Using the multilinear extension to study some probabilistic power indices

Josep Freixas and Montserrat Pons*

Abstract

We consider binary voting systems modeled by a simple game, in which voters vote independently of each other, and the probability distribution over coalitions is known. The Owen's multilinear extension of the simple game is used to improve the use and the computation of three indices defined in this model: the decisiveness index, which is an extension of the Banzhaf index, the success index, which is an extension of the Rae index, and the luckiness index. This approach leads us to prove new properties and inter–relations between these indices. In particular it is proved that the ordinal equivalence between success and decisiveness indices is achieved in any game if and only if the probability distribution is anonymous. In the anonymous case, the egalitarianism of the three indices is compared, and it is also proved that, for these distributions, decisiveness and success indices respect the strength of the seats, whereas luckiness reverses this order.

1 Introduction

The classical model of a binary voting scenario is a simple game $(N, \mathcal{W})$, defined by the set N of voters and by the voting rules given by the set $\mathcal{W}$ of winning coalitions. Owen (1972) defines the multilinear extension (MLE) of a simple game, which gives the expected utility of a random coalition, and proves that it uniquely characterizes the game, that is to say, the set of voting rules can be directly defined from $f_{\mathcal{W}}$. Since then the MLE has been used, with different goals, in the study of simple games. For instance, Owen himself uses it in (Owen, 1972) to compute the Shapley value (Shapley, 1953) and the Shapley–Shubik index (Shapley and Shubik, 1954), and in (Owen, 1975) to compute the Banzhaf value (Banzhaf, 1965). In (Alonso-Meijide et al, 2008) the multilinear extension, with some modifications, is used to compute the Johnston index (Johnston, 1978), the Deegan-Packel index (Deegan and Packel, 1978) and the Public Good index (Holler and Packel, 1983).

The decisiveness of a voter in a game $(N, \mathcal{W})$, that is, his/her possibility of influencing the final result, has been measured by different 'power indices'. Shapley and Shubik's interpretation of their index as the probability of being 'pivotal' in making a decision contributed to the association between 'power' and 'measure of decisiveness'. Banzhaf's index is also an evaluation of decisiveness. An alternative view is to measure the 'success' of a voter, that is, to focus in the likelihood

*The authors are with the Universitat Politècnica de Catalunya (Campus Manresa), in the Department of Mathematics; e-mails: josep.freixas@upc.edu, montserrat.pons@upc.edu; postal address: EPSEM, Avda. Bases de Manresa, 61-73, E-08242 Manresa, Spain.

of obtaining the result one votes for irrespective of whether one's vote is crucial for it or not. Rae (1969) was the first one to take an interest in a measure of success for symmetric simple games, or k-out-of-n games. Dubey and Shapley (1979) suggest that the index can be generalized to any simple game, leading to the definition of the Rae index. However, in simple games there is a linear relationship between the Banzhaf index and the Rae index so that these two measures become almost identical. Other power indices in the literature focus in aspects different from those of success or decisiveness as, for instance, luck (Holler and Packel, 1983), satisfaction (Brams and Lake, 1978; Van der Brink and Steffen, 2014; Davis et al, 1982), or inclusiveness (König and Bräuninger, 1998).

Decisiveness and success notions have been given a broader prospect by Laruelle and Valenciano (2005, 2008). In their work, additionally to the simple game $(N, \mathcal{W})$, it is assumed that a probability distribution over the coalitions of voters is known. In the present note we assume that, for any voter, we know – or at least have an estimate of – the probability p_i of voter i to vote affirmatively for the proposal and that each voter's vote is independent of the others. This information is captured in a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in (0,1)^n$. Clearly, if the vector $\mathbf{p}$ is known, then the probability of occurrence of each coalition is known. The model considered in this situation, named *assessed simple game* in (Carreras, 2004), is represented by the triple $(N, \mathcal{W}, \mathbf{p})$. This model, also known as generalized decision structure, has been studied by several other authors (Berg, 1999; Berg and Marañón, 2001; Dubey and Shapley, 1979; Straffin, 1994; Wilson and Pritchard, 2007).

In our work we use the MLE of the simple game to study the *decisiveness*, *success* and *luckiness* indices considered by Laruelle and Valenciano in this model. By using this technique we obtain new characterizations of these indices which simplifies their computation. We also analyze their properties and inter–relations, and we compare the rankings they induce on the set of voters, extending to more general voting contexts the comparative study of success and decisiveness done in (Freixas and Pons, 2015) for proper symmetric voting rules under independent and anonymous distributions. Other probabilistic power indices, i.e., power indices in assessed simple games, have been studied by using the MLE (Carreras, 2004, 2005; Freixas and Pons, 2005, 2008) but, as far as we know, the treatment of success, decisiveness and luckiness indices by using the MLE is novel and allows us to study new properties concerning these three indices: ordinal equivalence, egalitarianism and coherency with the strength of the seats.

The paper is organized as follows. In Section 2 we introduce known results directly linked with the context of the work. New expressions of success, decisiveness and luckiness indices, using the MLE, are included in Section 3. In Section 4 we compare the rankings given by the three indices, and in Section 5 the particular case of anonymous probability distributions is studied. The conclusions end the paper in Section 6.

2 Preliminaries

Assume that a proposal is put to the vote, that it must be either approved or rejected and that each voter can only vote "yes" or "no" (any alternative to voting "yes" is assimilated to voting "no"). A model for such a binary voting scenario is a *simple game*, that is to say, a pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of *voters* and $\mathcal{W}$ denotes the set of *winning coalitions*. A *coalition* is any subset $S \subseteq N$, and is interpreted as the set of voters which vote "yes" (those in $N \backslash S$ voting "no"). A coalition is winning when its occurrence causes the proposal to be accepted. Subsets of N that are not in $\mathcal{W}$ are called losing coalitions. A *simple game* is defined to be monotonic: if a voter joins a winning coalition then the resulting new coalition is also a winning one. It is assumed that coalitions N and $\emptyset$ are respectively winning and losing.

Before the votes are cast it is not possible to know which coalition will emerge, but we assume that an estimation p_i is made of the probability that voter i is in it, so that p_i is the estimated probability that voter i will vote in favor of the proposal.

Suppose now that the vote of every voter is *independent* from the vote of the remaining voters. In this case, the probability of coalition S to be formed, i.e., that the voters that vote 'yes' are precisely those of S, is given by

$$P(S) = \prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i). \tag{1}$$

In this way, a probability distribution on the set 2^N of all coalitions is determined by the probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and we will use $\mathbf{p}$ to refer interchangeably to both: the probability vector and the probability distribution defined on the set of coalitions. We will denote this model by $(N, \mathcal{W}, \mathbf{p})$ and it will be referred to as an *assessed simple game.* This is the context in which we work in this paper. From now on, the set of voters $N = \{1, \ldots, n\}$ is fixed, so that we will write $\mathcal{W}$ instead of $(N, \mathcal{W})$ to denote a simple game, and $(\mathcal{W}, \mathbf{p})$ instead of $(N, \mathcal{W}, \mathbf{p})$ to denote an assessed simple game.

