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STARK POINTS AND UNITS

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La conciencia no es un hotel de lujo, sino una pensión barata junto a una frontera.
Luis García Montero.

To my parents, for their love and support, and to Marta Pita, for so many Skype calls that have turned Boston into a neighborhood of Barcelona.

INTRODUCTION

During the last years, a great progress in the study of BSD and Block-Kato conjectures has been made. Here, we are interested in two different aspects: on the one hand, the conjecture suggested by Darmon, Lauder and Rotger in [DLR2] relating the value of a p-adic iterated integral (whose value may be encoded in one of Hida's p-adic L-functions) with Stark's units and the value of regulators defined in terms of p-adic logarithms of units in number fields (this can also be formulated for points over elliptic curves and gives interesting results for BSD in analytic rank 2, as it is explained in [DLR1]). On the other hand, several works of Bertolini, Darmon, Rotger and others (see for instance [DR2], [BDR2], [BCDDPR] or [KLZ]) allow us to construct families of cohomology classes satisfying good properties and related with special values of the p-adic L-functions. These classes are obtained via the image through étale and syntomic regulators of distinguished cycles in certain algebraic varieties.
In our work, we use the construction of families of cohomology classes interpolating two cuspidal forms to formulate a conjecture about the good behavior of these Kato classes and their relation with Ohta periods, that would imply the main result suggested in the paper of Darmon, Lauder and Rotger. These results are availabe in the forthcoming paper [RiRo].

However, the introduction of all these concepts is a complex matter that requires a solid background. For that reason, this thesis is structured in two parts: the first one, formed by the four first chapters, covers general facts about p-adic L-functions, overconvergent modular forms, Hida families and units in number fields and Stark conjectures, but always from the perspective of our future objectives. The second one includes first a motivation for the theory of Euler systems, including a review of Galois cohomology and p-adic Hodge theory. Then, in chapter 6 (that is maybe the core of this thesis), we explain the different constructions of cohomology classes that have been performed around these Euler systems of Rankin-Selberg type: Beilinson-Kato elements, Beilinson-Flach elements and Gross-Kudla-Schoen cycles; we focus on the connection with p-adic L-functions, in the possibility of interpolating these classes p-adically and we emphasize its importance in the study of BSD conjecture. Chapter 7 is a presentation of the material covered in [DLR1] and [DLR2], where the Elliptic Stark conjectures for points in elliptic curves and for units in number fields are properly formulated. Then, chapter 8 summarizes the main achievements of the preprint [RiRo], where some theoretical arguments relying on the good properties of the Beilinson-Flach elements are given to support the main conjecuture of [DLR2].

We will give a short overview of what our project represents, emphasizing our main contributions, that are collected basically at the end of chapter 6 and in chapter 8.
Along the last years, there have been great advances in the study of the so-called Euler systems of Garrett-Rankin-Selberg type. The most common situation is concerned with a triple (f, g, h) of eigenforms of weights k, l and m respectively with $k = l + m + 2r$, and $r \geq 0$, that involves the Petersson scalar product $I(f, g, h) := \langle f, g \times \delta_m^r h \rangle$ and relates the square of this quantity to the central critical value $L\big(f \otimes g \otimes h, \frac{k+l+m-2}{2}\big)$ of the convolution L-function. In almost all the situations that have been studied, f is a weight two modular form coming from an elliptic curve, and this provides satisfactory results for the study of the BSD conjecture (see [BDR2] and [DR2]), thanks to the

introduction of global cohomology classes varying in families that are related with families of p-adic L-series (Hida families). Let us recall some examples in the spirit of [BCDDPR]:

- When f, g and h are all cusp forms, the L-functoin is related to $\mathrm{AJ}_p(\Delta)(\eta_f \wedge \omega_g \wedge \omega_h)$, where $\mathrm{AJ}_p : \mathrm{CH}^2(X_1(N)^3)_0 \to \mathrm{Fil}^2(H^3_{\mathrm{dR}}(X_1(N)^3))^*$ is the p-adic Abel-Jacobi map, Δ is the Gross-Kudla-Schoen cycle and $\eta_f, \omega_g, \omega_h$ are suitable classes in the de Rham cohomology of the modular curves (the two latter belonging to the middle step of the Hodge filtration). In this case, the corresponding p-adic L-function is Hida-Harris-Tilouine triple product p-adic L-function and the p-adic Gross-Kudla formula as stated in [DR1] allows to prove the equivariant BSD conjecture for $\rho_1 \otimes \rho_2$, being ρ_1 and ρ_2 odd, irreducible, two-dimensional Artin representations of $\mathbb{Q}$ such that the tensor product has trivial determinant.

- When h is an Eisenstein series (and the other two are cuspidal), the geometric ingredient that comes into play now is $\mathrm{reg}_p(\Delta_{u_h})(\eta_f \wedge \omega_g)$, where Δ_{u_h} is a Beilinson-Flach element in $\mathrm{CH}^2(X_1(N)^2, 1)$ attached to the modular unit u_h viewed as a function on a diagonally embedded copy of $X_1(N) \subset X_1(N)^2$. Here, the link is made through Hida's p-adic Rankin L-function and it allows to prove equivariant BSD conjecture in analytic rank zero when ρ is an odd, irreducible, two-dimensional Artin representation. As before, the key is the construction of families of cohomology classes obtained via the Beilinson-Flach elements, varying p-adically.

- When g and h are Eisenstein series, the Mazur-Swinertonn-Dyer p-adic L-function is related with the regulator $\mathrm{reg}_p\{u_g, u_h\}(\eta_f)$, where u_g and u_h are the modular units whose logarithmic derivatives are equal to g and h. This p-adic regulator has a counterpart in p-adic étale cohomology, leading to a system of global cohomology classes.

The study of these different Euler systems lead to the formulation of the Elliptic Stark conjecture in [DLR1], relating the value of p-adic iterated integrals (that may be also seen in terms of p-adic L-functions) with a regulator defined in terms of global points in an elliptic curve. The authors provide numerical evidence and then, in [DR3], some theoretical evidence based on the good behavior of families of cohomology classes is given.

However, what has not been explored with so much deep is the case in which f is not a cuspidal form attached to an elliptic curve, say the case in which f is an Eisenstein series. This will allow us to change the setting of elliptic curves by that of units in number fields, as it is done for instance in [DLR2], where an analogue of the Elliptic Stark conjecture is formulated precisely for units in number fields. Therefore, I believe that the main novelty of this thesis is the exploration of this less well-known setting not related with elliptic curves, but with other arithmetic objects. Here, we plan to give some theoretical support to the main conjecture of [DLR2], via the study of the corresponding generalized cohomology classes, that will rest mainly on the results of [BDR2] and [KLZ] relating the image of these classes under Perrin-Riou big logarithm with special values of Hida's three variable p-adic L-function. In the setting of [BDR2], the situation consisted on the convolution product $f \otimes \mathbf{g}$, where f is the cuspidal eigenform attached to an elliptic curves and $\mathbf{g}$ is a Hida family interpolating a weight one modular form. Then, the works of [LLZ] and [KLZ] allow us to go a step beyond and consider a product of the form $\mathbf{g}\otimes\mathbf{h}$, where both $\mathbf{g}$ and $\mathbf{h}$ are Hida families interpolating weight one modular forms.

Since the topics covered in chapter 8 are still in progress, I plan to improve this part along the following months, towards having a clearer understanding of all the deep concepts involved there.

I would like to thank all the people from the Number Theory group in Barcelona for the good environment they have created and all the seminars and conferences we have organized. Thanks to Daniel Barrera, Adel Betina, Francesc Fité, Francesca Gatti, Xevi Guitart, Marc Masdeu, Santi Molina, Edu Soto, Xavier Xarles and many others for all the meetings discussing Hida families, overconvergent modular symbols or p-adic methods; thanks also to all the “senior” professors (that is not the same than old) that have made possible the wonderful Number Theory group at UPC: to Jordi Quer, Jordi Guàrdia, Joan Carles Lario, Luis Dieulefait, Pilar Bayer and a long etcetera. And last, but not least, thanks to Víctor Rotger, for convincing me each day that number theory is the most amazing branch of mathematics and for all the days we will work together in my forthcoming PhD thesis and the papers we will produce.

Contents

1 p-adic L-functions

There is not a unique approach to p-adic L-functions. Spivak, when writing about the multiple definitions of riemannian connections available in the literature, defended that the next one proposing an alternative one should be summarily executed. I believe that the same happens for the construction of p-adic L-functions. There are so many approaches that it is difficult to do a clear overview of the most relevant features in the topic. In this chapter, we will introduce the classical notion, based on the interpolation of the classical zeta function, and then we will construct the L-series of an elliptic curve (à la MTT). However, we will try to emphasize the different variants and possibilities one may have.
At the end, the main idea should be that, as classical L-functions are analytic objects encoding information about certain representations of the Galois group of $\mathbb{Q}$ (or a general number field), p-adic L-functions interpolate these representations in families. We will return to this topic in chapter 3 when discussing Hida families.

1.1 A first classical approach

We assume that the reader is familiarized with the basis of p-adic distributions and p-adic measures. For more references, see my expository notes *Distribuciones, medidas y álgebras de Tate* [R1] (in Spanish). There, and also in Koblitz [Ko], it is explained how to continuously interpolate $f(s) = n^s$ where $s \in \mathbb{Z}_p$ (this turns out to be also a very good motivation for the construction of the p-adic L-function and in general to illustrate the different patologies one finds when working p-adically); this can be done provided that n is a p-adic unit, showing first that $|n^s - n^{s'}|_p$ converges to zero when s and s' are congruent modulo $p-1$, but this suffices since

$$A_{s_0} := \{s \in \mathbb{Z}, s > 0 \mid s \equiv s_0 \pmod{p-1}\}$$

is dense in $\mathbb{Z}_p$.
With this in mind, our first aim is to give a construction of the so-called Kubota-Leopoldt p-adic zeta function ζ_p. In a first trial, one can think in imposing the interpolation property $\zeta_p(1-k) = \zeta(1-k)$, for all $k \in \mathbb{Z}^{>0}$. However, this does not work: one must remove the p-th Euler factor and focus only on those integer that are 0 modulo $k-1$. This is quite similar to the classical construction of the p-adic Gamma function, in which one interpolates the factorial deleting the multiples of p and doing a correction of sign. In other words, we want

$$\zeta_p(1-k) = (1-p^{k-1})\zeta(1-k), \qquad \text{for all } k \equiv 0 \mod (p-1).$$

For this process, it is convenient to recall that $\zeta(s)$ is the Mellin transform of the measure $dx/(e^x-1)$, or what is the same

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{dx}{e^x - 1}.$$

Further, the special values at the negative integers can be expressed in terms of Bernoulli numbers: for $k \in \mathbb{Z}^{>0}$,

$$\zeta(1-k) = -\frac{B_k}{k}.$$

In fact, we will be interested in a more general case, in which $\chi : (\mathbb{Z}/p^m\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ is a Dirichlet character (that can be extended to the whole $\mathbb{Z}$). For the sake of convenience, we will fix from now on an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}_p$. The L-function of χ, when $\Re(s) > 1$ is given by

$$L(s, \chi) = \prod_l \Big(1 - \frac{\chi(l)}{l^s}\Big).$$

When $k > 0$ is a positive integer, $L(1-k, \chi) = -\frac{B_{k,\chi}}{k}$, being $B_{k,\chi}$ the generalized Bernoulli numbers.

At the end, the result that interests us the most is the following one, that holds for general L-series:

Theorem 1 (Kubota-Leopold,Iwasawa)**.** *There exists a unique p-adic meromorphic (and analytic when χ is non-trivial) function $L_p(s, \chi)$, $s \in \mathbb{Z}_p$ such that for $k \in \mathbb{Z}^{>0}$,*

$$L_p(1-k, \chi) = L(1-k, \chi\omega^{-k}),$$

being ω the Teichmüller character.

For proving this, one must introduce different p-adic measures. In [Gui], it is possible to find a detailed discussion of this, but for our interests, we will just say that the properties satisfied by Bernoulli polynomials allow to define first a certain family of distributions and then, by a process called regularization, we can pass to measures (bounded distributions). We will refer to these measures as $\mu_{k,\alpha}$, where α is a certain p-adic number used in the regularization process and k is a positive integer. Namely, μ_k is defined in any ball as

$$\mu_k(a + (p^n)) := p^{n(k-1)} B_k\Big(\frac{a}{p^n}\Big),$$

for some $a \in \{0, 1, \ldots, p^n - 1\}$. μ_k extends to a distribution on $\mathbb{Z}_p$. When α is any integer not equal to one and such that p does not divide it, we define the k-th regularized Bernoulli distribution as

$$\mu_{k,\alpha}(U) = \mu_k(U) - \alpha^{-k}\mu_k(\alpha U).$$

$\mu_{k,\alpha}$ turns out to be a measure for $k \geq 1$. With this machinery, we can move into the construction of the p-adic L-function, that we sketch here first in the case of $\zeta_p(s)$.

Definition 1. *Let α be a rational integer that is not equal to one and not divisible by p. For any positive integer k, we define*

$$\zeta_p(1-k) = \frac{1}{\alpha^{-k} - 1} \int_{\mathbb{Z}_p^\times} x^{k-1} \mu_{1,\alpha}.$$

The proof that it is well-defined is not difficult using the following result whose proof is also in [Ko]. Basically, it states that the measures $\mu_{k,\alpha}$ satisfy some compatibility relations; in particular, when k is a positive integer and X is a compact-open in $\mathbb{Z}_p$, then,

$$\int_X \mu_{k,\alpha} = k \int_X x^{k-1} \mu_{1,\alpha}.$$

As we have mentioned, this is closely related with Bernoulli numbers, satisfying certain congruences properties of the type of those that frequently arise in the theory of modular forms. In particular, we will recover these congruences when studying Hida families.

Proposition 1. *$\zeta_p(1-k) = (1-p^{k-1})\Big(-\frac{B_k}{k}\Big)$, where B_k is the k-th Bernoulli number. Further, if $(p-1) \nmid k$, B_k/k is a p-adic integer and when we impose that $k \equiv k'$ $\pmod{(p-1)p^N}$ then*

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k'-1})\frac{B_{k'}}{k'} \pmod{p^{N+1}}.$$

Finally, we will fix $s_0 \in \{0, 1, \dots, p-2\}$. For a p-adic integer s there exists a sequence of positive integers $\{t_i\}_{i=1}^{\infty}$ converging to s, say $t_i = \sum_{j=0}^{i} a_j p^j$ if $s = \sum_{j=0}^{\infty} a_j p^j$. Therefore, unless s and s_0 are both zero, the following limit makes sense:

$$\zeta_{p,s_0}(s) := \lim_{i\to\infty}(1-p^{s_0+(p-1)t_i-1})\Big(\frac{-B_{s_0+(p-1)t_i}}{s_0+(p-1)t_i}\Big).$$

Hence, we can do the following definition:

Definition 2. *For $\alpha \in \mathbb{Z}$, with $\alpha \neq 1$ and $p \nmid \alpha$, and for fixed $s_0 \in \{0, 1, \dots, p-2\}$, we define*

$$\zeta_{p,s_0}(s) := \frac{1}{\alpha^{-(s_0+(p-1)s)}-1}\int_{\mathbb{Z}_p^\times} x^{s_0+(p-1)s-1}\mu_{1,\alpha},$$

for a p-adic integer s, except at $s=0$ when $s_0=0$.

It can be shown that $\zeta_{p,s_0}(k)$ is continuous (provided that s_0 and s are not both zero). Furthermore, it is true that $\zeta_p(1-k) = \zeta_{p,s_0}(k_0)$ where $k = s_0 + (p-1)k_0$, and $k_0 \in \mathbb{Z}$ with $k_0 > 0$. We also note that $\zeta_p(t)$ has a pole at $t=1$ by taking $k=0$ in $\zeta_p(1-k) = \zeta_{p,s_0}(k_0)$.

With this in mind, we would like to finally introduce general p-adic L-functions (this zeta function can be seen as the L-function for the trivial character). Towards, this purpose, we will use some of the lemmas stated in the expository paper [R1] and also in [Ko]. We content here with a brief remark to introduce some language that will be useful in next sections: let q be an integer that is equal to p if p is odd and equals 4 when $p=2$. Let a be an element in $\mathbb{Z}_p^\times$ and put $a = \sum_{i=0}^{\infty}\alpha_i p^i$. Assume first that $p \neq 2$. Since $\alpha_0^{p-1} \equiv 1 \pmod{p}$, we can identify α_0 with a primitive $p-1$-th root of unity and call $\omega(a)$ this representative. If $p=2$, $a = 1 + \alpha_1\cdot 2 + \sum_{i=2}^{\infty}\alpha_i 2^i$ with $\alpha_i \in \{0,1\}$, so we can identify $1+\alpha_1\cdot 2$ with $\{\pm 1\}$. Let $\langle a\rangle := a/\omega(a)$, so any p-adic unit can be written as $\omega(a)\langle a\rangle$. It is customary to put $\chi_k = \chi\cdot\omega^{-k}$.
Let $K = \mathbb{Q}_p(\chi)$ and define

$$b_k := (1-\chi_k(p)p^{k-1})B_{k,\chi_k} \text{ and } c_k := \sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}b_i.$$

Let $A_\chi(x) := \sum_{n=0}^{\infty} c_n\binom{x}{n}$. We are using the standard definition of

$$B_{k,\chi} = f^{k-1}\sum_{a=1}^{f}\chi(a)B_k\Big(\frac{a}{f}\Big),$$

where $B_k(x)$ is the k-th Bernoulli polynomial.

Definition 3. *The p-adic Dirichlet L-function is*

$$L_p(s,\chi) := \frac{1}{s-1}A_\chi(1-s).$$

The fact this is well defined and converges in $\{s \in \overline{\mathbb{Q}_p} \mid |s-1|_p < (p^{1/(p-1)})^{-1}|q|_p\}$ is also proved in the expository paper [R1]. Further, the following interpolation property is satisfied:

Proposition 2. *For a Dirichlet character χ and a positive integer k,*

$$L_p(1-k,\chi) = (1-\chi_k(p)p^{k-1})\Big(-\frac{B_{k,\chi_k}}{k}\Big).$$

This construction does not reveal the role played by these p-adic distributions, since everything is encoded in these ubiquous numbers, the so-called b_k and c_k. We refer the reader to any of the references of a deeper treatment, seeing for instance the relation with the p-adic Mellin transform.

In [Was] there is a construction of the p-adic L-function using the so called Stickelberger elements. It is a more conceptual approach, in which one must carefully use the properties of the cyclotomic extensions $K_n = \mathbb{Q}(\zeta_{q_n})$, where $q_n = dp^{n+1}$ for d coprime with p. In this setting, one can define in a natural way an element in the group ring, $\theta_n \in \mathbb{Q}_p[G_n]$. This element, together with some standard class field theory, allows us to define $L_p(s,\chi)$.

1.2 The p-adic L-function of an elliptic curve

In 1986, Mazur, Tate and Teitelbaum published in Inventione a paper called "On p-adic analogues of the conjectures of Birch and Swinertonn-Dyer" [MTT]. They comment in the introduction that since the p-adic analogue of the Hasse-Weil L-functon had been defined and also p-adic theories analogous to the theory of canonical height had been introduced, "it seemed to us to be an appropriate time to embark on the project of formulating a p-adic analogue of the conjecture of Birch and Swinertonn-Dyer, and gathering numerical data in its support [...] The project has proved to be anything but routine".
For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, write $\rho(A,z) = \rho(A) = \frac{(\det A)^{1/2}}{cz+d}$. Further, observe that if we denote by S_k the space of cuspidal modular forms of weight k and by $S_k(N,\epsilon)$ those of level N and nebentypus ϵ, $S_k = \oplus S_k(N,\epsilon)$, and in these spaces we have an action of $\mathrm{GL}_2(\mathbb{R})^+$ given by $(f|A)(\tau) = \rho(A)^k f(A\tau)$. Denote by $P_k(\mathbb{C})$ the space of polynomials of degree at most $k-2$; again, we can consider an action of $\mathrm{GL}_2(\mathbb{R})^+$ given by $(P|A)(z) = \rho(A)^{2-k}P(A(z))$. Let $\Delta = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ with the canonical action of $\mathrm{GL}_2(\mathbb{R})$. Finally, observe that $d(A\tau) = d\tau\rho(A)^2$ and therefore

$$(f|A)(P|A)d\tau = f(A(\tau))P(A(\tau))d(A\tau).$$

This gives us an application

$$\Phi : S_k \to \mathrm{Hom}_{\mathbb{Z}}(\Delta, P_k(\mathbb{C})^*),$$

sending f to the homomorphism such that, given $((a)-(b))$ maps it to an element acting on P as

$$\int_a^b f(\tau)P(\tau)d\tau.$$

We can also see this as a map

$$\phi : S_k \times P_k(\mathbb{C}) \times \mathbb{P}_1(\mathbb{Q}) \to \mathbb{C},$$

given by the formula

$$\phi(f,P,r) = 2\pi i \int_{\infty}^{r} f(z)P(z)dz = \begin{cases} 2\pi \int_0^\infty f(r+it)P(r+it)dt, & \text{if } r \in \mathbb{Q}, \\ 0, & \text{if } r = \infty. \end{cases}$$

If ϵ is a complex Dirichlet character, let $\mathbb{Z}[\epsilon] \subset \mathbb{C}$ be the subring of $\mathbb{C}$ generated by the values of ϵ. Let $A_j \in \mathrm{SL}_2(\mathbb{Z})$ be a coset of representatives of $\Gamma_0(N)$ in such a way that

$$\mathrm{SL}_2(\mathbb{Z}) = \sqcup_{j \in R} \Gamma_0(N) \cdot A_j,$$

where R is a finite set. Fix $f \in S_k$ and let $L_f \subset V$ be the $\mathbb{Z}$-module generated by the image of ϕ_f.

Proposition 3. *The $\mathbb{Z}$-module L_f is the $\mathbb{Z}[\epsilon]$-submodule of V generated by the elements*

$$\Phi(f)(A_i((\infty) - (0)))(z^j),$$

where $0 \le j \le k-2$, $i \in R$.

In general, we say that a modular integral Φ is a mapping

$$\Phi : S_k \times P_k(\mathbb{C}) \times \mathbb{P}_1(\mathbb{Q}) \to V,$$

where V is a complex vector space, and such that Φ is $\mathbb{C}$-bilinear in f and P and

$$\Phi(f|A, P|A, r) = \Phi(f, P, A(r)) - \Phi(f, P, A(\infty)).$$

The modular integral Φ can be used to define a modular symbol λ. For $a, m \in \mathbb{Q}$, $m > 0$, $f \in S_k$ and $P \in P_k(\mathbb{C})$, let

$$\begin{aligned} \lambda(f,P;a,m) &:= \Phi(f, P(mz+a), -a/m) \\ &:= m^{k/2-1}\Phi\Big(f, P\Big|\begin{pmatrix} m & a \\ 0 & 1 \end{pmatrix}, -a/m\Big) \\ &:= m^{k/2-1}\Phi\Big(f\Big|\begin{pmatrix} 1 & -a \\ 0 & m \end{pmatrix}, P, 0\Big). \end{aligned}$$

Proposition 4. *The modular symbol $\lambda(f,P;a,m)$ is $\mathbb{C}$-bilinear in (f,P). For fixed f, the modular symbol $\lambda_f(P;a,m)$ takes values in L_f. For fixed f and P, $\lambda_{f,P}(a,m)$ depends only on a modulo m.*

Recall that if $f \in S_k(N,\epsilon)$, for a prime l,

$$f|T_l := l^{k/2-1}\Big(\sum_{u=0}^{l-1} f\Big|\begin{pmatrix} 1 & u \\ 0 & l \end{pmatrix} + \epsilon(l)\cdot f\Big|\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\Big).$$

Proposition 5. *For $f \in S_k(N,\epsilon)$ and for any prime l, we have*

$$\lambda(f|T_l, P; a, m) = \sum_{u=0}^{l-1} \lambda(f, P; a-um, lm) + \epsilon(l) l^{k-2} \lambda(f, P; a, m/l).$$

Recall that when $f \in S_k(\epsilon)$ has Fourier expansion given by $\sum_{n\geq 1} a_n q^n$, its L-function is

$$L(f,s) := \sum_{n\geq 1} a_n n^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(it) t^s (dt/t).$$

Then,

$$\lambda(f, z^n, 0, 1) = \phi(f, z^n, 0) = i^n \frac{n!}{(2\pi)^n} L(f, n+1), \qquad \text{for } 0 \leq n \leq k-2.$$

We also define

$$f_\chi(z) := \sum_n \chi(n) a_n q^n.$$

In particular, when χ is primitive, we have Birch's lemma:

$$f_{\bar{\chi}}(z) = \frac{1}{\tau(\chi)} \sum_{a \bmod m} \chi(a) f\Big(z + \frac{a}{m}\Big).$$

Then,

$$\begin{aligned}\phi(f_{\bar{\chi}}, P, r) &= \frac{1}{\tau(\chi)} \sum_{a \bmod m} \chi(a) \phi\Big(f \Big| \begin{pmatrix} 1 & a/m \\ 0 & 1 \end{pmatrix}, P, r\Big) \\ &= \frac{1}{\tau(\chi)} \sum_{a \bmod m} \chi(a) \phi\Big(f, P \Big| \begin{pmatrix} 1 & -a/m \\ 0 & 1 \end{pmatrix}, r + a/m\Big)\end{aligned}$$

when χ is primitive. For the modular symbol,

$$\lambda(f_{\bar{\chi}}(z), P(mz); b, n) = \frac{1}{\tau(\chi)} \sum_{a \bmod m} \chi(a) \lambda(f, P; mb - na, mn),$$

for χ primitive modulo m.
When we put $b = 0$ and $n = 1$ we have for $0 \leq n \leq k-2$ an expression of the special values of the L-function of all twists of f in terms of the modular symbols for f, that is

$$L(f_{\bar{\chi}}, n+1) = \frac{1}{n!} \frac{(-2\pi i)^n}{m^{n+1}} \tau(\bar{\chi}) \sum_{a \bmod m} \chi(a) \lambda(f, z^n; a, m).$$

We will now mimic the process of constructing the p-adic L-function of a Dirichlet character for the case of modular forms.
Let p be a prime number. Let $f \in S_k(N, \epsilon)$ be an eigenform for T_p with eigenvalue a_p. Suppose that $X^2 - a_p X + \epsilon(p) p^{k-1}$ has a non-zero root. Let $v(m) = \mathrm{ord}_p(m)$. Then, define

$$\mu_{f,\alpha}(P; a, m) = \frac{1}{\alpha^{v(m)}} \lambda_{f,P}(a, m) - \frac{\epsilon(p) p^{k-2}}{\alpha^{v(m)+1}} \lambda_{f,P}(a, m/p),$$

for $a, m \in \mathbb{Z}$ and $m > 0$.

Proposition 6. *For $a, m \in \mathbb{Z}$ and $m > 0$ we have that*

$$\sum_{\substack{b \equiv a \bmod m \\ b \bmod pm}} \mu_{f,\alpha}(P; b, pm) = \mu_{f,\alpha}(P; a, m).$$

Further, if ψ is a Dirichlet character with conductor M relatively prime to p, for n prime to M,

$$\mu_{f\bar{\psi}, \alpha\bar{\psi}(p)}(P(Mz); b, n) = \frac{\psi(p)^{v(n)}}{\tau(\psi)} \sum_{a \bmod M} \psi(a) \mu_{f,\alpha}(P, Mb - na, Mn).$$

Fix now $M > 0$ prime to p. Set $\mathbb{Z}_{p,M} = \mathbb{Z}_p \times (\mathbb{Z}/M\mathbb{Z})$. We can view $\mathbb{Z}_{p,M}^\times$ as a p-adic analytic Lie group with a fundamental system of open disks

$$D(a,\nu) := a + p^\nu M\mathbb{Z}_{p,M},$$

where $n \geq 1$.
Let $\mathcal{O}_p \subset \mathbb{C}_p$ be the ring of integers. Fix a modular form $f \in S_k(N,\epsilon)$ and consider the finite dimensional $\mathbb{C}_p$-vector space $V_f := \mathbb{C}_p \otimes_{\bar{\mathbb{Q}}} L_f\bar{\mathbb{Q}}$ and the $\mathcal{O}_p$-lattice $\Omega_f \subset V_f$ generated by L_f. We can extend the definitions of $\phi(f,P,r), \lambda(f,P;a,m)$ and $\mu_{f,\alpha}(P;a,m)$ to the case where P has coefficients in $\mathbb{C}_p$ yielding values in V_f.
We would like to define now a V_f-valued integral $\int_U F$ where U is a compact open set of $\mathbb{Z}_{p,M}^\times$ and F is a locally analytic function on U. We expect, denoting by $x \mapsto x_p$ the projection of $\mathbb{Z}_{p,M}$ onto $\mathbb{Z}_p$,

$$\int_{D(a,\nu)} P(x_p) = \mu_{f,\alpha}(P, a, p^\nu M),$$

for $\nu, a \in \mathbb{Z}$, $\nu \geq 1$, $(a, pM) = 1$ and $P \in P_k(\mathbb{C}_p)$.

Theorem 2 (Vishik, Amice-Vélu)**.** *Fix an integer h such that $1 \leq h \leq k-1$. Suppose that the polynomial $X^2 - a_pX + \epsilon(p)p^{k-1}$ has a root $\alpha \in \mathbb{C}_p$ such that $\mathrm{ord}_p(\alpha) < h$. Then, there exists a unique V_f-valued integral satisfying the following axioms (with $\nu \geq 1$ and $a \in \mathbb{Z}$):*

1. *It is $\mathbb{C}_p$-linear in F and finitely additive in U.*
2. *(Evaluation on polynomials of small degree):*
$$\int_{D(a,\nu)} x_p^j = \mu_{f,\alpha}(z^j; a, p^\nu M), \qquad \textit{for } 0 \leq j < h.$$
3. *(Divisibility): For $n \geq 0$,*
$$\int_{D(a,\nu)} (x-a)_p^n \in \left(\frac{p^n}{\alpha}\right)^\nu \alpha^{-1}\Omega_f.$$
4. *If $F(x) = \sum_{n\geq 0} c_n(x-a)_p^n$ converges on $D(a,\nu)$, then*
$$\int_{D(a,\nu)} F = \sum_{n\geq 0} c_n \int_{D(a,\nu)} (x-a)_p^n.$$

Consider now a p-adic character (continuous homomorphism)

$$\chi : \mathbb{Z}_{p,M}^\times \to \mathbb{C}_p^\times,$$

for some p and M relatively prime. For example we can think of

$$\chi(x) = x_p^j\psi(x),$$

where $0 \leq j \leq k-2$ and ψ is a finite order character.
For $s \in \mathbb{Z}_p$, we can consider

$$\chi_s(x) := \langle x\rangle^s = \exp(s\log x) = \sum_{r=0}^\infty \frac{s^r}{r!}(\log\langle x\rangle)^r.$$

Definition 4. *If α is an allowable p-root of f, for a p-adic character χ we put*

$$L_p(f,\alpha,\chi) = \int_{\mathbb{Z}_{p,M}^{\times}} \chi d\mu_{f,\alpha},$$

were M is the p'-conductor of χ. Write $L_p(f,\alpha,\chi,s) := L_p(f,\alpha,\chi\chi_s)$.

Now, we just define the p-adic L-function of an elliptic curve as the p-adic L-function of the associated modular form.

The article of Mazur, Tate and Teitelbum then formulates a p-adic version of the BSD conjecture for $L_p(E,s)$, where in the multiplicative case the phenomenon of extra zeros arose. This means that whenever p is a prime of split multiplicative reduction for E, $L_p(E,s)$ vanishes to order $1+\mathrm{rank}(E(\mathbb{Q}))$, and that its leading term is a quantity that combines the p-adic regulator with the $\mathcal{L}$-invariant of $E/\mathbb{Q}_p$, defined by

$$\mathcal{L} = \frac{\log(q)}{\mathrm{ord}_p(q)},$$

being q Tate's p-adic period attached to $E/\mathbb{Q}_p$. A special case was proved by Greenberg and Stevens using Hida's theory.
The appearance of Tate's period in the derivative of $L_p(E,s)$ led Schneider to seek a purely p-adic analytic construction of $L_p(E,s)$ relying on a p-adic uniformization of E, $\mathbb{H}_p/\Gamma \to E(\mathbb{C}_p)$.
Motivated by this, Bertolini and Darmon did a parallel study in the anticyclotomic setting, in which the role of modular symbols is played by Heegner points attached to an imaginary quadratic field K. Then, Iovita and Spiess proposed a construction of the p-adic L-function following Schneider's approach. This clarified the role of p-adic integration and gave some insight into the obstruction that prevented Schneider's attemp from obtaining a satisfactory p-adic L-function in the cyclotomic setting.

1.3 The Rankin-Selberg method

We introduce in this section the Rankin-Selberg method, which provides results that will be very useful to explore Beillinson formulas that will be, in later chapters, expressed in a p-adic setting and will be very useful for our purposes.
We will begin by introducing the non-holomorphic Eisenstein series of weight 1, characters ψ and χ and parameter s, since it gives some hints around the ideas one must keep in mind. This Eiseinsten series can be analytically extended to a meromorphic function which is holomorphic when $\Re(s) > -1/2$, and thus we can obtain a modular form at $s=0$. Let ψ,χ be two Dirichlet characters modulo M and modulo N, respectively. For any integer $k \geq 1$, $z \in \mathbb{H}$, we put

$$\tilde{E}_k(z;\psi,\chi) = \sideset{}{'}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \frac{\psi(m)\chi(n)}{(mz+n)^k}.$$

The apostrophe means that the sum is over all pairs different from $(0,0)$. This series absolutely converges for $k \geq 3$, and we are interested in the case $k=1$.

Definition 5. *Let $z = x+iy \in \mathbb{H}$, $s \in \mathbb{C}$ and $k \in \mathbb{Z}$. We define*

$$\tilde{E}_k(z,s;\psi,\chi) = \sideset{}{'}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \frac{\psi(m)\chi(n)}{(mz+n)^k}\frac{y^s}{|mz+n|^{2s}}.$$

This function is called non-holomorphic Eisenstein series of weight k and characters ψ and χ.

When the characters are trivial, we frequently omit them in the notation. The series converges uniformly and absolutely for $k + 2\Re(s) \geq 2 + \epsilon$, so it is holomorphic when $\Re(s) > 1 - k/2$. But it is not holomorphic as a function of z. Let $\Gamma_0(M, N)$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ such that $c \equiv 0$ modulo N and $b \equiv 0$ modulo M. Then, $\Gamma_0(M, N)$ is a modular group, and for $\gamma \in \Gamma_0(M, N)$,

$$\tilde{E}_k(\gamma z, s; \psi, \chi) = \psi(d)\overline{\chi(d)}(cz + d)^k |cz + d|^{2s} \tilde{E}_1(z, s; \psi, \chi)$$

We assume that $\psi(-1)\chi(-1) = (-1)^k$ since elsewhere the value is 0. We want to study the case $k = 1$, but $\tilde{E}_1$ is not convergent at $s = 0$; however, it can be analytically continued.

Theorem 3. *The Eisenstein series $\tilde{E}_1(z, s; \psi, \chi)$ is analytically continued to a meromorphic function on the whole s-plane. If χ is non-trivial, $\tilde{E}_1(z, s; \psi, \chi)$ is an entire function, and elsewhere, it is holomorphic for $\Re(s) > -1/2$. At $s = 0$ we get a modular form of weight 1 for the modular group $\Gamma_0(M, N)$. Its Fourier expansion is given by*

$$\tilde{E}_1(z, s; \psi, \chi) = C + D + A \sum_{n=0}^{\infty} a_n q_N^n$$

The value at 0 will be 0 when ψ is the principal character and $2L(\chi, 1)$ elsewhere.
D is $-2\pi i L(\psi, 0) \prod_{p|M}(1 - p^{-1})$ when χ is trivial and 0 elsewhere. The other constants can be explicitely found in terms of the characters.

Definition 6. *Let χ be a character modulo N with $\chi(-1) = -1$. Let M be an integer such that $N|M$. Then, the non-holomorphic Eisenstein series of weight one, character χ and level M is*

$$\tilde{E}_1(z, s; \chi; M) := \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}}{}' \frac{\chi(n)}{Mmz + n} \frac{y^s}{|Mmz + n|^{2s}}.$$

We will write $\tilde{E}_1(z; \chi; N) := \tilde{E}_1(z, 0, \chi; N)$, that is an Einenstein series of weight one and character χ.

An almost immediate computation shows that

$$\tilde{E}_1(z, s; \chi; N) = \frac{1}{N^s} \tilde{E}_1(Nz, s; 1, \chi)$$

where 1 here is for the trivial character.
Consider $\tilde{E}_1(z, \chi, N) := \tilde{E}_1(z; 0, \chi; N)$, which is an Eisenstein series of weight one and character χ. It belongs to $M_1(\Gamma_0(N), \chi)$. We can explicitely compute its Fourier expansion as well as studying the functional equation it satisfies.
We introduce now the normalized Eisenstein series $E_1(z; \chi; N)$ that is related to $\tilde{E}_1(z; \chi; N)$ by the equation

$$\tilde{E}_1(z, s; \chi; N) = \frac{-4\pi i \tau(\chi)}{N} E_1(z, s; \chi; N).$$

Define also

$$\tilde{E}_1'(z,s;\chi;,N) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(Nm,n)=1}} \frac{\chi(n)}{Nmz+n}\frac{y^s}{|Nm'z+n'|^{2s}}.$$

Now, we we properly introduce the Rankin-Selberg method, that will make use of these different L-series we have presented. To motivate it, consider τ a continuous and irreducible finite-dimensional complex representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We can associate to τ an L-series $L(\tau,s)$; on the other hand, for an elliptic curve E we can consider the representation ρ_E associated to the p-adic Tate module of E in such a way that $L(\rho_E,s) = L(E,s)$. We can consider $L(\rho_E\otimes\tau,s)$, corresponding to the tensor product of the two representations. Rankin method will allow us to show that if τ arises from a modular form, then $L(\rho_E\otimes\tau,s)$ admits analytic continuation to $\mathbb{C}$.

The 2-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which are geometric are all expected to arise from modular forms, that is, if ρ is an odd 2-dimensional compatible system of l-adic representations, there is a modular form f and an integer j such that $L(\rho,s) = L(f,s+j)$. It is immediate that $L(V_1\oplus V_2,s) = L(V_1,s)L(V_2,s)$, but a much more interesting question is trying to construct the L-series corresponding to $V_1\otimes V_2$.

To begin with, let $f = \sum a_n q^n \in S_k(\Gamma_0(N),\chi_f)$ and $g = \sum b_n q^n \in S_l(\Gamma_0(N),\chi_g)$ be normalized eigenforms of level N. Assume that they are simultaneous eigenvectors for the Hecke operators T_r, with $(r,N)=1$, as well as the operators U_r attached to primes dividing N. Then, we can write

$$L(f,s) = \prod_{p|N}(1-a_pp^{-s})^{-1}\prod_{p\nmid N}(1-a_pp^{-s}+\chi_f(p)p^{k-1-2s})^{-1}$$

and consider $L_N(f,s) := \prod_{p\nmid N}(1-a_pp^{-s}+\chi_f(p)p^{k-1-2s})^{-1}$.
For each prime p, α_p and α_p' will be the roots of the Hecke polynomials $x^2-a_px+\chi_f(p)p^{k-1}$, and choose $(\alpha_p,\alpha_p') = (a_p,0)$ if $p|N$. Then,

$$L_N(f,s) = \prod_{p|N}(1-\alpha_pp^{-s})^{-1}(1-\alpha_p'p^{-s})^{-1}.$$

If $p|N$, $L_{(p)}(f,s) = (1-\alpha p^{-s})^{-1}$. We can shorten notations and put $L(f,s) = \prod_p L_{(p)}(f,s)$. We do an analogous treatment for g calling β_p and β_p' the corresponding roots.

Definition 7. *The Rankin L-series attached to (f,g) is $L(f\otimes g,s) = \prod_p L_{(p)}(f\otimes g,s)$, where*

$$L_{(p)}(f\otimes g,s) := (1-\alpha_p\beta_pp^{-s})^{-1}(1-\alpha_p\beta_p'p^{-s})^{-1}(1-\alpha_p'\beta_pp^{-s})^{-1}(1-\alpha_p'\beta_p'p^{-s})^{-1}.$$

Definition 8. *The modified Rankin function attached to f and g is defined by*

$$D(f,g,s) = \sum_{n=1}^{\infty}\frac{a_nb_n}{n^s}.$$

Considering $D_{(p)}(f,g,s) = \sum_{n=0}^{\infty} a_{p^n}b_{p^n}p^{-ns}$ and taking into account that $D(f,g,s)$ is absolutely convergent if $\Re(s) > (k+l)/2$, we can rearrange it and put

$$D(f,g,s) = \prod_p D_{(p)}(f,g,s).$$

Our aim is to relate now $D_{(p)}(f,g,s)$ and $L_{(p)}(f\otimes g,s)$.
We will now state some of the most remarkable theorems around these convolution products. Proofs are not rather conceptual and they are mostly computational. A good reference is [Sad].

Theorem 4. *Let $f \in S_k(\Gamma_0(N),\chi_f)$ and $g \in S_l(\Gamma_0(N),\chi_g)$. Then,*

$$L(f\otimes g,s) = L(\chi, 2s-k-l+2)D(f,g,s),$$

where $\chi = \chi_f\chi_g$. In particular, if $N=1$ and the characters are trivial, the L-factors corresponds to $\chi(2s-k-l+2)$.

Proposition 7. *Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_l(\mathrm{SL}_2(\mathbb{Z}))$. For $\Re(s) > 2-\frac{k-l}{2}$ we have that*

$$\langle \tilde{E}'_{k-l}(z,s)g(z), f^*(z)\rangle_k = \frac{2\Gamma(k+s-1)}{(4\pi)^{k+s-1}}D(f,g,k+s-1)$$

The formula even makes sense when $k=l$, in whose case we can consider the following Eisenstein series

$$\tilde{E}(z,s) = \tilde{E}_0(z,s) = {\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}}' \frac{y^s}{|mz+n|^{2s}},$$

which converges when $\Re(s)>1$. If we now define

$$\tilde{E}'(z,s) := \tilde{E}'_0(z,s) = \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}},$$

it turns out that $\tilde{E}'(z,s) = \zeta(2s)\tilde{E}'(z,s)$ and so $\tilde{E}(z,s)$ is a non-holomorphic Eisenstein series of weight 0.

Lemma 1. *Let $f,g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be two modular forms of the same weight. Then,*

$$\langle \tilde{E}(z,s)g(z), f(z)\rangle_k = \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L(f\otimes g, s+k-1).$$

With this in mind, and making use of Melling transforms, we can derive the following result:

Theorem 5. *Let $z\in\mathbb{H}$ be fixed. Then, $\tilde{E}(z,s)$ has a meromorphic continuation to the whole $\mathbb{C}$ and is entire but for a simple pole with residue π at $s=1$. Moreover, $G(z,s) := \frac{\Gamma(s)}{\pi^s}\tilde{E}(z,s)$ is holomorphic except for simple poles at $s=1$ and $s=0$ with residue 1 and -1, respectively. It satisfies that $G(z,s) = G(z,1-s)$.*

All in all, we have the integral representation for $L(f\otimes g, s+k-1)$ given by

$$\Lambda(f\otimes g,s) := \langle G(z,s-k+1)g, f\rangle_k = \frac{2\Gamma(s-k+1)\Gamma(s)}{4^s\pi^{s-k+1}}L(f\otimes g,s).$$

These function has several nice properties that we now summarize.

Proposition 8. *Let $f,g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be two modular forms of the same weight. Then, $\Lambda(f\otimes g,s)$ extends to a meromorphic function of s. It is holomorphic except at $s=k-1$ and $s=k$ where it has simple poles with residues $-\langle g,f\rangle$ and $\langle g,f\rangle$, respectively.*

Corollary 1. *Let $f,g \in S_k(\mathrm{SL}_2(\mathbb{Z})$ be two modular forms of the same weight. Then, $L(f\otimes g,s)$ extends to a meromorphic function of $s\in\mathbb{C}$ with a simple pole at $s=k$ if and only if $\langle f,g\rangle \neq 0$.*

This Rankin-Selberg method will be applied very often in subsequent chapters, since our usual setting will be concerned with either a pair or a triple of modular forms.

2 Overconvergent modular forms

There are several approaches to the theory of p-adic modular forms, initiated around the seventies by mathematicians such as Serre, Dwork or Katz. The first key observation was that one could think in the weights of classical modular forms as elements of $\mathbb{Z}_p^\times$ or, more generally, characters of $\mathbb{Z}_p^\times$. On the other hand, Katz noticed that one could define modular forms of integer weight with p-adic coefficients to be section of a line bundle (say ω) over a moduli space over $\mathbb{Z}_p$ of elliptic curves, and not as q-expansions with p-adic coefficients. From this viewpoint, Serre's p-adic modular forms of weight $k \in \mathbb{Z}$ were thought of as sections of $\omega^{\otimes k}$ over the elliptic curves with ordinary reduction. Katz's analysis of elliptic curves with "not too supersingular reduction" led to finer understanding of the U_p-operator and then, to the definition of p-adic modular forms with r-growth, what in the terminology introduced by Coleman would be r-overconvergent modular forms (in comparison with Serre's convergent p-adic modular forms of q-expansions).
There is still another important approach, that of Hida, that puts the emphasis on algebras of Hecke operators (or Hecke algebras). He observed that one could define an idempotent e on algebras $\mathbb{T}(N)$ of Hecke operators acting on spaces $S_2(\Gamma_1(Np^\infty))$ of cusp forms whose q-expansions have p-adic coefficients, and noted that it singled out subalgebras in which U_p is a unit. This will be explored when discussing Hida families. However, the most remarkable achievements in this theory arrived after Coleman had picked up the incomplete theory of Katz and had formulated a general framework of p-adic modular forms in which he coined the term overconvergent modular forms. In particular, he used Serre's theory of p-adic Banach spaces and defined an analogue of Hida's idempotent. This allowed him to construct families of finite slope overconvergent modular forms.

2.1 The classical view

We begin by recalling the different definitions of classical modular forms that we must bear in mind. They correspond to the case in which no level structure is present, that would only require some slight modifications.

Definition 9 (Version 1). *A modular form f of weight k over $\mathbb{C}$ is a function on lattices $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ such that:*

1. *$f(\mathbb{Z}\tau + \mathbb{Z})$ is holomorphic as a function of τ.*
2. *$f(\mu\Lambda) = \mu^{-k} f(\Lambda)$.*
3. *$f(\mathbb{Z}\tau + \mathbb{Z})$ is bounded as $\tau \to i\infty$.*

The classical relation between lattices and elliptic curves lead us to the following alternative definition.

Definition 10 (Version 2). *A modular form f of weight k over $\mathbb{C}$ is a function on pairs (E, ω) consisting of an elliptic curve E and a non-zero element $\omega \in H^0(E, \Omega_E^1)$ such that*

$$f(E, \mu\omega) = \mu^{-k} f(E, \omega),$$

and such that $f(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), dz)$ is bounded as $\tau \to i\infty$.

This definition can be formulated for general rings:

Definition 11 (Version 2a)**.** *A meromorphic modular form f of weight k over R is a function on pairs $(E/A, \omega)$ where ω is a nowhere vanishing section of $\Omega^1_{E/A}$ and A is an R-algebra such that:*

1. *$f(E/A, \omega)$ depends only on the A-isomorphism class of $(E/A, \omega)$.*
2. *$f(E, \mu\omega) = \mu^{-k} f(E, \omega)$ for any $\mu \in A^\times$.*
3. *If $\phi : A \to B$ is any map of rings, then $f(E/B, \omega_B) = \phi(f(E/A, \omega))$.*

The deficit of this definition is that it does not address the issue at the cusps, and for that it will be necessary to say something about Tate curves. Let $q = e^{2\pi i\tau}$. The exponential map induces an isomorphism

$$\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z} \to \mathbb{C}^\times/q^{\mathbb{Z}} = \mathbb{G}_m(\mathbb{C})/q^{\mathbb{Z}}.$$

Writing the Weierstrass parametrization in terms of the parameter q instead of τ (and changing the scaling by an appropriate factor of $2\pi i$), we find that a model for $\mathbb{C}^\times/q^{\mathbb{Z}}$ is given by

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

where

$$a_4 = -\sum \frac{n^3 q^n}{1-q^n}, \qquad a_6 = -\sum \frac{(5n^3 + 7n^5)q^n}{12(1-q^n)}$$

are both in $\mathbb{Z}[[q]]$.
This equation defines an elliptic curve over the Laurent series ring $\mathbb{Z}((q))$, that is called the Tate curve and is denoted by $T(q)$. It provides a description of the universal elliptic curve $\mathcal{E}/X$ over a punctured disc at the cusp ∞. We may associate to $T(q)$ a canonical differential

$$\omega_{\text{can}} := \frac{dt}{t} \in H^0(T(q), \Omega^1),$$

where $T(q) = \mathbb{G}_m/q^{\mathbb{Z}}$ and $\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[t, t^{-1}])$. In particular, given a meromorphic modular form f of weight k, we define the q-expansion of f to be

$$f(T(q), \omega_{\text{can}}) \in \mathbb{Z}((q)).$$

Definition 12 (Version 2b)**.** *A modular form f of weight k over R is a function on pairs $(E/A, \omega)$ where ω is a nowhere vanishing section of $\Omega^1_{E/A}$ and A is an R-algebra such that:*

1. *$f(E/A, \omega)$ depends only on the A-isomorphism class of $(E/A, \omega)$.*
2. *$f(E, \mu\omega) = \mu^{-k} f(E, \omega)$ for any $\mu \in A^\times$.*
3. *If $\phi : A \to B$ is any map of rings, then $f(E/B, \omega_B) = \phi(f(E/A, \omega))$.*
4. *We have $f(T(q), \omega_{\text{can}}) \in A[[q]]$.*

Another possibility is to understand modular forms as sections of a line bundle: how does $H^0(E, \Omega_E)$ vary as one winds around the curve $Y(\Gamma) = \mathbb{H}/\Gamma$? If we start with

$E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, together with its canonical differential dz, we can imagine moving in the upper half plane from τ to

$$\gamma\tau = \tau' = \frac{a\tau+b}{c\tau+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

which brings us back to the same elliptic curve $E_{\tau'} \simeq E_\tau$. The invariant differential varies continuously as we vary E, yet when we return to E we observe that ω_E and $\omega_{\gamma E}$ have changed:

$$dz \in H^0(\mathbb{C}/((a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z})), \Omega^1) \text{ goes to } (c\tau+d)dz \in H^0(\mathbb{C}(\mathbb{Z}+\tau'\mathbb{Z}), \Omega^1).$$

What occurs here is that the behavior of $(dz)^{\otimes k}$ as one winds around $Y(\Gamma)$ via γ exactly corrects the corresponding behavior of a modular form of weight k. This leads to the identification of modular forms as section of some line bundle $\mathcal{L}$ whose fibers at a point E hare naturally isomorphic to $H^0(E, \Omega^1_E)^{\otimes k}$.
To construct such a bundle, we want to interpolate the trivial sheaf Ω^1 as E varies over $Y(\Gamma)$. To do this, we can consider the sheaf of relative differential $\Omega^1_{\mathcal{E}/Y(\Gamma)}$ on $Y(\Gamma)$. If $\pi : \mathcal{E} \to Y(\Gamma)$ denotes the natural projection, then we set

$$\omega_Y := \pi_* \Omega^1_{\mathcal{E}/Y}.$$

We expect that the fiber of ω_Y at a point $E \in Y$ corresponding to an elliptic curve should be exactly $\Omega^1_E = H^0(E, \Omega^1_E)$. This is true and it only requires that the map π is proper:

Definition 13 (Version 3a)**.** *A meromorphic modular form f of weight k over R and level Γ is a section of $H^0(Y(\Gamma)_R, \omega^{\otimes k})$.*

We would like to understand what happens at the cusps, and the answer is that there is also a generalized elliptic curve $\mathcal{E}/X(\Gamma)$ and a corresponding local system ω_X on $X(\Gamma)$.

Definition 14 (Version 3b)**.** *A modular form f of weight k over R and level Γ is a section of $H^0(X(\Gamma)_R, \omega^{\otimes k})$.*

It is usually natural to assume for this that R is a $\mathbb{Z}[1/N]$-algebra, where N is the level of Γ. The R-modulo $H^0(X(\Gamma)_R, \omega^{\otimes k})$ is denoted as $M_k(\Gamma, R)$. Moreover, recall that in order to define ω_Y and ω_X one needs the existence of a universal generalized elliptic curve $\mathcal{E}$, which requires the moduli problem to be fine. This requires working with $X_1(N)$ rather than $X_0(N)$.

A classical view of modular forms of weight 2 over $\mathbb{C}$ arises from the fact that $f(\tau)d\tau$ is invariant under Γ, which leads us to suspect that $\Omega^1_X \simeq \omega_X^{\otimes 2}$. However, this is only correct along $Y(\Gamma)$, since $d\tau$ is not smooth at the cusps. In particular, a section of $H^0(X(\Gamma), \Omega^1_X)$ will be locally a multiple of dq, and so $f(\tau)dq = 2\pi i q f(\tau)d\tau$ will vanish at the cusp. In particular, the correct isomorphism is $\Omega^1_X(\infty) = \omega^2_X$, where $D(\infty)$ are the differential allowed to have poles of orders at most one at the cusps. These isomorphisms are referred as Kodaira-Spencer isomorphisms.

Before moving to our main issue of study, we remind the following useful result:

Proposition 9. *Let S be an R-algebra and suppose that N is invertible in R. Then, there is an isomorphism*

$$M_k(\Gamma(N), S) \simeq M_k(\Gamma(N), R) \otimes_R S,$$

when $N \geq 3$ and $k \geq 2$.

The interesting case comes when $R = \mathbb{Z}_p$ and $S = \mathbb{F}_p$ for a prime p not dividing N.

In the following sections we will discuss Serre construction of p-adic modular forms and then we will move to introduce the notion of overconvergence. To understand better all these concepts, it may be good to keep in mind that a connected modular curve $X(\Gamma)$ is determined by its q-expansion. Having a q-expansion is useful, for instance, when defining Hecke operators, that on modular forms of weight k can be defined as

$$T_p\Big(\sum a_n q^n\Big) = \sum (a_{np} + p^{k-1} a_{n/p}) q^n.$$

Alternatively, one can define it in terms of correspondences or isogenies. For more details, see [Cal].

2.2 Serre's construction

In this part, we review the basic steps of Serre's modular forms. It is a very elegant theory and in some sense, the origin of all further developments. We will restrict ourselves to classical modular forms defined over $\mathbb{Q}$ of level $N = 1$, but all the concepts naturally extend to forms of higher levels over number fields. A good reference to understand the different views of p-adic modular forms, including this, is the book [Gou].

Definition 15. *Let v_p be the usual p-adic valuation on $\mathbb{Q}_p$. Let $f = \sum a_n q^n \in \mathbb{Q}[[q]]$ be a formal power series in q over $\mathbb{Q}$. Define $v_p(f) = \inf_n v_p(a_n)$.*
Let now M_k be the space of modular forms of level one and weight k. A q-expansion $f = \sum a_n q^n \in \mathbb{Q}_p[[q]]$ is a Serre p-adic modular form if there exists a sequence of classical modular forms $f_i \in M_{k_i}$ such that $v_p(f - f_i) \to \infty$ as $i \to \infty$.

In our definition we do not require f_i to have fixed weight but the following result will have as a corollary that the weights k_i of the f_i converge p-adically to k.

Theorem 6. *Let f_1, f_2 (both non-zero) be two classical modular forms with coefficients in $\mathbb{Q}$ of weight k_1 and k_2 respectively. Assume that $v_p(f_1) = 0$. If there exists a positive integer m such that $v_p(f_1 - f_2) \geq m$, then $k_1 \equiv k_2 \pmod{p^{m-1}(p-1)}$ if $p \geq 3$ and $k_1 \equiv k_2 \pmod{2^{m-2}}$ if $p = 2$.*

We define now $X_n = \mathbb{Z}/p^{m-1}(p-1)\mathbb{Z}$ when $p \geq 3$ and $\mathbb{Z}/2^{m-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when $p = 2$. Let X be the projective limit of the X_m, which is equal to $\mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ (for instance by the Chinese remainder theorem). A direct corollary of the previous theorem is this first non-trivial property of Serre p-adic modular forms:

Corollary 2. *Let f be a Serre p-adic modular form and f_i a sequence of classical modular forms with coefficients in $\mathbb{Q}$ and weights k_i, converging p-adically to f. Then, there exists a unique $k \in X$ such that k_i converges to k. Moreover, it is independent of the choice of f_i. We call k the weight of f.*

Proof. Since $v_p(f - f_i) \to \infty$, the above theorem holds and we may set k as the projective limit of the k_i. Then, $k \in X$ and it is unique and independent of the specific choice of f_i. □

A useful result is that in order to construct a Serre p-adic modular form, it suffices to obtain a family f_i of classical modular forms of compatible weights whose a_n converge uniformly for $n \geq 1$.

Theorem 7 (Serre)**.** *Let $f_i = \sum_{n\geq 0} a_{i,n} q^n$ be a sequence of p-adic modular forms of weights k_i such that for $n \geq 1$, the $a_{i,n}$ converge uniformly to some $a_n \in \mathbb{Q}_p$ and k_i converge to some $k \in X$. Then, $a_{0,n}$ converge to some $a_0 \in \mathbb{Q}_p$ and $f = \sum_{n\geq 0} a_n q^n$ is a Serre p-adic modular form of weight k.*

This interpretation will be especially interesting for the posterior introduction of Hida families, a collection of modular forms varying continuously in the p-adic topology. In particular, in [Laf, Ch.4], we can see how towards the interpolation of a general Eisenstein series we can focus just in the non-constant terms and then use this last result of Serre.

