Dirichlet–to–Robin Matrix on networks

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Abstract

In this work, we define the Dirichlet–to–Robin matrix associated with a Schrödinger type matrix on general networks, and we prove that it satisfies the *alternating property* which is essential to characterize those matrices that can be the response matrices of a network. We end with some examples of the sign pattern behavior of the alternating paths.

Keywords: Response matrix, Schur complements, inverse problem, Dirichlet–to–Robin matrix, network
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1 Preliminaires

The Schur complement plays an important role in matrix analysis, statistics, numerical analysis, and many other areas of mathematics and its applications. Our goal is to introduce the Dirichlet–to–Robin matrix associated with a Schrödinger type matrix on general networks as the Schur complement of

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a Schrödinger type matrix with respect to an invertible submatrix defined troughout the interior vertices. Schur complement is a rich and basic tool in mathematical research and applications, so we display an important property that illustrates its power in solving the discrete inverse problem. A complete version of this work in terms of operators can be found in [1].

Let $\Gamma = (V, c)$ be a *network*; that is, a simple and finite connected graph where $V = \{1, 2, \ldots, \ell\}$ is the vertex set and $c : V \times V \longrightarrow \mathbb{R}^+$ is the *conductance* that defines the set of edges, E. We say that (i, j) is an edge if $c(i, j) = c_{ij} > 0$. Moreover, when $(i, j) \notin E$, then $c_{ij} = 0$, in particular $c_{ii} = 0$ for any $i = 1, \ldots, \ell$. The *(weighted) degree* of vertex i is defined as $\delta_i = \sum\limits_{j=1}^{\ell} c_{ij}$. If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. If F is a non–empty subset of V, its characteristic function is denoted by $\mathbf{1}_{_F}$. We denote by $N(i)$, the set of neighbours of $i \in V$; that is, the set of vertices adjacent to i.

Of course networks do not have boundaries by themselves, but starting from a network we can define a *network with boundary* as $\Gamma = (\bar{F}, c_{_F})$ where F is a proper subset and $c_{_F} = c \cdot \mathbf{1}_{_{(\bar{F}\times\bar{F})\setminus(\delta(F)\times\delta(F))}}$. From now on we will work with networks with boundary and we suppose that the vertices are labelled as $\delta(F) = \{1, \ldots, n\}$ and $F = \{n + 1, \ldots, n + m\}$. Moreover, for the sake of simplicity we denote $c = c_{_F}$.

Given $S = \{p_1, \ldots, p_k\}$ and $T = \{q_1, \ldots, q_k\}$ disjoint subsets of $\delta(F)$, there exist k paths, $\gamma_1, \ldots, \gamma_k$, such that γ_i starts at p_i ends at q_i and $\gamma_i \setminus \{p_i, q_i\} \subset F$, since F is connected. The pair $(S; T)$ is called *connected trough* Γ, when there exist k paths connecting S and T that are mutually disjoint.

The network $\Gamma = (\bar{F}, c)$ is called a *circular planar network* if it can be embedded in a closed disc D in the plane so that the vertices in F lie in $\overset{\circ}{D}$ and the vertices in $\delta(F)$ lie on the circumference $C = \partial D$. In this case, the vertices in $\delta(F)$ can be labelled in the clockwise circular order. The pair $(S; T)$ of boundary vertices is called a *circular pair* if the set $(p_1, \ldots, p_k; q_1, \ldots, q_k)$ is in circular order.

Given $\mathsf{u} \in \mathbb{R}^{n+m}$, the notation $\mathsf{u} \geq 0$, respectively $\mathsf{u} > 0$, means that $\mathsf{u}_i \geq 0$, respectively $\mathsf{u}_i > 0$, for any $i = 1, \ldots, n + m$. Any vector $\sigma \in \mathbb{R}^{n+m}$ such that $\sigma > 0$ and moreover $\sum\limits_{i=1}^{n+m} \sigma_i^2 = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. If $\sigma \in \Omega(\bar{F})$, $\sigma^{-1} \in \mathbb{R}^{n+m}$ is the vector whose entries are σ_i^{-1}, $i = 1, \ldots, n + m$.