In an assessed simple game $(\mathcal{W}, \mathbf{p})$, the probability of the proposal being accepted is

$$f_{\mathcal{W}}(\mathbf{p}) = \sum_{S \in \mathcal{W}} \prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i). \tag{2}$$

The function $f_{\mathcal{W}}$ is the *multilinear extension* (MLE) of the simple game $\mathcal{W}$ which was introduced by Owen (1972) in the general context of cooperative games. Note that $f_{\mathcal{W}}(\mathbf{p})$ is linear in each p_i so that, if $p_i = p$ for all $i \in N$ then $f_{\mathcal{W}}(\mathbf{p})$ is a polynomial function in p of degree n.

Using a conditional probability argument, given any voter $i \in N$ we can express the probability $f_{\mathcal{W}}(\mathbf{p})$ of acceptance of the proposal in terms of p_i by:

$$f_{\mathcal{W}}(\mathbf{p}) = p_i f_{\mathcal{W}}(1_i, \mathbf{p}) + (1 - p_i) f_{\mathcal{W}}(0_i, \mathbf{p}) \tag{3}$$

where $f_{\mathcal{W}}(1_i, \mathbf{p})$ is the conditional probability of the proposal being accepted if i votes "yes", that is to say, it is the value of $f_{\mathcal{W}}$ in $(p_1, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n)$. Similarly, $f_{\mathcal{W}}(0_i, \mathbf{p})$ is the probability of acceptance of the proposal if i votes against it, that is to say, it is the value of $f_{\mathcal{W}}$ in $(p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_n)$. By taking the partial derivative with respect to p_i in equation (3) we get:

$$\frac{\partial f_{\mathcal{W}}(\mathbf{p})}{\partial p_i} = f_{\mathcal{W}}(1_i, \mathbf{p}) - f_{\mathcal{W}}(0_i, \mathbf{p}) \tag{4}$$

Let us recall the notions of power index that are used in this work. Let $\mathcal{S}_N$ be the set of simple games on N and $\mathcal{AS}_N = \mathcal{S}_N \times (0,1)^n$ denote the set of assessed simple games on N. A power index h in $\mathcal{AS}_N$ is a map $h : \mathcal{AS}_N \to \mathbb{R}^n$ that assigns to every assessed simple game $(\mathcal{W}, \mathbf{p})$ a vector $h(\mathcal{W}, \mathbf{p})$, with components $h_i(\mathcal{W}, \mathbf{p})$ for all $i \in N$. Observe that for any $\mathbf{p} \in (0,1)^n$ this map h induces a power index $h^{\mathbf{p}}$ in the set of simple games $\mathcal{S}_N$, i.e., a map $h^{\mathbf{p}} : \mathcal{S}_N \to \mathbb{R}^n$ defined by $h^{\mathbf{p}}(\mathcal{W}) = h(\mathcal{W}, \mathbf{p})$. Notice that (4) defines a power index in $\mathcal{AS}_N$.

To end this section, we recall some definitions introduced by Laruelle and Valenciano (2005). For any $S \subseteq N$, let $P(S)$ denote the probability of S, that is to say, the probability that voters in S are precisely those that vote "yes". In an assessed simple game $(\mathcal{W}, \mathbf{p})$ the probability $P(S)$ is given in (1), and the three indices defined in Laruelle and Valenciano (2005) become power indices in the set of assessed simple games.

Definition 2.1 ***Decisiveness, Success and Luckiness indices***
Let $(\mathcal{W}, \mathbf{p})$ *be an assessed simple game and* $i \in N$*:*

i) Decisiveness index:

$$\Phi_i(\mathcal{W}, \mathbf{p}) = \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} P(S) + \sum_{\substack{S\,:\, i \notin S \notin \mathcal{W} \\ S \cup i \in \mathcal{W}}} P(S) \tag{5}$$

ii) Success index:

$$\Omega_i(\mathcal{W}, \mathbf{p}) = \sum_{S: i \in S \in \mathcal{W}} P(S) + \sum_{S: i \notin S \notin \mathcal{W}} P(S) \tag{6}$$

iii) Luckiness index:

$$\Lambda_i(\mathcal{W}, \mathbf{p}) = \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \in \mathcal{W}}} P(S) + \sum_{\substack{S\,:\, i \notin S \notin \mathcal{W} \\ S \cup i \notin \mathcal{W}}} P(S)$$

The Banzhaf index B and the Rae index R, defined in simple games, are induced, respectively, by the decisiveness index Φ and by the success index Ω defined in assessed simple games. Specifically, $B = \Phi^{\mathbf{p}^*}$ and $R = \Omega^{\mathbf{p}^*}$ where $\mathbf{p}^* = (0.5, \ldots, 0.5)$.

The following theorem was proved in (Laruelle et al, 2006).

Theorem 2.2 *Let* $\mathbf{p}^* = (0.5, \ldots, 0.5)$*. Then, if* $\mathbf{p} = (p_1, \ldots, p_n)$*,*

$$\left.\begin{array}{r} \Omega_i(\mathcal{W}, \mathbf{p}) = 0.5 + 0.5\ \Phi_i(\mathcal{W}, \mathbf{p}) \\ \\ \textit{for any } i \in N \textit{ and for any simple game } \mathcal{W} \end{array}\right\} \Leftrightarrow \mathbf{p} = \mathbf{p}^*.$$

Theorem 2.2 tells us that the relationship between Banzhaf and Rae indices

$$R(\mathcal{W}) = 0.5 + 0.5\, B(\mathcal{W}), \tag{7}$$

verified for any $\mathcal{W}$, cannot be generally extended to any assessed simple game $(\mathcal{W}, \mathbf{p})$.