2.3 p-adic modular forms

There are many references that are useful for this section, but we mainly follow the approaches of [Cal] and [Gou]. As we have seen, classical modular forms can be interpreted as function of triples $(E/A, \omega, \iota)$, composed of an elliptic curve E over A, a non-vanishing invariant differential ω and a level structure ι. Equivalently, they are global sections of certain invertible sheaves over the moduli space of elliptic curves with the given kind of level structure. Since we wish to obtain a p-adic theory of modular forms (that we pretend it reflects the p-adic topology), we cannot simply mimic the classical definition. One possibility, that we have already seen, is to do it in terms of Fourier expansions of modular forms. This produces an elementary theory with strong ties to the theory of congruences between classical modular forms, which turns out to be a special case of the modular theory developed by Katz in [Ka]. The main idea is to consider the rigid analytic space obtained by deleting p-adic disks around the supersingular points in the moduli space of elliptic curves with a $\Gamma_1(N)$-structure over $\mathbb{Z}_p$. We will begin the motivation of this introducing a very well-known object, the Hasse invariant, that will be very useful along our discussion.

Before moving to this, let S be a ring with $pS = 0$ and suppose that X/S is a scheme. On the one hand, we have the absolute Frobenius, that induces a map $F_{\text{abs}} : \text{Spec}(S) \to \text{Spec}(S)$ and $X \to X$. This map is given, locally on rings, as $x \mapsto x^p$. On the other hand, the relative Frobenius is a way of obtaining a morphism of schemes over S. Let $X^{(p)} = X \times_S S$, where S is thought of over S via F_{abs}. Then, the relative Frobenius is given by a map $F : X \to X^{(p)}$ satisfying that the compostion with the natural map $X^{(p)} \to X$ is F^*_{abs}.

Consider now S, a ring with $pS = 0$ and E/S an elliptic curve together with a differential ω_S generating $\Omega^1_{E/S}$. By Serre duality, we may associate to ω_S a dual basis element $\eta \in H^1(E, \mathcal{O}_E)$. The Frobenius map induces an application

$$F^*_{\text{abs}} H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E),$$

and we can write

$$F^*_{\text{abs}}(\eta) = A(E,\omega)\cdot\eta$$

for some $A(E,\omega) \in S$.

Lemma 2. *A is a meromorphic modular form of level one and weight $p-1$ over S.*

When S is a field, $A(E,\omega)$ is either zero or a unit (and it is zero when E is supersingular). We call A the Hasse invariant of E. It is a result of Deligne that when $R = \mathbb{F}_p((q))$,

$$A(T(q), \omega_{\text{can}}) = 1.$$

Further, when $p \geq 5$, A lifts to a modular form in characteristic zero. From the computation above, the q-expansion of any such lift is congruent to 1 modulo p. Hence, from Kummer congruences, we deduce the following:

Theorem 8. *Let $p \geq 5$. Then, A lifts to a modular form in characteristic zero. The modular form E_{p-1} is a lift of A such that $E_{p-1} \equiv A \mod p$. If $p = 2$ or $p = 3$, the modular forms E_4 and E_6 are lifts of $A^4 \mod 8$ and $A^3 \mod 9$ respectively.*

Let A be any lift of the Hasse invariant. Since $A \equiv 1 \mod p$, the powers of A are becoming more and more congruent to 1 modulo p. Hence, they converge to 1. Then, the powers A^{p^n-1} converge to A^{-1} (any lift of the Hasse invariant is invertible). We can then do the main definition. Observe first that we need to fix a congruence subgroup Γ of level prime to p. The definition of p-adic modular form and overconvergent p-adic modular form at level one are almost the same as the corresponding definition at level Γ; for that reason, we do not focus a lot in this latter case and work at level one.

Definition 16. *The p-adic modular functions on $X(\Gamma)$ are the functions which are well defined at all points of ordinary reduction.*

For instance, observe that the function $\bar{A}$ defined on pairs $(E/\mathbb{F}_p, \omega)$ to be 1 if E is ordinary and 0 if E is supersingular, defines a modular form over $\mathbb{F}_p$ with q-expansion identically 1. Recall that $E[p]$ over a field of characteristic p must be equal to $\mathbb{Z}/p\mathbb{Z}$ or 0, saying in the first case that E is ordinary and in the latter supersingular. $\bar{A}$ turns out to be the Hasse invariant of E.
From a fact on Bernoulli numbers, $E_{p-1} \equiv 1 \pmod p$ por $p \geq 5$, and so the reduction of E_{p-1} has the same q-expansion. By the q-expansion principle, $E_{p-1} \equiv \bar{A}$. The important fact here is the existence of such a lift of $\bar{A}$.

p-adic modular forms can also be interpreted using the previous general definitions.

Definition 17 (Version 2 revisited)**.** *A p-adic modular form f of weight k and level one over a p-adically complete algebra A is a function on pairs $(E/R,\omega)$ for a p-adically complete A-algebra R satisfying: ω is a nowhere vanishing section of $\Omega^1_{E/A}$; and $A(E/B,\omega_B)$ is invertible, where $B = A/p$. We require the following properties:*

1. *$f(E/A,\omega)$ depends only on the A-isomorphism class of $(E/A,\omega)$.*
2. *$f(E,\mu\omega) = \mu^{-k} f(E,\omega)$.*
3. *If $\phi : A \to B$ is a map of rings, then $f(E/B,\omega_B) = \phi(f(E/A,\omega))$.*

4. $f(T(q), \omega_{\mathrm{can}}) \in A[[q]]$.

In the case of higher levels, observe for instance that $Y_0(N)$ parametrizes elliptic curves with a subgroup of order N. Hence, modular forms should be defined to be functions on triples $(E/R, \alpha, \omega)$, with α a subgroup of order N.

This result gives a link with Serre's theory:

Lemma 3. *The closure of the set of classical modular forms over a p-adically complete ring R of all weights coincides with the set of p-adic modular forms over R of all weights.*

Beyond these definitions, one of the most interesting points lie in the theory of overconvergent modular forms. It turns out that modular forms are not good enough and we will be often interested in functions converging beyond the ordinary locus. Next sections are devoted to this.

2.4 Ordinary modular forms

The operator U_p plays a prominent role in all this theory. In terms of the q-expansion, it is defined as

$$U_p\Big(\sum a_n q^n\Big) = \sum a_{np} q^n,$$

which is the classical T_p-operator but removing the summand corresponding to $a_{n/p}$ when n divides p. In terms of isogenies, given an elliptic curve E with a distinguished p-isogeny $\eta : E \to B$, we define U_p by considering the maps $\phi : D \to E$ not equal to $\hat{\eta} : B \to E$, namely

$$U_p f(E, \eta : E \to B, \omega, \alpha) = p^{k-1} \sum_{\phi : D \to E}^{\phi \neq \hat{\eta}} f(D, \phi^*(\omega), \phi^* \alpha).$$

Definition 18. *Let $f \in S_k(\Gamma_0(pN))$ be a Hecke eigenform with U_p eigenvalue a_p. The slope of f is $v_p(a_p)$. f is said to be ordinary if it has slope 0. $M_k^{\mathrm{ord}}(\Gamma_0(pN))$ is the space of ordinary modular forms of weight k.*

Proposition 10. *The slope at p of an eigenform f of weight k and level $\Gamma_0(pN)$ is at most $k-1$.*

The basis of Hida's p-adic families that will be explored in subsequent chapters is the following result:

Theorem 9. *The dimension of $M_k^{\mathrm{ord}}(\Gamma_0(pN))$ is independent of the weight k modulo $p-1$.*

Since the dimension of spaces of classical modular forms grow linearly in k, we can expect something similar for non-ordinary forms. For this, Coleman's idea was to pass up to a space of infinite dimension. We will try later a first definition of modular forms in a more general setting.

Let X be a modular curve smooth over $\mathbb{Z}[1/p]$. It makes sense to talk about the ordinary locus of $X/\mathbb{F}_p$, since there are only finitely many supersingular points. It also makes sense to talk about the ordinary locus over $\mathbb{Z}_p$, since we would like to exclude all lifts of supersingular elliptic curves. Specifically, we would like to remove a unit ball around any supersingular point. Rigid analytic spaces provide the right context in which these constructions make sense, and such that the topology is fine enough to allow these constructions that does not work in the Zariski topology.

Definition 19. *The p-adic modular forms of weight k are the global section*

$$H^0(X^{\mathrm{rig}}[0], \omega^k),$$

where $X^{\mathrm{rig}}[0]$ is the ordinary locus of the rigid anlaytic space X^{rig}.

Let R be a p-adically complete ring, and let $M_k(\Gamma_0(p), R)$ be the space of classical modular forms, on which U_p acts. For being the space finite as an R-module, we can define $e_p := \lim_{\rightarrow} U_p^{n!}$.

Lemma 4. *e_p is an idempotent on $M_k(\Gamma_0(p), R)$, and projects onto the space generated by Hecke eigenforms on which U_p acts by a unit.*

U_p (and e_p) commutes with T_l for l prime to the level. Thus, e_p is a Hecke equivariant projection; when f is a Hecke eigenform with unit eigenvalue for U_p we say that f is ordinary. From the results of Hida that will be worked later, we have the following:

Theorem 10 (Hida)**.** *The operator e_p extends to an idempotent on $M_k(\Gamma, R, 0)$. Let $e_pM_k(\Gamma, R, 0)$ be its image. Then,*

1. *$e_pM_k(\Gamma, R, 0)$ is finite dimensional, and the dimension only depends on k modulo $p-1$.*
2. *If $k > 1$, then $e_pM_k(\Gamma, R, 0) \subset M_k(\Gamma_0(p), R, 0)$ is spanned by classical modular forms.*
3. *The minimal polynomial of U_p on $e_pM_k(\Gamma, R/p^n, 0)$ only depends on k modulo $p^{n-1}(p-1)$.*

The problem here arises in the fact that U_p contains a continuous spectrum on $M_k(\Gamma, R, 0)$ and this rules out the possibility of expressing a p-adic modular function into an infinite sum of eigenforms. The key point is that we must consider sections of X^{rig} converging beyond the ordinary locus $X^{\mathrm{rig}}[0]$. This is basically because we can pass between level 1 and level $\Gamma_0(p)$ (with p-adic modular forms) using the fact that an ordinary elliptic curve E/R comes with a canonical subgroup scheme $P \subset E[p]$ that comes from the kernel of the reduction map.
We consider (R, m), the ring of integers of a finite extension of $\mathbb{Q}_p$ and its maximal ideal. Let $k = R/m$ be the residue field and K the fraction field. Take a normalized valuation ($v(p) = 1$). If E/R is ordinary, $E(\bar{k}) = \mathbb{Z}/p\mathbb{Z}$. Consider the reduction map $E(\bar{K}) \to E(\bar{k})$; the kernel C of $E(\bar{K})[p] \to E(\bar{k})[p]$ is a cyclic subgroup of order p, canonically associated to E/R. The problem arises when E/k is supersingular, and so $E(\bar{k})[p]$ is trivial and $E(\bar{K})[p] = (\mathbb{Z}/p\mathbb{Z})^2$ contains $p+1$ subgroups C. In some cases there will be a canonical choice to make.

Theorem 11 (Lubin-Katz)**.** *Let R be a complete $\mathbb{Z}_p$-algebra. An elliptic curve E/R has a canonical subgroup of order p if and only if*

$$v(A) < \frac{p}{p+1},$$

where $A(E_S, \omega_S)$ is the Hasse invariant of E/S, with $S = R/p$. The elliptic curves that satisfy the hypothesis are called not too supersingular.

Fix a modular curve X of level prime to p and assume that X is a fine moduli space smooth over $\mathbb{Z}_p$. Let k be a finite extension of $\mathbb{F}_p$. The rigid analytic space X^{rig} admits a map

$$X^{\mathrm{rig}}(\mathbb{C}_p) \to X(\bar{k}).$$

The preimage of a point is an open disc and the complement of the open discs corresponding to the supersingular points is the ordinary locus. Let E/k be the elliptic curve corresponding to a supersingular point; since X is smooth at x, the completion of X at x is isomorphic to $W[[t]]$. We define $X^{\mathrm{rig}}[r]$ by removing from X^{rig} the open balls B of radius p^{-r} in the parameter t. When $r = 0$, we recover $X^{\mathrm{rig}}[0]$.

Lemma 5. *The definition does not depend of any choice provided that $r < 1$.*

Proof. Any different uniformizing parameter would be of the form $s = ap + ut$, where $a \in W, u \in W[[t]]^*$. We still have $v(s) = v(t)$, since either $v(s)$ or $v(t)$ is less than $v(p)$. □

Suppose that $r < \frac{p}{p+1}$ and consider $X^{\mathrm{rig}}[r]$. The key point is that for a subgroup scheme H of E of order p which is not the canonical subgroup, the valuation of $A(E/H, \hat{\phi}^*\omega)$ decreases as long as $0 < v(A) < p/(p+1)$.

Theorem 12 (Katz-Lubin)**.** *Let $(E/R, \omega)$ be an elliptic curve and suppose that*

$$v(A(E, \omega)) < \frac{p}{p+1}.$$

Let $H \subset E$ be a subgroup scheme of order p different from the canonical subgroup. Let $\phi : E \to E/H$ be the natural projection and $\hat{\phi} : E/H \to E$ the dual isogeny. Then,

$$v(A(E/H, \hat{\phi}^*\omega)) = \frac{v(A(E, \omega))}{p}.$$

The proof of this fact uses some properties of forrmal groups. The identification of $X^{\mathrm{rig}}[r]$ with the component of $X_0^{\mathrm{rig}}(p)[r]$ containing ∞ allows us to define an operator U_p on sections of $X^{\mathrm{rig}}[r]$; one simply takes the sum over all pairs (E, H) where H is not the canonical subgroup.

Theorem 13. *Let $0 < r < 1/(p+1)$. Suppose that f is a section of $H^0(X^{\mathrm{rig}}[r], \omega^k)$. Then, $U_p f$ extends to a function on $H^0(X^{\mathrm{rig}}[pr], \omega^k)$. In particular, U_p defines a map*

$$U_p : H^0(X^{\mathrm{rig}}[r], \omega^k) \to H^0(X^{\mathrm{rig}}[pr], \omega^k).$$

Proof. Let (E, ω) be an elliptic curve with $v(A(E, \omega)) < pr$. It suffices to show that we can extend $U_p f$ to (E, ω). By definition, to evaluate f on (E, ω) involves evaluating f on elliptic curves E/P as P runs over the p-subgroup schemes of $E[p]$ which are not the canonical subgroup. In particular, all those elliptic curves have Hasse invariant at most r and thus f is well defined.

The intuitive idea must be that U_p increases the convergence of an overconvergent modular form. □

Definition 20. *Let $0 < r < p/(p+1)$ be a rational number. The space of overconvergent modular forms of weight k, level Γ and radius r is defined to be*

$$M_k^\dagger(\Gamma, r) = H^0(X^{\mathrm{rig}}[r], \omega^k).$$

There are inclusions

$$M_k^\dagger(\Gamma, r) \hookrightarrow M_k(\Gamma, \mathbb{C}_p, 0) \hookrightarrow \mathbb{C}_p[[q]],$$

and hence overconvergent modular forms satisfy the q-expansion principle.

Lemma 6. *$M_0^\dagger(\Gamma, r)$ is a Banach space with respect to the supremum norm on $X^{\mathrm{rig}}[r]$.*

Theorem 14. *Suppose that $r < p/(p+1)$. Then, the map*

$$U : H^0(X^{\mathrm{rig}}[r], \omega^k) \to H^0(X^{\mathrm{rig}}[r], \omega^k)$$

is compact.

We finish this section with the so-called classicality theorem:

Theorem 15 (Coleman)**.** *An overconvergent modular form of weight k and level $\Gamma_1(N)$, which is an eigenform for U_p, is a classical modular form if the slope is strictly less than $k-1$. If its slope is $k-1$, unless it is in the image of the theta operator θ^{k-1}, it is also a classical modular form*

Coleman proved this using Hodge theory of modular curves; then, Payman Kassaei reproved the theorem by a different approach

2.5 Nearly holomorphic modular forms

This section is based on [Ur], although similar definitions are introduced in [DR1]. Along the thesis, some of these facts will appear again and we will be clarifying some of the concepts that may seem quite artificial at first glance.

Let f be a complex valued function on $\mathbb{H}$. For any integer $k \geq 0$ and $\gamma \in \mathrm{GL}_2(\mathbb{R})^+$, we set

$$f|_k\gamma(\tau) := \det(\gamma)^{k/2}(c\tau + d)^{-k} f(\gamma\tau).$$

Let $r \geq 0$ be another integer. We say that f is a nearly holomorphic modular form of weight k and order $\leq r$ for an arithmetic group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ if f satisfies the following properties:

- $f \in \mathcal{C}^\infty$ on $\mathbb{H}$.
- $f|_k\gamma = f$ for all $\gamma \in \Gamma$.
- There are holomorphic functions $f_0, \ldots, f_r$ on $\mathbb{H}$ such that

$$f(\tau) = f_0(\tau) + \frac{1}{y} f_1(\tau) + \ldots + \frac{1}{y^r} f_r(\tau).$$

- f has a finite limit at the cusps.

If $f \in \mathcal{C}^\infty(\mathbb{H})$, we denote

$$(\epsilon f)(\tau) := 8\pi i y^2 \frac{\partial f}{\partial \bar{\tau}}(\tau).$$

Then, the third condition is equivalent to $\epsilon^{r+1} f = 0$.
Further, if f is nearly holomorphic of weight k and of order $\leq r$, then ϵf is nearly holomorphic of weight $k-2$ and order $\leq r-1$. Write $\mathcal{N}_k^r(\Gamma, \mathbb{C})$ for the space of nearly holomorphic forms of weight k, order $\leq r$ and level Γ. For $r = 0$, it is just the space of

holmorphic modular forms of weight k and level Γ, $M_k(\Gamma,\mathbb{C})$.

On the space of nearly holomorphic modular forms, we have the Maass-Shimura differential operator δ_k, given by

$$\delta_k f := \frac{1}{2\pi i} y^{-k} \frac{\partial}{\partial \tau}(y^k f) = \frac{1}{2\pi i}\Big(\frac{\partial f}{\partial \tau} + \frac{kf}{2iy}\Big).$$

$\delta_k f$ is of weight $k+2$ and its degree of near holomorphy is increased by one. Define also for a positive integer s

$$\delta_k^s := \delta_{k+2s-2} \circ \ldots \circ \delta_k.$$

Lemma 7. *Let $f \in \mathcal{N}_k^r(\Gamma,\mathbb{C})$. Assume that $k > 2r$. Then, there exist $g_0,\ldots,g_r$ with $g_i \in M_{k-2i}(\Gamma,\mathbb{C})$ such that*

$$f = g_0 + \delta_{k-2} g_1 + \ldots + \delta_{k-2r}^r g_r.$$

We do now a few comments about the sheaf theoretic definition.
Let $Y = \Gamma\backslash\mathbb{H}$ and $X = \Gamma\backslash(\mathbb{H} \sqcup \mathbb{P}_1(\mathbb{Q}))$ be the open modular curve and the compactified modular curve of level Γ, respectively. Let $\mathcal{E} = \bar{\mathcal{E}} \times_{X_\Gamma} Y_\Gamma$ be the universal elliptic curve over Y_Γ and let $p : \bar{\mathcal{E}} \to X_\Gamma$ be the Kuga-Sato compactification of the universal elliptic curve over X_Γ. We consider the sheaf of invariant relative differential forms with logarithmic poles along $\partial\bar{\mathcal{E}} = \bar{\mathcal{E}}\backslash\mathcal{E}$, which is a normal crossing divisor of $\bar{\mathcal{E}}$. Let

$$\omega := p_* \Omega^1_{\bar{\mathcal{E}}/X}(\log(\partial\bar{\mathcal{E}})).$$

This is a locally free sheaf of rank one in the holomorphic topos of X. Let

$$H^1_{\mathrm{dR}} := \mathbb{R}^1 p_* \Omega^\bullet_{\bar{\mathcal{E}}/X}(\log(\partial\bar{\mathcal{E}}))$$

be the sheaf of relative degree one de Rham cohomology of $\bar{\mathcal{E}}$ over X with logarithmic poles along $\partial\bar{\mathcal{E}}$. The Hodge filtration induces an exact sequence

$$0 \to \omega \to H^1_{\mathrm{dR}} \to \omega^* \to 0,$$

that in the C^∞-topos splits and gives the decomposition $H^1_{\mathrm{dR}} = \omega \oplus \bar{\omega}$.

Proposition 11. *The Hodge decomposition induces a canonical isomorphism*

$$H^0(X_\Gamma, \mathbb{H}_k^r) \cong \mathcal{N}_k^r(\Gamma,\mathbb{C}).$$

In fact, let $\pi : \mathbb{H} \to Y_\Gamma$ and let $\pi^*\mathcal{E}$ be the pull-back of $\mathcal{E}$ by π. We have that $\pi^*\mathcal{E} = (\mathbb{C}\times\mathbb{H})/\mathbb{Z}^2$, where the action of $\mathbb{Z}^2$ is given by

$$(z,\tau)\cdot(a,b) = (z + a + b\tau, \tau).$$

The fiber $\mathcal{E}_\tau$ at $\tau \in \mathbb{H}$ can be identified with $\mathbb{C}/L_\tau$, where $L_\tau = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$. Further, $\pi^*\omega = \mathcal{O}_{\mathbb{H}} dz$, where $\mathcal{O}_{\mathbb{H}}$ is the sheaf of holomorphic functions on the upper-half place. Observe also that an explicit description of $\mathcal{E}$ can be given as

$$\mathcal{E} = \Gamma\backslash\mathbb{C}\times\mathbb{H}/\mathbb{Z}^2,$$

where the action of Γ on $\mathbb{C}\times\mathbb{H}/\mathbb{Z}^2$ is given by

$$\gamma\cdot(z,\tau) = ((c\tau+d)^{-1}z, \gamma\cdot\tau).$$

Then,

$$\gamma^* dz = (c\tau + d)^{-1} dz.$$

From here, together with the condition at the cusps, it turns out that

$$H^0(X_\Gamma, \omega^{\otimes k}) \cong M_k(\Gamma, \mathbb{C}).$$

The Hodge decomposition of $\pi^* \mathbb{H}^1_{\mathrm{dR}}$ readas as

$$\pi^* H^1_{\mathrm{dR}} \otimes \mathcal{C}^\infty_{\mathbb{H}} = \mathcal{C}^\infty_{\mathbb{H}} dz \oplus \mathcal{C}^\infty_{\mathbb{H}} d\bar{z},$$

and the Riemann-Hilbert correspondence implies that

$$\pi^* H^1_{\mathrm{dR}} = \mathcal{O}_{\mathbb{H}} \alpha \oplus \mathcal{O}_{\mathbb{H}} \beta,$$

where α, β is the basis of horizontal sections inducing on $H_1(E_\tau, \mathbb{Z}) = L_\tau$ the linear form $\alpha(a + b\tau) = a$ and $\beta(a + b\tau) = b$.

2.6 Overconvergent modular symbols

There are several approaches to the theory of modular symbols. The most classical one consists on fixing a congruence subgroup and consider $\pi : \mathbb{H}^* \to X(\Gamma)$ the natural projection map. When $r, s \in \mathbb{H}^*$, we can define the classical modular symbol $\{r, s\}$ for Γ as the element of $H_1(X(\Gamma), \mathbb{R})$ corresponding to $\omega \mapsto \int_r^s \pi^*(\omega) \in \mathrm{Hom}_{\mathbb{C}}(\Omega^1(X(\Gamma), \mathbb{C})$. Alternatively, they can be understood as elements in $\mathrm{Hom}(\Delta_0, \mathbb{C})$, where Δ_0 refers to the degree zero divisors and $\mathbb{C}$ can be replaced for a general abelian group. For more references, see [Da, Ch.2], although we will still provide more details in this section.

We recall that we had an identity relating spacial values of the L-function of a cusp form to an integral,

$$2\pi i \int_{i\infty}^0 z^{j-1} f(z) dz = \frac{(j-1)!}{(-2\pi i)^{j-1}} L(f, j).$$

We begin by focusing on the weight two case. There, what we observe is that evaluating the L-value is the same as sending the pair of cusps $\{i\infty, 0\}$ to the integral of $f(z)$. In general, we can consider

$$\phi_f : \{r, s\} \mapsto 2\pi i \int_r^s f(z) dz.$$

More precisely, we have a map from ordered pair of cusps (ordered pairs in $\mathbb{P}_1(\mathbb{Q})$) to degree zero divisors on $\mathbb{P}_1(\mathbb{Q})$. We can then see ϕ_f as a map from $\mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q}))$ to $\mathbb{C}$. Thus, each cusp form f gives rise to $\phi_f \in \mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q})), \mathbb{C})$.
We have an action of $\mathrm{SL}_2(\mathbb{Z})$ (and therefore of Γ) on the left on $\mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q}))$ given by

$$\gamma \cdot (\{s\} - \{r\}) = \{\gamma s\} - \{\gamma r\},$$

and then extended by linearity. This corresponds to a right action on the Hom space, given by $(\phi \cdot \gamma)(D) = \phi(\gamma D)$. A simple computation shows that for an element $\gamma \in \Gamma$, since f has weight two, $\int_{\gamma r}^{\gamma s} f(\gamma z) d(\gamma z) = \int_r^s f(z) dz$, and so ϕ_f is Γ-invariant. Therefore, a good definition is the following one:

Definition 21. *Let* Γ *be a congruence subgroup and* A *an abelian group. The space of* A*-valued modular symbols of level* Γ *is defined to be*

$$\mathrm{Symb}_\Gamma(A) = \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q})), \mathbb{C}),$$

the space of Γ*-invariant maps from the degree zero divisor group into* A*. We will be mainly interested (in this first approach) in the case* $A = \mathbb{C}$*.*

We have seen that $S_2(\Gamma) \hookrightarrow \mathrm{Symb}_\Gamma(\mathbb{C})$, but we can state a stronger result:

Theorem 16 (Eichler-Shimura)**.** *Suppose that* $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *normalizes* Γ *(this occurs for instance for* $\Gamma_0(N)$*). Then, there is an isomorphism*

$$S_2(\Gamma) \oplus M_2(\Gamma) \cong \mathrm{Symb}_\Gamma(\mathbb{C}).$$

One of the applications of the theory of modular symbols is the study of L-functions. We have for instance the following propostion:

Proposition 12. *Let* f *be a cusp form of weight two and level* Γ*. Then,*

$$L(f, 1) = \phi_f(\{0\} - \{\infty\}).$$

We can generalize the theory to higher weights. To motivate this, let P be a polynomial of degreee $\leq k-2$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Γ acts on such polynomials by the rule $(P|\gamma)(x) = (cx+d)^{k-2}P(\gamma x)$. The idea now is to choose our maps to arrive to some polynomial space, and define an action of Γ on the right of this.

Definition 22. *Let* $V_{k-2}(R)$ *be the space of homogenous polynomials of degree* $k-2$ *in two variables* X, Y *over a ring* R*.* $V_{k-2}(\mathbb{C})$ *has a right action of* Γ *given by*

$$(P|\gamma)(X, Y) = P(dX - cY, -bX + aY).$$

The space of weight k *modular symbols of level* Γ *is the space*

$$\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})) = \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q})), V_{k-2}(\mathbb{C}))$$

of Γ*-invariant maps, where the fact that* ϕ *is* Γ*-invariants means that* $\phi(D) = \phi(\gamma D)|\gamma$*, for all* $\gamma \in \Gamma$*.*

In the same way as before, to a cusp form we associate the map

$$\phi_f : \{s\} - \{r\} \mapsto 2\pi i \int_r^s (zX + Y)^{k-2} f(z) dz.$$

Let us check that the action of Γ on this space is such that $\phi_f(D)|\gamma^{-1} = \phi_f(\gamma D)$. To begin with, note that

$$(\gamma(z)X + Y)^{k-2} = ((az+b)X + (cz+d)Y)^{k-2}(cz+d)^{-(k-2)}.$$

Then, it follows that

$$\phi_f(\gamma(\{s\} - \{r\})) = 2\pi i \int_{\gamma r}^{\gamma s} (zX + Y)^{k-2} f(z) dz$$

$$= 2\pi i \int_r^s ((az+b)X + (cz+d)Y)^{k-2} f(z) dz = \phi_f(\{s\} - \{r\})|\gamma^{-1}.$$

Theorem 17 (Eichler-Shimura). *Suppose that* $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *normalizes* Γ. *Then,*

$$S_k(\Gamma) \oplus M_k(\Gamma) \cong \mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})).$$

Proposition 13. *Let f be a cusp form of weight k and level Γ, If we define c_j to be such that*

$$\phi_f(\{0\} - \{\infty\}) = \sum_{j=0}^{k-2} c_j X^j Y^{k-2-j},$$

then we have that

$$L(f, j+1) = \binom{k-2}{j}^{-1} \frac{c_j(-2\pi i)^j}{j!}.$$

The Eichler-Shimura isomorphism is also Hecke-invariant, whose meaning for modular symbols we recall now.

Definition 23. *For p prime, consider*

$$S_0(p) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \textit{ such that } p|c, p \nmid a, ad - bc \neq 0 \Big\}.$$

Let V be a right $\mathbb{Z}[\Gamma_0(pN)]$-module (a space with a right action of $\Gamma_0(pN)$). Suppose that V admits a right action of $S_0(p)$ and let $\phi \in \mathrm{Symb}_{\Gamma_0(pN)}(V)$. Then, if $l \nmid N$,

$$\phi | T_l = \phi \Big| \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{l-1} \phi \Big| \begin{pmatrix} 1 & a \\ 0 & l \end{pmatrix}.$$

If $q|N$, we consider

$$\phi | U_q = \sum_{a=0}^{q-1} \phi \Big| \begin{pmatrix} 1 & a \\ 0 & q \end{pmatrix}.$$

We need to introduce now some more notation. Let $\mathbb{A}$ be the ring of rigid analytic functions on the closed unit disk in $\mathbb{C}_p$ defined over $\mathbb{Q}_p$. Consider the group

$$\Sigma_0(p) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \text{ such that } p \nmid a, p|c, ad - bc \neq 0 \Big\}.$$

Define a left weight action of $\Sigma_0(p)$ on $\mathbb{A}$ by

$$\gamma \cdot_k f(x) = (cx + a)^k f\Big(\frac{dx + b}{cx + a}\Big),$$

and let $\mathbb{A}_k$ be the space $\mathbb{A}$ with this action. It is an exercice to check that this gives an action. Dualising, we get a space with a weight k right action, that we denote $\mathbb{D}_k$. Since polynomials are dense in $\mathbb{A}_k$, any distribution $\mu \in \mathbb{D}_k$ is determined by the values on the monomials X^j (the values $\mu(X^j)$ are usually referred as moments).

Definition 24. *For a congruence subgroup Γ, with $\Gamma \subset \Gamma_0(p)$, the space of overconvergent modular symbols of weight k is the space $\mathrm{Symb}_\Gamma(\mathbb{D}_{k-2})$.*

We have already seen that overconvergent Hecke eigenforms of small slope correspond to classical eigenforms. Here, instead of saying that every small slope eigensymbol is classical, we show a specialization map

$$\rho_{k-2} : \mathrm{Symb}_\Gamma(\mathbb{D}_{k-2}) \to \mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{Q}_p)),$$

and then we show that this is an isomorphism of symbols for small slope. Once we find a map $\lambda_{k-2} : \mathbb{D}_{k-2} \to V_{k-2}(\mathbb{Q}_p)$, composing with $\phi \in \mathrm{Symb}_\Gamma(\mathbb{D}_{k-2})$ we get the desired morphism. A map like this is given by

$$\lambda_{k-2}(\mu) = \int (Y - zX)^{k-2} d\mu(z).$$

This map is equivariant under the action of $\Sigma_0(p)$ and thus is Hecke invariant.

We have previously proven that every classical eigenform of level $\Gamma_0(pN)$ has slope $\leq k-1$. Via Eichler-Shimura, we can say that the maximal slope of a classical eigensymbol of level $\Gamma_0(pN)$ is also $k-1$ so any element of $\mathrm{Symb}_{\Gamma_0(pN)}$ with slope $> k-1$ will lie in the kernel of specialization. Stevens' control theorem tells us that outside the critical slope case this is what the kernel should look lie.

Theorem 18 (Stevens' Control theorem)**.** *The specialization map*

$$\rho_{k-2} : \mathrm{Symb}_{\Gamma_0(pN)}(\mathbb{D}_{k-2})^{<k-1} \to \mathrm{Symb}_{\Gamma_0(pN)}(V_{k-2})^{<k-1}$$

is an isomorphism.

Now, as a corollary, we have the following remarkable result:

Theorem 19. *Let f be an eigenform of weight k and level $\Gamma_0(pN)$ and non-critical slope at p (slope $< k-1$). Let Φ_f be the unique overconvergent lift of the corresponding classical modular symbol ϕ_f. Then, the restriction of the distribution*

$$L_p(f) = \Phi_f(\{0\} - \{\infty\})$$

to $\mathbb{Z}_p^\times$ is the p-adic L-function of f.

3 Hida families

The basic idea we will explore in this chapter is that p-adic modular forms may vary in families. We try to present the material in a such a way that is self-contained, but maybe a more exhaustive treatment, where the proofs are developed in more detail, can be found in [Laf] and also in my expository notes [R2].

We can consider $\Lambda = \mathbb{Z}_p[[X]]$ the usual Iwasawa algebra, and take $u = 1 + q$ (where $q = p$ in the odd case and $q = 4$ elsewhere). A Dirichlet character always admit a decomposition as $\chi = \chi_F \chi_S$ (see [Laf] and also [Was] for the precise definitions) which gives rise to a specialization map $\nu_{k,\chi} : \Lambda \to \overline{\mathbb{Q}_p}$ given by $X \mapsto \chi_S(u)u^k - 1$.

Definition 25. *Let N be a positive integer relatively prime with p and K a finite extension of the fraction field of Λ; let I be the integer closure of Λ in K. For any Dirichlet character χ of conductor Nqp^r, a Λ-adic modular form F of character χ and level Nqp^r is a formal q-expansion*

$$F(X) = \sum_{n=0}^{\infty} a_n(F)(X)q^n \in I[[q]],$$

such that for any integer $k > 1$ (but for a finite number),

$$\nu(F) = \sum_{n=0}^{\infty} \nu(a_n(F))q^n \in M_k(Nqp^r, \chi\omega^{-k}, \overline{\mathbb{Q}_p}),$$

for all $\nu \in A_k(\chi, I)$, the set of $\mathcal{O}$-algebras homomorphisms from I to $\overline{\mathbb{Q}_p}$.

From now on, let N be a positive integer and let p be an odd prime such that p divides N exactly once. Consider $\mathfrak{X} := \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$. Since p is odd, we have the identification of topological groups

$$\mathfrak{X} \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p,$$

and this can be used to define analytic functions on the weight space. If $U \subset \mathfrak{X}$ is an open subset, we let $A(U)$ denote the collection of analytic functions on U, that is, the collection of functions that are power series on each intersection $U \cap (\{a\} \times \mathbb{Z}_p)$. We will usually assume that U is contained in the residue disk of 2, and then $A(U)$ is simply the ring of power series that converge on an open subset of $\mathbb{Z}_p$. A Hida family is a formal q-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n q^n$$

such that there exists a neighborhood U of 2 in $\mathfrak{X}$ such that $a_n \in A(U)$ for all n and such that if $k \in U \cap \mathbb{Z}^{\geq 2}$, the weight k specialization

$$f_k := \sum_{n=1}^{\infty} a_n(k)q^n$$

is a normalized ordinary eigenform of weight k on $\Gamma_0(N)$. Weights in $\mathbb{Z}^{\geq 2} \subset \mathfrak{X}$ are called classical.

A modular form f of weight k on $\Gamma_0(N)$ lives in a Hida family if it is the weight k specialization of some f_∞. For instance, the p-th Fourier coefficient of the Eisenstein series $G_k(z)$ of level one is $1+p^{k-1}$ and this is not a p-adically continuous function of k. However, we can consider the p-stabilized series

$$G_k^*(z) = G_k(z) - p^{k-1}G_k(pz) = \frac{-(1-p^{k-1})B_k}{2k} + \sum_{n\geq 0}\Big(\sum_{d|n,(d,p)=1} d^{k-1}\Big)q^n.$$

This is an eigenform on $\Gamma_0(p)$; the Hecke operator U_p defined for N multiple of p acts as

$$U_p\Big(\sum_{n\geq 0} a_n q^n\Big) = \sum_{n\geq 0} a_{pn}q^n,$$

so $U_p(G_k^*(z)) = G_k^*(z)$. It is natural so to consider forms satisfying this condition that we have already explored when dealing with oveconvergent modular forms. Recall that an eigenform $f \in M_k(\Gamma_0(N))$ is ordinary at p if $U_p f = \lambda_p f$, for $\lambda_p \in \mathbb{Z}_p^\times$. $M_k(\Gamma_0(N))^{\mathrm{ord}}$ denotes the subspace spanned by the p-ordinary forms.

One of the settings in which we will be interested is that of an elliptic curve over $\mathbb{Q}$ of condutor $N = Mp$, with $(p, M) = 1$ and such that p is a prime of multiplicative reduction. Since E is modular, we can consider the attached normalized eigenform (of weight two) on $\Gamma_0(N)$ and repeat this same construction. If $f = \sum a_n q^n$, Hida's theory will associate to f a neighborhood U of $2 \in \mathfrak{X}$ and a formal q-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n(k)q^n, \quad a_1 = 1, \quad a_n \in A(U)$$

with two properties of its specializations $f_k := \sum a_n(k)q^n$ (for fixed k):

- For all integers $k \geq 2$ in U, the series f_k is the q-expansion of a normalized ordinary eigenform of weight k on $\Gamma_0(N)$.
- The weight 2 specialization f_2 is equal to f.

One of the main theorems in this theory is the following one:

Theorem 20. *Let $k \geq 2$ be an integer. Let $f \in S_k(\Gamma_0(N))$ be an eigenform and let p be a prime divisor of N. Assume that f is p-ordinary. Then, there exists a unique Hida family f_∞ which specializes to f at weight k.*

Sketch of the proof. Let $\mathbb{T}_k(N)$ be the full Hecke algebra of $S_k(N)$, generated by T_l, for $(l, N) = 1$, and by U_p, for p dividing N. Then, we have a natural perfect pairing

$$\mathbb{T}_k(N) \times S_k(N) \to \mathbb{C}$$

given by $(T, f) \mapsto a_1(Tf)$. This establishes a $\mathbb{C}$-linear isomorphism $S_k(N) \cong \mathbb{T}_k(N)^*$. Further, a modular form $f \in S_k(N)$ corresponds to a $\mathbb{C}$-algebra homomorphism via the above identification if and only if it is a normalized eigenform for all of the Hecke operators. Let $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ and let $\Lambda^\dagger$ be the ring of power series over $\mathbb{Z}_p$ converging on a neighbourhood of $2 \in \mathfrak{X}$. For proving the existence of the Hecke algebra we will go through four steps that will be carried out in the next section.

1. Define the ordinary Hecke algebra $\mathbb{T}_\infty^{\mathrm{ord}}$.
2. Show that an ordinary weight k eigenform f determines a map $\eta_f : \mathbb{T}_\infty^{\mathrm{ord}} \to E$ where E is the ring of integers of a finite extension of $\mathbb{Q}_p$.

3. Justify that this maps lifts to a Λ-algebra homomorphism

$$\eta_{f_\infty} : \mathbb{T}_\infty^{\mathrm{ord}} \to \Lambda^\dagger.$$

4. Show that it is possible to define a power series $a_n(k) = \eta_{f_\infty}(T_n)$, converging in a neighbourhood of $2 \in \mathfrak{X}$ such that $f_\infty = \sum_{n\geq 1} a_n(k)q^n$ has the desired properties.

$\square$

3.1 Hida theory

The main reference for this section will be the article [BD1], where they study a special case of the exceptional zero conjecture, in parallel with the setting developed by [GS]. Let $\tilde{\Lambda} = \mathbb{Z}_p[[\mathbb{Z}_p^*]]$ be the projective limit of $\mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^*]$, and $\Lambda = \mathbb{Z}_p[[(1+p\mathbb{Z}_p)^*]]$. These are usually referred as Iwasawa algebras. $\tilde{\Lambda}$ can be viewed as functions on the space of continuous $\mathbb{Z}_p$-algebra homomorphism, that we have denoted before as $\mathfrak{X}$.
With the standard notation of identifying an integer $k \geq 2$ with the character $x \mapsto x^{k-2}$, the element 2 corresponds to the augmentation map on $\tilde{\Lambda}$ and Λ (a word of caution must be said: this is just a convention that is not always followed; in many places, k is just identified with $x \mapsto x^k$).
For $n \geq 0$, we consider $Y(N, p^n)$, the open modular curve whose complex points are identified with $\mathbb{H}/(\Gamma_0(N) \cap \Gamma_1(p^n))$. Let

$$H_n := H_1(Y(N,p^n), \mathbb{Z}_p) = (\Gamma_0(N) \cap \Gamma_1(p^n))_{\mathrm{ab}} \otimes \mathbb{Z}_p.$$

The H_n are finitely generated $\mathbb{Z}_p$-modules equipped with an action of T_l (for $l \neq p$) and U_p. H_n is the direct sum of two spaces, $H_n = H_n^{\mathrm{ord}} \oplus H_n^{\mathrm{nil}}$, where U_p acts invertibly and topologically nilpotently, respectively.

Let H_∞^{ord} be the inverse limit of the ordinary parts. The main result is that this is a free Λ-module of finite rank with the property that

$$H_\infty^{\mathrm{ord}} \otimes_\Lambda \mathbb{Z}_p[\Gamma/\Gamma_r] = H_1(Y_1(p^r), \mathbb{Z}_p)^{\mathrm{ord}}$$

for all $r > 0$.
Let $\mathbb{T}_\infty^{\mathrm{ord}}$ be the Λ-algebra generated by the images of the Hecke operators acting on H_∞.

Theorem 21 (Hida)**.** *The algebra $\mathbb{T}_\infty^{\mathrm{ord}}$ is a free Λ-module of finite rank, unramified over the augmentation ideal of Λ.*

A normalised eigenform as that considered in the introduction of this chapter gives rise to an algebra homomorphism $\eta_f : \mathbb{T}_\infty^{\mathrm{ord}} \to \mathbb{Z}_p$ that sends T_n to $a_n(f)$ and U_p to $a_p(f)$. The restriction to Λ is the augmentation, corresponding to $2 \in \mathfrak{X}$.
We consider $\Lambda^\dagger \supset \Lambda$, the ring of power series which converge in some neighbourhood of $2 \in \mathfrak{X}$. Let $D_*^\dagger := D_* \otimes_\Lambda \Lambda^\dagger$. For being $\Lambda^\dagger$ henselian (and also since the augmentation ideal is unramified), η_f lifts uniquely to a Λ-algebra homomorphism $\eta_{f_\infty} : \mathbb{T}_\infty^{\mathrm{ord}} \to \Lambda^\dagger$. Considering $a_n(k) = \eta_{f_\infty}(T_n)$, consider

$$f_\infty := \sum_{n=1}^{\infty} a_n(k) q^n.$$

Fix a neighbourhood of $2 \in \mathfrak{X}$ on which the a_n converge (assume that is contained in the residue class of 2 modulo $p-1$). If $k \in U \cap \mathbb{Z}^{\geq 2}$, the weight k specialization is a normalized eigenform of weight k on $\Gamma_0(N)$, which is new at the primes dividing $M = N/p$ ([GS]). In particular, if $k \in U \cap \mathbb{Z}^{\geq 2}$, f_k arises from a normalized eigenform on $\Gamma_0(M)$ that we denote $f_k^\sharp$. Considering

$$1 - a_p(f_k^\sharp)p^{-s} + p^{k-1-2s} = (1-\alpha_p(k)p^{-s})(1-\beta_p(k)p^{-s})$$

we may order the roots $\alpha_p(k)$ and $\beta_p(k)$ in such a way that $\alpha_p(k) = a_p(f_k)$ and $\beta_p(k) = p^{k-1}a_p(f_k)^{-1}$. With this convention,

$$f_k(z) = f_k^\sharp(z) - \beta_p(k) f_k^\sharp(pz).$$

The field K_{f_k} generated by the Fourier coefficients of f_k is an extension of $\mathbb{Q}$ of finite degree, which we view as being embedded into both $\mathbb{C}$ and $\mathbb{C}_p$. For each k, we choose the Shimura periods $\Omega_k^+ := \Omega_{f_k}^+$ and $\Omega_k^- := \Omega_{f_k}^-$, requiring that

$$\Omega_2^+\Omega_2^- = \langle f, f\rangle, \qquad \Omega_k^+\Omega_k^- \langle f_k^\sharp, f_k^\sharp\rangle \quad (k > 2).$$

Since the Fourier coefficients of f_k vary p-adically analytically with k, it is natural to ask whether the functions I_{f_k}, which encode the Shimura periods of f_k, can be viewed as part of an analytically varying family. This aspect will be worked out in the following section.

3.2 Measure-valued modular symbols

The aim of this section is to develop the techniques needed to introduce the Mazur-Kitagawa p-adic L-function, one of the easiest and most well-known example of L-function in which there are two parameters, the usual variable s and the weight k at which a Hida family f_∞ especializes.
Let $L_* := \mathbb{Z}_p^2$ be the standard $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, and let L'_* be its set of primitive vectors (not divisible by p). The space of continuous $\mathbb{C}_p$-valued functions on L'_* is equipped with the right action of $\mathrm{GL}_2(\mathbb{Z}_p)$ defined by

$$(F|g)(x,y) := F(ax+by, cx+dy), \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Its continuous dual is the space of measures on L'_* and it is denoted by $\mathbb{D}_*$. The action of $\mathbb{Z}_p^*$ on L'_*, given by $\lambda(x,y) = (\lambda x, \lambda y)$ gives a natural $\tilde{\Lambda}$-module structure on $\mathbb{D}_*$, by writing

$$\int_{L'_*} F(x,y)d(a\cdot\mu)(x,y) := \int_{L'_*} F(ax,ay)d\mu(x,y),$$

for all $a \in \mathbb{Z}_p^*$. If X is a compact open subset of L'_* we understand that

$$\int_X F d\mu = \int_{L'_*} 1_X F d\mu.$$

The group $\mathrm{GL}_2(\mathbb{Z}_p)$ also acts on $\mathbb{D}_*$ on the left by translation

$$\int_X F d(\gamma\mu) = \int_{\gamma^{-1}X} (F|\gamma) d\mu.$$

Another important fact in $\mathbb{D}_*$ is that it is equipped, for all integers $k \geq 2$ with a $\Gamma_0(p\mathbb{Z}_p)$-equivariant homomorphism

$$\rho_k : \mathbb{D}_* \to V_k$$

defined by

$$\rho_k(\mu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x,y) d\mu(x,y).$$

This homomorphism ρ_k gives rise to a homomorphism

$$\rho_k : \mathrm{MS}_{\Gamma_0(M)}(D_*) \to \mathrm{MS}_{\Gamma_0(N)}(V_k).$$

If $\mu = \lambda_1\mu_1 + \ldots + \lambda_t\mu_t$, with $\lambda_i \in \Lambda^\dagger$ and $\mu_j \in \mathbb{D}_*$ is an element of $\mathbb{D}_*^\dagger$, there exists a neighbourhood U_μ of $2 \in \mathfrak{X}$ on which all the coefficients λ_j converge. Such a region will be called a neighbourhoof of regularity for μ.
For $k \in U_\mu$, we way that a continuous function F on L'_* is homogeneous of degree k if $F(\lambda x, \lambda y) = \lambda^{k-2} F(x,y)$ for all $\lambda \in \mathbb{Z}_p^\times$. For $k \in U_\mu$ and $F(x,y)$ homogenous of degree $k-2$, the function F can be integrated againts μ according to

$$\int_X F d\mu := \sum_{i=1}^{t} \lambda_i(k) \int_X F d\mu_i,$$

for $X \subset L'_*$ compact.
In $\mathrm{MS}_{\Gamma_0(M)}(\mathbb{D}_*)$ we can consider a natural action of the Hecke operators, including an operator U_p compatible with the specialization maps:

$$\int_X F d(U_p\mu)\{r \to s\} = \sum_{a=0}^{p-1} \int_{p^{-1}\gamma_a X} (F|p\gamma_a^{-1}) d\mu\{\gamma_a r \to \gamma_a s\}.$$

If $\mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbb{D}_*)$ denotes the ordinary subspace, [GS] asserts that this module is free and of finite rank over Λ.
Let $r, s \in \mathbb{P}_1(\mathbb{Q})$ and $\mu \in \mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}$; choose a common neighbourhood of regularity U_μ for the measures $\mu\{r \to s\}$ and this makes possible to define $\rho_k(\mu)$ for $k \in U_\mu \cap \mathbb{Z}^{\geq 2}$.

Theorem 22. *There exists a neighbourhood U of $2 \in \mathfrak{X}$ and a measure-valued symbol $\mu_* \in \mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbb{D}_*)^\dagger$ regular on U and satisfying:*

1. *$\rho_2(\mu_*) = I_f$.*
2. *For $k \in U \cap \mathbb{Z}^{\geq 2}$, there is a scalar $\lambda(k) \in \mathbb{C}_p$ such that $\rho_k(\mu_*) = \lambda(k) I_{f_k}$.*

Theorem 23. *There is a neighborhood U of $2 \in \mathfrak{X}$ with $\lambda(k) \neq 0$ for $k \in U \cap \mathbb{Z}^{\geq 2}$.*

Strongly related with these concepts, we have the Mazur-Kitagawa p-adic L-functions. The $\mathbb{D}_*^\dagger$-valued modular symbol μ_* can be used to define a two-variable p-adic L-function attached to f and a Dirichlet character χ (the so-called Mazur-Katagawa).

Definition 26. *Let χ be a primitive quadratic character of conductor m such that $\chi(-1) = w_\infty$. The Mazur-Kitagawa two-variable p-adic L-function attached to χ is the function of $(k,s) \in U \times \mathfrak{X}$ defined by the rule*

$$L_p(f_\infty, \chi, k, s) = \sum_{a=1}^{m} \chi(ap) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \left(x - \frac{pa}{m} y\right)^{s-1} y^{k-s-1} d\mu_*\{\infty \to pa/m\}.$$

It satisfies the following interpolation property with respect to special values of the classical L-functions $L(f_k, \chi, s)$:

Theorem 24. *Let $k \geq 2$ be an integer in U. Take $1 \leq j \leq k-1$ such that $\chi(-1) = (-1)^{j-1} w_\infty$. Then,*

$$L_p(f_\infty, \chi, k, j) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{j-1})L^*(f_k, \chi, j),$$

where

$$L_p^*(f_\infty, \chi, k) := \sum_{a=1}^{m} \chi(a) \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} y^{k-2} d\mu_*\{\infty \to a/m\}$$

is the so-called improved p-adic L-function, that extends to a p-adic analytic function of $k \in U$.

3.3 The structure of the ordinary subspaces

In this section, we recover some of the ideas we have explored in the previous chapter about modular forms. The idea is that working in families allows us to prove very remarkable results about the structure of the ordinary subspace of modular forms. The general philosophy is that those forms that lie in the ordinary subspace (U_p operator acting with unit eigenvalue) have better behavior concerning the uniqueness of the interpolating family.

Theorem 25. *For any $k \geq 2$ and any character χ modulo p^n we have that*

$$\operatorname{rank}_{\mathbb{Z}_p} S_k^{\mathrm{ord}}(\Gamma_0(p^n), \chi\omega^{-k}; \mathbb{Z}_p) = \operatorname{rank}_{\mathbb{Z}_p} S_2^{\mathrm{ord}}(\Gamma_0(p^n), \chi\omega^{-2}; \mathbb{Z}_p).$$

(The rank is constant as the weight varies).

Then, we can hope to p-adically interpolate the spaces $S_k^{\mathrm{ord}}(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p)$ as the weight varies, and we already have a candidate space for this: $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)$, the Λ-module of ordinary Λ-adic cusp forms (which is endowed with a unique idempotent e_p^Λ).

Theorem 26. *Let χ be a character of conductor p. Then,*

1. *The space $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)$ is free of finite rank over Λ, and in fact we have that*

$$\operatorname{rank}_\Lambda \mathbb{S}^{\mathrm{ord}}(\chi, \Lambda) = \operatorname{rank}_{\mathbb{Z}_p} S_2^{\mathrm{ord}}(\Gamma_0(p), \chi\omega^{-2}; \mathbb{Z}_p).$$

2. *After an extension of coefficients to a certain finite extension K of the fraction field of Λ, the space $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda) \otimes_\Lambda K$ has a basis consisting of Hecke eigenforms, and the specialization of this basis at weight k ($k \geq 2$) gives a basis of eigenforms for the space $S_k^{\mathrm{ord}}(\Gamma_0(p), \chi\omega^{-k}; \mathcal{O})$, where $\mathcal{O}$ is the ring of integers in some finite extension of $\mathbb{Q}_p$.*

As it happened with overconvergent modular forms and overconvergent modular symbols, we have here a control theorem for fixed level:

Theorem 27 (Control theorem)**.** *For each $k \geq 2$, let $\mathfrak{p}_k$ be the prime ideal of Λ generated by $X - (u^k - 1)$. Then, evaluation at $u^k - 1$ induces an isomorphism*

$$\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)/\mathfrak{p}_k \cong S_k^{\mathrm{ord}}(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p).$$

Hence, $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)$ is the interpolating space in the case of ordinary forms.

An important remark is that we can also consider variations in the level (until now, we only vary the weight). Such families also interpolate modular forms of varying p-power levels. Let ϵ be a finite order character of $1+p\mathbb{Z}_p$ factoring through the quotient $(1+p\mathbb{Z}_p)/(1+p\mathbb{Z}_p)^{p^\alpha}$, with α minimal. Then, if F is an ordinary Λ-adic modular form of character χ,

$$F(\epsilon(u)u^k-1) \in S_k^{\mathrm{ord}}(\Gamma_0(p^{\alpha+1}), \chi\epsilon\omega^{-k}; \mathbb{Z}_p[\epsilon]).$$

Hence, the control theorem can be reformulated as follows:

Theorem 28 (Control theorem, varying levels)**.** *Let $k \geq 2$ and ϵ as above. Let $\mathfrak{p}_{k,\epsilon}$ be the prime ideal of Λ generated by $X-(\epsilon(u)u^k-1)$. Then, evaluation at $\epsilon(u)u^k-1$ induces an isomorphism*

$$\mathbb{S}^{\mathrm{ord}}(\chi,\Lambda)/\mathfrak{p}_{k,\epsilon} \cong S_k^{\mathrm{ord}}(\Gamma_0(p^{\alpha+1}), \chi\epsilon\omega^{-k}; \mathbb{Z}_p[\epsilon]).$$

Thus, Λ-adic cusp forms interpolate modular forms on varying weights and levels.

3.4 Hida families and applications

At this point of the discussion it is convenient to reformulate the definition of Hida family in a more general framework, that will be the one we will be using to formulate and prove our main results, as it is done for instance in [DR1].
Our picture consists of the following ingredients:

- The Iwasawa algebra $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$.
- The weight space $\Omega = \mathrm{Hom}(\Lambda, \mathbb{C}_p) \subset \mathrm{Hom}((1+p\mathbb{Z}_p)^\times, \mathbb{C}_p^\times)$. Here, the integers form a dense subset via $k \mapsto (x \mapsto x^k)$.
- Classical weights: $\Omega_{\mathrm{cl}} = \mathbb{Z}^{\geq 2} \subset \Omega$.
- If $\tilde{\Lambda}$ is a finite flat extension of Λ, let $\tilde{\mathfrak{X}} = \mathrm{Hom}(\tilde{\Lambda}, \mathbb{C}_p)$ and let $\kappa : \tilde{\mathfrak{X}} \to \Omega$ be the natural projection to weight space. Then, $\tilde{\mathfrak{X}}_{\mathrm{cl}}$ is the set of x such that $\kappa(x) \in \Omega_{\mathrm{cl}}$.

Definition 27. *A Hida family of tame level N is a quadruple $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \mathbf{f})$, where:*

1. *Λ_f is a finite flat extension of Λ.*
2. *$\Omega_f \subset \mathfrak{X}_f = \mathrm{Hom}(\Lambda_f, \mathbb{C}_p)$ is a non-empty open subset for the p-adic topology.*
3. *$\mathbf{f} = \sum a_n q^n \in \Lambda_f[[q]]$ is a formal q-series such that $\mathbf{f}(x) := \sum x(a_n)q^n$ is the q-series of the ordinary p-stabilisation $f_x^{(p)}$ of a normalised eigenform (we write f_x) of weight $\kappa(x)$ on $\Gamma_1(N)$ for all $x \in \Omega_{f,\mathrm{cl}} := \Omega_f \cap \mathfrak{X}_{f,\mathrm{cl}}$.*

Hida's theorem, that has already been stated in a different way, says the following:

Theorem 29. *Let f be a normalized eigenform of weight $k \geq 1$ on $\Gamma_1(N)$ and $p \nmid N$ an ordinary prime for f. Then, there exists a Hida family $(\Lambda_f, \Omega_f, \mathbf{f})$ and a classical point $x_0 \in \Omega_{f,\mathrm{cl}}$ such that $\kappa(x_0) = k$ and $f_{x_0} = f$.*

By shrinking Ω_f if necessary, we can assume that $\kappa(x) = k$ modulo $p-1$ for all $x \in \Omega_f$. In particular, $\kappa(x)$ and k have the same parity for all classical $x \in \Omega_f$. It is also convenient to dispose of a more flexible notion of p-adic families of modular forms, interpolating classical modular forms not necessarily new, of even Hecke eigenvectors (and to allow Fourier coefficients to belong to more general rings). This leads us to the following definition:

Definition 28. *A Λ-adic modular form of tame level N is a quadruple $(R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}, \phi)$, where:*

1. *R is a complete, finitely generated (not necessarily finite), flat extension of Λ.*

2. *Ω_ϕ is an open subset of $\mathrm{Hom}(R, \mathbb{C}_p)$ and $\Omega_{\phi,\mathrm{cl}}$ is a dense subset of Ω_ϕ.*

3. *$\phi = \sum a_n q^n \in R[[q]]$ is a formal q-series with coefficients in R such that for all $x \in \Omega_{\phi,\mathrm{cl}}$, the power series*
$$\phi_x^{(p)} := \sum_{n=1}^{\infty} \mathbf{a}_n(x) q^n \in \mathbb{C}_p[[q]]$$
is the q-expansion of a classical ordinary cusp form in $S_{\kappa(x)}(\Gamma_1(N) \cap \Gamma_0(N); \mathbb{C}_p) := S_{\kappa(x)}(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{Q}) \otimes \mathbb{C}_p$.