Given $\mathsf{q} \in \mathbb{R}^{n+m}$ the *Schrödinger type matrix* on Γ with *potential* q is the matrix whose entries are $\mathcal{L}_{ij} = -c_{ij}$ for all $i \neq j$ and $\mathcal{L}_{ii} = \delta_i + q_i$. Therefore, for each vector $\mathsf{u} \in \mathbb{R}^{n+m}$ and for each $i = 1, \ldots, n+m$,

$$(\mathcal{L}\mathsf{u})_i = (\delta_i + \mathsf{q}_i)\mathsf{u}_i - \sum_{j=1}^{n+m} c_{ij} u_j = \sum_{j=1}^{n+m} c_{ij}(\mathsf{u}_i - \mathsf{u}_j) + \mathsf{q}_i \mathsf{u}_i.$$

Observe that $\mathsf{q} = \mathsf{0}$ corresponds with the so–called *combinatorial Laplacian* that will be denoted by $\mathcal{L}^0$ throughout this work. Moreover,

$$\mathsf{L} = \begin{bmatrix} \mathsf{D} & -\mathsf{C}(\delta(F); F) \\ -\mathsf{C}(\delta(F); F)^\top & \mathcal{L}(F; F) \end{bmatrix}$$

where D is the diagonal matrix of order n whose diagonal entries are given by $\delta + \mathsf{q}$ on $\delta(F)$ and $\mathsf{C}(\delta(F); F) = (c_{ij})_{i \in \delta(F), j \in F}$. In general, given a matrix M and A, B sets of indexes, the matrix $\mathcal{M}(A; B)$ will denote the matrix obtained from M with rows indexed by A and columns indexed by B.

For any weight $\sigma \in \Omega(\bar{F})$, the so–called *potential associated with* σ is the vector $\left(\mathsf{q}_\sigma\right)_i = -\sigma_i^{-1}\left(\mathcal{L}^0 \sigma\right)_i$. The authors proved in [1] the following result.

Corollary 1.1 *If there exist $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$ such that $\mathsf{q} = \mathsf{q}_\sigma + \lambda \mathsf{1}_{\delta(F)}$, then the corresponding Schrödinger type matrix is positive semi–definite. Moreover, it is not strictly definite iff $\lambda = 0$, in which case the eigenvectors are $\mathsf{v} = a\sigma$, $a \in \mathbb{R}$.*

From now on, we will work with potentials given by a weight $\sigma \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$ such that $\mathsf{q} = \mathsf{q}_\sigma + \lambda \mathsf{1}_{\delta(F)}$; so that the corresponding Schrödinger type matrix is positive semi–definite. Observe that in this case

$$\left(\mathcal{L}\sigma\right)_i = 0, i = n+1, \cdots, n+m \quad \text{and} \quad \left(\mathcal{L}\sigma\right)_i = \lambda \sigma_i, i = 1, \ldots, n. \tag{1}$$

In [2, Proposition 4.10], some of these authors proved the following version of the minimum principle that will be useful in what follows.

Proposition 1.2 (Monotonicity) *If $\mathsf{u} \in \mathbb{R}^{n+m}$ is such that $\mathcal{L}\mathsf{u} \geq 0$ on F and $\mathsf{u} \geq 0$ on $\delta(F)$, it is verified that either $\mathsf{u} > 0$ on F or $\mathsf{u} = \mathsf{0}$ on $\bar{F}$.*

