theta$  then

$$\left| \int_1^{\infty} A(x) x^{-(s+1)} dx \right| \leq \int_1^{\infty} |A(x)| x^{-(\sigma+1)} dx \ll \int_1^{\infty} x^{\theta-\sigma-1} dx < \infty$$

and the integral in (2.6) converges absolutely and  $|A(N)N^{-s}| \rightarrow 0$ .

So the integral in (2.6) converges to  $\alpha(s)/s$  as  $N \rightarrow \infty$  if  $\sigma > \phi$ .

Therefore we have established (2.3) if  $\phi = \sigma_c$ .

- Now we prove that  $\phi = \sigma_c$  for  $\sigma_c \geq 0$ .
  - By definition 2.3 we have that  $\sum_{n=1}^{\infty} a_n n^{-s}$  diverges when  $\sigma < \sigma_c$ . Then  $\phi \geq \sigma_c$ , because we have shown in the steps above that the Dirichlet series for  $\alpha$  converge for  $\sigma > \phi$ .

- We prove now that  $\phi \leq \sigma_c$ . Let  $\sigma_0 > \sigma_c$ . If we define  $R(u) = \sum_{n>u} a_n n^{-\sigma_0}$  then by using a similar argument to that in (2.6) one has

$$A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_0^N R(u)u^{\sigma_0-1} du \quad (2.7)$$

We observe that  $R(N)$  is bounded because  $\sigma_0 > \sigma_c$ , and we are only ignoring a finite number of terms of  $\alpha$ . From (2.7) we can deduce that  $A(N) \ll N^{\sigma_0}$ . Therefore  $\phi \leq \sigma_0$  for all  $\sigma_0 > \sigma_c$ , so  $\phi < \sigma_c$ .

The proof is complete.  $\square$

## 2.6 Perron's formula

Perron's formula basically states that one can invert the inverse Mellin transform of theorem 2.7, and was first proved rigorously by Perron (1908). Riemann also used inverse Mellin transform but was Hjalmar Mellin who first described the functions which can be inverted by Perron's formula.

That is, we claim that under mild conditions on  $\alpha$  and  $A$  we have

$$\alpha(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} A(x)x^{-(s+1)} dx \quad (2.8)$$

$$A(x) = \sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s} ds \quad (2.9)$$

where  $\sigma_0 > \max(0, \sigma_c)$ .

The expression in (2.8) is already established by theorem 2.7. Naively, to prove (2.9) we should be able to exchange sum and integral. To solve this difficulty, we will split  $\alpha$  in two terms and exchange a finite sum with an integral in the first term, apply next lemma 2.8 and bound the second term.

Essentially, the lemma is used to extract the coefficients  $a_n$  from  $\alpha$ .

**Lemma 2.8.** *Let  $c, y \in \mathbb{R}^+$ . Then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds = \begin{cases} 0 & \text{if } 0 < y < 1 \\ \frac{1}{2} & \text{if } y = 1 \\ 1 & \text{if } y > 1 \end{cases} \quad (2.10)$$

*Proof.*

1. Suppose  $y > 1$ . We choose the rectangular contour  $C_T$  with vertices  $c - iT, c + iT, -T + iT, -T - iT, c - iT$ , where  $T \in \mathbb{R}^+$ . The integrand  $\frac{y^s}{s}$  has a unique pole at  $s = 0$  with residue 1 so by Cauchy's theorem

$$\int_{C_T} \frac{y^s ds}{s} = 2\pi i = \int_{c-iT}^{c+iT} + \int_{c+iT}^{-T+iT} + \int_{-T+iT}^{-T-iT} + \int_{-T-iT}^{c-iT} = I_1 + I_2 + I_3 + I_4$$

Observe that  $I_2$  and  $I_4$  have essentially the same behavior. We will show  $I_2, I_3, I_4$  tend to 0 as  $T \rightarrow \infty$ . It follows from

$$I_4 = \int_{-T-iT}^{c-iT} \frac{y^s ds}{s} = \int_{-T}^c \frac{y^{\sigma-iT} d\sigma}{\sigma - iT} \ll \frac{1}{T} \int_{-T}^c y^\sigma d\sigma = \frac{y^c - y^{-T}}{T \log y} \rightarrow 0$$

because  $y > 1$ . Also

$$I_3 = \int_{-T+iT}^{-T-iT} \frac{y^s ds}{s} = \int_T^{-T} y^{-T+it} \frac{dt}{-T+it} \ll \frac{1}{T} \int_T^{-T} y^{-T} dt \leq \frac{2T}{T} y^{-T} \rightarrow 0$$

2. Suppose  $y = 1$ . Choose the branch of the complex logarithm in  $\mathbb{C} - \mathbb{R}^+ = \mathbb{C} - \{s : \sigma \geq 0\}$  such that  $\log i = \frac{\pi i}{2}$ . Then

$$\int_{c-iT}^{c+iT} \frac{ds}{s} = \int_{-T}^T \frac{dt}{c+it} = \log \left( \frac{c+iT}{c-iT} \right) \rightarrow \pi i$$

because  $\frac{c+iT}{c-iT} \rightarrow -1$  as  $T \rightarrow \infty$ .

3. If  $0 < y < 1$  we choose the rectangular contour  $C_T$  with vertices  $c-iT, c+iT, T+iT, T-iT, c-iT$ . The bounds on the corresponding  $I_2, I_3, I_4$  are similar to those for the case  $y > 1$ , but here  $C_T$  does not contain any pole, so  $\int_{C_T} \frac{y^s ds}{s} = 0$

□

**Theorem 2.9** (Perron's formula). Let  $\alpha = \sum a_n n^{-s}$  be a Dirichlet series and set  $x > 0$  and  $\sigma_0 > \max(0, \sigma_c)$ . Then

$$\sum'_{n \leq x} a_n = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s) \frac{x^s}{s} ds \quad (2.11)$$

The prime indicates that when  $x$  is an integer, the last term in the sum has coefficient  $\frac{1}{2}$ .

*Proof.* Choose  $N > 2x + 2$ . We split  $\alpha$  in two terms

$$\alpha(s) = \sum_{n \leq N} a_n n^{-s} + \sum_{n > N} a_n n^{-s} = \alpha_1(s) + \alpha_2(s)$$

1.  $\alpha_1$  is a finite sum. Lemma 2.8 with  $c = \sigma_0$  yields

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_1(s) \frac{x^s}{s} ds = \sum_{n \leq x} \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} a_n \left( \frac{x}{n} \right)^s \frac{ds}{s} = \sum'_{n \leq x} a_n \quad (2.12)$$

because  $y = \frac{x}{n} \geq 1$  for the terms with  $n \leq x$ .

2. On the other hand,  $\alpha_2$  can be expressed by a Riemann-Stieltjes integral, which we defined in 1.9.

$$\alpha_2(s) = \sum_{n>N} a_n n^{-s} = \int_N^\infty u^{-s} d(A(u) - A(N))$$

Integrating by parts yields

$$\alpha_2(s) = - \int_N^\infty (A(u) - A(N)) d(u^{-s}) = s \int_N^\infty (A(u) - A(N)) u^{-(s+1)} ds$$

Let  $\theta$  be such that  $\sigma_0 > \theta > \max(0, \sigma_c)$ . By equation (2.4) of theorem 2.7 we have that  $A(u) - A(N) \ll u^\theta$ . Hence

$$\alpha_2(s) \ll |s| \int_N^\infty u^{\theta-\sigma-1} du = |s| \left. \frac{u^{\theta-\sigma}}{\theta-\sigma} \right|_{u=N}^{u=\infty} = |s| \frac{N^{\theta-\sigma}}{\sigma-\theta} \quad (2.13)$$

For (2.11) to hold, the term  $\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_2(s) \frac{x^s}{s} ds$  must be small, i.e. we want to prove that the main contribution is from  $\alpha_1$ .

Set  $S_T$  to be the rectangular contour with vertices at  $\sigma_0-iT, T-iT, T+iT, \sigma_0+iT$ . Then by Cauchy theorem

$$0 = \int_{S_T} \alpha_2(s) \frac{x^s}{s} ds = - \int_{\sigma_0-iT}^{\sigma_0+iT} + \int_{\sigma_0-iT}^{T-iT} + \int_{T-iT}^{T+iT} + \int_{T+iT}^{\sigma_0+iT} = -I + I_1 + I_2 + I_3$$

so

$$I = \int_{\sigma_0-iT}^{\sigma_0+iT} = \int_{\sigma_0-iT}^{T-iT} + \int_{T-iT}^{T+iT} + \int_{T+iT}^{\sigma_0+iT} = I_1 + I_2 + I_3$$

These integrals  $I_j$  can be bounded:

- (a) By using (2.13)

$$\begin{aligned} -I_3 &= \int_{\sigma_0+iT}^{\sigma_0+iT} \alpha_2(s) \frac{x^s}{s} ds \ll \int_{\sigma_0+iT}^{T+iT} |s| \frac{N^{\theta-\sigma}}{\sigma-\theta} \frac{x^\sigma ds}{|s|} \leq \frac{N^\theta}{\sigma_0-\theta} \int_{\sigma_0}^\infty \left(\frac{x}{N}\right)^\sigma d\sigma \\ &= \frac{N^{\theta-\sigma_0}}{\sigma_0-\theta} \frac{x^{\sigma_0}}{\log(N/x)} \end{aligned} \quad (2.14)$$

- (b) The integral  $I_2$  runs over  $s = T + it$  so here  $ds = dt$  and  $\sigma = T$  is fixed hence

$$I_2 = \int_{T-iT}^{T+iT} \alpha_2(s) \frac{x^s}{s} ds \ll \int_{T-iT}^{T+iT} |s| \frac{N^{\theta-\sigma}}{\sigma-\theta} \frac{x^\sigma ds}{|s|} \leq 2T \frac{N^{\theta-\sigma_0}}{T-\theta} x^T \ll N^{\theta-\sigma_0} x^{\sigma_0} \quad (2.15)$$

- (c) The integral  $I_1$  has a similar behavior to  $I_3$ .

$N$  was arbitrary and the integrals in (2.14) and (2.15) tend to 0 as  $N \rightarrow \infty$  because  $\sigma_0 > \theta$  so  $I \rightarrow 0$ . Therefore theorem 2.9 holds because of (2.12).  $\square$

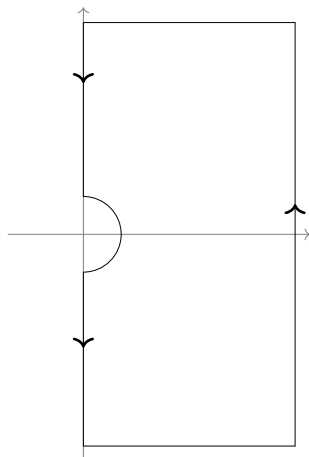


Figure 3: The contour used in step 3 of lemma 2.11.

## 2.7 Perron's formula with error term

Although theorem 2.9 is useful it is not sufficiently quantitative because it is an statement about a limit, and not about the integral  $\int_{\sigma_0-iT}^{\sigma_0+iT}$ .

A good way to solve this problem is to find a better error term, but now we will have to enforce  $\sigma_0 > \sigma_a > \sigma_c$  to have absolute convergence. It involves the sine integral function defined as

$$\text{si}(x) = - \int_x^\infty \frac{\sin u}{u} du$$

and an easy bound for  $\text{si}(x)$  we prove now:

**Lemma 2.10** (A bound for  $\text{si}(x)$ ). *For  $x \geq 1$  the following bound*

$$\text{si}(x) \ll \min\left(1, \frac{1}{x}\right)$$

holds.

*Proof.*

1. If  $x$  is small, say  $x < 1$  then clearly  $\text{si}(x)$  is bounded.
2. If  $x \geq 1$  then integration by parts yields

$$- \int_x^\infty \frac{\sin u}{u} du = -\frac{\cos x}{x} + \int_x^\infty \frac{\cos u}{u^2} du$$

3. This last integral can be bounded  $|\int_x^\infty \frac{\cos u}{u^2} du| \leq |\int_x^\infty \frac{du}{u^2}| = \frac{1}{x}$  and  $|\frac{\cos x}{x}| \leq \frac{1}{x}$
4. Hence  $\text{si}(x) \ll \min(1, \frac{1}{x})$

□



We establish a similar lemma previously as we did in lemma 2.8.

**Lemma 2.11.** *We have*

$$\int_{\sigma_0-iT}^{\sigma_0+iT} y^s \frac{ds}{s} = \begin{cases} 1 + O\left(\frac{y^{\sigma_0}}{T}\right) & \text{if } y \geq 2 \\ 0 + O\left(\frac{y^{\sigma_0}}{T}\right) & \text{if } y \leq \frac{1}{2} \\ 1 + \frac{1}{\pi} \operatorname{si}(T \log y) + O\left(\frac{2^{\sigma_0}}{T}\right) & \text{if } 1 < y \leq 2 \\ 0 - \frac{1}{\pi} \operatorname{si}\left(T \log \frac{1}{y}\right) + O\left(\frac{2^{\sigma_0}}{T}\right) & \text{if } \frac{1}{2} \leq y < 1 \end{cases}$$

Note that we chose 2 and  $\frac{1}{2}$  as the limit cases, but this is arbitrary.

*Proof.*

1. Suppose  $y \geq 2$  and choose the infinite rectangular contour  $C_T$  with vertices  $-\infty - iT, \sigma_0 - iT, \sigma_0 + iT, -\infty + iT$  so the real part is mainly negative. The integrand has a pole at  $s = 0$  with residue 1 so by Cauchy's theorem

$$\int_{C_T} \frac{y^s}{s} ds = I_1 + 2\pi i I_T + I_2 = 2\pi i$$

We are interested in  $I_T$ , so it suffices to bound  $I_1$  and  $I_2$ . Observe they have a similar behavior, so we only have to bound  $I_2$

$$I_2 = \int_{\sigma_0+iT}^{-\infty+iT} y^{\sigma+iT} \frac{d\sigma}{\sigma+iT} \ll \frac{1}{T} \int_{\sigma_0}^{-T} y^{\sigma} d\sigma = -\frac{y^{\sigma_0}}{T \log y} \ll \frac{y^{\sigma_0}}{T}$$

Hence  $I_T = 1 + O\left(\frac{y^{\sigma_0}}{T}\right)$

2. Suppose  $y \leq \frac{1}{2}$  and choose the infinite rectangular contour  $C_T$  with vertices  $\infty - iT, \sigma_0 - iT, \sigma_0 + iT, \infty + iT$  so the real part is mainly positive. The integrand now has no poles by Cauchy's theorem  $\int_{C_T} \frac{y^s}{s} ds = I_1 + 2\pi i I_T + I_2 = 0$ . The bounds on  $I_1$  and  $I_2$  are done similarly and remain valid, so  $I_T = 0 + O\left(\frac{y^{\sigma_0}}{T}\right)$ .
3. Suppose now  $1 < y \leq 2$  and choose the rectangular contour  $C_{T,\epsilon}$  with vertices  $-i\epsilon, -iT, \sigma_0 - iT, \sigma_0 + iT, iT, i\epsilon$  with a circular indentation of radius  $\epsilon$  at  $s = 0$  (see figure 3). The integrand has no poles there so by Cauchy theorem

$$\int_{C_{T,\epsilon}} = \int_{-i\epsilon}^{-iT} + \int_{-iT}^{\sigma_0-iT} + \int_{\sigma_0-iT}^{\sigma_0+iT} + \int_{\sigma_0+iT}^{iT} + \int_{iT}^{i\epsilon} + \int_{C_\epsilon} = 0 = I_1 + I_2 + I_T + I_3 + I_4 + I_\epsilon$$

It is clear that  $I_2$  and  $I_3$  have similar behavior.

(a) We bound  $I_3$

$$I_3 = \int_{\sigma_0+iT}^{iT} y^{\sigma+iT} \frac{d\sigma}{\sigma+iT} \ll \frac{1}{T} \int_0^{\sigma_0} y^{\sigma} d\sigma = \frac{y^{\sigma_0}}{T \log y} \ll \frac{2^{\sigma_0}}{T}$$

(b) We can expand  $y^s$  locally at  $s = 0$  to be  $y^s = 1 + O(|s|)$  so  $\lim_{\epsilon \rightarrow 0} I_\epsilon = \lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} \frac{y^s}{s} ds = \frac{-1}{2}$

(c) It turns out  $I_1 + I_4$  can be expressed in terms of  $\text{si}(x)$  because

$$\begin{aligned} \frac{1}{2\pi i}(I_1 + I_4) &= \frac{-1}{2\pi i} \int_\epsilon^T (y^{i\eta} - y^{-i\eta}) \frac{d\eta}{\eta} = \frac{-1}{\pi} \int_{\epsilon \log y}^{T \log y} \frac{\sin \eta}{\eta} d\eta \\ &= \frac{1}{\pi} \left( \text{si}(\epsilon \log y) - \text{si}(T \log y) \right) \end{aligned} \quad (2.16)$$

Letting  $\epsilon \rightarrow 0$  we have that  $\lim_{\epsilon \rightarrow 0} (I_1 + I_4) = \frac{-1}{2} - \frac{1}{\pi} \text{si}(T \log y)$  because  $\text{si}(0) = -\frac{\pi}{2}$ .

Hence  $\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = 1 + \frac{1}{\pi} \text{si}(T \log y) + O\left(\frac{2^{\sigma_0}}{T}\right)$

4. If  $\frac{1}{2} \leq y < 1$  we choose the same contour we did for  $1 \leq y \leq 2$ , but now  $\log y < 0$  so the integral in expression 2.16 is

$$\frac{-1}{\pi} \int_{\epsilon \log y}^{T \log y} \frac{\sin \eta}{\eta} d\eta = \frac{1}{\pi} \int_{T \log y}^{\epsilon \log y} \frac{\sin \eta}{\eta} d\eta \rightarrow \frac{1}{2} + \frac{1}{\pi} \text{si}(T \log \frac{1}{y})$$

and now the constant terms cancel and we are left with  $-\frac{1}{\pi} \text{si}(T \log \frac{1}{y}) + O\left(\frac{2^{\sigma_0}}{T}\right)$ . □

**Theorem 2.12** (Perron's formula with error term). *If  $\sigma_0 > \max(0, \sigma_a)$  and  $x > 0$*

$$\sum'_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds + R(T, x) \quad (2.17)$$

where

$$R = \sum_{n \leq x \leq 2n} a_n \frac{1}{\pi} \text{si}(T \log y) - \sum_{\frac{n}{2} \leq x \leq n} a_n \frac{1}{\pi} \text{si}(T \log \frac{1}{y}) + O\left(\frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}\right) \quad (2.18)$$

*Proof.*

- For simplicity we suppose  $x$  is not an integer. This case is done analogously.
- The series  $\alpha$  are uniformly convergent for  $\sigma_0 + it$  with  $-T < t < T$ , because we can open a circular sector containing this region and use proposition 2.2. Therefore one can exchange sum with integral and

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds = \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \sum_{n=1}^{\infty} a_n I_{n,x} \quad (2.19)$$

- Now we let  $y = \frac{x}{n}$  run over  $n$  and use lemma 2.11 and we get, after some calculations

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n I_{n,x} &= \sum_{x \geq 2n} a_n \left(1 + O\left(\frac{y^{\sigma_0}}{T}\right)\right) + \sum_{n \leq x \leq 2n} a_n \left(1 + \frac{1}{\pi} \operatorname{si}(T \log y) + O\left(\frac{2^{\sigma_0}}{T}\right)\right) \\
&\quad + \sum_{\frac{n}{2} \leq x \leq n} a_n \left(-\frac{1}{\pi} \operatorname{si}(T \log \frac{1}{y}) + O\left(\frac{2^{\sigma_0}}{T}\right)\right) + \sum_{x \leq \frac{n}{2}} a_n O\left(\frac{y^{\sigma_0}}{T}\right) \\
&= \dots = \sum_{n \leq x} a_n + \sum_{\frac{x}{2} \geq n} a_n O\left(\frac{y^{\sigma_0}}{T}\right) + \sum_{n \leq x \leq 2n} a_n O\left(\frac{2^{\sigma_0}}{T}\right) + \sum_{\frac{n}{2} \leq x \leq n} a_n O\left(\frac{2^{\sigma_0}}{T}\right) \\
&\quad + \sum_{2x \leq n} a_n O\left(\frac{y^{\sigma_0}}{T}\right) + \sum_{n \leq x \leq 2n} a_n \frac{1}{\pi} \operatorname{si}(T \log y) - \sum_{\frac{n}{2} \leq x \leq n} a_n \frac{1}{\pi} \operatorname{si}(T \log \frac{1}{y}) \\
&= \sum_{n \leq x} a_n + \sum_{n \leq x \leq 2n} a_n \frac{1}{\pi} \operatorname{si}(T \log y) - \sum_{\frac{n}{2} \leq x \leq n} a_n \frac{1}{\pi} \operatorname{si}(T \log \frac{1}{y}) + O\left(\frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}\right)
\end{aligned}$$

- This last term  $\frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}$  comes from grouping the terms

$$\begin{aligned}
&\sum_{\frac{x}{2} \geq n} a_n O\left(\frac{y^{\sigma_0}}{T}\right) + \sum_{n \leq x \leq 2n} a_n O\left(\frac{2^{\sigma_0}}{T}\right) + \sum_{\frac{n}{2} \leq x \leq n} a_n O\left(\frac{2^{\sigma_0}}{T}\right) + \sum_{2x \leq n} a_n O\left(\frac{y^{\sigma_0}}{T}\right) = \\
&= \sum_{\frac{x}{2} \geq n \text{ or } 2x \leq n} a_n O\left(\frac{x^{\sigma_0}}{T n^{\sigma_0}}\right) + \sum_{n \leq x \leq 2n} a_n O\left(\frac{2^{\sigma_0}}{T}\right) + \sum_{\frac{n}{2} \leq x \leq n} a_n O\left(\frac{2^{\sigma_0}}{T}\right) = O\left(\frac{(4x)^{\sigma_0}}{T}\right) \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}
\end{aligned}$$

because in the sums with terms having either  $n \leq x \leq 2n$  or  $\frac{n}{2} \leq x \leq n$  one can say that  $2^{\sigma_0} \leq \left(4\frac{x}{n}\right)^{\sigma_0}$

□

In this last proof we had uniform convergence and it was immediate to prove the claim, but it was tedious to find the error term. On the other hand, in theorem 2.9 the difficult step was proving the claim because there was no error term involved.