When x varies over $\Omega_{f,\mathrm{cl}}$, the corresponding specializations f_x give rise to a p-adically coherent collection of classical newforms on $\Gamma_1(N)$. One of the aims that we will explore in later chapters is how to construct p-adic L-functions interpolating classical special values attached to these eigenforms. For instance, we want them to interpolate critical values like
$$\frac{L(f_x \otimes g_y, j)}{\Omega(f_x, g_y, j)} \in \bar{\mathbb{Q}},$$
where $(x, y, j) \in \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \mathbb{Z}$.
This kind of formulas, together with the relation between the L-values at special points and the images under étale and syntomic regulators of certain elements in the higher Chow groups will be developed along this thesis. Now, let us fix just some notation. Let $(\Lambda_g, \Omega_g, \mathbf{g})$ and $(\Lambda_h, \Omega_h, \mathbf{h})$ be Λ-adic modular forms of tame level N. Let $\Lambda_{gh} = \Lambda_g \otimes_{\mathcal{O}} \Lambda_h$ be the finitely generated Λ-algebra equipped with the natural diagonal embedding $\Lambda \hookrightarrow \Lambda_g \otimes \Lambda_h$ sending $[a] \in \Lambda$ to $[a] \otimes [a]$. Set $\Omega_{gh} := \Omega_g \times \Omega_h$ and $\Omega_{gh,\mathrm{cl}} := \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$. Then, $(\Lambda_{gh}, \Omega_{gh}, \Omega_{gh,\mathrm{cl}}, e_{\mathrm{ord}}(\mathbf{g} \times \mathbf{h}))$, where $\mathbf{g} \times \mathbf{h}$ is viewed as an element of $\Lambda_{gh}[[q]]$ is an example of a Λ-adic modular form with Fourier coefficients in Λ_{gh}.
Assume now that $p \nmid N$, and let $a_p(\mathbf{g}) \in \Lambda_g$, $a_p(\mathbf{h}) \in \Lambda_h$ denote the Hecek eigenvalues associated to T_p. Consider now
$$\mathbf{g}^{[p]} := (1 - VU)\mathbf{g} = \sum_{p \nmid n} a_n(\mathbf{g}) q^n.$$
Its specialization $g_y^{[p]}$ at $y \in \Omega_{g,\mathrm{cl}}$ can either be viewed as a p-adic modular form of tame level N or as a classical modular form of level Np^2. The fact that $\mathbf{g}^{[p]}$ has Fourier coefficients supported on the integers prime to p allows the forml q-series
$$d^\bullet \mathbf{g}^{[p]} := \sum_{p \nmid n} [n] a_n(\mathbf{g}) q^n$$
to be seen as an element of $\Lambda \otimes_{\mathcal{O}} \Lambda_g[[q]]$. The specialization at $(t, y) \in \Omega_{\mathrm{cl}} \times \Omega_{g,\mathrm{cl}}$ is just $d^t g_y^{[p]}$.
Let $R_{gh} := \Lambda \otimes_{\mathcal{O}} \Lambda_g \otimes_{\mathcal{O}} \Lambda_h$, so that the map from $\mathrm{Hom}(R_{gh}, \mathbb{C}_p) = \Omega \times \Omega_g \times \Omega_h$ to the weight space sends $(t, y, z) \in \mathbb{Z}^{\geq 0} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$ to $\kappa(y) + \kappa(z) + 2t$. The specialization of $e_{\mathrm{ord}}(d^\bullet \mathbf{g}^{[p]} \times \mathbf{h})$ at (t, y, z) is equal to
$$e_{\mathrm{ord}}(d^\bullet \mathbf{g}^{[p]} \times \mathbf{h})_{t,y,z} = e_{\mathrm{ord}}(d^t g_y^{[p]} \times h_z^{(p)}) = e_{\mathrm{ord}}(d^t g_y^{[p]} \times h_z).$$
The following proposition will play a key role in the construction of the triple Garrett-Rankin p-adic L-function.

Proposition 14. *Let*

$$\Omega_{gh,\mathrm{cl}} := \{(t,y,z) \in \mathbb{Z} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}, \quad t > -\min(\kappa(y),\kappa(z)).\}$$

The quadruple

$$e_{\mathrm{ord}}(d^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h}) := (R_{gh}, \Omega \times \Omega_g \times \Omega_h, \Omega_{gh,\mathrm{cl}}, e_{\mathrm{ord}}(d^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h}))$$

is an ordinary Λ-adic modular form of tame level N. For $(t,y,z) \in \Omega_{gh,\mathrm{cl}}$, the specialization at (t,y,z) is the classical modular form $e_{\mathrm{ord}}(d^t g_y^{[p]} \times h_z)$, that belongs to $S_k(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{C}_p)$.

Typically, we denote by $\mathbb{S}^{\mathrm{ord}}(N;R)$ the space of Λ-adic modular forms with coefficients in the Λ-algebra R. A Hida family $(\Lambda_f, \Omega_f, \mathbf{f})$ of eigenforms gives rise to a subspace

$$\mathbb{S}^{\mathrm{ord}}(N;\Lambda_f)[\pi_f] := \left\{\tilde{\mathbf{f}} \in \mathbb{S}^{\mathrm{ord}}(N;\Lambda_f) \text{ such that } T_n\tilde{\mathbf{f}} = \mathbf{a_n}\tilde{\mathbf{f}}, \text{ for all } (n,N)=1\right\}.$$

Let $\phi = (R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}) \in \mathbb{S}^{\mathrm{ord}}(N;R)$ be a Λ-adic modular form, and let $(x,y) \in \Omega_{f,\mathrm{cl}} \times \Omega_{\phi,\mathrm{cl}}$ be a pair of points with $\kappa(x) = \kappa(y)$. The projection $e_{f_x}\phi_y^{(p)}$ is the p-stabilization of a classical modular form, that we will denote by $\phi_{x,y}$.

Lemma 8. *For all $\tilde{\mathbf{f}} \in \mathbb{S}^{\mathrm{ord}}(N;\Lambda_f)[\pi_f]$ and all $\phi = (R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}, \phi) \in \mathbb{S}^{\mathrm{ord}}(N;R)$, there exists a unique $J(\tilde{\mathbf{f}},\phi) \in \Lambda'_f \otimes_\Lambda R$ such that, for $(x,y) \in \Omega_{f,\mathrm{cl}} \times \Omega_{\phi,\mathrm{cl}}$,*

$$J(\tilde{\mathbf{f}},\phi)(x,y) = \frac{\langle f_x, \phi_{x,y}\rangle_N}{\langle f_x, f_x\rangle_N} = \langle \eta_{f_x}, \phi_{x,y}\rangle.$$

An alternative point of view in which we will be frequently interested is the Galois theoretic approach. For this introductory approach, we will use the less standard notations (taken from [Em]) $S_l = \langle l\rangle l^{k-2}$ and we will write $\mathbb{T}_k(N)$ for the Hecke algebra generated by the operators lS_l and T_l for all $l \nmid N$. When f is a Hecke eigenform, there is a ring homomorphism $\lambda : \mathbb{T}_k \to \mathbb{C}$ such that $Tf = \lambda(T)f$, for all $T \in \mathbb{T}_k$. We refer to such a homomorphism λ as a system of Hecke eigenvalues.
Let us recall some basic facts about Galois representations: fix $k \geq 1$ and $N \geq 1$ and choose a prime number p. If λ is a system of Hecke eigenvalues appearing in $M_k(N)$, since we know that λ takes values in the ring $\overline{\mathbb{Z}}$ of algebraic itegers, we may compose it with a fixed embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ and so regard λ as taking values in $\overline{\mathbb{Z}_p}$. For the remainder, we will regard the systems of Hecke eigenvalues as being $\overline{\mathbb{Z}_p}$-valued, and if we write $\lambda : \mathbb{T}_k \to \bar{\mathbb{Z}}_p$ for the system of Hecke eigenvalues we can consider the composition with the reduction map $\bar{\lambda} : \mathbb{T}_k \to \overline{\mathbb{F}_p}$.

Let Σ denote the set of primes dividing Np, let $\mathbb{Q}_\Sigma$ be the maximal algebraic extension of $\mathbb{Q}$ unramified outside Σ and let $G_{\mathbb{Q},\Sigma}$ be its Galois group.

Theorem 30. *There is a continuous, semi-simple representation $\rho_\lambda : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ uniquely determined (up to equivalence) by the condition that for each prime $l \nmid Np$, the matrix $\rho_l(\mathrm{Frob}_l)$ has characteristic polynomial equal to $X^2 - \lambda(T_l)X + \lambda(lS_l)$.*

Given $\lambda : \mathbb{T}_k(N) \to \mathbb{C}$, for $l \nmid N$, the l-th Hecke polynomial of λ is $X^2 - \lambda(T_l)X + \lambda(lS_l)$. Moreover, a remarkable fact is that when λ arises from a cuspform, the representation ρ_λ is irreducible.
A common notation is to write $\mathbb{T}_k^{(p)}$ for the subalgebra of $\mathbb{T}_k$ generated by lS_l and T_l for l not dividing Np. As a ring, $\mathbb{T}_k^{(p)}$ has finite index in $\mathbb{T}_k$. On the other hand, $\mathbb{T}_{\leq k}^{(p)}(N)$ will refer to the $\mathbb{Z}$-algebra of endomorphism of $\oplus_{i=1}^{i} M_i(N)$ generated by lS_l and T_l, for $l \nmid Np$. Finally, we define the p-adic Hecke algebra $\mathbb{T}(N)$ (or just $\mathbb{T}$ when the level is understood) as the projective limit $\mathbb{T} := \lim_{\leftarrow} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_{\leq k}^{(p)}$. This ring $\mathbb{T}$ has very nice properties, namely, it is a p-adically complete, noetherian $\mathbb{Z}_p$-algebra and it is the product of finitely many complete noetherian local $\mathbb{Z}_p$-algebras.

Definition 29. *A p-adic system of Hecke eigenvalues is a homomorphism of $\mathbb{Z}_p$-algebras $\xi : \mathbb{T} \to \overline{\mathbb{Z}_p}$.*

Theorem 31. *If $\xi : \mathbb{T} \to \overline{\mathbb{Z}_p}$ is a p-adic system of Hecke eigenvalues, there is a continuous, semi-simple representation*

$$\rho_\xi : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$$

uniquely determined (up to equivalence) by the condition that for each prime $l \nmid Np$, the matrix $\rho_\xi(\mathrm{Frob}_l)$ has characteristic polynomial equal to $X^2 - \xi(T_l)X + \xi(lS_l)$.

There is a canonical map $\operatorname{Spec} \mathbb{T} \to \operatorname{Spec} \mathbb{Z}[[T]]$. Set $\Gamma = 1 + q\mathbb{Q}_p$ and let $\mathcal{L} := \{l \text{ prime } \mid l \equiv 1 \bmod Nq\}$, which is dense in Γ; the map $\mathcal{L} \to \mathbb{T}$ given by $l \mapsto S_l$ extends uniquely to a continuous group homomorphism $\Gamma \to \mathbb{T}^\times$.

This theory allows us to understand some basic examples of one-dimensional families of system of Hecke eigenvalues prametrized by weight, as the Eisenstein family. But until now, we have omitted the Hecke operators S_p and T_p. For example, for the Eisenstein case, it turns out that $\lambda_k(S_p) = p^{k-2}$ and $\lambda_k(T_p) = 1 + p^{k-1}$, which do not interpolate well as p-adic functions of k. Its Hecke polynomial is $(X-1)(X-p^{k-1})$ and while the first root interpolates well, the second one does not. The idea must be to consider points not in $\operatorname{Spec} \mathbb{T}$, but in $\operatorname{Spec} \mathbb{T} \times \mathbb{G}_m$.

Definition 30. *Let $\mathcal{X}$ denote the set of $\overline{\mathbb{Q}_p}$-valued points of $\operatorname{Spec} \mathbb{T} \times \mathbb{G}_m$ consisting of pairs (ξ, α), where $\xi : \operatorname{Spec} \mathbb{T} \to \overline{\mathbb{Z}_p}$ is classical, attached to some system λ of Hecke eigenvalues, and α is a root of the p-th Hecke polynomial. We write $\mathcal{X}^{\mathrm{ord}}$ for the subset of $\mathcal{X}$ consisting of pairs (ξ, α) for which $\alpha \in \overline{\mathbb{Z}_p}^\times$.*

The following result of Hida will allow us to describe the interpolation of the points in $\mathcal{X}^{\mathrm{ord}}$:

Theorem 32. *The Zariski closure $\mathcal{C}^{\mathrm{ord}}$ of $\mathcal{X}^{\mathrm{ord}}$ in $\operatorname{Spec} \mathbb{T} \times \mathbb{G}_m$ is one-dimensional; the composite $\mathcal{C}^{\mathrm{ord}} \to \operatorname{Spec} \mathbb{T} \times \mathbb{G}_m \to \operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ is finite and étale in the neighborhood of those points of $\mathcal{X}^{\mathrm{ord}}$ attached to system of Hecke eigenvalues appearing in weight $k \geq 2$. We refer to $\mathcal{C}^{\mathrm{ord}}$ as the Hida family of tame level N.*

Theorem 33 (Coleman-Mazur)**.** *The rigid analytic Zariski closure $\mathcal{C}$ of $\mathcal{X}$ in $(\operatorname{Spec} \mathbb{T} \times \mathbb{G}_m)^{\mathrm{an}}$ is one-dimension and the composite $\mathcal{C} \hookrightarrow (\operatorname{Spec} \mathbb{T} \times \mathbb{G}_m)^{\mathrm{an}} \to (\operatorname{Spec} \mathbb{Z}_p[[\Gamma]])^{\mathrm{an}}$ is flat and has discrete fibers. Further, for any positive constant C, there are finitely many points $(\xi\alpha)$ in any given fiber with $\mathrm{ord}_p(\alpha) \leq C$. The curve $\mathcal{C}$ is called the eigencurve of tame level N.*

The following result, due to Mazur and Wiles, gives a Galois theoretic interpretation of the points of $\mathcal{C}^{\mathrm{ord}}$.

Theorem 34. *If (ξ, α) is a $\overline{\mathbb{Z}_p}$-valued point in $\mathcal{C}^{\mathrm{ord}}$, then $\rho_\xi|_{G_{\mathbb{Q}_p}}$ admits a one-dimensional unramified quotient on which* Frob_p *acts with eigenvalue α.*

The eigencurve turns out to be a very important objects that has plenty of applications in all these works. We refer the reader to [Bel2] for a better understanding, and also to [BeDi] for the latest achievements on that field.

3.5 Hida-Rankin's p-adic L-functions

We explore in this section the p-adic L-function introduced in the article of Bertolini and Darmon [BD2] devoted to the study of Kato's Euler systems arising from p-adic families of Beilinson elements in the K-theory of modular curves. We also discuss the modifications we need to present the Hida-Rankin p-adic L-function of [BDR1], that will be related with p-adic families of Beilinson-Flach elements (see Chapter 6 for details).
This section, together with the following one, introduces different types of p-adic L-function with at least one parameter varying in a family of weights, that will turn out to be closely related with global cohomology classes and that will be the main inspiration for the main results we will develop along this thesis. In a simple setting, Kato's construction yields a global class $\kappa \in H^1(\mathbb{Q}, V_p(E))$, that is crystalline (and hence belongs to the p-adic Selmer group of E) precisely when $L(E,s)$ vanishes at $s=1$. The goal of [BD2] is a proof of a p-adic Beilinson formula relating the syntomic regulators of certain distinguished elements in the K-theory of modular curves to the special values at integer points ≥ 2 of the MSD p-adic L-function attached to a cusp form f of weight 2. This formula, which will be presented in the following chapters, is based on the direct evaluation of the p-adic Rankin L-function attached to a Hida family interpolating f.
In general, the common strategy for the study of this type of special formulas arising from Euler systems of Garrett-Rankin-Selberg type consists on three steps: first of all, introducing the appropriate p-adic L-function, typically in terms of some interpolating property; then, defining the "geometric" and "cohomological" ingredients that come into play, establishing how they vary in p-adic families. Finally, the aim is to prove some kind of relation between these two different objects.
This section tries to be self contained and for that reason we will recover some of the results we have already seen in previous chapters concerning nearly holomorphic modular forms and the Rankin-Selberg method.

We will work with f, g, two normalized newforms of weights k, l, levels N_f, N_g and nebentypus χ_f, χ_g; we put $\chi = \chi_f^{-1}\chi_g^{-1}$. If N is the least common multiple of N_f, N_g we can replace χ_f and χ_g by their counterparts of modulus N and we also replace f and g by normalized eigenforms of level N with the same eigenvalues for the good Hecke operators T_r (with $\gcd(r,N)=1$) and such that they are eigenvectors for the U_r attached to the primes $r|N$.
As we have already seen, the non-holomorphic Eisenstein series of weight k and level N attached to a primitive character χ of conductor N is

$$\tilde{E}_{k,\chi}(z,s) = \sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi^{-1}(n)}{(mz+n)^k} \cdot \frac{y^s}{|mz+n|^{2s}},$$

where the apostrophe means the sum over the non-zero lattice vectors. It converges when $\Re(s) > 1 - k/2$. If $k > 2$, $\tilde{E}_{k,\chi}(z) := \tilde{E}_{k,\chi}(z, 0)$ belongs to $M_k(N, \chi)$.
The normalized Eisenstein series $E_{k,\chi}(z)$ is

$$E_{k,\chi}(z) = \frac{1}{2N^{-k}\tau(\chi^{-1})} \cdot \frac{(k-1)!}{(-2\pi i)^k} \tilde{E}_{k,\chi}(z).$$

The Shimura-Maass derivative operator was defined as

$$\delta_k := \frac{1}{2\pi i}\Big(\frac{d}{dz} + \frac{ik}{2y}\Big).$$

It sends (real analytic) modular forms of weight k to forms of weight $k+2$. There are different relations between Eisenstein series linked via this operator; for instance

$$\delta_k \tilde{E}_{k,\chi}(z, s) = -\frac{(s+k)}{4\pi} \tilde{E}_{k+2,\chi}(z, s-1).$$

At this point of the discussion, we need to recover the more general Eisenstein series $E_k(\chi_1, \chi_2) \in M_k(N, \chi_1\chi_2)$, namely

$$E_k(\chi_1, \chi_2)(z) = \delta_{\chi_1} L(\chi_1^{-1}\chi_2, 1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(\chi_1, \chi_2)(n) q^n,$$

being $\delta_{\chi_1} = 1/2$ when $N_1 = 1$ and 0 otherwise.
The following property will be very useful, since it is concerned with a factorization of the L-function that will appear again when proving the relation of the L-function with Beilinson-Kato elements:

$$L(E_k(\chi_1, \chi_2), s) = L(\chi_1, s) L(\chi_2, s-k+1).$$

The prototypical formula one must bear in mind, and in which many of the following results will be based on, is that of the Rankin-Selberg method, that is proved for instance in [Hi] basically via a direct computation:

$$L(f \otimes g, s) = \frac{1}{2} \frac{(4\pi)^s}{\Gamma(s)} \Big\langle f^*(z), \tilde{E}_{k-l,\chi}(z, s-k+1) \cdot g(z) \Big\rangle_{k,N}.$$

There are also different results concerning factorization for critical values of the L-series of the convolution product. Assume that $l < k$; j is critical if and only if it lies in $[l, k-1]$ (here, there are several results of Deligne involved). Fix $j \in [(l+k-1)/2, k-1]$, let $k \geq 0$ and $m \geq 1$ defined as

$$t := k - 1 - j, \qquad m := k - l - 2t.$$

If $m \leq 2$, assume also that χ is non trivial. Let

$$\Xi(f, g, j) := \delta_m^t E_{m,\chi} \times g,$$

a nearly holmorphic modular form of nebentypus χ_f^{-1}. Its image under the holomorphic projection Π_N^{hol} is in $S_k(N, \chi_f^{-1}; K_{fg})$. We have that

$$L(f \otimes g, j) = C(f, g, j) \cdot \Big\langle f^*(z), \delta_m^t E_{m,\chi}(z) \times g(z) \Big\rangle_{k,N}.$$

In particular,

$$L^{\mathrm{alg}}(f\otimes g,j):=C(f,g,j)^{-1}\frac{L(f\otimes g,j)}{\langle f^*,f^*\rangle_{k,N}}\in K_{fg}.$$

We will work with g the Eisenstein series $E_l(\chi_1,\chi_2)$. Hence, by the factorization we have seen,

$$L(f\otimes E_l(\chi_1,\chi_2),c)=L(f,\chi_1,c)\cdot L(f,\chi_2,c-l+1).$$

One must do some technical assumptions towards deriving the main results. In particular, we will assume that $l=m$, χ_f is trivial, χ is primitive and $(N_1,N_2)=1$. Then, f is an eigenform of weight $k=2l+2t$; if $t\geq 0$, then $c=k/2+l-1$ is a critical point for $L(f\otimes E_l(\chi_1,\chi_2),s)$ and

$$L(f\otimes E_l(\chi_1,\chi_2),k/2+l-1)=L(f,\chi_1,k/2+l-1)\cdot L(f,\chi_2,k/2).$$

We can normalize the L-function and consider

$$L^*(f,\psi,j)=\frac{(j-1)!\tau(\bar{\psi})}{(-2\pi i)^{j-1}\Omega_f^{\epsilon}}L(f,\psi,j)\in\mathbb{Q}_{f,\psi},$$

where $\Omega_f^{\pm}$ are complex periods defined in [BD1]. Then, combining the previous expressions,

$$L^*(f,\chi_1,k/2+l-1)\cdot L^*(f,\chi_2,k/2)=C_{f,\chi_1,\chi_2}\cdot\frac{\left\langle f,(\delta_l^{k/2-l}E_{l,\chi})\cdot E_l(\chi_1,\chi_2)\right\rangle_{k,N}}{\langle f,f\rangle_{k,N}}.$$

When $k=l=2$ ($c=2$ and $s=-1$), it is possible to deduce a complex Beilinson formula for the non critical value of $L(f,s)$ at $s=2$. By the Rankin method,

$$L(f\otimes E_2(\chi_1,\chi_2),2)=\frac{1}{2}(4\pi)^2\Big\langle f(z),\tilde{E}_{0,\chi}(z,1)\cdot E_2(\chi_1,\chi_2)(z)\Big\rangle_{2,N}.$$

Choose units u_χ and $u(\chi_1,\chi_2)$ with logarithmic derivatives equal to $E_{2,\chi}$ and $E_2(\chi_1,\chi_2)$. Then, the equation can be written as

$$L(f\otimes E_2(\chi_1,\chi_2),2)=16\pi^3N^{-2}\tau(\chi^{-1})\langle f(z),\log|u_\chi(z)|\cdot\mathrm{dlog}(u(\chi_1,\chi_2)(z))\rangle_{2,N}.$$

With the aim of giving a more algebraic description of the previous formulas, we can introduce the non-holomorphic differential attached to f as

$$\eta_f^{\mathrm{ah}}:=\frac{\bar{f}(z)d\bar{z}}{\langle f,f\rangle_{2,N}}.$$

Moreover, another ingredient that will be of great importance is the modular form

$$\Xi_{k,l}(\chi_1,\chi_2):=(\delta_l^{k/2-l}E_{l,\chi})\cdot E_l(\chi_1,\chi_2).$$

It belongs to the space of nearly holomorphic modular forms $M_k^{\mathrm{nh}}(N,\mathbb{Q}_{\chi_1\chi_2})$, and we can also consider its image under the holomorphic projection Π_N^{hol}, that we write $\Xi(f,g,j)^{\mathrm{hol}}$.

Let $p\geq 3$ be a prime, and fix an embedding of K into $\mathbb{C}_p$. Assume that f is an eigenform ordinary at p, such that $p\nmid N$. The f-isotypic part of the exact sequence

$$0\to H^0(X_K,\omega^r\otimes\Omega_X^1)\to H^1_{\mathrm{dR},c}(Y_K,L_r,\nabla)\to H^1(X_K,\omega^{-r}\otimes I)\to 0$$

(being I the ideal sheaf of the cusps) admits a canonical unit root splitting arising from the action of the Frobenius, provided that $K = \mathbb{C}_p$. Let η_f^{ur} be the lift of η_f to the unit root subspace $H^1_{\mathrm{dR},c}(Y_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\mathrm{ur}}$. It can be checked that

$$\frac{\langle f, \Xi_{k,l}(\chi_1,\chi_2)\rangle_{k,N}}{\langle f, f\rangle_{k,N}} = \frac{\langle f, \Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)\rangle_{k,N}}{\langle f, f\rangle_{k,N}} = \langle \eta_f, \Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)\rangle_{k,Y}$$

$$= \langle \eta_f^{\mathrm{ur}}, \Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)\rangle_{k,Y}.$$

Viewing $\Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)$ as an overconvergent p-adic modular form, we can identify its ordinary projection $e_{\mathrm{ord}}\Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)$ with a cohomology class in $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla)^{\mathrm{ord}}$. Then,

$$\langle \eta_f^{\mathrm{ur}}, \Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)\rangle_{k,Y} = \langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}\Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2)\rangle_{k,Y}.$$

But $e_{\mathrm{ord}}\Xi^{\mathrm{hol}}_{k,l}(\chi_1,\chi_2) = e_{\mathrm{ord}}((d^{k/2-l}E_{l,\chi})\cdot E_l(\chi_1,\chi_2))$, and $E_l(\chi_1,\chi_2)$ is the ordinary p-stabilization of a Hida family of Eisenstein series, $\mathbf{E}(\chi_1,\chi_2)$.
Define $\Xi^{\mathrm{ord},p}_{k,l}(\chi_1,\chi_2) := e_{\mathrm{ord}}((d^{k/2-l}E^{[p]}_{l,\chi})\cdot E_l(\chi_1,\chi_2))$, whose Fourier coefficients extend analytically to $U_{\mathbf{f}} \times (\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p)$, as functions in k and l.

Proposition 15. *Let e_{f_k} be the projector to the f_k-isotypic subspace $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla)^{f_k}$. For all $k \geq 2$ and $2 \leq l \leq k/2$,*

$$e_{f_k}\Xi^{\mathrm{ord},p}_{k,l}(\chi_1,\chi_2) = \frac{\mathcal{E}(f_k,\chi_1,\chi_2,l)}{\mathcal{E}(f_k)} \cdot e_{f_k}\Xi^{\mathrm{ord}}_{k,l}(\chi_1,\chi_2),$$

where $\mathcal{E}(f,g,j)$ and $\mathcal{E}(f)$ are explicit factors.

Setting now $\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 p^{1-k}$, we can define the L-function

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1,\chi_2))(k,l) := \frac{1}{\mathcal{E}^*(f_k)} \langle \eta^{\mathrm{ur}}_{f_k}, \Xi^{\mathrm{ord},p}_{k,l}(\chi_1,\chi_2)\rangle_{k,Y}.$$

It is defined for $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $2 \leq l \leq k/2$ and it can be extended to an analytic function, that we denote by $L_p(\mathbf{f}, \mathbf{E}(\chi_1,\chi_2))$ on $U_{\mathbf{f}} \times (\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p)$.

Fix $k_0 \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$. Recall the Mazur-Kitagawa two-variable p-adic L-function $L_p(\mathbf{f},\psi)(k,s)$. It satisfies

$$L_p(\mathbf{f},\psi)(k,s) = \lambda^{\pm}(k) L_p(f_k,\psi,s),$$

for $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$. Here, $\lambda^{\pm}(k) \in \mathbb{C}_p$ is a p-adic period.

Theorem 35. *There exists an analytic function $\eta(k)$ on a neighborhood $U_{\mathbf{f},k_0}$ of k_0 such that for all $(k,l) \in U_{\mathbf{f},k_0} \times (\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p)$,*

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1,\chi_2))(k,l) = \eta(k) \times L_p(\mathbf{f},\chi_1)(k, k/2+l-1) \times L_p(\mathbf{f},\chi_2)(k,k/2).$$

We will now consider $f \in S_2(\Gamma_1(N),\chi_f)$ and $g \in S_2(\Gamma_1(N),\chi_g)$. We will apply the same formalism with the p-adic family $\mathbf{E}_\chi$ replaced by the Hida family $\mathbf{g}$ interpolating the cusp form g. We take the characters χ_f, χ_g to have modulus N, and f, g are normalized eigenforms of level N (including for the operators U_p with $p|N$).
Let $\mathbf{f}$ and $\mathbf{g}$ be Hida families of ordinary p-adic modular forms of tame level N indexed by weight variables k, l in suitable neighbourhood $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$ of $\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p$, contained in a single residue class modulo $p-1$. Assume also that $j = k-1-t$ belongs

to a single residue class modulo $p-1$, so that the same holds for $m = k-l-2t$.
Let $f_k \in S_k(N, \chi_f)$ and $g_l \in S_l(N, \chi_g)$ be the classical cusp forms whose p-stabilizations are the weight k and l specializations of $\mathbf{f}$ and $\mathbf{g}$ respectively. We also introduce, following the same notations as before, the cusp form

$$\Xi(f, g, j) := \delta_m^t E_{m,\chi} \times g \in S_k^{\mathrm{nh}}(N, \chi_f^{-1}),$$

where $t := k-1-j$ and $m := l-l-2t$. We can define in the same way its holomorphic and ordinary projection. Finally, let

$$\Xi(f, g, j)^{\mathrm{ord},p} := e_{\mathrm{ord}}(d^t E_{m,\chi}^{[p]} \cdot g).$$

The collection of p-adic modular forms $\Xi(f_k, g_l, j)^{\mathrm{ord},p}$ indexed by

$$\{(k, l, j), \qquad k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}, \quad l \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}, \quad \frac{l+k-1}{2} \leq j \leq k-1\}$$

has Fourier coefficients which extend analytically to $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$, as functions in k, l and j. Hence, it can be seen as a three-variable Λ-adic family of modular forms of level N. Write

$$\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 \chi_f^{-1}(p) p^{1-k}.$$

We have that

$$L_p(\mathbf{f}, \mathbf{g})(k, l, j) := \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta_{f_k}^{\mathrm{ur}}, \Xi(f_k, g_l, j)^{\mathrm{ord},p} \right\rangle_{k,X},$$

defined on (k, l, j) in the previous set.
Alternatively, for all (k, l, j) in the range of classical interpolation, it satisfies the interpolation property

$$L_p(\mathbf{f}, \mathbf{g})(k, l, j) = \frac{\mathcal{E}(f_k, g_l, j)}{\mathcal{E}^*(f_k)\mathcal{E}(f_k)} L^{\mathrm{alg}}(f_k \otimes g_l, j).$$

If we generalize our setting, we could have not assumed that $g \in S_2(N, \chi_g)$ is ordinary, so g may not necessarily be viewed as the weight 2 specialization of a Hida family. In this case, the above construction allows us to define a two-variable p-adic L-function $L_p(\mathbf{f}, g)(k, j)$ on $U_{\mathbf{f}} \times \mathbb{Z}_p$ by the equation

$$L_p(\mathbf{f}, g)(k, j) := \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\mathrm{ur}}, \Xi(f_k, g, j)^{\mathrm{ord},p} \rangle_{k,X},$$

for $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $(k+1)/2 \leq j \leq k-1$. Here, as usual, L^{alg} is the normalized L-function ("deleting the transcendental part") and $\mathcal{E}^*(f_k)$ is an explicit factor. Again, see [BD2] for more references.

3.6 Garrett-Rankin triple product L-functions

We present now the third kind of L-function that will be related with the theory of Euler systems of Rankin-Selberg type. The idea developed first by Hida and then extended by Harris and Tilouine was to construct three distinct p-adic L-function of three variables, $\mathscr{L}_p{}^f(f, g, h)$, $\mathscr{L}_p{}^g(f, g, h)$ and $\mathscr{L}_p{}^h(f, g, h)$ interpolating the square roots of the central critical values of the classical L-function $L(f_x, g_y, h_z, s)$, as (x, y, z) ranges over a particular region of interpolation. The special values of this function will be related with the image under the Abel-Jacobi map of diagonal cycles, that will be described later.

We begin by considering $f \in S_k(N_f, \chi_f), g \in S_l(N_g, \chi_g), h \in S_m(N_h, \chi_h)$ to be a triplet of normalized primitive cuspidal eigenforms such that $\chi_f \chi_g \chi_h = 1$. Let $\mathbb{Q}_{f,g,h}$ be the field generated by the Fourier coefficients of f, g and h and let $N = \mathrm{lcm}(N_f, N_g, N_h)$. The Garrett-Rankin triple product L-function $L(f, g, h; p^{-s})$ is defined by

$$L(f, g, h; s) = \prod_p L^{(p)}(f, g, h; p^{-s})^{-1},$$

where for $p \nmid N$

$$\begin{aligned} L^{(p)}(f, g, h; T) = (1 - \alpha_{f,p}\alpha_{g,p}\alpha_{h,p}T) \times (1 - \alpha_{f,p}\alpha_{g,p}\beta_{h,p}T) \\ \times(1 - \alpha_{f,p}\beta_{g,p}\alpha_{h,p}T) \times (1 - \alpha_{f,p}\alpha_{g,p}\beta_{h,p}T) \\ \times(1 - \beta_{f,p}\alpha_{g,p}\alpha_{h,p}T) \times (1 - \beta_{f,p}\alpha_{g,p}\beta_{h,p}T) \\ \times(1 - \beta_{f,p}\beta_{g,p}\alpha_{h,p}T) \times (1 - \beta_{f,p}\beta_{g,p}\beta_{h,p}T). \end{aligned}$$

A precise expression for the primes dividing N can be given, and once we add the archimedian primes, we have a completed L-function satisfying the equation

$$\Lambda(f, g, h; s) = \mathcal{E}(f, g, h)\Lambda(f, g, h; k + l + m - 2 - s),$$

where $\mathcal{E}(f, g, h)$ is the sign of the functional equation that determines the order of vanishing at the center c. The sign coincides with the sign of infinity when the triples is unbalanced and differs when it is balanced. We now recall the main properties of this p-adic L-function in the spirit of [DR1].

Definition 31. *Let $k = l + m + 2t$ with $t \geq 0$. The trilinear period attached to*

$$(\tilde{f}, \tilde{g}, \tilde{h}) \in S_k[N][\pi_f] \times S_l(N)[\pi_g] \times S_m(N)[\pi_h]$$

is the expression

$$I(\tilde{f}, \tilde{g}, \tilde{h}) := \langle \tilde{f}^*, \delta_l^t \tilde{g} \times \tilde{h} \rangle_N.$$

Theorem 36. *Let (f, g, h) be a triple of modular forms of unbalanced weights (k, l, m) with $k = l + m + 2t$ and $t \geq 0$. Then, there exist*

- *holomorphic modular forms $\tilde{f} \in S_l(N)[\pi_f], \tilde{g} \in S_l[N][\pi_g], \tilde{h} \in S_m(N)[\pi_h]$,*
- *for each prime $q|N\infty$ a constant $C_q \in \mathbb{Q}_{f,g,h}$ depending only on the local components at q of $\tilde{f}, \tilde{g}, \tilde{h}$ in the admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to π_f, π_g, π_h (the automorphic representations attached to the modular forms f, g, h)*

such that

$$\frac{\prod_{q|N\infty} C_q}{\pi^{2k}} \cdot L(f, g, h, c) = |I(\tilde{f}, \tilde{g}, \tilde{h})|^2.$$

Further, there is a choice of the vectors for which all C_q are all non-zero.

Let $\iota_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ and assume that f, g, h are ordinary with respect to ι_p.
Fix a triple (f, g, h) of newforms such that $\chi_f \chi_g \chi_h = 1$, $\epsilon_q(f, g, h) = +1$ for all finite primes $q|N$ and such that the triple of weights (k, l, m) is unbalanced. Let $\tilde{f} \in S_K(N; K_f)[\pi_f]$, $\tilde{g} \in S_l(N; K_g)[\pi_g]$ and $\tilde{h} \in S_m(N; K_h)[\pi_h]$ be test vectors for which the local constants C_q of the previous theorem are non-zero and the central critical value of $L(\pi_f, \pi_g, \pi_h, s)$ is (up to elementary non-zero fudge factors) equal to the square of $I(\tilde{f}, \tilde{g}, \tilde{h})$.
Assume further that f, g, h are ordinary with respect to ι, and let $\tilde{\mathbf{f}} = (\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \tilde{\mathbf{f}})$, $\tilde{\mathbf{g}} = (\Lambda_g, \Omega_g, \Omega_{g,\mathrm{cl}}, \tilde{\mathbf{g}})$ and $\tilde{\mathbf{h}} = (\Lambda_h, \Omega_h, \Omega_{h,\mathrm{cl}}, \tilde{\mathbf{h}})$ be the Hida families of forms on $\Gamma_1(N)$ interpolating the test vectors in weights k, l, m. Let

$$\Sigma := \{(x, y, z) \in \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}\},$$

and set

$$\Sigma_f = \{(x, y, z) \in \Sigma, \text{ such that } 2t := \kappa(x) - \kappa(y) - \kappa(z) \geq 0\},$$
$$\Sigma_{\mathrm{bal}} = \{(x, y, z) \in \Sigma, \text{ such that } (\kappa(x), \kappa(y), \kappa(z)) \text{ is balanced. }\}$$

For $(x, y, z) \in \Sigma_f$, consider the algebraic number

$$J(\tilde{f}_x, \tilde{g}_y, \tilde{h}_z) := \frac{I(\tilde{f}_x, \tilde{g}_y, \tilde{h}_z)}{\langle f_x^*, f_x^* \rangle_N} \in \bar{\mathbb{Q}}.$$

The natural approach to interpolate the p-adic L-function attached to $\pi_f \otimes \pi_g \otimes \pi_h$ would be to interpolate the ratios $J(\tilde{f}_x, \tilde{g}_y, \tilde{h}_z)$ as (x, y, z) ranges over Σ_f. For that, let $\phi := e_{\mathrm{ord}}(d^\bullet \tilde{\mathbf{g}}^{[p]} \times \tilde{\mathbf{h}})$ be the ordinary family of modular forms attached to $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$.
Further, let $\tilde{\mathbf{f}}^* = \tilde{\mathbf{f}} \otimes \chi_f^{-1}$. Observe that $(\tilde{f}^*)_x = (\tilde{f}_x)^*$ for all $x \in \Omega_{f,\mathrm{cl}}$. A classical point is nothing but a character of the form $\gamma \mapsto \gamma^k$ for some $k \in \mathbb{Z}^{\geq 2}$.

Definition 32. *The Garrett-Rankin triple product p-adic L-function attached to the triple $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ of Λ-adic modular forms is the element*

$$\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}}) := J(\tilde{\mathbf{f}}^*, e_{\mathrm{ord}}(d^\bullet \tilde{\mathbf{g}}^{[p]} \times \tilde{\mathbf{h}})) \in \Lambda_f' \otimes_\Lambda (\Lambda_g \otimes \Lambda_h \otimes \Lambda)$$

attached to the families $\mathbf{f}$ *and* $\phi = e_{\mathrm{ord}}(d^\bullet \tilde{\mathbf{g}}^{[p]} \times \tilde{\mathbf{h}})$.

Any element $\mathscr{L} \in \Lambda_f' \otimes (\Lambda_g \otimes \Lambda_h \otimes \Lambda)$ has poles at (x, y, z) for only finitely many $x \in \Omega_f$, and hence is completely determined by its values at the points of $\Omega_{f,\mathrm{cl}} \times (\Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}} \times \Omega)$ where it is defined. Furthermore, it is always defined at (x, y, z) if $x \in \Omega_{f,\mathrm{cl}}$. For $(x, y, z) \in \Sigma_f$, setting $\kappa(x) = \kappa(y) + \kappa(z) + 2t$,

$$\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})(x, y, z) = \frac{\langle \tilde{f}_x^{*(p)}, e_{\mathrm{ord}}(d^t \tilde{g}_y^{[p]} \times \tilde{h}_z) \rangle_{N,p}}{\langle f_x^{*(p)}, f_x^{*(p)} \rangle_{N,p}}.$$

In particular, this value is algebraic and belongs to the field K generated by α_{f_x} and the Fourier coefficients of $\tilde{f}_x, \tilde{g}_y$ and $\tilde{h}_z$.
The following proposition gives a formula for the p-adic special values of the function:

Proposition 16. *For $(x, y, z) \in \Sigma_{\mathrm{bal}}$, let $(f, g, h) := (f_x, g_y, h_z)$ and define (k, l, m, t) by*

$$(k, l, m) = (\kappa(x), \kappa(y), \kappa(z)), \qquad k = l + m - 2 - 2t.$$

Then,

$$\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})(x, y, z) = \mathcal{E}_0(f)^{-1} \langle \eta_{\tilde{f}_x}^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1-t} \tilde{g}_y^{[p]} \times \tilde{h}_z) \rangle,$$

where:

- $\mathcal{E}(f) := (1 - \beta_f^2 \chi_f^{-1}(p) p^{1-k})$,
- $\eta_{\tilde{f}_x}^{\mathrm{ur}} \in H^1_{\mathrm{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)^{\mathrm{ur}}$ *is the unique lift to the unit root subspace of the cohomology class in* $H^1(X_{K_{f_x}}, \omega^{-r})$ *attached to* $\tilde{f}_x^*$. *These concepts will be developed in the next chapters.*

The following theorem explores the result of evaluating $\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ at points $(x, y, z) \in \Sigma_f$ and relates these values to certain complex periods.

Theorem 37. *Let* (x, y, z) *be a point of* Σ_f *and set*

$$(f, g, h) := (f_x, g_y, h_z), \quad (\tilde{f}, \tilde{g}, \tilde{h}) := (\tilde{f}_x, \tilde{g}_y, \tilde{h}_z), \quad (k, l, m) = (\kappa(x), \kappa(y), \kappa(z)).$$

Then,

$$\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})(x, y, z) = \frac{\mathcal{E}(f, g, h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \times \frac{I(\tilde{f}, \tilde{g}, \tilde{h})}{\langle f^*, f^* \rangle_N},$$

where

$$\mathcal{E}(f, g, h) = (1 - \beta_f \alpha_g \alpha_h p^{-c})(1 - \beta_f \alpha_g \beta_h p^{-c})(1 - \beta_f \beta_g \alpha_h p^{-c})(1 - \beta_f \beta_g \beta_h p^{-c}),$$

$$\mathcal{E}_0(f) = (1 - \beta_f^2 \chi_f^{-1}(p) p^{1-k}),$$

$$\mathcal{E}_1(f) := (1 - \beta_f^2 \chi_f^{-1}(p) p^{-k}.$$

We could have adopted a somewhat more flexible point of view (as in [DR2]) and define

$$\mathscr{L}_p{}^{f_\alpha}(\tilde{\mathbf{f}}^*, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})(x, y, z) := \tilde{\mathbf{f}}_x^*(e_{\mathrm{ord}}(d^t \tilde{g}_y^{[p]} \times \tilde{h}_z)), \quad t := (k - l - m)/2,$$

as (x, y, z) ranges over the dense set of points in $\Omega_f \times \Omega_g \times \Omega_h$ of integral weights (k, l, m), with $k \equiv l + m \pmod 2$. We do identical definitions for $\mathscr{L}_p{}^{g_\alpha}(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}^*, \tilde{\mathbf{h}})$ and $\mathscr{L}_p{}^{h_\alpha}(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}}^*)$ with the difference that their regions of classical interpolation are given by $l \geq k + m$ and $m \geq k + l$ respectively.
The point $(x_0, y_0, z_0) \in \Omega_f \times \Omega_g \times \Omega_h$ of weight $(2, 1, 1)$ for which $(f_{x_0}, g_{y_0}, h_{z_0}) = (f_\alpha, g_\alpha, h_\alpha)$ lies within the region of interpolation defining $\mathscr{L}_p{}^{f_\alpha}(\tilde{\mathbf{f}}^*, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ and its value at the point is given by

$$\mathscr{L}_p{}^{f_\alpha}(\tilde{f}^*, \tilde{g}_\alpha, \tilde{h}_\alpha) := \mathscr{L}_p{}^{f_\alpha}(\tilde{\mathbf{f}}^*, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})(x_0, y_0, z_0) = \tilde{f}_{x_0}^*(e_{\mathrm{ord}}(\tilde{g}_\alpha^{[p]} \times \tilde{h}_\alpha)).$$

4 Units in number fields and Stark's conjectures

There are plenty of analogies between elliptic curves and units in number fields; in fact, they are present all over this thesis: in which concerns L-functions, when comparing circular and elliptic units with Heegner points, in the finiteness results... For instance, one of the motivations for the BSD conjecture is the fact that the rank of the group of units of a number field is finite (as it occurs with the Mordell-Weil group of an elliptic curve), and the residue of the L-function at $s = 1$ encodes the value of this rank, namely $r_1 + r_2 - 1$, where r_1 is the number of real embeddings of the number field and $2r_2$ is the number of complex embeddings. We will make use of this fact together with its counterpart for S-units: the group of S-units is finitely generated, with rank equal to $r_1 + r_2 - 1 + s$, where $s = |S|$ (an element $x \in K$ in an S-unit if the principal fractional ideal (x) is a product of primes in S, to positive or negative powers). BSD aspires to give a similar result for elliptic curves, namely, we want to "read" the rank of the elliptic curve in the L-function. However, we focus now in an alternative subset of conjectures, those due to Stark and whose main ingredients are these units of number fields.

The most basic example of interaction between the analytic and the algebraic point of view is via the Dedekind zeta-function $\zeta_K(s)$ of a number field, namely

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathbb{N}\mathfrak{p}^{-s}},$$

that converges for $\Re(s) > 1$. As we where saying, Dirichlet proved the class number formula:

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2r_1(2\pi)^{r_2} h_K R_K}{\sqrt{|D_K|}e_K},$$

where h_K is the class number, R_K the regulator, D_K the discriminant and e_K the number of roots of unity of K. Alternatively, this means (via the functional equation) that the first non-zero term in the Taylor series of $\zeta_K(s)$ at $s = 0$ is $-\frac{h_K R_K}{e_K} s^{r_1+r_2-1}$. When L/K is a finite Galois extension with Galois group G and χ is the character of an odd, irreducible, finite-dimensional Artin representation of G, we ask ourselves if there is an analogous formula for the first non-zero coefficient in the Taylor expansion of $L_{L/K}(s, \chi)$ at $s = 0$. This is one of the starting points for dealing with Stark conjectures.

4.1 An introduction to the Stark conjecture

Let K/F be an abelian extension of number fields with associated rings of integers $\mathcal{O}_K$ and $\mathcal{O}_F$. Let S be a finite set of places of F containing the archimedean places and those ramifying in K. Assume that S contains at least one place v that splits completely in K and that $|S| \geq 2$. For each ideal $\mathfrak{n} \subset \mathcal{O}_F$ not divisible by a prime ramifying in K, we denote by $\sigma_{\mathfrak{n}}$ the associated Frobenius element in $G := \mathrm{Gal}(K/F)$. For $\sigma \in G$, we can define the partial zeta function

$$\zeta_{K/F,S}(\sigma, s) := \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_F \\ (\mathfrak{n}, S)=1, \sigma_{\mathfrak{n}}=\sigma}} \frac{1}{\mathbb{N}\mathfrak{n}^s}, \quad s \in \mathbb{C}, \Re(s) > 1.$$

Each function $\zeta_{K/F,s}(\sigma, 0) = 0$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. The fact that S contains a place v that splits completely in K ensures that

$\zeta_{K/F,s}(\sigma,0)=0$ for all $\sigma\in G$. Denote by e the number of roots of unity in K. Further, let $U_{v,S}=U_{v,S}(K)$ be the set of elements $u\in K^{\times}$ such that:

- If $|S|\geq 3$, then $|u|_{w'}=1$ for all $w'\nmid v$.
- If $S=\{u,v'\}$, then $|u|_{w'}$ is constant over all w' above v' and $|u|_{w'}=1$ for all $w'\notin S$.

Conjecture 1 (Rank one abelian Stark conjecture)**.** *Fix a place w of K lying above v. Then, there exists a $u\in U_{v,S}$ such that*

$$\zeta'_{K/F,S}(\sigma,0)=-\frac{1}{e}\log|u^{\sigma}|_w \quad \textit{for all } \sigma\in G$$

and such that $K(u^{1/e})/F$ is an abelian extension.

Observe that the conditions $u\in U_{v,s}$ together with the formula given by the conjecture specify the absolute value of u at every place of K. Hence, if the unit u exists it is unique up to multiplication by a root of unity in $K^{\times}$. Tate has given some alternative formulation of the statement that need the introduction of a finite set T of primes of F; we write $S\cap T=\phi$. The smoothed zeta functions $\zeta_{K/F,S,T}(\sigma,s)$ are defined by the group ring equation

$$\sum_{\sigma\in G}\zeta_{K/F,S,T}(\sigma,s)[\sigma^{-1}]=\prod_{\mathfrak{c}\in T}(1-[\sigma_{\mathfrak{c}}^{-1}]\mathbb{N}\mathfrak{c}^{1-s})\sum_{\sigma\in G}\zeta_{K/F,S}(\sigma,s)[\sigma^{-1}]$$

in $\mathbb{C}[G]$. For instance, when $T=\{\mathfrak{c}\}$, then

$$\zeta_{K/F,S,T}(\sigma,s)=\zeta_{K/F,S}(\sigma,s)-\mathbb{N}\mathfrak{c}^{1-s}\zeta_{K/F,S}(\sigma\sigma_{\mathfrak{c}}^{-1},s).$$

We will denote by $U_{v,S,T}$ the finite index group of $U_{v,S}$ consisting of the $u\in U_{v,S}$ congruent with 1 modulo $\mathfrak{c}O_K$ for every $\mathfrak{c}\in T$. Assume that there are no non-trivial roots of unity in $U_{v,S,T}$. The condition is authomatically satisfied if either T contains two distinct primes with different residue characteristics or one prime with residue characteristic at least 2 plus its absolute ramification index.

Conjecture 2 (Tate)**.** *Fix a place w of K above v. There exists $u_T\in U_{v,S,T}$ such that*

$$\zeta'_{K/F,S,T}(\sigma,0)=-\log|u_T^{\sigma}|_w \quad \textit{for all } \sigma\in G.$$

If u_T exists, it is unique.

Observe that as we have pointed out, Stark conjecture can be seen as a generalization of the Dirichlet class number formula. For a finite set of places S of F containing the infinite places, the S-imprimitive Dedekind zeta function of F is the special case of the function $\zeta_{K/F,s}$, that is,

$$\zeta_{F,S}(s):=\sum_{\substack{\mathfrak{n}\subset\mathcal{O}_F\\(n,S)=1}}\frac{1}{\mathbb{N}\mathfrak{n}^s}=\prod_{\mathfrak{p}\notin S}(1-\mathbb{N}\mathfrak{p}^{-s})^{-1},\quad \Re(s)>1.$$

Theorem 38. *The Taylor series of $\zeta_{F,S}(s)$ at $s=0$ begins*

$$\zeta_{F,S}(s)=-\frac{h_SR_S}{e_F}s^{|S|-1}+O(s^{|S|}),$$

where h_S and R_S are the S-class number and S-regulator of F, and e_F is the number of roots of unity in F.

Recall that the order of vanishing of $\zeta_{F,S}(s)$ at $s = 0$ is the rank $r_S = |S| - 1$ of the group of S-units $\mathcal{O}_{F,S}^\times$, as given by the Dirichlet unit theorem. The S-class number of F is defined as $h_S = |\operatorname{Cl}(\mathcal{O}_{F,S})|$, the class number of the ring of S-integers of F. This group, $\operatorname{Cl}(\mathcal{O}_{F,S})$ may be identified with the quotient of $\operatorname{Cl}(\mathcal{O}_F)$ by the subgroup generated by the images of the finite primes in S.
To define the S-regulator of F, we proceed as follows: let $u_1, \dots, u_{r_S}$ be a basis for the quotient of $O_{F,S}^\times$ by its torsion subgroup. Call the elements of S $v_0, v_1, \dots v_{r_S}$. Then, the S-regulator of F is the absolute value of the determinant of a certain $r_S \times r_S$ matrix:

$$R_S = |\det(\log(|u_i|_{v_j}))_{1 \le i,j \le r_S}|.$$

It can be checked that the definition of R_S is independent of the various choices we have made.
In the case of a finite extension of number fields K/F, for each character $\chi : G \to \mathbb{C}^\times$ we define an associated L-function by the formula

$$L_S(\chi, s) = \sum_{\sigma \in G} \chi(\sigma)\zeta_{K/F,S}(\sigma, s) = \sum_{\substack{\mathfrak{n} \subset O_F \\ (\mathfrak{n},S)=1}} \frac{\chi(\sigma)}{\mathbb{N}\mathfrak{n}^s},$$

where the second formula is true for $\Re(s) > 1$. In this case, we have also an Euler product

$$L_S(\chi, s) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s})^{-1}.$$

Further, there is a functional equation relating $L_S(\chi, s)$ and $L_S(\bar{\chi}, 1-s)$, and an explicit formula for the order of vanishing of $L_S(\chi, s)$ at $s = 0$, given by $r_S(\chi) = |\{v \in S \mid \chi(G_v) = 1\}|$ when $\chi \neq 1$ and by $|S| - 1$ when $\chi = 1$. We denote by S_K the set of places of K above the places in S and by G_v the decomposition group of v. $r_S(\chi)$ is also equal to the dimension over $\mathbb{C}$ of

$$(\mathcal{O}_{S_K}^\times \otimes \mathbb{C})^{\chi^{-1}} := \{x \in \mathcal{O}_{S_K}^\times \otimes \mathbb{C} \mid \sigma(x) = \chi^{-1}(\sigma)x \text{ for all } \sigma \in G\}.$$

The zeta function $\zeta_{K,S_K}(s)$ can be factored in terms of the L-function associated to the abelian extension K/F:

$$\zeta_{K,S_K}(s) = \prod_{\chi \in \hat{G}} L_S(\chi, s).$$

The factor on the right corresponding to $\chi = 1$ is $\zeta_{F,S}(s)$. The formula is consistent with the orders of vanishing at $s = 0$, i.e.,

$$|S_K| - 1 = |S| - 1 + \sum_{\chi \neq 1} |\{v \mid \chi(G_v) = 1\}|.$$

Stark's motivations for the conjecture was the idea that the leading term $-h_{S_K}R_{S_K}/e_K$ of $\zeta_{K,S_K}(s)$ at $s = 0$ should factor in a nice way over the various characters χ. In particular, the leading term of $L_S(\chi, s)$ at $s = 0$ should be expressed as a rational number times the determinant of an $r_S(\chi) \times r_S(\chi)$-matrix whose entries depend linearly of logarithms of $(\mathcal{O}_{S_K}^\times \otimes \mathbb{C})^{\chi^{-1}}$.

There are different versions of Stark's conjecture. The most relevant one is that concerning the rank one setting, related only with the first derivative of $L_S(\chi, s)$ at $s = 0$ in the case $r_S(\chi) \ge 1$ for all χ. Using the properties of the L-functions we have seen, we arrive to the following alternative formulation of the conjecture:

Conjecture 3 (Stark)**.** *Suppose that $v \in S$ splits completely in K and fix a place $w \in S_K$ above v. Then, there exists a $u \in U_{v,S}$ such that*

$$L'_S(\chi, 0) = -\frac{1}{e} \sum_{\sigma \in G} \chi(\sigma) \log |u^\sigma|_w \textit{ for all } \chi \in \hat{G}$$

and such that $K(u^{1/e})/F$ is an abelian extension.

Observe that the element

$$u^{\chi^{-1}} := \sum_{\sigma \in G} u^\sigma \otimes \chi(\sigma) \in \mathcal{O}_{K,S}^\times \otimes \mathbb{C}$$

lies in $(\mathcal{O}_{K,S}^\times \otimes \mathbb{C})^{\chi^{-1}}$ and that the sum in the conjecture is simply the value of the linear extension of $\log |\cdot|_w$ to $O_{K,S}^\times \otimes \mathbb{C}$ evaluated at $u^{\chi^{-1}}$.

Stark's conjecture is known to be true in the cases where one has an explicit class theory. When $F = \mathbb{Q}$ and v is the infinite prime, the proof involves the cyclotomic units; when $F = \mathbb{Q}$ and v is a finite prime p, the conjecture follows from Stickelberger's theorem. When F is quadratic imaginary, Stark proved it using the theory of elliptic units and Kronecker's second limit formula.
Since the rank one abelian Stark conjecture holds when S contains two primes that split completely in K, we need only to consider the setting where S contains exactly one prime v that splits completely in K. Since complex places split completely in every extension, we are left with the following three possibilities:

- Case TR_∞: F is a totally real field and the place v is real. The places of K above v are real and all other archimedean places are complex.
- Case ATR: F is almost totally real, that is, it has one complex place v and all other places are real. The field K is totally complex.
- Case TR_p: F is totally real and the place v is finite. The field K is totally complex.

Let us say something about the class field theory for TR_∞. The general idea is to start in the base field F and find some nice extension of the Hilbert class field H to which Stark's conjecture apply (and then hope we can figure out H from this). Let $L_i = H(\sqrt{\alpha})$ for some $\alpha \in F$.

- We need L_i/F to be abelian.
- We need to arrange the problem of having one real place split and the rest become complex. For this, it is enough to choose α to be positive at one real place and negative at all the others.
- Then, apply Stark's conjecture to see that there exists $\epsilon \in L$ whose logarithms of conjugates give the L-series

$$L'_S(O, \chi) = -\frac{1}{e} \cdot \sum_{\sigma \in G} \chi(\sigma) \cdot \log |\epsilon^\sigma|_w.$$

- It is enough to compute the L-series for all the characters χ and then Fourier-invert and exponentiate to get ϵ and its conjugates.
- Once we have an approximate value for ϵ, we can use numerical methods to find what ϵ's minimal polynomial should be, and hence get an exact value for α.
- Finally take the trace, which will lie in H. In many cases only one Stark unit is needed to generate H, but in general we may need to compute several ϵ_i in order to generate H.