2 Dirichlet–to–Robin matrix

Let us consider the following *Dirichlet problem*: Given $\mathsf{f} \in \mathbb{R}^m$ and $\mathsf{g} \in \mathbb{R}^n$ find $\mathsf{u} \in \mathbb{R}^{n+m}$ satisfying

$$\begin{bmatrix} \mathsf{I} & \mathsf{0} \\ -\mathcal{C}(\delta(F);F)^\top & \mathcal{L}(F;F) \end{bmatrix} \begin{bmatrix} \mathsf{u} \end{bmatrix} = \begin{bmatrix} \mathsf{g} \\ \mathsf{f} \end{bmatrix}. \tag{2}$$

The existence and uniqueness of solution for System (2) were proved in [2]. In fact, the *Dirichlet Principle* tell us that for any data $\mathsf{f} \in \mathbb{R}^m$ and $\mathsf{g} \in \mathbb{R}^n$, Problem (2) has a unique solution.

Associated with the Dirichlet problem we can consider the following semi homogenous problems, that allow us to introduce the concept of Green and Poisson matrices. Given $\mathsf{f} \in \mathbb{R}^m$ find $\mathsf{u}_f \in \mathbb{R}^m$ satisfying

$$\mathcal{L}(F;F)\mathsf{u}_f = \mathsf{f} \tag{3}$$

and given $\mathsf{g} \in \mathbb{R}^n$ find $\mathsf{v}_g \in \mathbb{R}^{n+m}$ satisfying

$$\begin{bmatrix} \mathsf{I} & \mathsf{0} \\ -\mathcal{C}(\delta(F);F)^\top & \mathcal{L}(F;F) \end{bmatrix} \begin{bmatrix} \mathsf{v}_g \end{bmatrix} = \begin{bmatrix} \mathsf{g} \\ \mathsf{0} \end{bmatrix}. \tag{4}$$

The existence and uniqueness of solution for System (3) implies that matrix $\mathcal{L}(F;F)$ is invertible and its inverse is called *Green matrix* for F and it is denoted by $\mathcal{G}$. Observe that $\mathcal{G}$ is a symmetric matrix.

On the other hand, we define the *Poisson matrix* for F as the matrix of order $(n+m) \times n$ given by

$$\mathcal{P}(F;\delta(F)) = \mathcal{G} \cdot \mathcal{C}(\delta(F);F)^\top \quad \text{and} \quad \mathcal{P}(\delta(F);\delta(F)) = \mathcal{I}.$$

Notice that for any $\mathsf{g} \in \mathbb{R}^n$, the unique solution of System (4) is $\mathsf{v}_g = \mathcal{P}\mathsf{g}$. Moreover, from Equation (1), we get that $\mathcal{P}\sigma_{\delta(F)} = \sigma$.

Kirkhoff's law say that the sum of the currents flowing out of each interior vertex is zero, as state by System (4). If a vector g is assigned at the boundary vertices, the network Γ will acquire a unique harmonic vector v_g, with $(\mathsf{v}_g)_i = \mathsf{g}_i$ for each $i = 1, \ldots, n$. The vector v_g is called the *potential due* to g.

The function v_g determines a current through each boundary node,

$$\left(\mathcal{L}\mathsf{v}_g\right)_i = \sum_{j=n+1}^{n+m} c_{ij}\left[\mathsf{g}_i - \left(\mathsf{v}_g\right)_j\right].$$

Now, we are ready to define the *Dirichlet–to–Robin matrix* on general networks and to study its main properties. This map is naturally associated to a Schrödinger type matrix, and generalizes the concept of *Dirichlet–to–Neumann map* for the case of the combinatorial Laplacian matrix.

The *Dirichlet–to–Robin matrix*, denoted by Λ, is the Schur complement of $\mathcal{L}(F;F)$ in $\mathcal{L}$; that is,

$$\Lambda = \mathsf{L}/\mathcal{L}(F;F) = \mathsf{D} - \mathsf{C}(\delta(F);F)\cdot\mathcal{G}\cdot\mathsf{C}(\delta(F);F)^\top.$$

Observe that for any $\mathsf{g}\in\mathbb{R}^n$, $\Lambda\mathsf{g} = \mathsf{D}\mathsf{g} - \mathsf{C}(\delta(F);F)\mathsf{v}_{g_{|F}} = \mathcal{L}(\delta(F);\bar{F})\mathsf{v}_g$.