On the other hand, it is obvious that for any $(\mathcal{W}, \mathbf{p})$ we have

$$\Omega(\mathcal{W}, \mathbf{p}) = \Phi(\mathcal{W}, \mathbf{p}) + \Lambda(\mathcal{W}, \mathbf{p}). \tag{8}$$

and, by defining $L = \Lambda^{\mathbf{p}^*}$, with $\mathbf{p}^* = (0.5, \ldots, 0.5)$, we obtain Barry's equation (Barry, 1980a,b), verified for any simple game $\mathcal{W}$:

$$R(\mathcal{W}) = B(\mathcal{W}) + L(\mathcal{W}). \tag{9}$$

We conclude this preliminary section of well-known results by noticing that decisiveness of a voter only depends on the other voters' behavior, not on his/her own, since voter i's decisiveness can be written as

$$\Phi_i(\mathcal{W}, \mathbf{p}) = \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} (P(S) + P(S \setminus \{i\})). \tag{10}$$

3 The three indices expressed in terms of the MLE

The aim of this section is to express the indices of success, decisiveness and luckiness in terms of $f_{\mathcal{W}}(\mathbf{p})$ (2). The formulations in Proposition 3.1 are fundamental to achieve the results in the two next sections. Notice that, taking into account (4), the part a) of the next proposition tells us that $\Phi_i(\mathcal{W}, \mathbf{p})$ coincides with the partial derivative of $f_{\mathcal{W}}(\mathbf{p})$ with respect to p_i.

Proposition 3.1 *Let $(\mathcal{W}, \mathbf{p})$ an assessed simple game and $i \in N$:*

a) $\Phi_i(\mathcal{W}, \mathbf{p}) = f_{\mathcal{W}}(1_i, \mathbf{p}) - f_{\mathcal{W}}(0_i, \mathbf{p})$,

b) $\Omega_i(\mathcal{W}, \mathbf{p}) = p_i f_{\mathcal{W}}(1_i, \mathbf{p}) + (1 - p_i)(1 - f_{\mathcal{W}}(0_i, \mathbf{p}))$,

c) $\Lambda_i(\mathcal{W}, \mathbf{p}) = p_i f_{\mathcal{W}}(0_i, \mathbf{p}) + (1 - p_i)(1 - f_{\mathcal{W}}(1_i, \mathbf{p}))$.

Proof:

a) From (1), (5), and (10) it follows

$$\begin{aligned}
\Phi_i(\mathcal{W}, \mathbf{p}) &= \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \left(\prod_{j \in S} p_j \prod_{j \in N \setminus S} (1 - p_j) + \prod_{j \in S \setminus i} p_j \prod_{j \in (N \setminus S) \cup i} (1 - p_j) \right) \\
&= \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} (p_i + (1 - p_i)) \left(\prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j) \right) \\
&= \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j)
\end{aligned}$$

On the other hand, from (2):

$$f_{\mathcal{W}}(1_i, \mathbf{p}) = \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j) \quad + \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \in \mathcal{W}}} \prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j)$$

and

$$f_{\mathcal{W}}(0_i, \mathbf{p}) = \sum_{\substack{S\,:\, S \in \mathcal{W} \\ i \notin S}} \prod_{j \in S} p_j \prod_{j \in N \setminus (S \cup i)} (1 - p_j) \quad = \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus i \in \mathcal{W}}} \prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j)$$

Thus, the second term in $f_{\mathcal{W}}(1_i, \mathbf{p})$ simplifies with $f_{\mathcal{W}}(0_i, \mathbf{p})$ in $f_{\mathcal{W}}(1_i, \mathbf{p}) - f_{\mathcal{W}}(0_i, \mathbf{p})$ and therefore $\Phi_i(\mathcal{W}, \mathbf{p})$ coincides with $f_{\mathcal{W}}(1_i, \mathbf{p}) - f_{\mathcal{W}}(0_i, \mathbf{p})$.

b) From (6) in Definition 2.1 it is clear that success can be written

$$\Omega_i(\mathcal{W}, \mathbf{p}) = p_i \left(\sum_{\substack{S\,:\, S \in \mathcal{W} \\ i \in S}} \prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j) \right) + (1 - p_i) \left(\sum_{\substack{S\,:\, S \notin \mathcal{W} \\ i \notin S}} \prod_{j \in S} p_j \prod_{j \in N \setminus (S \cup i)} (1 - p_j) \right)$$

But from (2):

$$f_{\mathcal{W}}(1_i, \mathbf{p}) = \sum_{\substack{S\,:\,S\,\in\,\mathcal{W}\\ i\,\in\,S}} \prod_{j\in S\setminus i} p_j \prod_{j\in N\setminus S} (1-p_j)$$

and

$$1 - f_{\mathcal{W}}(0_i, \mathbf{p}) = 1 - \sum_{\substack{S\,:\,S\,\in\,\mathcal{W}\\ i\,\notin\,S}} \prod_{j\in S} p_j \prod_{j\in N\setminus(S\cup\, i)} (1-p_j) = \sum_{\substack{S\,:\,S\,\notin\,\mathcal{W}\\ i\,\notin\,S}} \prod_{j\in S} p_j \prod_{j\in N\setminus(S\cup\, i)} (1-p_j),$$

because, for any vector $\mathbf{p}$, we have:

$$\begin{aligned} 1 = \quad & \sum_{\substack{S\,:\,S\,\in\,\mathcal{W}\\ i\,\in\,S}} \prod_{j\in S} p_j \prod_{j\in N\setminus S} (1-p_j) + \sum_{\substack{S\,:\,S\,\in\,\mathcal{W}\\ i\,\notin\,S}} \prod_{j\in S} p_j \prod_{j\in N\setminus S} (1-p_j) + \\ & \sum_{\substack{S\,:\,S\,\notin\,\mathcal{W}\\ i\,\in\,S}} \prod_{j\in S} p_j \prod_{j\in N\setminus S} (1-p_j) + \sum_{\substack{S\,:\,S\,\notin\,\mathcal{W}\\ i\,\notin\,S}} \prod_{j\in S} p_j \prod_{j\in N\setminus S} (1-p_j) \end{aligned}$$

but, if $p_i = 0$ then the first and the third addends are zero and the factor $1 - p_i$ does not appear in the remaining addends.

c) Taking into account (8) and applying parts *a*) and *b*), we get the result.

□

We remark that Theorem 2.2 can easily be proved by using the expressions of success and decisiveness established in Proposition 3.1.

4 Comparing rankings

The main purpose of this section is to study the ordinal equivalence of decisiveness, success and luckiness indices in assessed simple games. A similar study was done in (Freixas et al, 2012) for the Shapley, Banzhaf and Johnston indices in the context of simple games.

Definition 4.1 *Let g and h be two power indices in $\mathcal{AS}_N$.*
i) *g and h are* ordinally equivalent *in $(\mathcal{W}, \mathbf{p})$ if for all $i, j \in N$:*

$$g_i(\mathcal{W}, \mathbf{p}) > g_j(\mathcal{W}, \mathbf{p}) \iff h_i(\mathcal{W}, \mathbf{p}) > h_j(\mathcal{W}, \mathbf{p})$$

which implies that $g_i(\mathcal{W}, \mathbf{p}) = g_j(\mathcal{W}, \mathbf{p})$ if and only if $h_i(\mathcal{W}, \mathbf{p}) = h_j(\mathcal{W}, \mathbf{p})$.
ii) *g and h are* ordinally opposite *in $(\mathcal{W}, \mathbf{p})$ if for all $i, j \in N$:*

$$g_i(\mathcal{W}, \mathbf{p}) > g_j(\mathcal{W}, \mathbf{p}) \iff h_i(\mathcal{W}, \mathbf{p}) < h_j(\mathcal{W}, \mathbf{p})$$

which also implies that $g_i(\mathcal{W}, \mathbf{p}) = g_j(\mathcal{W}, \mathbf{p})$ if and only if $h_i(\mathcal{W}, \mathbf{p}) = h_j(\mathcal{W}, \mathbf{p})$.