Note the similitudes between this and the theory of Heegner points, where you also build points in a larger extension and then you take traces.

In each of the three cases, $\mathrm{TR}_\infty, \mathrm{TR}_p$ and ATR, there is a setting where classical explicit class field theory provides a concrete construction of Stark units and gives a very elegant solution to our motivating question. That is,

- TR_∞ when $F = \mathbb{Q}$. Here, Stark units are given by cyclotomic units.
- TR_p when $F = \mathbb{Q}$. Here, Stark units are given by Gauss sums.
- ATR when F is quadratic imaginary. Stark units are given by elliptic units via the theory of complex multiplication.

The Stark unit in these cases is given by an explicit formula that can be seen as the value of a certain analytic function at a particular point. For example, in the first case units are given by the values of the function $f(x) = 2-2\cos(2\pi x)$ at rational arguments x. These cases are easier to handle since $\mathcal{O}_F^\times$ is finite. In general, the fact that class field theory is very well understood for both $\mathbb{Q}$ and quadratic imaginary fields is not a coincidence, since these groups are distinguished by the fact that their unit groups are finite. In general, the special values of partial zeta functions of a number field F can be expressed as periods parameterized by the unit group of F (bear in mind that a period is like the integral of a differential r-form along an r-cycle, being r the rank of the unit group of F.
To overcome these problems, there are two remarkable methods that we will not deal with but that mention for the sake of completeness.

- **Shintani domain:** one chooses a fundamental domain for the units acting on F, cut out by a union of simplicial cones. After a conjectural construction of Stark units, we must go back and check that the construction does not depend on the choice of fundamental domain taken.
- **Arithmetic cohomology:** in this setting we do not look at values of functions, but at specializations of group cohomology classes. The classes will be in $H^r(G)$ for an arithmetic group G, with r the rank of $\mathcal{O}_F^\times$ and such that G is equipped with a homomorphism $\phi : \mathcal{O}_F^\times \to G$. The specialization will be the value of the class on the image of a basis of units under ϕ. For instance, consider TR_p, when F is real quadratic and p is an inert prime in F. Let $X = \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ be the space of primitive vectors in $\mathbb{Z}_p^2$. Let $M(X)$ be the space of $\mathbb{Z}$-valued measures on X with total cohomology zero. We can construct a certain Eisenstein group cohomology class $[\kappa] \in H^1(\mathrm{GL}_2(\mathbb{Z}), M(X))$. This class will not depend on the real quadratic field. Choose an embedding $\phi : \mathcal{O}_F \hookrightarrow M_2(\mathbb{Z})$ restricting to a

group homomorphism $O_F^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Z})$. Let $\tau \in F - \mathbb{Q}$ and consider the group of $\epsilon \in O_F^\times$ such that $\phi(\epsilon)$ has $\binom{\tau}{1}$ as an eigenvector. This is a finite index subgroup of $\mathcal{O}_F^\times$. If ϵ_τ is a generator, we conjecture that up to an explicit power of p, the p-adic number

$$\int_X (x - y\tau) d\kappa(\phi(\epsilon_\tau))(x, y) \in F_p^\times$$

is in the narrow Hilbert class field K/F and is a Stark unit for the extension.

In 1988, Gross stated a conjectural refinement of Stark's conjecture. For that purpose, consider the abelian extension K/F and finite sets of primes S and T of F as before, with the place $v \in S$ splitting completely in K. Assume that Stark conjectures holds. Let L be a finite abelian extension of F containing K and unramified outside S. Since v splits completely in K and w is a place of K above v, there is a canonical isomorphism of completions $F_v \cong K_w$. Let

$$\mathrm{rec}_w : K_w \to A_K^\times \to \mathrm{Gal}(L/K)$$

denote the Artin reciprocity map of local class field theory. Since $K^\times \subset K_w^\times$ we may evaluate rec_w on $K^\times$ and we have the following conjecture:

Conjecture 4 (Gross, strong form). *Let $u_T \in U_{v,S,T} \subset K^\times$ be Stark's unit satisfying Stark's conjecture. Then,*

$$\mathrm{rec}_w(u_T^\sigma) = \prod_{\substack{\tau \in \mathrm{Gal}(L/F) \\ \tau|_K = \sigma}} \tau^{-\zeta_{L/F,S,T}(\tau,0)}$$

in $\mathrm{Gal}(L/K)$ *for each* $\sigma \in G$.

Gross also conjectured a formula for the expected leading term at $s = 0$ of the Deligne-Ribet p-adic L-function associated to a totally even character ψ of a totally real field F. Basically, it said that after scaling by $L(\psi\omega^{-1}, 0)$, this value is equal to a p-adic regulator of units in the abelian extension of F cut out by $\psi\omega^{-1}$. This conjecture was proved in 2016 by Dasgupta, Kakde and Ventullo.

4.2 Two words about Birch and Swinertonn-Dyer

One of the aims of this thesis is the presentation of the Elliptic Stark conjecture, which has a deep connection with BSD in rank 2. Not only this, the leitmotiv of all the works around Euler systems that we will present is the conjecture of Birch and Swinertonn-Dyer. The aim of chapter 7 will be to formulate an analogue of the Stark conjecture in the setting of global points on elliptic curves. This conjecture is offered as a more constructive alternative to the BSD conjecture, offering an efficient analytic computation of p-adic logarithms of global points. The following two sections of this chapter are mainly descriptive, since we would like to offer a nice presentation of this conjecture. Let me recall that BSD in its basic formulation states the following:

Conjecture 5 (BSD). *The L-function $L(E, s)$ extends to an entire function on $\mathbb{C}$ and the rank r of $E(\mathbb{Q})$ is equal to the order of vanishing at $s = 1$.*

In fact, an explicit expression for $\lim_{\to 1} \frac{L(E,s)}{(s-1)^r}$ can be given, in terms of the Shafarevich group and the regulator of E.

We can however state a more general version, what is called the equivariant BSD conjecture. For this, let $E/\mathbb{Q}$ be an elliptic curve and let

$$\rho_F : G_{\mathbb{Q}} \to \mathrm{Aut}(V_\rho)$$

be an Artin representation, that therefore factors through a finite extension $H/\mathbb{Q}$ and takes values in a number field L, i.e.,

$$\rho : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_n(L) \simeq \mathrm{Aut}(V_\rho).$$

We can consider $E(H) \otimes L$, that is a finite-dimensional L-vector space by the Mordell-Weil theorem. Let $r(E, \rho)$ be the multiplicity of V_ρ in $E(H) \otimes L$, that when the dimension is irreducible is nothing but

$$\dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\rho, E(H) \otimes L).$$

On the other hand, we can consider the L-series $L(E, \rho, s) = L(f \otimes g, s)$, where $f \in S_2(N)$ is the eigenform attached to f by the modularity theorem and g is a weight one modular form attached to the representation ρ by the results of Deligne. This L-series converges for $\Re(s) > 3/2$ and it is expected that it can be analytically continued to the whole complex plane.

Conjecture 6 (Equivariant BSD)**.** *We have that*

$$\mathrm{ord}_{s=1} L(E, \rho, s) = \dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\rho, E(H) \otimes L).$$

The right hand side is precisely the multiplicity of V_ρ in $E(H) \otimes L$ when the representation is irreducible.

Some known results for this conjecture are the following ones:

1. When $K/\mathbb{Q}$ is imaginary quadratic and

   $$\chi : \mathrm{Gal}(H/\mathbb{Q}) \to \mathbb{C}^\times$$

   is a ring class character, if $\rho = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ the conjecture is true provided that the analytic rank is ≤ 1 (Gross-Zagier, Kolyvagin).

2. If the analytic rank is 0 and ρ is a Dirichlet character, the work of Kato implies the conjecture.

3. If the analytic rank is 0 and ρ is an odd, irreducible, two-dimensional representation satisfying some mild conditions, it is proved in the work of Bertolini, Darmon and Rotger.

4. If the analytic rank is 0 and ρ is an irreducible, self-dual constituent of $\rho_1 \otimes \rho_2$, where ρ_1, ρ_2 are odd, two-dimensional representations, it is proved in the work of Darmon and Rotger.

When the analytic rank is > 0, we also want some tool to construct points on

$$(E(H) \otimes L)^\rho := \sum \phi(V_\rho),$$

where the sum is over a basis of $\mathrm{Hom}_{G_{\mathbb{Q}}}(V_\rho, E(H) \otimes L)$.
Recall that in analytic rank one, Heegner points are the main ingredient for that. Let

$E/\mathbb{Q}$ be an elliptic curve and let K be an imaginary quadratic field such that (E, K) satisfies the Heegner hypothesis. Let $\{P_n\} = \{\Phi_N(\tau_n)\}$ be the usual Heegner system, as constructed for instance in [Da]. Let

$$P_K = \mathrm{Tr}_{H_1/K}(P_1) \in E(K)$$

be the trace of a Heegner point of conductor 1 defined over the Hilbert class field of K. In general, if $\chi : \mathrm{Gal}(H_n/K) \to \mathbb{C}^\times$ is any primitive ring class character, let

$$P_n^\chi = \sum_{\sigma \in \mathrm{Gal}(H_n/K)} \bar{\chi}(\sigma) P_n^\sigma \in E(H_n) \otimes \mathbb{C}.$$

Theorem 39 (Gross-Zagier,Zhang)**.** *Let $\langle , \rangle_n$ be the canonical Néron-Tate height on $E(H_n)$ extended by linearity to a hermitian pairing on $E(H_n) \otimes \mathbb{C}$. Then,*

1. $\langle P_K, P_K \rangle = *L'(E/K, 1)$.
2. $\langle P_n^\chi, P_n^{\bar{\chi}} \rangle = *L'(E/K, \chi, 1)$.

Here, $*$ means a non-zero fudge factor (that can be made explicit).

This is the main tool for proving the following remarkable result:

Theorem 40. *If E is an elliptic curve over $\mathbb{Q}$ and $\mathrm{ord}_{s=1} L(E, s) \leq 1$, then*

$$r(E(\mathbb{Q})) = \mathrm{ord}_{s=1} L(E, s)$$

and the Shafarevich group is finite.

This can be extended to the equivariant case.

The elliptic Stark conjecture suggested in [DLR] will relate points in $(E(H) \otimes L)^\rho$ and p-adic iterated integrals of a triple of modular forms (f, g, h) of weights $(2, 1, 1)$ when the analytic rank is two, and this is one of the reasons for its importance: it is one of the first results for analytic rank greater than one.

4.3 Stark's conjectures and its generalizations

As we know, Stark's conjectures give complex analytic formulae for units in number fields (their logarithms) in term of the leading terms of Artin L-functions at $s = 0$. We ask ourselves if there are similar formulae for algebraic points on elliptic curves, bearing in mind that Heegner points are analogous to circular or elliptic units.
Let $g = \sum a_n(g) q^n$ be a cusp form of weight one, level N and odd character χ. Recall the following (already classical) modularity results:

Theorem 41 (Deligne-Serre)**.** *There is an odd two-dimensional Artin representation*

$$\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$$

attached to g and satisfying that $L(\rho_g, s) = L(g, s)$.

Theorem 42 (Buzzard-Taylor, Khare,Wintenberger)**.** *Conversely, if ρ is an odd, irreducible, two-dimensional Artin representation, there is a weight one newform g such that $L(\rho, s) = L(g, s)$.*

Now, let us consider H_g, the field cut out by an Artin representation ρ_g and $L \subset \mathbb{Q}(\zeta_n)$, the field generated by the Fourier coefficients of g. V_g will be the vector space underlying ρ_g. Let us recall from the first section of this chapter Stark conjecture in a more convenient formulation:

Conjecture 7 (Stark)**.** *Let g be a cuspidal newform of weight one with Fourier coefficient in L. Then, there is a modular unit $u_g \in (O_{H_g}^{\times} \otimes L)^{\sigma_\infty = 1}$ (σ_∞ for the complex conjugation) such that*

$$L'(g, 0) = \log(u_g).$$

There are some cases (the reducible one, the imaginary dihedral case), where it has been proved. The general result is still unknown to be true.

We are going to motivate now the generalization done in [DLR1] of the Stark conjecture to the p-adic setting. We present now some results without giving a solid background on them, and later they will be covered with more detail.

Consider an elliptic curve E attached to $f \in S_2(\Gamma_0(N))$. Towards extending Stark's conjecture to elliptic curves, we replace Artin L-series by Hasse-Weil-Artin L-series

$$L(E, \rho_g, s) = L(f \otimes g, s).$$

p-adic logarithms of global points usually arise as leading terms of p-adic L-series attached to elliptic curves, as we have seen in previous sections: in the Katz p-adic L-function, in the MSD p-adic L-funcion, in various types of p-adic Rankin L-functions and in many others.
Recall Atkin-Serre d-operator (which raises the weight by two), $d = q\frac{d}{dq}$. When $f \in M_2^{\mathrm{oc}}(N)$, we can consider

$$F := d^{-1} f \in M_0^{\mathrm{oc}}(N).$$

If $h \in M_k(Np, \chi)$, then $F \times h \in M_k^{\mathrm{oc}}(N, \chi)$, $e_{\mathrm{ord}}(F \times h) \in M_k(Np, \chi) \otimes \mathbb{C}_p$, where e_{ord} is Hida's ordinary projector.

Definition 33. *Let $f \in S_2(N), h \in M_k(N, \chi), \gamma \in M_k(Np, \chi)^*$. The p-adic iterated integral of f and h along γ is*

$$\int_\gamma f \cdot h := \gamma(e_{\mathrm{ord}}(F \times h)) \in \mathbb{C}_p.$$

In the most usual setup, $f \in S_2(N)$ will correspond to an elliptic curve E and $g \in M_1(N, \chi^{-1}), h \in M_1(N, \chi)$ are classical weight one eigenforms. Let $V_{gh} := V_g \otimes V_h$, which is a four-dimensional self-dual Artin representation. Let H_{gh} be the field cut out by it. Take $g_\alpha \in M_1(Np, \chi^{-1})$, an ordinary p-stabilisation of g attached to a root α_g of its Hecke polynomial. Take $\gamma_{g_\alpha} \in M_k(Np, \chi)^*[g_\alpha]$. We would like to give an arithmetic interpretation to

$$\int_{\gamma_{g_\alpha}} f \cdot h, \qquad \text{as } \gamma_{g_\alpha} \in M_1(Np, \chi)^*[g_\alpha].$$

We will need for this purpose certain assumptions that we will be clarifying along the chapter.

1. We will require certain local sign in the functional equation for $L(E, V_{gh}, s)$ to be 1, and in particular, that this L-function vanishes to even order at $s = 1$.

2. $V_{gh} = V_1 \oplus V_2 \oplus W$, where $\mathrm{ord}_{s=1} L(E, V_1, s) = \mathrm{ord}_{s=1} L(E, V_2, s) = 1$ and $L(E, W, 1) \neq 0$.

3. The Frobenius Frob_p acting on V_1 (V_2) has the eigenvalue $\alpha_g \alpha_h$ ($\alpha_g \beta_h$).

4. These eigenvalues do not arise in (V_1, V_2) at the same time.

Under the previous assumptions and in favorable circumstances, we will express the elliptic Stark conjecture in the following way:

$$\int_{\gamma_{g_\alpha}} f \cdot h = \frac{\log_{E,p}(P_1) \log_{E,p}(P_2)}{\log_p(u_{g_\alpha})},$$

where P_j is in the V_j-isotypic component of $E(H_{gh}) \otimes L$ and

$$\mathrm{Frob}_p P_1 = \alpha_g \alpha_h \cdot P_1, \qquad \mathrm{Frob}_p P_2 = \alpha_g \beta_h \cdot P_2.$$

Further, by u_{g_α} we mean the Stark unit in $\mathrm{Ad}^0(V_g)$-isotypic part of $(O^*_{H_g}) \otimes L$;

$$\mathrm{Frob}_p u_{g_\alpha} = \frac{\alpha_g}{\beta_g} \cdot u_{g_\alpha}.$$

Let us briefly recall what do we mean when we talk about isotypic component in this context. Let X be a modular curve (we generally have in our mind the cases $X_0(N)$ or $X_1(N)$). For that purpose, let $f \in S_2(X)$ be an eigenform for the Hecke algebra, whose q-expansion is give by $f = \sum_{n \geq 1} a_n q^n$ (then, $T_l(f) = a_l f$). We consider $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}_p)$, which is a $\mathbb{Z}_p$-module, that is free and of rank $2g$; further, it comes equipped with a linear action of $G_{\mathbb{Q}}$. The f-isotypic component of $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}_p)$ (or directly, the f-isotypic component of f by an abuse of notation), is

$$V_f := H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}_p)[f] = \{v \in H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}_p) \mid T_l v = a_l \cdot v \text{ for all } l\}.$$

This will be a two-dimensional $\mathbb{Z}_p$-module.

We will see in Chapter 7 that there is no great theoretical evidence for the Elliptic Stark conjecture, and one of the few cases where it is proved is when g and h are theta series attached to the same imaginary quadratic field K and the prime p splits in K. In this case, the points P_1 and P_2 can be expressed in terms of Heegner points and the unit u_{g_α} in terms of elliptic units. Let us do a survey through the pieces the proof decomposes:

1. The relation established in [DR] between $\int_{\gamma_{g_\alpha}} f \cdot h$ and the Garrett-Rankin L-function $L_p(f \otimes g \otimes h)$.

2. When $g = \theta_{\psi_g}$ and $h = \theta_{\psi_h}$ are theta series, we have a factorization

   $$L_p(f \otimes g \otimes h) = L_p(f \otimes \theta_{\psi_1}) L_p(f \otimes \theta_{\psi_2}) \times \eta^{-1},$$

   being $\psi_1 = \psi_g \psi_h, \psi_2 = \psi_g \psi_h'$ and η a certain ratio of periods.

3. The p-adic Gross-Zagier formula of [BDP] relating $L_p(f \otimes \theta_{\psi_j})$ to Heegner points over certain ring class fields of K.

4. The interpretation of η as a value of the Katz p-adic L-function for K (the Stark unit of the denominator comes from Katz's p-adic variant of the Kronecker limit formula.

The condition of p being split in K is used both in the p-adic Gross-Zagier formula of [BDP] and in Katz's p-adic Kronecker limit formula. Therefore, the conjecture makes sense when p is inert in K. Further, Lauder's algorithms give support to the conjecture in different instances, what invites to believe that it must be true.

4.4 Classification of weight one modular forms

This section, that follows [DLR2] is included here for the sake of convenience so as to recall some preliminaries about group theory with the aim of having a best understanding of the main theorems we have already presented and the ones that we will expose later. Let V_g and V_h two dimensional L-vector spaces realising ρ_g and ρ_h respectively. Let $V_{gh} = V_g \otimes V_h$ be the L-vector spaces realising ρ_{gh}.
The image of ρ_g is a finite subgroup $G \subset \mathrm{GL}_2(\mathbb{C})$. This has been studied in several places, and the natural image Γ of G in $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to one of the following:

- A cyclic group if g is an Eisenstein series.
- A dihedral group if g is the theta series attached to a finite order character of a real or imaginary quadratic field.
- The group A_4, S_4 or A_5. In this last case, g is said to be exotic.

The group G is then isomorphic to

$$G = (Z \times \tilde{\Gamma})/\langle \pm 1 \rangle,$$

where $Z = G \cap \mathbb{C}^\times$ is the center of G and $\tilde{\Gamma} = G \cap \mathrm{SL}_2(\mathbb{C})$ is the universal central extension of Γ. The irreducible representations of G are obtained in a simple way from those of $\tilde{\Gamma}$: if $\rho_1, \dots, \rho_t$ is a complete list of the irreducible representations of $\tilde{\Gamma}$, those of G are of the form $\chi_Z \otimes \rho_j$, where χ_Z is character of Z such that $\chi_Z(-1) = \rho_j(-1)$.

When ρ_g has cyclic projective image, it decomposes as a direct sum of two one-dimensional representations attached to Dirichlet characters χ_1 and χ_2, and consequently $g = E_1(\chi_1, \chi_2)$ is the weight one Eisenstein series attached to this pair of characters.
When ρ_g has dihedral projective image, it is induced from a one-dimensional character $\psi_g : G_K \to \mathbb{C}^\times$ of a quadratic field K, and g is the theta series attached to this character. The representation

$$V_{\psi_g} := \mathrm{Ind}_K^{\mathbb{Q}}(\psi_g) : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$$

is irreducible when restricted to K, where it is isomorphic to $L(\psi_g) \oplus L(\psi'_g)$ (ψ'_g being the character obtained by composing ψ_g with the automorphism of $K/\mathbb{Q}$). Further, V_{ψ_g} if and only if either:

1. K is an imaginary quadratic field.
2. K is a real quadratic field and ψ_g is of mixed signature (even at one archimedian place of K and odd at the other).

Observe that V_{ψ_g} is irreducible (and hence g is a cusp form) when $\psi_g \neq \psi'_g$. The most basic invariant of V_{gh} is its decomposition into irreducible representations, $V_{gh} = V_1 \oplus \ldots \oplus V_t$. Let $d = (d_1, \dots, d_t)$ be the dimensions of the V_j arranged in non-increasing order. We have five possibilities.
If g and h are both irreducible (Eisenstein series), then $d = (1, 1, 1, 1)$. Now, assume that at least one of g or h is a cusp form. The following lemmas will summarize the main results concerning the classification of weight one modular forms.

Lemma 9. *The representation V_{gh} contains a subrepresentation of dimension 1 if and only if h is a twist of $\bar{g}$, that is, $V_h = V_g^*(\iota)$, for some Dirichlet character ι. In that case,*

- *$d = (3, 1)$ if and only if g is exotic.*
- *$d = (2, 1, 1)$ if and only if g is dihedral and $\psi_g^- := \psi_g/\psi_g'$ is not a quadratic (genus) character.*
- *$d = (1, 1, 1, 1)$ if and only if g is dihedral and ψ_g^- is a genus character.*

Lemma 10. *The representation V_{gh} decomposes as a sum of two irreducible representations of dimension two if and only if:*

1. *Exactly one of g or h is a weigh one Eisenstein series, say $h = E_1(\chi_1, \chi_2)$, in which case*
$$V_{gh} = V_g(\chi_1) \oplus V_g(\chi_2).$$
2. *Both g and h are theta series attached to characters ψ_g and ψ_h of the same quadratic field K for which neither of the characters $\psi_1 := \psi_g\psi_h$, $\psi_2 = \psi_g\psi_h'$ is equal to its conjugate over K. In this case,*
$$V_{gh} = V_{\psi_1} \oplus V_{\psi_2}.$$
The representations V_{ψ_1} and V_{ψ_2} are non-isomorphic, and if K is real quadratic, they are of opposite (pure) signatures.

The generic case where $d = (4)$ contains all scenarios where ρ_g and ρ_h cut out linearly disjoint extensions of $\mathbb{Q}$. When it is irreducible, the Artin representation V_{gh} occurs with multiplicity two in $L \otimes \mathcal{O}_H^\times$.

The reader interested in a more detailed presentation of the Elliptic Stark conjecture can move directly to chapter 7, since chapters 5 and 6 are devoted to introduce the most technical aspects from the theory of Euler systems, and this is not strictly necessary in a first reading of this thesis.

4.5 The Gross-Stark unit attached to a weight one form

We now explain, following [DLR1] how to associate to a p-stabilized eigenform g_α satisfying certain hypothesis a global element $u_{g_\alpha} \in H^\times \otimes L$ (well defined up to scaling by $L^\times$), whose p-adic logarithm will have important properties. The hypothesis we require, and that will be explored later on, is the so-called classicality property for g_α, that forces g to satisfy one of the following explicit conditions:

- It is a cusp form which is regular at the prime p (i.e., ρ_g is irreducible and the Frobenius element at p acts on it with two distinct eigenvalues) and it is not the theta series of a character of a real quadratic field in which p splits.
- It is an Eisenstein series which is irregular at p (alternatively, ρ_g is the direct sum of two characters χ_1 and χ_2 such that $\chi_1(p) = \chi_2(p)$.

This element will belong to the Ad_g-isotypic subspace of $H^\times \otimes L$, where $\mathrm{Ad}_g := \mathrm{End}_0(V_g)$ is the adjoint representation attached to V_G, that is, $\mathrm{End}(V_g)$ is a 4-dimensional space and we restrict here to those of trace zero, on which $G_{\mathbb{Q}}$ acts by conjugation via ρ_g.
If σ acts on V, then it also acts on $\mathrm{End}(V)$, via

$$\sigma(f(v)) = \sigma f(\sigma^{-1} v)$$

and it is not difficult to check that if the eigenvalues of σ acting on V are α, β, then those of σ acting on Ad_g are $1, \alpha/\beta, \beta/\alpha$.
In defining u_{g_α} it is convenient to replace the field H by the smaller field H_g cut out by Ad_g, that is the same through which factors the representation into the projectivization of the vector space. We also replace L by any field over which the representation Ad_g can be defined. Roughly speaking, $L \cdot u_{g_\alpha}$ is expected to be contained in $(\mathcal{O}_{H_g}^\times)_L^{\mathrm{Ad}_g}$ when g is cuspidal and in $(\mathcal{O}_{H_g}[1/p]^\times)_L^{\mathrm{Ad}\, g}$ when g is Eisenstein. Further, the arithmetic Frobenius element Frob_p must act on u_{g_α} with eigenvalue α_g/β_g.
Write $\sigma_\infty \in G := \mathrm{Gal}(H_g/\mathbb{Q})$ for the involution given by complex conjugation induced by the fixed choice of complex embedding of $H_g \subset \bar{\mathbb{Q}}$. Then, Ad_g decomposes as a direct sume of $+$ and $-$ eigenspaces for the action of σ_∞, denoted as $\mathrm{Ad}_g^+, \mathrm{Ad}_g^-$ that are of dimension 1 and 2 respectively.

We begin dealing with the case in which ρ_g is irreducible, i.e., g is a cuspidal eigenform of weight one.

Proposition 17. *The vector space* $\mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{Ad}_g, \mathcal{O}_{H_g}^\times \otimes L)$ *is* 1*-dimensional. Further, the Artin L-function* $L(\mathrm{Ad}_g, s)$ *has a simple zero at* $s = 0$.

Proof. $G = \mathrm{Gal}(H_g/\mathbb{Q})$ acts faithfully on Ad_g and is identified with the image of the projective representation attached to V_g. $G_\infty = \langle \sigma_\infty \rangle$ is of order two (Ad_g is not totally even) and Dirichlet's unit theorem determines the structure of $O_{H_g}^\times \otimes L$ as an $L[G]$-module, namely

$$\mathcal{O}_{H_g}^\times \otimes L = (\mathrm{Ind}_{G_\infty}^G L) - L,$$

where the two occurrences of L refer to trivial representations. The irreducibility of V_g implies that Ad_g contains no G-invariant vectors (by Schur's lemma), and hence

$$\mathrm{Hom}_G(\mathrm{Ad}_g, \mathcal{O}_{H_g}^\times \otimes L) = \mathrm{Hom}_G(\mathrm{Ad}_g, \mathrm{Ind}_{G_\infty}^G L).$$

By Frobenius reciprocity,

$$\mathrm{Hom}_g(\mathrm{Ad}_g, \mathrm{Ind}_{G_\infty}^G L) = \mathrm{Hom}_{G_\infty}(\mathrm{Ad}_g, L) = \mathrm{Hom}(\mathrm{Ad}_g^+, L).$$

Now, the first part follows from the dimension of Ad_g^+ being 2; the second, from the spape of the Γ-factors in the functional equation of $L(\mathrm{Ad}_g, s)$. $\square$

Choose now a basis ϕ for the one-dimensional vector space $\mathrm{Hom}_G(\mathrm{Ad}_g, \mathcal{O}_{H_g}^\times \otimes L)$, and let

$$(\mathcal{O}_{H_g}^\times)_L^{\mathrm{Ad}_g} := \phi(\mathrm{Ad}_g) \subset \mathcal{O}_{H_g}^\times \otimes L.$$

The $L[G]$-module $(\mathcal{O}_{H_g}^\times)_L^{\mathrm{Ad}_g}$ is of dimension ≤ 3 over L and abstractly isomorphic to the unique irreducible subrepresentation of Ad_g containing Ad_g^+.
The Frobenius Frob_p acts on Ad_g with eigenvalues $1, \alpha_g/\beta_g, \beta_g/\alpha_g$. Then, define

$$U_{g_\alpha} := \{u \in (\mathcal{O}_{H_g}^\times)_L^{\mathrm{Ad}_g} \text{ such that } \sigma_p(u) = (\alpha_g/\beta_g)u\}.$$

Lemma 11. *Assume that V_g is regular. If ρ_g is induced from a character of a real quadratic field in which p splits, then $U_{g_\alpha} = 0$. Otherwise, $\dim_L(U_{g_\alpha}) \geq 1$ with equality if either:*

1. $\alpha_g \neq -\beta_g$ *(Ad_g is regular).*
2. ρ_g *is induced from a character of a quadratic field in which p is inert.*

The proof of the lemma is a case by case analysis, according to the fact that V_g may have imaginary dihedral projective image, real dihedral projective image or in the case that g is an exotic weight one form, then it has projective image A_4, S_4 or A_5.

Definition 34. *A non-zero element u_{g_α} of U_{g_α} is called a Stark unit attached to g_α.*

We now move to the case where ρ_g is reducible. That is, g corresponds to the Eisenstein series $E_1(\chi_1, \chi_2) \in M_1(N, \chi)$, where χ_1 and χ_2 are odd and even Dirichlet characters satisfying $\chi_1(p) = \chi_2(p)$. In that case,

$$V_g = L(\chi_1) \oplus L(\chi_2), \qquad \mathrm{Ad}_g = L \oplus L(\eta) \oplus L(\eta^{-1}),$$

where $\eta := \chi_1/\chi_2$. The condition $\chi_1(p) = \chi_2(p)$ implies $\eta_p(1)$ and hence p splits completely in the abelian extension H_η cut out by η.
To define the Stark unit, consider a prime ideal $\mathfrak{P}$ of O_{H_η} over p inducing the embedding $H_\eta \hookrightarrow \mathbb{C}_p$, and let $u_\mathfrak{P} \in O_{H_\eta}[1/p]^\times$ be a generator of the principal ideal $\mathfrak{P}^h$, where h is the class number of H_η:

$$\mathfrak{P}^h = (u_\mathfrak{P}), \qquad u_\mathfrak{P} \in (O_{H_\eta}[1/\mathfrak{P}])^\times \pmod{O_{H_\eta}^\times}.$$

Definition 35. *Consider*

$$e_\eta := \frac{1}{|G|}\sum_{\sigma \in G} \eta^{-1}(\sigma)\sigma, \qquad e_{\eta^{-1}} := \frac{1}{|G|}\sum_{\sigma \in G} \eta(\sigma)\sigma.$$

Set

$$u_{g_\alpha} = e_\eta u_\mathfrak{P} + e_{\eta^{-1}} u_\mathfrak{P}.$$

The unit u_{g_α} is a linear combination of Gross-Stark units attached to the characters η and η^{-1}, in such a way that

$$\log_\mathfrak{P}(u_{g_\alpha}) = \frac{1}{|G|}\sum_{\sigma \in \mathrm{Gal}(H_\eta/\mathbb{Q})} (\eta(\sigma) + \eta^{-1}(\sigma)) \log_\mathfrak{P}(\sigma(u_\mathfrak{P})).$$

Here, we have denoted by $\log_\mathfrak{P}$ the Iwasawa's logarithm, composed with the p-adic embedding of H_η attached to $\mathfrak{P}$.

More details can be found in [DLR].

5 Euler systems

Let $E/\mathbb{Q}$ be an elliptic curve. BSD conjecture relates two apparently different worlds: on the one hand, that of group of cycles (arithmetic) and on the other, that of L-functions (analytic). Kato's idea was to consider some kind of arithmetic shape of the L-functions: these are the Euler systems. Its definition is a general machine in which many people have contributed: Kolyvagin, Perrin-Riou, Kato, Rubin,... For establishing its connection with L-functions we need p-adic Hodge theory, that is the key tool towards formulating explicit reciprocity laws.

The rough idea is the following: let T be a continuous geometric representation of $G_{\mathbb{Q}}$ on a finite free $\mathbb{Z}_p$-module. An Euler system for T will be a collection of cohomology classes $c_m \in H^1(F_m, T)$, (the F_m are number fields, the easiest case will be just $H^1(\mathbb{Q}(\zeta_m), T)$ in the case of cyclotomic units) satisfying certain compatibility relations. The Euler system for T will control the Selmer group associated with $T^*(1)$, which is a subgroup of $H^1(\mathbb{Q}, T^*(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$. Along this chapter we will go deeper into these ideas, but keeping in mind Colmez's view that "the construction of Euler systems is a totally craft activity".

We will begin by introducing the basic ideas of Galois cohomology to understand the concepts of Selmer groups and Euler systems. We will cover part of the material of [Bel], that constitutes the theoretical background that we need to carry out our study. Further, we also revisit the first part of [BCDDPR], where the most classical examples of this theory are presented with a connection with L-functions, namely, Leopoldt's formula (relating the value at $s = 1$ of the Kubota-Leopoldt p-adic L-function with the p-adic logarithm of circular units), Katz's p-adic Kronecker limit formula (relating the values of the two variable p-adic L-function of a quadratic imaginary field at finite order characters with p-adic logarithms of associated elliptic units) and also a more recent formula in which Heegner points arise, by replacing Einsenstein series by cusp forms. With all this machinery, in the next chapter we will move to the study of the Euler system of Beilinson-Flach elements, that is one of the main topics of this thesis.

5.1 Galois cohomology

Along this part, we will assume certain familiarity with the basic theory of group representations and group cohomology. We will need the language of étale cohomology. The construction and definitions are done in any standard book in the topic. For our interests, we will quote (without proof) those properties that will be more relevant. In this section, some ubiquous ring of periods will appear. Its meaning will be clarified in subsequent sections. This material is mostly taken from [Bel]

Let K be a number field, and let X be a proper and smooth variety over K of dimension n, i an integer and p a prime number. Then,

$$H^i(X, \mathbb{Q}_p) = \lim_{\leftarrow} H^1_{\text{et}}(X \times \bar{K}, \mathbb{Z}/p^n\mathbb{Z}).$$

This space is endowed with a $\mathbb{Q}_p$ linear action of G_K. Some of its important features are the following ones:

1. $H^i(X, \mathbb{Q}_p)$ is finite dimensional and the dimension is independent of p. The action of G_K is continuous.

2. $X \mapsto H^i(X, \mathbb{Q}_p)$ is a contravariant functor from the category of proper and smooth varieties over K to the cateogry of p-adic representations of G_K.

3. $H^i(X, \mathbb{Q}_p) = 0$ if $i < 0$ or $i > 2n$. When X is (geometrically) connected, $H^0(X, \mathbb{Q}_p) = \mathbb{Q}_p$ (with trivial action) and $H^{2n}(X, \mathbb{Q}) = \mathbb{Q}_p(-n)$.

4. There is a functorial cup product map of G_K-representations.

5. Let v be a finite place of K prime to p. If X has good reduction at v, then $H^i(X, \mathbb{Q}_p)$ does not ramify at v. Frob_v acting on $H^i(X, \mathbb{Q}_p)$ has its coefficients in $\mathbb{Z}$ and is independent of p.

6. Let v be a place of K dividing p. Then, as a representation of G_v, $H^i(X, \mathbb{Q}_p)$ is de Rham. If X has good reduction at v, $H^i(X, \mathbb{Q}_p)$ is crystalline.

7. If Z is a subvariety of codimension q, there is associated to Z a cohomology class $\eta(Z) \in H^{2q}(X, \mathbb{Q}_l)(q)$ invariant by G_K. This extends by linearity to cycles and rationally equivalent cycles have the same cohomology class. Intersection of cycles becomes cup-product of cohomology cycles.

Let V be an irreducible p-adic representation of G_K. V is said to **come from geometry** if there is an integer i, an integer n and a proper and smooth variety X over K such that V is isomorphic to a quotient of $H^i(X, \mathbb{Q}_p)(n)$. When V is a p-adic semi-simple representation of G_K, we say that it is **geometric** if it is unramified at almost all places and de Rham at all those that divide p. The famous Fontaine-Mazur conjecture asserts that when V is geometric, then it comes from geometry.
We continue with more terminology: let V be a representation of G_K unramified outside a finite set of places Σ. Then, it is **algebraic** if there is a finite set of places Σ' containing Σ such that the characteristic polynomial of Frob_v on V has coefficients in $\overline{\mathbb{Q}}$ when $v \notin \Sigma'$. Finally, for $w \in \mathbb{Z}$ we say that V is **pure of weight** w if there is a finite set of places Σ' containing Σ such that V is Σ'-rational and all the roots of the characteristic polynomial of Frob_v have complex absolute values $q_v^{-w/2}$. For V pure of weight w, we call w the motivic weight of V.

From now on, G will be a profinite group and p a prime. We say that condition $\mathrm{Fin}(p, i)$ is verified if for every open subgroup U of G, the set $H^i(U, \mathbb{Z}/p\mathbb{Z})$ is finite. Further, condition $\mathrm{Fin}(p)$ is satisfied when we have $\mathrm{Fin}(p, i)$ for all $i \geq 0$.
Our interest is in the continuous group cohomology $H^i(G, V)$ of profinite groups G satisfying $\mathrm{Fin}(p)$ with values in finite dimensional $\mathbb{Q}_p$-vector spaces V with a continuous action of G. Tools such as Shapiro's lemma, inflation-restriction or long exact sequences work in the same way. Many of the basic results in Galois cohomology are proved with discrete coefficients, and to pass to p-adic coefficients, we need the following result of Tate.

Proposition 18 (Tate). *Let G be a profinite group satisfying* $\mathrm{Fin}(p)$ *and let V be a continuous representation of G. Let Λ be a $\mathbb{Z}_p$ lattice in V stable by G.*

1. *The continuous cohomology group $H^i(G, \Lambda)$ is a finite $\mathbb{Z}_p$-module and we have a canonical isomorphism*

$$H^i(G, V) \simeq H^i(G, \Lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

2. *We have a canonical isomorphism $H^i(G,\Lambda) = \lim_{\leftarrow} H^i(G,\Lambda/p^n\Lambda)$, where $\Lambda/p^n\Lambda$ is a finite group endowed with the discrete topology.*

An important tool to build elements of H^1 is the Kummer construction. We are going to provide three different ways to proceed:

1. Let K be a field and A be a commutative group scheme over K such that multiplicaton by p, $[p] : A \to A$ is surjective and finite. Then, multiplication by p induces surjective homomorphisms $A[p^{n+1}] \to A[p^n]$ and hence surjective homomorphisms $A[p^{n+1}](\bar{K}) \to A[p^n](\bar{K})$ compatible with the action of G_K. Let $T_p(A)$ be the usual Tate module and $V_p(A) = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The space $V_p(A)$ is a p-adic representation of G_K. It is clear that we have an injective homomorphism
$$\kappa_n : A(K)/p^nA(K) \to H^1(G_K, A[p^n](\bar{K})).$$

2. Let $x \in A(K)$. Since multiplication by p^n is surjective, there exists $y \in A(\bar{K})$ such that $p^n y = x$. Choose such a y and define $c_y(g) = g(y) - y$ for all $g \in G_K$. Then,
$$p^n c_y(g) = p^n(g(y) - y) = g(p^n y) - p^n y = g(x) - x = 0,$$
and hence $c_y(g) \in A[p^n](\bar{K})$. It is straightforward that $g \mapsto c_y(g)$ is a 1-cocycle from G_K to $A[p^n](\bar{K})$ and then, it has a class $\bar{c}_y$ in $H^1(G_K, A[p^n](\bar{K}))$. It can be checked that this class does not depend on the choice of y, but only on x.
We have constructed a map $x \mapsto \bar{c}_y$, $A(K) \to H^1(G_K, A[p^n](\bar{K}))$. This is a morphism of groups sending $p^nA(K)$ to zero, and hence we have a map
$$A(K)/p^nA(K) \to H^1(G_K, A[p^n](\bar{K})),$$
that is the same κ_n constructed before.

3. The third way is the most technical one. We assume for the sake of simplicity that K is perfect. Take $x \in A(K)$ and consider $T_{n,x}$, the fiber at x of the map $[p^n]$, that is a finite subscheme of A over which there is an algebraic action of the commutative group scheme $A[p^n]$. This is a morphism of K-schemes $A[p^n] \times T_{n,x} \to T_{n,x}$ which on R-points is a group action of the group $A[p^n](R)$ on the set $T_{n,x}(R)$ for all K-algebras R. The map sends (z, y) to $z + y$. Observe that $T_{n,x}$ is isomorphic to $A[p^n]$ itself with its right translation action. This implies that $T_{n,x}$ is what is called a K-torsor under $A[p^n]$, locally trivial for the étale topology. This torsors are classified by $H^1_{\text{et}}(\operatorname{Spec} K, A[p^n]) = H^1(G_K, A[p^n](\bar{K}))$. In particular, $T_{n,x}$ defines an element of $H^1(G_K, A[p^n](\bar{K}))$, that is $\kappa_n(x)$.

Since these maps for various n are compatible, we have a map
$$\lim_{\leftarrow} A(K) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} \to \lim_{\leftarrow} H^1(G, A[p^n](K)).$$
The LHS is the p-adic completion of $A(K)$, $\widehat{A(K)}$. The RHS is $H_1(G, T_p(A))$.
Let K be a finite extension of $\mathbb{Q}_l$ and V a p-adic representation of G_K. From the standard results of Tate for Galois cohomology, we have the following:

Proposition 19. *The* **cohomological dimension** $H^i(G_K, V) = 0$ *if* $i > 2$.
In addition, there is a notion of **duality***: we have a canonical isomorphism* $H^2(G_K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$ *and the pairing* $H^i(G_K, V) \times H^{2-i}(G_K, V^*(1)) \to H^2(G_K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$ *given by the cup product is a perfect pairing for* $i = 0, 1, 2$.
Finally, the **Euler-Poincaré** *relation states that* $\dim H^0(G_K, V) - \dim H^1(G_K, V) + \dim H^2(G_K, V) = 0$ *when* $l \neq p$ *and* $[K : \mathbb{Q}_p] \dim V$ *when* $l = p$.

An important concept that will arise frequently is the following one:

Definition 36. *The unramified H^1 is $H^1_{\mathrm{ur}}(G_K, V) = \ker(H^1(G_K, V) \to H^1(I_K, V))$.*

Proposition 20. *The following properties hold:*

1. $\dim H^1_{\mathrm{ur}}(G_K, V) = \dim H^0(G_K, V)$.
2. *An element of $H^1(G_K, V)$ that corresponds to the extension $0 \to V \to W \to \mathbb{Q}_p \to 0$ is in $H^1_{\mathrm{ur}}(G_K, V)$ if and only if the sequence $0 \to V^{I_K} \to W^{I_K} \to \mathbb{Q}_p \to 0$ is exact.*
3. *If $l \neq p$, the orthogonal to $H^1_{\mathrm{ur}}(G_K, V)$ is $H^1_{\mathrm{ur}}(G_K, V^*(1))$.*

When $p \neq l$, the isomorphism κ identifies the subspace $\widehat{\mathcal{O}_K^*} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of $\widehat{K^*} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ with the subspace $H^1_{\mathrm{ur}}(G_K, \mathbb{Q}_p(1))$ of $H^1(G_K, \mathbb{Q}_p(1))$. This trivial result is the shadow in characteristic 0 of a remarkable result with torsion coefficients, namely, that κ_n maps $\mathcal{O}_K^* \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ into $H^1_{\mathrm{ur}}(G_K, \mu_{p^n}(\bar{K}))$, which is defined as

$$\ker(H^1(G_K, \mu_{p^n}(\bar{K})) \to H^1(I_K, \mu_{p^n}(\bar{K}))).$$

Now, we are going to take a number field K and a prime number p. Σ will denote a finite set of primes of K containing all the primes above p. For v a place of K, we denote by G_v the absolute Galois group of the completion K_v of K at v. Let V be a p-adic representation of $G_{K,\Sigma}$, that is a representation of G_K unramified at all primes not in Σ. For global Galois cohomology, we still have a simple Euler-Poincaré formula:

$$\dim H^0(G_{K,\Sigma}, V) - \dim H^1(G_{K,\Sigma}, V) + \dim H^2(G_{K,\Sigma}, V)$$
$$= \sum_{v|\infty} H^0(G_v, V) - [K : \mathbb{Q}] \dim V.$$

The notion of Selmer group plays a prominent role in the whole theory.

Definition 37. *Let V be a p-adic representation of G_K unramified almost everywhere. A Selmer structure $\mathcal{L} = (L_v)$ for V is a collection of subspace L_v of $H^1(G_v, V)$ for all finite places v of K such that for almost all v, $L_v = H^1_{\mathrm{ur}}(G_v, V)$.*
The Selmer group attached to $\mathcal{L}$ is the subspace $H^1_{\mathcal{L}}(G_K, V)$ of elements $x \in H^1(G_K, V)$ such that for all finite places v, the restriction x_v of x ot $H^1(G_v, V)$ is in L_v.

However, these concepts we have been studied turn much more complicated at places dividing p. We would like to have a subspace L of $H^1(G_K, V)$ analogue to $H^1_{\mathrm{ur}}(G_{K'}, V)$ where K' is a finite extension of $\mathbb{Q}_l$ and V a p-adic representation, $p \neq l$. The answer $H^1_{\mathrm{ur}}(G_K, V)$ is not satisfying, since the p-adic analog of being unramified is being crystalline. Hence, we have to do another definition in terms of some rings of periods that will be introduced in the next section.

Definition 38. *We set $H^1_f(G_K, V) = \ker(H^1(G_K, V) \to H^1(G_K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})$.*
In the same way, we define $H^1_g(G_K, V) = \ker(H^1(G_K, V) \to H^1(G_K, V \otimes B_{\mathrm{dR}}))$ and $H^1_e(G_K, V) = \ker(H^1(G_K, V) \to H^1(G_K, V \otimes B_{\mathrm{crys}}^{\phi=1}))$. Since $B_{\mathrm{crys}}^{\phi=1} \subset B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$, we have that

$$H^1_e(G_K, V) \subset H^1_f(G_K, V) \subset H^1_g(G_K, V).$$

These structures satisfy different properties that will interest us in some moments. The idea we can keep in mind is that $H^1_f(G_K, V)$ is the analog of $H^1_{\mathrm{ur}}(G_K, V)$. For instance, if V is de Rham, the orthogonal of $H^1_f(G_K, V)$ is $H^1_f(G_K, V^*(1))$, and the Kummer map $\kappa : \widehat{K^*} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^1(G_K, \mathbb{Q}_p(1))$ identifies $\mathcal{O}^*_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $H^1_f(G_K, \mathbb{Q}_p(1))$.

We will state now some properties that it is useful to keep in mind, all of them referring to the case in which K is a finite extension of $\mathbb{Q}_p$. Far from being a list of isolated theorems, these facts will appear again in chapter eight and they have an importance that not even the most accurate of the adjectives would be able to describe:

1. An element of $H^1(G_K, V)$ corresponding to the extension $0 \to V \to W \to \mathbb{Q}_p \to 0$ is in $H^1_f(G_K, V)$ if and only if when applying the functor D_{crys} is still exact.

2. If V is de Rham, considering $D^+_{\mathrm{dR}} = (V \otimes B^+_{\mathrm{dR}})^{G_K}$, then
$$\dim_{\mathbb{Q}_p} H^1_f(G_K, V) = \dim_{\mathbb{Q}_p}(D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V)) + \dim_{\mathbb{Q}_p} H^0(G_K, V).$$

3. If V is de Rham, for the duality between $H^1(G_K, V)$ and $H^1(G_K, V^*(1))$, the orthogonal of $H^1_f(G_K, V)$ is $H^1_f(G_K, V^*(1))$.

4. When E is an elliptic curve over K, the Kummer isomorphism is an isomorphism $E(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^1_f(G_K, V_p(E))$.

5. When V is de Rham,
$$\dim H^1_e(G_K, V) = \dim D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V) + \dim H^0(G_K, V) - \dim D_{\mathrm{crys}}(V)^{\phi=1},$$
$$\dim H^1_g(G_K, V) = \dim D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V) + \dim H^0(G_K, V) + \dim D_{\mathrm{crys}}(V^*(1))^{\phi=1}.$$

6. If V is de Rham, the orthogonal of $H^1_e(G_K, V)$ is $H^1_g(G_K, V^*(1))$ and the orthogonal of $H^1_g(G_K, V)$ is $H^1_e(G_K, V^*(1))$.

We finish by introducing the global Bloch-Kato Selmer group when K is a number field and V is a geometric p-adic representation of G_K. $H^1_f(G_K, V)$ is the subspace of elements x of $H^1(G_K, V)$ such that for all finite places v of K, the restriction x_v belongs to $H^1_f(G_K, V)$. $H^1_{f,S}(G_K, V)$ is the subspace of those elements such that x_v is in $H^1_f(G_v, V)$ when $v \notin S$ and is in $H^1_g(G_K, V)$ if $v \in S$. We call $H^1_g(G_K, V)$ the union of all $H^1_{f,S}(G_K, V)$ when S runs over all finite set of primes of K.

Proposition 21. *The Kummer map κ realizes an isomorphism*
$$\mathcal{O}^\times_K \otimes_{\mathbb{Z}} \mathbb{Q}_p \to H^1_f(G_K, \mathbb{Q}_p(1)).$$

The proof would show us that $H^1_{f,p}(G_K, \mathbb{Q}_p(1)) \simeq \mathcal{O}^\times_{K,p} \otimes \mathbb{Q}_p$, where $\mathcal{O}^\times_{K,p}$ is the group of elements x in $H^1(G_K, v)$ such that $x_p \in H^1_g(G_{\mathbb{Q}_p}, V)$ and $x_v \in H^1_f(G_{\mathbb{Q}_v}, V)$ for $v \neq p$. We will use this fact later on.

5.2 p-adic Hodge theory

p-adic Hodge theory is a topic that has an interest by itself. Beyond the rings of periods that appeared in the previous section, there are many results (most of them from the vast Langland's programme) in which they are a key tool; not only this: p-adic Hodge theory is the first step to understand the ubiquous perfectoid spaces, a topic of maximum interest nowadays. We give here some ideas of the main results.
Our first aim will be to develop p-adic comparison theorems analog to the ones we have in the complex setting:

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) \cong H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C},$$

whenever X is a smooth, projective variety over $\mathbb{C}$.
Let us recall some of these classical results:

Theorem 43 (de Rham)**.** *Let X be a smooth, projective variety over $\mathbb{C}$. Then,*

$$H^i(X(\mathbb{C}), \mathbb{C}) := H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i_{\mathrm{dR}}(X(\mathbb{C})/\mathbb{C}).$$

The next step is the Hodge theorem. Recall that de Rham cohomology $H^i_{\mathrm{dR}}(X(\mathbb{C})/\mathbb{C})$ is formed by classes of forms, and that forms can be of type (p, q), with $p + q = i$. Let $H^{p,q}(X)$ be the subspace of (p, q)-forms.

Theorem 44 (Hodge)**.** *We have a canonical decomposition*

$$H^i_{\mathrm{dR}}(X(\mathbb{C})/\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X),$$

and further $\overline{H^{p,q}(X)} = H^{q,p}(X)$, where $\bar{H}$ means complex conjugation.

In this setting, it is useful to recall Dolbeaut's theorem:

Theorem 45. *We have a canonical and functorial isomorphism*

$$H^{p,q}(X) \cong H^p(X(\mathbb{C}), \Omega^q).$$

All in all, what we have is that

$$H^i(X, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X/\mathbb{C}) \cong \bigoplus_{p+q=i} H^p(X, \Omega^q).$$

Then, the natural question is how can we extend these results to other fields. In particular, we will consider from now on a finite extension of $\mathbb{Q}_p$, say K. Consider a variety X over $\bar{K}$. We expect to obtain analogues to the classical Hodge theory, and this is precisely what étale cohomology does. In next sections we will introduce its most relevant aspects.
One of the cornerstones of this theory is the following:

Theorem 46 (Faltings)**.** *Let X be a smooth, projective variety over K. Then, there is a G_K-invariant canonical isomorphism*

$$H^i(\bar{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{p+q=i} \Big(H^p(X, \Omega^q_{X/K}) \otimes_K \mathbb{C}_p(-p)\Big).$$

Alternatively,
$$H^i_{\text{et}}(\bar{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}} \cong H^i_{\text{Hod}}(X/K) \otimes_K B_{\text{HT}},$$
where $H^i_{\text{Hod}} = \bigoplus_{p+q=i} H^p(X, \Omega^q)$ and B_{HT} is the ring of periods given by

$$B_{\text{HT}} := \sum_{j \in \mathbb{Z}} \mathbb{C}_p(j).$$

All in all, we can say that the aim of p-adic Hodge theory is to understand p-adic representations V of G_K. If $l \neq p$, the l-adic representations of G_K are well understood but when $l = p$ we get many more representations as the topologies of the $\mathbb{Q}_p$-vector spaces and G_K are compatible, so we need more theory to study these representations. The main strategy is to construct rings of periods B, like B_{HT}, equipped with a Galois action, such that $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$ is an invariant of the representation V. We require, at least, that B must be a domain, that $\text{Frac}(B)^{G_K} = B^{G_K}$ (and in particular B^{G_K} is a field), and that for $y \in B$ with $y\mathbb{Q}_p \subset B$ stable under G_K, then $y \in B^\times$.

Definition 39. *If V is a p-adic representation of G_K, we define*

$$D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Since B is a domain, we have a natural injective map

$$B \otimes_{B^{G_K}} D_B(V) \to B \otimes \mathbb{Q}_p V.$$

From here, we see that $\dim_{B^{G_K}}(D_B(V)) \leq \dim_{\mathbb{Q}_p}(V)$. When we have equality of dimensions, we say that V is B-admissible.

One of the main examples of a field of periods was constructed by Fontaine; it is B_{dR}. It is the field of fractions of a complete valuation ring B^+_{dR} which has residue field $\mathbb{C}_p$. Some of the most remarkable properties are:

1. The maximal ideal of B^+_{dR} is generated by an element t, which depends on a choice ϵ of compatible p-power roots of unity.

2. The action of $G_{\mathbb{Q}_p}$ on t is via the cyclotmic character χ: $g(t) = \chi(g)t$ for $g \in G_{\mathbb{Q}_p}$.

3. We have a descending filtration $\text{Fil}^i = t^i B^+_{\text{dR}}$, which is stable by the action of $G_{\mathbb{Q}_p}$.

4. Since B^+_{dR} is a complete discrete valuation ring, we ca use Hensel to see that $\overline{\mathbb{Q}_p} \subset B^+_{\text{dR}}$, and this is compatible with the action of $G_{\mathbb{Q}_p}$.

5. $(B_{\text{dR}})^{G_K} = (B^+_{\text{dR}})^{G_K} = K$. Thus, if V is a p-adic representation of G_K, $D_{\text{dR}}(V) = D_{B_{\text{dR}}}(V)$ is a filtered K-vector space.

We will say that a representation V of G_K is de Rham if it is B_{dR}-admissible. That is, we have an isomorphism

$$B_{\text{dR}} \otimes_K D_{\text{dR}}(V) \cong B_{\text{dR}} \otimes_{\mathbb{Q}_p} V.$$

Theorem 47. *Let X/K be a smooth proper variety, and let $V = H^i_{\text{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$. Then, V is de Rham, and there is a natural isomorphism of filtered K-vector spaces*

$$D_{\text{dR}}(V) \cong H^i_{\text{dR}}(X/K).$$

The property of being crystalline, for a p-adic representation V is analogous to the property of being unramified for an l-adic representation ($l \neq p$). For instance, if V is the p-adic Tate module of an abelian variety A/K, then V is crystalline if and only if V has good reduction.

Theorem 48. *Let k be a perfect field of characteristic p, let $\mathcal{O}_F = W(k)$ and $F = \mathrm{Frac}(\mathcal{O}_F)$. Let $X/\mathcal{O}_F$ be a smooth, proper scheme, geometrically irreducible. Then, for every $i \geq 0$ there is a functorial isomorphism*

$$H^i_{\mathrm{et}}(X_{\bar{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \cong H^i_{\mathrm{crys}}(X_k/\mathcal{O}_F) \otimes_{O_F} B_{\mathrm{crys}}.$$

We are going to make some comments about the construction of the de Rham ring of periods.
First of all, consider

$$R = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} = \{(x_0, x_1, \ldots) \mid x_{i+1}^p = x_i\}.$$

By this process, we get a perfect ring, that can also be understood as

$$\lim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p} = \{(x_0, x_1, \ldots) \mid x_{i+1}^p = x_i\}.$$

It is customary to denote R as $\tilde{E}^+$. It is a valuation ring with valuation $v_E(x) = v_p(x_0)$ and maximal ideal $\mathfrak{m}_{\tilde{E}^+} = \{x \in \tilde{E}^+ \mid v_E(x) > 0\}$.
In the first ring we have considered, R, addition and multiplication are defined componentwise, while in this latter we have the structure inherited by the first ring through the obvious isomorphism, and then sum is not componentwise. We may choose once for all

$$\epsilon = (1, \epsilon_1, \ldots) \in R, \quad \epsilon_q \neq 1.$$

Let $\tilde{\pi} = \epsilon - 1 \in R$, whose valuation is $\frac{p}{p-1} > 0$. As we have said, χ, the cyclotomic character, acts as

$$g(\epsilon) = \epsilon^{\chi(g)} = \sum_{k=0}^{\infty} \binom{\chi(g)}{k} \tilde{\pi}^k.$$

Let

$$\theta : R \to \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$$

be the projection onto the first component. Define also W, the ring of Witt vectors of R, as the elements of the form $\{\sum_{k \geq 0}[r_k]p^k \mid r_k \in R\}$, where $[r_k]$ is the so-called Teichmüller lift (in the case where $R = \mathbb{F}_p$, $W = \mathbb{Z}_p$ and we recover the usual Teichmüller character). Observe also that $W/pW = R$.
It satisfies a universal property: any morphism $\phi : R \to S$ of perfect rings lifts to $\Phi : W_R \to W_S$. Hence, we may define another morphism (we will call it θ by an abuse of notation)

$$\theta : W \to \mathcal{O}_{\mathbb{C}_p}$$
$$\sum p^k [r_k] \mapsto \sum p^k [r_k]_0.$$

In this same way, we can also extend the morphism and say

$$\theta : W\left[\frac{1}{p}\right] \to \mathbb{C}_p.$$

It is not difficult to check that θ is surjective and the kernel is principal, $\ker \theta = tW[1/p]$ for some $t \in W[1/p]$.