Hence, Λ sends boundary Dirichlet date g to boundary Robin currents $\mathcal{L}\mathsf{v}_g$. The inverse problem is to recover the conductances $\mathcal{C}$ form Λ, see [1,3,4]. In this work we are not worried about this problem, but in studying some properties of Λ. The following ones are a direct consequence of the expression of Λ and of some properties for Schur complements of symmetric matrices, see [5, Theorem 1.12]

Proposition 2.1 *The Dirichlet–to–Robin matrix is symmetric, negative off–diagonal, positive on the diagonal and positive semi–definite. Moreover, λ is the lowest eigenvalue of Λ and its associated eigenvectors are multiple of $\sigma_{|_{\delta(F)}}$.*

Now we show that the Dirichlet–to–Robin matrix has the alternating property, which may be considered as a generalization of the monotonicity property; see [6, Theorem 2.1] for the continuous version of this property.

Theorem 2.2 (Alternating paths) *Suppose that $\delta(F) = A\cup B$, where A and B are disjoint subsets. Let $\mathsf{g}\in\mathbb{R}^n$ such that $\mathsf{g}_i \neq 0$ iff $i\in B$ and $p_1,\dots,p_k\in A$ such that*

$$(-1)^{i+1}\Big(\Lambda\mathsf{g}\Big)_{p_i} > 0. \tag{5}$$

Then, there exist $q_1,\dots,q_k\in B$ such that

$$\Big(\Lambda\mathsf{g}\Big)_{p_i}\mathsf{g}_{q_i} < 0. \tag{6}$$

Moreover, for any $i = 1, \ldots, k$, there exists a path from p_i to q_i such that $\gamma_i \setminus \{p_i, q_i\} \subset F$ and $\mathsf{g}_{q_i}\mathsf{v}_{g_{|(\gamma_i \setminus p_i)}} > 0$, where $\mathsf{v}_g = \mathcal{P}\mathsf{g}$.

Proof. As $p_1 \in A$, from (5), we have that $0 < \Big(\Lambda \mathsf{g}\Big)_{p_1} = -\sum_{i=1}^{m} c_{p_1 n+i}(\mathsf{v}_g)_{n+i}$. Then, there exists $t \in F \cap N(p_1)$ such that $(\mathsf{v}_g)_t < 0$.

Let W be the connected component of $\{k \in F : (\mathsf{v}_g)_k < 0\}$ containing t. Suppose that $\overline{W} \cap B = \emptyset$; that is, $\overline{W} \subset F \cup A$. We consider $\mathsf{u} = \Big(\mathsf{v}_g\Big)_{|\overline{W}}$; then $\mathcal{L}\mathsf{u} = 0$ on W, $\mathsf{u} \geq 0$ on $\delta(W)$, then from the monotonicity principle $\mathsf{u} \geq 0$ on W which is a contradiction. Therefore, $\overline{W} \cap B \neq \emptyset$ and hence $\mathsf{v}_g \geq 0$ on $\delta(W) \cap F$. If $\mathsf{v}_g \geq 0$ on $\delta(W) \cap B$, we get that $\mathcal{L}\mathsf{v}_g = 0$ on W, $\mathsf{v}_g \geq 0$ on $\delta(W)$, so $\mathsf{v}_g \geq 0$ on W applying again the monotonicity principle which is a contradiction. So, there exists $q_1 \in \delta(W) \cap B$ such that $(\mathsf{v}_g)_{q_1} < 0$. As $q_1 \in \delta(W)$, there exists $z_1 \in W$, so $(\mathsf{v}_g)_{z_1} < 0$, such that $q_1 \sim z_1$. As W is a connected subset we can join q_1 and j_1 by a path $\gamma_1 = \{p_1 \sim t \sim \ldots \sim z_1 \sim q_1\}$ such that $\{t, \ldots, z_1\} \subset W$ and hence $\mathsf{v}_{g_{|(\gamma_1 \setminus p_1)}} < 0$.