Remark 4.2
If g and h are ordinally equivalent in $(\mathcal{W}, \mathbf{p})$ then the induced indices $g^{\mathbf{p}}$ and $h^{\mathbf{p}}$ verify

$$g_i^{\mathbf{p}}(\mathcal{W}) > g_j^{\mathbf{p}}(\mathcal{W}) \iff h_i^{\mathbf{p}}(\mathcal{W}) > h_j^{\mathbf{p}}(\mathcal{W})$$

for all $i, j \in N$*, and we say that* $g^{\mathbf{p}}$ *and* $h^{\mathbf{p}}$ *are ordinally equivalent in* $\mathcal{W}$*.*
Similarly, if g *and* h *are ordinally opposite in* $(\mathcal{W}, \mathbf{p})$ *then the induced indices* $g^{\mathbf{p}}$ *and* $h^{\mathbf{p}}$ *verify*

$$g_i^{\mathbf{p}}(\mathcal{W}) > g_j^{\mathbf{p}}(\mathcal{W}) \iff h_i^{\mathbf{p}}(\mathcal{W}) < h_j^{\mathbf{p}}(\mathcal{W})$$

for all $i, j \in N$*, and we say that* $g^{\mathbf{p}}$ *and* $h^{\mathbf{p}}$ *are ordinally opposite in* $\mathcal{W}$*.*

A clear consequence of (7) is that the Banzhaf and Rae indices are ordinally equivalent in any simple game. Moreover, by combining equations (7) and (9) we deduce that, for any simple game $\mathcal{W}$, it is:

$$L(\mathcal{W}) = 0.5(1 - B(\mathcal{W})). \tag{11}$$

Hence, the index L is ordinally opposite to the Banzhaf and Rae indices in any simple game $\mathcal{W}$.

These relations among rankings verified when $\mathbf{p} = \mathbf{p}^\star$ can not be generally extended to other probability distributions. The following lemma will be used to compare the rankings given by decisiveness, success and luckiness indices for $\mathbf{p} \neq (0.5, \ldots, 0.5)$. The notation $f_{\mathcal{W}}(x_i, y_j, \mathbf{p})$ is used to denote the value of $f_{\mathcal{W}}$ on a vector whose components are the same as those of $\mathbf{p}$ except components i and j whose values are, respectively, x and y. The value of $f_{\mathcal{W}}(x_i, y_j, \mathbf{p})$ is the conditional probability of the proposal being accepted if i votes "yes" with probability x and j votes "yes" with probability y.

Lemma 4.3 *Let* $(\mathcal{W}, \mathbf{p})$ *be an assessed simple game and* i, j *be different voters in* N*. Then,*

a) $$\begin{aligned}\Phi_i(\mathcal{W}, \mathbf{p}) - \Phi_j(\mathcal{W}, \mathbf{p}) = \; & (p_j - p_i)[f_{\mathcal{W}}(1_i, 1_j, \mathbf{p}) + f_{\mathcal{W}}(0_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p})] \\ & + [f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p})]\end{aligned}$$

b) $$\Omega_i(\mathcal{W}, \mathbf{p}) - \Omega_j(\mathcal{W}, \mathbf{p}) = 2p_i(1 - p_j) f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - 2p_j(1 - p_i) f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) + p_j - p_i$$

c) $$\Lambda_i(\mathcal{W}, \mathbf{p}) - \Lambda_j(\mathcal{W}, \mathbf{p}) = 2[p_i f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - p_j f_{\mathcal{W}}(0_i, 1_j, \mathbf{p})] + (p_i - p_j)(b_{ij} - 1) - a_{ij}(1 + 2p_i p_j)$$

$$\textit{where:} \quad \begin{cases} a_{ij} = f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) \\ b_{ij} = f_{\mathcal{W}}(1_i, 1_j, \mathbf{p}) + f_{\mathcal{W}}(0_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}). \end{cases}$$

Proof:
Expressions $a)$ and $b)$ are obtained by using (3) in Proposition 3.1. Part $c)$ is deduced from them by using (8). □

Theorem 4.4 *Let i, j be different voters in N and $\mathbf{p} \in (0,1)^n$. Then,*

a) If $p_i = p_j$ then, for any game $\mathcal{W}$,

$$\Omega_i(\mathcal{W},\mathbf{p}) > \Omega_j(\mathcal{W},\mathbf{p}) \iff \Phi_i(\mathcal{W},\mathbf{p}) > \Phi_j(\mathcal{W},\mathbf{p}) \iff \Lambda_i(\mathcal{W},\mathbf{p}) < \Lambda_j(\mathcal{W},\mathbf{p}).$$

b) If $N = \{i, j\}$ then, for any game $\mathcal{W}$,

$$\Omega_i(\mathcal{W},\mathbf{p}) > \Omega_j(\mathcal{W},\mathbf{p}) \iff \Phi_i(\mathcal{W},\mathbf{p}) > \Phi_j(\mathcal{W},\mathbf{p}) \iff \Lambda_i(\mathcal{W},\mathbf{p}) \leq \Lambda_j(\mathcal{W},\mathbf{p}) .$$

c) If $p_i \neq p_j$ and there is a voter $k \neq i, j$ with $p_k \neq 0.5$, then we can find a game $\mathcal{W}$ such that

$$[\Omega_i(\mathcal{W},\mathbf{p}) - \Omega_j(\mathcal{W},\mathbf{p})]\,[\Phi_i(\mathcal{W},\mathbf{p}) - \Phi_j(\mathcal{W},\mathbf{p})] < 0.$$

d) If $p_i \neq p_j$ and for any voter $k \neq i, j$ it is $p_k = 0.5$, then we can find a game $\mathcal{W}$ such that

$$\Phi_i(\mathcal{W},\mathbf{p}) \neq \Phi_j(\mathcal{W},\mathbf{p}) \text{ and } \Omega_i(\mathcal{W},\mathbf{p}) = \Omega_j(\mathcal{W},\mathbf{p}).$$

e) If $n \geq 3$ and $p_i \neq p_j$ then we can find a game $\mathcal{W}$ such that

$\Phi_i(\mathcal{W},\mathbf{p}) - \Phi_j(\mathcal{W},\mathbf{p})$, $\Omega_i(\mathcal{W},\mathbf{p}) - \Omega_j(\mathcal{W},\mathbf{p})$, and $\Lambda_i(\mathcal{W},\mathbf{p}) - \Lambda_j(\mathcal{W},\mathbf{p})$ have the same sign.