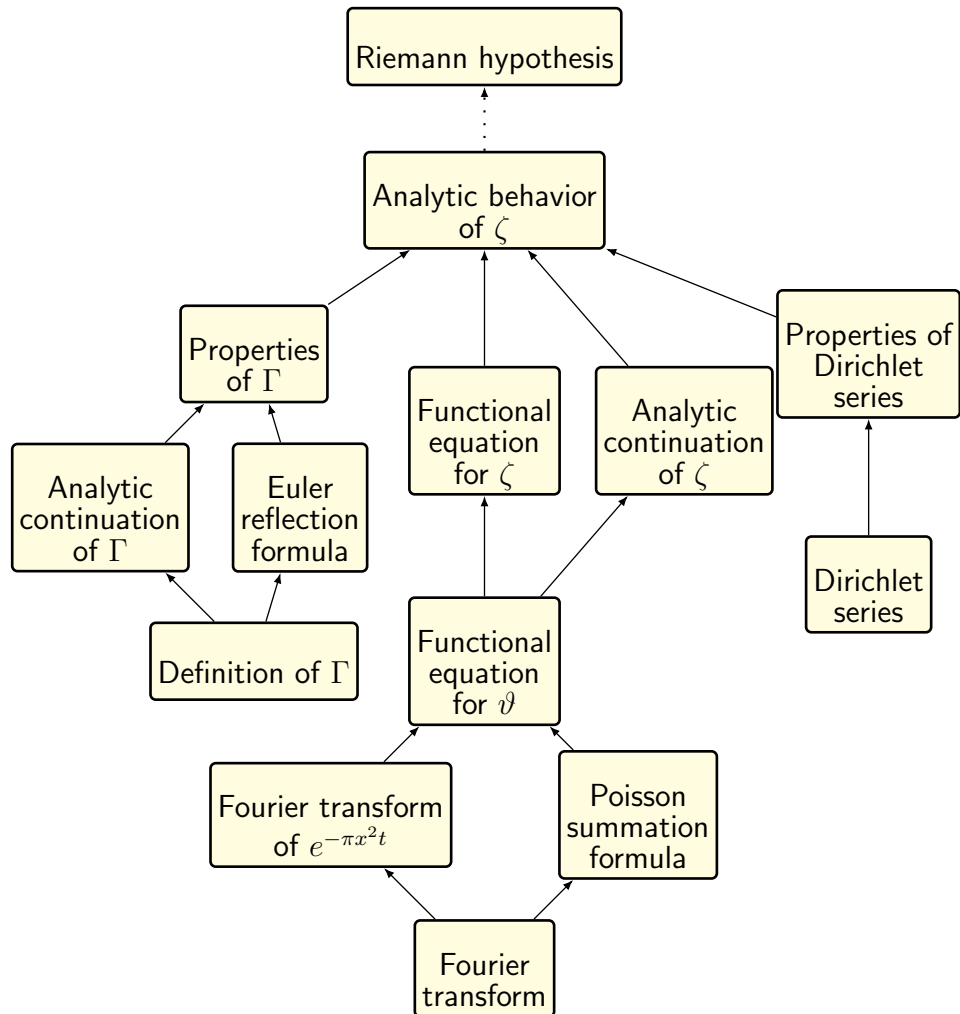
### 2.7.1 Simplification of the error term

The error term  $R(T, x)$  can be simplified because of lemma 2.10 by observing that  $\frac{n}{x} = 1 + \frac{n-x}{x}$  and  $|\log(1 + \epsilon)| \ll \epsilon$  if  $-\frac{1}{2} \leq \epsilon \leq 1$ . Applying this argument to the  $\operatorname{si}(\dots)$  terms we get immediately

**Corollary 2.13** (Practical bound for the error term in Perron's formula).

$$R(T, x) \ll \sum_{\frac{x}{2} < n < 2x} |a_n| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}$$

### 3 The Riemann zeta function



In this section we introduce the Riemann zeta function ( $\zeta$ ). It is essential to understand the rest of the thesis because  $\zeta$  lies at the heart of the proof of the prime number theorem (theorem 5.17).

The results we expose here are classical. Some of them were proven by Riemann, like the analytic continuation of  $\zeta$  and the functional equation. These results essentially described and controlled completely  $\zeta$  almost everywhere, but in the so called **critical region**. The Riemann hypothesis (conjecture 3.17) tries to solve this problem, although it remains unproven.

We begin by giving some historical context: who defined  $\zeta$ , what results were known before Riemann and so on. The rest of the section contains bounds and properties of  $\zeta$  (the analytic continuation and the functional equation) we will use later.

A section one may skip is where we talk about the distribution of the zeros in the

critical region (section 3.6) because this result tries to fill the gap in our knowledge on the critical region, although is not essential.

For the sake of clearness of exposition, the way the properties of  $\zeta(s)$  are deduced is reflected in the diagram above.

### 3.1 History and motivation

$\zeta(s)$  was in fact originally defined by Euler, the great Swiss mathematician, who proved

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

This type of formula is called an **Euler product** for  $\zeta(s)$ . As we discussed already in the introduction, the formula was used by Euler to provide an alternate proof of the infinitude of primes in  $\mathbb{Z}$ . The main argument is that  $\zeta(s)$  does not have a finite value at  $s = 1$ , that is, the harmonic series diverges. If there is only a finite number of primes the left hand side of the above equation is a rational number, which is impossible.

Euler calculated all values of  $\zeta(s)$  at the even numbers:

$$\zeta(2n) = -\frac{(2\pi i)^{2n} B_{2n}}{2(2n)!} \quad (3.1)$$

The case  $n = 1$  is known as the **Basel problem**. These values are a great proportion of the known values. In fact, it is unknown in general whether  $\zeta(2k + 1)$  is rational or not. It has been proven several years ago by Apéry (1978) that  $\zeta(3)$  is an irrational number. The proof uses a well known irrationality criterion by Dirichlet, which in turn can be proved with the pigeonhole principle.

It has been proven recently (2000) that infinitely many numbers of the form  $\zeta(2n+1)$  are irrational, and that at least one of the numbers  $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$  is irrational[11, 16]. A breakthrough is expected soon in this research area.

However, particular values of  $\zeta$  were known before Euler. The fact that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges was proven by Nicole Oresme, but his result did not have much repercussion among his contemporaries.

#### 3.1.1 Riemann's memoir

Curiously enough, Riemann only published one paper[10] dealing with the Riemann zeta function. The results and ideas in his paper were in the direction of proving the prime number theorem (theorem 5.17).

As noted in [4], Riemann showed  $\zeta$  to be very useful to understand prime numbers. He proved that

1. The function  $\zeta$  admits a meromorphic extension to  $\mathbb{C}$ . We prove this in theorem 3.15.
2. It also satisfies a functional equation, as it can be seen in corollary 3.3.

On the other hand, he conjectured several results

1. There are infinitely many zeros in the critical strip, in the region  $0 < \Re s < 1$  (and they are symmetrically distributed in some sense by corollary 3.16).
2. If we denote by  $N(T)$  the number of zeros  $\rho = \sigma + it$  in the critical strip with  $0 < t \leq T$  then

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

as seen in theorem 3.18

3. The validity of an explicit formula for  $\pi(x) - \text{li}(x)$ . That is, an expression that is valid gives the value of  $\pi(x)$  with arbitrary precision involving the zeros of  $\zeta$ . The full proof of this result was given by von Mangoldt in 1895. We can see a proof sketch in section 5.3

## 3.2 The Riemann zeta function: definition

This is the original definition of  $\zeta$ :

**Definition 3.1.** Riemann  $\zeta$  function for  $\Re s > 1$ :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Euler discovered an important fact about  $\zeta$ :

**Theorem 3.2** (Euler product for  $\zeta$ ). *If  $\Re s > 1$  then*

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Moreover, the above product is uniformly convergent over compact subsets of  $\{\Re s > 1\}$ .

*Proof.*

- This follows easily from the argument seen in section 2.4 by observing that  $\zeta$  can be given by a Dirichlet series in this region

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

with  $a_n = 1$  for all  $n$ . These coefficients are obviously multiplicative.

- Now we prove the second claim: we define  $P_N(s)$  to be as follows

$$P_N(s) = \prod_{p \leq N} (1 - p^{-s})^{-1}$$

- Clearly,  $P_N(s) \neq 0$  for all  $s$ . Hence one may take logarithms, because of the non-vanishing of  $P_N$ .
- We observe that the Taylor expansion of  $-\log(1 - z)$  at  $z = 0$  implies that  $|\log(1 - z)| \ll |z|$  uniformly in the closed disk  $z \in D_{1/2}$ .
- Suppose now that  $K$  is a compact subset of  $\{\Re s > 1\}$  and  $s \in K$ . In particular  $\Re s \geq \sigma_0 > 1$ .
- Given  $\epsilon' > 0$  and  $M, N$  with  $M > N$  we have by the integral test that

$$\begin{aligned} \left| \log \frac{P_M}{P_N}(s) \right| &= \left| \log P_N(s) - \log P_M(s) \right| = \left| \sum_{N < p \leq M} -\log(1 - p^{-s}) \right| \\ &\ll \sum_{j=N+1}^M j^{-\sigma} \ll \frac{M^{1-\sigma} - N^{1-\sigma}}{1-\sigma} \end{aligned}$$

This last quantity can be made  $< \epsilon'$  uniformly in  $K$  by letting  $M, N \rightarrow \infty$ , because  $\sigma \geq \sigma_0$  and  $(1 - \sigma_0) < 0$ .

- Therefore, given  $\epsilon > 0$  one can enforce  $|P_M(s) - P_N(s)| < \epsilon$  uniformly in  $K$  by letting  $M, N \rightarrow \infty$ , because of the steps above and the continuity of the exponential function.

□

### 3.3 Analytic continuation for $\zeta$

We expose Riemann's method for extending  $\zeta$ . It uses some properties of the Jacobi theta function. Those were proved in section 1.5.

- Define  $\psi$  to be the following auxiliary function

$$\psi(u) = \frac{\vartheta(u) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}$$

where  $\vartheta(u) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 u}$ .

- $\psi$  satisfies

$$|\psi(u)| < C u^{-\frac{1}{2}}, 0 < u < 1 \quad (3.2)$$

$$|\psi(u)| < C' e^{-\pi u}, u \geq 1 \quad (3.3)$$

The first bound in (3.2) is a consequence of proposition 1.21, because  $\vartheta \rightarrow 0$  as  $t \rightarrow \infty$ .

The claim in (3.3) follows if we bound  $\psi$  with a geometric series with ratio  $e^{-\pi u}$ :

$$\psi(u) = \sum_{n=1}^{\infty} e^{-\pi n^2 u} < \sum_{n=1}^{\infty} e^{-\pi n u} = \frac{e^{-\pi u}}{1 - e^{-\pi u}} = (e^{\pi u} - 1)^{-1} \ll e^{-\pi u}$$

- Changing variables  $v = \pi n^2 u$ ,  $dv = \pi n^2 du$  yields

$$\begin{aligned} \int_0^{\infty} e^{-\pi n^2 u} u^{\frac{s}{2}-1} du &= \int_0^{\infty} e^{-v} \left(\frac{v}{\pi n^2}\right)^{\frac{s}{2}-1} \frac{dv}{\pi n^2} = (\pi n^2)^{-\frac{s}{2}} \int_0^{\infty} e^{-v} v^{\frac{s}{2}-1} dv \\ &= \pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) \end{aligned} \quad (3.4)$$

- Now sum for all  $n$  in (3.4):

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 u} u^{\frac{s}{2}-1} du = \sum_{n=1}^{\infty} \pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) \quad (3.5)$$

- In the region where  $\zeta$  is now defined ( $\Re s > 1$ ), we can exchange limits with integrals in (3.5) because of (3.2) and (3.3). In order to define  $\zeta$  in a larger domain, we must split the integral in two parts and study them carefully.

$$\int_0^{\infty} \psi(u) u^{\frac{s}{2}-1} du = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^1 \psi(u) u^{\frac{s}{2}-1} du + \int_1^{\infty} \psi(u) u^{\frac{s}{2}-1} du = (I) + (II)$$

- Now we deal with (I):

- Change variables in (I) with  $v = \frac{1}{u}$ ,  $dv = -\frac{1}{u^2} du$

$$(I) = \int_0^1 \psi(u) u^{\frac{s}{2}-1} du = - \int_{\infty}^1 \psi\left(\frac{1}{v}\right) v^{1-\frac{s}{2}} \frac{1}{v^2} dv \quad (3.6)$$

- (3.6) can be simplified. From proposition 1.21 we deduce that  $\psi$  satisfies

$$2\psi(u) = \vartheta(u) - 1 = \sqrt{\frac{1}{u}} \left(\vartheta\left(\frac{1}{u}\right) - 1\right) + \sqrt{\frac{1}{u}} - 1$$

- But using the definition of  $\psi(1/u)$  in  $\vartheta(1/u) - 1$  yields

$$\sqrt{\frac{1}{u}} \left(\vartheta\left(\frac{1}{u}\right) - 1\right) + \sqrt{\frac{1}{u}} - 1 = 2\sqrt{\frac{1}{u}} \psi\left(\frac{1}{u}\right) + \sqrt{\frac{1}{u}} - 1$$



– If we use these two last expressions to put  $\psi(1/u)$  in terms of  $2\psi(u)$  we get

$$\frac{\sqrt{u}}{2}(2\psi(u) - \sqrt{\frac{1}{u}} + 1) = \psi\left(\frac{1}{u}\right) = \sqrt{u}\psi(u) - \frac{1}{2} + \frac{\sqrt{u}}{2} \quad (3.7)$$

– Now we apply (3.7) in (3.6)

$$\begin{aligned} (I) &= \int_1^\infty \psi\left(\frac{1}{v}\right)v^{-1-\frac{s}{2}}dv = \int_1^\infty v^{-1-\frac{s}{2}}\left(\sqrt{v}\psi(v) - \frac{1}{2} + \frac{\sqrt{v}}{2}\right)dv \\ &= \int_1^\infty \left(v^{-\frac{1}{2}-\frac{s}{2}}\psi(v) - \frac{1}{2}v^{-1-\frac{s}{2}} + \frac{1}{2}v^{-\frac{1}{2}-\frac{s}{2}}\right)dv = \int_1^\infty v^{-\frac{1}{2}-\frac{s}{2}}\psi(v)dv - \frac{1}{s} - \frac{1}{1-s} \\ &= \int_1^\infty v^{-\frac{1}{2}-\frac{s}{2}}\psi(v) + \frac{1}{s(s-1)} \end{aligned}$$

• Finally

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = (I) + (II) = \int_1^\infty \psi(u)\left(u^{\frac{s}{2}-1} + u^{-\frac{1}{2}-\frac{s}{2}}\right)du + \frac{1}{s(s-1)} \quad (3.8)$$

• The integral in (3.8) converges absolutely for all  $s \neq 0, 1$  because of the exponential decay for  $\psi$  in (3.3) (by a similar argument to that of step 1a of theorem 1.23), hence defines an entire function. We extend  $\zeta$ :

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left(\int_1^\infty \psi(u)\left(u^{\frac{s}{2}-1} + u^{-\frac{1}{2}-\frac{s}{2}}\right)du + \frac{1}{s(s-1)}\right) \quad (3.9)$$

The point  $s = 0$  is not a pole, it is a point singularity as we will see in theorem 3.15. It can be defined to be  $\zeta(0) = -\frac{1}{2}$ .

The analytic continuation of  $\zeta$  is useful to assign values to some divergent series in a consistent way. For example, the series

$$1 + 2 + 3 + \dots \stackrel{\text{formally, definition 3.1}}{=} \zeta(-1)$$

does not converge to a complex number (if we endow  $\mathbb{C}$  with the euclidean topology). It can be shown that

$$\zeta(-1) = -\frac{1}{12}$$

### 3.3.1 Functional equation for $\zeta$

From (3.9) we define

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \psi(u)\left(u^{\frac{s}{2}-1} + u^{-\frac{1}{2}-\frac{s}{2}}\right)du + \frac{1}{s(s-1)}$$

valid for  $s \neq 0, 1$ . Observe  $\xi(s)$  remains invariant by  $s \mapsto 1 - s$  because the right hand side does. This implies

**Corollary 3.3.** *Functional equation: For all  $s \neq 0, 1$*

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s)$$

Although the function  $\xi(s)$  tells us a lot of  $\zeta$  already, in most references it is redefined so that it is an entire function, say

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad (3.10)$$

We will adopt this last definition. Observe  $\xi$  is still invariant by  $s \mapsto 1-s$ .

The functional equation (corollary 3.3) shows that  $\xi$  is symmetric with respect to 1 in the complex plane.

### 3.4 Properties of $\zeta$

We prove several results on  $\zeta$  that are used throughout the next sections: the Prime Number Theorem (theorem 5.17) and Dirichlet's theorem on arithmetic progressions (theorem 4.20). One can imagine how does  $\zeta$  look in the complex plane by looking at figure 4.

#### 3.4.1 Behavior of $\zeta$ at $s = 1$ (the simple pole)

We now examine the behavior of  $\zeta$  at  $s = 1$  by proving that the residue of  $\zeta$  at  $s = 1$  is 1:

**Proposition 3.4.**  $\zeta(s) \approx \frac{1}{s-1}$  as  $s \rightarrow 1$

*Proof.* Essentially, we extend  $\zeta$  again in a convenient way that shows without great effort the behavior at  $s = 1$ . This is not a problem because all analytic continuations agree.

- We rewrite  $(s-1)^{-1}$  carefully

$$\frac{1}{s-1} = \int_1^\infty t^{-s} dt = \sum_{n=1}^\infty \int_n^{n+1} t^{-s} dt$$

- Reordering the series is valid

$$\frac{1}{s-1} = \sum_{n=1}^\infty \int_n^{n+1} (t^{-s} - n^{-s} + n^{-s}) dt = - \sum_{n=1}^\infty \int_n^{n+1} (n^{-s} - t^{-s}) dt + \zeta(s) =$$

because the Dirichlet series of definition 3.1 converges absolutely in  $\Re s > 1$  ( $\sigma_a > 1$ ). Therefore

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^\infty \int_n^{n+1} (n^{-s} - t^{-s}) dt = \frac{1}{s-1} + \sum_{n=1}^\infty f_n(s) = \frac{1}{s-1} + H(s)$$

Now we prove the expression above is valid in a larger region.