We can repeat this argument to produce paths γ_j such that γ_j joins p_j to a point $q_j \in B$ such that $\gamma_j \setminus \{p_j, q_j\} \subset F$ and $(-1)^j(\mathsf{v}_g)_z < 0$ for all $z \in \gamma_j \setminus p_j$. $\square$

Corollary 2.3 *Suppose that the network is circular planar and $\delta(F) = A \cup B$, where A and B are disjoint subsets. Let $\mathsf{g} \in \mathbb{R}^n$ such that $\mathsf{g}_i \neq 0$ iff $i \in B$ and $p_1, \ldots, p_k \in A$ in circular order such that $(-1)^{i+1}\Big(\Lambda \mathsf{g}\Big)_{p_i} > 0$. Then, there exist $q_1, \ldots, q_k \in B$ in circular order such that $\Big(\Lambda \mathsf{g}\Big)_{p_i} \mathsf{g}_{q_i} < 0$. Moreover, for any $i = 1, \ldots, k$, there exists a path from p_i to q_i such that $\gamma_i \setminus \{p_i, q_i\} \subset F$ and $\mathsf{g}_{q_i}\mathsf{v}_{g_{|(\gamma_i \setminus p_i)}} > 0$ such that $(p_1, \ldots, p_k; q_1, \ldots, q_k)$ is connected through Γ.*

The following result can be obtained from Theorem 2.2 by a slightly modification of the proof.

Theorem 2.4 (Strong alternating paths) *Suppose that $\delta(F) = A \cup B$, where A and B are disjoint subsets. Let $\mathsf{g} \in \mathbb{R}^n$ such that $\mathsf{g}_i \neq 0$ iff $i \in B$ and $p_1, \ldots, p_k \in A$ such that $\Big(\Lambda \mathsf{g}\Big)_{p_i} = 0$, then there is a sequence of points $q_1, \ldots, q_k \in B$ such that $(-1)^i \mathsf{g}_{q_i} \geq 0$. Moreover, for any $i = 1, \ldots, k$, there exists a path from $p_i \sim x_1^i \sim \ldots \sim x_{n_i}^i \sim q_i$ such that $P_i \setminus \{p_i, q_i\} \subset F$ and there exists $m_i \in \{1, \ldots, n_i + 1\}$ such that $(\mathsf{v}_g)_{x_\ell} = 0$ for all $\ell = 0, \ldots, m_i - 1$ and $\mathsf{g}_{\ell_i}(\mathsf{v}_g)_{x_\ell} > 0$ for all $\ell = m_i, \ldots, n_i$.*

The following examples show the behavior of the paths described in the above results.

1. Consider the Spider graph displayed in Figure 1 (left), see [1] for the definition. With the following weights and parameters: $\sigma = \frac{1}{10}$ on $\delta(F) \cup \{x_{00}\}$ and $\sigma = \frac{1}{5}$ on $F \setminus \{x_{00}\}$, where x_{00} is the central vertex and $\lambda = 2$. Moreover, all the conductances equal 1 on the edges in the radius and equal 2 on the edges of the circles. Then, the Dirichlet–to–Robin matrix is