Proof:

a) If $p_i = p_j = p$ then, for any game $\mathcal{W}$, using lemma 4.3 we have,

$$\begin{aligned}
\Phi_i(\mathcal{W},\mathbf{p}) - \Phi_j(\mathcal{W},\mathbf{p}) &= f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) \\
\Omega_i(\mathcal{W},\mathbf{p}) - \Omega_j(\mathcal{W},\mathbf{p}) &= 2p(1-p)[f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p})] \\
\Lambda_i(\mathcal{W},\mathbf{p}) - \Lambda_j(\mathcal{W},\mathbf{p}) &= (-2p^2 + 2p - 1)[f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p})].
\end{aligned}$$

The fact that $0 < p < 1$ leads to the conclusion.

b) If $N = \{i, j\}$ then there are only four possible games, and we see that in all of them the statement is true.

$f_{\mathcal{W}}(\mathbf{p})$	$\Phi_i(\mathcal{W},\mathbf{p})$	$\Phi_j(\mathcal{W},\mathbf{p})$	$\Omega_i(\mathcal{W},\mathbf{p})$	$\Omega_j(\mathcal{W},\mathbf{p})$	$\Lambda_i(\mathcal{W},\mathbf{p})$	
p_i	1	0	1	$1 - p_i - p_j + 2p_ip_j$	0	$1 -$
p_j	0	1	$1 - p_i - p_j + 2p_ip_j$	1	$1 - p_i - p_j + 2p_ip_j$	
p_ip_j	p_j	p_i	$1 - p_i + p_ip_j$	$1 - p_j + p_ip_j$	$1 - p_i - p_j + p_ip_j$	$1 -$
$p_i + p_j - p_ip_j$	$1 - p_j$	$1 - p_i$	$1 - p_j + p_ip_j$	$1 - p_i + p_ip_j$	$1 - p_i - p_j + p_ip_j$	$1 -$

c) Assume, without loss of generality, that $p_i < p_j$, and let $k \neq i, j$ be such that $p_k \neq 0.5$.

- If $p_k < 0.5$ then consider the game $\mathcal{W}$ with $f_{\mathcal{W}}(\mathbf{p}) = p_ip_k + p_jp_k - p_ip_jp_k$. In this case it is $f_{\mathcal{W}}(1_i, 1_j, \mathbf{p}) = p_k$, $f_{\mathcal{W}}(0_i, 0_j, \mathbf{p}) = 0$ and $f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) = f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) = p_k$. Thus, from Lemma 4.3,

$$\Phi_i(\mathcal{W},\mathbf{p}) - \Phi_j(\mathcal{W},\mathbf{p}) = (p_j - p_i)(-p_k) < 0$$

$$\Omega_i(\mathcal{W},\mathbf{p}) - \Omega_j(\mathcal{W},\mathbf{p}) = (p_j - p_i)(-2p_k + 1) > 0$$

- If $p_k > 0.5$ then consider the game $\mathcal{W}$ with $f_{\mathcal{W}}(\mathbf{p}) = p_k + p_i p_j - p_i p_j p_k$. In this case it is $f_{\mathcal{W}}(1_i, 1_j, \mathbf{p}) = 1$ and $f_{\mathcal{W}}(0_i, 0_j, \mathbf{p}) = f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) = f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) = p_k$. Thus, from Lemma 4.3,
$$\Phi_i(\mathcal{W}, \mathbf{p}) - \Phi_j(\mathcal{W}, \mathbf{p}) = (p_j - p_i)(1 - p_k) > 0$$
$$\Omega_i(\mathcal{W}, \mathbf{p}) - \Omega_j(\mathcal{W}, \mathbf{p}) = (p_j - p_i)(1 - 2p_k) < 0$$

d) Taking any $k \neq i, j$, the game $\mathcal{W}$ with $f_{\mathcal{W}}(\mathbf{p}) = p_k + p_i p_j - p_i p_j p_k$ verifies that

$$\Phi_i(\mathcal{W}, \mathbf{p}) - \Phi_j(\mathcal{W}, \mathbf{p}) = 0.5(p_j - p_i) \text{ and } \Omega_i(\mathcal{W}, \mathbf{p}) - \Omega_j(\mathcal{W}, \mathbf{p}) = 0$$

and the statement is proved.

e) Let $\mathcal{W}$ be the unanimity game, for which $f_{\mathcal{W}}(\mathbf{p}) = \prod_{k=1}^{n} p_k$. Then, from Lemma 4.3,

$$\Lambda_i(\mathcal{W}, \mathbf{p}) - \Lambda_j(\mathcal{W}, \mathbf{p}) = (p_i - p_j)\left(\prod_{k \neq i,j} p_k - 1\right),$$

$$\Phi_i(\mathcal{W}, \mathbf{p}) - \Phi_j(\mathcal{W}, \mathbf{p}) = (p_j - p_i)\left(\prod_{k \neq i,j} p_k\right),$$

$$\Omega_i(\mathcal{W}, \mathbf{p}) - \Omega_j(\mathcal{W}, \mathbf{p}) = p_j - p_i.$$

Thus, since $\mathbf{p} \in (0,1)^n$, the three differences have the same sign.

□

As a consequence of part *a*) in Theorem 4.4, for a probability distribution with coincident components, i.e., $\mathbf{p} = (p, \ldots, p)$, success and decisiveness indices are ordinally equivalent, and ordinally opposite to the luckiness index, in $(\mathcal{W}, \mathbf{p})$ for any game $\mathcal{W}$. In the next section we study in more detail the rankings given by the three indices in this kind of probability distributions.

5 Anonymous probability distributions

Definition 5.1 *A probability distribution* $\mathbf{p} = (p_1, \ldots, p_n)$ *is* anonymous *if* $p_1 = \cdots = p_n$, *i.e., all voters have the same probability of voting 'yes'. In this case, the probability of forming a coalition* $S \subseteq N$ *depends only on the number of voters* s *in* S, *and is given by* $P(S) = p^s(1-p)^{n-s}$, *where* p *is the common value for all* p_i $(1 \leq i \leq n)$. *Since in this case we have* $\mathbf{p} = (p, p, \ldots, p)$, *the corresponding assessed simple game will be denoted by* $(\mathcal{W}, p)$.