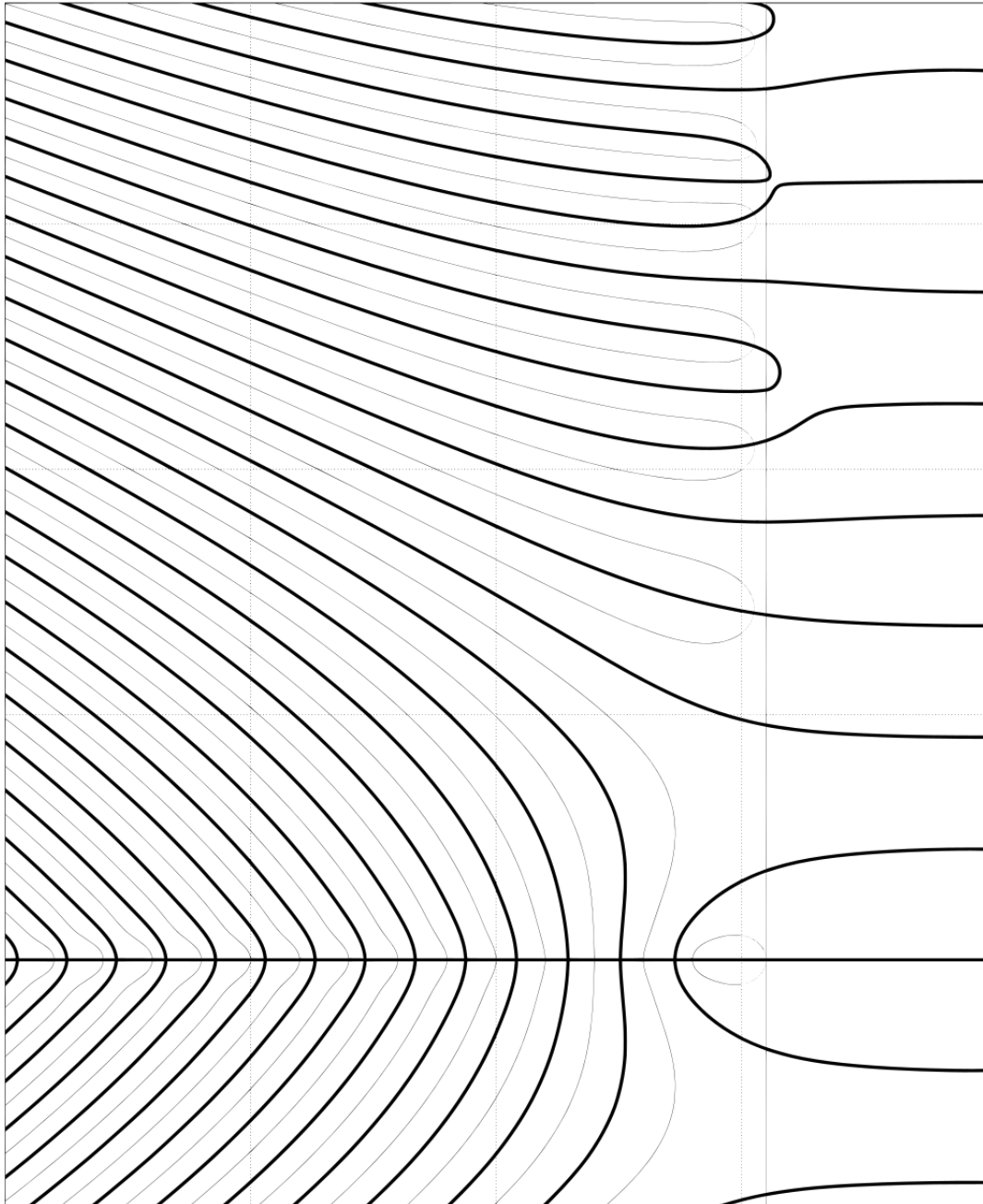


Figure 4: A plot of the Riemann zeta function.  $\zeta$  is real-valued along the thick lines and  $\zeta/i$  is real-valued along the thin lines, so if two lines cross  $\zeta$  vanishes at that point. Image from [2]

- $H(s)$  is holomorphic, because the series  $\sum_n f_n(s)$  defines an holomorphic function: each  $f_n$  is holomorphic and for every compact subset

$$\epsilon \leq \Re s, \Im s \leq 1/\epsilon$$

with  $\epsilon > 0$  one has

$$\left| \sum_{n=1}^{\infty} f_n(s) \right| \leq \sum_{n=1}^{\infty} |f_n(s)| \leq |s| \sum_{n=1}^{\infty} \frac{1}{n^{\epsilon+1}} = |s| \zeta(1 + \epsilon)$$

because  $|f_n(s)| \leq \int_n^{n+1} |n^{-s} - t^{-s}| dt \leq |s|/n^{\epsilon+1}$ .

The claim follows by theorem 1.3.

- Now let  $\epsilon \rightarrow 0$ .

The proof is complete. □

### 3.4.2 Non-vanishing of $\zeta$ in the half-plane $\Re s > 1$

We recall some well-known concepts:

**Definition 3.5** (The Möbius function). It is defined to be

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is square-free (not divisible by any square)} \\ 0 & \text{otherwise} \end{cases}$$

where  $k$  is the number of different prime factors of  $n$ .

We observe  $\mu$  is multiplicative. If  $n, m$  are relatively prime they do not have common prime factors so  $\mu(nm) = \mu(n)\mu(m)$ , and if one of them was not square-free so is the product.

**Definition 3.6** (Euler's  $\varphi(n)$  function). It is defined to be the number of integers smaller than  $n$  which are coprime with  $n$ .

The following result is a consequence of the Inclusion-Exclusion Principle or Möbius inversion formula:

**Theorem 3.7.**

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \sum_{d|n} \frac{\mu(d)}{d}$$

*Observation 3.8.* These two functions can be given similar probabilistic interpretations. It is clear that  $\frac{\varphi(n)}{n}$  is the probability some number between 1 and  $n$  is coprime with  $n$ . The probability of not being divisible by  $p$  is  $(1 - \frac{1}{p})$  and some of these factors appear in 3.7. The events "being divisible by  $p$ " are independent when we consider all natural numbers and the probability of  $k + 1$  integers being relatively prime is

$$\prod_p \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k)}$$

using 3.2. These events are not independent if we are dealing with a finite number of integers, and we can think of  $\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$  as an heuristic approach leading to a correct result. In this context the fact that  $\zeta(1)$  is not defined and its partial sums diverge to  $\infty$  tells us that all integers are divisible by some prime number, because the probability for not being divisible by any is  $1/\zeta(1) = 0$ .

Last observation could lead us to think that  $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} =? \frac{1}{\zeta(s)}$ . It is in fact true for  $\Re s > 1$  and has an important consequence, essential in our proof of the Prime Number Theorem (theorem 5.17).

In the proof we construct a multiplicative inverse, although this is unnecessary because of the Euler product expression for  $\zeta$  (theorem 3.2).

**Theorem 3.9.**  $\zeta$  does not vanish in the half-plane  $\sigma > 1$

*Proof.*

1. Consider the series

$$\alpha(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and

$$\beta(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

By the triangle inequality, these series both converge absolutely in the region  $\Re s > 1$ .

2. We can then (step 1) multiply  $\alpha$  and  $\beta$  and reorder terms. For Dirichlet series, this calculation is usually called **Dirichlet product**. Using that  $\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor$  results in

$$\zeta(s)\beta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \sum_{d|n} \frac{\mu(d)}{n^s} = 1$$

3. We constructed a multiplicative inverse in this region. Then  $\zeta$  cannot vanish there. □

### 3.4.3 Some useful expressions and bounds for $\zeta$

**Theorem 3.10.** For  $\sigma, x > 0$  and  $s \neq 1$  we have

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^\infty \{u\} u^{-(s+1)} du \quad (3.11)$$

where  $\{x\}$  denotes the fractional part of  $x$ .

*Proof.*

- It is clear that for  $\sigma > 1$

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \sum_{n > x} n^{-s}$$

From (1.1) it follows that  $\sum_{n > x} n^{-s} = \int_x^\infty u^{-s} d[u]$ , where  $[u] + \{u\} = u$  denote the floor and fractional part functions. This observation and integration by parts yields

$$\begin{aligned} \int_x^\infty u^{-s} d[u] &= \int_x^\infty u^{-s} du - \int_x^\infty u^{-s} d\{u\} \\ &= \left( \frac{x^{1-s}}{s-1} \right) + \left( \frac{\{x\}}{x^s} - s \int_x^\infty \{u\} u^{-(s+1)} du \right) \end{aligned} \quad (3.12)$$

because the first term above is just  $\left. \frac{u^{1-s}}{1-s} \right|_x^\infty = -\frac{x^{1-s}}{1-s}$  when  $\sigma > 1$  and  $x > 0$ .

- Observe that the integrand in (3.12) is holomorphic with respect to  $s$  and continuous with respect to  $u$  when  $\sigma > 0$ . Hence the integral in (3.12) defines an holomorphic function when  $\sigma > 0$  by theorem 1.5.
- Therefore, theorem 3.10 holds and defines an extension for  $\sigma > 0$  (although we already extended  $\zeta$  everywhere).

□

**Corollary 3.11.** We have the inequalities

$$\frac{1}{\sigma-1} < \zeta(\sigma) < \frac{\sigma}{\sigma-1} \quad (3.13)$$

for all  $0 < \sigma < 1$ . In particular,  $\zeta$  does not vanish in the segment  $s = \sigma + it$  with  $t = 0$  and  $\sigma \in (0, 1)$ .

*Proof.*

- Using 3.10 and setting  $x = 1$  we have that for  $\sigma > 0$

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \{u\} u^{-(s+1)} du$$

- By taking absolute values

$$0 \leq \int_1^\infty \{u\}u^{-(\sigma+1)} \leq \int_1^\infty u^{-(\sigma+1)} du = \frac{1}{\sigma}$$

- Hence

$$\frac{\sigma}{\sigma-1} = 1 + \frac{1}{\sigma-1} - 0\sigma \geq \zeta(\sigma) \geq \frac{1}{\sigma-1} + 1 - \frac{\sigma}{\sigma} = \frac{1}{\sigma-1}$$

□

**Corollary 3.12.** Fix  $\delta > 0$ . For  $\sigma \geq \delta$  and  $s \neq 1$  we have

$$\sum_{n \leq x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(\tau x^{-\sigma}) \quad (3.14)$$

where  $\tau = |t| + 1$ ,  $s = \sigma + it$ .

*Proof.*

- We will apply theorem 3.10 and then bound each term in (3.11). Observe that  $\frac{\{x\}}{x^s} \ll K$  is bounded, because  $x \geq 1$  and  $\sigma \geq \delta > 0$ . The integral term verifies

$$|s \int_x^\infty \{u\}u^{-(s+1)} du| \leq |s| \int_x^\infty u^{-(\sigma+1)} du = |s| \frac{x^{-\sigma}}{\sigma}$$

- But  $\frac{|s|}{\sigma} = \frac{\sqrt{\sigma^2+t^2}}{\sigma} = \sqrt{1 + (\frac{t}{\sigma})^2} = O(\tau)$  because of  $\sigma \geq \delta > 0$ .

The proof is complete. □

*Observation 3.13.* One could actually compute  $\zeta(s)$  everywhere using (3.14), because the functional equation (corollary 3.3) relates the values  $\zeta(1-s)$  and  $\zeta(s)$ . However, this is a hopeless method because by (3.14)

$$\zeta(s) = \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(\tau x^{-\sigma})$$

so if  $s = 0.5 + it$ , in order to compute  $\zeta(s)$  to 5 significant figures  $x$  must be  $\approx 10^{10}$ . An efficient method involves advanced techniques[6].

**Corollary 3.14.** Fix  $\delta > 0$ . If  $\delta \leq \sigma \leq 2$  and  $x \geq 2$  then

$$\zeta(s) \ll (1 + \tau^{1-\sigma}) \min\left(\frac{1}{|\sigma-1|}, \log \tau\right)$$

where  $\tau = |t| + 1$ .

*Proof.*

1. By the triangle inequality and the integral test one has

$$S_{x,s} = \sum_{n \leq x} n^{-s} \ll \sum_{n \leq x} n^{-\sigma} \ll 1 + \int_1^x u^{-\sigma} du = 1 + I_{x,\sigma}$$

Now we deal with  $I_{x,\sigma}$ :

- If  $\sigma > 1$  then  $I_{x,\sigma} < \int_1^\infty u^{-\sigma} du = \frac{1}{\sigma-1}$
- If  $\sigma = 1$  then  $I_{x,\sigma} = \log x$
- If  $0 \leq \sigma < 1$  then  $I_{x,\sigma} < \frac{x^{1-\sigma}}{1-\sigma}$

2. Hence

$$\begin{aligned} S_{x,s} &\ll 1 + \min\left(\frac{1}{|\sigma-1|}, \log x, \frac{x^{1-\sigma}}{|\sigma-1|}\right) \leq 1 + \min\left(\frac{1+x^{1-\sigma}}{|1-\sigma|}, \log x\right) \\ &\leq 1 + (1+x^{1-\sigma}) \min\left(\frac{1}{|\sigma-1|}, \log x\right) \ll (1+x^{1-\sigma}) \min\left(\frac{1}{|\sigma-1|}, \log x\right) \end{aligned}$$

because  $\delta \leq \sigma \leq 2$  and  $x \geq 2$

3. By putting  $x = \tau$  in (3.14) of corollary 3.12 we have the result. □

### 3.5 Analytic properties of $\zeta$

We deduce from previous results the following properties of  $\zeta$ :

**Theorem 3.15** (Analytic properties of  $\zeta$ ).  $\zeta$  is a meromorphic function with zeros at the negative even numbers, the **trivial zeros** and it has a simple pole at  $s = 1$ .

*Proof.*

- Corollary 3.3 for  $\sigma < 0$  yields

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s)$$

The poles of  $\Gamma(\frac{s}{2})$  become zeros for  $\zeta$  at the even negative integers for  $s \neq 0$ , because  $\pi^{s-1/2} \Gamma(\frac{1-s}{2}) \zeta(1-s)$  is holomorphic and does not vanish by theorem 3.9 and proposition 1.24.

- By proposition 1.23 and proposition 3.4, it is clear that the point singularity at  $s = 0$  does not become a pole nor a zero because  $\Gamma(\frac{s}{2}) \approx \frac{2}{s}$  by (1.5) and  $\zeta(1-s) \approx -\frac{1}{s}$  when  $s \rightarrow 0$ . This proves that  $\zeta(0) = -\frac{1}{2}$ . □



### 3.5.1 Riemann's hypothesis

Corollary 3.3 links the values of  $\zeta$  in  $s$  and  $1 - s$ . In fact,

**Corollary 3.16.** *If  $\rho = \sigma + it$  is a non-trivial zero of  $\zeta(s)$  then  $1 - \rho = 1 - \sigma - it$  is a zero.*

We also observe that the critical strip  $0 < \Re s < 1$  remains invariant by  $s \rightarrow 1 - s$ . For a particular value of  $t$ , there could be more than one zero with  $\Im \rho = t$ , unless all zeros have real part equal to  $\frac{1}{2}$ . Riemann claimed:

**Conjecture 3.17** (Riemann hypothesis). *All non-trivial zeros of  $\zeta$  have real part  $\frac{1}{2}$  (they all lie on the **critical line (CL)**).*

Riemann (1859) said :

... it is very probable that all roots are real. Of course one would wish for a rigorous proof here; I have for the time being, after some fleeting vain attempts, provisionally put aside the search for this, as it appears dispensable for the immediate objective of my investigation.

Its proof would lead to unconditional proofs of some statements and better bounds in several known results. For example, under the Riemann hypothesis one can say that

$$\pi(x) = \text{li}(x) + O(\sqrt{x} \log x)$$

which is a stronger statement than the prime number theorem (see theorem 5.17) because the error term there is  $O(x \exp(-c\sqrt{\log x}))$ .

There have been several attempts to prove RH. One remarkable result is Hardy's theorem (see section 6): there are infinitely many zeros on the critical line.

The best result in this direction was given by N. Levinson in [8], that a fraction of 0.4077 of non-trivial zeros lie in the critical line. The methods used in the proof are technical and beyond the scope of this thesis.

These results try to relate Riemann hypothesis with random matrix theory. It has been checked for a large number of non-trivial zeros that these behave as the eigenvalues of a random unitary matrix.

This is closely related with the **Hilbert-Pólya conjecture**: it states that one can find an unbounded self-adjoint operator whose eigenvalues are precisely the imaginary parts  $\gamma_n$  of the non-trivial zeros of  $\zeta$ . Furthermore, it states that the Riemann hypothesis could have some physical meaning: that one can find a physical system whose Hamiltonian with eigenvalues being the  $\gamma_n$ . There is ongoing research on this topic, although it remains conjectural.

It has been checked that the first  $10^{13}$  zeros of  $\zeta$  verify the Riemann hypothesis by X. Gourdon [6], and some results like the weak Goldbach conjecture, that every sufficiently large odd number is the sum of three primes, were first proven conditionally on the RH and then unconditionally. This can be seen as a weak evidence for the RH.

### 3.6 Distribution of zeros in the critical strip

As exposed in [4, 9] if we denote by  $N(T)$  the number of zeros  $\rho = \sigma + it$  in the critical strip with  $0 < t \leq T$  then

**Theorem 3.18.**

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + E(T)$$

where  $E(T)$  is an error term such that  $\lim_{T \rightarrow \infty} \frac{E(T)}{T \log T} = 0$ . This proves that  $\zeta$  has infinitely many zeros in the critical strip. This bound is also used to prove that some error terms or sums involving the zeros of  $\zeta$  in explicit formulas are convergent.

#### 3.6.1 A first approximation for $N(T)$

It is not hard to prove a weaker result, that  $N(T) \ll T \log T$ :

**Theorem 3.19** (Asymptotics for  $N(T)$ ). *We have for  $T \geq 0$*

$$N(T) \ll T \log T$$

*Proof.*

1. From corollary 3.14 we have that for  $1/2 \leq \sigma \leq 2$  and  $|t| \geq 1$

$$\zeta(s) \ll \tau^{1/2} \log \tau$$

where as usual  $\tau = |t| + 1$ .

2. Recall that  $\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}}$  is an entire function we defined in (3.10). It follows from Stirling's formula (theorem 1.26) that

$$\xi(s) \ll |s^2(\frac{s}{2})^{s/2-1/2} e^{-s}|s|^{1/2} \log |s| e^{-\frac{s}{2} \log \pi} \ll e^{|s| \log |s|}$$

for  $\sigma \geq 1/2$  and  $|s| \geq 2$ . Observe that we choose  $|s| \geq 2$  in order to impose  $|t| \geq 1$ .

3. By the functional equation for  $\xi$  as seen in corollary 3.3 this last bound holds for any  $|s| \geq 2$ , because the reflection of the condition  $\sigma \geq 1/2$  is  $\sigma \leq 1/2$ .
4. By the general analytic properties of zeta (theorem 3.15) we know that  $\zeta(0) = -\frac{1}{2}$ . Hence  $\xi(0) = \frac{1}{2}$  because the factor  $s$  in  $\xi$  cancels the simple pole of  $\Gamma$  at  $s = 0$ .

So  $\xi$  is an entire function verifying the hypotheses of theorem 1.28. Therefore, by Jensen's inequality (theorem 1.28) by choosing  $R = 2|s|$ ,  $r = |s|$  and  $s$  with  $|s| = iT$  we have

$$N(T) \ll \log e^{|s| \log |s|} = |s| \log |s| = T \log T$$

because it is clear that the zeros of  $\zeta$  in the critical strip are the only zeros of  $\xi$ .  $\square$

### 3.6.2 Logarithmic derivative of $\xi$

We will apply the argument principle to  $\xi$  to give an asymptotic formula for  $N(T)$ . Clearly, the quantity  $\xi'(s)/\xi(s)$  will appear in this calculation. So it is natural to compute its logarithmic derivative. From

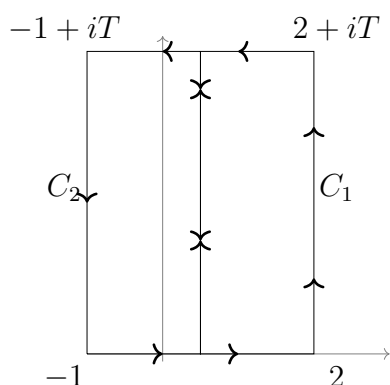
$$\xi(s) = \frac{1}{2}s(1-s)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

we have

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{1-s} + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{2} \log \pi \quad (3.15)$$

*Observation 3.20.* We remark taking the logarithmic derivative of  $\xi$  is valid when  $s$  is real and  $s > 0$  or  $\Re s > 1$ , because of the non-vanishing of  $\zeta$  in these regions (theorem 3.9).