Definition 40. *We define B_{dR}^+, the ring of Fontaine's de Rham periods as*

$$\varprojlim_{n\geq 0} W[1/p]/t^n W[1/p] = \{\sum_{n\geq 0} b_n t^n \mid b_n \in W[1/p]\}.$$

B_{dR}^+ is a complete discrete valuation ring equipped with an action of $G_{\mathbb{Q}_p}$. There is in some way a canonical generator of the kernel of θ, attached to a collection $\{\zeta_r\}_{r\geq 0} \in R$ of roots of unity satisfying $\zeta_r^p = \zeta_{r-1}$. This canonical uniformizer $t \in B_{\mathrm{dR}}^+$ satisfies that $\sigma(t) = \chi_{\mathrm{cyc}}(\sigma) \cdot t$, and this t can be seen as some kind of logarithm in $W[1/p]$.

Definition 41. $B_{\mathrm{dR}} := B_{\mathrm{dR}}^+[1/t]$ *is a field with a decreasing filtration given by*

$$\mathrm{Fil}^n(B_{\mathrm{dR}}) = t^n B_{\mathrm{dR}},$$

where $n \in \mathbb{Z}$. Recall that

$$B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+/\ker(\theta) \cong \mathbb{C}_p.$$

Furthermore, for any $n \in \mathbb{Z}$,

$$\mathrm{Fil}^n(B_{\mathrm{dR}})/\mathrm{Fil}^{n+1}(B_{\mathrm{dR}}) = t^n B_{\mathrm{dR}}/t^{n+1}B_{\mathrm{dR}} \cong \mathbb{C}_p,$$

where the latter is an isomorphism of $\mathbb{C}_p$-vector spaces. Observe that in $t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+$ there is a $\mathbb{C}_p(n)$-action (that is, given by the n-power of the cyclotomic character). An important remark is that $G_{\mathbb{Q}_p}$ preserves the filtration on B_{dR} and

$$B_{\mathrm{dR}}^{G_{\mathbb{Q}_p}} = \mathbb{C}_p^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p.$$

The same occurs when $K/\mathbb{Q}_p$ is a finite extension.

Inside B_{dR}^+ there is an important ring: $B_{\mathrm{crys}}^+ \subset B_{\mathrm{dR}}^+$. On R, there is a Frobenius map $\phi(x) = x^p$. By the universal property of Witt vectors, there is a lifting of these Frobenius $\Phi : W[1/p] \to W[1/p]$, but the kernel is not preserved by it and is not compatible with the Galois action. Hence, we must construct a smaller ring where the Frobenius element acts. $t \in B_{\mathrm{crys}}^+$, and in general this ring is formed by elements satisfying a certain growth condition. If $\pi = \epsilon - 1$, $\pi_1 = \epsilon^{1/p} - 1$ and $\omega = \pi/\pi_1$, then

$$B_{\mathrm{crys}}^+ := \big\{\sum_{n\geq 0} a_n \frac{\omega^n}{n!}, \quad a_n \in \tilde{B}^+, \lim_{n\to\infty} a_n = 0\big\}.$$

For instance, $\sum \frac{1}{p^{n^2}} t^n \in B_{\mathrm{dR}}^+ \backslash B_{\mathrm{crys}}^+$. This new ring is no more a discrete valuation ring, not even a valuation ring. Hence, $B_{\mathrm{crys}} := B_{\mathrm{crys}}^+[1/t]$ is just a ring, not a field. Consequently, is not endowed with a canonical filtration. On the counterpart, there is a natural lift of the Frobenius. In fact,

$$B_{\mathrm{crys}}^{G_K} = K_0,$$

where K_0 is the maximal unramified extension of $\mathbb{Q}_p$ inside K. The Frobenius is compatible with the Galois action of $G_{\mathbb{Q}_p}$.

Proposition 22. *There is a short exact sequence given by*

$$0 \to \mathbb{Q}_p \to B_{\mathrm{crys}}^{\phi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0.$$

Let now V be a $\mathbb{Q}_p$-vector space of dimension $d < +\infty$ with a linear action of $\mathbb{Q}_p$.

Definition 42. *The de Rham module attached to V is*

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$$
$$= \{\sum v \otimes b \mid \sum \sigma(v) \otimes \sigma(b) = \sum v \otimes b \text{ for all } \sigma \in G_{\mathbb{Q}_p}\}.$$

In an analogous way, we define $D_{\mathrm{crys}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}) \subset D_{\mathrm{dR}}(V)$.

Lemma 12. $\dim_{\mathbb{Q}_p}(D_{\mathrm{crys}}(V)) \leq \dim_{\mathbb{Q}_p}(D_{\mathrm{dR}}(V)) \leq d = \dim_{\mathbb{Q}_p}(V)$.

Definition 43. *We say that V is a de Rham p-adic representation if* $\dim_{\mathbb{Q}_p}(D_{\mathrm{dR}}(V) = d$.
If $\dim_{\mathbb{Q}_p}(D_{\mathrm{crys}}(V)) = d$, *the representation is called crystalline. Observe that in particular a crystalline representation is de Rham.*

From now on, we will assume that V is de Rham. Then, $D_{\mathrm{dR}}(V)$ inherits a filtration from B_{dR}, say

$$D_{\mathrm{dR}}(V) = \mathrm{Fil}^0 D_{\mathrm{dR}}(V) \supset \ldots \supset \mathrm{Fil}^n D_{\mathrm{dR}}(V) \supset \mathrm{Fil}^{n+1} D_{\mathrm{dR}}(V) \supset \ldots \supset 0.$$

Let $h_1, \ldots, h_n$ the (possibly repeated) indexes such that $\mathrm{Fil}^{-h_i} D_{\mathrm{dR}}(V) \not\supset \mathrm{Fil}^{-h_i+1} D_{\mathrm{dR}}(V)$. These numbers are called the Hodge-Tate weights of V.

Lemma 13. *We have the following isomorphism of Galois modules*

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \sum_{i=1}^{d} \mathbb{C}_p(h_i).$$

Proof. We have that

$$\frac{\mathrm{Fil}^i D(V)}{\mathrm{Fil}^{i+1} D(V)} = \frac{(t^i B \otimes V)^{G_{\mathbb{Q}_p}}}{(t^{i+1} B \otimes V)^{G_{\mathbb{Q}_p}}} = \Big(\frac{\mathrm{Fil}^i(B)}{\mathrm{Fil}^{i+1}(B)} \otimes V\Big)^{G_{\mathbb{Q}_p}} = (\mathbb{C}_p(i) \otimes V)^{G_{\mathbb{Q}_p}}.$$

In particular, this is different from zero if and only if there is an element $v \in V$ such that

$$\sigma(v) = \chi_{\mathrm{cyc}}^{-i}(\sigma) \cdot v,$$

or what is the same if there exists a copy of $\mathbb{C}_p(-i)$ in V (in general, we have so many independent elements $v \in V$ in the quotient of filtered modules as copies of $\mathbb{C}_p(-i)$ in V). □

Theorem 49 (Faltings, Tsuji)**.** *Let $X/\mathbb{Q}_p$ be a smooth projective variety (with either good or bad reduction at p). Then, we have the following isomorphism between $G_{\mathbb{Q}_p}$-filtered modules*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^i_{\mathrm{et}}(X_{\bar{\mathbb{Q}_p}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes H^i_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

Moreover, if X has good reduction at p, the same holds for B_{crys} (and further, the isomorphism can also be seen as an isomorphism of Frobenius modules).

Corollary 3. *If we take $G_{\mathbb{Q}_p}$-invariants in the previous theorem, it yields that*

$$D_{\mathrm{dR}}(H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)) \cong H^i_{\mathrm{dR}}(X/\mathbb{Q}_p),$$

seen as an isomorphism of filtered spaces

Corollary 4. *Considering now the quotient* $\mathrm{Fil}^0 / \mathrm{Fil}^1$*, we have that*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \cong \mathbb{C}_p \otimes_{\mathbb{Q}_p} H^1_{\mathrm{dR}}(X/\mathbb{Q}_p) \cong \sum_{i=1}^{d} \mathbb{C}_p(h_i),$$

as $G_{\mathbb{Q}_p}$*-representations.*

In general, we also have the following spaces that have already appeared in the previous section:

$$H^1_e(\mathbb{Q}_p, V) \subset H^1_f(\mathbb{Q}_p, V) \subset H^1_g(\mathbb{Q}_p, V) \subset H^1(G_{\mathbb{Q}_p}, V) = \mathrm{Ext}^1(V, \mathbb{Q}_p),$$

where

$$H^1_g(\mathbb{Q}_p, V) := \ker(H^1(G_{\mathbb{Q}_p}, V) \to H^1(\mathbb{Q}_p, B_{\mathrm{dR}} \otimes V)),$$
$$H^1_g(\mathbb{Q}_p, V) := \ker(H^1(G_{\mathbb{Q}_p}, V) \to H^1(\mathbb{Q}_p, B_{\mathrm{crys}} \otimes V)),$$
$$H^1_g(\mathbb{Q}_p, V) := \ker(H^1(G_{\mathbb{Q}_p}, V) \to H^1(\mathbb{Q}_p, B_{\mathrm{crys}}^{\phi=1} \otimes V)).$$

Tensoring now with V the exact sequence

$$0 \to \mathbb{Q}_p \to B_{\mathrm{crys}}^{\phi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+} \to 0,$$

we obtain

$$0 \to V \to B_{\mathrm{crys}}^{\phi=1} \otimes V \to \frac{B_{\mathrm{dR}} \otimes V}{B_{\mathrm{dR}}^{+} \otimes V} \to 0.$$

We now take the long exact sequence in cohomology and we obtain

$$\frac{(B_{\mathrm{crys}}^{\phi=1} \otimes V)^{G_{\mathbb{Q}_p}}}{D_{\mathrm{crys}}^{\phi=1}} \to \frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0(D_{\mathrm{dR}}(V))} \to H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, B_{\mathrm{crys}}^{\phi=1} \otimes V) \to \dots$$

From here, the following isomorphism, denoted as $\exp_{\mathrm{BK}}$, follows immediate from the definition:

$$\exp_{\mathrm{BK}} : \frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V) + D_{\mathrm{crys}}^{\phi=1}(V)} \to H^1_e(\mathbb{Q}_p, V).$$

The inverse of this map is denoted as $\log_{\mathrm{BK}}$.

We have an étale regulator map

$$\mathrm{reg}_{\mathrm{et}} : \mathrm{CH}^c(X, n) \to H^1(G_{\mathbb{Q}}, V), \qquad V = H^{2c-n-1}_{\mathrm{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(c)),$$

that in the next chapter will be explicitly described for some particular case.
When I have a variety $X/\mathbb{Q}$, it is a conjecture (proved in some cases, like for modular curves by Saito), that the image of the étale regulator is in $H^1_f(\mathbb{Q}_p, V) \subset H^1(G_{\mathbb{Q}_p}, V)$, and in favorable circumstances (say for modular curves), $H^1_e(\mathbb{Q}_p, V) = H^1_f(\mathbb{Q}_p, V)$. In general, these two modules are not equal when we have non-trivial invariants and $H^0(G_{\mathbb{Q}_p}, V)$ or $H^0(G_{\mathbb{Q}_p}, V(1))$ are non-zero.
Then, we can apply the Bloch-Kato logarithm that maps $H^1_e(\mathbb{Q}_p, V)$ to $\frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0 D_{\mathrm{dR}} V}$ (when $H^1_f = H^1_e$ the other term of the quotient vanishes) and then via Faltings to $\frac{H^{2c-n-1}_{\mathrm{dR}}(X/\mathbb{Q}_p)}{\mathrm{Fil}^c H_{\mathrm{dR}}(X/\mathbb{Q}_p)}$.

Summing a map, we will have a map, called syntomic regulator

$$\mathrm{CH}^c(X,n) \to \frac{H^{2c-n-1}_{\mathrm{dR}}(X/\mathbb{Q}_p)}{\mathrm{Fil}^c H_{\mathrm{dR}}(X/\mathbb{Q}_p)}.$$

We will explain now the concept of dual exponential map. We will describe first in more detail the filtered ϕ-module $D_{\mathrm{crys}}(\mathbb{Q}_p(j))$. The choice of $\epsilon = (\epsilon_n)$ determines the element $t \in B_{\mathrm{dR}}$ and a basis e_j of $\mathbb{Q}_p(j)$ for each j, such that $e_j \otimes e_{j'} = e_{j+j'}$. Further, the element $t^{-j}e_j \in B_{\mathrm{crys}} \otimes \mathbb{Q}_p(j)$ is Galois invariant and determines a canonical basis for $D_{\mathrm{crys}}(\mathbb{Q}_p(j))$.
The dual exponential map of V is the map directly obtained by duality and given by

$$\exp^*_{K,V} : H^1_e(K, V^*(1)) \to \mathrm{Fil}^0 D_{\mathrm{dR}}(V^*(1)).$$

Observe that by the orthogonality properties we have worked before, $H^1_e(K,V^*(1)) \simeq H^1(K,V)/H^1_g(K,V)$, and we can understand the dual exponential map as an application from $H^1(K,V)$ to $\mathrm{Fil}^0 D_{\mathrm{dR}}(V^*(1))$ such that a class lies in the kernel if and only if it is de Rham.

Let us finish this section saying a few words about Iwasawa cohomology and the construction of Perrin-Riou big logarithm that interpolates both the Bloch-Kato logarithm and the dual exponential map; more material in this direction can be found in [Col]. As a motivation for that, consider a one-dimensional representation given by $V = \mathbb{Q}_p(\psi\chi^j_{\mathrm{cyc}})$, where ψ is a non-trivial, unramified character and j is any integer. If ψ were trivial, this would require the use of some of the more mysterious H_e and H_g. In this favorable case, we can see $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V)$ as the quotient $H^1(\mathbb{Q}_p,V)/H^1_f(\mathbb{Q}_p,V)$ (and in general, we will have to take care of H_e and H_g). Recall that the orthogonal of $H^1_e(G_K,V)$ is $H^1_g(G_K,V^*(1))$ and the orthogonal of $H^1_g(G_K,V)$ is $H^1_e(G_K,V^*(1))$. It turns out that, when $j \leq 0$, $H_f(\mathbb{Q}_p,V) = 0$ and $H^1(\mathbb{Q}_p,V) = H^1_{\mathrm{sing}}(\mathbb{Q}_p) \xrightarrow{\exp^*} \mathbb{Q}_p$. However, if $j \geq 0$, $H^1(\mathbb{Q}_p,V) = H^1_f(\mathbb{Q}_p,V) \xrightarrow{\log} \mathbb{Q}_p$. In any case, $H^1(\mathbb{Q}_p,V)$ is isomorphic to $\mathbb{Q}_p$, but this isomorphism is given up to a certain value by the dual exponential and then by the logarithm. It is natural to think that it would be desirable to have some interpolation map. This is Perrin-Riou big logarithm, that will be recovered in the last section of Chapter 6.

Let V be a p-adic representation of G_K of dimension d and $T \subset V$ a $\mathbb{Z}_p$-lattice stable by G_K. Recall the classical notations $K_n = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p)$, $K_\infty = \cup K_n$ and $\Gamma_n = \mathrm{Gal}(K_\infty/K_n) = \gamma_n^{\mathbb{Z}_p}$, where γ_n is a topological generator of Γ_n and $\gamma_n = \gamma_1^{p^{n-1}}$. Shapiro's lemma gives, for any $\mathbb{Z}_p[G_K]$-module M, an isomorphism

$$H^i(G_{K_n}, M) \to H^i(G_K, \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \otimes M).$$

The Iwasawa cohomology of V is

$$H^i_{\mathrm{Iw}}(K,V) = \mathbb{Q}_p \otimes_{\mathbb{Q}_p} \varprojlim H^i(K_n, T),$$

where the inverse limit is taken with respect to the corestriction map.
The ring $\mathbb{Q}_p \otimes \Lambda_K$ can be identified with the space of p-adic measures on Γ_K,

$$\mathbb{Q}_p \otimes \Lambda_K = \mathrm{Hom}(C(\Gamma_K, \mathbb{Q}_p), \mathbb{Q}_p),$$

where $C(\Gamma_K, \mathbb{Q}_p)$ is the space of continuous $\mathbb{Q}_p$-valued functions.

Theorem 50 (Perrin-Riou)**.** *We have that $H^i_{\mathrm{Iw}}(K,V)=0$ for $i\neq 1,2$. Moreover,*

1. *The torsion submodule of $H^1_{\mathrm{Iw}}(K,V)$ is a $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Lambda_K$-module isomorphic to V^{H_K}, and $H^1_{\mathrm{Iw}}(K,V)/V^{H_K}$ is a free $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Lambda_K$-module of rank $[K:\mathbb{Q}_p]d$.*
2. *$H^2_{\mathrm{Iw}}(K,V)=(V^*(1)^{H_K})^*$.*

We can finally introduce the Perrin-Riou regulator map. Let p be an odd prime, and $F/\mathbb{Q}_p$ a finite unramified extension. Let V be a crystalline representation of G_F with non-negative Hodge-Tate weights and no quotient isomorphic to the trivial representation. Further, let $\mathcal{H}_{\mathbb{Q}_p}(\Gamma_F)$ be the algebra of $\mathbb{Q}_p$-valued distributions on Γ_F. The logarithm of Perrin-Riou is a map

$$\mathcal{L}_{V,F}:H^1_{\mathrm{Iw}}(F,V)\to\mathcal{H}(\Gamma_F)\otimes_{\mathbb{Q}_p}D_{\mathrm{crys}}(V).$$

This map interpolates the Bloch-Kato dual exponential and logarithm maps for twists of V in the cyclotomic power. The main idea we must bear in mind is that it allows us to construct p-adic L-functions from Euler systems. It is customary to write $\int_\Gamma \eta\nu$ for $\nu(\eta)$, when $\nu\in\mathcal{H}(\Gamma_K)\otimes_{\mathbb{Q}_p}D_{\mathrm{crys}}(V)$ and $\eta:\Gamma_K\to\mathbb{Q}_p^\times$ is a character. Then, we have this result:

Theorem 51. *Let V be a crystalline representation of $G_{\mathbb{Q}_p}$ with non-negative Hodge-Tate weights and no quotient isomorphic to $\mathbb{Q}_p$. Let $\eta=\chi^j_{\mathrm{cyc}}\epsilon$, where ϵ is a finite order character of conductor n. Then, for $x\in H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)$, we have that:*

- *If $j\geq 0$, then $\int_\Gamma \eta\mathcal{L}_v(x)$ equals a explicit constant (that may be zero) times $\exp_{\mathbb{Q}_p,V(\eta^{-1})*(1)}(x_{n,0})\otimes t^{-j}\epsilon_j$.*
- *If $j<0$, then $\int_\Gamma \eta\mathcal{L}_V(x)$ equals a explicit constant (that may be zero) times $\log_{\mathbb{Q}_p,V(\eta-1)(1)}(x_{n,0}\otimes t^{-j}e_j)$.*

These concepts will be reformulated when introducing global cohomology classes coming from Beilinson-Flach elements.

5.3 What is an Euler system?

In this section, we will present a possible "official" definition of what an Euler system is. However, we will not restrict too much to it and we will work these systems through the (few) examples available in the literature.

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Fix also a rational prime p and a p-adic representation T of G_K with coefficients in the ring of integers $\mathcal{O}$ of some finite extension Φ of $\mathbb{Q}_p$. Further, assume that T does not ramify outside a finite set of primes of K.

Let $\mathfrak{q}$ be a prime of K not dividing p, and such that T is unramified at $\mathfrak{q}$. $K(\mathfrak{q})$ will denote the maximal p-extension of K inside the ray class field of K modulo $\mathfrak{q}$ and $\mathrm{Frob}_\mathfrak{q}$ is the Frobenius of $\mathfrak{q}$ in G_K. Define

$$P(\mathrm{Frob}_\mathfrak{q}^{-1}|T^*;x)=\det(1-\mathrm{Frob}_\mathfrak{q}|T^*)\in\mathcal{O}[x].$$

Recall that the determinant is well-defined because T^* is unramified at $\mathfrak{q}$.

Definition 44. *Let $\mathcal{K}$ be an (infinite) abelian extension of K and $\mathcal{N}$ an ideal of K divisible by p and by all primes where T ramifies such that:*

1. $\mathcal{K}$ contains $K(\mathfrak{q})$ for every prime $\mathfrak{q}$ of K not dividing $\mathcal{N}$.

2. $\mathcal{K}$ contains an extension K_∞ of K such that $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^d$ for some $d \geq 1$ and such that no (finite) prime of K splits completely in K_∞/K.

A collection of cohomology classes

$$\mathfrak{c} = \{\mathfrak{c}_F \in H^1(F,T) \mid K \subset_f F \subset \mathcal{K}\}$$

is an Euler system for $(T, \mathcal{K}, \mathcal{N})$ if whenever $K \subset_f F \subset_f F' \subset \mathcal{K}$, then

$$\mathrm{Cor}_{F'/F}(c_{F'}) = \Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)} P(\mathrm{Frob}_{\mathfrak{q}}^{-1} |T^*; \mathrm{Frob}_{\mathfrak{q}}^{-1})\Big) c_F,$$

where $\Sigma(F'/F)$ is the set of finite primes of K not dividing $\mathcal{N}$ which ramify in F' but not in F.
We say that a collection $c = \{c_F \in H^1(F,T)\}$ is an Euler system for T if c is an Euler system for $(T, \mathcal{K}, \mathcal{N})$ for some choice of $\mathcal{N}$ and $\mathcal{K}$ as above.
If K_∞ is a $\mathbb{Z}_p^d$-extension of K in which no finite prime of K splits completely, we say that $c = \{c_F \in H^1(F,T)\}$ is an Euler system for (T, K_∞) if c is an Euler system for $(T, \mathcal{K}, \mathcal{N})$ for some choice of $\mathcal{N}$ and $\mathcal{K}$ containing K_∞ as above.

For a first example, take $K = \mathbb{Q}$. For every extension F of $\mathbb{Q}$,

$$H^1(F, \mathbb{Z}_p(1)) = \varprojlim H^1(F, \mu_{p^n}) = \varprojlim F^\times/(F^\times)^{p^n} = \widehat{F^\times}.$$

Fix a collection $\{\zeta_m \mid m \in \mathbb{Z}\}$ of compatible primitive roots of unity ($\zeta_{mn}^n = \zeta_m$ for every m and n). For all $m \geq 1$ and every prime l we have that

$$\mathbb{N}_{\mathbb{Q}(\mu_{ml})/\mathbb{Q}(\mu_m)}(\zeta_{ml} - 1) = \begin{cases} (\zeta_m - 1) & \text{if } l|m, \\ (\zeta_m - 1)^{1-\mathrm{Frob}_l^{-1}} & \text{if } l \nmid m \text{ and } m > 1 \\ (-1)^{l-1} l & m = 1 \end{cases}$$

where Frob_l is the Frobenius of l in $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$.
Now, for every $m \geq 1$, we consider

$$\tilde{c}_{m\infty} = \mathbb{N}_{\mathbb{Q}(\mu_{mp})/\mathbb{Q}(\mu_m)}(\zeta_{mp} - 1) \in \mathbb{Q}(\mu_m)^\times \subset H^1(\mathbb{Q}(\mu_m), \mathbb{Z}_p(1)),$$

and $\tilde{c}_m = \mathbb{N}_{\mathbb{Q}(\mu_m)/\mathbb{Q}(\mu_m)^+}(\tilde{c}_{m\infty})$, where $\mathbb{Q}(\mu_m)^+$ is the maximal real subfield of $\mathbb{Q}(\mu_m)$. The previous relations show that

$$\{\tilde{c}_{m\infty}, \tilde{c}_m \mid m \in \mathbb{Z}^+\}$$

is an Euler system for $(\mathbb{Z}_p(1), \mathbb{Q}^{\mathrm{ab}}, p)$.
Another well known instance of this theory is the Euler system of Heegner points, that we will later revisit.

5.4 Circular units

Recall that one of the motivations Bryan Birch and Peter Swinertonn-Dyer had before formulating its celebrated BSD conjecture was to look for some analogy to the class number formula, that asserts that the zeta function of a number field, $\zeta_K(s)$, converges absolutely for $\Re(s) > 1$ and extends to a meromorphic function defined for all complex s with only one simple pole at $s = 1$, with residue

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot h_K\cdot \mathrm{Reg}_K}{w_K\cdot\sqrt{|D_K|}}.$$

The conjecutre of Birch and Swinertonn-Dyer, that we have explained in detail in chapter 4, gives a similar result for the L-function of an elliptic curve. I would like to establish an analogy between analogy (class number formula versus BSD conjecture) and the less known analogy between the classical Euler systems of circular and elliptic units and the more modern ones more related with BSD. Let us begin with a short review of some facts concerning these circular units.

We begin by taking $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, a primitive, non-trivial, even Dirichlet character of conductor N. At the negative odd integers, the values of $L(s,\chi)$ belong to $\mathbb{Q}_\chi$, what can be seen by writing $L(1-k,\chi)$ for even $k \geq 2$ as the constant term of the holomorphic Eisenstein series

$$E_{k,\chi}(q) := L(1-k,\chi) + 2\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n$$

of weight k, level N and character χ (and invoking then the q-expansion principle to argue that the constant term inherits the rationality properties of the coefficients $\sigma_{k-1,\chi}(n)$).

If p is any prime, the ordinary p-stabilisation

$$E^{(p)}_{k,\chi}(q) := E_{k,\chi}(q) - \chi(p)p^{k-1}E_{k,\chi}(q^p)$$

has a similar Fourier expansion, where now it appears the term

$$L_p(1-k,\chi) = (1-\chi(p)p^{k-1})L(1-k,\chi)$$

and other terms of the form $\sigma^{(p)}_{k-1,\chi}(n) = \sum_{p\nmid d|n}\chi(d)d^{k-1}$:

$$E^{(p)}_{k,\chi}(q) = L_p(1-k,\chi) + 2\sum_{n=1}^{\infty}\sigma^{(p)}_{k-1,\chi}(n)q^n.$$

For each $n \geq 1$, the function on $\mathbb{Z}$ sending k to the n-th Fourier coefficient $\sigma^{(p)}_{k-1}(n)$ extends to a p-adic analytic function of $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$, and the constant term inherits the same property. The resulting extension of $L_p(s,\chi)$, that had been defined as a function on the negative odd integers, is the Kubota-Leopoldt p-adic L-function attached to χ.

The collection of eigenforms $E^{(p)}_{k,\chi}$ is an example of p-adic family of modular forms (the specialisations at negative even integers admit a geometric interpretation as p-adic modular forms of weight k and level N_0). If $k = 0$, they are rigid analytic functions on the ordinary locus $A \subset X_1(N_0)(\mathbb{C}_p)$ obtained by deleting the residue discs attached to supersingular elliptic curves in characteristic p.

We can proceed in a different way using the Siegel units $g_a \in \mathcal{O}^*_{Y_1(N)}$ attached to a fixed choice of primitive N-th root of unity ζ and parameter $1 \leq a \leq N-1$, such that the q-expansion is

$$g_a(q) = q^{1/12}(1-\zeta^a)\prod_{n>0}(1-q^n\zeta^a)(1-q^n\zeta^{-a}).$$

Let Φ be the canonical lift of Frobenius on A, sending $(E,t) \in A$ to $(E/C, t+C)$, where $C \subset E(\mathbb{C}_p)$ is the canonical subgroup of order p in E. The rigid analytic function

$$g_a^{(p)} := \Phi^*(g_a)g_a^{-p} = g_{pa}(q^p)g_a(q)^{-p}$$

maps the ordinary locus A to the residue disc of 1 in $\mathbb{C}_p$ and therefore $\log_p g_a^{(p)}$ is a rigid analytic function on A with q-expansion

$$\log_p g_a^{(p)} = \log_p\Big(\frac{1-\zeta^{ap}}{(1-\zeta^a)^p}\Big) + p\sum_{n=1}^{\infty}\Big(\sum_{p\nmid d|n}\frac{\zeta^{ad}+\zeta^{-ad}}{d}\Big)q^n$$

Theorem 52 (Leopoldt)**.** *Let χ be a non-trivial even primitive Dirichlet character of conductor N. Then,*

$$L_p(1,\chi) = -\frac{(1-\chi(p)p^{-1})}{\mathfrak{g}(\chi^{-1})}\sum_{a=1}^{N-1}\chi^{-1}(a)\log_p(1-\zeta^a).$$

The expressions of the form $(1-\zeta^a)$ when N is composite and $\frac{1-\zeta^a}{1-\zeta^b}$ when N is prime, are called circular units. Let F_χ be the field cut out by χ, and let $\mathbb{Z}_\chi$ be the ring generated by its values. Then,

$$u_\chi := \prod_{a=1}^{N-1}(1-\zeta^a)^{\chi^{-1}(a)} \in (\mathcal{O}_{F_\chi}^\times \otimes \mathbb{Z}_\chi)^\chi$$

is a distinguished unit in F_χ lying in the χ-eigenspace for the natural action of the absolute Galois group of $\mathbb{Q}$.
The unit u_χ acts as a universal norm over the tower of cyclotomic fields $F_{\chi,n} = F_\chi(\mu_{p^n})$. That is, we have that

$$u_{\chi,n} = \prod_{a=1}^{N-1}(1-\chi_{Np^n}^a)^{\chi^{-1}(a)} \in (\mathcal{O}_{F_{\chi,n}}^\times \otimes \mathbb{Z}_\chi)^\chi$$

and also that

$$\mathrm{Norm}_{F_{\chi,n}}^{F_{\chi,n+1}}(u_{\chi,n+1}) = \begin{cases} u_{\chi,n} & \text{if } n \geq 1 \\ u_\chi \otimes (\chi(p)-1) & \text{if } n = 0. \end{cases}$$

After viewing χ as a $\mathbb{C}_p$-valued character, let $\mathbb{Z}_{p,\chi}$ be the ring generated over $\mathbb{Z}_p$ by the values of χ, with the trivial Galois action. Let $\mathbb{Z}_{p,\chi}$ be the free module of rank one over $\mathbb{Z}_{p,\chi}$ on which the Galois group acts via χ. We define in the same way $\mathbb{Q}_{p,\chi}$ and $\mathbb{Q}_{p,\chi}(\chi)$. The images

$$\kappa_{\chi,n} := \delta u_{\chi,n} \in H^1(F_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^\chi = H^1(F_n, \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))$$

under the connecting homomorphism of Kummer theory,

$$\delta : (F_{\chi,n}^\times \otimes \mathbb{Z}_\chi)^\chi \to H^1(F_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^\chi,$$

can be patched together in an element $\kappa_{k,\infty} := (\kappa_{\chi,n})_{n\geq 0}$ that belongs to

$$\varprojlim H^1(F_n, \mathbb{Z}_{p,\chi}(1)(\chi^{-1})) = H^1(\mathbb{Q}, \Lambda_{\mathrm{cyc}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p,\chi}(1)(\chi^{-1})),$$

where Λ_{cyc} is the completed group ring of $\mathbb{Z}_p^\times$ endowed with the tautological action of $G_{\mathbb{Q}}$. Later on, we will construct similar cohomology classes like that associated to modular forms, say $\kappa(g,h)$, that will be useful for instance in the study of the Elliptic Stark conjecture.

Given $k \in \mathbb{Z}$ and a Dirichlet character χ of p-power conductor, let $\nu_{k,\xi} : \Lambda \to \mathbb{Z}_{p,\chi}$ be the ring homomorphism sending $a \in \mathbb{Z}_p^\times$ to $a^{k-1}\xi^{-1}(a)$. It induces a specialization map $\nu_{k,\chi} : \Lambda_{\mathrm{cyc}} \to \mathbb{Q}_{p,\xi}(k-1)(\xi^{-1})$ giving rise to a collection of global cohomology classes

$$\kappa_{k,\chi\xi} := \nu_{k,\xi}(\kappa_{\chi,\infty}) \in H^1(\mathbb{Q}, \mathbb{Q}_{p,\chi\xi}(k)((\chi\xi)^{-1})).$$

For a Dirichlet character η with $\eta(p) \neq 1$, let $F_{p,\eta}$ be the finite extension of $\mathbb{Q}_p$ cut out by the corresponding Galois character and $G_\eta = \mathrm{Gal}(F_{p,\eta}/\mathbb{Q}_p)$ its Galois group. We have that

$$H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(1)(\eta)) = (\mathcal{O}_{F_{p,\eta}}^\times \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\eta}(\eta))^{G_\eta}.$$

The p-adic logarithm $\log_p : \mathcal{O}_{F_{p,\eta}} \to F_{p,\eta}$ gives a map

$$\log_\eta : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(1)(\eta)) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_{p,\eta}} \mathbb{Q}_{p,\eta}(\eta))^{G_{\mathbb{Q}_p}}.$$

The last identification comes from the Tate-Sen isomorphism $F_{p,\eta} = \mathbb{C}_p^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_{p,\eta})}$. Leopoldt's theorem can be re-phrased as the following relation between the classes $\kappa_{1,\chi\xi}$ and the values of the Kubota-Leopoldt L-function at $s=1$:

$$L_p(1,\chi\xi) = -\frac{(1-\chi\xi(p)p^{-1})}{1-(\chi\xi)^{-1}(p)} \times \frac{\log_{\chi\xi}(\kappa_{1,\chi\xi})}{\mathfrak{g}(\chi\xi)}.$$

In particular, these classes $\kappa_{1,\chi\xi}$ determine the Kubota-Leopoldt L-function completely, since an element of the Iwasawa algebra has finitely many zeros (Weierstrass preparation). This is the first main example of a bunch of formulas we will be studying along this thesis: the special value of a p-adic L-function equals the logarithm of a certain cohomology class.

5.5 Elliptic units

Apart from the cusps, modular curves also have another distinguished class of algebraic points, the CM points attached to the moduli of elliptic curves with complex multiplication by an order in a quadratic imaginary field K. At such points, the values of modular units give rise to units in abelian extensions of K, called elliptic units, which play the same role for abelian extensions of K as circular units in the study of cyclotomic fields. Writing $q = e^{2\pi i\tau}$, the Eisenstein series is given by

$$E_{k,\chi}(\tau) = N^k \mathfrak{g}(\bar{\chi})^{-1}\frac{(k-1)!}{(2\pi i)^k} \sideset{}{'}\sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}} \frac{\bar{\chi}(n)}{(m\tau+n)^k}.$$

Assume that K has class number one, trivial unit group and odd discriminant $D<0$, and also that there is $\mathfrak{n} \in \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. In this case we will say that

$E_{k,\chi}$ satisfy the Heegner hypothesis relative to K. With these assumptions, the even character χ gives rise to a finite order character $\chi_{\mathfrak{n}}$ of conductor $\mathfrak{n}$ on the ideals of K given by

$$\chi_{\mathfrak{n}}((\alpha)) := \bar{\chi}(\alpha \mod \mathfrak{n}).$$

This theorem of Katz is the analogous of Leopoldt's formula replacing cusps by CM points and circular units by elliptic units.
Considering $\tau_{\mathfrak{n}} = \frac{b+\sqrt{D}}{2N}$, where $\mathfrak{n} = \mathbb{Z}N + \mathbb{Z}\frac{b+\sqrt{D}}{2}$, we have that

$$E_{k,\chi}(\tau_{\mathfrak{n}}) = N^k \mathfrak{g}(\bar{\chi})^{-1} \frac{(k-1)!}{(2\pi i)^k} L(K, \chi_n, k, 0),$$

where for $k_1, k_2 \in \mathbb{Z}$ with $k_1 + k_2 > 2$,

$$L(K, \chi_{\mathfrak{n}}, k_1, k_2) := \sum_{\alpha \in O_K}{}' \chi_{\mathfrak{n}}(\alpha) \alpha^{-k_1} \bar{\alpha}^{-k_2}.$$

We observe that $L(K, \chi_{\mathfrak{n}}, s) := \frac{1}{2} L(K, \chi_{\mathfrak{n}}, s, s)$, viewed as a function of a complex variable s, is the so-called Hecke L-function attached to the finite order character $\chi_{\mathfrak{n}}$.

The following result of Katz is the direct counterpart of Leopoldt's formula in which cusps are replaced by CM points and circular units by elliptic units, that are nothing but expressions of the form

$$u_{a,\mathfrak{n}} := g_a(A, t_{\mathfrak{n}}) = g_a(\tau_{\mathfrak{n}}), \qquad u_{a,\mathfrak{n}}^{(p)} := g_a^{(p)}(A, t_{\mathfrak{n}}) = g_a^{(p)}(\tau_{\mathfrak{p}\mathfrak{n}}) = u_{a,\mathfrak{n}}^{\sigma_{\mathfrak{p}} - p},$$

where $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(K_{\mathfrak{n}}/K)$ is the Frobenius at $\mathfrak{p}$.

Theorem 53 (Katz)**.** *Let χ be a non-trivial even primitive Dirichlet character of conductor N and let K be a quadratic imaginary field equipped with an ideal $\mathfrak{n}$ satisfying $O_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. Let $\chi_{\mathfrak{n}}$ be the ideal character of K attached to the pair $(\chi, \mathfrak{n})$. Then,*

$$L_p(K, \chi_{\mathfrak{n}}, 0) = -\frac{(1 - \chi_{\mathfrak{n}}(\mathfrak{p})p^{-1})}{\tau(\bar{\chi})} \times \sum_{a=1}^{N-1} \chi^{-1}(\mathfrak{a}) \log_p u_{a,\mathfrak{n}}.$$

A key tool for proving this result is the introduction of the Shimura-Maass derivative operator that already arose in the study of nearly holomorphic modular forms,

$$\delta_k = \frac{1}{2\pi i}\Big(\frac{d}{d\tau} + \frac{k}{\tau - \bar{\tau}}\Big),$$

that sends real analytic modular forms of weight k to real analytic modular forms of weight $k+2$. More precisely, if

$$\delta_k^r = \delta_{k+2r-2} \circ \ldots \circ \delta_{k+2} \circ \delta_k,$$

then $\delta_k^r E_{k,\chi} = E_{k+r,-r,\chi}$. Recall that a nearly holomorphic modular form of weight k on $\Gamma_1(N)$ is nothing but a linear combination

$$f = \sum_{i=1}^{t} \delta_{k-2j_i}^{j_i} f_i, \qquad f_i \in M_{k-2j_i}(\Gamma_1(N)),$$

where the f_i are classical modular forms of weight $k-2j_i$ on $\Gamma_1(N)$.
By a result of Shimura, nearly holomorphic modular forms of weight k defined over $\bar{\mathbb{Q}}$ take algebraic values at CM triples like $(A, t_{\mathfrak{n}}, \omega_A)$. More precisely,

$$f(A, t_{\mathfrak{n}}, \omega_A) := \frac{f(\tau_{\mathfrak{n}})}{(\bar{n}\Omega_K)^k} \in LK_{\mathfrak{n}}.$$

This comes from the relationship between nearly holomorphic modular forms of weight k and global sections of an algebraic vector bundle (arising from the relative de Rham cohomology on the universal elliptic curve over $Y_1(N)$), and on the interpretation of the Shimura-Maass derivative in terms of the Gauss-Manin connection on this vector bundle. Further, we have that

$$\delta_k^r f(A, t_{\mathfrak{n}}, \omega_A) = d^r f(A, t_{\mathfrak{n}}, \omega_A).$$

In this setting it is convenient to introduce Katz two-variable p-adic L-function. First of all, let

$$E_{k,\chi}^{[p]}(\tau) = E_{k,\chi}(\tau) - (1+\chi(p)p^{k-1})E_{k,\chi}(p\tau) + \chi(p)p^{k-1}E_{k,\chi}(p^2\tau).$$

It turns out that $d^r E_{k,\chi}^{[p]} = \sum_{p\nmid n}^{\infty} n^r \sigma_{k-1,\chi}(n) q^n$, and the coefficients of the expansion extend to p-adic analytic functions of k and r on the weight space. This suggests the definition of a function $L_p(K, \chi_{\mathfrak{n}}, k_1, k_2)$, such that for $k \geq 2, r \geq 0$ satisfies the interpolation property

$$\frac{L_p(K, \chi_{\mathfrak{n}}, k+r, -r)}{\Omega_p^{k+2r}} := d^r E_{k,\chi}^{[p]}(A, t_{\mathfrak{n}}, \omega_A),$$

and then extends to an analytic function of $(k,r) \in \mathcal{W}^2$. In particular, $L_p(K, \chi_{\mathfrak{n}}, k, 0) = (1-\chi_{\mathfrak{n}}(\mathfrak{p})\mathfrak{p}^{-k})L_p(K, \chi_{\mathfrak{n}}, k)$. We finish this section with a theorem of Katz:

Theorem 54 (Katz)**.** *With the previous notations,*

$$L_p(K, \chi_{\mathfrak{n}}^{-1}, 1, 1) = (1-\chi_{\mathfrak{n}}^{-1}(\bar{\mathfrak{p}}))(1-\chi_{\mathfrak{n}}(\mathfrak{p})/p) \times \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p u_{a,\mathfrak{n}}^{\times}.$$

5.6 Heegner points

We will say just a few words about Heegner points, one of the most prototypical examples of Euler systems that have been widely studied. In general, when K is an imaginary quadratic extension of $\mathbb{Q}$ and write H_n for the ring class field of K of conductor n, a Heegner system attached to (E,K) is a collection of points $P_n \in E(H_n)$ indexed by integers n prime to N satisfying certain (explicit) norm compatibility properties. When (E,K) satisfies the Heegner hypothesis (all primes dividing the conductor of E split in $K/\mathbb{Q}$), then there is a non-trivial Heegner system attached to (E,K). Let $\{P_n\}_n$ be such Heegner system and let $P_K = \mathrm{Trace}_{H_1/K}(P_1) \in E(K)$. More generally, consider $\chi : \mathrm{Gal}(H_n/K) \to \mathbb{C}^\times$ a primitive character of a ring class field extension of K of conductor n and let

$$P_n^\chi = \sum_{\sigma \in \mathrm{Gal}(H_n/K)} \bar{\chi}(\sigma) P_n^\sigma \in E(H_n) \otimes \mathbb{C}.$$

The following formula provides the relation between the Heegner system $\{P_n\}$ and the special values of the complex L-series $L(E/K,s)$ and its twists.

Theorem 55 (Gross-Zagier)**.** *Let $\langle , \rangle_n$ be the canonical Néron-Tate height on $E(H_n)$ extended by linearity to a Hermitian pairing on $E(H_n) \otimes \mathbb{C}$. Then,*

1. $\langle P_K, P_K \rangle = *L'(E/K, 1)$.
2. $\langle P_n^{\chi}, P_n^{\bar{\chi}} \rangle = *L'(E/K, \chi, 1)$.

Here, $*$ means equality up to a non-zero factor that can be explicitly described.

This allows to prove the theorem of Gross-Zagier and Kolyvagin, a proof of BSD for analytic rank ≤ 1.
In this analogy with circular and elliptic units, what we are doing is to replace the Eisenstein series $E_{k,\chi}$ by a cusp form of weight k. We consider $f \in S_k(N)$, a normalized cuspidal eigenform of even weight k on $\Gamma_1(N)$ with rational Fourier coefficients and trivial nebentypus character. Let K be a quadratic imaginary field satisfying the hypothesis of the previous section, and assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ is a rational prime splitting in K and which does not divide N. Consider the quantities $\delta_k^r f(A, t_n, \omega_A) = d^r f(A, t_n, \omega_A)$, which are in K_n for $r \geq 0$. In this setting, there is a formula of Waldspurger relating the L-function of f twisted by certain Hecke characters with the values of $\delta_k^r f$. If ϕ is such a character, then the L-series $L(f, K, \phi, s)$ of f/K twisted by ϕ is defined by

$$L(f, K, \phi, s) = \prod_{\mathfrak{l}} [(1 - \alpha_{\mathbb{N}\mathfrak{l}}(f) \cdot \phi(\mathfrak{l}) \mathbb{N}\mathfrak{l}^{-s})(1 - \beta_{\mathbb{N}\mathfrak{l}}(f) \cdot \phi(\mathfrak{l}) \mathbb{N}\mathfrak{l}^{-s})]^{-1},$$

where the product is over the primes in O_K and $\alpha_l(f), \beta_l(f)$ are the roots of the Hecke polynomial $x^2 - a_l(f) + l^{k-1}$ for f at l and we write $\alpha_{\mathbb{N}\mathfrak{l}} := \alpha_l(f)^t$ if $\mathbb{N}\mathfrak{l} = l^t$. Rankin's method can be used to analytically continue $L(f, K, \phi, s)$ to the whole complex plane.

If k_1, k_2 are integers with the same parity, let ϕ_{k_1,k_2} be the unramified Hecke character of K of infinity type (k_1, k_2) defined on fractional ideals by the rule

$$\phi_{k_1,k_2}((\alpha)) = \alpha^{k_1} \bar{\alpha}^{k_2},$$

and define now

$$L(f, K, k_1, k_2) := L(f, K, \phi_{k_1,k_2}^{-1}, 0).$$

We will finish the chapter stating the main result of [BDP] in the case that $k = 2$ and f is attached to an elliptic curve E of conductor N. Let $P_K \in J_0(N)(K)$ be the class of the degree 0 divison $(A, t_{\mathfrak{n}}) - (\infty)$ in the Jacobian variety $J_0(N)$ of $X_0(N)$ and let $P_{f,K}$ denote its image in $E(K)$ under the modular parametrization $\phi_E : J_0(N) \to E$ arising from the modular form f, and let ω_E be the regular differential on E, such that $\phi_E^*(\omega_E) = \omega_f := 2\pi i f(\tau) d\tau$.

Theorem 56. *Let $f \in S_2(N)$ be a normalized cuspidal eigenform of level $\Gamma_0(N)$ with N prime to p and let K be a quadratic imaginary field equipped with an integral ideal $\mathfrak{n}$ satisfying $O_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. Then,*

$$L_p(f, K, 1, 1) = \Big(\frac{1 - a_p(f) + p}{p}\Big)^2 \log_p(P_{K,f})^2.$$

Although BSD conjecture is not our main concern in this thesis, it is a matter of justice to recognize that many of this work comes with the motivation of finding evidence towards the proof of BSD conjecture. In my expository paper [R3] I explain how the p-adic L-funcion is involved in a different conjecture where the so-called exceptional zero phenomenon (related with the fact that the elliptic curve may have split multiplicative reduction) arises. In [MTT], a p-adic L-function $L_p(g, s)$ is associated to any ordinary eigenform g of even weight $k \geq 2$, and to a choice of a complex period Ω_g. In [GS], under the condition that the sign of E (as an elliptic curve over $\mathbb{Q}$) has sign 1), it is seen how the fact that $L_p(f_\infty; k, s)$ vanishes on the critical line $s = k/2$ means that

$$\frac{\partial}{\partial s} L_p(f_\infty; 2, 1) = -2\frac{\partial}{\partial k} L_p(f_\infty; 2, 1),$$

and a further study of the factorization of $L_p(f_\infty, k, s)$, leads to

$$L_p'(f, 1) = -2a_p'(2)\frac{L(f, 1)}{\Omega_f}.$$

This will imply that $L_p'(f, 1) = \frac{\log(q)}{\mathrm{ord}_p(q)}\frac{L(f,1)}{\Omega_f}$, that was a conjecture in [MTT].

Heegner points are useful in the parallel study in which the sign of E is -1, and in this case $L(E, 1) = 0$. The main result of the paper [BD] is the following:

Theorem 57. *Suppose that E has at least two primes of semistable reduction. Then:*

1. *There is a global point $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ and a scalar $l \in \mathbb{Q}^\times$ such that*

$$\frac{d^2}{dk^2} L_p(f_\infty; k, k/2)_{k=2} = l \cdot \log_E(P)^2.$$

2. *The point P is of infinite order if and only if $L'(E, 1) \neq 0$.*

The proof of this theorem exhibits P as a Heegner point arising from an appropriate Shimura curve parametrization.

6 Euler systems of Rankin-Selberg type

In this chapter we will introduce some of the main ideas of this thesis. Previously, we have discussed some formulas describing the p-adic logarithm of circular units, elliptic units or Heegner points in terms of values of the associated p-adic L-functions at points outside the range of classical interpolation. Now we have several pretensions: on the one hand, develop a new type of Euler systems in which we will be concerned about triples of modular forms, say (f, g, h). This will be our first aim, that we will motivate by recalling the celebrated Beilinson conjecture. After a short presentation of the de Rham cohomology of curves over p-adic rings, we will move to a more detailed description of each of the possibilities arising here: Beilinson-Kato elements, Beilinson-Flach elements and Gross-Kudla-Schoen elements, dealing first with the geometric constructions and then with the L-functions. We will comment their applications to the BSD conjectures and also, emphasize the role played by the different actors (cohomology groups, regulators and L-functions) involved in this comedy.

6.1 Beilinson conjecture

Let $\mathbf{f}$ be a Hida family of tame level N and tame nebentypus $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to L^\times \subset \mathbb{C}^\times$. With this, we mean that there is a finite flat extension Λ_f of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$ and a formal q-expansion $\mathbf{f}(q) = \sum_{n\geq 0} a_n(\mathbf{f})q^n \in \Lambda_f[[q]]$ such that when μ is a classical weight lying above μ_{k,ϵ_r},

$$\mu(\mathbf{f}) = \sum \mu(a_n(f))q^n$$

is a classical modular form of level Np^r and nebentypus $\chi\epsilon_r\omega^{-k}$. In this section, we use the convention that μ_{k,ϵ_r} corresponds to the map $z \mapsto z^k\epsilon_r(z)$.
Geometrically, we can understand this as a finite covering

$$W_{\mathbf{f}} = \mathrm{Spf}(\Lambda_{\mathbf{f}}) \twoheadrightarrow W = \mathrm{Spf}(\Lambda).$$

We have the following result concerning the Mazur-Kitagawa p-adic L-function, that we will analyze later in more detail:

Theorem 58. *There exists a function $L_p(\mathbf{f}) \in \Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \Lambda$ which can be seen as a rigid analytic function,*

$$L_p(\mathbf{f}) : W_{\mathbf{f}} \times W \to \mathbb{C}_p$$

$$(\mu, \nu) \mapsto L_p(\mathbf{f})(\mu, \nu) := (\mu, \nu)(L_p(\mathbf{f})) \in \mathbb{C}_p,$$

such that for all (μ, ν) with μ lying above μ_{k,ϵ_r} ($k \geq 2, r \geq 1$) and $\nu = \nu_{j,\epsilon_s}$ ($0 \leq j \leq k-2, s \geq 2$),

$$L_p(\mathbf{f})(\mu, \nu) = \frac{e_p(\mu, \nu)L(f_{k,\epsilon_r}, \nu_{j,\bar{\epsilon}_s}, 1)}{\Omega^{\pm}_{f_k,\epsilon_r}} = \frac{e_p(\mu, \nu)L(f_{k,\epsilon_r}, \bar{\epsilon}_s, 1+j)}{\Omega^{\pm}_{f_k,\epsilon_r}}.$$

Observe that in the special case that $\chi\epsilon_r\omega^{-k}\bar{\epsilon}_s$ is trivial or quadratic, then $L(f_{k,\epsilon_r}, \bar{\epsilon}_s, s)$ has sign ± 1 and $L(f_{k,\epsilon_r}, \bar{\epsilon}_s, k/2) = 0$ when the sign is -1.
Recall that the integers satisfying either $1 + j \geq k$ or $1 + j \leq k - 1$ are dense on the whole plan $W_{\mathbf{f}} \times W$, so in particular points of the form (μ_k, ν_j) are dense, and also those of the form $(\mu_{2,\epsilon_r}, \nu_{-1,\epsilon_s})$.

Let X be a smooth proper variety of dimension $n \geq 0$ over a number field F. For $0 \leq i \leq 2n$, the étale cohomology gives rise to

$$\{H^1_{\text{et}}(X_{\bar{F}}, \mathbb{Q}_l)\}_l,$$

a compatible system of Galois representations of G_F.
Let $L(X, i, s) = L(H^i(X), s) = \prod_{\mathfrak{p} \subset \mathcal{O}_F}(\det(\text{Frob}_\mathfrak{p} - x)^{-1})|_{x = N\mathfrak{p}^{-s}}$, where $\text{Frob}_\mathfrak{p}$ is an endomorphism of $(H^i(X_{\bar{F}}, \mathbb{Q}_l)^{I_\mathfrak{p}}$. This function converges for $\Re(s) > i/2 + 1$ and satisfies a functional equation relating the values at s and at $i + 1 - s$.

Conjecture 8 (Beilinson)**.** *Let F be a number field and X/F be a variety of dimension d. Take $0 \leq i \leq 2d$, and $c < i/2 + 1$. Then, the order of vanishing*

$$\text{ord}_{s=c} L(H^i(X), s)$$

coincides with the rank of the higher Chow group $\text{CH}^{-c+i+1}(X, -2c + i + 1)$.

Consider the simple case in which $X = \text{Spec}(F)$, with F a number field and $i = 0$. Then,

$$L(H^0(\text{Spec}(F)), s) = \zeta_F(s).$$

This function has a pole at $s = 1$ and for $c < 1$ we have that the order of $\zeta_F(s)$ at $s = c$ is:

- $r_1 + r_2 - 1$ when $c = 0$.
- $r_1 + r_2$ when c is negative and even.
- r_2 when c is negative and odd.

This quantity always coincides with the rank of $K_{1-2c}(\mathcal{O}_F)$ by a theorem of Borel.

6.2 Higher Chow groups: a quick overview

We will now explain the definition of the higher Chow groups and their relation with the previous material. For this, let

$$\Delta_n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{A}^{n+1} \mid x_0 + x_1 + \ldots + x_n = 1\}$$

be the usual n-dimensional symplex.
We will have a chain of groups of the form

$$\text{CH}^c(X, n+1) \to \text{CH}^c(X, n) \to \ldots \to \text{CH}^c(X, 1) \to \text{CH}^c(X, 0) \to H_{2d-2c}(X) \cong H^{2c}(X),$$

where the maps are denoted by $\partial_i : \text{CH}^c(X, i) \to \text{CH}^c(X, i - 1)$, and in the last step, $H^{2c}(X)$ denotes any of the standard cohomologies attached to X (étale, de Rham, ...).

Definition 45. $\widetilde{\text{CH}}^c(X, n) = \{\sum n_i Z_i \mid n_i \in \mathbb{Z}\}$*, where $Z_i \subset X \times \Delta_n$ is an irreducible subvariety of dimension $d + n - c$ (codimension c in $X \times \Delta_n$) intersecting property with all the subfaces of Δ_n (that is, all the subsets $X \times F$, where F is a subface of Δ_n).*

Definition 46. *We define the boundary maps*

$$\delta_n : \widetilde{\mathrm{CH}}^c(X,n) \to \widetilde{\mathrm{CH}}^c(X,n-1)$$

$$Z \mapsto \sum_{k=0}^{n} (-1)^k (Z \cap (X \times \Delta_n^k)),$$

where $\Delta_n^i = \{(x_0,\ldots,x_n) \in \Delta_n \mid x_i = 0\}$.
The map $\partial_0 : \widetilde{\mathrm{CH}}^c(X,0) \to H_{2d-2c}(X)$ *is defined by sending* $Z \mapsto [Z]$*, since* $Z = \sum n_i Z_i$*, where the* Z_i *have complex dimension* $d-c$*, and hence real dimension* $2(d-c)$.

It is not complicated to check that this defines a complex of abelian groups and hence we can consider the associated cohomology groups, namely

$$\mathrm{CH}^c(X,n) := \frac{\ker \partial_n}{\mathrm{Im}\ \partial_{n+1}}.$$

There are other notations for this, more in the realm of K-theory, such as $K_n^{(c)}(X)$, and also in the motivic setting, $H_{\mathrm{mot}}^{2c-n}(X, \mathbb{Z}(c))$.

Let us consider some examples.

1. $\mathrm{CH}^c(X,0)$ is nothing but the set of elements of the form $Z = \sum n_i Z_i$, where the Z_i have codimension c in X and are homologically trivial, considered modulo the equivalence relation induced by the image of ∂_1. Roughly speaking, two elements of $\mathrm{CH}^c(X,0)$ are the same when we can continuously deform one into the other. When X is a curve, we recover the jacobian: $\mathrm{CH}^1(X,0) = \mathrm{Jac}(X)$, that is, the formal sum of $\sum n_i P_i$ (with $n_i \neq 0$ only for a finite number of points), under the relation that $D \sim D'$ if $D - D' = \mathrm{Div}(u)$ where $u \in K(X)^\times$.

2. Consider now a variety X and its first Chow group $\mathrm{CH}^1(X,1)$. It consists on elements $Z \subset X \times \mathbb{A}^1$ of codimension 1. The relevant fact here is that there is a natural inclusion
$$\mathcal{O}(X)^\times \hookrightarrow \mathrm{CH}^1(X,1)$$
that in general will be an isomorphism. It is given by sending u to Z_u, the graph of u. u is a function from X to $\mathbb{P}^1$ and hence it can be mapped to $\{(x, u(x) \mid x \in X\}$. We observe that $\partial_1(Z_u) = Z_u|_{X\times\{0\}} - Z_u|_{X\times\{\infty\}} = 0$, since u has neither zeros nor poles.

3. Another remarkable example occurs when X is an algebraic surface and we consider $\mathrm{CH}^2(X,1)$. In this case, we can understood it as elements of the form $\sum n_i(C_i, u_i)$, where C_i is a curve in X and $u_i \in K(C_i)$ with $\sum n_i \mathrm{Div}(u_i) = 0$ (modulo an equivalence relation). The set of these elements can be embedded in $\mathrm{CH}^2(X,1)$ by sending (C,u) to the graph $Z_{(C,u)} = \{(c, u(c) \in X \times \mathbb{A}^1\}$ (c being a point in the curve).