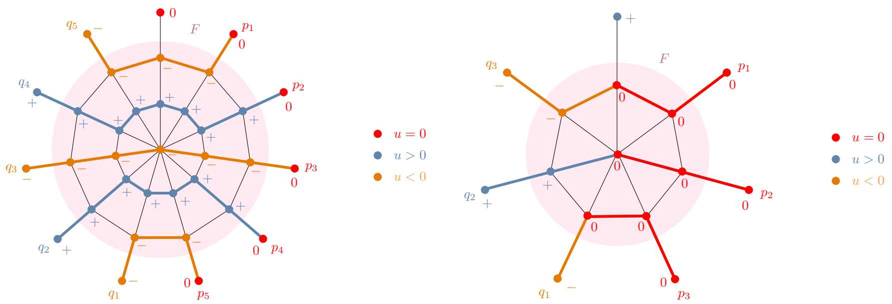

Fig. 1. Sign pattern in a Spider.

$$\Lambda = \frac{1}{110384474959} \begin{pmatrix} a & b & c & d & e & f & f & e & d & c & b \\ b & a & b & c & d & e & f & f & e & d & c \\ c & b & a & b & c & d & e & f & f & e & d \\ d & c & b & a & b & c & d & e & f & f & e \\ e & d & c & b & a & b & c & d & e & f & f \\ f & e & d & c & b & a & b & c & d & e & f \\ f & f & e & d & c & b & a & b & c & d & e \\ e & f & f & e & d & c & b & a & b & c & d \\ d & e & f & f & e & d & c & b & a & b & c \\ c & d & e & f & f & e & d & c & b & a & b \\ b & c & d & e & f & f & e & d & c & b & a \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} a = 395732805366 \\ b = -28317414524 \\ c = -19609504324 \\ d = -15073456676 \\ e = -12739926180 \\ f = -11741626020. \end{array}$$

Finally, for $\mathsf{g} = (-4, 21, -37.2, 26.38, -6.29519)^T$, we get the following sign pattern for v_g depicted in Figure 1.

2. Consider now the spider network displayed in Figure 1 (right). In this case, $\sigma = \frac{1}{6}$ on $\delta(F) \cup \{x_{00}\}$ and $\sigma = \frac{1}{3}$ on $F \setminus \{x_{00}\}$ and $\lambda = 2$. Moreover, the conductances equal 1 on the edges in the radius and equal 2 on the edges of the circles. Then, the Dirichlet–to–Robin matrix is

$$\Lambda = \frac{1}{889} \begin{pmatrix} a & b & c & d & d & c & b \\ b & a & b & c & d & d & c \\ c & b & a & b & c & d & d \\ d & c & b & a & b & c & d \\ d & d & c & b & a & b & c \\ c & d & d & c & b & a & b \\ b & c & d & d & c & b & a \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} a = 3128 \\ b = -281 \\ c = -211 \\ d = -183. \end{array}$$

Finally, for $\mathsf{g} = (-1, 3.5, -3.5, 1)^T$ we get the following sign pattern for v_g depicted in Figure 1(right).

3. Consider the network displayed in Figure 3. With the following weights and parameters: $\sigma = \frac{1}{4}$ on $A \cup F$ and $\sigma = \frac{1}{2}$ on B and $\lambda = 2$. Moreover, the conductances equal 2 on $F \times F$ and equal 1 otherwise. Then, the Dirichlet–to–Robin matrix is

$$\Lambda = \frac{1}{24}\begin{pmatrix} 105 & -13 & -15 & -13 & -3 & -5 \\ -13 & 105 & -13 & -15 & -5 & -3 \\ -15 & -13 & 105 & -13 & -3 & -5 \\ -13 & -15 & -13 & 105 & -5 & -3 \\ -3 & -5 & -3 & -5 & 57 & -1 \\ -5 & -3 & -5 & -3 & -1 & 57 \end{pmatrix}.$$

Finally, for $\mathsf{g} = (1, -1.5)^T$ we get the following sign pattern for v_g depicted in Figure 2.

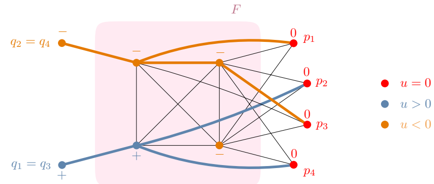

Fig. 2. Sign pattern in a non–planar network

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