### 3.6.3 Proof sketch of theorem 3.18



Let  $T \in \mathbb{R}$  and  $C$  be the rectangular contour described by the four points  $-1, 2, 2 + iT, -1 + iT$ . Observe that inside  $C$  the entire function  $\xi$  redefined in (3.3) has only non-trivial zeros and no poles because  $\zeta$  did not have any pole in this region. Also,  $C$  does not touch any pole or zero because the nearest trivial zeros of  $\xi$  are  $-2$  and  $3$ . Therefore, by the argument principle and (1.2) we have

$$N(T) = \frac{1}{2\pi i} \int_C \frac{\xi'(s)}{\xi(s)} ds \quad (3.16)$$

- Define the rectangular contours, with the following vertices and orientations

$$\begin{cases} C_1 : 1/2 \rightarrow 2 \rightarrow 2 + iT \rightarrow 1/2 + iT \\ C_2 : 1/2 + iT \rightarrow -1 + iT \rightarrow -1 \rightarrow 1/2 \\ C_3 : 1/2 - iT \rightarrow 2 - iT \rightarrow 2 \rightarrow 1/2 \\ C_4 : 1/2 - iT \rightarrow 2 - iT \rightarrow 2 + iT \rightarrow 1/2 + iT \end{cases}$$

- Observe that  $C_3$  is also the image of  $C_2$  by the map  $s \mapsto 1 - s$ , and  $\overline{C_1} = C_3$  are conjugates. Also, we have  $C_4 = C_3 + C_1$  as the segment  $1/2 \rightarrow 2$  cancels because of the orientations. Using the functional equation (corollary 3.3) yields

$$\frac{\xi'(s)}{\xi(s)} = -\frac{\xi'(1-s)}{\xi(1-s)} \rightarrow \int_{C_2} \frac{\xi'(s)}{\xi(s)} ds = \int_{C_2} -\frac{\xi'(1-s)}{\xi(1-s)} ds = \int_{C_3} \frac{\xi'(s)}{\xi(s)} ds$$

so (3.16) becomes

$$N(T) = \frac{1}{2\pi i} \left( \int_{C_1} + \int_{C_3} \right) \frac{\xi'(s)}{\xi(s)} ds = \frac{1}{2\pi i} \int_{C_4} \frac{\xi'(s)}{\xi(s)} ds \quad (3.17)$$

- Applying (3.15) to (3.17) yields

$$N(T) = \frac{1}{2\pi i} \left( \log s + \log(s-1) + \log \zeta(s) + \log \Gamma\left(\frac{s}{2}\right) - \frac{s}{2} \log \pi \right) \Big|_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \quad (3.18)$$

- $\frac{1}{2} + iT$  and  $\frac{1}{2} - iT$  are complex conjugates, so Schwarz reflection principle (theorem 1.14) can be applied to compute last expression. That is, we only take into account imaginary parts in the logarithms because the real parts will cancel. The imaginary part of the branches of the complex logarithm we have chosen is the argument of each term. By symmetry, the arguments double so

$$\begin{aligned} (3.18) &= \frac{1}{2\pi i} \left( 2i \operatorname{Arg}\left(\frac{1}{2} + iT\right) + 2i \operatorname{Arg}\left(-\frac{1}{2} + iT\right) + 2i \operatorname{Arg} \log \zeta\left(\frac{1}{2} + iT\right) \right. \\ &\quad \left. + 2i \operatorname{Arg} \log \Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - 2\frac{iT}{2} \log \pi \right) \\ &= \frac{1}{\pi} \left( \pi + \operatorname{Arg} \log \zeta\left(\frac{1}{2} + iT\right) + \operatorname{Arg} \log \Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \frac{T}{2} \log \pi \right) \\ &= 1 + Z(T) + G(T) - \frac{T}{2\pi} \log \pi \end{aligned} \quad (3.19)$$

where  $Z(T) = \operatorname{Arg} \log \zeta(1/2 + iT)$  and  $G(T) = \operatorname{Arg} \log \Gamma(1/4 + iT/2)$ .

- It can be shown that by applying Stirling's formula for  $\Gamma(s)$  (theorem 1.26) to  $G(T)$  in (3.19) yields that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + Z(T) + O(1/T)$$

- We state without proof a result that is beyond the scope of this thesis: Under the Riemann hypothesis, one has that

$$\operatorname{Arg} \zeta(s) \ll \frac{\log \tau}{\log \log \tau}$$

where  $\tau = |t| + 2$ . Applying this to  $Z(T)$  results in  $Z(T) = O(\log T / \log \log T)$ . Therefore

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T / \log \log T)$$

## 4 Dirichlet's theorem

The aim of this section is to prove Dirichlet's theorem on arithmetic progressions (theorem 4.20). We begin by proving results about the dual group of a finite abelian group, because the orthogonality relations are essential in our proof of theorem 4.20. Next we define what is a Dirichlet character and show that we can associate a Dirichlet series to a Dirichlet character.

The important step is to prove that if  $(k, m) = 1$  the following series

$$\sum_{p \equiv k \pmod{m}} \frac{1}{p}$$

diverges. From this it follows that there are infinitely many primes in arithmetic progressions, although we do not give a distribution of these primes in the same spirit of the prime number theorem (theorem 5.17). We will define the natural and Dirichlet densities, which given a set measures in some sense how the elements of the set are distributed in the natural numbers.

In fact our proof is a result about the Dirichlet density of the set of primes in arithmetic progressions, although it can be proven that the natural density and the Dirichlet density are the same in this particular case, but this is beyond the scope of this thesis.

In this section we follow [12], although we can find an alternative proof in [1]. This latter is in fact more resemblant to Dirichlet's original proof, which used a more involved argument.

### 4.1 Orthogonality relations

Let  $G$  be a finite abelian group. We use multiplicative notation to denote the product of  $G$ .

**Definition 4.1** (Characters of an abelian group). We say  $f : G \rightarrow \mathbb{C}^*$  is a character of  $G$  if it is a group morphism.

The characters of  $G$  form a group with respect multiplication because  $\mathbb{C}^*$  is a group. Denote by  $\hat{G} = \text{Hom}(G, \mathbb{C}^*)$  the group of characters of  $G$ .

By Lagrange theorem, given  $H \subset G$  a subgroup of  $G$  and for any  $x \in G$  there exists a positive integer  $h$  such that  $x^h \in H$ . Denote by  $h_H(x)$  the least of such positive integers  $h$  verifying  $x^h \in H$ .

**Proposition 4.2** (Construction of subgroups of abelian groups). *Let  $G$  be a finite abelian group. If  $H$  is a proper subgroup of  $G$  and  $x \in G - H$ , then one can construct a subgroup  $H' = \{wx^i : w \in H, 0 \leq i < h_H(x)\}$  of  $G$  containing  $H$  verifying  $|H'| = h_H(x)|H|$ .*

*Proof.* Let  $h = h_H(x)$ . It is clear that  $H$  is a subset of  $H'$ , because any element  $h$  of  $H$  is of the form  $hx^0 \in H'$ . We must check whether  $H'$  is a subgroup of  $G$ .

1. Existence of identity element:  $1 \in H'$  because  $H \subset H'$  and  $H$  is a subgroup of  $G$ .
2. Closed by multiplication: Let  $wx^i$  and  $yx^j$  be elements of  $H'$ . Then  $wx^i yx^j = wyx^{i+j} = wyzx^k$  for some  $z \in H$ . Clearly  $wyz \in H$  because  $H$  is a subgroup.
3. Existence of inverses: Given  $wx^i \in H'$  take  $w^{-1}x^{h-i} \in H'$ .

The claim  $|H'| = h|H|$  is because the expression for each element of  $H'$  is unique. Suppose  $wx^i = yx^j \in H'$  with  $0 \leq i, j < h$ . Then  $wy^{-1} = x^{j-i} \in H$  because both  $w$  and  $y$  are in  $H$  and  $H$  is a subgroup. By writing  $j - i = qh + r$  one has  $x^{j-i} = x^{qh+r} = x^{qh}x^r = wy^{-1}$ . Hence  $x^r = wy^{-1}x^{-qh} \in H$ , so  $r = 0$  by the minimality of  $h$  and the inequalities  $0 \leq r < h$ .

Each element of  $H'$  is of the form  $wx^j$ , where  $w$  and  $j$  run over  $|H|$  and  $h$  different values respectively.  $\square$

Denote by  $\langle H; x \rangle_G$  the subgroup of  $G$  we constructed from  $H$  and  $x$ .

**Theorem 4.3** (A finite abelian group and its dual have the same cardinal). *Let  $G$  be a finite abelian group. Then  $|G| = |\hat{G}|$*

*Proof.*

1. Start with the trivial group  $H_0 = \{1\}$  and construct a sequence  $\{H_i\}_i$  of subgroups of  $G$ , where  $H_i = \langle H_{i-1}, x_i \rangle_G$  and  $x_i \in G - H_{i-1}$  and

$$H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_i \subsetneq H_{i+1} \subsetneq \cdots$$

Clearly this is a finite sequence because  $G$  is finite and  $|H_i| = h_i |H_{i-1}|$  with  $h_i = h_G(x_i)$ . Hence each subgroup  $H_i \supset H_{i-1}$  and  $H_i \neq H_{i-1}$  because  $h_i \neq 1$ .

2. We prove the claim by induction on  $i$ . If  $i = 0$  it is clear.
3. Suppose the induction hypothesis is true for  $i \leq k$ . Hence  $|H_k| = |\hat{H}_k|$ .
4. It is clear that if  $\bar{f}$  is a character of  $H_{k+1}$  its restriction in  $H_k$  is a character of  $H_k$ . Then, if we prove each element of  $\hat{H}_k$  can be extended in  $h_k$  different ways the claim is proven for  $i = k$ , that is  $|\hat{H}_{k+1}| = h_k |\hat{H}_k| = h_k |H_k| = |H_{k+1}|$ , because an extension of a character of  $H_k$  to a character of  $H_{k+1}$  is a character of  $H_{k+1}$  and every character of  $H_{k+1}$  is an extension of its restriction to  $H_k$  by definition of extension and restriction.
5. Let  $f \in \hat{H}_k$ . If  $\bar{f}$  extends  $f$  to  $H_{k+1}$ , then  $\bar{f}(wx_k^{h_k}) = \bar{f}(w)\bar{f}(x_k^{h_k}) = f(w)f(x_k^{h_k}) = f(w)\bar{f}(x_k)^{h_k}$ . Hence this proves a necessary condition for  $\bar{f}$  to be an extension of  $f$  is that  $\bar{f}(x_k)$  is a  $h_k$ -th root of  $f(x_k^{h_k})$  in  $\mathbb{C}$ . But in  $\mathbb{C}$  there are  $h_k$  such roots for any non-zero complex number.
6. We observe that if  $f$  is a character and  $H_k$  is finite then  $f \neq 0$ , because for every  $\alpha \in H_k$  there exists  $n$  such that  $\alpha^n = 1$  so  $f(\alpha^n) = f(1) = 1 = f(\alpha)^n$ . That is,  $|f(\alpha)| = 1$ .

7. By defining  $\bar{f}(wx_k^j) := f(w)\bar{f}(x_k)^j$  where  $\bar{f}(x_k)$  is a  $h_k$ -th root of  $f(x_k^{h_k})$  in  $\mathbb{C}$  we obtain a character for  $H_{k+1}$  that extends  $f$ , because  $\bar{f}(wx_k^j y x_k^l) = \bar{f}(wyx_k^{j+l}) = f(wy)\bar{f}(x_k)^{j+l} = f(w)f(y)\bar{f}(x_k)^j \bar{f}(x_k)^l = \bar{f}(wx_k^j)\bar{f}(yx_k^l)$ .

The proof is complete.  $\square$

The fact that  $\hat{G}$  is finite implies that we can order the characters  $\hat{G} = \{\chi_0, \chi_1, \dots, \chi_{n-1}\}$ , where  $\chi_0$  is the trivial character.

It is clear the complex conjugate of a characters is a character: because of  $|\chi(\alpha)| = 1$  we have  $\bar{\chi}(\alpha) = \chi(\alpha)^{-1} = \chi(\alpha^{-1})$ .

**Lemma 4.4.** *Let  $\chi \in \hat{G}$ . Then*

$$\sum_{j=0}^{n-1} \chi_i(\alpha_j) = \begin{cases} 0, & \text{if } \chi \neq 1 \\ n, & \text{if } \chi = 1 \end{cases}$$

*Proof.* If  $\chi = 1$  each term in the sum is 1 and there are  $n$  of them. If  $\chi \neq 1$  there exists  $g \in G$  such that  $\chi(g) \neq 1$ .  $G$  acts by multiplication on itself, so  $G$  is invariant by multiplication by  $g$  so

$$S_\chi = \sum_{j=0}^{n-1} \chi_i(\alpha_j) = \sum_{j=0}^{n-1} \chi_i(g\alpha_j) = \chi(g) \sum_{j=0}^{n-1} \chi_i(\alpha_j) = \chi(g)S_\chi$$

Then  $S_\chi = 0$  because  $\chi(g) \neq 1$ .  $\square$

Consider the square matrix  $X = \{X_{i,j}\}_{i,j}$  where  $X_{i,j} = \chi_i(\alpha_j)$  and  $\alpha_j$  runs over  $G$ .

**Theorem 4.5** (Orthogonality relations).  $X\bar{X}^t = n \text{ Id}$ , where  $\text{Id}$  is the identify matrix and  $n = |G|$ . That is,

$$\sum_{j=0}^{n-1} \chi_i(\alpha_j)\bar{\chi}_k(\alpha_j) = \begin{cases} 0, & \text{if } i \neq k \\ n, & \text{if } i = k \end{cases}$$

$$\sum_{j=0}^{n-1} \bar{\chi}_j(\alpha_i)\chi_j(\alpha_k) = \begin{cases} 0, & \text{if } i \neq k \\ n, & \text{if } i = k \end{cases}$$

*Proof.*

- $\hat{G}$  is a group, so  $\chi_i\bar{\chi}_k = \chi_l \in \hat{G}$  for some  $\chi_l$ .
- Moreover, define  $S_{i,k}$  to be the following sum

$$\sum_{j=0}^{n-1} \chi_i(\alpha_j)\bar{\chi}_k(\alpha_j) = \sum_{j=0}^{n-1} \chi_l(\alpha_j) = S_{i,k}$$

- Therefore,
  - If  $i = k$  then  $\chi_l = 1$  and  $S_{i,k} = n$  by lemma 4.4.
  - If  $i \neq k$  then  $\chi_l \neq 1$  and  $S_{i,k} = 0$  by lemma 4.4.

- Now we observe a matrix commutes with its inverse, that is  $X\bar{X}^t = \bar{X}^tX = n Id$ . Therefore, the second sum in the claim is  $n$  times the coefficient  $i, k$  of the identity matrix.

□

## 4.2 L-functions

Our goal is to prove that

$$\sum_{p \equiv k \pmod{m}} \frac{1}{p^s} = L(s, \chi = 1)$$

(with  $(k, m) = 1$ ) does not have a finite value at  $s = 1$ : we must define what is  $\chi$  and  $L(s, \chi)$ .

**Definition 4.6** (Dirichlet characters). We say  $\chi$  is a **Dirichlet character modulo  $m$**  if  $\chi$  is a character of  $(\mathbb{Z}/m\mathbb{Z})^*$ . Every  $\chi$  can be extended to  $\mathbb{N}$  as follows:

$$\chi(n) = \begin{cases} \chi([n]), & \text{if } (n, m) = 1 \\ 0, & \text{otherwise} \end{cases}$$

where  $[n]$  is the class of  $n$  modulo  $m$  in  $(\mathbb{Z}/m\mathbb{Z})^*$ . We use the same notation for the character  $\chi$  and its extension to  $\mathbb{N}$ .

These extensions are still completely multiplicative, because the characters were:  $\chi(nm) = \chi(n)\chi(m)$ .

**Definition 4.7** (Principal characters). Let  $\chi$  be a Dirichlet character modulo  $m$ . We say  $\chi$  is the **principal character modulo  $m$**  if

$$\chi(n) = \begin{cases} 1 & \text{for all } n \text{ coprime to } m \\ 0 & \text{otherwise} \end{cases}$$

The L-functions will help us prove Dirichlet's theorem:

**Definition 4.8** (L-function). We define  $L(s, \chi)$  for  $\sigma > \sigma_c$  as follows:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$



**Euler product for  $L(s, \chi)$ :** We have an Euler product for  $L(s, \chi)$ , because  $\chi$  is completely multiplicative (see section 2.4).

$$L(s, \chi) = \prod_{p \nmid m} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (4.1)$$

In the product above  $p \nmid m$  so we are only avoiding a finite number of terms, those corresponding to the prime factors of  $m$ .

When  $\chi$  is the principal character we can say that

$$L(s, \chi) = \prod_{p \nmid m} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p|m} \left(1 - \frac{1}{p^s}\right) \quad (4.2)$$

**Convergence for  $\Re s > 0$ :** From the fact that we can express  $L(s, \chi)$  for  $\chi$  principal in terms of  $\zeta(s)$  and a finite product, we can then extend  $L(s, \chi)$  to the whole complex plane. Therefore, we have

**Corollary 4.9** (Analytic continuation of  $L(s, \chi)$ ).  *$L(s, \chi)$  can be extended to a meromorphic function in  $\mathbb{C}$ , with a single simple pole at  $s = 1$  when  $\chi$  is principal.*

*Proof.* If  $\chi$  is the principal character, then by (4.2)

$$L(s, \chi) = \zeta(s) \prod_{p|m} \left(1 - \frac{1}{p^s}\right)$$

This expression is defined everywhere but at  $s = 1$  and agrees with definition 4.8 for  $\Re s > 1$  by construction and because  $\zeta(s)$  has a simple pole at  $s = 1$ . The finite product does not affect the behavior.  $\square$

For  $\chi$  not principal,  $L(s, \chi)$  can also be extended, but we do not require a full analytic continuation defined in  $\mathbb{C}$  for proving Dirichlet's theorem. We only need to prove that the expression in definition 4.8 is well-defined for  $\Re s > 0$ , because essentially the proof uses the behavior of the functions  $L(s, \chi)$  at  $s = 1$ .

**Proposition 4.10** (Convergence of the series for  $L(s, \chi)$  for  $\Re s > 0$ ). *For non principal  $\chi$ , the series for  $L(s, \chi)$  is convergent for  $\Re s > 0$ .*

*Proof.*

- We will apply Abel's criterion (proposition 1.13) to prove that the series converges for  $\Re s > 0$  and use proposition 2.2.
- Using the same notation  $a_n = \chi(n)$ ,  $b_n = n^{-s}$ , we observe that

$$\left| \sum a_n \right| \leq \varphi(m)$$

if  $\chi$  is a non principal character modulo  $m$ . This follows from the orthogonality relations in theorem 4.5: we have

$$\sum_{n=0}^m \chi(n) = 0$$

Therefore the function  $F(N) = \sum_{n=0}^N \chi(n)$  is periodic with period at most  $m$ . Also  $|a_n| = 1$  for all  $n$  with  $(n, m) = 1$ , and there are  $\varphi(m)$  of such integers  $n$ .

- The sequence  $b_n$  is not real-valued, but it can be bounded by a real-valued sequence  $|b_n| = |n^{-s}| = n^{-\sigma}$  that is strictly decreasing.

□

### 4.3 Dirichlet density and natural density

We now define Dirichlet's density and natural density:

**Definition 4.11** (Dirichlet density). Let  $C$  be a subset of the prime numbers. Define **Dirichlet's density of  $C$**  ( $\delta_C$ ) as

$$\lim_{s \rightarrow 1^+} \frac{\sum_{p \in C} \frac{1}{p^s}}{\log\left(\frac{1}{s-1}\right)} = \lim_{s \rightarrow 1^+} \frac{\sum_{p \in C} \frac{1}{p^s}}{\sum_p \frac{1}{p^s}} = \delta_C$$

It is clear that if  $C$  has a positive Dirichlet's density then  $C$  is infinite. Otherwise the series in the numerator would not diverge because it would be a finite sum.

Although this definition may seem arbitrary it makes sense in the simplest case:

**Theorem 4.12.** *The set of all prime numbers has Dirichlet's density 1, that is*

$$\lim_{s \rightarrow 1^+} \frac{\sum_p \frac{1}{p^s}}{\log\left(\frac{1}{s-1}\right)} = 1$$

*Proof.* We only have to prove the following:

$$\log \zeta(s) \approx \log\left(\frac{1}{s-1}\right)$$

as  $s \rightarrow 1$ . From this, it will follow that

$$\sum_p \frac{1}{p^s} \approx \log\left(\frac{1}{s-1}\right)$$

- The first claim follows by taking logarithms in proposition 3.4.