The next proposition shows that, for more than two voters, anonymous probability distributions are the only ones for which the indices induced in simple games by success and decisiveness are ordinally equivalent in any game, and also the only ones for which the index induced by luckiness is ordinally opposite to both of them in any game.

Proposition 5.2 *Let* $n \geq 3$. *Then, the following statements are equivalent:*

a) $\mathbf{p}$ *is anonymous.*

b) Φ *and* Ω *are ordinally equivalent in* $(\mathcal{W}, \mathbf{p})$ *for all* $\mathcal{W}$.

c) Ω *and* Λ *are ordinally opposite in* $(\mathcal{W}, \mathbf{p})$ *for all* $\mathcal{W}$.

d) Λ *and* Φ *are ordinally opposite in* $(\mathcal{W}, \mathbf{p})$ *for all* $\mathcal{W}$.

Proof: From Theorem 4.4-a) it is clear that if $\mathbf{p}$ is anonymous then parts b), c) and d) are verified. Conversely, if $p_i \neq p_j$ for some components $i, j \in N$ then, from Theorem 4.4-c), d) we can find a game $\mathcal{W}$ for which $\Phi_i(\mathcal{W}, \mathbf{p}) > \Phi_j(\mathcal{W}, \mathbf{p})$ but $\Omega_i(\mathcal{W}, \mathbf{p}) \not> \Omega_j(\mathcal{W}, \mathbf{p})$, and, from Theorem 4.4-$e$), we can find a game $\mathcal{W}$ for which $\Phi_i(\mathcal{W}, \mathbf{p}) > \Phi_j(\mathcal{W}, \mathbf{p})$, $\Omega_i(\mathcal{W}, \mathbf{p}) > \Omega_j(\mathcal{W}, \mathbf{p})$ and $\Lambda_i(\mathcal{W}, \mathbf{p}) > \Lambda_j(\mathcal{W}, \mathbf{p})$. □

As a consequence of Proposition 5.2 we see that, for anonymous probability distributions, the opposite effect that luckiness show compared with decisiveness has no additive effect on success.

Two power indices can rank voters in the same (or opposite) way but taking extremely different numerical values. In what follows we compare the range of values of the three considered indices for anonymous probability distributions.

Definition 5.3 *Let* g *and* h *be two power indices in* $\mathcal{AS}_N$.

i) g *is* more egalitarian than h *in* $(\mathcal{W}, \mathbf{p})$ *if for all* $i, j \in N$

$$|g_i(\mathcal{W}, \mathbf{p}) - g_j(\mathcal{W}, \mathbf{p})| < |h_i(\mathcal{W}, \mathbf{p}) - h_j(\mathcal{W}, \mathbf{p})|.$$

We can also say that h *is* less egalitarian than g *in* $(\mathcal{W}, \mathbf{p})$.

ii) g *is a* k-reduction *of* h *in* $(\mathcal{W}, \mathbf{p})$ *if there exists* k $(0 < k < 1)$ *such that, for all* $i, j \in N$

$$|g_i(\mathcal{W}, \mathbf{p}) - g_j(\mathcal{W}, \mathbf{p})| = k|h_i(\mathcal{W}, \mathbf{p}) - h_j(\mathcal{W}, \mathbf{p})|$$

k *is said to be the reduction coefficient.*

i) If g *is a* k*-reduction of* h *in* $(\mathcal{W}, \mathbf{p})$ *then,*
if g *and* h *are ordinally equivalent in* $(\mathcal{W}, \mathbf{p})$ *we say that* g *is a* direct k-reduction *of* h *in* $(\mathcal{W}, \mathbf{p})$,
if g *and* h *are ordinally opposite in* $(\mathcal{W}, \mathbf{p})$ *we say that* g *is an* inverse k-reduction *of* h *in* $(\mathcal{W}, \mathbf{p})$.

Remark 5.4 *In any assessed simple game* $(\mathcal{W}, \mathbf{p})$:

- *If* g *is a* k*-reduction of* h *for some* k *then* g *is more egalitarian than* h.
- *If* g_1 *is a* k_1*-reduction of* h *and* g_2 *is a* k_2*-reduction of* h*, then* $k_1 < k_2$ *implies that* g_1 *is more egalitarian than* g_2.
- *If* g *is a direct* k*-reduction of* h *then, for all* $i, j \in N$ *it is* $g_i(\mathcal{W}, \mathbf{p}) - g_j(\mathcal{W}, \mathbf{p}) = k(h_i(\mathcal{W}, \mathbf{p}) - h_j(\mathcal{W}, \mathbf{p}))$.
- *If* g *is an inverse* k*-reduction of* h *then, for all* $i, j \in N$ *it is* $g_i(\mathcal{W}, \mathbf{p}) - g_j(\mathcal{W}, \mathbf{p}) = k(h_j(\mathcal{W}, \mathbf{p}) - h_i(\mathcal{W}, \mathbf{p}))$.

Remark 5.5 *All the notions introduced in Definition 5.3 about two power indices g and h in $\mathcal{AS}_N$ have an immediate translation on the power indices they induce in simple games. In particular:*

- *If g is more egalitarian than h in $(\mathcal{W}, \mathbf{p})$ then, for all $i, j \in N$ it is*

$$|g_i^{\mathbf{p}}(\mathcal{W}) - g_j^{\mathbf{p}}(\mathcal{W})| < |h_i^{\mathbf{p}}(\mathcal{W}) - h_j^{\mathbf{p}}(\mathcal{W})|$$

and we say that $g^{\mathbf{p}}$ is more egalitarian than $h^{\mathbf{p}}$ in $\mathcal{W}$.

- *If g is a direct k-reduction of h in $(\mathcal{W}, \mathbf{p})$ then, for all $i, j \in N$ it is*

$$g_i^{\mathbf{p}}(\mathcal{W}) - g_j^{\mathbf{p}}(\mathcal{W}) = k(h_i^{\mathbf{p}}(\mathcal{W}) - h_j^{\mathbf{p}}(\mathcal{W}))$$

and we say that $g^{\mathbf{p}}$ is a direct k–reduction of $h^{\mathbf{p}}$ in $\mathcal{W}$.

- *If g is an inverse k-reduction of h in $(\mathcal{W}, \mathbf{p})$ then, for all $i, j \in N$ it is*

$$g_i^{\mathbf{p}}(\mathcal{W}) - g_j^{\mathbf{p}}(\mathcal{W}) = k(h_j^{\mathbf{p}}(\mathcal{W}) - h_i^{\mathbf{p}}(\mathcal{W}))$$

and we say that $g^{\mathbf{p}}$ is an inverse k–reduction of $h^{\mathbf{p}}$ in $\mathcal{W}$.