These groups are endowed with a rich structure given by the intersection (cup) product:

$$\mathrm{CH}^c(X,n) \times \mathrm{CH}^{c'}(X,n') \to \mathrm{CH}^{c+c'}(X,n+n').$$

In particular, we can consider $\mathrm{CH}^1(X,1) \times \mathrm{CH}^1(X,1) \to \mathrm{CH}^2(X,2)$, that sends a pair of modular units (u,v) to $\{u,v\}$.
Another important fact is the existence of an excision exact sequence: if $Y \subset X$ is a subvariety, there is an exact sequence

$$\ldots \to \mathrm{CH}^c(X\backslash Y, n) \to \mathrm{CH}^c(X,n) \to \mathrm{CH}^c(Y,n) \to \mathrm{CH}^c(X\backslash Y, n-1) \to \ldots$$

In particular, we have that a part of this exact sequence is

$$\mathrm{CH}^1(X,1) \to \mathrm{CH}^1(Y,1) \to \mathrm{CH}^1(X\backslash Y, 0)$$

Inside $\mathrm{CH}^1(X,1)$, I have the elements of $\mathcal{O}(X)^\times$, that are only the constant functions. Inside $\mathrm{CH}^1(Y,1)$, we have $\mathcal{O}(Y)^\times$; when some of these elements maps to zero, it must come from a non-trivial element in $\mathrm{CH}^1(X,1)$, and this is a remarkable way of detecting non-trivial classes in these Chow groups.

We cannot forget that our ultimate goal will be to give formulas for the values

$$L_p(\mathbf{f})(\mu_{k,\epsilon_r}, \nu_{j,\epsilon_s}),$$

for $k \geq 2$ with $j < 0$ or $j \geq k-1$. Since

$$L_p(\mathbf{f})(\mu_{k,\epsilon_r}, \nu_{j,\epsilon_s}) = * L_p(\mathbf{f})(\mu_{k,\epsilon_r}, \nu_{k-j-2,\epsilon_s}),$$

(where $*$ is a non-zero simple algebraic factor), the values at $j<0$ are directly related to those of the form $j \geq k-1$ and we will focus only on $j < 0$.
Morally, we expect a relation between $L_p(\mathbf{f})(\mu_{k,\epsilon_r}, \nu_{j,\epsilon_s})$ and

$$L(f_k, 1+j) = * \frac{\Gamma(k-1-j)}{\Gamma(1+j)} L(f_k^*, k-1-j) = \frac{(k-j-2)!}{\text{simple pole}} L(f_k^*, k-1, j) = 0.$$

We therefore have that the order of $L(f_k, s)$ at $s = c = 1+j$ is always ≥ 1 and typically one.
Let $k \geq 2$ and $j < 0$. The Beillinson conjecture predicts (taking $c = 1+j$, $i = k-1$) that

$$\mathrm{ord}_{s=1+j} L(f_{k,\epsilon_r}, \bar{\epsilon_s}, s) = \mathrm{rank}\, \mathrm{CH}^{k-j-1}(W_k, k-2j-2).$$

In this setting, since the order of vanishing of the L-function is ≥ 1, it is natural to ask ourselves if there is a canonical choice of an element in the Chow group of the variety W_k. Not only this, we expect to relate $L'(f_{k,\epsilon_r}, \bar{\epsilon_s}, s)$ with some complex invariant of Δ and $L_p(f)((k,\epsilon_r),(j,\epsilon_s))$ with some p-adic invariant of Δ.
It is for that reason that we recover the Abel-Jacobi maps, also known in this context as étale regulators. Let X/F be a d-dimensional variety. Then, there is a homomorphism

$$\mathrm{reg}_{\mathrm{et}} : \mathrm{CH}^c(X,n) \to H^1(G_F, V)$$

where $V = H^{2c-n-1}_{\mathrm{et}}(X_{\bar{F}}, \mathbb{Z}_p(c))$. Recall that if V is a G_F-module, then $V(c)$ is the G_F-module where σ acts in $v \in V$ by $\sigma *_c v = \chi^c_{\mathrm{cyc}}(\sigma)\sigma(v)$ (this is what we call a twist by the cyclotomic character).
In general, we have that $H^1(G_F, V) \cong \mathrm{Ext}^1(V, \mathbb{Z}_p)$. Observe that for instance, given an extension

$$0 \to V \to E \xrightarrow{\pi} \mathbb{Z}_p \to 0$$

we can associate to it the cocycle that sends σ to $\sigma(\tilde{v})-\tilde{v}$, where $\tilde{v}$ is a preimage of 1 by the application $\pi : E \to \mathbb{Z}_p$.

We now explain how to build this étale regulator map when $n=0$. Let $Z=\sum_{i=1}^r n_i Z_i$. Recall the excision exact sequence

$$H_{\mathrm{et}}^{i-2c}(Z,V) \to H_{\mathrm{et}}^i(X,V) \to H_{\mathrm{et}}^i(X-Z,V) \to H_{\mathrm{et}}^{i+1-2c}(Z,V),$$

and in the particular case that $i=2c-1$, it yields that

$$0 \to H_{\mathrm{et}}^{2c-1}(X,V) \to H_{\mathrm{et}}^{2c-1}(X-Z,V) \to H_{\mathrm{et}}^0(Z,V) \simeq \mathbb{Z}_p^r \to H_{\mathrm{et}}^{2c}(X,V) \to \dots$$

We can see $\mathbb{Z}_p$ inside $\mathbb{Z}_p^r$ by considering the map given by $1 \mapsto (n_1,\dots,n_r)$ and then extended by linearity. This element is 0 in $H_{\mathrm{et}}^{2c}(X,V)$ by hypothesis, since in the Chow group there are only homologically trivial elements. Further, if $\pi : H^{2c-1}(X-Z,V) \to \mathbb{Z}_p^r$, let $E_z = \pi^{-1}(\langle (n_1,\dots,n_r)\rangle)$. Then, I obtain the exact sequence

$$0 \to V \hookrightarrow E_z \to \mathbb{Z}_p \to 0,$$

which is an element in $H^1(G_F,V)$, as desired (this will be the image of Z).
Let us consider a concrete example. Let X/F be a curve; the étale regulator, or étale Abel-Jacobi, is given by

$$\mathrm{reg}_{\mathrm{et}} : \mathrm{CH}^1(X.0) \simeq \mathrm{Jac}(X)(F) \to H^1(G_F, T_p(\mathrm{Jac}(X))) = H^1(G_F, H^1(X_{\bar{F}}, \mathbb{Z}_p(1))).$$

Take $z=D=[\sum n_i x_i]$ with $\sum n_i = 0$. Then, we can apply the same ideas as before and associate to an element in the Chow group the exact sequence

$$0 \to T_p(\mathrm{Jac}\,X) \to E_D \to \mathbb{Z}_p \to 0,$$

where we can visualize the idea that points on elliptic curves can be seen as extensions of the Tate module by $\mathbb{Z}_p$.

We can repeat this in the setting of de Rham cohomology. When we take $X=X_1(M)$ and $Y=Y_1(M)$ we have that the excision exact sequence reads as

$$0 \to H_{\mathrm{dR}}^1(X_1(M),\mathbb{C}) \to H_{\mathrm{dR}}^1(Y_1(M),\mathbb{C}) \to \oplus_{i=1}^r H_{\mathrm{dR}}^0(c_i) \to H_{\mathrm{dR}}^2(X_1(M),\mathbb{C}) \to 0,$$

where c_i refers to the cusps.
The group $H_{\mathrm{dR}}^1(X_1(M),\mathbb{C})$ plays a role comparable with that of the Tate module, since it is isomorphic to $S_2(M) \oplus S_2(M)^*$. If instead of using coefficients in $\mathbb{C}$ we had used V_{k-2} we would have recovered $S_k(M) \oplus S_k(M)^*$.
On the other hand, $H_{\mathrm{dR}}^1(Y_1(M),\mathbb{C}) \simeq H_c^1(\Gamma_1(M),\mathbb{C}) =: \mathrm{MS}_{\Gamma_1(M)}(\mathbb{C})$, and there is a surjection of this cohomology group onto $\mathrm{Div}^0(\mathrm{cusps}) \simeq \mathbb{C}^{r-1}$.
We are going to see that

$$\mathrm{Eis}_2(M) \cong \mathrm{Div}^0(\mathrm{cusps}) \otimes \mathbb{C},$$

that is, that I can recover Eisenstein series from the cusps. This would be the key tool for proving Eichler-Shimura isomorphism (the space of modular symbols consists on twice the cuspidal forms together with the Eisenstein series).
Given $D \in \mathrm{Div}^0(\mathrm{cusps})$, we get that $[D] \in \mathrm{Jac}(X)$ is torsion by the Manin-Drinfeld

theorem. Then, for some $m \geq 1$, $mD = \mathrm{Div}(u)$, where $u \in K(X)^\times$. In fact, $u \in \mathcal{O}(Y)^\times$. In this setting, it is natural to define what will be the Eisenstein series

$$E_u = \mathrm{dlog}(u) = \frac{u'}{u} = \left(\frac{1}{u}\right)u' \in \Omega^1(Y),$$

obtaining that way a differential on Y (and a meromorphic differential on X). It is not difficult to check that u has a simple pole at all the cusps in the support of D.
We are going to do now a sketch of the construction of the Beilinson-Kato element $\Delta_{(k,\epsilon_r),(j,\epsilon_s)}$ for the case $k = 2, j = -1$. Assume for the sake of simplicity that $r \leq s$ (this is already dense in $W_{\mathbf{f}} \times W$).
Let $W = X_1(Np^s)$. Recall that

$$\Delta = \Delta_{(2,\epsilon_r),(-1,\epsilon_s)} \in \mathrm{CH}^2(X_1(Np^s), 2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Observe also that we have an excision exact sequence and an intersection product

$$\mathrm{CH}^1(Y,1) \times \mathrm{CH}^1(Y,1) \to \mathrm{CH}^2(Y,2).$$

Note that for the case of two Dirichlet characters χ_1, χ_2, we can consider

$$E_2(\chi_1,\chi_2) = \sum_{n \geq 0} a_n q^n,$$

where for $n \geq 1$,

$$a_n = \sum_{d|n} \chi_1(n/d)\chi_2(d)d.$$

Then, $E_2(\chi_1,\chi_2) = \mathrm{dlog}(u)$, for some $u = u(\chi_1,\chi_2) \in \mathcal{O}(Y)^\times$.

In general, when we consider an arbitrary Dirichlet character χ of conductor N, since $\mathcal{O}(Y)^\times$ is embedded in the first Chow group, we can take

$$u_1 = u(\chi, \epsilon_r\epsilon_s), \qquad u_2 = u(1, \chi\bar{\epsilon_r}\bar{\epsilon_s})$$

and consider its cup product, that lies in $\mathrm{CH}(Y,2)$. We would like to define now $\Delta_\chi \in \mathrm{CH}^2(X,2)$ as something of the form

$$\Delta_\chi = \{u_1, u_2\} + \text{ auxiliary terms},$$

where $\{u_1, u_2\}$ is the element of $\mathrm{CH}^2(Y,2)$ corresponding to the cup product. This comes from the existence of the following exact sequence

$$0 = K_2(\mathbb{Q}(\zeta_{Np^s})) \otimes \mathbb{Q} = \mathrm{CH}^1(\mathrm{cusps}, 2) \to \mathrm{CH}^2(X,2) \to \mathrm{CH}^2(Y,2) \to \mathrm{CH}^1(\mathrm{cusps}, 1),$$

so we can take Δ_χ to be $\{u_1, u_2\}$ minus the value of its image in $\mathrm{CH}^1(\mathrm{cusps}, 1)$.

From p-adic Hodge theory, we have another different map, called syntomic regulator

$$\mathrm{CH}^c(X,n) \to \frac{H^{2c-n-1}_{\mathrm{dR}}(X/\mathbb{Q}_p)}{\mathrm{Fil}^c H_{\mathrm{dR}}(X/\mathbb{Q}_p)}.$$

In particular, we can apply it to the Kato element $\Delta_\chi \in \mathrm{CH}^2(X_1(Np^s), 2)$.

Theorem 59 (Kato)**.** *We have the following equality, where $*$ is a factor depending on χ:*

$$L_p(\mathbf{f})((2,\epsilon_r),(-1,\epsilon_s)) = *\langle \mathrm{reg}_{\mathrm{syn}}(\Delta_\chi), \eta_{f_{2,\epsilon_r}} \rangle.$$

We will turn to analyze some of these aspects later on and explore the role played by L-functions more deeply.

6.3 The de Rham cohomology of curves over p-adic rings

In the next sections, we will construct cohomology classes with several arithmetic applications. To understand the construction, it is crucial to have a general view of some results concerning both the cohomology of curves over p-adic rings and some facts about nearly holomorphic forms that already appeared in chapter 2.

Let $\mathcal{X}$ be a smooth proper curve over $\mathrm{Spec}(\mathbb{Z}_p)$, and let $\tilde{X}$ and X be its special and generic fiber. Let $\{P_1,\dots,P_s\} \subset \tilde{X}(\bar{\mathbb{F}}_p)$ be a non-empty collection of closed points stable under the action of $G_{\mathbb{F}_p}$. Since $\tilde{X}$ is smooth, these points admit lifts $\tilde{P}_1,\dots,\tilde{P}_s \in \mathcal{X}(\mathcal{O}_{\mathbb{C}_p})$, and we will fix one of these lifts stable under the natural action of $G_{\mathbb{Q}_p}$, which determines the affine scheme $\mathcal{X}' = \mathcal{X} - \{\tilde{P}_1,\dots,\tilde{P}_s\}$ over $\mathrm{Spec}(\mathbb{Z}_p)$. We call $\tilde{X}'$ and X' to its special and generic fiber, respectively. We have a natural identification $X(\mathbb{C}_p) = \mathcal{X}(\mathcal{O}_{\mathbb{C}_p})$ and a reduction map $\mathrm{red} : X(\mathbb{C}_p) \to \tilde{X}(\bar{\mathbb{F}}_p)$. Let $\mathcal{A} := \mathrm{red}^{-1}(\tilde{X}'(\bar{\mathbb{F}}_p))$ be the standard affinoid attached to X'. It is a connected affinoid region obtained by deleting a collection of s disjoint residue disks. For $j = 1,\dots,s$ choose a local coordinate λ_j at $\tilde{P}_j$; this gives rise to a family of wide open neighbourhood of $\mathcal{A}$, indexed by a real parameter $\epsilon > 0$,

$$\mathcal{W}_\epsilon := \mathcal{A} \cup \bigcup_{j=1}^{s} \{x \in \mathrm{red}^{-1}(P_j) \text{ such that } \mathrm{ord}_p \lambda_j(x) < \epsilon\}.$$

We clearly have $\mathcal{A} \subset \mathcal{W}_\epsilon \subset X'(\mathbb{C}_p)$ and $\mathcal{W}_{\epsilon_1} \subset \mathcal{W}_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$.

If K is a complete subfield of $\mathbb{C}_p$, let $\Omega^1(\mathcal{W}_\epsilon/K)$ be the rigid differentials on $\mathcal{W}_\epsilon$ defined over K. Let $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K) := \frac{\Omega^1(\mathcal{W}_\epsilon/K)}{d\mathcal{O}_{\mathcal{W}_\epsilon/K}}$. Because X' is affine, $H^1_{\mathrm{dR}}(X'/K) = \frac{\Omega^1(X'/K)}{d\mathcal{O}_{X'}}$. The natural restriction map $\Omega^1(X'/K) \to \Omega^1(\mathcal{W}_\epsilon/K)$ sends exact forms to exact forms and it induces a map

$$\mathrm{comp}_\epsilon : H^1_{\mathrm{dR}}(X'/K) \to H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K).$$

For each annulus in $\mathcal{W}_\epsilon$, say $V_1,\dots,V_s$ there is a residue map

$$\mathrm{res}_{V_j} : \Omega^1(\mathcal{W}_\epsilon/K) \to K(-1).$$

This map vanishes on $d\mathcal{O}_{\mathcal{W}_\epsilon}$ and hence it is well-defined on cohomology. Further, it results that for any $\epsilon > 0$, comp_ϵ is an isomorphism of K-vector spaces.
This will help us to understand the action of the Frobenius on de Rham cohomology. Let $\sigma \in \mathrm{Gal}(\bar{K}/\mathbb{Q}_p)$ be a Frobenius automorphism and let $\Phi : A \to A$ be a characteristic zero lift on the special fiber $\tilde{X}'$. It extends to a morphism $\Phi : \mathcal{W}_\epsilon \to \mathcal{W}_{\epsilon'}$ for $0 < \epsilon < \epsilon'$ and it induces linear maps $\Phi : \mathcal{O}(\mathcal{W}_{\epsilon'}/K) \to \mathcal{O}(\mathcal{W}_\epsilon/K)$ and $\Phi : \Omega^1(\mathcal{W}_{\epsilon'}/K) \to \Omega^1(\mathcal{W}_\epsilon/K)$. This Frobenius gives rise to an endomorphism on $H^1_{\mathrm{dR}}(X'/K)$, also denoted by Φ; it is defined as the unique endomorphism obtained from

$$\Phi : H^1_{\mathrm{rig}}(\mathcal{W}_{\epsilon'}/K) \to H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K)$$

composing with comp at each side.

$H^1_{\mathrm{dR}}(X_K)$ is defined as the K-vector subspace on which Φ acts via multiplication by a p-adic unit (unit root subspace, $H^1_{\mathrm{dR}}(X_K)^{\mathrm{ur}}$). In general, $H^1_{\mathrm{dR}}(X_K)^{\Phi,t}$ is the subspace spanned by vectors on which Φ acts with slope t. The de Rham cohomology $H^1_{\mathrm{dR}}(X_K)$

is equipped with the usual alternating Poincaré duality, that is compatible with the Frobenius:

$$\langle \Phi\xi_1, \Phi\xi_2 \rangle = \Phi\langle \xi_1, \xi_2 \rangle = p\langle \xi_1, \xi_2 \rangle.$$

For the case of elliptic curves, let $\mathcal{X}_1(N)$ be the modular curve over $\mathrm{Spec}(\mathbb{Z}[1/N])$ classifying generalized elliptic curves endowed with an embedding of μ_N. If $p \nmid N$, let $\mathcal{X} := \mathcal{X}_1(N) \times_{\mathrm{Spec}\,\mathbb{Z}[1/N]} \mathbb{Z}_p$ and $X = \mathcal{X} \times \mathrm{Spec}(\mathbb{Q}_p)$ and consider the supersingular points $P_1, \ldots, P_s \in \tilde{\mathcal{X}}(\mathbb{F}_{p^2})$ of the special fiber. These points are the zeros of the Hasse invariant, introduced in chapter 2. We can reproduce the previous procedure, and we will have $\mathcal{X}' = \mathcal{X} - \{\tilde{P}_1, \ldots, \tilde{P}_s\}$, $\mathcal{A} = \mathrm{red}^{-1}(\tilde{X}'(\bar{\mathbb{F}}_p))$. $\mathcal{A}$ is called the ordinary locus. Recall further that whenever $\mathrm{ord}\, E_{p-1}(x) < \frac{p}{p+1}$, the elliptic curve A_x admits a canonical subgroup Z_x which makes possible to choose a canonical lift of the Frobenius.

Before going on, it is convenient to recall the sheaf interpretation of modular forms, that we will freely use along the next pages (these concepts had already appeared in Chapter 2): a modular form ϕ on $\Gamma_1(N)$ of weight $k = r+2$ with Fourier coefficients in K is a global section of $\omega^{r+2} = \omega^r \otimes \Omega^1_X(\log \mathrm{cusps})$ over the base change X_K of X to K. ω^r is a subsheaf of $\mathcal{L}_r := \mathcal{R}^1\pi_*(\mathcal{E} \to Y)$, where we have the exact sequence

$$0 \to \omega \to \mathcal{L} \to \omega^{-1} \to 0.$$

$\mathcal{L}_r$ is a coherent sheaf over X of rank $r+1$ endowed with the Gauss-Manin connection. We have a notion of Poincaré duality, that induces a perfect pairing

$$\langle , \rangle_{k,X} : H^1(X_K, \omega^{-r}) \times H^0(X_K, \omega^r \otimes \Omega^1_X) \to K.$$

We will set, as usual, $\omega_f = f(z)dz$ and $\bar{\omega}_f = \bar{f}^*(z)d\bar{z}$. The antiholomorphic differential is

$$\eta_f^{\mathrm{ah}} := \frac{\bar{\omega}_f}{\langle \bar{\omega}_f, \omega_f \rangle_{k,X}},$$

and this gives rise to a class in $H^1_{\mathrm{dR}}(X_{\mathbb{C}}, \mathcal{L}_r, \nabla)$, whose image η_f in $H^1(X_{\mathbb{C}}, \omega^{-r})$ belongs to $H^1(X_K, \omega^{-r})$.

When $K = \mathbb{C}$, Hodge theory gives a canonical splitting $\mathrm{Spl}_{\mathrm{hdg}} : \mathcal{L} \to \omega$ on the previous exact sequence. We will denote by the same symbol the associated map $\mathcal{L}^k \to \omega^k$, as well as the resulting map

$$\mathrm{Spl}_{\mathrm{hdg}} : H^0(X_{\mathbb{C}}, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \to H^0(Y(\mathbb{C})_{\mathrm{an}}, \omega^r \otimes \Omega^1_X).$$

The image of $\mathrm{Spl}_{\mathrm{hdg}}$ is called the space of nearly holomorphic cusp forms of weight $k = r+2$ on $\Gamma_1(N)$.

Let us see some applications of this theory to the construction of Beilinson-Kato elements.
Given modular units u_1 and u_2 in $\mathcal{O}(\bar{Y}_1)^\times$, we consider an element $\{u_1, u_2\} \in \mathrm{CH}^1(Y, 1)$ (as we have seen, this gives rise to an element in $\mathrm{CH}^1(X, 1)$) that is called the associated Steinberg element. In this setting it is important the description done by Besser of the p-adic regulator $\mathrm{reg}_p\{u_1, u_2\} \in H^1_{\mathrm{dR}}(Y_1)$. It is a rigid morphism on a system $\{\mathcal{W}_\epsilon\}$ of wide open neighborhoods of the ordinary locus $\mathcal{A} \subset Y_1$ obtained by deleting from Y_1 both the supersingular and the cuspidal residue discs. Let $\Phi_{12} = (\Phi_{Y_1}, \Phi_{Y_1})$ be the corresponding lift of Frobenius on $Y_1 \times Y_1$ and let $P \in \mathbb{Q}[x]$ be any polynomial

satisfying that $P(\Phi)$ annihilates the class of $\frac{du_1}{u_1} \otimes \frac{du_2}{u_2}$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$ and that $P(\Phi_{Y_1})$ acts invertibly on $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$. This choice of P gives a rigid 1-form on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$, say $\rho_P(u_1, u_2)$, such that

$$d\rho_P(u_1, u_2) = P(\Phi_{12})\Big(\frac{du_1}{u_1} \otimes \frac{du_2}{u_2}\Big).$$

It is well defined up to closed rigid one-forms. Choose now a base point $x \in \mathcal{W}_\epsilon$ and let δ, i_x and j_x denote the diagonal, horizontal and vertical inclusions, respectively. Set

$$\bar{\xi}_{P,x}(u_1, u_2) := (\delta^* - i_x^* - j_x^*)(\rho_P(u_1, u_2)) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon).$$

Its natural image in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$, that we will denote as $\xi_{P,x}(u_1, u_2)$ does not depend on the choice of one-form ρ_P. The conditions we have imponed on P allow us to define the class

$$\xi_x(u_1, u_2) := P(\Phi_{Y_1})^{-1} \xi_{P,x}(u_1, u_2) \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon).$$

This class does not depend neither on P nor on the choice of the base point x. We can then define $\mathrm{reg}_p\{u_1, u_2\} := \xi(u_1, u_2)$. We therefore have

$$\mathrm{reg}_p\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}) := \langle \eta_f^{\mathrm{ur}}, \mathrm{reg}_p\{u_\chi, u(\chi_1, \chi_2)\}\rangle_{2,Y}.$$

This kind of results, that may seem mysterious at first sight, will have a great importance later on.

6.4 Beilinson-Flach elements

In this section, we explore the construction of the Beilinson-Flach elements, following [BDR1] and [BDR2]. This is the first step towards our results of chapter eight, where we will see several arithmetic applications of this cohomology construction. We are going to consider f and g, two normalized newforms of weights k, l, levels N_f, N_g and nebentypus χ_f, χ_g, respectively.
The classical Beilinson formula relates, on the one hand, the Rankin L-series $L(f \otimes g, s)$ evaluated at $s = 2$, and on the other, the image under the complex regulator of certain explicit elements in $\mathrm{CH}^2(X_1(N)^2, 1) \otimes \mathbb{Q}$. In the p-adic setting, the complex L-series will be replaced by Hida's p-adic Rankin L-series and the role of the complex regulator wil be played by the p-adic syntomic regulator we have already presented. Along this section, we keep the notations introduced in chapter 3 when discussing the L-series associated to Hida families.

Let S be a quasi-projective variety over a field K and let $K_j(S)$ be Quillen's algebraic K-groups of S. The motivic cohomology groups $H_{\mathcal{M}}(S, \mathbb{Q}(n)) = K^{(n)}_{2n-1}(S)$ of S are just the n-th graded piece of the Adams filtration on $K_{2n-i}(S) \otimes \mathbb{Q}$. In a parallel setting, Bloch introduced the higher Chow groups, $\mathrm{CH}^i(S, n)$ of S or in general, as we have seen, of a more general variety. We will follow this approach.

First of all, we will consider the smooth projective surface $S := X \times X$, where X is the modular curve over the field K generated by the Fourier coefficients of the two modular forms f and g. Then, we will move to $S_s := X_0(Np) \times X_1(Np^s)$ and eventually to $S_{r,s} := X_1(Np^r) \times X_1(Np^s)$. The higher Chow group $\mathrm{CH}^2(S, 1)$ can be described as the first homology of the Gersten complex

$$K_2(K(S)) \xrightarrow{\partial} \oplus_{Z \subset S} K(Z)^\times \xrightarrow{\mathrm{Div}} \oplus_{P \in S} \mathbb{Z},$$

where $K_2(K(S))$ is the second Milnor K-group of the rational function field $K(S)$; ∂ is the map whose component at Z is the tame symbol attached to the valuation ord_Z; $\Theta := \oplus_{Z \subset S} K(Z)^\times$ is the set of finite formal linear combinations $\sum_i (Z_i, u_i)$, with Z_i an irreducible curve in S and u_i a rational function on Z_i.

Let F the the field generated over K by the values of the Dirichlet characters of modulus N, where N is the least common multiple of N_f and N_g. Recall that we have begun by considering the case $S = X \times X$. An element of Θ of the form $(\{P\} \times X, u)$ is called vertical, and one of the form $(X \times \{P\}, u)$ is said to be horizontal. A linear combination of vertical and horizontal terms is said to be negligible. Let $\Delta \subset S$ the diagonal embedding of the curve X in S. Let F be the field of definition of our surface. We claim the following:

Lemma 14. *There exists a negligible element $\theta_u \in \Theta \otimes F$ such that*

$$\mathrm{Div}(\theta_u) = \mathrm{Div}(\Delta, u).$$

Proof. We will make use of Manin-Drinfeld theorem, considering for that the element $D_u = \mathrm{Div}(\Delta, u) \in \sqcup_{P \in S} F$, the image of $(\Delta, u) \in \Theta$ under the divisor map. Since D_u is an F-linear combination of elements of the form $(c_1, c_1) - (c_2, c_2)$, where c_1 and c_2 are cusps of the modular curve X_K, it is enough to construct $\theta \in \Theta \otimes \mathbb{Q}$ such that $\mathrm{Div}(\theta) = (c_1, c_1) - (c_2, c_2)$, and by Manin-Drinfeld, there is $\alpha \in \mathcal{O}(Y_K)^\times \mathbb{Q}$ whose divisor is $c_1 - c_2$ and the negligible element given by

$$\theta = (\{c_1\} \times X, \alpha) + (X \times \{c_2\}, \alpha)$$

satisfies the desired requirements. □

Then, to an element $(\Delta, u) \in \Theta \otimes F$ we can associate it the class of

$$\Delta_u = [(\Delta, u) - \theta_u] \in \mathrm{CH}^2(S, 1).$$

We want to explore the meaning of the p-adic regulator in this setting. Let us give a feeling of what is happening with it. Let $\mathcal{X}$ denote the smooth model of X over $\mathcal{O}_p$, the ring of integers of $\mathbb{K}_p$ (a finite extension of $\mathbb{Q}_p$). Let $\tilde{\mathcal{X}}$ be the special fiber and consider $S = \mathcal{X} \times \mathcal{X}$. The p-adic syntomic regulator is, as we have seen, a map

$$\mathrm{reg}_p : \mathrm{CH}^2(S_{K_p}, 1) \to (\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/K_p))^*.$$

After possibly enlarging K_p, let $\{P_1, \ldots, P_t\} \subset \mathcal{X}(\mathcal{O}_p)$ be a set of points consisting of the cusps and of a choice of a lift of every supersingular point in $\tilde{\mathcal{X}}(\bar{\mathbb{F}}_p)$.

We will use the same notations as before for the rigid cohomology of a modular curve. In particular, we will recover the polynomial $P(x) \in \mathbb{C}_p[x]$ defined in the previous section, as well as the rigid analytic one-form $\rho_P \in \Omega^1(\mathcal{W}_\epsilon^2)$. From now on, we will fix both P and ρ_P.

Let $P_g(t) \in \mathbb{C}_p[t]$ be such that $P_g(\Phi)$ annihilates the class of ω_g in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$, say $P_g(t) := t^2 - a_p(g)t + \chi_g(p)p$ and let $F_g \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ be a Coleman primitive of ω_g. In the same way, let $P_{E_\chi}(t)$ be a polynomial such that $P_{E_\chi}(\Phi)$ annihilates the class of E_χ, say $P_{E_\chi}(t) := t^h - p^h$, where h is the order of the root of unity $\chi(p)$. Then, $F_{E_\chi} := p^{-h} P_{E_\chi}(\phi) \log(u_\chi) \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ is a Coleman integral of E_χ.

Let $\Delta \subset \mathcal{W}_\epsilon^2$ denote the diagonal and define $\xi'_P := [\rho_{P|\Delta}] \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \simeq H^1_{\mathrm{dR}}(X')$. It can be checked that this class is well-defined. Moreover, we can set $\xi' := P(\Phi)^{-1} \cdot \xi'_P \in H^1_{\mathrm{dR}}(X')$. Finally, let $\mathrm{Spl}_X : H^1_{\mathrm{dR}}(X') \to H^1_{\mathrm{dR}}(X)$ be the Frobenius equivariant splitting of

$$0 \to H^1_{\mathrm{dR}}(X) \to H^1_{\mathrm{dR}}(X') \to K_p(-1)^{t-1} \to 0$$

and put $\xi := \mathrm{Spl}_X(\xi') \in H^1_{\mathrm{dR}}(X)$.

Proposition 23. *We have that* $\mathrm{reg}_p(\Delta_{u_X})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \langle \eta_f^{\mathrm{ur}}, \xi \rangle$.

In next sections we will explore the connection between the regulator map and L-functions.

We will now move to the case of $S_s := X_0(Np) \times X_s$ (with $X_s = X_1(Np^s)$) and $S_{r,s} := X_r \times X_s$. For the first case (that is worked out in [BDR2]), let

$$\tilde{\pi}_s : X_s \to X_0(Np)$$

be the natural forgetful projection of modular curves compatible with the U_p correspondence acting on both curves. Let

$$\iota_s = (\tilde{\pi}_s, \mathrm{Id}) : X_s \hookrightarrow S_s$$

be the closed embedding given by $\iota_s(x) = (\tilde{\pi}_s(x), x)$. Finally, let $\Delta_s := \iota_s(X_s) \subset S_s$ be the resulting embedded curve in S_s.

For $S_{r,s}$, with $r \leq s$ we could consider

$$\pi_{r,s} : X_s \to X_r$$

the natural forgetful projection of modular curves, and then define

$$\iota_{r,s} = (\tilde{\pi}_{r,s}, \mathrm{Id}) : X_s \hookrightarrow S_{r,s}.$$

$\Delta_{r,s} := \iota_{r,s}(X_s) \subset S_{r,s}$ would be the diagonal in $S_{r,s}$.

Lemma 15. *For any $u \in \mathcal{O}_{Y_s}^\times$ there is a negligible element $\theta_s \in \mathrm{CH}^2(S_s, 1)_F$ such that*

$$\mathrm{Div}(\theta_s) = \mathrm{Div}(\Delta_s, u).$$

Hence, we can define

$$\mathrm{BF}(u) := [(\Delta_s, u) - \theta_s] \in \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1)_{\mathbb{Q}}(\mathbb{Q}),$$

that is called the Beilinson-Flach element attached to $u \in \mathcal{O}_{Y_s}^\times$. In general, we can consider $\mathrm{BF}(a; Np^s) := \mathrm{BF}(\mathfrak{g}^w_{a;Np^s})$ and $\mathrm{BF}_s := \mathrm{BF}(1; Np^s)$, where we have used the notation $\mathfrak{g}_{a;M}$ for the Siegel unit

$$\mathfrak{g}_{a;M} := (1 - \zeta_M^a) q^{1/12} \prod_{n=1}^{\infty} (1 - \zeta_M^a q^n)(1 - \zeta_M^{-a} q^n).$$

If we introduce now the cohomology modules

$$V_0(Np) := H^1_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p)(1), \qquad V_s = H^1_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p)(1) \text{ for } s \geq 1,$$

and natural maps

$$\mathrm{pr}_{1,1} : H^2_{\mathrm{et}}(\bar{S}_s, \mathbb{Z}_p)(2) \to V_0(Np) \otimes V_s$$

$$\mathrm{reg}_{\mathrm{et}} : \mathrm{CH}^2(S_s, 1)(\mathbb{Q}) \to H^1(\mathbb{Q}, V_0(Np) \otimes V_s),$$

we will be able to define the so-called Beilinson-Flach cohomology classes of level Np^s, that are

$$\kappa(a; Np^s) := \mathrm{reg}_{\mathrm{et}}(\mathrm{BF}(a; Np^s)) \in H^1(\mathbb{Q}, V_0(Np) \otimes V_s),$$

$$\kappa_s := \kappa(1; Np^s).$$

A study of the restriction to $G_{\mathbb{Q}_p}$ of the classes κ_s would be the key for proving the relation with the values of the Hida-Rankin p-adic L-function associated to Hida families passing through f and g.
Although we will not discuss this in detail, we will say a few words about the key ingredients involved in the discussion. In this description, one must use p-adic Hodge theory and introduce the Dieudonné module of the Kummer dual $V^*_{fg}(1) = V_{fg^*}(-1)$ of V_{fg}, denoted as $D_{fg^*}(-1)$ and canonically identified with D_{fg^*}. Since in this case $H^1_e(\mathbb{Q}_p(\mu_{p^s}), V_{fg}) = H^1_f(\mathbb{Q}_p(\mu_{p^s}), V_{fg})$, the Bloch-Kato logarithm gives an isomorphism between $H^1_f(\mathbb{Q}_p(\mu_{p^s}), V_{fg})$ and $\mathrm{Fil}^0(D_{fg*}(-1))^*$. Now, one can define suitable vectors in $\mathrm{Fil}^0(D_0(Np)[f] \otimes D_s(-1)[g^*])$ constructing explicit elements $\eta_f \in D_0(Np)[f]$ and $\omega_{g^w} \in D_s[g^*]$. More details are given in [BDR2].

The most remarkable aspect of all this theory is that the classes we have constructed form a norm-compatible system of elements. This is relevant since our main contribution will be based on the idea of taking advantage of these p-adic variations and proving nice properties about these Λ-adic classes.
We will begin with the simpler case of X_s. For $s \geq 0$, the U_p compatible projections

$$\pi_{s+1,s} : X_{s+1} \to X_s$$

of modular curves give rise to maps

$$\pi_{s+1,s} : S_{s+1} \to S_s$$

on the associated surfaces. Write also $\pi_{s+1,s} : \mathcal{O}^\times_{X_{s+1}} \to \mathcal{O}^\times_{X_s}$ for the norm maps on units.

Proposition 24. *For all $s \geq 1$,*

$$\pi_{s+1,s}(\mathfrak{g}^s_{a;Np^{s+1}}) = \mathfrak{g}^w_{a;Np^s}.$$

Write $\pi_{s+1,s} : \mathrm{CH}^2(S_{s+1}, 1) \to \mathrm{CH}^2(S_s, 1)$ for the norm maps on higher Chow groups induced by push-forward under the maps $\pi_{s+1,s}$. It preserves the subspaces of negligible classes and then induces a well-defined map

$$\pi_{s+1,s} : \mathrm{CH}^2_{\mathrm{neg}}(S_{s+1}, 1) \to \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1).$$

These norm compatibilities of the units $\mathfrak{g}_{a;Np^s}$ are inherited by the associated Beilinson-Flach elements.

Proposition 25. *For all $s \geq 1$,*

$$\pi_{s+1,s}(\mathrm{BF}(a; Np^{s+1})) = \mathrm{BF}(a; Np^s), \qquad \pi_{s+1,s}(\mathrm{BF}_{s+1}) = \mathrm{BF}_s,$$

the equalities in $\mathrm{CH}^2_{\mathrm{neg}}(S_s, 1)(\mathbb{Q})$.

Now, we can formulate as a corollary one the main results of [BDR2]:

Corollary 5. *The Beilinson-Flach cohomology classes κ_s are compatible under the norm map*

$$\pi_{s+1,s}(\kappa_{s+1}) = \kappa_s.$$

In particular, the classes κ_s can be packaged together into the inverse limit class

$$\kappa_\infty := (\kappa_s)_{s\geq 1} \in H^1(\mathbb{Q}, V_0(Np) \otimes \mathbb{V}_\infty),$$

where

$$\mathbb{V}_\infty := \lim_{\leftarrow, s} V_s.$$

Recall that we did something similar in the presentation of circular units: analogies with those simple settings are continuously present.

We would like to repeat this same construction for the case of $X_{r,s}$. When r is fixed, we can consider the projection

$$\pi^r_{s+1,s} : X_{r,s+1} \to X_{r,s},$$

and then with s fixed we could take

$$\pi^{r+1,r}_s : X_{r+1,s} \to X_{r,s}.$$

Both maps should be well-behaved with the different structures involved (following the same reasonings than before) and we coud finally consider a global class

$$\kappa_\infty := (\kappa_{r,s})_{r,s\geq 1} \in H^1(\mathbb{Q}, \mathbb{V}_\infty \otimes \mathbb{V}_\infty).$$

Coming back to the X_s-case, it will be convenient to replace the module $\mathbb{V}_\infty$ for its image under the ordinary projection. Let

$$V_s^{\text{ord}} := e_{\text{ord}} V_s, \qquad V_\infty^{\text{ord}} := e_{\text{ord}} \mathbb{V}_\infty.$$

The action of the group D_s of diamond operators on X_s endows the $G_\mathbb{Q}$-module $\mathbb{V}_\infty^{\text{ord}}$ with a natural structure of module over the Iwasawa algebra $\tilde{\Lambda} = \mathbb{Z}_p[[D_\infty]]$. A result of Hida assures that this module is finitely generated and locally free over this algebra. Its Hecke eigenspaces realise the Λ-adic representations attached to ordinary families of eigenforms.
The Λ-adic Beilinson-Flach cohomology class is defined as

$$\kappa_s^{\text{ord}} := e_{\text{ord}} \kappa_s \in H^1(\mathbb{Q}, V_0(Np) \otimes V_s^{\text{ord}}),$$

$$\kappa_\infty^{\text{ord}} := e_{\text{ord}} \kappa_\infty \in H^1(\mathbb{Q}, V_0(Np) \otimes \mathbb{V}_\infty^{\text{ord}}.$$

Let $f = q + \sum_{n\geq 2} a_n(f) q^n \in S_2(Np)$ be the cusp form on which U_p acts with eigenvalue α_f. Let $\psi_f : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be the unramified character of $G_{\mathbb{Q}_p}$ such that $\psi_f(\text{Frob}_p) = \alpha_f$. Then, there is an exact sequence

$$0 \to V_f^+ \to V_f \to V_f^- \to 0,$$

with $V_f^+ \simeq \mathcal{O}(\psi_f^{-1}\epsilon_{\text{cyc}})$ and $V_f^- \simeq \mathcal{O}(\psi_f)$. Let $\bar{\epsilon}_{\text{cyc}}$ be the Λ-adic cyclotomic character satisfying

$$\nu_{l,\epsilon} \circ \bar{\epsilon}_{\text{cyc}} = \epsilon \epsilon_{\text{cyc}}^{l-1} \omega^{1-l},$$

for any $l \geq 1$ and any Dirichlet character ϵ of p-power conductor.
We can define a projection $\pi_f : V_0(Np)\otimes\mathcal{O} \to V_f$. For that, let $\pi_d : X_1(Np) \to X_1(N_f)$ be the degeneracy map induced by multiplication by d on the upper half plane (d is any positive divisor of Np/N_f). Let $\pi_{d*} : V_0(Np)[f_0] \to V_f$ be the map in étale cohomology induced by π_d and define $\pi_f := \sum \lambda_d \pi_{d*}$.

In the same way, let $\mathbf{g} = \sum_{n\geq 1} \mathbf{a_n}(\mathbf{g})q^n$ be a Λ-adic cuspidal eigenform of tame level N and tame character χ. By the results of Hida and Wiles there is an associated two-dimensional Galois representation $\mathbb{V}_{\mathbf{g}}$ over $\Lambda_{\mathbf{g}}$, characterized by the property that the characteristic polynomial of Frob_l for $l \nmid Np$ is $T^2 - \mathbf{a_l}(\mathbf{g})T + \epsilon_{\mathrm{cyc}}(l)$. As before, we have the exact sequence

$$0 \to \mathbb{V}_{\mathbf{g}}^+ \to \mathbb{V}_{\mathbf{g}} \to \mathbb{V}_g^- \to 0,$$

with $\mathbb{V}_g^+ \simeq \Lambda_g(\psi_g^- \chi\epsilon_{\mathrm{cyc}})$ and $\mathbb{V}_{\mathbf{g}}^+ \simeq \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}})$. Again, $\psi_{\mathbf{g}} : G_{\mathbb{Q}_p} \to \Lambda_{\mathbf{g}}^\times$ is the unramified character sending Frob_p to $\mathbf{a_p}(\mathbf{g})$.
The Λ-adic form $\mathbf{g}$ gives rise to an epimorphism $\pi_{\mathbf{g}} : \mathbb{V}_\infty^{\mathrm{ord}} \to \mathbb{V}_{\mathbf{g}}$ of $\tilde{\Lambda}[G_{\mathbb{Q}}]$-modules. Set

$$\mathbb{V}_{f,\mathbf{g}} := V_f \otimes \mathbb{V}_{\mathbf{g}}, \qquad \pi_{f,\mathbf{g}} := \pi_f \otimes \pi_g : V_0(Np) \otimes \mathbb{V}_\infty^{\mathrm{ord}}\mathbb{V}_{f,\mathbf{g}}.$$

In the setting of $X_{r,s}$ we would have to consider modules $\mathbb{V}_{\mathbf{f},\mathbf{g}}$ and a projection $\pi_{\mathbf{f},\mathbf{g}} : \mathbb{V}_\infty^{\mathrm{ord}} \otimes \mathbb{V}_\infty^{\mathrm{ord}} \to \mathbb{V}_{\mathbf{f},\mathbf{g}}$.

Definition 47. *The Λ-adic cohomology class attached to $(f, \mathbf{g})$ is*

$$\kappa(f, \mathbf{g}) = \pi_{f,\mathbf{g}}(\kappa_\infty^{\mathrm{ord}}) \in H^1(\mathbb{Q}, \mathbb{V}_{f,\mathbf{g}}).$$

In the same way we would have defined $\kappa(\mathbf{f}, \mathbf{g})$.
Let $l \geq 1$ be an integer and ϵ a Dirichlet character of conductor p^s for some $s \geq 1$. Let y be a point in $\Omega_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}})$ such that $\mathrm{wt}(y) = \nu_{l,\epsilon}$, and assume that the specialization g_y of $\mathbf{g}$ at y is a classical eigenform (which is true if $l \geq 2$). Then, g_y belongs to $M_l(Np^s, \chi\epsilon\omega^{1-l})$ and is a cuspform if $l \geq 2$. Define $\kappa(f, g_y) \in H^1(\mathbb{Q}, V_{f,g_y})$ to be the specialization at y.

One of the main results in the theory of Beilinson-Flach elements, that will appear again, is the following connection with L-functions. Basically, it states that the logarithm of the local class $\kappa_p(f, g_y)$ equals the p-adic L-function evaluated at y (multiplying by some explicit factors).

Theorem 60. *Let $y \in \Omega_{\mathbf{g}}$ be an arithmetic point of weight-character $\nu_{2,\epsilon}$ for some character ϵ of conductor p^s. Then,*

$$\log_p \kappa_p(f, g_y)(\eta_f \otimes \omega_{g_y^w}) = \mathcal{G}(\chi\epsilon\omega^{-1}) \cdot \frac{\alpha_g^{s-1}}{1 - \chi(p)^{-1}\alpha_f^{-1}\alpha_{g_y}^{-1}} \cdot L_p(f, \mathbf{g})(y),$$

where $\mathcal{G}(\chi) = \sum_{a=1}^M \chi(a)\zeta_M^a$, being M the conductor of χ.

The last aim of the section is to provide a sketch of how to derive an explicit reciprocity law, that will have a parallelism in the setting of Gross-Kudla-Schoen elements and that is useful for establishing the connection with L-functions. The objective is to emphasize the general philosophy of how to reach results like the last theorem of this section, in which we establish a connection between the vanishing of the L-function in the central point and the fact that a cohomology class is de Rham. In all these results we see the shadow of the ubiquous Perrin-Riou's big logarithm. These results will reach a great importance at the end of the chapter, where we will summarize the recent results of [KLZ] that are used in [RiRo].

The $\Lambda_{\mathbf{g}}$-module $\mathbb{V}_{f\mathbf{g}}$ admits a $G_{\mathbb{Q}_p}$-stable filtration, given by

$$\mathbb{V}_{f\mathbf{g}}^{++} := V_f^+ \otimes V_{\mathbf{g}}^+ \subset \mathbb{V}_{f\mathbf{g}}^0 := V_f \otimes \mathbb{V}_{\mathbf{g}}^+ + V_f^+ \otimes \mathbb{V}_{\mathbf{g}} \subset \mathbb{V}_{f\mathbf{g}}.$$

Observe that $\mathbb{V}_{f\mathbf{g}}^{++}$ and $V_{f\mathbf{g}}^+$ have rank 1 and 3 over $\Lambda_{\mathbf{g}}$ respectively. More details can be found in [BDR2] and some of these ideas will be recovered at the end of Chapter 7 when exploring the canonical structures associated to a Hida family.

Lemma 16. *There is an isomorphism of $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-modules between $\mathbb{V}_{f\mathbf{g}}$ and $\Lambda_g(\psi_f\psi_{\mathbf{g}})$. Further, the quotient $\mathbb{V}_{f\mathbf{g}}/\mathbb{V}_{f\mathbf{g}}^{++}$ decomposes as a $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-module as*

$$\mathbb{V}_{f\mathbf{g}}^{+}/\mathbb{V}_{f\mathbf{g}}^{++} \simeq \mathbb{V}_{f\mathbf{g}}^{f} \oplus \mathbb{V}_{f\mathbf{g}}^{g},$$

where $\mathbb{V}_{f\mathbf{g}}^{f} = \Lambda_{\mathbf{g}}(\psi_f\psi_g^{-1}\chi \cdot \bar{\epsilon}_{\text{cyc}})$ and $\mathbb{V}_{f\mathbf{g}}^{g} = \Lambda_{\mathbf{g}}(\psi_f^{-1}\psi_{\mathbf{g}}\epsilon_{\text{cyc}})$.

The natural inclusion between $\mathbb{V}_{f\mathbf{g}} \hookrightarrow \mathbb{V}_{f\mathbf{g}}$ induces a homomorphism $H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{g}}^+) \hookrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{g}})$ which is injective since $H^0(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{g}}/\mathbb{V}_{f\mathbf{g}}^+) = 0$.

Lemma 17. *The local class $\kappa_p(f, \mathbf{g})$ belongs to $H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{g}}^+) \subset H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{g}})$.*

Consider now $\Psi := \psi_f\psi_{\mathbf{g}}^{-1}\chi$.
The previous results allow us to define $\kappa_p^f(f, \mathbf{g}) \in H^1(\mathbb{Q}_p, \Lambda_g(\Psi \cdot \epsilon_{\text{cyc}}))$ as the projection of $\kappa_p(f, \mathbf{g})$ to the first factor $\mathbb{V}_{f\mathbf{g}}^f$. Further, for a classical point y, let $\Psi_y : G_{\mathbb{Q}_p} \to K_y^\times$ be the specialization at y of the unramified Λ-adic character Ψ.

Lemma 18. *For all arithmetic points y of weight $\nu_{l,\epsilon}$ with $l \geq 2$, the local cohomology group $H^1(\mathbb{Q}_p, K_y(\Psi_y\epsilon\epsilon_{\text{cyc}}^{l-1}\omega^{1-l}))$ is one-dimensional over K_y and is equal to $H_e^1(\mathbb{Q}_p, K_y(\Psi_y\epsilon\epsilon_{\text{cyc}}^{l-1}\omega^{1-l}))$.*

These tools, combined with some (hard) p-adic Hodge theory, lead us to the following remarkable results about the dual exponential map of Bloch and Kato, that will finish this section:

Theorem 61. *Let $y \in \Omega_{\mathbf{g}}$ be the point over $\nu_{1,1}$ corresponding to the classical p-stabilized weight one form $g_y = g_\alpha \in S_1(Np, \chi)$. Then,*

$$\exp_p^*(\kappa_p^f(f, g_\alpha)) \neq 0 \quad \textit{if and only if} \quad L(f \otimes g, 1) \neq 0.$$

6.5 Gross-Kudla-Schoen cycles

Until now, we have explored two different settings in which we have done some geometric constructions attached to modular forms: that of Beilinson-Kato elements and that of Beilinson-Flach elements. Now, we want to explore a different one, in which we will consider the image under the p-adic Abel-Jacobi map (regulator) of certain generalized Gross-Kudla-Schoen cycles in the product of three Kuga-Sato varieties. In subsequent sections, we will derive the relations of all these geometric constructions with L-functions. In this case, we will obtain a relation with the special value of the

p-adic L-function attached to the Garrett-Rankin triple convolution of three Hida families of modular forms, at a point lying outside its region of interpolation.
To begin with, consider $f \in S_k(N_f, \chi_f)$, $g \in S_l(N_g, \chi_g)$ and $h \in S_m(N_h, \chi_h)$. The triple is said to be balanced if the largest weight is strictly smaller than the sum of the other two. It is customary to assume that the local root number ϵ_v at all the finite primes $v|N$ are equal to $+1$. This assumption holds in many cases of arithmetic interest. It implies that

$$\epsilon = \epsilon_\infty = \begin{cases} -1 & \text{if } (k,l,m) \text{ is balanced;} \\ -1 & \text{ if } (k,l,m) \text{ is unbalanced.} \end{cases}$$

In particular, when the triple is balanced, the L-function must vanish at the central point.

Let $\mathcal{E}$ be the universal generalized elliptic curve fibered over $X = X_1(N)$; for $n \geq 0$, let $\mathcal{E}^n$ be the n-th Kuga-Sato variety over $X_1(N)$, that is an $n+1$-dimensional variety. The p-adic Galois representation $V_p(f,g,h)$ occurs in the cohomology of the triple product $W := \mathcal{E}^{k-2} \times \mathcal{E}^{l-2} \times \mathcal{E}^{m-2}$. In the balanced case, it is conjectured (Bloch-Kato) that there should exist a non-trivial cycle in the Chow group $\mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ of rational equivalence classes of null-homologous cycles of codimension c on the variety W. The construction follows the same spirit as those of previous sections.

The case $(k,l,m) = (2,2,2)$ has been done by Gross and Kudla. In general, we always have the projection $\pi : \mathcal{E} \to X$. Also, a generic point in $\mathcal{E}^r$ is $(x; P_1, \ldots, P_r)$, where $x \in X$ and P_i are points in the fiber $\mathcal{E}_x$. Consider

$$\epsilon_{\mathrm{sym}} = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma)\sigma \in \mathrm{Corr}(\mathcal{E}^r) \otimes \mathbb{Q},$$

$$\epsilon_{\mathrm{inv}} := \Big(\frac{1-u_1}{2}\Big) \otimes \ldots \otimes \Big(\frac{1-u_r}{2}\Big) \in \mathrm{Corr}(\mathcal{E}^r) \otimes \mathbb{Q},$$

being u_i the involution on the i-th factor in the fibration. Let $\epsilon_r = \epsilon_{\mathrm{sym}}\epsilon_{\mathrm{inv}}$ be the composition of both maps. Further, put $(k,l,m) = (r_1+2, r_2+2, r_3+2)$, with $r_3 \geq r_2 \geq r_1 \geq 0$, and set $r = \frac{r_1+r_2+r_3}{2}$.

Now, we will carry on our main objective: define a generalized Gross-Kudla-Schoen cycle $\Delta_{k,l,m}$ of codimension $r+2$ in the $(2r+3)$-dimensional variety $W = \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}$. It shall be regarded as an element in $\mathrm{CH}^{r+2}(W)$.

Definition 48. *Let $(k,l,m) = (2,2,2)$. For any non-empty subset $I \subset \{1,2,3\}$, let*

$$X_I = \{(P_1,P_2,P_3) \in X^3 \mid P_i = P_j \textit{ for all } \{i,j\} \subset I, P_j = o \textit{ for } j \notin I\}.$$

Then, the Gross-Kudla-Schoen diagonal cycle is

$$\Delta_{2,2,2} = X_{123} - X_{12} - X_{23} - X_{31} + X_1 + X_2 + X_3 \in \mathrm{CH}^2(X_1 \times X_2 \times X_3).$$

For the remaining cases, let $A = \{a_1, \ldots, a_{r_1}\}$, $B = \{b_1, \ldots, b_{r_2}\}$ and $C = \{c_1, \ldots, c_{r_3}\}$ be subsets of $\{1, \ldots, r\}$ such that $A \cap B \cap C$ is empty. We can consider the closed embeddings

$$\phi_{ABC} : \mathcal{E}^r \to \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}, \quad (x; P_1, \ldots, P_r) \mapsto ((x; P_{a_i}), (x; P_{b_i}), (x; P_{c_i}))$$

$$\phi_{BC} : \mathcal{E}^r \to \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}, \quad (x; P_1, \ldots, P_r) \mapsto ((x; P_{b_i}), (x; P_{c_i})).$$

Definition 49. *If $(k,l,m) = (2,l,l)$ for $l = r+2 > 2$, the generalized Gross-Kudla-Schoen is*

$$\Delta_{2,l,l} = (\mathrm{Id}, \epsilon_{r_2}, \epsilon_{r_3})(\phi_{ABC}(\mathcal{E}^r) - \{o\} \times \phi_{BC}(\mathcal{E}^r)) \in \mathrm{CH}^{r+2}(X \times \mathcal{E}^r \times \mathcal{E}^r).$$

Definition 50. *If $k,l,m > 2$ the generalized Gross-Kudla-Schoen cycle is*

$$\Delta_{k,l,m} = (\epsilon_{r_1}, \epsilon_{r_2}, \epsilon_{r_3})\phi_{ABC}(\mathcal{E}^r) \in \mathrm{CH}^{r+2}(W).$$

By examining the image of the cycle class map in each of the Künneth components of the complex of the de Rham cohomology group $H^{2r+4}_{\mathrm{dR}}(W/\mathbb{C})$ of the variety W, it follows that $\Delta_{k,l,m}$ is null-homologous, that is,

$$\Delta_{k,l,m} \in \mathrm{CH}^{r+2}(W)_0 := \ker(\mathrm{cl} : \mathrm{CH}^{r+2}(W) \to H^{2r+4}_{\mathrm{dR}}(W)).$$

We will be interested in the p-adic regulator

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W)_0 \to \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W)^*.$$

In particular, we want to provide two formulas for $\mathrm{AJ}_p(\Delta_{k,l,m})$, one in terms of Coleman integration and the other in terms of p-adic modular forms. For the first one, just recalling the previous definitons, we can derive some technical lemmas. Recall that $\mathcal{L}_r := \mathrm{sym}^r \mathcal{L}$, being $\mathcal{L} := \mathbb{R}^1\pi_*\Omega^\bullet_{\mathcal{E}/Y}$. Keep also the same notations concerning the polynomial P introduced in previous sections.

Lemma 19. *There exists a real $\epsilon > 0$ and an $\mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3}$-valued rigid one-form $\rho(P, \omega_2, \omega_3)$ on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$ such that*

$$\nabla\rho(P, \omega_2, \omega_3) = P(\Phi)(\omega_2 \wedge \omega_3).$$

Here, ∇ is the Gauss-Manin connection of the variety.

Write $\phi^*_{23} = \iota^*_{23} - \iota^*_2 - \iota^*_3$ for the pullbacks of the inclusions.

Lemma 20. *The map $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon) \to H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ induced by ϕ^*_{23} is the zero map.*

The map ϕ_{BC} combined with Poincaré duality on the fibers of $\mathcal{E}^r \to \mathcal{E}^{r_1}$ gives rise to a pullback on sheaves $\mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3} \to \mathcal{L}_A(-t)$, being $t := |A'| = r - r_1$. Then, we have a map

$$\phi^*_{A,BC} : \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon, \mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3}) \to \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_A(-t)).$$

Finally, let

$$\phi^* := \begin{cases} \phi^*_{23} & \text{if } (k,l,m) = (2,2,2), \\ \phi^*_{A,BC} & \text{otherwise.} \end{cases}$$

Lemma 21. *If $\sigma \in \Omega^1(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon, \mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3})$ is ∇-closed, then $\phi^*(\sigma)$ is ∇-exact on $\mathcal{W}_\epsilon$.*

Proposition 26. *The element $\xi(P, \omega_2, \omega_3)$, defined as the class of $\phi^*\rho(P, \omega_2, \omega_3)$ in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_A(-t))$ does not depend on the choice of the rigid differential $\rho(P, \omega_2, \omega_3)$ and has vanishing annular residues. In particular, it belongs to $H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$.*

The Frobenius Φ acts on $H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$ with eigenvalues of absolute value $\sqrt{p}^{1+r_2+r_3}$. Since the roots of P have absolute value either p^{r+1} or $\sqrt{p}^{2+r_2+r_3}$, the endomorphism $P(\Phi)$ acts invertibly on $H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$. In particular, for any $\eta \in H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$, the class $P(\Phi)^{-1}\eta$ is well-defined and

$$\xi(\omega_2, \omega_3) := P(\Phi)^{-1}\xi(P, \omega_2, \omega_3) \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_A(-t))$$

does not depend on P.