- By theorem 3.2 we have

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

This product converges uniformly over compact subsets of  $U = \{\Re s > 1\}$ . Therefore, we can take logarithms, use the Taylor expansion of  $\log(1 + x)$  with  $|x| < 1$  and reorder terms:

$$\log \zeta(s) = - \sum_p \log(1 - p^{-s}) = \sum_{k=1}^{\infty} \sum_p \frac{1}{kp^s}$$

- Therefore

$$\log \zeta(s) = \sum_{k,p} \frac{1}{kp^{sk}} = \sum_p \frac{1}{p^s} + \sum_{k \geq 2} \sum_p \frac{1}{kp^{sk}}$$

- The dominant term is  $\sum_p p^{-s}$ , because

$$\left| \sum_{k \geq 2} \sum_p \frac{1}{kp^{sk}} \right| \leq \frac{1}{2} \sum_p \frac{1}{|p^s(p^s - 1)|} \leq \frac{1}{2}$$

and the second claim follows.

- The limit is done with  $s \rightarrow 1^+$ , because the series in the numerator is defined only for  $\Re s > 1$  and because the product expression we used for  $\zeta$  is only valid for  $\Re s > 1$ . These facts justify somehow the definition of Dirichlet's density.

□

**Definition 4.13** (Natural density). The **natural density** of  $C$  is defined by

$$\lim_{n \rightarrow \infty} \frac{|\{x \in C : x \leq n\}|}{n} = \Delta_C$$

These two definitions of density are not equivalent in general. Moreover, there are subsets of the prime number having Dirichlet's density but no natural density. As noted in [15], it turns out that natural density and Dirichlet density are particular cases of the following situation:

**Definition 4.14** (Density of a subset with respect to a sequence). Let  $\lambda = \{\lambda_n\}_{n \geq 1}$  be a real-valued sequence  $\lambda_n \geq 0$  satisfying  $\sum_{n=1}^{\infty} \lambda_n = \infty$ .

Let  $A \subset \mathbb{N}$ . Put

$$D(x, \lambda, A) = \frac{\sum_{n \leq x \text{ and } n \in A} \lambda_n}{\sum_{n \leq x} \lambda_n}$$

We define the **density of  $A$  with respect to  $\lambda$**  to be the limit

$$d(A, \lambda) = \lim_{x \rightarrow \infty} D(x, \lambda, A)$$

if it exists. Similarly, one defines **upper and lower density of  $A$  with respect to  $\lambda$**  to be

$$\bar{d}(A, \lambda) = \limsup_{x \rightarrow \infty} D(x, \lambda, A)$$

$$\underline{d}(A, \lambda) = \liminf_{x \rightarrow \infty} D(x, \lambda, A)$$

respectively.

It is easy to check that  $\delta_C = d(C, \lambda)$  with

$$\lambda_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

and natural density verifies  $\Delta_C = d(C, \lambda)$  with  $\lambda_n = 1$  for all  $n$ . It is not hard to see, using the properties of the Riemann-Stieltjes integral (see definition 1.9) that  $\Delta_C$  is stronger than  $\delta_C$  in some sense:

**Proposition 4.15.** *For any  $A \subset \mathbb{N}$  one has*

$$\underline{\Delta}_C \leq \underline{\delta}_C \leq \bar{\delta}_C \leq \bar{\Delta}_C$$

This implies that if the natural density exists then Dirichlet density also exists and they are equal. However, the proof of Dirichlet's theorem on arithmetic progressions we are exposing only shows that a Dirichlet density exists, although it can be proven that the particular set of primes we talking about does have a natural density, and it is equal to the Dirichlet density.

#### 4.4 Dirichlet theorem on arithmetic progressions

We define yet another function, that controls the behavior of all the  $L(s, \chi)$ :

**Definition 4.16.** Let  $\zeta_m(s)$  be defined as follows:

$$\zeta_m(s) = \prod_{\chi} L(s, \chi)$$

where  $\chi$  runs over the Dirichlet characters modulo  $m$ .

$\zeta_m(s)$  verifies an important property:

**Lemma 4.17.**

$$\zeta_m(s) = \prod_{\chi} \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \prod_p \left(1 - p^{-s \text{ord}(p)}\right)^{-\frac{\varphi(m)}{\text{ord}(p)}}$$

where  $\text{ord}(p)$  is the order of  $p$  modulo  $m$ .

*Proof.*

- We only have to prove that for any  $T \in \mathbb{C}$  and for any  $p$  prime

$$\prod_{\chi} (1 - \chi(p)T) = (1 - T^{\text{ord}(p)})^{\frac{\varphi(m)}{\text{ord}(p)}}$$

and then put  $T = p^{-s}$  in the expression above, let  $p$  run over the prime numbers and use the Euler product for  $L(s, \chi)$  (see (4.1)).

- Let  $\text{ord}(p)$  be the order of  $p$  modulo  $m$ . Therefore, there are  $\varphi(m)/\text{ord}(p)$  characters modulo  $m$  such that  $\chi(p) = \omega$ , where  $\omega$  is any  $\text{ord}(p)$ -th root of unity in  $\mathbb{C}$ :  $\chi(p^{\text{ord}(p)}) = \chi(1) = 1 = \chi(p)^{\text{ord}(p)}$  so  $\chi(p)$  is a  $\text{ord}(p)$ -th root of unity, and if we act on the set of characters modulo  $m$  evaluated at  $p$  by the quotient group  $(\mathbb{Z}/(m\mathbb{Z}))^*/\langle p \rangle$  by taking powers is clear that this is a transitive, faithful and well-defined action.
- By the orbit-stabilizer theorem, the claim on the number of characters follows.
- The left hand side vanishes if and only if  $T$  is a  $\text{ord}(p)$ -th root of unity, and if we regard it as a polynomial in  $T$  it must have degree  $\varphi(m)/\text{ord}(p)$  because there are this number of characters modulo  $m$ ; the right hand side vanishes if and only if  $T$  is a  $\text{ord}(p)$  root of unity, and has the same degree as the left hand side with the same roots when regarded as a polynomial in  $T$ .

□

The product can be simplified further, because the terms in the product are equal to 1 if  $p$  divides  $m$ , because  $\chi(p)$  vanishes. The product in lemma 4.17 defines a Dirichlet series converging in the half-plane  $\Re s > 1$ , because all the terms  $L(s, \chi)$  are Dirichlet series and in this region we can multiply and reorder terms arbitrarily, and there is only a finite number of  $\chi$  characters.

The following step is crucial for theorem 4.20:

**Theorem 4.18.**

$$L(1, \chi) \neq 0$$

for non principal  $\chi$ . Therefore,  $\zeta_m(s)$  has a simple pole at  $s = 1$ .

*Proof.*

- Suppose  $L(1, \chi) = 0$  for some non-principal  $\chi$ . Then  $\zeta_m(s)$  would become holomorphic at  $s = 1$  because of corollary 4.9 and proposition 4.10:  $L(s, \chi)$  has a simple pole at  $s = 1$  for  $\chi$  principal, and is bounded for non principal  $\chi$ . The pole would cancel out.
- Then  $\zeta_m$  would be holomorphic for  $\Re s > 0$ :

- Using lemma 4.17 we have that for  $\Re s > 1$

$$\zeta_m(s) = \prod_p \left(1 - p^{-s \operatorname{ord}(p)}\right)^{-\frac{\varphi(m)}{\operatorname{ord}(p)}}$$

This is a finite product of converging geometric series, each with ratio  $p^{-s \operatorname{ord}(p)}$  and to the power of  $\varphi(m)/\operatorname{ord}(p)$  (an integer). We can multiply and reorder terms arbitrarily. This will result in a Dirichlet series with positive terms, because all the factors are Dirichlet series with positive terms.

- All the coefficients are positive, so by corollary 2.6 the series we found in last step converge for  $\Re s > 0$ .  
 – Also, for  $\sigma > 0$  we have

$$\begin{aligned} \zeta_m(\sigma) &= \prod_p \left(\frac{1}{1 - \frac{1}{p^{\sigma \operatorname{ord}(p)}}}\right)^{\frac{\varphi(m)}{\operatorname{ord}(p)}} > \prod_p \frac{1}{1 - \frac{1}{p^{\sigma \varphi(m)}}} \\ &= \sum_{(n,m)=1} \frac{1}{n^{\sigma \varphi(m)}} = L(\sigma \varphi(m), 1) \end{aligned}$$

because for every prime  $p$  and  $\sigma > 0$

$$\left(1 - \frac{1}{p^{\sigma \operatorname{ord}(p)}}\right)^{-\frac{\varphi(m)}{\operatorname{ord}(p)}} = \left(\sum_{j=0}^{\infty} p^{-j \sigma \operatorname{ord}(p)}\right)^{\varphi(m)/\operatorname{ord}(p)} > \left(1 - \frac{1}{p^{\sigma \varphi(m)}}\right)^{-1}$$

But  $L(\sigma \varphi(m), 1) = \infty$  diverges for  $\sigma = \frac{1}{\varphi(m)} > 0$  by corollary 4.9.

Contradiction. Therefore  $L(1, \chi) \neq 0$  for all non principal  $\chi$ .

- $L(1, \chi) \neq 0$  for all non principal  $\chi$  implies that  $\zeta_m$  has a pole at  $s = 1$ , because of lemma 4.17: there is no pole/zero cancellation and there is a finite number of characters.

□

Define  $\Omega_\chi(s)$  to be as follows (this is not standard notation):

$$\Omega_\chi(s) = \sum_p \frac{\chi(p)}{p^s}$$

**Corollary 4.19** (Behaviour of  $L(s, \chi)$  when  $s \rightarrow 1$ ).

$$\Omega_\chi(s) = \sum_p \frac{\chi(p)}{p^s} = \begin{cases} \approx \log \frac{1}{s-1} & \text{if } \chi \text{ is principal} \\ \text{bounded as } s \rightarrow 1 & \text{if } \chi \text{ is not principal} \end{cases}$$

*Proof.*

- Because of equation 4.2 we have

$$\log L(s, \chi) = \log \zeta(s) + \log \prod_{p|m} \left(1 - \frac{1}{p^s}\right)$$

- The dominant term in this last expression is  $\log \zeta(s)$ : there is only a finite number of primes dividing  $m$ , and we can apply corollary 4.12.
- If  $\chi$  is non principal we apply proposition 4.10; the series converge.

□

**Theorem 4.20** (Dirichlet's theorem). *Let  $k$  be such that  $(k, m) = 1$  and let  $C = \{ \text{prime numbers } \equiv k \pmod{m} \}$ .*

*Then the function*

$$D_{m,k}(s) = \sum_{p \in C} \frac{1}{p^s}$$

*has a simple pole at  $s = 1$ , and*

$$\delta_C = \frac{1}{\varphi(m)}$$

*Therefore,  $|C| = \infty$ .*

*Proof.*

- Note that  $\chi(k) \neq 0$  because  $(k, m) = 1$ . We will use the orthogonality relations applied to the Dirichlet characters modulo  $m$  (corollary 4.5):

$$\begin{aligned} D_{m,k}(s) &= \sum_{p \equiv k \pmod{m}} \frac{1}{p^s} = \frac{1}{\varphi(m)} \varphi(m) \sum_{p \equiv k \pmod{m}} \frac{1}{p^s} = \frac{1}{\varphi(m)} \sum_{\chi} \chi(k^{-1}p) \sum_p \frac{1}{p^s} \\ &= \frac{1}{\varphi(m)} \sum_{\chi} \chi(k)^{-1} \sum_p \frac{\chi(p)}{p^s} = \frac{1}{\varphi(m)} \sum_{\chi} \chi(k)^{-1} \Omega_{\chi}(s) \end{aligned} \quad (4.3)$$

We observe that absolute convergence was not used because we did not reorder any terms.

- The main argument is that (4.3) concludes the proof, because the terms with non principal  $\chi$  are bounded and the term with  $\chi$  principal behaves like  $\approx \log\left(\frac{1}{s-1}\right)$  by corollary 4.19.
- This proves that the Dirichlet's density for this set of prime numbers is exactly  $\frac{1}{\varphi(m)}$ , because

$$\lim_{s \rightarrow 1^+} \frac{\frac{1}{\varphi(m)} \sum_{\chi} \chi(k)^{-1} \Omega_{\chi}(s)}{\log \frac{1}{s-1}} = \frac{1}{\varphi(m)}$$

□

## 5 The prime number theorem

The aim of this section is to prove the Prime Number Theorem which states that  $\pi(x) =$  the number of primes up to  $x$ , verifies the following asymptotic relation:

$$\pi(x) \approx \frac{x}{\log x}$$

In fact we prove a stronger statement: There exists  $c > 0$  so that

$$\pi(x) = \text{li}(x) + O(x \exp(-c\sqrt{\log x}))$$

where  $\text{li}(x) = \int_2^x \frac{du}{\log u}$  is called the **logarithmic integral function**.

We begin with some historical remarks including some classical results on Tchebychev functions, although one may skip this section because it is not essential. Next we move to the preliminaries: the relationship between  $\pi$  and  $\psi$ , between  $\psi$  and  $\zeta$ , the zero-free regions and bounds of  $\zeta$  in the critical strip. We end the section with the proof of the prime number theorem and a proof sketch of von Mangoldt's formula (which is a stronger statement) and an heuristic argument which gives the correct order of magnitude.

### 5.1 Some historical remarks

Based on large prime tables, Legendre conjectured in 1797 the following distribution

$$\pi(x) \approx \frac{x}{\log(x) - 1.08} \quad (5.1)$$

which is very close to the real result, although Gauss gave a better approximation (in 1792, when he was a boy)

$$\pi(x) \approx \int_2^x \frac{dt}{\log(t)} = \text{li}(x) \quad (5.2)$$

In 1848 Tchebychev tried to prove the prime number theorem, and in his attempt he introduced and proved several results about the Tchebychev functions  $\psi$  and  $\vartheta$ . These were the first rigorous arguments in the direction of the PNT. He was also able to prove Bertrand's postulate (see section 5.1.3): there is a prime between  $n$  and  $2n$  for  $n \geq 1$ .

In 1859 Riemann wrote his memoir "On the number of primes less than a given magnitude", where he related the Riemann zeta function and the prime number theorem in a precise way (see the section dealing with the Riemann zeta function, section 3 and the rest of this section).

His ideas were extended, and Hadamard and de la Vallée-Poussin independently gave two proofs of the prime number theorem in 1896. The proof we expose in theorem 5.17, although it is not exactly the same, resembles de la Vallée-Poussin method, which gives an error term.



### 5.1.1 Tchebychev functions

**Definition 5.1** (The von Mangoldt function). We define  $\Lambda(n)$  to be

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is a prime power} \\ 0 & \text{otherwise} \end{cases}$$

The function is named after Hans Carl Friedrich von Mangoldt (1854-1925). He contributed to the solution of the PNT.

**Definition 5.2** (Tchebychev's functions). Define

$$\psi(x) = \sum_{p^k \leq x} \log p = \sum_{n \leq x} \Lambda(n)$$

$$\vartheta(x) = \sum_{p \leq x} \log p$$

The sums are over prime powers and over primes respectively.

$\vartheta$  can be seen as a smoothed version of  $\pi(x) = \sum_{p \leq x} 1$ , that assigns an increasing weight to every prime according to its size. Clearly, the inequality  $\psi(x) \geq \vartheta(x)$  holds because all terms in  $\vartheta$  are in  $\psi$  by definition.

Curiously enough,  $\psi(x)$  is the logarithm of the least common multiple of all the integers between 1 and  $x$  because the smallest exponent a prime  $p \leq x$  has to have in order to be divisible by any other number less than  $x$  is clearly  $\lfloor \log x / \log p \rfloor$ . That is to say,

$$\psi(x) = \sum_p \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \quad (5.3)$$

One can also relate  $\psi$  and  $\vartheta$ : to sum  $\log p$  for every power of  $p$  is equivalent to sum  $\log p$  for primes less than  $x$ , sum  $\log p$  for primes less than  $x^{1/2}$ , because its square becomes  $\leq x$ , and so on.

$$\psi(x) = \vartheta(x) + \vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \dots \quad (5.4)$$

By the Möbius inversion formula of last expression 5.4 one gets

$$\vartheta(x) = \sum_{i=1}^{\infty} \mu(i) \psi(x^{1/i})$$

We observe that in any case these sums are in fact finite.

### 5.1.2 Classical results on Tchebychev functions

We include some classical results involving the Tchebychev functions that use similar or the same techniques Tchebychev used. Although these results have mathematical and historical significance, they are not essential in our proof of the Prime Number Theorem.

On the other hand, they motivate the main idea of the proof: Rather than dealing directly with  $\pi(x)$  one can try to find the asymptotic behavior of  $\psi$  or  $\vartheta$ .

**Proposition 5.3.** *If all these following limits exist then*

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = \lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = \lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} \quad (5.5)$$

*Proof.*

- Denote the limits of the expression in 5.5 by  $\lambda_1, \lambda_2, \lambda_3$  respectively. It is clear that one has  $\lambda_1 \geq \lambda_3$  because  $\psi(x) \geq \vartheta(x)$ . By the expression 5.3 for  $\psi(x)$  one has

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) \leq \sum_{p \leq x} \frac{\log(x)}{\log(p)} \log(p) = \pi(x) \log(x)$$

Hence  $\lambda_1 \leq \lambda_2$ .

- Introduce the free variable  $0 < u < 1$  and bound  $\vartheta$  below

$$\vartheta(x) \geq \sum_{x^u < p \leq x} \log(p) = (\pi(x) - \pi(x^u)) \log(x^u) = u \log(x) (\pi(x) - \pi(x^u))$$

- Then

$$\lambda_3 = \lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} \geq u \lim_{x \rightarrow \infty} \left( \frac{\pi(x) \log x}{x} - \frac{\pi(x^u) \log x}{x} \right) = u \lambda_2$$

But  $u$  was arbitrary so  $\lambda_3 \geq \lambda_2$ . The inequalities imply  $\lambda_1 = \lambda_2 = \lambda_3$ .

□

*Observation 5.4.* It is clear that in 5.5 one could have supposed that  $\lambda_1 = \limsup_{x \rightarrow \infty} \frac{\vartheta(x)}{x}$  or  $\lambda_1 = \liminf_{x \rightarrow \infty} \frac{\vartheta(x)}{x}$  exist and the argument that proves these  $\lambda_i$  are equal is the same. Hence the version using inferior and superior limits also holds.

The following theorem has historical significance. Essentially, Tchebychev tried to prove the Prime Number Theorem by proving that

$$A \frac{x}{\log x} \leq \pi(x) \leq B \frac{x}{\log x}$$

for some constants  $A, B$ , and his proof involves careful bounds for central binomial numbers.

**Theorem 5.5.** [7]

$$\limsup_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} \leq 4 \log 2 \quad (5.6)$$

$$\liminf_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} \geq \log 2 \quad (5.7)$$

*Proof.*

1. Consider the binomial coefficient  $N = \frac{(2n)!}{n!^2}$ . It is clear that  $N < 2^{2n} < (2n+1)N$  by looking at the expansion of  $(1+1)^{2n}$  by Newton's theorem on binomials. Also,  $N$  is divisible by all the primes  $n < p \leq 2n$ . By combining this fact with the last stated inequalities one has

$$2n \log 2 > \log N \geq \sum_{n < p \leq 2n} \log p = \vartheta(2n) - \vartheta(n)$$

By summing the inequalities as  $n$  runs over powers of 2 we obtain a telescopic sum. If  $n$  runs over from  $n = 1$  up to  $2^{s+1} > x \geq 2^s$  one has

$$4x \log 2 \geq 2 \cdot 2^{s+1} \log 2 > \vartheta(2^{s+1}) \geq \vartheta(x) \Rightarrow \limsup_{x \rightarrow \infty} \frac{\vartheta(x)}{x} \leq 4 \log 2$$

By using last proposition 5.5 and observation 5.4 we deduce the first inequality 5.6.

2. We know by 5.3 that  $\psi(2n)$  is

$$e^{\psi(2n)} = \prod_{p \leq 2n} p^{\lfloor \frac{\log 2n}{\log p} \rfloor} = \text{lcm}\{1, 2, 3, \dots, 2n\}$$

Also,  $N$  is a product of primes between 1 and  $2n$ .