According to equations (7) and (11) we deduce that the success index Ω is a direct 0.5-reduction of the decisiveness index Φ, and the luckiness index Λ is an inverse 0.5-reduction of it, in $(\mathcal{W}, \mathbf{p}^*)$ for any $\mathcal{W}$. In other words, the Rae index is a direct 0.5-reduction of the Banzhaf index, and the index L is an inverse 0.5-reduction of it, in any simple game $\mathcal{W}$. Thus, in particular, both indices are more egalitarian than Banzhaf's in any $\mathcal{W}$. Next corollary is a generalization of this result for anonymous probability distributions, and it follows from Lemma 4.3.

Corollary 5.6 *Let $\mathcal{W}$ be a simple game, $p \in (0, 1)$, and $n \geq 3$. Then,*

a) Ω is a direct k–reduction of Φ in $(\mathcal{W}, p)$ with $k = 2p(1 - p)$.

b) Λ is an inverse k–reduction of Φ in $(\mathcal{W}, p)$ with $k = 1 + 2p(p - 1)$.

c) Ω and Λ are both more egalitarian than Φ in $(\mathcal{W}, p)$.

From this corollary, for any anonymous probability distribution $\mathbf{p} = (p, \ldots, p)$, the indices Ω^p and Λ^p, induced in simple games respectively by success and luckiness, are more egalitarian than the index Φ^p, induced by decisiveness, in any game $\mathcal{W}$.

Notice that for any $p \in (0, 1)$ it is $2p(1-p) \leq 0.5 \leq 1+2p(p-1)$, so that, for any simple game $\mathcal{W}$ and $i, j \in N$ we have

$$| \Omega_i^p(\mathcal{W}) - \Omega_j^p(\mathcal{W}) | \leq 0.5 \mid \Phi_i^p(\mathcal{W}) - \Phi_j^p(\mathcal{W}) | \leq | \Lambda_i^p(\mathcal{W}) - \Lambda_j^p(\mathcal{W}) |,$$

and these inequalities become equalities if and only if $p = 0.5$.

When the probability distribution is anonymous, it seems intuitive that those voters that occupy a stronger seat should be more powerful. A tool to compare the strength of the seats of two voters in a simple game $\mathcal{W}$ is the weak desirability relation $\succsim_{\mathcal{W}}$ (Carreras and Freixas, 2008).

Definition 5.7 ***Weak desirability relation***
Let $\mathcal{W}$ be a simple game and $i, j \in N$.

$$i \succsim_{\mathcal{W}} j \quad \text{iff} \quad |\mathcal{C}_i(k)| \geq |\mathcal{C}_j(k)| \quad \text{for all} \quad k = 1, 2, \ldots, n,$$

where $\mathcal{C}_i(k)$ is formed by the coalitions of k voters for which i is crucial, i.e., winning coalitions which become losing if voter i leaves them.
The relation $\succsim_{\mathcal{W}}$ is a preorder on N. We write $i \succ_{\mathcal{W}} j$ when $i \succsim_{\mathcal{W}} j$ and $j \not\succsim_{\mathcal{W}} i$. When the weak desirability relation $\succsim_{\mathcal{W}}$ is total, i.e., for all $i, j \in N$ it is either $i \succsim_{\mathcal{W}} j$ or $j \succsim_{\mathcal{W}} i$, we say that the simple game $\mathcal{W}$ is weakly complete.

This relation is weaker than the well–known desirability relation (Isbell, 1958) in the sense that if a voter i is at least as desirable as a voter j then $i \succsim_{\mathcal{W}} j$ (Diffo Lambo and Moulen, 2002), and the class of weakly complete games includes the class of complete games, i.e. the games for which the desirability relation is complete.

In (Freixas, 2010) the following proposition was proved.

Proposition 5.8 *Let $\mathcal{W}$ be a simple game, $i, j \in N$ and $\mathbf{p} = (p, \ldots, p)$ with $p \in (0, 1)$. Then,*

a) $i \succsim_{\mathcal{W}} j \quad \Rightarrow \quad f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) \geq 0.$

b) $i \succ_{\mathcal{W}} j \quad \Rightarrow \quad f_{\mathcal{W}}(1_i, 0_j, \mathbf{p}) - f_{\mathcal{W}}(0_i, 1_j, \mathbf{p}) > 0.$

Thus, from Lemma 4.3 and Proposition 5.8 the next corollary follows.

Corollary 5.9 *Let $\mathcal{W}$ be a simple game, $i, j \in N$ and $\mathbf{p} = (p, \ldots, p)$ with $p \in (0, 1)$. Then,*

a) *If* $i \succsim_{\mathcal{W}} j$ *then* $\Phi_i(\mathcal{W}, p) \geq \Phi_j(\mathcal{W}, p)$, *which is equivalent to:*
$\Omega_i(\mathcal{W}, p) \geq \Omega_j(\mathcal{W}, p)$ *and* $\Lambda_i(\mathcal{W}, p) \leq \Lambda_j(\mathcal{W}, p)$.

b) *If* $i \succ_{\mathcal{W}} j$ *then* $\Phi_i(\mathcal{W}, p) > \Phi_j(\mathcal{W}, p)$, *which is equivalent to:*
$\Omega_i(\mathcal{W}, p) > \Omega_j(\mathcal{W}, p)$ *and* $\Lambda_i(\mathcal{W}, p) < \Lambda_j(\mathcal{W}, p)$.

Thus, for anonymous probability distributions the strategic part of the model $(\mathcal{W}, p)$, which is captured by the vector $\mathbf{p} = (p, \ldots, p)$, is neutral since all voters have a common probability. Therefore, what really matters is the strength of the seats that they occupy. Next result follows from the previous corollary.

Corollary 5.10 *Let $\mathcal{W}$ be a weakly complete simple game and $\mathbf{p} = (p, \ldots, p)$ with $p \in (0, 1)$. Then, the following items are equivalent:*

a) $1 \succsim_{\mathcal{W}} 2 \succsim_{\mathcal{W}} \cdots \succsim_{\mathcal{W}} n$,

b) $\Phi_1(\mathcal{W}, p) \geq \Phi_2(\mathcal{W}, p) \geq \cdots \geq \Phi_n(\mathcal{W}, p)$,

c) $\Omega_1(\mathcal{W}, p) \geq \Omega_2(\mathcal{W}, p) \geq \cdots \geq \Omega_n(\mathcal{W}, p)$,

d) $\Lambda_1(\mathcal{W}, p) \leq \Lambda_2(\mathcal{W}, p) \leq \cdots \leq \Lambda_n(\mathcal{W}, p)$.

As a consequence of this corollary, the decisiveness and the success indices rank the voters in the same way as the weak desirability relation in any weakly complete game $\mathcal{W}$, for any $p \in (0, 1)$, while luckiness rank them in an opposite way. That is, the less significant is the seat of a voter the more lucky the voter is.