Consider $\iota_A : \mathcal{E}^A \to \mathcal{E}^{r_1}$, that gives rise to an isomorphism of sheaves between $\mathcal{L}_{r_1}$ and $\mathcal{L}_A$. With this in mind, we want to describe the restriction of $\mathrm{AJ}_p(\Delta_{k,l,m})$ to

$$H^1_{\mathrm{dR}}(X, \mathcal{L}_{r_1}) \otimes H^0(X, \omega^{r_2} \otimes \Omega^1_X) \otimes H^0(X, \omega^{r_3} \otimes \Omega^1_X) \subset H^{2r+3}_{\mathrm{dR}}(W).$$

Theorem 62. *Let $\eta \otimes \omega_2 \otimes \omega_3$ be any class in the previous space. Then,*

$$\mathrm{AJ}_p(\Delta_{k,l,m})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \iota_A^* \eta, \xi(\omega_2, \omega_3) \rangle,$$

where $\langle , \rangle$ is the pairing arising from Poincaré duality.

We will now use the common notations of g_α and g_β for the p-stabilizations, on which U_p acts with eigenvalues α_g, β_g respectively. Denote by η_f^{ur} the unique lift to the unit root subspace of the cohomology class in $H^1(X_{K_f}, \omega^{-r_1})$ attached to f. Set $e_{f^*,\mathrm{ord}} = e_{f^*} e_{\mathrm{ord}}$.

Theorem 63. *Suppose that $k = l = m = 2$ (in this case, $\xi(\omega_g, \omega_h)$ is just an overconvergent modular form of weight 2). Then,*

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h)) = -\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\mathrm{ord}}(d^{-1} g^{[p]} \times h).$$

For the general case, we write $k = l + m - 2 - 2t$, with $t \geq 0$, and c for the central point $c = (k + l + m - 2)/2$, so that $\xi(\omega_g, \omega_h)$ is a class in $H^1_{\mathrm{dR}}(X, \mathcal{L}_{r_1}(-t))$.

Theorem 64. *The projection $e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h))$ is represented by the classical modular form*

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h)) = -\frac{(-1)^t \cdot t! \cdot \mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\mathrm{ord}}(d^{-1-t} g^{[p]} \times h).$$

These results allows us to provide a formula for the image of $\Delta_{k,l,m}$ under the p-adic Abel-Jacobi map

$$\mathrm{AJ}_p(\Delta_{k,l,m})(\eta_f^{\mathrm{ur}}, \omega_g, \omega_h) = (-1)^{t+1} \frac{t! \cdot \mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \langle \eta_f^{\mathrm{ur}}, d^{-1-t} g^{[p]} \times h \rangle.$$

Once we have given the basis about these generalized Gross-Kudla-Schoen cycles, we would like to recover the results we have derived for Beilinson-Flach elements varying in families, that can be rephrased for these diagonal cycles, working now with a triple of forms (f, g, h).

The p-adic Abel-Jacobi map will have the feature of factoring through the restriction to $G_{\mathbb{Q}_p}$. This will allow us to consider, as before, distinguished global Galois cohomology classes $\kappa(f_x, g_y, h_z) \in H^1(\mathbb{Q}, V_{f_x g_y h_z}(N))$, where f_x, g_y and h_z are classical specializations of weights k, l and m which are balanced and $V_{f_x g_y h_z}(N)$ is the Kummer self-dual twist of the direct sum of several copies of the tensor product of the p-adic representations V_{f_x}, V_{g_y}, V_{h_z} of $G_{\mathbb{Q}}$ attached by Eichler-Shimura and Deligne to these forms, occuring in the middle cohomology of a Kuga-Sato variety. We will come back to a more precise definition later on.

Further, we will formulate a relationship between the Bloch-Kato p-adic logarithms of cohomology classes and the special values of Garrett-Hida p-adic L-functions at points outside the range of classical interpolation. Moreover, we want a relation between $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})(N) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}}(N)$ and Hida's p-adic L-functions.

Let us sketch how the definition of these Λ-adic classes goes. Let $f \in S_2(N_f)$ be the newform attached to an elliptic curve E, and let $\mathbf{g}$ and $\mathbf{h}$ be two Λ-adic newforms of tame level N_g and N_h, and tame character χ and χ^{-1} respectively. We assume that p does not divide the least common multiple of N_f, N_g and N_h.

The following paragraphs, that follow [DR2], are very dense in which concerns notation, so they can be skipped in a first reading. The objective is to establish the existence of a Λ-adic class $\kappa_\infty \in H^1(\mathbb{Q}, \mathbb{V}_\infty^{\mathrm{ord}})$.
Let as usual $X_s = X_1(Np^s)$, that can be seen as classifiying triples (A, i_N, i_p), where i_N and i_p are embeddings of finite group schemes. A key ingredient will be the curve $X_s^\dagger/\mathbb{Q}$, which arises as the moduli space associated to the problem of classifying triples (A, i_N, P), where (A, i_N) is as above and P is a point of exact order p^s. The curves X_s form projective systems in two different ways, relative to the collections of degeneracy maps

$$\pi_1, \pi_2 : X_{s+1} \to X_s, \qquad \begin{cases} \pi_1(A, i_N, i_p) = (A, i_N, p \cdot i_p), \\ \pi_2(A, i_N, i_p) = (A, i_N, i_p)/C, \quad C := i_p(\mu_p). \end{cases}$$

We will play also with $X_s^\flat$, the modular curve attached to $\Gamma_1(Np^s) \cap \Gamma_0(p^{s+1})$.
For an integer $s \geq 1$, define

$$W_{s,s} := X_0(Np) \times X_1(Np^s) \times X_1(Np^s) = X_0(Np) \times X_s \times X_s,$$

and following an analogous construction, $W_{s,s}^\flat$ and $W_{s,s}^\dagger$.
Denote by δ the natural diagonal embeddings of X_s and $X_s^\flat$ in the triple product of identical curves. Choosing a system $\{\zeta_s\}$ of compatible p^s-th roots of unity, set

$$\Delta_{s,s}^\flat := (j_1 \circ \pi_2^{s-1} \circ \omega_s, \mathrm{Id}, \omega_s)_* \delta_*(X_s) \in C^2(W_{s,s})(\mathbb{Q}(\zeta_s)),$$

where $\omega_s(A, i_N, i_p) = (A/C_p, i_N, i_p')$, with $C_p := \langle i_p(\zeta_s) \rangle$. Further, j_1 is the natural map from X_1 to $X_0(Np)$. It can be easily checked that

$$\pi_{22,*}(\Delta_{s+1,s+1}^\flat) = p \cdot (\mathrm{Id}, U_p, \mathrm{Id})_*(\Delta_{s,s}^\flat).$$

Let $\mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^2(W_s^\dagger)_0(\mathbb{Q}) \to H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{W}_s^\dagger, \mathbb{Z}_p)(2))$ be the étale regulator map (where $W_s^\dagger$ is a quotient of $W_{s,s}^\dagger$ by the action of a certain group D_s of diamond operators), and consider $\kappa_s^{(1)} := \mathrm{AJ}_{\mathrm{et}}(\Delta_s)$. Consider also the Künneth decomposition

$$H^d_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p) = \bigoplus_{i+j+k=d} H^i_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p) \otimes H^j_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p) \otimes H^k_{\mathrm{et}}(\bar{X}_s^\dagger, \mathbb{Z}_p),$$

and also the projection $\mathrm{pr}_s : W_{s,s}^\dagger \to W_s^\dagger$, that induces functorial maps between the corresponding étale cohomology groups

$$\mathrm{pr}_s^* : H^3_{\mathrm{et}}(\bar{W}_s^\dagger, \mathbb{Z}_p) \to H^3_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p)^{D_s}, \quad \mathrm{pr}_{s,*} : H^3_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p)_{D_s} \to H^3_{\mathrm{et}}(\bar{W}_s^\dagger, \mathbb{Z}_p).$$

We have a map $H^3_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p)$ onto the $(1,1,1)$-component, and also a $G_{\mathbb{Q}}$-equivariant map to V_s, call it pr_{111}. Consider $\kappa_s^{(3)} := \mathrm{pr}_{111}(\kappa_s^{(2)}) \in H^1(\mathbb{Q}, V_s)$, where $\kappa_s^{(2)}$ is defined in terms of some correspondence between the modular curves. Since for $s \geq 1$,

$$\pi_{22,*}(\kappa_{s+1}^{(3)}) = (1 \otimes U_p \otimes 1)(\kappa_s^{(3)}),$$

it makes sense to define

$$\kappa_\infty := \varprojlim \kappa_s \in H^1(\mathbb{Q}, \mathbb{V}_\infty^{\mathrm{ord}}).$$

Here, we have used the usual notations

$$\mathbb{V}_1^{\mathrm{ord}}(Np^\infty) := \varprojlim V_1^{\mathrm{ord}}(Np^s), \qquad \mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^\dagger := \varprojlim V_1^{\mathrm{ord}}(Np^s)^\dagger,$$

$$\mathbb{V}_\infty^{\mathrm{ord}} := \varprojlim V_s^{\mathrm{ord}}.$$

Finally, take the Λ-adic representation $\mathbb{V}_\infty^{\mathrm{ord}} := V_0(Np) \otimes (V_1^{\mathrm{ord}}(Np^\infty) \otimes_{\tilde{\Lambda}} \mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^\dagger)$ and also

$$\mathbb{V}_{f\mathbf{gh}} := V_f \otimes (\mathbb{V}_\mathbf{g} \otimes_\Lambda \mathbb{V}_\mathbf{h}^*), \quad \mathbb{V}_{f\mathbf{gh}}(N) := V_f(Np) \otimes (\mathbb{V}_\mathbf{g}(N) \otimes_\Lambda \mathbb{V}_\mathbf{h}^*(N)).$$

These are modules over $\Lambda_{f\mathbf{gh}} := \mathcal{O}_f \otimes (\Lambda_\mathbf{g} \otimes_\Lambda \Lambda_\mathbf{h})$. We can consider now the corresponding specialization morphisms. $V_{fg_yh_z}$ is a Galois representation of rank 8 over $\mathcal{O}$, and there are natural identifications

$$V_{fg_yh_z} = V_f \otimes V_{g_y} \otimes V_{h_z}^*, \quad V_{fg_yh_z}(N) := V_f(Np) \otimes V_{g_y}(Np^s) \otimes V_{h_z}^*(Np^s).$$

The canonical projections π_f, $\pi_\mathbf{g}$ and $\pi_\mathbf{h}^*$ associated to f, $\mathbf{g}$ and $\mathbf{h}$ give rise to a surjective Λ-module homomorphism

$$\pi_{f,\mathbf{g},\mathbf{h}} : \mathbb{V}_\infty^{\mathrm{ord}} \to \mathbb{V}_{f\mathbf{gh}}(N).$$

Definition 51. *The one-variable Λ-adic cohomology class attached to $(f, \mathbf{g}, \mathbf{h})$ is the class*

$$\kappa(f, \mathbf{gh}) := \pi_{f\mathbf{gh}}(\kappa_\infty) \in H^1(\mathbb{Q}, \mathbb{V}_{f\mathbf{gh}}(N)).$$

Our next aim will be to analyze a little the restrictions to $G_{\mathbb{Q}_p}$ of the Galois representations $\mathbb{V}_{f\mathbf{gh}}$ and of the Λ-adic cohomology class $\kappa(f, \mathbf{gh})$, with the aim of deriving some kind of reciprocity law and several properties about the local classes.

Lemma 22. *The Galois representation $\mathbb{V}_{f\mathbf{gh}}$ is endowed with a four-step filtration*

$$0 \subset \mathbb{V}_{f\mathbf{gh}}^{++} \subset \mathbb{V}_{f\mathbf{gh}}^{+} \subset \mathbb{V}_{f\mathbf{gh}}^{-} \subset \mathbb{V}_{f\mathbf{gh}}$$

by $G_{\mathbb{Q}_p}$-stable $\Lambda_{f\mathbf{gh}}$-submodules of rank $0, 1, 4, 7$ and 8. $G_{\mathbb{Q}_p}$ acts on the quotients as a direct sum of $\Lambda_{f\mathbf{gh}}$-adic characters.

In particular, let $\mathbb{V}_{f\mathbf{gh}}^f(N) \oplus \mathbb{V}_{f\mathbf{gh}}^g(N) \oplus \mathbb{V}_{f\mathbf{gh}}^h(N) := \mathbb{V}_{f\mathbf{gh}}^+(N)/\mathbb{V}_{f\mathbf{gh}}(N)^{++}$. Let

$$\kappa_f(f, \mathbf{gh}) := \mathrm{res}_p(\kappa(f, \mathbf{gh})) \in H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{gh}})$$

be the image of the global class in the local cohomology at p, and consider

$$\xi_{f\mathbf{gh}} = (1 - \alpha_f a_p(\mathbf{g}) a_p(h)^{-1} \chi^{-1}(p))(1 - \alpha_f a_p(\mathbf{g})^{-1} a_p(\mathbf{h}) \chi(p)).$$

Proposition 27. *The class $\xi_{f\mathbf{gh}} \cdot \kappa_p(f, \mathbf{gh})$ belongs to the natural image of $H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{gh}}^+(N))$ in $H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{gh}}(N))$.*

In this setting, it is convenient to replace the ring $\Lambda_{f\mathbf{gh}}$ and all modules over it with their localizations by the multiplicative set generated by $\xi_{f\mathbf{gh}}$. Since this element is non-zero at all classical points of weight $l \geq 1$, we can still specialize the Λ-adic class $\kappa(f, \mathbf{gh})$ at these points.

Corollary 6. *The class $\kappa_p(f, \mathbf{gh})$ belongs to the image of $H^1(\mathbb{Q}_p, \mathbb{V}^+_{f\mathbf{gh}}(N))$ in the module $H^1(\mathbb{Q}_p, \mathbb{V}_{f\mathbf{gh}}(N))$ under the map induced from the inclusion $\mathbb{V}^+_{f\mathbf{gh}}(N) \hookrightarrow \mathbb{V}_{f\mathbf{gh}}(N)$.*

Basically, we are interested in analyzing the specializations

$$\kappa(f, g_y, h_z) := \kappa(f, \mathbf{gh})_{(y,z)} \in H^1(\mathbb{Q}, V_{fg_yh_z}(N)),$$

where $(y, z) \in \Omega_g \times \Omega_h$ is a classical point of weight l and character ϵ.
We can consider an analogous four step filtration and observe that when $l = 2$, the classes $\kappa(f, g_y, h_z)$ are directly related to the étale Abel-Jacobi images of twisted diagonal cycles.

Proposition 28. *Assume that the classical point (y, z) have weight $l = 2$ and character ϵ of conductor p^s. Then,*

$$\kappa(f, g_y, h_z) = \alpha_{g_y}^{-s} \omega_{fg_yh_z}(\mathrm{AJ}_{\mathrm{et}}(\Delta_s)) \in H^1(\mathbb{Q}, V_{fg_yh_z}(N)).$$

Corollary 7. *For all classical points $(y, z) \in \Omega_g \times \Omega_h$ of weight two and character ϵ of conductor p^s, the class $\kappa_p(f, g_y, h_z)$ belongs to the image of $H^1(\mathbb{Q}_p(\zeta_s), V^+_{fg_yh_z}(N))$ in $H^1(\mathbb{Q}_p(\zeta_s), V_{fg_yh_z}(N))$ under the map induced from the inclusion.*

We finish the section with the case that interests us the most, the weight one setting. A priori, we have four global classes $\kappa(f, g_\alpha, h_\alpha)$, $\kappa(f, g_\alpha, h_\beta)$, $\kappa(f, g_\beta, h_\alpha)$ and $\kappa(f, g_\beta, h_\beta)$. For primes $l \neq p$, it is a known result that $H^1(\mathbb{Q}_l, V_{fgh}(N)) = 0$ and hence the restriction to $G_{\mathbb{Q}_l}$ of the above classes are all trivial. This is not the case for $l = p$.
The space $V_{gh}(N)$, that is four-dimensional, can be decomposed into four one-dimensional subspace according to the eigenvalues of the Frobenius. From previous results, it turns out that the local class $\kappa_p(f, g_\alpha, h_\alpha)$ belongs to the kernel of

$$H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}(N)) \to H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}^{\alpha\alpha}(N)).$$

More details about this will be sketched when discussing the applications to BSD conjecture and are available in [DR2]. In particular, the article mimics the study of [BDR2], analyzing the restriction of the classes to $\mathbb{Q}_p$ and its image under Perrin Riou's Λ-adic logarithm. There, the algebraic geometry involved is more subtle, and in particular one must take care of $\mathcal{X}_1(Np^s)$, the proper, flat, regular model of X_s over $\mathbb{Z}_p[\zeta_s]$. Its special fiber turns out to be the union of a finite number of reduced Igusa curves over $\mathbb{F}_p$, meeting at their supersingular points. In the following sections all these classes will appear in new formulas involving also the triple product L-functions.

6.6 p-adic L-functions and Euler systems

As we have anticipated, the main example of Euler system of Rankin-Selberg type will be concerned with a triple (f, g, h) of eigenforms of weights k, l, m respectively with $k - l - m = 2r$, $r \geq 0$. We define first

$$I(f, g, h) := \langle f, g \times \delta_m^r h \rangle$$

and we will relate the square of this quantity with the central critical value $L(f \otimes g \otimes h, \frac{k+l+m-2}{2})$ of the convolution L-function attached to f, g and h. We will see along the following sections how the quantity $I(f, g, h)$ can be p-adically interpolated as f, g, h vary over a set of classical specializations of Hida families. In fact, when f, g, h are of weight two (forcing r to tend to -1 in weight space), the p-adic limit of $I(f, g, h)$, denoted $I_p(f, g, h)$, can be interpreted as the Bloch-Kato p-adic logarithm of a global cohomology class arising from a geometric construction:

- If g, h are Eisenstein series, $I_p(f,g,h)$ is related with the p-adic regulator

$$\mathrm{reg}_p\{u_g, u_h\}(\eta_f),$$

where u_g, u_h are modular units with logartihmic derivatives equal to g and h respectively and η_f is the suitable class already defined in $H^1_{\mathrm{dR}}(X_1(N))$ attached to f.

- When only h is Eisenstein, $I_p(f,g,h)$ is related to the p-adic regulator of the form

$$\mathrm{reg}_p(\Delta_{u_h})(\eta_f \wedge \omega_g),$$

where Δ_{u_h} is a Beilinson-Flach element in $\mathrm{CH}^2(X_1(N)^2, 1)$ attached to u_h.

- When both g and h are cusps forms, $I_p(f,g,h)$ is related to

$$\mathrm{AJ}_p(\Delta)(\eta_f \wedge \omega_g \wedge_h),$$

where Δ is the Gross-Kudla-Schoen cycle.

We begin by exploring first the Kato case that already arised in the introduction when discussing Beilinson conjecture. Let $f \in S_2(N)$ be a cuspidal eigenform on $\Gamma_0(N)$ and let p be an odd prime not dividing N. Assume also that p is ordinary for f relative to a fixed embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. Consider $L_p(f,s)$, the MSD p-adic L-function attached to f. We plan to explain the connection between the value of $L_p(f,s)$ at $s=2$ and the image of the Beilinson-Kato elements by the p-adic syntomic regular on the K_2 of the modular curve of level N. We write Y for the open modular curve $Y_1(N)$ over $\mathbb{Q}$ and X for the canonical compactification. $\bar{Y}$ and $\bar{X}$ denote the extensions to $\bar{\mathbb{Q}}$. When F is a field of characteristic 0, $\mathrm{Eis}_2(\Gamma_1(N), F)$ will denote the F-vector space of weight 2 Eisenstein series on $\Gamma_1(N)$ with Fourier coefficients in F.
Let

$$\mathrm{dlog} : \mathcal{O}_{\bar{Y}}^{\times} \otimes F \to \mathrm{Eis}_2(\Gamma_1(N), F)$$

be the (surjective) homomorphism sending a modular unit u to the Eisenstein series $\frac{1}{2\pi i}\frac{u'(z)}{u(z)}$. Recall that we write, given $u_1, u_2 \in \mathcal{O}_{\bar{Y}}^{\times}$, $\{u_1, u_2\} \in K_2(\bar{Y}) \otimes \mathbb{Q}$ for the Steinberg symbol in the second K-group of $\bar{Y}$.

Let $\alpha_p(f)$ and $\beta_p(f)$ the unit and non unit root respectively of the Frobenius polynomial associated to f, and let $H^1_{\mathrm{dR}}(X)^{f,\mathrm{ur}}$ be the unit root subspace of the f isotypic part of $H^1_{\mathrm{dR}}(X)$, on which Frobenius acts as multiplication by $\alpha_p(f)$. Attach to f a canonical element η_f^{ur} of $H^1_{\mathrm{dR}}(X)$: first, let as usual $\eta_f^{\mathrm{ah}} := \langle f, f\rangle_{2,N}^{-1} \bar{f}(z) d\bar{z}$. This gives rise to a class in $H^1_{\mathrm{dR}}(X_{\mathbb{C}})$ whose natural image in $H^1(X_{\mathbb{C}}, \mathcal{O}_X)$ is in fact defined over $\bar{\mathbb{Q}}$. Using now the embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ we obtain a class $\eta_f \in H^1(X, \mathcal{O}_X)$ and a lift η_f^{ur} of η_f to $H^1_{\mathrm{dR}}(X)^{f,\mathrm{ur}}$.

Theorem 65. *The following equality holds:*

$$L_p(f,2) \cdot \frac{L(f,\chi,1)}{\Omega_f^+} = (2iN^{-2}\tau(\chi))(1-\beta_p(f)p^{-2})(1-\beta_p(f)) \cdot \mathrm{reg}_p\{u_{\chi^{-1}}, u_{\chi}\}(\eta_f^{\mathrm{ur}})$$

where Ω_f^+ is the real period attached to f.

The proof follows these three steps:

- The p-adic approximation of $L_p(f,2)$ by means of values in the range of classical interpolation of the Mazur-Kitagawa p-adic L-function.
- The decomposition of the Mazur-Kitagawa p-adic L-function as a factor of a p-adic Rankin L-series $L_p(\mathbf{f},\mathbf{E}_\chi)$ associated to the convolution of $\mathbf{f}$ and $\mathbf{E}_\chi$ interpolating in weight two f and $E_{2,\chi}$ respectively.
- The explicit evaluation of $L_p(\mathbf{f},\mathbf{E}_\chi)$ at the weights $(2,2)$, which yields an expression related to the p-adic regulator $\mathrm{reg}_p\{u_{\chi^{-1}},u_\chi\}(\eta_f^{\mathrm{ur}})$.

I would like to point out several interesting aspects of the proof.
Let $U_{\mathbf{f}}$ be the weight space attached to $\mathbf{f}$, and let $f_k\in S_k(N)$ be its ordinary p-stabilisation in weigh k, for k in the space of classical weights (in particular $f_2=f$). Let $L_p(f_k,\rho,s)$ be the Mazur-Swinertonn-Dyer p-adic L-function of [MMT] associated to f_k and to a Dirichlet character ρ, that here is 1 or χ. Thus, $L_p(f_k,\rho,s)$ interpolates the values $L_p(f_k\otimes\xi\rho,j)$ for $1\le j\le k-1$ and ξ in the set of Dirichlet characters of p-power conductor. As k varies, the p-adic L-functions $L_p(f_k,\rho,s)$ can be patched together to yield the Mazur-Kitagawa two variable p-adic L-function $L_p(\mathbf{f},\rho)(k,s)$ defined on $U_{\mathbf{f}}\times\mathbb{Z}_p$. For $k\in U_{\mathbf{f}}$, we have

$$L_p(\mathbf{f},\rho)(k,s)=\lambda(k)\cdot L_p(f_k,\rho,s),$$

where $\lambda(k)$ is a p-adic period equal to 1 at $k=2$ and non-vanishing in a neighborhood of $k=2$. Then, $L_p(f,2)$ is the p-adic limit as (k,l) tends to $(2,2)$ of the values $L_p(\mathbf{f},1)(k,k/2+l-1)$ occurring in the range of classical interpolation for $L_p(\mathbf{f},1)$.

Now, we should do a part of tedious computations using Shimura's formulae to derive the relation

$$L_p(\mathbf{f},\mathbf{E}_\chi)=\frac{1}{1-\beta_p(f)^2p^{-1}}\langle\eta_f^{\mathrm{ur}},e_{\mathrm{ord}}(d^{-1}E^{[p]}_{2,\chi^{-1}}\cdot E_{2,\chi})\rangle_Y,$$

which together with

$$L(f_k\otimes E_{l,\chi},k/2+l-1)=L(f_k,k/2+l-1)\cdot L(f_k,\chi,k/2),$$

yields

$$L_p(\mathbf{f},\mathbf{E}_\chi)(k,l)=\eta(k)\cdot L_p(\mathbf{f},1)(k,k/2+l-1)\cdot L_p(\mathbf{f},\chi)(k,k/2),$$

where $\eta(k)$ is a p-adic analytic function whose exact value at 2 can be explicitly found.

Take now F a p-adic field containing the values of χ, and let $\delta_{\chi^\pm}$ be the image of $u_{\chi^\pm}$ in $H^1_{\mathrm{et}}(Y,F(1))$ arising from Kummer theory. The p-adic regulator can be seen as

$$\mathrm{reg}_{\mathrm{et}}\{u_{\chi^{-1}},u_\chi\}:=\delta_{\chi^{-1}}\cup\delta_\chi\in H^2_{\mathrm{et}}(Y,F(2)).$$

Considering now the isomorphism given by the inverse of the Bloch-Kato exponential composed with the comparison theorem between étale and de Rham cohomology

$$\log_{Y,2}:H^1(\mathbb{Q}_p,H^1_{\mathrm{et}}(\bar{Y},F(2)))\to D_{\mathrm{dR}}(H_{\mathrm{et}}(\bar{Y},F(2)))=H^1_{\mathrm{dR}}(Y/F),$$

we have that $\log_{Y,2}(\mathrm{reg}_{\mathrm{et},p}\{u_{\chi^{-1}},u_\chi\})=\mathrm{reg}_p\{u_{\chi^{-1}},u_\chi\}$ and the previous theorem can also be seen as a relation between $L_p(f,2)$ and the Bloch-Kato logarithm of the étale regulator.

We want to establish now a link between Beilinson-Flach elements and Hida's three variable L-function, following the notations we have introduced at the end of Chapter 3. As before, we can provide a formula establishing a link between the p-adic regulator of the Beilinson-Flach element and the special values of this Hida L-function. More references about the proof and possible extensions of the result can be found in either [BDR2] or [BCDDPR].

Theorem 66. *We have the following formula:*

$$\mathscr{L}_p{}^f(f \otimes g, 2) = \frac{\mathcal{E}(f,g,2)}{\mathcal{E}(f)\mathcal{E}^*(f)} \operatorname{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}),$$

where

$$\mathcal{E}(f,g,2) = (1-\beta_p(f)\alpha_p(g)p^{-2})(1-\beta_p(f)\beta_p(g)p^{-2})$$
$$\times(1-\beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1-\beta_p(f)\beta_p(g)\chi(p)p^{-1}),$$
$$\mathcal{E}(f) = 1-\beta_p(f)^2\chi_f^{-1}(p)p^{-2},$$
$$\mathcal{E}^*(f) = 1-\beta_p(f)^2\chi_f^{-1}(p)p^{-1}.$$

The proof of the theorem is more or less the same than the one of the previous section, with the Hida family $\mathbf{E}_\chi$ replaced by $\mathbf{g}$. We should use Shimura's generalization of the previous formula for the critical values of $L(f_k \otimes g_l, s)$ and interpret the right side of this formula in terms of the Poincaré pairing on algebraic de Rham cohomology.

We finally move to the setting of diagonal cycles. For that, we analyze what happens in the setting in which $f, g, h \in S_2(N)$ is a triple of normalized cuspidal eigenforms of weight 2, level N and nebentypus χ_f, χ_g, χ_h in such a way that $\chi_f\chi_g\chi_h$ is the trivial character and then $V_{f,g,h} := V_f \times V_g \times V_h$ is self-dual and the Garret-Rankin L-function $L(f,g,h,s)$ of $V_{f,g,h}$ satisfies a functional equation relating the values s and $4-s$. We omit many details and refer the reader to [DR1] and [BCDDPR].
Let $p \nmid N$ and fix an embedding $\mathbb{Q}_{f,g,h} \hookrightarrow \mathbb{C}_p$ for which the three newforms are ordinary, and let $\mathbf{f} : \Omega_f \to \mathbb{C}_p[[q]]$ be the Hida family of overconvergent p-adic modular forms through f. Single out one of the eigneforms, say f. Consider the Garrett-Rankin triple product L-function introduced in chapter three

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}) : \Omega_f \times \Omega_g \times \Omega_h \to \mathbb{C}_p.$$

This p-adic L-function interpolates the square-root of the central critical value of the complex L-function $L(f_x, g_y, h_z, s)$ as (x,y,z) ranges over $\Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$, where $\kappa(x) \geq \kappa(y) + \kappa(z)$.

In this setting, we want to describe $\mathscr{L}_p{}^f(f,g,h)(2,2,2)$ as the image of a certain cycle on the cube of the modular curve $X = X_1(N)/\mathbb{Q}$ under the p-adic syntomic Abel-Jacobi map. The cycle in question is essetialy the diagonal X_{123} in X^3, but modified to make it null-homologous, that is,

$$X_{123} - X_{12} - X_{23} - X_{31} + X_1 + X_2 + X_3.$$

The map

$$\mathrm{AJ}_{\mathrm{syn},p}(\Delta) : \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3) \to \mathbb{Q}_p$$

can be described purely in terms of Coleman integration.

Theorem 67. *For each $\phi \in \{f, g, h\}$ let $\omega_\phi \in \Omega^1(X)$ be the regular 1-form associated to ϕ, and $\eta_f^{\mathrm{ur}} \in H^1_{\mathrm{dR}}(X)^{f,\mathrm{ur}}$ be the unique clas in the unit root subspace of the f-isotypical component of $H^1_{\mathrm{dR}}(X)$ such that $\langle \omega_f, \eta_f^{\mathrm{ur}} \rangle = 1$. Let also $\alpha_p(\phi), \beta_p(\phi)$ be the two roots of the Hecke polynomial $x^2 - a_p(\phi)x + p$ labelled in such a way that $\alpha_p(\phi)$ is a p-adic unit. Then, the following inequality holds:*

$$\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \frac{\mathcal{E}(f, g, h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \mathrm{AJ}_p(\Delta)(\eta_f^{\mathrm{ur}} \times \omega_g \times \omega_h).$$

6.7 Applications to BSD conjecture

The aim of this section is to say a few words about how all these results can be used to derive several cases of the BSD conjecture. In particular, we will focus on the approach followed both in [BDR2] and [DR2], that are the settings we have studied deeper. The approach in the Beilinson-Kato case was the construction of an element $\kappa_{f,\infty} \in H^1(\mathbb{Q}, \Lambda_{\mathrm{cyc}} \otimes V_f(2))$ such that

$$\kappa_{f,\xi}(k) := \nu_{k,\xi}(\kappa_{f,\infty}) \in H^1(\mathbb{Q}, \mathbb{Q}_{p,\xi} \otimes V_f(1+k)(\xi^{-1})).$$

Kato's explicit reciprocity law describes $L_p(f, \xi, 1)$ in terms of $\exp^*_{0,\xi}(\kappa_{f,\xi}(0))$ and allows us to prove that this is non-zero if and only if $L_p(f, \xi, 1) \neq 0$ and again if and only if $L(f, \xi, 1) \neq 0$. When combined with Kolivagin's theory, implies that the non-vanishing of $L(f, \xi, 1)$ implies that $\mathrm{Hom}(\mathbb{C}(\xi), E(\bar{\mathbb{Q}} \otimes \mathbb{C}) = 0$.

Our plan is to sketch the particular characteristics of the Beilinson-Flach elements arising in [BDR2], and how the regulator maps gives the Λ-adic classes that are used for the proof of a particular case of BSD conjecture. Recall that

$$E(H)^\rho_L := \mathrm{Hom}_{G_\mathbb{Q}}(V_\rho, E(H) \otimes L) = (E(H) \otimes \rho^*)^{\mathrm{Gal}(H/\mathbb{Q})}.$$

The main results concerning BSD in that article are the following:

Theorem 68. *Let E be an elliptic curve over $\mathbb{Q}$ and let ρ be an odd, irreducible, two-dimensional Artin representation. Assume that the conductors of E and ρ are prime to each other. Then, if $L(E, \rho, 1) \neq 0$, $E(H)^\rho_L$ is trivial.*

In addition to Artin representations with projective image isomorphic to A_4, S_4 and A_5, the previous theorem also applies to a large class of two-dimensional representations induced from general ray class characters of quadratic fields:

Theorem 69. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N, let K be a quadratic field of discriminant D and let $\psi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$ be a ray class character of conductor $\mathfrak{f}$. Assume that $\gcd(N, D \cdot \mathbb{N}(\mathfrak{f})) = 1$ and that ψ is of mixed signature if K is real quadratic. Then, if $L(E/K, \psi, 1) \neq 0$, $E(H)^\psi = 0$.*

The strategy of the proof is the following:

1. Embed $E(H)^\rho_L$ in an appropriate Selmer group attached to the choice of a rational prime p (that we assume prime to $N_f N_g$, and ordinary for f). Then, let $V_f := V_p(E)$ and let V_g be the Artin representation attached to the modular form g. Then, the connecting homomorphism of Kummer theory composed with the inverse of the restriction map from $\mathbb{Q}$ to H gives a linear injection of L_p-vector spaces
$$E(H)^\rho_{L_p} := E(H)^\rho_L \otimes_L L_p \hookrightarrow H^1(\mathbb{Q}, V_f \otimes V_g^*).$$

2. In particular, for each place v of $\mathbb{Q}$, the corresponding map on local points yields an injection $\delta_v : E(H_v)^{\rho}_{L_p} \hookrightarrow H^1(\mathbb{Q}_v, V_f \otimes V_g^*)$. The image of δ_v is which we have called $H^1_{\text{fin}}(\mathbb{Q}_v, V_f \otimes V_g^*)$. We can define in the same way $H^1_{\text{fin}}(\mathbb{Q}_v, V_f \otimes V_g)$. In this case, the condition $p \nmid N_f N_g$ means that the Dieudonné module of V has no ϕ-invariant vectors and thus $\delta_p(E(H_p)^{\rho}_{L_p})$ agrees with the exponential, the finite and the geometric representations.

3. Writing $\text{res}_p : H^1(\mathbb{Q}, V_f \otimes V_g) \to H^1_{\text{sing}}(\mathbb{Q}_p, V_f \otimes V_g)$ for the natural projection attached to $V_f \otimes V_{g,\sigma}$, we have that if this map is surjective (as a map of L_p-vector spaces) for all $\sigma : L \to L_p$, then $E(H)^{\rho}_L = 0$.

4. Then, the key is constructing two independent classes in $H^1(\mathbb{Q}, V_f \otimes V_g)$ whose images generate the singular quotient $H^1_{\text{sing}}(\mathbb{Q}_p, V_f \otimes V_g)$. This follows the lines of [BDR2] of constructing global cohomology classes. These ideas will appear again in the last chapter.

We will now make some brief comments about the new results that are proved in [DR2]. The first main theorem is the following:

Theorem 70. *If $L(E, \rho, 1) \neq 0$, then $E(H)^{\rho} = 0$.*

For any ring class character ψ of a quadratic field K (of conductor relatively prime to N_f), if H/K is the ring class field cut out by it, consider

$$E(H)^{\psi} = \{P \in E(H) \otimes L \text{ such that } P^{\sigma} = \psi(\sigma)P \text{ for all } \sigma \in \text{Gal}(H/K)\}.$$

When K is a real quadratic field, we must impose some extra-hypothesis. In this setting, the theorem implies that if $L(E/K, \psi, 1) \neq 0$, then $E(H)^{\psi} = 0$.

The second important result of the article is concerned with the case where $L(E, \rho, s)$ vanishes at $s = 1$ (and hence to order at least 2). Fix p odd not dividing $N := \text{lcm}(N_f, N_g, N_h)$ at which f is not Eisenstein and fix an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Let Frob_p be the Frobenius element at p induced by the embedding, and let L_p be the completion of L in $\bar{\mathbb{Q}}_p$. We assume that $\alpha_g \neq \beta_g$ and $\alpha_h \neq \beta_h$, where as usual (α_g, β_g) and (α_h, β_h) are the pairs of eigenvalues of $\rho_g(\text{Frob}_p)$ and $\rho_h(\text{Frob}_p)$. We also assume that E is ordinary at p.

Let $\text{Sel}_p(E, \rho) := H^1_{\text{fin}}(\mathbb{Q}, V_p(E) \otimes V_\rho \otimes_L L_p)$ denote the ρ-isotypic component of the Bloch-Kato Selmer group of E/H.

Theorem 71. *If $L(E, \rho, 1) = 0$ and ${\mathscr{L}_p}^{g_\alpha}(\tilde{f}, \tilde{g}^*, \tilde{h}) \neq 0$ for some choice of test vectors, then*

$$\dim_{L_p} \text{Sel}_p(E, \rho) \geq 2.$$

In fact, two of the Kato classes we have constructed will be shown to be linearly independent.

The proofs of these theorems rest precisely on the system of global cohomology classes for the Rankin convolution of three modular forms of weights $(2, 1, 1)$ we have previously constructed. These classes arise from generalized Gross-Kudla-Schoen cycles in the product of three Kuga-Sato varieties fibered over a classical modular curve and their variation in Hida families. As we know, the extensions of p-adic Galois representations associated to these classes arise from geometry, since they are realised in the p-adic étale cohomology of an open subvariety of the product of Kuga-Sato varieties with good reduction at p, and in particular their restrictions to a decomposition group at p are crystalline. Hence, what we have is a direct relationship between the Bloch-Kato p-adic logarithms of the class $\kappa(f_x, g_y, h_z)$ and the special values of Garrett-Hida p-adic L-functions at points outside their range of classical interpolation. The aim is to p-adically interpolate $\kappa(f_x, g_y, h_z)$ as the triple (f_x, g_y, h_z) is made to vary over the classical, balanced specializations of ordinary Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$. A p-adic interpolation like that is described by the global cohomology class already studied

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, V_{\mathbf{fgh}}(N)).$$

The first idea then is to attach a one-variable cohomology class to a triple $(f, \mathbf{g}, \mathbf{h})$ formed by an eigenform f of weight two and trivial nebentype and two Hida families of tame characters χ and χ^{-1}. The class $\kappa(f, \mathbf{gh}) \in H^1(\mathbb{Q}, \mathbb{V}_{f,\mathbf{gh}}(N))$ gives rise to a collection of global classes $H^1(\mathbb{Q}, V_{fg_yh_z}(N))$ varying p-adic analytically as (g_h, h_z) varies over pairs of specializations of $\mathbf{g}$ and $\mathbf{h}$ with **common weight an nebentype character at** p.
That way, we obtain four relevant cohomology classes

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta).$$

One of the main ingredients toward the proof of the main theorems of the article is:

Theorem 72. *The generalized Kato class $\kappa(f, g_\alpha, h_\alpha)$ is crystalline at p if and only if $L(E, \rho, 1) = 0$.*

When $L(E, \rho, 1) = 0$ and hence the analytic rank is ≥ 2, the four classes are in $\mathrm{Sel}_p(E, \rho)$. The submodule generated by these four classes is expected to be non-trivial when this analytic rank is 2, i.e., when $L''(E, \rho, 1) \neq 0$.

Theorem 73. *If $L(E, \rho, 1) = 0$ and $\mathscr{L}_p{}^{g_\alpha}(\tilde{f}, \tilde{g}^*, \tilde{h}) \neq 0$ for some choice of test vectors, there exist $G_{\mathbb{Q}}$-equivariant projections $j_\alpha, j_\beta : V_{fgh}(N) \to V_p(E) \otimes V_\rho$ such that the classes $\kappa_{\alpha\alpha} = j_\alpha(\kappa(f, g_\alpha, h_\alpha))$ and $\kappa_{\alpha\beta} = j_\beta(\kappa(f, g_\alpha, h_\beta))$ are linearly independent in $\mathrm{Sel}_p(E, \rho)$.*

There are many other applications to BSD that passes through the theory of Euler systems. Recently, Venerucci gives new insight into the exceptional zero conjecture using similar tecniques. Given an elliptic curve E, let $\mathbf{f}$ be the Hida family whose weight two specialization is the modular form f attached to E via modularity. Then, by results of Kato and Ochiai, there is a two-variable Euler system $\kappa(\mathbf{f}) \in H^1(\mathbb{Q}, V_{\mathbf{f}} \otimes \Lambda)$ such that $\mathcal{L}(\kappa(\mathbf{f})) = *L_p(\mathbf{f}, k, s)$. In particular, for $k \geq 2$ and $1 \leq j \leq k-1$, it interpolates the dual exponential map and for $1 \leq j \leq k-1$, the logarithm. In the case that $k = 2$ and $j = 1$, we write $\Delta := \kappa(f)(2, 1) \in H^1(\mathbb{Q}, V_p(E))$. It lies in $H^1_{\mathrm{fin}}(\mathbb{Q}, V_p(E))$ and hence in the kernel of $\exp^*$. Further, inside the H^1_{fin} we have (via de Kummer map), a copy of $E(\mathbb{Q}) \otimes \mathbb{Q}_p$, that in the rank-one setting is an isomorphism (finiteness of the Shafarevich group). Then, $\Delta = \mathbb{Q}_p \cdot P$, where P is a Heegner point; Venerucci uses [BD1] to prove that $\mathbb{Q}_p = \log(P)$ and from here he shows that $L_p(E, s)$ has order of vanishing at less 2 at $s = 1$.

6.8 Rankin-Eisenstein classes and explicit reciprocity laws

One of the aims of this thesis is to recall the construction of canonical cohomology classes in $H^1(\mathbb{Q}, V_{gh}(1))$, since these classes will provide theoretical evidence for the main conjecture of [DLR2]. Roughly speaking, the idea consists on applying a étale regulator map to distinguished elements in the higher Chow group of a surface and let these elements vary in families, to obtain a compatible collection of cohomology classes that can be encoded as a global class κ in $H^1(\mathbb{Q}, \mathbb{V}(1))$.
We begin by clarifying the meaning of $V_{gh} = V_g \otimes V_h$.
If $\mathbf{g} = \{g_k\}$ is a cuspidal Hida family, the Galois representation for ρ_{g_k} when $k \in \mathbb{Z}^{\geq 1}$ is given by:

- If $k = 2$, $V_p = \lim_{\leftarrow} \mathrm{Jac}(X)(\bar{\mathbb{Q}})[p^r] = H^1_{\mathrm{et}}(\bar{X}_1(N), \mathbb{Z}_p)(1)$, and then
$$V_{g_2} = \cap_l \ker(T_l - a_l : V_p \to V_p).$$
- If $k \geq 2$, we must consider $\mathcal{E}$, the universal elliptic curve over X and in particular $\mathcal{E}^{k-2} = \{(A, x_1, \ldots, x_{k-2})\}$. Then, we take the endomorphism
$$T_l \in \mathrm{End}\ H^{k-1}_{\mathrm{et}}(\bar{\mathcal{E}}^{k-2}, \mathbb{Z}_p(k-1)),$$
where T_l is the usual Hecke operator, and define
$$V_{g_k} = \cap_l \ker(T_l - a_l).$$
- For $k = 1$, we recall that by a result of Hida and Wiles, there exists a finite extension $\Lambda_{\mathbf{g}}$ of the Iwasawa algebra and a map $\mathbb{V}_{\mathbf{g}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\mathbb{V}_{\mathbf{g}}) = \mathrm{GL}_2(\Lambda_{\mathbf{g}})$ such that $\nu_k(\rho_{\mathbf{g}}) = V_{g_k}$ whenever this expression makes sense ($k \geq 2$). Hence, for $k = 1$, we define
$$V_{g_1} = \nu_1(\mathbb{V}_{\mathbf{g}}),$$
that can be understood as the limit when V_{g_k} tends to 1 in the weight space.

In the case that we have two Hida families of cuspidal modular forms, we perform exactly the same game considering now the tensor product of the two Hida families, say $\mathbf{g} \otimes \mathbf{h}$. Hence, it makes sense to talk about a map $\mu_{g_l} \otimes \nu_{h_m}$.

The aim now would be to construct classes in $H^1(\mathbb{Q}, V_{gh}(1))$, where g and h are of weight one and V_{gh} is defined as we have just explained; the 1 makes reference to the twist by the cyclotomic character.

The main source for this is the work of Kings, Loeffler and Zerbes [KLZ]. There, they study what they call the étale Rankin-Eisenstein class, $\mathrm{Eis}^{[k,k',j]}_{\mathrm{et},1,N}$, defined as the image of the motivic Rankin-Eisenstein class $\mathrm{Eis}^{[k,k',j]}_{\mathrm{mot},1,N}$ in étale cohomology. For eigenforms g, h of weights $k+2$ and $k'+2$ and levels dividing N, they project this étale Rankin-Eisenstein class into the (g,h)-isotypical component, obtaining a class with values in the first cohomology group (they call it $H^1_{\mathrm{et}}(\mathbb{Z}[1/Np], M_{L_{\mathfrak{P}}}(g \otimes h)^*(-j)))$, that is just what interests us. Their aim is to interpolate $\mathrm{Eis}^{[g,h,j]}_{\mathrm{et}}$ in all three variables, changing g and h by Hida families (and the twist j by the universal character of the cyclotomic Iwasawa algebra). That way, we obtain a family of classes $\kappa(g_l, h_m) \in H^1_{\mathrm{et}}(G_{\mathbb{Q}}, V_{g_l h_m})$, where $V_{g_l h_m} = V_{g_l} \otimes V_{h_m}$ corresponds to the construction we have explained before. The following result, that directly follows from [KLZ, Theorem 8.1.3] (what they call Theorem A), summarizes what will be our starting point.

Theorem 74. *Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra and $\Lambda_{\mathbf{gh}}$ the extension of Λ that corresponds to consider Hida families interpolating g and h. Then:*

1. *There exists a $\Lambda_{\mathbf{gh}} \times \Lambda_{\mathbf{gh}}$-module $\mathbb{V}$ such that $\mu_l \otimes \nu_m(\mathbb{V}) = V_{g_l h_m}$ for all $l, m \geq 1$, where μ_l and ν_m are the specialization maps for the Hida families attached to g and h.*

2. *There exists $\kappa(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}(1))$, satisfying the following interpolation property: $\kappa(g_l, h_m) := \mu_l \otimes \nu_m(\kappa(\mathbf{g}, \mathbf{h})) \in H^1_{\mathrm{fin}}(\mathbb{Q}, V_{lm}(1))$ for all $l, m \geq 2$.*

Definition 52. *$\kappa(g_\alpha, h_\alpha)$ is defined as $\kappa(g_\alpha, h_\alpha) := \mu_1 \otimes \nu_1(\kappa(\mathbf{g}, \mathbf{h})) \in H^1(\mathbb{Q}, V_{gh}(1))$, where $\kappa(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}(1))$ is the global class of the previous theorem.*

However, the main interest of this construction lies in the connection of these classes with p-adic L-functions. To properly state the results we will use later, we need to recall Perrin-Riou's big logarithm. First of all, with the standard notations in p-adic Hodge theory (see for instance [BDR2]), recall that the Bloch-Kato logarithm is an isomorphism between

$$\log_{\mathrm{BK}} : H^1_e(\mathbb{Q}_p, V) \to \frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V) + D_{\mathrm{crys}}^{\phi=1}(V)},$$

where V is any representation of $G_{\mathbb{Q}_p}$. The inverse of this map is called the Bloch-Kato exponential map. Dualizing this last application, we obtain the so-called dual exponential map

$$\exp^*_{\mathrm{BK}} : H^1_e(\mathbb{Q}_p, V^*) \to \mathrm{Fil}^0 D_{\mathrm{dR}}(V^*),$$

and since by [Bel] we know that $H^1_e(\mathbb{Q}_p, V^*) \simeq H^1(\mathbb{Q}_p, V)/H^1_g(\mathbb{Q}_p, V)$ we can directly see the dual exponential as an application from $H^1(\mathbb{Q}_p, V)$ to $\mathrm{Fil}^0 D_{\mathrm{dR}}(V^*)$ with kernel $H^1_g(\mathbb{Q}_p, V)$.

From now on, we will use the standard notation $D(M) = \left(M \hat{\otimes} \widehat{\mathbb{Q}}_p^{\mathrm{ur}}\right)^{G_{\mathbb{Q}_p}}$. To state the results of [KLZ], where we work with modules over $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, we must understand both the integers and the Dirichlet characters of p-power conductor as subsets of the characters of $\mathbb{Z}_p^\times$, that will be written additively. We will work over the ring $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda$, and we will write $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{j}$ for the canonical characters of each factor ($\mathbf{k}$ and $\mathbf{k}'$ for weights and $\mathbf{j}$ for cyclotomic twists). The two first factors will be referred as Λ_D (D for diamond operators) and the last one by Λ_Γ, where Γ represents the cyclotomic Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. Finally, we put $\mathbb{V}_{\mathbf{f}}^-$ for the unramified $G_{\mathbb{Q}_p}$-module quotient of $\mathbb{V}_{\mathbf{f}}$. On the other hand, $\mathbb{V}_{\mathbf{f}}^+$ is the flat, locally free, rank one $\Lambda_{\mathbf{f}}[G_{\mathbb{Q}_p}]$-module sitting in the short exact sequence

$$0 \to \mathbb{V}_{\mathbf{g}}^+ \to \mathbb{V}_{\mathbf{g}} \to \mathbb{V}_{\mathbf{g}}^- \to 0.$$

Further, $\mathbb{V}_{\mathbf{gh}}^{-+} := \mathbb{V}_{\mathbf{g}}^- \otimes \mathbb{V}_{\mathbf{h}}^+$.

Theorem 75. *Let $D(\mathbb{V}_{\mathbf{gh}}^{-+})$ be the module $D(\mathbb{V}_{\mathbf{gh}}^{-+}(-1-\mathbf{k}'))$, equipped with the non-trivial action of Γ given by the character $1 + \mathbf{k}'$. Consider also $\mathbf{g}$ and $\mathbf{h}$, two Hida families interpolating weight one cuspidal modular forms. Then, there exists a Λ-linear map*

$$\mathcal{L} : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}}^{-+} \hat{\otimes} \Lambda_\Gamma(-j)) \to D(\mathbb{V}_{\mathbf{gh}}^{-+}) \hat{\otimes} \Lambda_\Gamma$$

with the following property: for all classical especializations f, g of $\mathbf{f}, \mathbf{g}$ and all characters of Γ of the form $\tau = j + \eta$ with η of finite order and $j \in \mathbb{Z}$, and for $\mathcal{Z} \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}}^{-+} \hat{\otimes} \Lambda_\Gamma(-j))$:

1. *when* $j \leq k'$, $\tilde{\nu}_\tau(\mathcal{L}(\mathcal{Z})) = * \log(\nu_\tau(\mathcal{Z}))$,

2. *when* $j > k'$, $\tilde{\nu}_\tau(\mathcal{L}(\mathcal{Z})) = * \exp^*(\nu_\tau(\mathcal{Z}))$,

where we have used the following notations:

- $\nu_\tau : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}}^{-+} \hat{\otimes} \Lambda_\Gamma(-j)) \to H^1(\mathbb{Q}_p, V_{gh}^{-+}(-j-\eta))$ *for the classical specialization.*

- $\tilde{\nu}_\tau : D(\mathbb{V}_{\mathbf{gh}}^{-+}) \hat{\otimes} \Lambda_\Gamma \to D_{\mathrm{crys}}(V_{gh})(-\epsilon_{g,p})$, *where* $\epsilon_{g,p}$ *is the* p*-part of the nebentypus of* g.

- $*$ *refers to explicit fudge factors.*

Via the results of [KLZ], we can map $\kappa(\mathbf{g}, \mathbf{h})$ into $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}}^{-+} \hat{\otimes} \Lambda_\Gamma(-\mathbf{j}))$ and hence consider $\mathcal{L}(\kappa(\mathbf{g}, \mathbf{h}))$.

This map we have just defined will be closely related with p-adic L-functions, but before stating our main result we need to establish the possibility of interpolating the differentials ω_g and η_g in families:

Proposition 29. *Let* $\mathbf{f}$ *be a Hida family of tame level* N*. Then, there is a canonical isomorphism of* $\Lambda_{\mathbf{f}}$*-modules*

$$\omega_{\mathbf{f}} : D(\mathbb{V}_{\mathbf{f}}(1-k-\epsilon_{\mathbf{f}})) \to \Lambda_{\mathbf{f}}^{\mathrm{cusp}},$$

where $\Lambda_{\mathbf{f}}^{\mathrm{cusp}}$ *is the quotient of* $\Lambda_{\mathbf{f}}$ *acting faithfully on cuspidal* Λ*-adic modular forms, such that for every cuspidal specialization* f_α *of weight* $k+2 \geq 2$ *and level* Np^r $(r \geq 1)$ *the map obtained by specializing* $\omega_{\mathbf{f}}$ *coincides with that given by pairing with* ω_{f_α}*.*
In the same way, there is a morphism of $\Lambda_{\mathbf{f}}$*-modules and a fractional* $\tilde{\Lambda}_{\mathbf{f}}$*-ideal* $I_{\mathbf{f}}$ *such that*

$$\eta_{\mathbf{f}} : D(\mathbb{V}_{\mathbf{f}}^-) \otimes_{\Lambda_{\mathbf{f}}} \tilde{\Lambda}_{\mathbf{f}} \to I_{\mathbf{f}}$$

interpolates the pairing with the class η_{f_α} *provided that* f_α *is new at* p*.*

The content of the following result can be roughly synthesized under the following sentence: "the image of $\kappa(g, h)$ under Perrin-Riou's big logarithm is Hida's p-adic Rankin-Selberg L-function".

Theorem 76. *Let* $\mathbf{g}$ *and* $\mathbf{h}$ *be two Hida families corresponding to the interpolation of two cuspidal weight-one modular forms. Then,*

$$\langle \mathcal{L}(\kappa(\mathbf{g}, \mathbf{h})), \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle = * L_p(\mathbf{g}, \mathbf{h}, 1+\mathbf{j}).$$

Observe the similitudes of this work with that of [DR2] or [BDR2]: in the former, the interpolation was over the classes $\kappa(f, g_k, h_k)$ (just moving one parameter), since the key was the construction of diagonal cycles in the product of the Kuga-Sato variety $X_0(Np) \times X_1(Np^s) \times X_1(Np^s)$; in the latter, the interpolation was done over $\kappa(g, h_k)$, moving again one parameter. It would be desirable to mimic these constructions in our case and adapt the proof for deriving the results established [KLZ] by different methods.

The construction we have explained will be used later to define four cohomology classes attached to the p-stabilizations of g and h.

7 Stark points and units

In chapter 4 we have done a rough presentation of the different Stark conjectures, and motivated the study of the so-called Elliptic Stark conjecture. The aim of this chapter is to state that conjecture both in the setting of points in elliptic curves and of units in number fields; this will lead us to the introduction of the p-adic iterated integrals, that are connected with p-adic L-functions, as we will see.

7.1 Stark points and p-adic iterated integrals

Recall that when E is an elliptic curve over $\mathbb{Q}$ with Galois representation $V_p(E) := H^1_{\text{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(1))$ and $\rho : G_{\mathbb{Q}} \to \text{Aut}(V_\rho) \simeq \text{GL}_n(L)$, with $\dim_L(V_\rho) = n \geq 1$ is an Artin representation with coefficients in some finite extension L, the L-series $L(E, \rho, s)$ is the L function of the compatible system $V_p(E) \otimes V_\rho$ of p-adic representations of $G_{\mathbb{Q}}$. Defining

$$r_{\text{an}}(E, \rho) := \text{ord}_{s=1} L(E, \rho, s),$$

$$r(E, \rho) := \dim_L \text{Hom}_{G_{\bar{\mathbb{Q}}}}(V_\rho, E(H) \otimes L),$$

the equivariant version of BSD states that $r_{\text{an}}(E, \rho) = r(E, \rho)$. This must be our main source of inspiration throughout the following part.