If we express  $k!$  in his prime factorization, the exponent of  $p$  in such factorization is

$$\nu_p(k) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{p^i} \right\rfloor$$

because between 1 and  $k$  there are  $\lfloor \frac{k}{p^i} \rfloor$  multiples of  $p^i$ , so when  $i$  terms are summed only the multiples of  $p^i$  contribute  $i$ . This sum is finite with  $\lfloor \frac{\log k}{\log p} \rfloor$  terms. Therefore

$$N = \frac{(2n)!}{n!^2} = \prod_{p \leq 2n} p^{\nu_p(2n) - 2\nu_p(n)}$$

The exponents can be expressed like this

$$e_p(N) = \nu_p(2n) - 2\nu_p(n) = \sum_{i=1}^{\lfloor \log 2n / \log p \rfloor} \left( \left\lfloor \frac{2n}{p^i} \right\rfloor - 2 \left\lfloor \frac{n}{p^i} \right\rfloor \right)$$

Each term is a value of the real-valued function  $\phi(y) = \lfloor 2y \rfloor - 2\lfloor y \rfloor$ , and  $\phi(y) \in \{0, 1\}$ . So  $e_p(N) \leq \lfloor \log 2n / \log p \rfloor$ . Hence  $N | e^{\psi(2n)}$  and in particular  $N \leq e^{\psi(2n)}$  so

$$\frac{2^{2n}}{2n+1} < N \leq e^{\psi(2n)} \Rightarrow \psi(2n) > 2n \log 2 - \log(2n+1)$$

Choose  $n = \lfloor x/2 \rfloor$  and

$$\psi(x) \geq \psi(2\lfloor x/2 \rfloor) > 2\lfloor x/2 \rfloor \log 2 - \log(2\lfloor x/2 \rfloor + 1)$$

Hence

$$\liminf_{x \rightarrow \infty} \frac{\psi(x)}{2\lfloor x/2 \rfloor} = \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} > \liminf_{x \rightarrow \infty} \left( \log 2 - \frac{\log(2\lfloor x/2 \rfloor + 1)}{2\lfloor x/2 \rfloor} \right) = \log 2$$

□

### 5.1.3 Weak Bertrand's postulate

If we take the prime number theorem (theorem 5.17) for granted, we can prove a result resembling Bertrand's postulate:

**Theorem 5.6.** *Given  $\epsilon > 0$ , there exists  $N_\epsilon$  such that for every  $x$  with  $x \geq N_\epsilon$  there is a prime number between  $x$  and  $(1 + \epsilon)x$ .*

*Proof.*

- If one proves that

$$A \frac{x}{\log x} \leq \pi(x) \leq B \frac{x}{\log x} \tag{5.8}$$

for sufficiently large  $x$  it is then easier to prove that there is a prime between  $n$  and  $Cn$  for sufficiently large  $n$ , where  $C > B/A$ .

- Because

$$\pi(x) \leq B \frac{x}{\log x}$$

and

$$\pi(Cx) \geq A \frac{Cx}{\log x + \log C}$$

so

$$\pi(Cx) - \pi(x) \geq AC \frac{x}{\log x + \log C} - B \frac{x}{\log x}$$

- But it is clear that for any  $C'$  with  $C > C' > B/A$  one has that

$$C \frac{x}{\log x + \log C} > C' \frac{x}{\log x}$$

for  $x$  sufficiently large, so we have

$$\pi(Cx) - \pi(x) > (AC' - B) \frac{x}{\log x} > 1$$

if  $x$  is sufficiently large, so there is a prime in that interval.

- Theorem 5.17 proves in particular that given  $\delta > 0$  one has expression 5.8 for  $x$  sufficiently large and for  $A < 1 + \delta$  and  $B > 1 - \delta$ , and we can choose  $\delta$  such that  $C = 1 + \epsilon > A/B$ .

□

## 5.2 Preliminaries and proof

### 5.2.1 Outline of the proof

Our proof of the prime number theorem is due de la Vallée Poussin (theorem 5.17). Essentially, we compare two contour integrals. One is along a rectangle and the other is a linear segment. The proof sketch is the following:

- Let  $x \geq 2$  and consider the following contour integral depending on  $x$ , where  $L$  is the linear segment  $[\sigma_0 - iT, \sigma_0 + iT]$

$$I(x) = \int_{\sigma_0 - iT}^{\sigma_0 + iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds = \int_L -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds$$

with  $\sigma_0 > 1$ . To calculate  $I(x)$  we proceed as follows:

- We use the fact that

$$\alpha(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = -\frac{\zeta'}{\zeta}(s)$$

is the Dirichlet series associated to the sequence  $a_n = \Lambda(n)$ , where  $\Lambda(n)$  is the von Mangoldt function. This expression is valid along  $L$  because we choose  $\sigma_0 > 1$  ( by lemma 5.9).

- Applying Perron's formula with error term (theorem 2.12) yields

$$I(x) = \sum_{n \leq x} a_n + \text{error term} = \sum_{n \leq x} \Lambda(n) + \text{error term} = \psi(x) + \text{error term}$$

- Now consider the following contour integral, where  $R$  is a rectangular contour we will choose later:

$$J(x) = \int_R -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds$$

To compute  $J(x)$  we proceed as follows:

- We can choose  $R$  so that  $R$  contains no zeros of the Riemann zeta function, and only contains the simple pole of  $\zeta$  at  $s = 1$ . Moreover, we can choose  $R$  so that the right side of the rectangle is the linear segment  $L = [\sigma_0 - iT, \sigma_0 + iT]$ , the same we used in the steps above. This is done using the zero-free regions (theorem 5.15 ).

- Applying Cauchy theorem and the argument principle to  $J(x)$  yields

$$J(x) = x$$

- The rest of the proof is showing that the main contribution to the integral above is along  $L$ . This will show that

$$J(x) = x = \int_L -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds + \text{error terms}$$

- Hence

$$I(x) = \psi(x) + \text{error term} = \int_L = J(x) - \text{error terms} = x - \text{error terms}$$

If we show that the error terms are small, this proves that

$$\psi(x) = x + \text{error terms}$$

- From this, we will show that  $\pi(x) = \text{li}(x) + O(\exp(-c\sqrt{\log x}))$ , using theorem 5.7 and the Riemann-Stieltjes integral.

### 5.2.2 The relationship between $\psi$ and $\pi$

The following theorem is essential in our proof of the prime number theorem (theorem 5.17). Loosely speaking, it shows that if we want to estimate  $\psi$  we can ignore prime powers, and gives the asymptotic behavior of  $\pi$  in terms of  $\psi$ :

**Theorem 5.7.**

$$\vartheta(x) = \psi(x) + O(x^{\frac{1}{2}}) \quad (5.9)$$

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \quad (5.10)$$

*Proof.*

- From 5.4 it follows that

$$\psi(x) - \vartheta(x) = \sum_{k=2}^{\infty} \vartheta(x^{\frac{1}{k}}) = \vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \dots$$

- By theorem 5.5, we know that  $\psi(x) \leq 4 \log(2)x$ . Hence  $\vartheta(x^{1/2}) \leq \psi(x^{1/2}) \leq 4 \log(2)x^{1/2}$ .
- The sum  $\sum_{k \geq 3}^{\infty} \vartheta(x^{1/k}) \leq 3\psi(x^{1/3})$  can be bounded, and again by theorem 5.5 we have  $\psi(x^{1/3}) \leq 4 \log(2)x^{1/3}$ .

- Therefore

$$\psi(x) - \vartheta(x) \leq 4 \log 2(x^{1/2} + 3x^{1/3}) \ll x^{1/2}$$

- To prove 5.10, we express  $\pi(x)$  as a Riemann-Stieltjes integral involving  $\vartheta$  and integrate by parts

$$\pi(x) = \int_{2^-}^x (\log u)^{-1} d\vartheta(u) = \frac{\vartheta(x)}{x} - \int_2^x \vartheta(u) d(\log u)^{-1} = \frac{\vartheta(x)}{x} + \int_2^x \frac{\vartheta(u)}{u(\log u)^2} du$$

Note that we integrate from  $2^-$  to capture the jump at  $x = 2$ , because  $\vartheta(u)$  is not continuous (the jumps are important only if we are dealing with Riemann-Stieltjes integrals of the form  $\int u dv$ , where  $v$  is not continuous).

- This last integral can be bounded using 5.9

$$\int_2^x \frac{\vartheta(u)}{u(\log u)^2} du \ll \int_2^x \frac{du}{(\log u)^2} \ll \frac{x}{(\log x)^2}$$

□

*Observation 5.8.* We said that theorem 5.5 is not essential for the proof of theorem 5.17. This is so because, although we used theorem 5.5 to make the proof of theorem 5.5 self-contained, the important step to prove (5.9) is showing that  $\psi(x) \approx x$ , and this information is known when we apply theorem 5.7 in step 8 of theorem 5.17. Therefore, theorem 5.5 is still not essential.

### 5.2.3 The relationship between $\zeta$ and $\psi$

The following lemma is essential in our proof of the Prime Number Theorem.

**Lemma 5.9.** [Relationship between  $-\frac{\zeta'}{\zeta}$  and  $\psi$ ] If  $\Re s > 1$  we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Moreover, these series converge absolutely for  $\Re s > 1$

*Proof.*

- We will use the Euler product for  $\zeta$  (theorem 3.2). This infinite product converges uniformly over compact subsets of  $\{\Re s > 1\}$ , so one may take logarithms and derive termwise:

$$\log \zeta(s) = - \sum_p \log \left( 1 - \frac{1}{p^s} \right) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{sn}}$$

- Therefore

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_p \sum_{n=1}^{\infty} \frac{\log(p)}{p^{sn}} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

- To prove the second claim, we argue as follows; given  $\epsilon > 0$ , one has by l'Hopital's rule that

$$\lim_{x \rightarrow \infty} x^{-\epsilon} \log x = \lim_{x \rightarrow \infty} \frac{1}{\epsilon} x^{-1+1-\epsilon} = 0$$

This implies that  $\Lambda(n) \leq \log n \ll n^\epsilon$ .

- Therefore for  $\Re s > 1$ ,

$$\left| - \frac{\zeta'}{\zeta}(s) \right| \ll \sum_{n=1}^{\infty} n^{-(\sigma-\epsilon)} = \zeta(\sigma - \epsilon) < \infty$$

if  $\epsilon$  is sufficiently small, and the second claim follows. □

#### 5.2.4 An expression for $\zeta$ near $\Re s = 1$

The following proposition is used to find the zero-free region for  $\zeta$ . It expresses the logarithmic derivative of  $\zeta$  in terms of a sum over a subset  $Z_{t_0}$

$$Z_{t_0} = \{\text{non-trivial zeros that lie inside a ball centered at } 3/2+it_0 \text{ with radius } 5/6\}$$

of the non-trivial zeros. In fact, Riemann hypothesis (conjecture 3.17) implies  $Z_{t_0} = \emptyset$  because  $\frac{3}{2} - \frac{5}{6} = \frac{2}{3} > \frac{1}{2}$ . Therefore under RH one can think  $\frac{\zeta'}{\zeta}(s) = O(\log \tau)$ .

**Proposition 5.10.** *Let  $\sigma, t \in \mathbb{R}$  be fixed such that  $|t_0| \geq \frac{7}{8}$  and  $\frac{5}{6} \leq \sigma \leq 2$ . Let  $Z_{t_0} = \{\rho \in \mathbb{C} : \zeta(\rho) = 0 \text{ and } |\rho - (3/2 + it)| \leq 5/6\}$ . Then*

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho \in Z_{t_0}} \frac{1}{s - \rho} + O(\log \tau)$$

where as usual  $\tau = |t| + 2$ .

*Proof.* We will use corollary 1.30:

1. Set  $f(z) = \zeta(s + (\frac{3}{2} + it_0)z)$  and let  $R = \frac{5}{6}$  and  $r = \frac{2}{3}$  so  $0 < r < R < 1$ . We chose these values for convenience although other values could be set for  $r, R$ .
2. Then clearly  $f(0) \neq 0$  by the Euler product for  $\zeta$  (theorem 3.2) or by theorem 3.9.
3. By corollary 3.14 we have that  $|f(z)| \ll \tau$  for  $|z| \leq 1$ . Hence

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho \in Z_t} \frac{1}{s - \rho} + O(\log \tau)$$



4. We observe that the implicit constant in  $O()$  depends only on our election of  $r, R$ , as we remarked in corollary 1.30.

□

*Observation 5.11.* We chose these values of  $r, R$  in order to maintain  $f$  entire for any election of  $t_0$  and  $\sigma$ , because  $\frac{7}{8} - \frac{5}{6} = \frac{1}{24} > 0$  so the little circle we used to apply Jensen's theorem does not contain the simple pole of  $\zeta$  at  $s = 1$ .

### 5.2.5 Zero-free regions for $\zeta$

In the proof of the prime number theorem ( theorem 5.17 ) we will integrate along several rectangular-shaped contours in a neighborhood of the line  $\Re s = 1$  involving  $\frac{\zeta'}{\zeta}$ , so it is essential to know where are the non-trivial zeros to avoid them. Under the Riemann hypothesis (see conjecture 3.17) this is not necessary and the theorems about zero-free regions are useless.

**Lemma 5.12.** For  $s$  with  $\sigma > 1$

$$C(\sigma, t) = \log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 0$$

*Proof.* We observe that for real  $c$

$$\log |\zeta(\sigma + ict)^k| = k \log \Re \zeta(\sigma + ict) = k \sum_{p,n} \frac{\cos(cnt \log n)}{np^{\sigma n}} \quad (5.11)$$

Using the trigonometric identity

$$3 + 4 \cos(\alpha) + \cos(2\alpha) = 2 \cos^2(\alpha) - 1 + 3 + 4 \cos(\alpha) = 2(\cos(\alpha) + 1)^2 \geq 0$$

and the fact that the series in expression 5.11 are absolutely convergent one has

$$C(\sigma, t) = \sum_{p,n} 2 \frac{(\cos(nt \log(n)) + 1)^2}{np^{\sigma n}} \geq 0$$

□

Lemma 5.12 is used to prove the following important property of  $\zeta$

**Theorem 5.13.**  $\zeta$  does not vanish for  $s$  with  $\sigma = 1$

*Proof.*

1. Suppose  $\zeta(1 + it) = 0$  for some  $t \neq 0$ . We say  $t > 0$  because  $\zeta$  has a simple pole at  $s = 1$ .

2. Fix  $t$  and observe that

$$|\zeta(\sigma + it)| < C|\sigma - 1|$$

because of the holomorphicity of  $\zeta$  at  $s = 1 + it$ .

3. The simple pole at  $s = 1$  lets us write

$$|\zeta(\sigma)| < K|\sigma - 1|^{-1}$$

4.  $\zeta$  has no other poles so it is holomorphic at  $s = 1 + 2it$  and in particular is bounded in a neighborhood of  $s$ ,

$$|\zeta(\sigma + 2it)| < M$$

5. By applying these last three inequalities we obtain contradiction because

$$C(\sigma, t) = \log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| < \log(K^3 C^4 M |\sigma - 1|) \rightarrow -\infty$$

as  $\sigma \rightarrow 1$ .

6. This clearly contradicts lemma 5.12, that  $C(\sigma, t) > 0$ .

□

**Corollary 5.14.** For  $\Re s = \sigma > 1$  one has

$$\Re\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it)\right) \geq 0$$

*Proof.* The argument is similar to that of lemma 5.12 because we can use the same trigonometric identity

$$\Re\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it)\right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma+1}} (3 + 4\cos(t \log n) + \cos(2t \log n)) \geq 0$$

□

**Theorem 5.15** (Zero-free region for  $\zeta$ ). *There exists  $c > 0$  such that*

$$\zeta(s) \neq 0 \text{ for } \sigma > 1 - \frac{c}{\log \tau}$$

where  $\tau = |t| + 2$ .

*Proof.*

1. By using theorem 3.10 one has that for  $\sigma > 0$

$$\left| \zeta(s) - \frac{s}{s-1} \right| \leq |s| \int_1^\infty u^{-\sigma-1} du = \frac{|s|}{\sigma}$$

Hence

$$\left| \left| \zeta(s) \right| - \frac{|s|}{|s-1|} \right| \leq \left| \zeta(s) - \frac{s}{s-1} \right| \leq \frac{|s|}{\sigma} \rightarrow |s| \left( \frac{1}{|s-1|} - \frac{1}{\sigma} \right) \leq |\zeta(s)| \leq |s| \left( \frac{1}{\sigma} + \frac{1}{|s-1|} \right)$$

If  $|s| \left( \frac{1}{|s-1|} - \frac{1}{\sigma} \right) > 0 \leftrightarrow |s-1| < \sigma \leftrightarrow (\sigma-1)^2 + t^2 < \sigma^2 \leftrightarrow \frac{1+t^2}{2} < \sigma$  then  $\zeta(s) \neq 0$ .

So there is a zero-free parabolic region for  $\zeta$  (see figure 5).

2. Suppose  $|t| \geq \frac{7}{8}$ . The rest of the argument relies on the expression in proposition 5.10, and essentially uses the same linear combination of values of  $\frac{\zeta'}{\zeta}$  we used in lemma 5.12:

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho \in Z_t} \frac{1}{s-\rho} + O(\log \tau) \quad (5.12)$$

(a) It is easy to see that one has  $\Re\left(\frac{1}{s-\rho'}\right) > 0$  for any non-trivial zero  $\rho'$  of  $\zeta$  whenever  $\Re s = \sigma > 1$ :

$$\Re \frac{1}{\sigma + it - \beta' - i\gamma'} = \frac{\sigma - \beta'}{(\sigma - \beta')^2 + (t - \gamma')^2} > 0$$

(b) Let  $\rho = \beta + i\gamma$  be a fixed zero of  $\zeta$  with  $\frac{5}{6} \leq \beta \leq 1$  and put  $s = 1 + \delta + i\gamma$ , where  $\delta$  is small and to be determined later and  $\gamma = \Im \rho$ .

(c) One can examine the behavior of  $\frac{\zeta'}{\zeta}$  for points  $s$  that are nearby to  $\Re s = 1$  with fixed imaginary part by using expression (5.12):

$$-\Re \frac{\zeta'}{\zeta}(1 + \delta + i\gamma) \leq -\frac{1}{1 + \delta - \beta} + c \log(|\gamma| + 2)$$

Observe we ignored the terms  $\frac{1}{s-\rho'}$  for  $\rho' \neq \rho$  because of point 2a. The constant  $c$  exists because of proposition 5.10.

(d) Similarly, by ignoring all the terms one gets

$$-\Re \frac{\zeta'}{\zeta}(1 + \delta + 2i\gamma) \leq c \log(|2\gamma| + 2) \leq c' \log(|\gamma| + 2)$$

Also by proposition 3.4

$$-\frac{\zeta'}{\zeta}(1 + \delta) = \frac{1}{\delta} + O(1)$$

(e) Hence by corollary 5.14

$$0 \leq \Re\left(-3\frac{\zeta'}{\zeta}(1+\delta) - \frac{\zeta'}{\zeta}(1+\delta+i\gamma) - \frac{\zeta'}{\zeta}(1+\delta+2i\gamma)\right) \leq \frac{3}{\delta} - \frac{4}{1+\delta-\beta} + C \log(|\gamma|+2)$$

and this holds for any  $\delta > 0$  because  $\Re(1+\delta) > 1$

(f) We choose

$$\delta = \frac{1}{2C \log(|\gamma|+2)}$$

so that

$$6C \log(|\gamma|+2) + C \log(|\gamma|+2) = 7C \log(|\gamma|+2) \geq \frac{4}{1+\delta-\beta}$$

Therefore

$$1 - \beta + \delta \geq \frac{8}{7}\delta \rightarrow 1 - \beta \geq \frac{1}{7}\delta > 0 \rightarrow \beta \leq 1 - \frac{1}{7}\delta$$

This bounds from above the real part of a zero of  $\zeta$  as wanted.

□

The constant in step 2c can be made explicit, but we do not need it for our purposes (the proof of the prime number theorem, theorem 5.17). It is clear that by putting a smaller constant we worsen the zero-free region, although it remains valid. That is, if  $c' < c$  then

$$\sigma > 1 - \frac{c'}{\log \tau} > 1 - \frac{c}{\log \tau}$$

So we can change  $c$  and make it sufficiently small and the result remains valid (see figure 5). Some points with  $|t| \leq 7/8$  are not covered by these regions. This is mainly because of our election of the radius in proposition 5.10. We could change the  $r$  and  $R$  in proposition 5.10 to cover an arbitrary small neighborhood of the line  $\Re s = 1$  but a neighborhood of the point  $s = 1$ . We say this because we need to be able to include rectangular shaped contours in the zero-free region.

There is no problem in fact because the first non-trivial zero of the Riemann  $\zeta$  function is approximately[3]

$$\rho \approx \frac{1}{2} \pm 14.134725 \dots$$

The zero-free regions of this kind are called of **de la Vallée Poussin** type[5], because Poussin (1866-1962) firstly proved a theorem similar to theorem 5.15.