6 Conclusions

In this paper we use the multilinear extension of simple games to study three power indices in assessed simple games. An assessed simple game $(N, \mathcal{W}, \mathbf{p})$ is defined by a simple game $(N, \mathcal{W})$, which model a voting scenario, and a vector $\mathbf{p} = (p_1, \ldots, p_n) \in (0,1)^n$. Each component p_i is the assumed a priori probability that voter $i \in N$ votes in favor of a proposal, independently of each other. The multilinear extension turns out to be a useful tool to compute the indices of decisiveness, success and luckiness in assessed simple games, and to establish some new properties covering the issues of ordinal equivalence and egalitarianism.

In particular, the rankings among voters given by the success index Ω, the decisiveness index Φ and the luckiness index Λ in $(N, \mathcal{W}, \mathbf{p})$, are compared by using the multilinear extension $f_{\mathcal{W}}$ of the game $(N, \mathcal{W})$. In Theorem 4.4 it is proved that if two voters i and j have the same probability of voting 'yes', i.e., $p_i = p_j$ in the vector $\mathbf{p}$, then decisiveness and success indices rank them in the same way in $(N, \mathcal{W}, \mathbf{p})$ for any simple game $\mathcal{W}$, while luckiness rank them in the reverse order. It is also proved that, for $n > 2$, if $p_i \neq p_j$ we can always find games for which this relationship between rankings is not maintained. From these results we prove that success and decisiveness are ordinally equivalent in $(N, \mathcal{W}, \mathbf{p})$ for any $\mathcal{W}$ if and only if $\mathbf{p}$ is anonymous. And it is also proved that anonymous probability distributions are the only ones for which success (decisiveness) is ordinally opposite to luckiness in any game.

In the anonymous case, the egalitarianism of the three studied indices is compared and we prove that, for any pair of these indices, one of them is a k–reduction of the other for some k. It is also proved that the common ranking given by success and decisiveness coincides with the ranking given by the weak desirability relation in weakly complete simple games, that is to say, the success or the decisiveness of voters depend only on their position in the game. Instead, luckiness give a ranking of voters opposite to that given by the weak desirability relation.

Weakly complete games may not be the only kind of games for which all anonymous and independent probability distributions induce the same ranking among the voters. An open question to be studied is to characterize the simple games with this property.

References

Alonso-Meijide J, Casas-Méndez B, Holler M, Lorenzo-Freire S (2008) Computing power indices: Multilinear extensions and new characterizations. European Journal of Operational Research 188(2):540–554

Banzhaf J (1965) Weighted voting doesn't work: a mathematical analysis. Rutgers Law Review 19(2):317–343

Barry B (1980a) Is it better to be powerfull or lucky? part i. Political Studies 28(2):183–194

Barry B (1980b) Is it better to be powerfull or lucky? part ii. Political Studies 28(3):338–352

Berg S (1999) On voting power indices and a class of probability ddistribution: with applications to EU data. Group Decision and Negotiation 8:17–31

Berg S, Marañón A (2001) Collective decisional skill and decisive voting games. Group Decision and Negotiation 10:389–403

Brams S, Lake M (1978) Power and satisfaction in a representative democracy. In: Ordeshook P (ed) Game Theory and Political Science, New York University Press, pp 529–562

Van der Brink R, Steffen F (2014) On the measurement of success and satisfaction. In: Fara R, Leech D, Salles M (eds) Voting power and procedures, Springer, pp 41–64

Carreras F (2004) α-Decisiveness in simple games. Theory and Decision 56(1-2):77–91

Carreras F (2005) A decisiveness index for simple games. European Journal of Operational Research 163(2):370–387

Carreras F, Freixas J (2008) On ordinal equivalence of power measures given by regular semivalues. Mathematical Social Sciences 55:221–234

Davis M, Straffin P, Brams S (1982) Power and satisfaction in an ideologically divided voting body. In: Holler M (ed) Power, voting and voting power, Physica Verlag, Warzburg–Wien, pp 239–253

Deegan J, Packel E (1978) A new index of power for simple n-person games. International Journal of Game Theory 7:113–123

Diffo Lambo L, Moulen J (2002) Ordinal equivalence of power notions in voting games. Theory and Decision 53(4):313–325

Dubey P, Shapley L (1979) Mathematical properties of the Banzhaf power index. Mathematics of Operations Research 4(2):99–131

Freixas J (2010) On ordinal equivalence of the Shapley and Banzhaf values for cooperative games. International Journal of Game Theory 39(4):513–527

Freixas J, Pons M (2005) Two measures of circumstantial power: influences and bribes. Homo Oeconomicus 22(4):569–588

Freixas J, Pons M (2008) Circumstantial power: optimal persuadable voters. European Journal of Operational Research 186(3):1114–1126

Freixas J, Pons M (2015) Success and decisiveness on proper symmetric games. Central European Journal of Operations Research 23(4):779–794

Freixas J, Marciniak D, Pons M (2012) On the ordinal equivalence of the Johnston, Banzhaf and Shapley power indices. European Journal of Operational Research 216(2):367–375

Holler M, Packel E (1983) Power, luck and the right index. Journal of Economics 43:21–29

Isbell J (1958) A class of simple games. Duke Mathematics Journal 25(3):423–439

Johnston R (1978) On the measurement of power: Some reactions to Laver. Environment and Planning A 10(8):907–914

König T, Bräuninger T (1998) The inclusiveness of European decision rules. Journal of Theoretical Politics 10(1):125–142

Laruelle A, Valenciano F (2005) Assessing success and decisiveness in voting situations. Social Choice and Welfare 24(1):171–197

Laruelle A, Valenciano F (2008) Voting and Collective Decision–Making. Cambridge University Press

Laruelle A, Martínez R, Valenciano F (2006) Success versus decisiveness, conceptual discussion and case study. Journal of Theoretical Politics 18(2):185–205

Owen G (1972) Multilinear extensions of games. Management Science 18(5):64–79

Owen G (1975) Multilinear extensions and the Banzhaf value. Naval Research Logistics Quarterly 22(4):741–750

Rae D (1969) Decision rules and individual values in constitutional choice. American Political Science Review 63(1):40–56

Shapley L (1953) A value for n-person games. In: Tucker A, Kuhn H (eds) Contributions to the theory of games II, Princeton University Press, Princeton, USA, pp 307–317

Shapley L, Shubik M (1954) A method for evaluating the distribution of power in a committee system. American Political Science Review 48(3):787–792

Straffin P (1994) Power and stability in politics. In: SHart PA (ed) Handbook of Game Theory with Economic Applications, vol 2, Elsevier, pp 1127–1151

Wilson M, Pritchard G (2007) Probability calculations under the IAC hypothesis. Mathematical Social Sciences 54:244–256