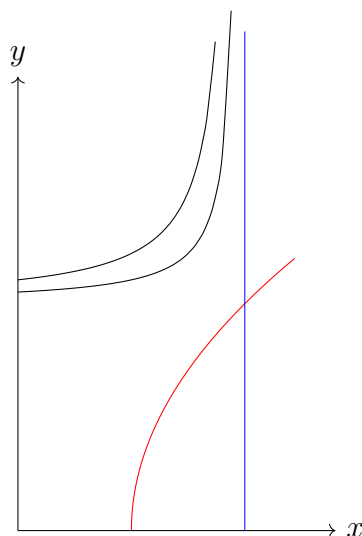


Figure 5: The zero-free regions in theorem 5.15. The blue line is  $\Re s = 1$ , and the figure shows the critical strip. We show two regions with different  $c$  (We have chosen an arbitrary constant to draw this figure)

### 5.2.6 Bounds for $\zeta'/\zeta$ near $\Re s = 1$

In order to prove the prime number theorem (theorem 5.17), we need bounds for  $\frac{\zeta'}{\zeta}$  near the line  $\Re s = 1$  in the zero-free region. These bounds will be used in step 5 of theorem 5.17 to bound the integrals along some sides of the rectangle-shaped contour we use (see figure 6).

**Theorem 5.16** (Bounds for  $\zeta'/\zeta$  near  $\Re s = 1$ ). *If  $|t| \geq 7/8$  and  $5/6 \leq \sigma \leq 2$  then*

$$\frac{\zeta'}{\zeta}(s) \ll \log \tau$$

where as usual  $\tau = |t| + 1$ .

*Proof.*

1. Recall that we impose  $|t| \geq 7/8$  somewhat arbitrarily because of our election of radius in proposition 5.10, as we noted in observation 5.11. This is because we must avoid the pole of  $\zeta$  at  $s = 1$ , where it is impossible to bound.
2. Put

$$\begin{cases} s = \sigma + it \\ s' = 1 + \frac{1}{\log \tau} + it \end{cases}$$

3. We split the proof in two cases:

- (a) Suppose that  $\sigma \geq 1 + \frac{1}{\log \tau}$ . Then the Dirichlet series for  $\zeta'/\zeta$  converge and we have by lemma 5.9

$$\left| \frac{\zeta'}{\zeta}(s) \right| = \left| \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} = -\frac{\zeta'}{\zeta}(\sigma)$$

Using that we know the residue of  $\zeta$  at  $s = 1$  we have that  $-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$  in a neighborhood of  $s = 1$  so

$$\left| \frac{\zeta'}{\zeta}(s) \right| \leq -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma-1} \leq \log \tau$$

by using the hypothesis 3a. In fact we do not need  $|t| \geq 7/8$  here.

- (b) Suppose now  $1 - \frac{c}{2}(\log \tau)^{-1} \leq \sigma \leq 1 + (\log \tau)^{-1}$ , where  $c$  is the constant in theorem 5.15.

We will try to bound  $\zeta'/\zeta$  inside the critical strip using the bounds from the outside ( $\Re s > 1$ ) and proposition 5.10.

- i. For  $|t| \geq 7/8$  and by proposition 5.10:

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s') = \sum_{\rho} \left( \frac{1}{s-\rho} - \frac{1}{s'-\rho} \right) + O(\log \tau)$$

where  $\rho \in Z_t$  is the set of zeros of  $\zeta$  defined in proposition 5.10, because  $\Im s = \Im s' = t$ .

- ii. Then

$$\left| \frac{1}{s-\rho} - \frac{1}{s'-\rho} \right| = \frac{|s'-s|}{|s-\rho||s'-\rho|} \leq \left(1 + \frac{c}{2}\right) \frac{1}{|s-\rho||s'-\rho| \log \tau}$$

- iii. But  $\frac{1}{|s-\rho||s'-\rho|} \leq K \frac{1}{|s'-\rho|^2}$ : by the triangle inequality

$$\left| \frac{s'-\rho}{s-\rho} \right| \leq 1 + \frac{|s'-s|}{|s-\rho|} \leq 1 + \frac{(1+c/2) \frac{1}{\log \tau}}{(c/2) \frac{1}{\log \tau}} = K = O(1)$$

because  $|s'-s| \leq (1+c/2)/\log \tau$  and  $|s-\rho| \geq c/(2\log \tau)$  (we can say this because we halved the constant  $c$ ).

- iv. Hence

$$\left| \frac{1}{s-\rho} - \frac{1}{s'-\rho} \right| \ll \frac{1}{|s'-\rho|^2 \log \tau} \ll \Re \frac{1}{s'-\rho}$$

because if we put  $s'-\rho = \alpha + i\beta$  then  $(s'-\rho)^{-1} = (\alpha - i\beta)(\alpha^2 + \beta^2)^{-1}$  and

$$\frac{1}{|s'-\rho| \log \tau} = \frac{1}{\alpha^2 + \beta^2} \frac{1}{\log \tau} \leq \frac{\alpha}{\alpha^2 + \beta^2} = \Re \frac{1}{s'-\rho}$$

since  $\alpha \geq \frac{1}{\log \tau}$ .

v. Therefore

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s') + \sum_{\rho \in Z_t} \left( \frac{1}{s - \rho} - \frac{1}{s' - \rho} \right) + O(\log \tau) \ll \log \tau$$

by steps 3(b)i, 3a and 3(b)iv.

□

### 5.2.7 Proof of the prime number theorem

**Theorem 5.17** (Prime number theorem). *There exists a constant  $c > 0$  such that for  $x \geq 2$*

$$\pi(x) = li(x) + O\left(x \exp(-c\sqrt{\log x})\right)$$

where

$$li(x) = \int_2^x \frac{du}{\log u}$$

is the logarithmic integral function.

*Proof.*

1. We recall that by lemma 5.9 one has for  $\Re s = \sigma > 1$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

2. We start integrating along a linear segment, as we pointed out in section 5.2.1. Observe that for  $\Re s > 1$  the function  $-\zeta'/\zeta$  can be given as the value of an absolutely converging Dirichlet series with coefficients  $a_n = \Lambda(n)$ . Then, using the same notation we used in the section concerning Dirichlet series (section 2.1) we put

$$A(x) = \sum_{n \leq x} a_n = \sum_{n \leq x} \Lambda(n) = \psi(x)$$

and

$$\alpha(s) = -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

So by Perron's formula with its error term (theorem 2.12) we have that for any  $\sigma_0 > 1$  (the Dirichlet series for  $\alpha(s)$  converges absolutely for  $\sigma > 1$ ) and  $T \geq 1$

$$\begin{aligned} A(x) = \psi(x) &= \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds + R(x, T) \\ &= \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + R(x, T) \end{aligned}$$

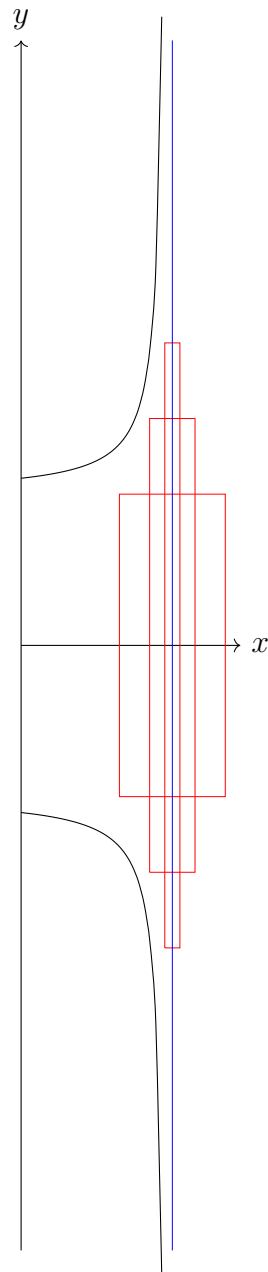


Figure 6: The rectangular contours used in theorem 5.17, step 5. The contours are inside the zero-free region and include only the simple pole at  $s = 1$



where

$$R(x, T) \ll \sum_{\frac{x}{2} < n < 2x} \Lambda(n) \min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0}} = A + B$$

We split this error term in  $A$  and  $B$ , where  $B = (4x)^{\sigma_0}/T \sum_{n=1}^{\infty} \Lambda(n)n^{-\sigma_0}$ .

3. We bound  $R(x, T)$ :

(a) For the terms in  $A$  (the first sum), either  $|x-n| < 1$  or  $|x-n| \geq 1$ . Set  $m = \min(1, x/(T|x-n|))$ :

i. In the first case  $m = 1$ .

ii. Otherwise  $m = x/(T|x-n|)$ . We see that  $|x-n|$  runs over  $x$  and  $-x/2$  (by evaluating at the limits  $x/2$  and  $2x$ ) and the sum of  $1/|x-n|$  can be bounded by an harmonic number  $H_x = \sum_{k \leq x} \frac{1}{k}$ , which in turn  $H_x \ll \log x$ .

(b) Also  $\Lambda(n) \leq \log n \leq \log(2x) \ll \log x$ .

(c) Hence

$$A \ll \log x \left(1 + \frac{x}{T} \sum_{k \leq x} \frac{1}{k}\right) \ll \log x + \frac{x}{T} (\log x)^2$$

(d)  $B$  is just  $-\frac{\zeta'}{\zeta}(\sigma_0) < \infty$ , which is a constant multiplied  $\frac{(4x)^{\sigma_0}}{T}$ .

(e) Given  $x$  we choose  $\sigma_0 = 1 + \frac{1}{\log x}$  and suppose  $2 \leq T \leq x$  because they are arbitrary. Then

$$(4x)^{\sigma_0} = e^{(\log 4 + \log x)(1 + \frac{1}{\log x})} = 4xe^{\frac{\log 4}{\log x} + 1} \leq Cx \ll x$$

because  $x \geq 2$ .

Hence

$$R \ll A + B \ll \log x + \frac{x}{T} (\log x)^2 + x \ll \frac{x}{T} (\log x)^2$$

4. Now we integrate along a rectangle, as we pointed out in 5.2.1. The argument principle (section 1.3) yields

$$\frac{1}{2\pi i} \int_C -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds = -(-1) \frac{x^s}{s} \Big|_{s=1} = x$$

where  $C$  is a contour containing only the pole at  $s = 1$  we will choose now.

5. Put  $\sigma_1 = 1 - \frac{c}{\log T}$  where  $c > 0$  and let  $C$  be the rectangular contour with vertices at  $\sigma_0 - iT, \sigma_0 + iT, \sigma_1 + iT, \sigma_1 - iT$  (see figure 6). By theorem 5.15 there are no zeros inside  $C$ , and there cannot be more poles because of theorem 3.15.

We will show that the main contribution is along the segment  $[\sigma_0 - iT, \sigma_0 + iT]$ , the same we used to apply Perron's formula in step 2. This is because:

- (a) The integrals along the horizontal segments have the same behavior, because our contour is symmetrical with respect to complex conjugation and Schwarz reflection principle (section 1.4) so  $\zeta(\bar{s}) = \overline{\zeta(s)}$ . Then by theorem 5.16 and the definition of  $\sigma_0$  and  $\sigma_1$  (steps 3e and 5) we have

$$\int_{\sigma_0+iT}^{\sigma_1+iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds \stackrel{\text{(theorem 5.16)}}{\ll} \log T \int_{\sigma_0+iT}^{\sigma_1+iT} \frac{x^s}{s} ds$$

$$\stackrel{(\sigma_0 > \sigma_1)}{\leq} x^{\sigma_0} \log T \int_{\sigma_0+iT}^{\sigma_1+iT} \frac{ds}{s} \stackrel{(|\sigma_0+iT| \geq T)}{\ll} \frac{\log T}{T} x^{\sigma_0} (\sigma_0 - \sigma_1) \ll \frac{x^{\sigma_0}}{T} \ll \frac{x}{T}$$

because  $\sigma_0 - \sigma_1 = O((\log T)^{-1})$  and  $x^{\sigma_0} \ll x$ . Therefore, the integral along the horizontal segments is  $\ll x/T$ .

- (b) We treat now the vertical segment  $[\sigma_1 + iT, \sigma_1 - iT]$ , by bounding the integrand for large imaginary part (theorem 5.16) and in a neighborhood of the pole at  $s = 1$ , where  $\zeta'/\zeta(s) \ll (s-1)^{-1}$ :

$$\int_{\sigma_1+iT}^{\sigma_1-iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds \ll x^{\sigma_1} \log T \int_{-T}^T \frac{dt}{|\sigma_1 + it|} + x^{\sigma_1} \int_{-1}^1 \frac{dt}{|\sigma_1 + it - 1|}$$

$$\ll x^{\sigma_1} (\log T)^2 + x^{\sigma_1} \int_{-1}^1 \frac{dt}{|\sigma_1 - 1|}$$

$$= x^{\sigma_1} (\log T)^2 + \frac{2x^{\sigma_1}}{|\sigma_1 - 1|} \ll x^{\sigma_1} (\log T)^2 + \frac{2x^{\sigma_1}}{c} \log T \ll x^{\sigma_1} (\log T)^2$$

because  $\sigma_1 - 1 = c/\log T$ . Therefore, the integral along the left vertical segment of  $C$  is  $\ll x^{\sigma_1} \log T$ .

6. Then by step 5

$$x = \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds + O\left(\frac{x}{T} + x^{\sigma_1} (\log T)^2\right)$$

and by steps 2 and 3

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds + O\left(\frac{x}{T} (\log x)^2\right)$$

Hence

$$\psi(x) = x + O\left(\frac{x}{T} (\log x)^2 + \frac{x}{T} + x^{\sigma_1} (\log T)^2\right)$$

7. We can simplify the error term because:

$$E(x, T) = O\left(\frac{x}{T} (\log x)^2 + \frac{x}{T} + x^{\sigma_1} (\log T)^2\right) = O\left(x (\log x)^2 \left(\frac{1}{T} + x^{-c/\log T}\right)\right)$$

- One can impose that  $1/T = x^{-c/\log T}$  to simplify the bound above for  $E(x, T)$ . This results in  $(\log T)^2 = c \log x$
- Then have two possibilities  $T = \exp(\pm\sqrt{c \log x})$ , but we choose the sign that makes  $T \leq x$  so  $T = \exp(-\sqrt{c \log x})$ .
- If we suppose that  $c \in (0, 1)$  (the constant in step 2c of theorem 5.15) then

$$E(x, T) = O\left(x(\log x)^2 \exp(-\sqrt{c \log x})\right) = O\left(x \exp(-c\sqrt{\log x})\right)$$

because  $\sqrt{c} - c > 0$ .

Finally we apply this to  $\vartheta$  and  $\pi$ .

8. By step 7 and theorem 5.7 we have that

$$\vartheta(x) = x + O\left(x \exp(-c\sqrt{\log x})\right) + O(x^{1/2}) = x + O\left(x \exp(-c\sqrt{\log x})\right)$$

9. By applying the same expression for  $\pi(x)$  we used in theorem 5.7 and integrating by parts

$$\begin{aligned} \pi(x) &= \int_{2^-}^x \frac{d\vartheta}{\log u} = \int_{2^-}^x \frac{du}{\log u} + \int_{2^-}^x \frac{d(\vartheta(u) - u)}{\log u} = \text{li}(x) + \frac{\vartheta(u) - u}{\log u} \Big|_{2^-}^x \\ &\quad + \int_{2^-}^x \frac{\vartheta(u) - u}{u(\log u)^2} du = \text{li}(x) + A(x) + B(x) \end{aligned}$$

10.  $A(x)$  and  $B(x)$  can be bounded by  $O\left(x \exp(-c\sqrt{\log x})\right)$

$$\begin{aligned} A(x) &= \frac{\vartheta(u) - u}{\log u} \Big|_{2^-}^x = \frac{\vartheta(x) - x}{\log x} + O(1) \stackrel{\text{step 8}}{=} O\left(\frac{x}{\log x} \exp(-c\sqrt{\log x})\right) \\ &= O\left(x \exp(-c\sqrt{\log x})\right) \end{aligned}$$

and

$$\begin{aligned} B(x) &= \int_{2^-}^x \frac{\vartheta(u) - u}{u(\log u)^2} du = \int_{2^-}^x \frac{\vartheta(u)}{u(\log u)^2} du - \int_{2^-}^x \frac{du}{(\log u)^2} \\ &= O\left(x \exp(-c\sqrt{\log x})\right) + O(1) = O\left(x \exp(-c\sqrt{\log x})\right) \end{aligned}$$

11. Hence by steps 9 and 10

$$\pi(x) = \text{li}(x) + O(\exp(-c\sqrt{\log x}))$$

The proof is complete. □

### 5.3 Von Mangoldt's explicit formula

We expose some of the arguments Riemann gave when he tried to prove the prime number theorem. Von Mangoldt (1854-1925) proved the following expression:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log(1 - x^{-2})$$

where the sum runs over the non-trivial zeros in a sense we will specify later.

These expressions (involving sums over non-trivial zeros) are usually called **explicit formulae** and a variation of the expression above was used by Riemann in his argument. It will become clear that proving the explicit formula would prove the prime number theorem.

Perron's formula (theorem 2.9) yields for  $\sigma_0 > 1$

$$\psi(x) = \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds$$

The explicit formula is derived from the expression above, and proves in some sense that given a sufficiently large number of non-trivial zeros of  $\zeta$  one can calculate  $\psi(x)$ .

This is interesting, because  $\psi$  was defined to be

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

That is, it seems that if we want to compute  $\psi$  we should find all prime powers less than  $x$ . There are fast algorithms to find prime numbers (**primality tests**) but it is a computationally difficult problem.

One can relate  $\psi(x)$  with  $\pi(x)$  directly, as Riemann did. Hence, by knowing the zeros of  $\zeta$  one can calculate  $\pi(x)$ .

We give a proof sketch of the explicit formula:

- Let

$$f(s) = -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s}$$

- Consider a rectangle-shaped contour  $C_K$  with vertices at  $-K - iK, \sigma_0 - iK, \sigma_0 + iK, -K + iK$ . Then by the calculus of residues one has that

$$\int_{C_K} f(s) ds = \sum_{\rho=0 \text{ or zero or pole of } \zeta \text{ inside } C_K} \text{Res}_{s=\rho}(f)$$

- Assuming that  $\frac{\zeta'}{\zeta}$  tends to zero fast enough one can argue that  $f$  tends to zero as  $s \rightarrow -\infty$  fast enough, so that the only contribution to the integral  $\int_{C_K}$  is  $\int_{\sigma_0 - iK}^{\sigma_0 + iK}$  when  $K \rightarrow \infty$ .

- One can calculate the residues at these points:
  - Recall that the trivial zeros are the negative even integers (see theorem 3.15). The trivial zeros are simple (they have multiplicity 1) so by arguments similar to those given in the section dedicated to the argument principle (section 1.3) one can show that when  $\rho = -2n$  is a trivial zero then

$$\operatorname{Res}_{s=\rho}(f) = -\left(1 \cdot \frac{x^{-2n}}{-2n}\right) = \frac{x^{-2n}}{2n}$$

Observe that we do not need to know the residue of  $\zeta$  at  $\rho$ .

- By theorem 3.15 again,  $\zeta$  has only one pole. That is,  $\zeta$  has multiplicity  $-1$  at  $s = 1$ , so  $\rho = 1$  implies

$$\operatorname{Res}_{s=\rho}(f) = -\left(-1 \cdot \frac{x^1}{1}\right) = x$$

- When  $\rho$  is a non-trivial zero, we do not know whether  $\rho$  is a simple zero or not. Suppose that the multiplicity is  $m_\rho$ . Then

$$\operatorname{Res}_{s=\rho}(f) = -\left(m_\rho \cdot \frac{x^\rho}{\rho}\right) = -m_\rho \frac{x^\rho}{\rho}$$

- There is still one case, when  $\rho = 0$ . 0 is not a pole nor a zero of  $\zeta$  but it is a simple pole of the integrand  $f$ . Then

$$\operatorname{Res}_{s=0}(f) = -\frac{\zeta'}{\zeta}(0)$$

In fact, one can show that  $\frac{\zeta'}{\zeta} = \log(2\pi)$ .

- By the above we can say that

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n}$$

where  $\rho$  runs over the non-trivial zeros counted with multiplicity. That is, given  $\rho$  the corresponding term appears as many times as  $m_\rho$ .

One can simplify the formula above by saying that  $\sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} = -\frac{1}{2} \log(1 - x^{-2})$ . This is valid because  $x \geq 2$ , and we can use the Taylor series of the logarithm at  $x = 0$ :

$$-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Then for  $x \geq 2$  one has

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log(1 - x^{-2}) \quad (5.13)$$

- The contour is important in a rigorous proof, because the sum over the non-trivial zeros is not absolutely convergent. That is, the sum is written in the following sense

$$\sum_{\rho} \frac{x^{\rho}}{\rho} = \lim_{K \rightarrow \infty} \sum_{\rho \in Z_K} \frac{x^{\rho}}{\rho}$$

where  $Z_K = \{ \text{non-trivial zeros with } |\Im \rho| \leq K \}$ .

- Instead of proving the prime number theorem directly, we could have proven expression 5.13 because it can be shown that the fact that the real parts  $\Re \rho$  of the non-trivial zeros are between 0 and 1 (by theorem 3.15) implies that asymptotically  $\psi(x) \approx x$ .

Curiously enough, although it may seem hard or even impossible to deal with such sums involving non-trivial zeros it has been shown that, for example

$$\sum_{\rho} \frac{1}{\rho} = \gamma - \frac{1}{2} \log \pi - \log 2 + 1$$

This fact was known by Riemann.

## 5.4 An heuristic argument for PNT

We can find in [9] a well-known heuristic argument concerning the prime number theorem that uses probability. The argument is not heuristic because of the use of probability, but because the assumptions are made without further justification.

The probability of some number being divisible by  $p$  is  $\frac{1}{p}$ . Denote by  $f(n)$  the number of prime numbers smaller than  $n$ . Therefore we can expect that

$$f(n) \approx \prod_{p < n} \left(1 - \frac{1}{p}\right) \quad (5.14)$$

This approximation is in fact incorrect because it was proved in 1874 by Mertens that the product in expression 5.14 behaves like  $\frac{e^{-\gamma}}{\log(n)}$ . Surprisingly, although the argument is now invalid we get a correct answer, because if we assume expression 5.14 is true then we can relate  $f(n+1)$  and  $f(n)$ . If  $n$  is prime then

$$f(n+1) = \left(1 - \frac{1}{n}\right) f(n) \quad (5.15)$$

and if  $n$  is not prime  $f(n+1) = f(n)$ . Combining these cases we can expect that

$$f(n+1) \approx \left(1 - \frac{f(n)}{n}\right) f(n) \quad (5.16)$$

Suppose the approximation  $f(x+1) - f(x) \approx f'(x)$  is valid. Then  $f$  is an approximate solution of

$$f'(x) = -\frac{f(x)}{x^2}$$

Solving this differential equation gives  $f = \frac{1}{\log x}$ . This heuristic argument gives the right answer, but it is not satisfying because it is invalid.

## 6 Hardy's theorem

Hardy's theorem is closely related to the Riemann Hypothesis. The statement is that there are infinitely many zeros lying on the critical line ( $\Re s = 1/2$ ). This is not a proof of the Riemann hypothesis, because there could be non-trivial zeros not lying in this line. The proof uses carefully some properties of the  $\Gamma$  function like Stirling's formula, that we will take for granted to be true not because it is an immediate or easy result (which it is not) but because its proofs are not specially illuminating. A proof of this fact that uses Euler-Maclaurin formula, a powerful method in analysis that can be found in [9].

The best result in this direction was given by N. Levinson in [8], that a fraction of 0.4077 of non-trivial zeros lie in the critical line. The methods used in the proof are technical and beyond the scope of this thesis.

The results proven in this section are interesting because the first lemma is just an example of calculation of the moments of the Riemann  $\zeta$  function. That is, estimating or computing the value of integrals of the form

$$\int_{-\infty}^{\infty} |\zeta(1/2 + it)|^k dt$$

These results try to relate Riemann hypothesis with random matrix theory. It has been checked for a large number of non-trivial zeros that these behave as the eigenvalues of a random unitary matrix.

This is closely related with the **Hilbert-Pólya conjecture** (see conjecture 3.17, the section dealing with the Riemann hypothesis): that one can find an operator whose eigenvalues are the imaginary parts of the non-trivial zeros of the Riemann  $\zeta$  function.

### 6.1 Some preliminary lemmas

**Lemma 6.1.** *For all  $\epsilon > 0$  and  $T \geq 2$  we have*

$$\int_1^T \zeta(1/2 + it) dt = T + O(T^{1/2+\epsilon})$$

*Proof:*

We choose  $C_T$  to be the rectangular contour with vertices  $\frac{1}{2} + i$ ,  $2 + i$ ,  $2 + iT$ ,  $\frac{1}{2} + iT$ .  $\zeta$  has no singularities inside  $C_T$  so  $\int_{C_T} \zeta(s) ds = 0$  and

$$\int_{C_T} = \int_{1/2+i}^{2+i} + \int_{2+i}^{2+iT} + \int_{2+iT}^{1/2+iT} + \int_{1/2+iT}^{1/2+i} = I_1 + I_2 + I_3 + I_4$$

1.  $I_1$  is a constant so  $I_1 = O(1)$

2. By using corollary 3.14 we have for all  $\epsilon > 0$

$$I_3 = \int_{2+iT}^{1/2+iT} \zeta(s) ds \ll \int_{1/2}^2 (\log T)(1 + T^{1-\sigma}) d\sigma \ll T^{1/2+\epsilon}$$

This is valid because  $T/\tau \rightarrow 1$  and  $T \geq 2$  and the interval of integration is compact.

3. Hence  $I_2 = I_4 + O(T^{1/2+\epsilon})$  but  $I_4$  can be computed because of absolute convergence

$$I_4 = \int_1^T \sum_{n=1}^{\infty} \frac{1}{n^{2+it}} dt = T + \sum_{n=2}^{\infty} n^{-2} \int_1^T n^{-it} dt = T + \sum_{n=2}^{\infty} \frac{n^{-i} - n^{-iT}}{in^2 \log n} = T + O(1)$$

4. Therefore  $I_2 = \int_{1/2+i}^{1/2+iT} \zeta(s) ds = \int_1^T \zeta(1/2 + it) dt = T + O(T^{1/2+\epsilon})$   $\square$

We state without proof another result related to the  $\Gamma$  function:

**Theorem 6.2** (Mellin's theorem). *Let  $w \in \mathbb{C}$  be such that  $\Re w > 0$  and fix  $\sigma_0 > 0$ . Then*

$$\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s/2) w^{-s/2} ds = 2e^{-w}$$

*Proof sketch:*

1. By using Stirling's formula ( theorem 1.26 ) one can prove that it is valid to apply Cauchy's theorem to the family of rectangular contours with vertices at  $-K - iK, \sigma_0 - iK, \sigma_0 + iK, K + iK, -K + iK$  and letting  $K \rightarrow \infty$ , so that the integrals along the sides with endpoints at  $-K - iK, \sigma_0 - iK, \sigma_0 + iK, K + iK$  and  $K + iK, -K + iK$  tend to zero as  $K \rightarrow \infty$ .
2. Then by calculus of residues and changing variables  $s/2 \mapsto u$  the integrand has a pole at the negative integers  $u = -n$  with residue  $\frac{(-1)^n}{n!}$  ( see section 1.6, definition 1.5 ) so

$$\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s/2) w^{-s/2} \frac{ds}{2} = \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(u) w^{-u} du = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} w^n = e^{-w}$$

$\square$

**Lemma 6.3.** *Let  $z \in \mathbb{C}$  be such that  $\Re z > 0$  and  $\sigma_0 > 1$ . Then*

$$\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta(s) \Gamma(s/2) (\pi z)^{-s/2} ds = 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z}$$

*Proof:*



We will use Mellin's theorem (theorem 6.2) and Weierstrass M-test (theorem 1.8):

1. By theorem 6.2 we have that for  $w$  such that  $\Re w > 0$  and fixed  $\sigma_0 > 1 > 0$

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s/2) w^{-s/2} ds = 2e^{-w}$$

2. Putting  $w = \pi n^2 z$  and summing over  $n$  results in the following expression:

$$\sum_{n=1}^{\infty} 2e^{-\pi n^2 z} = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s/2) (\pi z)^{-s/2} n^{-s} ds$$

- (a) Fix  $\delta > 0$  and let  $z$  be such that  $\Re z > \delta > 0$ .  
 (b) Fix  $s \in [\sigma_0 - i\infty, \sigma_0 + i\infty]$  and let

$$f_n(z) = \Gamma(s/2) (\pi z)^{-s/2} n^{-s}$$

- (c) Then

$$|f_n(z)| \leq |\Gamma(s/2)| \pi^{-\sigma_0/2} |z^{-s/2}| n^{-\sigma_0}$$

- (d) Put  $z = |z|e^{i\theta}$  with  $|\theta| < \pi/2$  ( because  $\Re z > \delta > 0$  ) and  $s = \sigma_0 + i\beta$ .  
 Then

$$|z^{-s/2}| = \left| |z|^{-s/2} e^{i\theta s/2} \right| = |z|^{-\sigma_0/2} e^{\theta\beta}$$

- (e) From corollary 1.27 one has that

$$|\Gamma(s/2)| = \sqrt{2\pi} e^{-\pi|\beta|/4} \left| \frac{\beta}{2} \right|^{\sigma_0 - 1/2} (1 + R(y))$$

where  $|R| \rightarrow 0$  as  $|y| \rightarrow \infty$ .

- (f) Therefore there exists  $C_\delta$  such that

$$|f_n(z)| \leq \sqrt{2\pi} \left( (\pi|z|)^{-\sigma_0/2} e^{\theta\beta/2 - \pi|\beta|/4} \left| \frac{\beta}{2} \right|^{\sigma_0 - 1/2} \right) n^{-\sigma_0} \leq C_\delta n^{-\sigma_0}$$

for any  $n \geq 1$  and  $z$  with  $\Re z > \delta > 0$ .

- (g) Then  $f_n \rightarrow 0$  uniformly for  $\Re z > \delta > 0$ . Then  $\sum_{n=1}^{\infty} f_n(z)$  converges uniformly by Weierstrass M-test ( theorem 1.8 ). So one can exchange summation with integration for any fixed  $\delta$ .

3. Then the expression in step 2 becomes

$$2 \sum_{n=1}^{\infty} e^{-\pi n^2 z} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \sum_{n=1}^{\infty} \Gamma(s/2) (\pi z)^{-s/2} n^{-s} ds = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \zeta(s) \Gamma(s/2) (\pi z)^{-s/2} ds$$

because the Dirichlet series for  $\zeta$  converge for  $\sigma_0 > 1$ . □

## 6.2 Proof

**Definition 6.4** (Hardy's  $Z$  function). Define  $Z(t)$  to be the following real-valued function

$$Z(t) = \zeta(1/2 + it) \frac{\Gamma(1/4 + it/2) \pi^{-1/4 - it/2}}{|\Gamma(1/4 + it/2) \pi^{-1/4 - it/2}|}$$

Because of the real-valuedness of  $Z(t)$  one can say that given  $T$ ,  $Z(t)$  vanishes for some  $t \in (T, 2T)$  if

$$\left| \int_T^{2T} Z(t) dt \right| < \int_T^{2T} |Z(t)| dt$$

That is,  $Z(t)$  is not of constant sign. One could prove the expression above holds for large  $T$ , so there are infinitely many zeros, although a proof of this fact is beyond the scope of this thesis.

However, it is possible to establish a weaker result: a weighted variant of the expression above.

We now prove Hardy's theorem, although one may skip some of the details.

**Theorem 6.5** (Hardy's theorem). *There are infinitely many zeros of  $\zeta$  on the critical line.*

*Proof.*

1. We may see that  $Z(t)$  is indeed real-valued:

(a) We can write the functional equation for  $\zeta$  evaluated at  $s = 1/2 + it$  (corollary 3.3):

$$\begin{aligned} \xi(1/2 + it) &= \xi(1/2 - it) = \pi^{-1/4 - it/2} \Gamma(1/4 + it/2) \zeta(1/2 + it) \\ &= \pi^{-1/4 + it/2} \Gamma(1/4 - it/2) \zeta(1/2 - it) \end{aligned}$$

(b) By using step 1a and Schwarz reflection principle (section 1.4) one can check that  $Z(t)$  is indeed real-valued by taking the complex conjugate:

$$\begin{aligned} \overline{Z(t)} &= \overline{\zeta(1/2 + it) \frac{\Gamma(1/4 + it/2) \pi^{-1/4 - it/2}}{|\Gamma(1/4 + it/2) \pi^{-1/4 - it/2}|}} \stackrel{1.4}{=} \zeta(1/2 - it) \frac{\Gamma(1/4 - it/2) \pi^{-1/4 + it/2}}{|\Gamma(1/4 + it/2) \pi^{-1/4 - it/2}|} \\ &\stackrel{1.4}{=} \zeta(1/2 - it) \frac{\Gamma(1/4 - it/2) \pi^{-1/4 + it/2}}{|\Gamma(1/4 - it/2) \pi^{-1/4 + it/2}|} \stackrel{\text{step 1a}}{=} Z(t) \end{aligned}$$

2. Lemma 6.3 states that for  $z$  with  $\Re z > 0$  and  $\sigma_0 > 1$  one has

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} f(s) ds = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \zeta(s) \Gamma(s/2) (\pi z)^{-s/2} ds = 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z}$$

We are interested in moving the contour  $[\sigma_0 - i\infty, \sigma_0 + i\infty]$  to a new contour  $[1/2 - i\infty, 1/2 + i\infty]$ , the critical line. This can be done because:

- (a) Let  $L_K$  be the contour with endpoints at  $\sigma_0 - iK, \sigma_0 + iK$ .  
 (b) By definition

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \zeta(s) \Gamma(s/2) (\pi z)^{-s/2} ds = \lim_{K \rightarrow \infty} \frac{1}{2\pi i} \int_{L_K} \zeta(s) \Gamma(s/2) (\pi z)^{-s/2} ds$$

- (c) Consider the rectangular contour  $P_K$  with vertices at  $1/2 - iK, \sigma_0 - iK, \sigma_0 + iK, 1/2 + iK$ . Our integrand, namely  $f(s) = \zeta(s) \Gamma(s/2) (\pi z)^{-s/2}$  has a unique simple pole at  $s = 1$  with residue

$$\text{Res}_{s=1} f(s) = \lim_{s \rightarrow 1} (s-1) f(s) = 1 \cdot \Gamma(1/2) (\pi z)^{-1/2} = z^{-1/2}$$

because  $\Gamma(1/2) = \sqrt{\pi}$  and by proposition 3.4 ( $\text{Res}_{s=1} \zeta(s) = 1$ ) and the properties of  $\zeta$  and  $\Gamma$  (theorem 3.15 and proposition 1.23).

- (d) Consider the contour  $C_K$  with endpoints at  $1/2 - iK, 1/2 + iK$  (a linear segment). Then

$$\int_{C_K \cup P_K} = \int_{W_K} = \int_{1/2 - iK}^{\sigma_0 - iK} + \int_{\sigma_0 - iK}^{\sigma_0 + iK} + \int_{\sigma_0 + iK}^{1/2 + iK} = (I) + (II) + (III)$$

because the contributions along  $C_K$  cancel.

- (e) The fact that  $(I), (III) \rightarrow 0$  as  $K \rightarrow \infty$  follows from choosing  $x = 1$  in corollary 3.12:

$$\zeta(s) = \sum_{n \leq 1} n^{-s} + \frac{1^{1-s}}{s-1} + \frac{\{1\}}{1^s} - s \int_1^{\infty} \{u\} u^{-(s+1)} du \ll |s|$$

- i. We can observe further, that  $\zeta(s) \ll |s| \ll |\Im s| \leq \tau$  uniformly because  $1/2 \leq \Re s \leq \sigma_0$ , where  $\tau = |\Im s| + 1$ .  
 ii. By using corollary 1.27, that

$$|\Gamma(x + iy)| = \sqrt{2\pi} e^{-\pi|y|/2} |y|^{x-1/2} \left(1 + r(x, y)\right)$$

we see that

$$|(I)|, |(III)| \rightarrow 0$$

as  $K \rightarrow \infty$  because of the exponential decay of  $|\Gamma|$  and step 2(e)i, and the fact that the measure of the contours  $1/2 \pm iK, \sigma_0 \pm iK$  is finite.

3. By the above, we can say that  $\int_{L_K} f = \int_{C_K \cup P_K} f$  so

$$\begin{aligned} 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z} - z^{-1/2} &= \lim_{K \rightarrow \infty} \left( \int_{L_K} f(s) ds - \int_{P_K} f(s) ds \right) \\ &= \lim_{K \rightarrow \infty} \int_{C_K} f(s) ds = \int_{1/2 - i\infty}^{1/2 + i\infty} f(s) ds \end{aligned}$$

4. Multiplying both sides of the expression above by  $z^{1/4}$  and changing variables  $s \mapsto 1/2 + it$  results in

$$\begin{aligned} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2} ds &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(1/2+it)\Gamma(1/4+it/2)\pi^{-1/4-it/2} z^{-it/2} dt = \\ &= -z^{-1/4} + 2z^{1/4} \sum_{n=1}^{\infty} e^{-\pi n^2 z} \end{aligned}$$

5. We can find a weight such that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(1/2 + it)\Gamma(1/4 + it/2)\pi^{-1/4-it/2} z^{-it/2} dt = \int_{-\infty}^{\infty} W(t)Z(t)dt$$

Here  $W(t) = \frac{1}{2\pi}\pi^{-1/4}z^{-it/2}|\Gamma(1/4 + it/2)|$

- (a) We want  $W$  to be real-valued and positive. Otherwise, there could be in principle some other additional cancellations so we could not prove  $Z(t)$  vanishes. We put  $z = e^{i\theta}$  where  $\theta = \pi/2 - \delta$  and  $\delta > 0$  is small to enforce this.

So  $z^{-it/2} = e^{\theta t/2}$ .

- (b) Corollary 1.27 implies that for  $s = 1/4 + it$  and  $\tau = |t| + 1$  one has

$$|\Gamma(1/4 + it)| \asymp \exp\left(-\frac{\pi|\tau|}{4}\right)|\tau|^{-1/4}$$

so

$$|W(t)| \asymp \exp\left(\frac{\pi}{4}(t - \tau) - \delta\frac{t}{2}\right)\tau^{-1/4} \asymp \exp\left(-\delta\frac{t}{2}\right)\tau^{-1/4} = b(t)$$

- (c)  $b(t)$  is strictly decreasing. If  $0 \leq t \leq \delta^{-1}$  then

$$e^{-1/2}\tau^{-1/4} \leq b(t) \leq \tau^{-1/4}$$

so  $b(t) \asymp \tau^{-1/4}$  uniformly for  $t \in [0, \delta^{-1}]$  (the implicit constants in  $\asymp$  are independent of  $\delta$  if we restrict  $t \in [0, \delta^{-1}]$ ), and this implies that

$$|W(t)| \asymp \tau^{-1/4}$$

6. By step 5 and lemma 6.1 we have that

$$\begin{aligned} \int_{-\infty}^{\infty} W(t)|Z(t)| &\gg \delta^{1/4} \int_{(2\delta)^{-1}}^{\delta^{-1}} |Z(t)| = \delta^{1/4} \int_{(2\delta)^{-1}}^{\delta^{-1}} |\zeta(1/2 + it)| dt \\ &\gg \delta^{1/4}\delta^{-1} = \delta^{-3/4} \end{aligned}$$

7. By step 4, the triangle inequality, the integral test and the fact that  $z = ie^{-i\delta} = \sin \delta + i \cos \delta$  we have that

$$\begin{aligned} \left| \int_{-\infty}^{\infty} W(t)Z(t)dt \right| &\ll \sum_{n=1}^{\infty} \exp(-\pi n^2 \sin \delta) \ll \int_0^{\infty} \exp(-\pi u^2 \sin \delta) du \\ &\ll \sqrt{(\sin \delta)^{-1}} \ll \delta^{-1/2} \end{aligned}$$

8. Suppose that  $Z(t)$  has a finite number of zeros. Let  $M$  be the last zero of  $Z$ ,

$$M = \max\{|t| : Z(t) = 0\}$$

Then

$$\int_{-\infty}^{\infty} W(t)Z(t)dt = \int_{-M}^M W(t)(Z(t) - |Z(t)|)dt \pm \int_{-\infty}^{\infty} W(t)|Z(t)|dt$$

The sign in the integral above depends on whether  $Z(t) > 0$  or  $Z(t) < 0$  for  $|t| > M$ . This implies that

$$\left| \int_{-\infty}^{\infty} W(t)Z(t)dt \right| = O(1) + \int_{-\infty}^{\infty} W(t)|Z(t)|dt$$

9. Then we would have that

$$\delta^{-1/2} \gg \left| \int_{-\infty}^{\infty} W(t)Z(t)dt \right| = O(1) + \int_{-\infty}^{\infty} W(t)|Z(t)|dt \gg \delta^{-3/4}$$

We obtain contradiction by letting  $\delta \rightarrow 0$  because  $\delta^{-3/4} > \delta^{1/2}$ .

Therefore,  $Z(t)$  has infinitely many zeros. □

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