From now on, we will assume that ρ is an irreducible component of $\rho_\flat \otimes \rho_\sharp$, where both are odd, two-dimensional Artin representations that are self-dual,

$$\chi := \det(\rho_\flat)^{-1} = \det(\rho_\sharp)$$

and that $r_{\text{an}}(E, \rho) = 1$ or 2. The objective is then to relate global points in $E(H)^\rho_L$ to p-adic iterated integrals. By modularity results, we can attach a normalized weight two newform $f \in S_2(N_f)_{\mathbb{Q}}$ to E and weight one newforms $g \in M_1(N_g, \chi^{-1})_L, h \in M_1(N_h, \chi)_L$ to $\rho_\flat$ and $\rho_\sharp$, where N_g and N_h are their respective conductors. Let $\rho_{gh} = \rho_g \otimes \rho_h : \text{Gal}(H/\mathbb{Q}) \to \text{SL}_4(L)$.
When either g or h is Eisenstein, the Rankin-Selberg method yields an analytic continuation and functional equation for the L-function $L(E, \rho_{gh}, s) = L(f \otimes g \otimes h, s)$ that relates the values at s and $2 - s$. The result was extended by Garrett to the cuspidal case.
By the self duality condition, the root number $\epsilon(E, \rho_{gh})$ that appears in the functional equation is ± 1 and it can be written as $\epsilon(E, \rho_{gh}) = \prod_{v|N\infty} \epsilon_v(E, \rho_{gh})$, where $N = \text{lcm}(N_f, N_g, N_h)$ and $\epsilon_\infty(E, \rho_{gh}) = 1$. We do the following hypothesis:

Hypothesis A (local sign hypothesis): for all finite places $v|N$, $\epsilon_v(E, \rho_{gh}) = +1$. This forces that $L(E, \rho_{gh}, s)$ vanishes to even order at $s = 1$. Under these hypothesis, we can relate $L(f \otimes g \otimes h, 1)$ with the values of the trilinear form

$$I : S_2(N)_{\mathbb{C}} \times M_1(N, \chi^{-1})_{\mathbb{C}} \times M_1(N, \chi)_{\mathbb{C}} \to \mathbb{C}$$

defined by $I(\tilde{f}, \tilde{g}, \tilde{h}) := \langle \tilde{f}, \tilde{g}\tilde{h} \rangle$. This form already appeared in the previous chapter in a wider setting, being this a degeneracy instance in which the sum of two weights equals the third.
Let $\mathbb{T}_N$ be the Hecke algebra generated by the T_n with $n \nmid N$. If M is a $\mathbb{T}_N$-module

and $\phi \in M$ is a simultaneous eigenform, let I_ϕ be the ideal associated to that system of eigenvalues and let

$$M[\phi] = \ker(I_\phi) = \{m \in M \mid (T_l - a_l(\phi))m = 0 \text{ for all } l \nmid N\}.$$

In [HaKu] it is proved that the restriction

$$I_{fgh} : S_2(N)[f] \times M_1(N, \chi^{-1}[g] \times M_1(N, \chi)[h] \to \mathbb{C}$$

of I is identically zero if and only if the central critical value of the L-function vanishes.

Hypothesis B (global vanishing hypothesis): the L-function $L(E, \rho_{gh}, s)$ vanishes at $s = 1$ and hence the trilinear form I_{fgh} is identically zero.
Consider Serre's differential operator $d = q\frac{d}{dq}$ from $M_k^{(p)}(N, \chi) \to M_{k+2}^{(p)}(N, \chi)$. For $\tilde{f} \in S_2^{\mathrm{oc}}(N)$, the overconvergent primitive of $\tilde{f}$ is

$$\tilde{F} := d^{-1}\tilde{f} := \lim_{t \to -1} d^t \tilde{f} \in S_0^{\mathrm{oc}}(N).$$

The limit here is taken over positive integers tending to -1 in weight space.
The p-adic iterated integral attached to

$$(\tilde{f}, \tilde{\gamma}, \tilde{h}) \in S_2(Np)_L \times M_k^{\mathrm{ord}}(Np, \chi)_L^* \times M_k(Np, \chi)_L$$

is defined to be

$$\int_{\tilde{\gamma}} \tilde{f} \cdot \tilde{h} := \tilde{\gamma}(e_{\mathrm{ord}}(\tilde{F} \cdot \tilde{h})) \in \mathbb{C}_p.$$

The setting $k = 2$ is more or less understood and these iterated integrals can be used to construct the so called Chow-Heegner points on E. The case $k = 1$ is more mysterious and the p-adic integrals do not have a meaning in terms of the cohomology of Kuga-Sato varieties.
Consider, apart from $M[\phi] = \ker(T_l - a_l(\phi))$,

$$M[[\phi]] := \cup_{n \geq 1} \ker((T_l - a_l(\phi))^n).$$

Hypothesis C (classicality property for g_α): the overconvergent cuspidal generalized eigenspace $S_1^{\mathrm{oc,ord}}(N, \chi)_{\mathbb{C}_p}[[g_\alpha^*]]$ is non-trivial and consists only of classical forms.
It is however for us more convenient to assume also another hypothesis that can be considered to be more or less equivalent to the previous one:
Hypothesis C': The modular form g satisfies one of the following properties:

- It is a cusp form regular at p and it is not the theta series of a character of a real quadratic field in which p splits.
- It is an Eisenstein series irregular at p, i.e., ρ_g is the direct sum of χ_1 and χ_2 with $\chi_1(p) = \chi_2(p)$.

In [DLR1], the authors explain, following results of Cho-Vastal and Bellaiche-Dimitrov, why Hypothesis C is frequently satisfied in practice:

Proposition 30. *Let $g \in S_1(N, \chi^{-1})$ be a cusp form of weight one which is regular at p, and let g_α denote one of its p-stabilizations. Then, the natural inclusion*

$$S_1(Np, \chi)_{\mathbb{C}_p}[g_\alpha^*] \hookrightarrow S_1^{\mathrm{oc,ord}}(N, \chi)[[g_\alpha^*]]$$

is an isomorphism of $\mathbb{C}_p$-vector spaces if and only if ρ_g is not induced from a character of a real quadratic field in which p splits.

Hypothesis C and C' are expected to be equivalent when g is a cusp form. When $g = E_1(\chi_1, \chi_2)$ is a weight one Eisenstein series with $\alpha_g := \chi_1(p)$ and $\beta_g := \chi_2(p)$, consider $g_\alpha(q)$ and $g_\beta(q)$, its (not necessarily distinct) p-stabilizations. In this setting, we have the following result:

Proposition 31. *The cuspidal generalized eigenspace $S_1^{\mathrm{oc,ord}}(N,\chi)[[g_\alpha^*]]$ is non-trivial (the eigenform g_α is a p-adic cusp form if and only if $\chi_1(p) = \chi_2(p)$).*

Now, consider the logarithms $\log_p : \mathcal{O}_F^\times \to F$ and $\log_{E,p} : E(F) \to F$. Via the embedding of H into $\mathbb{C}_p$, we can have isomorphisms

$$\log_p : (\mathcal{O}_H)_L^\times \to H_p \otimes L, \qquad \log_{E,p} : E(H)_L \to H_p \otimes L.$$

When $r(E, \rho_{gh}) = 2$, let (Φ_1, Φ_2) be an L-vector basis for $\mathrm{Hom}_{G_{\bar{\mathbb{Q}}}}(V_{gh}, E(H)_L)$.
With hypothesis C' in mind, we can make a choice of a one dimensional $G_{\mathbb{Q}_p}$-stable subspace of V_g, that we will denote $V_g^{g_\alpha}$, according to the following constraint:

1. When it is a cusp form, the arithmetic Frobenius Frob_p acts on V_g with distinct eigenvalues and then $V_g = V_g^{g_\alpha} \oplus V_g^{g_\beta}$ is a decomposition into one-dimensional eigenspaces for Frob_p, with eigenvalues $\alpha_g \chi(p)$ and $\beta_g \chi(p)$.

2. When g is Eisenstein, let $V_g^{g_\alpha}$ be any one-dimensional subspace of the reducible representation V_g not stable under $G_{\mathbb{Q}}$.

We are going to explain now what is the regulator matrix.
Let $V_{gh}^{g_\alpha} = V_g^{g_\alpha} \otimes V_h \subset V_{gh}$, where the arithmetic Frobenius acts on $V_g^{g_\alpha}$ with eigenvalue β_g. Then, we have a basis $v_{\beta\alpha}, v_{\beta\beta}$ in which the Frobenius acts with eigenvalues $\beta_g \alpha_h$ and $\beta_g \beta_h$.
Let now Φ_1, Φ_2 be a basis of $\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}, E(H) \otimes L)$.

Definition 53. *The regulator matrix is*

$$R_{g_\alpha}(E, \rho_{gh}) = \begin{pmatrix} \log_E \Phi_1(v_{\beta\alpha}) & \log_E \Phi_2(v_{\beta\alpha}) \\ \log_E \Phi_1(v_{\beta\beta}) & \log_E \Phi_2(v_{\beta\beta}) \end{pmatrix}.$$

The determinant of this matrix is

$$\mathrm{Reg}_{g_\alpha}(E, \rho_{gh})$$

and the Frobenius acts with eigenvalue $\beta_g \alpha_h \beta_g \beta_h = \frac{\beta_g}{\alpha_g}$.

We would like to relate the value of the regulator with the p-adic iterated integral, but for this we need to divide by some quantity with eigenvalue β_g/α_g, and it is for that reason that we must divide by a Stark unit.
For defining this Stark unit, let $\mathrm{Ad}_g := \mathrm{Hom}^0(V_g, V_g)$ be the three-dimensional adjoint representation attached to ρ_g. Attached to g_α there is a Stark unit $u_{g_\alpha} \in (\mathcal{O}_{H_g}[1/p]^\times)_L^{\mathrm{Ad}_g}$, where H_g is the number field cut out by Ad_g. The main conjecture of [DLR] is:

Conjecture 9 (Elliptic Stark conjecture)**.** *Assume hypothesis $A, B, C - C'$ hold. If $r(E, \rho_{gh}) > 2$, then I_p' is identically zero. If $r(E, \rho_{gh}) = 2$, then there exist test vectors*

$$(\tilde{f}, \tilde{\gamma}, \tilde{h}) \in S_2(Np)_L[f] \times M_1(Np, \chi)_L^*[g_\alpha] \times M_1(Np, \chi)_L[h],$$

for which

$$\int_{\tilde{\gamma}} \tilde{f} \cdot \tilde{h} = \frac{\mathrm{Reg}_{g_\alpha}(E, \rho_{gh})}{\log_p(u_{g_\alpha})}.$$

One of the motivations for studying the conjecture is the connection between p-adic Rankin L-values and generalized Kato classes, elements of the form $\kappa(f, g_\alpha, h_\alpha) \in H^1(\mathbb{Q}, V_p(E) \otimes V_{gh})$, that were constructed as p-adic limits of étale Abel-Jacobi images of Gross-Kudla-Schoen diagonal cycles. When the L-function vanishes, these Kato classes can be seen as substitutes of Heegner points in settings of analytic rank two.

7.2 Gross-Stark units and p-adic iterated integrals

In this section we explore an analogue to the Elliptic Stark for points in elliptic curves, as it is done in [DLR2], where the main objects are units in number fields. In [DLR1], a conjectural expression for the p-adic iterated integrals attached to a triple (f, g, h) of classical eigenforms of weights $(2, 1, 1)$ is proposed. Since f was a cusp form, the expression involved the p-adic logarithms of Stark points (defined over the modular abelian variety attached to f and over the number field cut out by the Artin representations attached to g and h). Here, we replace f by a weight two Eisenstein series rather than a cusp form. In this setting, the formula will involve the p-adic logarithms of units and p-units in suitable number fields.
Let $g \in M_1(N_g, \chi_g), h \in M_1(N_h, \chi_h)$ be classical eigenforms of weight one. In [DLR] there was a crucial self-duality assumption: it must happen that $\chi_{gh} = \chi_g \chi_h$ were trivial, and thus $\rho_{gh} := \rho_g \otimes \rho_h$ was a contragradient representation. In the Eisenstein setting, this self-duality assumption is not required any more and the character χ_{gh} can now be an arbitrary Dirichlet character dividing $N_{gh} := \mathrm{lcm}(N_g, N_h)$. Let $p \nmid N_{gh}$ and let $\alpha_g, \alpha_h; \beta_g, \beta_h$ be the roots of the corresponding Hecke polynomials. As usual, let $L \subset \mathbb{C}$ be the field over which the representations ρ_g and ρ_h are defined, and enlarging it if necessary, we can assume that $\alpha_g, \beta_g, \alpha_h, \beta_h$ are in L. Denote by g_α and g_β the p-stabilizations of g which are eigenvectors for U_p (with eigenvalues α_g and β_g). Let

$$f := E_2(1, \chi_{gh}^{-1}) \in M_2(N_{gh}, \chi_{gh}^{-1})$$

be the weight two Eisenstein series with Fourier expansion

$$f(q) := c_0 + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \chi_{gh}^{-1}(d) d \Big) q^n.$$

Consider also

$$F := d^{-1} f = E_0^{[p]}(\chi_{gh}^{-1}, 1),$$

the overconvergent Eisenstein series of weight zero attached to the pair $(\chi_{gh}^{-1}, 1)$ of Dirichlet characters, with Fourier expansion

$$F(q) := \sum_{p \nmid n} \Big(\sum_{d|n} \chi_{gh}^{-1}(n/d) d^{-1} \Big) q^n.$$

We introduce the following object, that will play a crucial role (observe the parallelism between these definitions and the ones of the previous section):

$$\Xi(g_\alpha, h) := e_{g_\alpha^*} e_{\mathrm{ord}}(Fh),$$

where e_{ord} is Hida's ordinary projection on the space of overconvergent modular forms of weight one and $e_{g_\alpha^*}$ is the Hecke equivariant projection to the generalized eigenspace attached to the system of Hecke eigenvalues for the dual form g_α^* of g_α.

While $\Xi(g_\alpha, h)$ is always overconvergent and ordinary, it is not necessary classical, and for that reason we will assume hypothesis C of the previous section (classicality property) for g_α. This implies that the inclusion

$$M_1(Np, \chi_g^{-1})_{\mathbb{C}_p}[g_\alpha^*] \hookrightarrow M_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})[[g_\alpha^*]]$$

of the eigenspace attached to g_α, that consists only of classical forms, into the generalized eigenspace of p-adic modular forms, maps $M_1(Np, \chi_g^{-1})_{\mathbb{C}_p}[g_\alpha^*]$ into $S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})[[g_\alpha^*]]$ of p-adic cusp forms and gives rise to the isomorphism

$$M_1(Np, \chi_g^{-1})_{\mathbb{C}_p}[g_\alpha^*] \simeq S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})[g_\alpha^*]$$

As before, given γ in the L-linear dual space $M_1(N_{gh}, \chi_g^{-1})_L[g_\alpha^*]^*$, the p-adic iterad integral attached to (γ, h) is

$$\int_\gamma f \cdot h := \gamma(\Xi(g_\alpha, h)).$$

To study this integral, consider the four-dimensional tensor product

$$\rho_{gh} := \rho_g \otimes \rho_h.$$

Let H be the smallest number field through which ρ_{gh} factors. Let

$$U_{gh} := L \otimes \mathcal{O}_H^\times, \qquad U_{gh}^{(p)} := L \otimes (\mathcal{O}_H[1/p]^\times / p^{\mathbb{Z}}),$$

understood as finite dimensional L-linear representations of $G_{\mathbb{Q}}$.
The following result, stated in [DLR2], and whose proof is given for instance in [Das], will play a crucial role:

Lemma 23. *Let s be the multiplicity of the trivial representation in ρ_{gh}. Let*

$$d_{gh} := \dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_{gh}, U_{gh}), \qquad d_{gh}^{(p)} := \dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_{gh}, U_{gh}^{(p)}).$$

Then,

1. $d_{gh} = 2 - s$.
2. $d_{gh}^{(p)} = 2 + \dim_L(\rho_{gh})^{G_{\mathbb{Q}_p}} - 2s.$.

The Galois element Frob_p acts on ρ_g (resp. on ρ_h) with eigenvalues α_g and β_g (resp. α_h and β_h). Write the corresponding decompositions as

$$\rho_g := \rho_g^\alpha \oplus \rho_g^\beta, \qquad \rho_h := \rho_h^\alpha \oplus \rho_h^\beta.$$

For g cuspidal, we can attach to g_α a two-dimensional subspace of the representation ρ_{gh} setting

$$\rho_{gh}^{g_\alpha} := \rho_g^\beta \otimes \rho_h.$$

When $d_{gh}^{(p)} = 2$, we can associate to the pair (g_α, h) a p-adic regulator choosing by one side an L-basis (Φ_1, Φ_2) for $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_{gh}, U_{gh}^{(p)})$ and for the other, an L-basis (v_1, v_2) for $\rho_{gh}^{g_\alpha}$. Then,

$$R_{g_\alpha}(\rho_{gh}) := \begin{pmatrix} \log_p(\Phi_1(v_1)) & \log_p(\Phi_1(v_2)) \\ \log_p(\Phi_2(v_1)) & \log_p(\Phi_2(v_2)) \end{pmatrix}.$$

The matrix is well-defined up to left and right multiplication by invertible matrices with entries in L, and hence its determinant is well-defined modulo $L^\times$. This determinant belongs to $\mathbb{Q}_p^{\mathrm{ur}} \otimes L$ and Frob_p acts on it with eigenvalue $\beta_g\alpha_h\beta_g\beta_h = \frac{\beta_g}{\alpha_g} \cdot \chi_{gh}(p)$.
If χ is a Dirichlet character of conductor m, let

$$\mathfrak{g}(\chi) = \sum_{a=1}^{m} \chi^{-1}(a)e^{2\pi i a/m}$$

be the usual Gauss sum, on which $G_{\mathbb{Q}}$ acts through χ and thus Frob_p acts with eigenvalue $\chi(p)$.

Conjecture 10 (Elliptic Stark conjecture for units)**.** *Under the previous assumptions, when $d_{gh}^{(p)} = 2$, the following holds: there exists a choice of test vectors*

$$(\tilde{f}, \tilde{g}_\alpha, \tilde{h}) \in M_2(Np, \chi_{gh}^{-1})_L[f] \times S_1(Np, \chi_g)_L[g_\alpha] \times S_1(Np, \chi_h)_L[h]$$

for which

$$\int_{\tilde{\gamma}_\alpha} \tilde{f} \cdot \tilde{h} =:= \Xi(\tilde{g}_\alpha, \tilde{h}) = \frac{\mathrm{Reg}_{g_\alpha}(\rho_{gh})}{\mathfrak{g}(\chi_{gh})\log_p(u_{g_\alpha})},$$

where $\tilde{\gamma}_\alpha$ is the dual element corresponding to the Hecke equivariant projection $e_{\tilde{g}_\alpha^*}$.

An important aspect shared by both settings is the presence in the conjecture of an iterated integral. We will try to understand now the relation between iterated integrals and L-functions and for that purpose, we will do the following assumption: there exists a point $x \in U_{\mathbf{g}}^0$ of weight $\kappa_{\mathbf{g}} = 1$ such that the specialization $g_\alpha := x(\mathbf{g}) \in M_1(N_g p, \chi_g)$ satisfies Hypothesis C of the previous section.

Proposition 32. *There exists a linear form*

$$\tilde{\gamma}_\alpha : S_1(Np, \chi_g^{-1})_L[g_\alpha^*] \to L$$

with $L = \mathbb{Q}(g_1, f_2, h_1)$ such that

$$\mathscr{L}_p(\mathbf{g}, \mathbf{f}, \mathbf{h})(1, 2, 1) = \int_{\gamma_\alpha} f_2 \cdot h_1.$$

Proof. Fix a finite flat extension $\Lambda^\dagger$ of Λ, such that it contains the coefficients of all the Λ-adic modular forms we have considered. Let $\mathbf{S}^{\mathrm{ord}}(N; \Lambda^\dagger)$ be the space of Λ-adic modular forms with coefficients in $\Lambda^\dagger$. The Hida family $\mathbf{g}$ gives rise to the subspace

$$\mathbf{S}^{\mathrm{ord}}(N; \Lambda^\dagger)[g] := \big\{\mathbf{g} \in \mathbf{S}^{\mathrm{ord}}(N; \Lambda^\dagger) \text{ such that } T_n\mathbf{g} = a_n(\mathbf{g})\mathbf{g}, \text{ for all } (n, N) = 1\big\}.$$

Let $\Lambda^\dagger$ be the fraction ideal of $\Lambda^\dagger$; then, $\mathbf{S}^{\mathrm{ord}}(N; \Lambda^\dagger)[\mathbf{g}]$ is finite-dimensional over $\mathcal{L}^\dagger$ and has for basis the set $\{\mathbf{g}(q^d)\}_{d|(N/N_g)}$ of Λ-adic forms. Moreover, it can be checked that there exists a linear operator

$$J(\mathbf{g}^*) : \mathbf{S}^{\mathrm{ord}}(N, \Lambda^\dagger) \to \Lambda^\dagger \otimes_\Lambda \Lambda^\dagger, \quad \phi \mapsto J(\mathbf{g}^*, \phi),$$

characterized by the property that for every point in $U_{\mathbf{g}}^0 \cap U_\phi^0$ of weight $k \geq 2$, the specialization $J(\mathbf{g}^*)$ is regular and described by

$$\nu_k(J(\mathbf{g}^*)) : S_k^{\mathrm{ord}}(N, \chi_g^{-1})_{\mathbb{Q}_p(g_k)} \to \mathbb{Q}_p(g_k), \quad \phi \mapsto \frac{\langle g_k^*, \phi\rangle}{\langle g_k^*, g_k^*\rangle}.$$

We now analyze the specialization in weight 1. For $d|(N/N_g)$, let $c_d : \mathbf{S}^{\mathrm{ord}}(N, \mathcal{L}^\dagger) \to \mathcal{L}^\dagger$ be the function which associated to ϕ its coefficients in $\mathbf{g}(q^d)$ with respect to the previously described basis. It turns out that

$$J(\mathbf{g}^*) = \sum_{d|(N/N_g)} \lambda_d \cdot c_d,$$

where $\lambda_d \in \mathcal{L}^\dagger$ are elements which, as functions on $\mathbb{Z}^{\geq 2}$ (by the rule $k \mapsto \nu_k(\lambda_d)$) can be expressed as polynomials in q^k, $a_q(g_k)$, $1/q$ and $1/(q+1)$, as q ranges over the divisors of N. Then, the specialization in weight 1 is also regular and gives a linear operator

$$\gamma_\alpha := \nu_1(J(\mathbf{g}^*)) : S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})_{L_p} \to L_p.$$

It factors through $S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})_{L_p}[[g_\alpha^*]]$, which by assumption is isomorphic to the space of classical forms $S_1(Np, \chi_g^{-1})_{L_p}[g_\alpha^*]$.
Hypothesis C equips $S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})_{L_p}[[g_\alpha^*]]$ with an L-rational structure, that we will put $S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})_L[[g_\alpha^*]]$. Then, it follows that $\nu_1(\lambda_d) \in L$ and hence γ_α is L-rational (it belongs to $S_1(Np, \chi_g^{-1})_L[g_\alpha^*]^* = S_1^{\mathrm{oc,ord}}(N, \chi_g^{-1})_L[[g_\alpha^*]]^*$).
Let $e_{\mathrm{ord}}(d^\bullet f_2^{[p]} \times h_1)$ be the Λ-adic modular form whose specialization in weight k is $e_{\mathrm{ord}}(d^{\frac{k-3}{2}} f_2^{[p]} \times h_1)$ for all $k \geq 2$. By our previous results, $\mathscr{L}_p{}^g(\mathbf{g}, f_2, h_1) = J(\mathbf{g}, e_{\mathrm{ord}}(d^\bullet f_2^{[p]} \times h_1))$ and by construction

$$\mathscr{L}_p(\mathbf{g}, f_2, h_1)(1) = \gamma_\alpha(e_{\mathrm{ord}}(d^{-1} f_2^{[p]} \times h_1)) = \int_{\gamma_\alpha} f_2 \cdot h_1.$$

□

Further, we also have the following result with a high resemblance to the previous one:

Lemma 24. *Let $g_\alpha^* = g_\alpha \otimes \chi_g^{-1}$ denote the twist of g_α under the inverse of its nebentypus. Then, there exists a linear function*

$$\gamma_\alpha : M_1(N_{gh}p, \chi_g^{-1})_L[g_\alpha^*] \to L$$

such that

$$\mathfrak{g}(\chi_{gh})^{-1} \times \mathscr{L}_p(\mathbf{g}, h)(1) = \int_{\gamma_\alpha} (f \cdot h).$$

7.3 Known results about the Elliptic Stark conjecture

The Elliptic Stark conjecture for points in elliptic curves is basically just proved when both g and h are theta series attached to the same imaginary quadratic field K in which p splits. We would like to give some ideas around the proof, as it is done in [DLR].
Let $-D_K$ be the discriminant of the field, and let E be an elliptic curve over $\mathbb{Q}$ with associated newform $f \in S_2(N_f)$. Given $\psi : G_K \to \mathbb{C}^\times$ a finite order character, let $\theta_\psi \in M_1(D_K \cdot \mathbb{N}_{K/\mathbb{Q}}(\mathfrak{c}_\psi), \chi)$ be the theta series attached to ψ, and let $V_\psi := \mathrm{Ind}_K^{\mathbb{Q}} \psi$ be the two-dimensional induced representation of ψ from G_K to $G_{\mathbb{Q}}$. θ_ψ is Eisenstein if and only if V_ψ is reducible, that is, $\psi = \psi'$, being ψ' the character of G_K defined by $\psi'(\sigma) = \psi(\sigma_0 \sigma \sigma_0^{-1})$, where $\sigma_0 \in G_{\mathbb{Q}} \backslash G_K$.
We will fix ψ_g and ψ_h, two finite order characters of conductor $\mathfrak{c}_g$ and $\mathfrak{c}_h$ in such a way that the central character χ of ψ_h is inverse to that of ψ_g. Set $g := \theta_{\psi_g} \in M_1(N_g, \bar{\chi})$ and

$h := \theta_{\psi_h} \in M_1(N_h, \chi)$. Put $\psi_1 = \psi_g\psi_h$ and $\psi_2 = \psi_g\psi_h'$; both are ring class characters of K associated to orders $\mathcal{O}_{c_1}$ and $\mathcal{O}_{c_2}$ in $\mathcal{O}_K$ of conductors c_1 and c_2 respectively. There is a decomposition and in parallel to that a factorization of L-series in the following way:

$$V_{gh} = V_{\psi_1} \oplus V_{\psi_2}, \qquad L(E, \rho_{gh}, s) = L(E/K, \psi_1, s) \cdot L(E/K, \psi_2, s).$$

If we assume that N_f is relatively prime with $\mathfrak{c}_g\mathfrak{c}_h$, for all places of K above any place $v|N\infty$ of $\mathbb{Q}$, the local signs of $L(E/K, \psi_1, s)$ and $L(E/K, \psi_2, s)$ are equal, and hence $\epsilon_v(E, \rho_{gh}) = (\pm 1)^2 = 1$. This implies the first hypothesis. Let H be the ring class field of K of conductor c, the least common multiple of c_1 and c_2. The field L of coefficients of the previous L-series can be taken to be any finite extension of the field generated by the traces of V_{ψ_1} and V_{ψ_2}.

Now, it turns out that in this setting, when $r(E, V_{\psi_1}) = r(E, V_{\psi_2}) = 1$, the global points and units that arise in the Elliptic Stark conjecture are expressible in terms of Heegner points and elliptic units. In [DLR] there is a detailed explanation of how to obtain the explicit relations needed for the proof. Let us focus now on the L-functions involved in the proof, introducing first the Katz two-variable p-adic L-function of an imaginary quadratic field. Let $\mathfrak{c} \subset O_K$ be an integral ideal of the imaginary quadratic field K and let Σ denote the set of Hecke characters of K of conductor dividing $\mathfrak{c}$. Write $\Sigma_K = \Sigma_K^{(1)} \cup \Sigma_K^{(2)} \subset \Sigma$ for the disjoint union of the sets

$$\Sigma_K^{(1)} = \{\psi \in \Sigma \text{ of infinity type } (\kappa_1, \kappa_2) \text{ with } \kappa_1 \leq 0, \kappa_2 \geq 1\},$$

$$\Sigma_K^{(2)} = \{\psi \in \Sigma \text{ of infinity type } (\kappa_1, \kappa_2) \text{ with } \kappa_1 \geq 1, \kappa_2 \leq 0\}.$$

For all $\psi \in \Sigma_K$, $s = 0$ is a critical point for the Hecke L-function $L(\psi^{-1}, s)$, and Katz's p-adic L-function is constructed interpolating the value $L(\psi^{-1}, 0)$ as ψ ranges over $\Sigma_K^{(2)}$.

In fact, let $\hat{\Sigma}_K$ be the completion of $\Sigma_K^{(2)}$ with respect to the compact open topology on the space of O_{L_p}-valued functions on a certain subset of $A_K^\times$. Then, by the results of Katz, there is a p-adic analytic function

$$L_p(K) : \hat{\Sigma}_K \to \mathbb{C}_p$$

uniquely characterized by the property that, for all $\psi \in \Sigma_K^{(2)}$ of infinity type (κ_1, κ_2),

$$L_p(K)(\psi) = \mathfrak{a}(\psi) \times \mathfrak{e}(\psi) \times \mathfrak{f}(\psi) \times \frac{\Omega_p^{\kappa_1 - \kappa_2}}{\Omega^{\kappa_1 - \kappa_2}} \times L_c(\psi^{-1}, 0),$$

where we have used the following notations:

- $\mathfrak{a}(\psi) = (\kappa - 1)!\pi^{-\kappa_2}$, $\mathfrak{e}(\psi) = (1 - \psi(\mathfrak{p})p^{-1})(1 - \psi^{-1}(\bar{\mathfrak{p}}))$ and $\mathfrak{f}(\psi) = D_K^{\kappa_2/2} \cdot 2^{-\kappa_2}$.
- $\Omega_p \in \mathbb{C}_p^\times$ is a p-adic period attached to K.
- $\Omega \in \mathbb{C}^\times$ is the complex period associated to K.
- $L_c(\psi^{-1}, s)$ is the Hecke L-function attached to ψ^{-1} with the Euler factors at primes dividing c removed.

We now recall the role played by the p-adic L-function attached to a cusp form and an imaginary quadratic field. For any Hecke character ψ of K of infinity type (κ_1, κ_2), let

$$L(f, \psi, s) := L\Big(\pi_f \times \pi_\psi, s - \frac{\kappa_1 + \kappa_2 + 1}{2}\Big)$$

denote the L-series associated with the product of the global automorphic representations attached to the weight two cuspforms f and the Hecke character ψ. Fix a positive integer $c \geq 1$ relatively prime to pN_f. Let $\Sigma_{f,c}$ be the set of Hecke characters $\psi \in \Sigma$ of conductor c and trivial central character for which $L(f, \psi^{-1}, s)$ is self-dual and has $s = 0$ as the central point. This set is the disjoint union of three disjoint subsets

$$\Sigma_{f,c}^{(1)} = \{\psi \in \Sigma_{f,c} \text{ of infinity type } (1,1)\},$$

$$\Sigma_{f,c}^{(2)} = \{\psi \in \Sigma_{f,c} \text{ of infinity type } (\kappa + 2, -\kappa), \kappa \geq 0\},$$

$$\Sigma_{f,c}^{(2')} = \{\psi \in \Sigma_{f,c} \text{ of infinity type } (-\kappa, \kappa + 2), \kappa \geq 0\}.$$

These sets are all dense in the completion $\hat{\Sigma}_{f,c}$ with respect to the p-adic compact open topology. There exists a unique p-adic analytic function

$$L_p(f, K) : \hat{\Sigma}_{f,c} \to \mathbb{C}_p$$

interpolating the values $L_p(f, \psi^{-1}, 0)$ for $\psi \in \Sigma_{f,c}^{(2)}$. It is referred as the p-adic Rankin L-function attached to (f, K). By results of Bertolini, Darmon and Prasanna, for any $\psi \in \Sigma_{f,c}^{(2)}$ of infinity type $(\kappa + 2, -\kappa)$,

$$L_p(f, K)(\psi) = *L(f, \psi^{-1}, 0),$$

where $*$ is a constant that can be made explicit.

Now, the main idea passes for establishing a comparision between the Garrett-Hida p-adic L-function and these two Katz-style p-adic L-functions we have just presented.

7.4 Hida families and periods for weight one forms

Many of the most relevants theorems not only around BSD, but in the theory of modular forms, require the introduction of Hida families. A good starting point is, for instance, the notes of Lafferty [Laf]. We recall here the most interesting structures toward our purposes.

Let $g \in S_1(N, \chi)$ be a newform of weight one, level N and Fourier coefficients in L. Let

$$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(V) \simeq \mathrm{GL}_2(L)$$

be the Artin representation associated to it. We can view ρ as acting on a two dimensional L-vector space V, where L can be chosen to be contained in a cyclotomic field. Let H be the number field cut out by ρ, so ρ factors through $\mathrm{Gal}(H/\mathbb{Q})$. Fix a rational prime p and let $\mathfrak{p}$ a prime of H above p, what defines a canonical inclusion

$$H \subset H_p \subset \overline{\mathbb{Q}_p}$$

of H in its completion H_p at $\mathfrak{p}$. We must assume that the pair (ρ, p) satisfies the following conditions.

1. The prime p splits completely in $L/\mathbb{Q}$ so that L is equipped with an embedding into $\mathbb{Q}_p$ which will be fixed from now on. This allows ρ to be viexed as a $\mathbb{Q}_p$ linear representation via the natural action of $G_{\mathbb{Q}}$ on the $\mathbb{Q}_p$-vector space $V \otimes_L \mathbb{Q}_p$.

2. V is unramified at p, and then there is a well defined arithmetic Frobenius element $\mathrm{Frob}_p \in \mathrm{Gal}(H/\mathbb{Q})$ acting canonically on V and the characteristic polynomial of $\rho(\mathrm{Frob}_p)$ is equal to the Hecke polynomial attached to g.

3. The modular form g is regular at p, that is, $\alpha_g \neq \beta_g$. After possibly enlarging L, we can assume that this coefficient field contains the roots of unity α_g and β_g.

4. ρ_g is not induced from a character of a real quadratic field K in which the prime p splits. The reason of why this is needed is explained in [DLR].

Recall that the Artin representation V decomposes naturally as a direct sum $V = V^\alpha \oplus V^\beta$, where V^α and V^β are the one-dimensional eigenspaces for Frob_p, with eigenvalues α_g and β_g, respectively.
By the theorems of Hida, we know that there exists a finite flat extension Λ_g of the Iwasawa algebra Λ and a Hida family $\mathbf{g} \in \Lambda_g[[q]]$ of tame level N and tame character χ passing through the p-stabilized weight one eigenform g_α. When g is cuspidal, the family is unique and comes equipped with the following canonical structures:

1. There is a locally free Λ_g-module $\mathbb{V}_g$ of rank two, affording Hida's ordinary Λ-adic Galois representation $\rho_{\mathbf{g}} : G_{\mathbb{Q}} \to \mathrm{Aut}_{\Lambda_g}(\mathbb{V}_g)$, realized in the inverse limit of ordinary étale cohomology groups associated to the tower $X_1(Np^r)$ of modular curves. Therefore, this representation interpolates the Galois representations associated to the classical specializations of $\mathbf{g}$.

2. The restriction of $\mathbb{V}_g$ to $G_{\mathbb{Q}_p}$ admits a stable filtration
$$0 \to \mathbb{U}_g \to \mathbb{V}_g \to \mathbb{W}_g \to 0,$$
where both $\mathbb{U}_g$ and $\mathbb{W}_g$ are flat $\Lambda_g[G_{\mathbb{Q}_p}]$-modules locally free of rank one over $\Lambda_{\mathbf{g}}$, and $\mathbb{W}_g$ is unramified with Frob_p acting on it as multiplication by $a_p(\mathbf{g})$. It is also customary to write $\mathbb{V}_g^+ := \mathbb{U}_g$ and $\mathbb{W}_g := \mathbb{V}_g^-$.

3. Let $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Then, there is a Λ_g-adic period $\omega_{\mathbf{g}} \in D(\mathbb{W}_g) := (\widehat{\mathbb{Q}_p^{\mathrm{ur}}} \hat{\otimes} \mathbb{W}_g)^{G_{\mathbb{Q}_p}}$, corresponding to the normalized Λ-adic eigenform $\mathbf{g}$.

4. There is a natural perfect Galois-equivariant duality
$$\mathbb{U}_g \times \mathbb{W}_g \to \Lambda_g(\det(\rho_{\mathbf{g}})),$$
where $G_{\mathbb{Q}}$ acts on Λ_g by the right-hand side via multiplication by $\rho_{\mathbf{g}}$.

Let $y_g : \Lambda_g \to \mathbb{Q}_p$ be the specialization map attached to g_α. Specializing the structures above attached to $\mathbf{g}$ via y_g we obtain:

1. A non-canonical isomoprhism of $\mathbb{Q}_p[G_{\mathbb{Q}}]$-modules, given by
$$\Phi_{g_\alpha} : V_g := \mathbb{V}_g \otimes_{y_g} \mathbb{Q}_p \to V \otimes_L \mathbb{Q}_p.$$

2. A non-trivial $G_{\mathbb{Q}_p}$-stable filtration

$$0 \to U_g \to V_g \to W_g \to 0$$

of V_g by one-dimensional subspaces, where $U_g := \mathbb{U}_g \otimes_{y_g} \mathbb{Q}_p$ and $W_g := \mathbb{W}_g \otimes_{y_g} \mathbb{Q}_p$. Frob_p acts on W_g and U_g as multiplication by α_g and β_g respectively. Since the eigenvalues are assumed to be distinct, the exact sequence splits canonically, and then $U_g = V_g^{\beta}$, $W_g = V_g^{\alpha}$, $V_g = U_g \oplus W_g$.

3. Specialising Ohta's period leads to a canonical element

$$\omega_{g_\alpha} := y_g(\omega_{\mathbf{g}}) \in D(V_g^{\alpha}) := (\mathbb{Q}_p^{\mathrm{ur}} \otimes V_g^{\alpha})^{G_{\mathbb{Q}_p}} = (H_p \otimes V_g^{\alpha})^{G_{\mathbb{Q}_p}}.$$

4. The duality specialises via y_g to a canonical pairing of $\mathbb{Q}_p$-vector spaces

$$\langle , \rangle : V_g^{\beta} \times V_g^{\alpha} \to \mathbb{Q}_p(\chi),$$

which induces a pairing by functoriality

$$\langle , \rangle : D(V_g^{\beta}) \times D(V_g^{\alpha}) \to D(\mathbb{Q}_p(\chi)).$$

If the pairing is perfect, we can define a period $\eta_{g_\alpha} \in D(V_g^{\beta})$ as the unique element satisfying $\langle \eta_{g_\alpha}, \omega_{g_\alpha} \rangle = \tau(\chi) \otimes 1$, where $\tau(\chi)$ is the usual Gauss sum attached to χ, seen as an element of H_p.

We now introduce certain p-adic periods associated to g and the choice of a L-structure on V_g. We assume, for the sake of simplicity, that g is a cusp form.
Take a $G_{\mathbb{Q}}$-equivariant isomoprhism $j_g : V \otimes_L \mathbb{Q}_p \to V_g$ and let $V_g^L := j_g(V)$ (well defined up to scaling in $\mathbb{Q}_p^{\times}$ by Schur's lemma). j_g induces isomorphisms $V_g^{\alpha} \simeq V^{\alpha} \otimes_L \mathbb{Q}_p$ and $V_g^{\beta} \simeq V^{\beta} \otimes_L \mathbb{Q}_p$, and then we may choose L-bases $v_g^{\alpha}, v_g^{\beta}$ for $V_g^L \cap V_g^{\alpha}$ and $V_g^L \cap V_g^{\beta}$ in such a way that

$$V_g^L \cap V_g^{\alpha} = \langle v_g^{\alpha} \rangle_L \text{ and } V_g^L \cap V_g^{\beta} = \langle v_g^{\beta} \rangle_L.$$

Define p-adic periods

$$\Omega_{g_\alpha} = \Omega_{g_\alpha}(V_g^L) \in H_p^{\mathrm{Frob}_p = \alpha_g^{-1}}, \qquad \Xi_{g_\alpha} = \Xi_{g_\alpha}(V_g^L) \in H_p^{\mathrm{Frob}_p = \beta_g^{-1}}$$

setting

$$\Omega_{g_\alpha} \otimes v_g^{\alpha} = \omega_{g_\alpha}, \qquad \Xi_{g_\alpha} \otimes v_g^{\beta} = \eta_{g_\alpha}.$$

These periods depend on the choice of the basis for V_g^L but only up to multiplication by $L^{\times}$. Further,

$$\Omega_{g_\alpha}(\mu V_g^L) = \mu^{-1} \cdot \Omega_{g_\alpha}(V_g^L), \qquad \Xi_{g_\alpha}(\mu V_g^L) = \mu^{-1} \cdot \Xi_{g_\alpha}(V_g^L),$$

and hence $\mathcal{L}_{g_\alpha} := \Omega_{g_\alpha} / \Xi_{g_\alpha} \in (H_p)^{\mathrm{Frob}_p = \beta_g \alpha_g^{-1}}$ is in $H_p^{\times}$ and well defined up to multiplication by $L^{\times}$. This expression is a canonical p-period attached to g_α that can be viewed as a p-adic avatar of the Petersson norm of g.
It seems that these periods are in close relation with the previously defined unit u_{g_α}, as it suggests the following conjecture of [DR2.5]:

Conjecture 11. *The period $\mathcal{L}_{g_\alpha}$ satisfies*

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g\alpha}) \pmod{L^{\times}}.$$

8 Generalized cohomology classes and Stark conjecture

The aim of this last chapter is to present the results obtained trying to use Beilinson-Flach elements to give theoretical support to the Elliptic Stark conjecture. This follows the preprint [RiRo].

8.1 Generalized cohomology classes

In [DR3], the Kato classes defined in [DR2] are used to formulate a conjecture relating the periods $\omega_{g_\alpha}, \omega_{h_\alpha}$, the Kato classes and a new kind of regulators (enhanced regulators), and that would imply the main result of [DLR1]. We now do an analogous treatment, but working in the setting of Gross-Stark units instead of points (f Eisenstein), and we will formulate a conjecture that implies the main result of [DLR2].
Here, since f will be Eisenstein, we cannot use the same classes, so we will have to work with a different kind of Euler system, that is closely related to the one in [BDR2], but with the difference that we want to interpolate $\kappa(g_l, h_m)$, allowing both weights to move at the same time.
Let ρ_1, ρ_2 be an odd, irreducible, two-dimensional Artin representations of $G_{\mathbb{Q}}$

$$\rho_1, \rho_2 : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(L),$$

where L and H are finite extensions of $\mathbb{Q}$ (L is chosen to be contained in a cyclotomic field). Let V_1, V_2 be $L[\mathrm{Gal}(H/\mathbb{Q})]$-modules, two dimensional over L and realizing ρ_1 and ρ_2 respectively. Fix also a rational prime p such that (ρ_1, p) and (ρ_2, p) satisfy the hypothesis of the previous section.

By modularity results, we can attach weight one cusp forms to ρ_1 and ρ_2, in such a way that the L-functions of the representation and the series coincide (and further, they admit functional equations and analytic continuations to the entire complex plane).

Then, consider as in previous sections, $g \in S_1(N_g, \chi_g)$, $h \in S_1(N_h, \chi_h)$ be such that $V_1 \otimes_L \mathbb{Q}_p \simeq V_g$, $V_2 \otimes_L \mathbb{Q}_p \simeq V_h$ (defined up to multiplication by a scalar in $\mathbb{Q}_p^\times$). Let $V_{gh} := V_g \otimes V_h$, and consider also $j_{gh} : V_{12} \otimes_L \mathbb{Q}_p \to V_{gh}$. We will write $V_{gh}^L := j_{gh}(V_{12})$.
Further, let $f := E_2(1, \chi_{gh}^{-1}) \in M_2(N_{gh}, \chi_{gh}^{-1})$.

Finally, let $N = \mathrm{lcm}(N_g, N_h)$, and assume that p does not divide N.
Recall that $g_\alpha = g(z) - \beta_g g(pz)$ and define in the same way g_β, h_α and h_β. We can choose L large enough to contain the eigenvalues of the Frobenius $\alpha_g, \beta_g, \alpha_h, \beta_h$ which therefore belong to $\mathbb{Q}_p$. Let

$$V_g^\alpha, V_g^\beta \subset V_g, \qquad V_h^\alpha, V_h^\beta \subset V_h$$

be the eigenspaces in V_g and V_h respectively associated to these eigenvalues, and set

$$V_{gh}^{\alpha\alpha} = V_g^\alpha \otimes V_h^\alpha, \quad V_{gh}^{\alpha\beta} = V_g^\alpha \otimes V_h^\beta, \quad V_{gh}^{\beta\alpha} = V_g^\beta \otimes V_h^\alpha, \quad V_{gh}^{\beta\beta} = V_g^\beta \otimes V_h^\beta.$$

Although V_g and V_h are both assumed to be regular at p, this is not necessarily true for V_{gh}, and we have the decomposition

$$V_{gh} = V_{gh}^{\alpha\alpha} \oplus V_{gh}^{\alpha\beta} \oplus V_{gh}^{\beta\alpha} \oplus V_{gh}^{\beta\beta}.$$

To begin with, we will study the singular quotient of the local cohomology at p. Recall that for any representation V of $\mathbb{Q}_p$,

$$H^1_{\mathrm{fin}}(\mathbb{Q}_p, V) := \ker\big(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}})\big) = \mathrm{Ext}^1_{\mathrm{crys}}(\mathbb{Q}_p, V),$$

$$H^1_g(\mathbb{Q}_p, V) := \ker\big(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}})\big) = \mathrm{Ext}^1_{\mathrm{dR}}(\mathbb{Q}_p, V).$$

The dimension of $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V)$ is always smaller or equal than that of $H^1_g(\mathbb{Q}_p, V)$. In particular, combining [Bel, 2.8 and 2.21],

$$\dim_{\mathbb{Q}_p} H^1_g(\mathbb{Q}_p, V) - \dim_{\mathbb{Q}_p} H^1_{\mathrm{fin}}(\mathbb{Q}_p, V) = \dim D_{\mathrm{crys}}(V^*(1))^{\phi=1}.$$

In what follows, we will denote by

$$H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{gh}(1)) = H^1(\mathbb{Q}_p, V_{gh}(1))/H^1_g(\mathbb{Q}_p, V_{gh}(1)),$$

the singular quotient of the local cohomology at p. Observe that typically H^1_{sing} is defined as the quotient of the H^1 by H^1_{fin}; in part, this is due to the fact that $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V) = H^1_g(\mathbb{Q}_p, V)$ for most representations, and the discrepancy in these two quantities is related with the presence of p-adic invariants.
Recall that we are restricting $H^1(\mathbb{Q}, V_{gh}(1))$ to $H^1(\mathbb{Q}_p, V_{gh}(1))$ and we will denote by res_p the projection to that quotient.

Observe that we always have four canonical generalized Kato classes (a priori distinct) attached to the p-stabilizations of g and h, namely

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta) \in H^1(\mathbb{Q}, V_{gh}(1)),$$

In the setting we will present later on, it will occur that $L(g \otimes h, 1) = 0$ and this will imply that the classes lie in $H^1_g(\mathbb{Q}_p, V_{gh}(1))$.

These classes are obtained as p-adic limits $\kappa(g_\alpha, h_\alpha) := \lim_{k \to 1} \kappa(g_k, h_k)$ as (g_k, h_k) ranges over the classical specializations of weight $k \geq 2$ of Hida families specializing to g_α and h_α at weight one.

Let $V_g(N)$ be the g-isotypic component of $H^1(X_0(N))$ which is non-canonically isomorphic to a finite number of copies of V_g indexed by the positive divisors of N/N_g. Define in the same way V_h. A pertinent observation is that the classes we have initially constructed took values in the Galois representation $V_{gh}(N) = V_g(N) \otimes V_h(N)$ and then the ones we are interested in ($\kappa(g_\alpha, h_\alpha)$ and so on) are obtained via a natural $G_{\mathbb{Q}}$-equivariant projection

$$\pi : V_{gh}(N) \to V_{gh}$$

such that $\pi^*(V_{gh}^*)$ is an eigen-subspace of $V_{gh}(N)^*$ for all (good and bad) Hecke operators. Further, the projection can be chosen to be compatible with the L-structure, filtration, Ohta periods and dualities.

The generalized classes belong to the global cohomology group

$$H^1(\mathbb{Q}, V_{gh}(1)) = \mathrm{Ext}^1_{G_{\mathbb{Q}}}(\mathbb{Q}_p, V_{gh}(1)),$$

where $\mathbb{Q}_p$ refers to the one-dimensional p-adic representation of $G_{\mathbb{Q}}$ with trivial action and the Ext group is taken in the category of finite dimensional $\mathbb{Q}_p$-vector spaces

equipped with a continuous $G_{\mathbb{Q}}$-action (the restriction to $G_{\mathbb{Q}_p}$ is not necessarily de Rham).
The p-adic Selmer group

$$H^1_{\text{fin}}(\mathbb{Q}, V_{gh}(1)) := \text{Ext}^1_{\text{crys}}(\mathbb{Q}_p, V_{gh}(1))$$

attached to $V_{gh}(1)$ is the group of extensions of $\mathbb{Q}_p$ by $V_{gh}(1)$ in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$ such that the restriction to $\mathbb{Q}_l$ lies in $H^1_{\text{fin}}(\mathbb{Q}_l, V)$ for all primes l (in particular, it needs to be crystalline at p). We give now a definition that will be very useful for our purposes.

Definition 54. *$H^1_{\text{fin},p}(\mathbb{Q}, V_{gh}(1))$ is the group of extensions that are de Rham at p (the restriction to $\mathbb{Q}_p$ lies in $H^1_g(\mathbb{Q}_p, V_{gh}(1))$) and crystalline at all primes $l \neq p$ (their restrictions to $l \neq p$ lie in H^1_{fin}).*

Since we will work in the setting in which $L(V_{gh}, 1) = 0$, this will allow us to assume that the Kato classes are de Rham at p and thus belong to $H^1_{\text{fin},p}(\mathbb{Q}, V_{gh}(1))$.

The Selmer group has a natural L-rational structure, and we have (from [Bel, Proposition 2.12]) the following two relations:

$$\text{Hom}_{G_{\mathbb{Q}}}(V_{12}, U^{(p)}_{gh}) \otimes \mathbb{Q}_p = H^1_{\text{fin},p}(\mathbb{Q}, V_{gh}(1));$$

$$\text{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}) \otimes \mathbb{Q}_p = H^1_{\text{fin}}(\mathbb{Q}, V_{gh}(1)).$$

Hence, we will consider $\text{Hom}_{G_{\mathbb{Q}}}(V_{12}, U^{(p)}_{gh})$ since the assumption we will do later will allow us to assume that $\kappa(g_\alpha, h_\alpha) \in H^1_{\text{fin},p}(\mathbb{Q}, V_{gh}(1))$. Moreover, we are interested in working in the situation that the dimension of this space is two (and consequently, that of $H^1_{\text{fin}}(\mathbb{Q}, V_{gh}(1))$ is at most two, but it can be smaller). The following results summarize our situation:

Proposition 33. *The following are equivalent:*

1. *The generalized Kato classes belong to*

$$H^1_g(\mathbb{Q}_p, V_{gh}(1)),$$

   that is, their images in $H^1_{\text{sing}}(\mathbb{Q}_p, V_{gh}(1))$ are trivial.

2. *The central critical value $L(V_{gh}, 1)$ vanishes.*

Proof. This follows from the results of the last section of Chapter 6, in the particular case that we consider weight one specializations for **g** and **h**, $j = 0$ and $k' = -1$ $(-1 + 2 = 1)$. This means that $L_p(g, h, 1) = *\exp^*(\kappa(g_\alpha, h_\alpha))$. Then, we have that $L_p(g, h, 1) = 0$ if and only if $\kappa(g_\alpha, h_\alpha)$ lies in the kernel of the dual exponential map, that we have seen that happens if and only if the class belongs to $H^1_g(\mathbb{Q}_p, V_{gh}(1))$. ☐

Consequently, when $L(V_{gh}, 1) = 0$, the generalised Kato classes will belong to the homomorphism group previously considered. We are interested in the rank 2 situation, where we can formulate the following conjectural fact.

Conjecture 12. *The generalized Kato classes previously described generate a non-trivial subgroup of the Selmer group of V_{gh} if and only if the following equivalent conditions are satisfied:*

1. *The L-series $L(V_{gh}, s)$ has a double zero at $s = 1$.*

2. *The homomorphism group $\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)})$ is two-dimensional over L.*

The equivalence of these two properties follows from [Das]. From now on, we will assume that these conditions hold throughout the exposition.

Consider again $\kappa_p(g_\alpha, h_\alpha) = \mathrm{res}_p(\kappa(g_\alpha, h_\alpha)$, the image of the global class in the local cohomology group. Our point is that the image of the global class $\kappa(g_\alpha, h_\alpha)$ is controlled by suitable p-adic avatars of the second derivative of the classical L-series $L(V_{gh}, s)$ at the central critical point $s = 0$. These p-adic values, that already appeared for instance in [BDR2] and [DLR2] are precisely the L-series previously presented,

$$\mathscr{L}_p{}^{g_\alpha}(\tilde{g}^*, \tilde{h}), \qquad \mathscr{L}_p{}^{g_\beta}(\tilde{g}^*, \tilde{h}), \qquad \mathscr{L}_p{}^{h_\alpha}(\tilde{g}, \tilde{h}^*), \qquad \mathscr{L}_p{}^{h_\beta}(\tilde{g}, \tilde{h}^*).$$

They depend on the choice of certain test vectors

$$(\tilde{g}, \tilde{h}) \in S_1(N, \chi_g; L) \times S_1(N, \chi_h; L)$$

with the same system of Hecke eigenvalues as g and h respectively and with Fourier coefficients in L, and also on the choice of dual test vectors

$$(\tilde{g}^*, \tilde{h}^*) \in \mathrm{Hom}(S_1(N, \chi_g^{-1}; L), L) \times \mathrm{Hom}(S_1(N, \chi_h^{-1}; L), L)$$

with the same system of Hecke eigenvalues as g and h.
We remark that these p-adic L-values are defined essentially as the p-adic limit of central critical values of the L-function associated to $V_{g_l} \otimes V_h$ as g_l ranges over the specializations of odd weight $l \geq 3$ of the Hida family $\mathbf{g}$ specializing to g_α in weight one (or those associated to $V_g \otimes V_{h_l}$).

Choose a basis of V_{gh} over $\mathbb{Q}_p$ compatible with the previous decomposition:

$$e_{\beta\beta} \in V_{gh}^{\alpha\alpha}, \qquad e_{\beta\alpha} \in V_{gh}^{\alpha\beta}, \qquad e_{\alpha\beta} \in V_{gh}^{\beta\alpha}, \qquad e_{\alpha\alpha} \in V_{gh}^{\beta\beta}.$$

Recall that the Frobenius acts in $V_{gh}^{\alpha\alpha}$ with eigenvalue $\beta_g\beta_h$ and similarly for the others. We can consider the dual basis to this: in particular, the element $\phi \in V_{gh}^*$ corresponding to $e_{\beta\beta}$ will have Frobenius eigenvalue $\chi_{gh}^{-1}(p)\alpha_g\alpha_h$. Then, call the elements of the dual basis $\{\phi_{\alpha\alpha}, \phi_{\alpha\beta}, \phi_{\beta\alpha}, \phi_{\beta\beta}\}$, where the Frobenius acts on $\phi_{\alpha\alpha}$ with eigenvalue $\chi_{gh}^{-1}(p)\alpha_g\alpha_h$.
Write

$$\kappa_p(g_\alpha, h_\alpha) = R_{\beta\beta}\phi_{\alpha\alpha} + R_{\beta\alpha}\phi_{\alpha\beta} + R_{\alpha\beta}\phi_{\beta\alpha} + R_{\alpha\alpha}\phi_{\beta\beta},$$

where $R_{\beta\beta} \in U_{gh}^{(p)}$ is the image by κ_p of $e_{\alpha\alpha}$ and the Frobenius acts on the coordinate $R_{\alpha\alpha}$ with eigenvalue $\alpha_g\alpha_h$ (and similarly for the others). Observe that this makes sense for the identification we have made between $\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}) \otimes \mathbb{Q}_p$ and $H^1_{\mathrm{fin},p}(\mathbb{Q}, V_{gh}(1))$.

Let

$$\log_p : \mathcal{O}(H_p)_{\mathbb{Q}_p}^\times \to H_p$$

be the formal group logarithm. We formulate here the following remarkable result whose proof will be our next objective.

Theorem 77. *When $L(V_{gh},0)=0$, there exist test vectors for g and h such that the previously chosen coordinates satisfy*

$$\log_p(R_{\alpha\beta}) \sim \mathscr{L}_p{}^{g_\alpha}(\tilde{g}^*,\tilde{h}), \qquad \log_p(R_{\beta\alpha}) \sim \mathscr{L}_p{}^{h_\alpha}(\tilde{g},\tilde{h}^*), \qquad \log_p(R_{\beta\beta})=0,$$

where $\sim$ means equality up to a non-zero p-adic period in $H_p^\times$.

After proving the theorem, this corollary will be a direct consequence.

Corollary 8. *If $L(V_{gh},0)=0$ and $\mathscr{L}_p{}^{g_\alpha}(\tilde{g}^*,\tilde{h})\neq 0$ for a suitable choice of test vectors, then the two global classes*

$$\kappa_p(g_\alpha,h_\alpha), \qquad \kappa_p(g_\alpha,h_\beta)$$

are linearly independent in the Selmer group $H^1_g(\mathbb{Q},V_{gh}(1))$ attached to V_{gh}.

We will explain how to proceed now to prove the previous theorem and the corollary. Recall that we are working under the hypothesis that $L(V_{gh},1)$ vanishes at order 2, and in particular $\kappa(g_\alpha,h_\alpha)$ and $\kappa(g_\alpha,h_\beta)$ belong to $H^1_{\mathrm{fin},p}(G_{\mathbb{Q}},V_{gh}(1))$ (or to the corresponding homomorphism group).
Hence, we can consider the p-adic class $\kappa_p(g_\alpha,h_\alpha)$ that lies in the direct sum of these four spaces:

$$H^1_g(\mathbb{Q}_p,V_{gh}(1)) = H^1_g(\mathbb{Q}_p,V_{gh}^{\alpha\alpha}(1))\oplus H^1_g(\mathbb{Q}_p,V_{gh}^{\alpha\beta}(1))\oplus H^1_g(\mathbb{Q}_p,V_{gh}^{\beta\alpha}(1))\oplus H^1_g(\mathbb{Q}_p,V_{gh}^{\beta\beta}(1)).$$

Lemma 25. *$\kappa_p(g_\alpha,h_\alpha)(e_{\beta\beta})=0$, where we see $\kappa_p(g_\alpha,h_\alpha)$ as an element in the space of $\mathrm{Hom}(V_{gh},O(H)_L^\times)$.*

Proof. We have that

$$\kappa_p(g_\alpha,h_\alpha)=\lim_{l\to 1}\kappa_p(g_l,h_l)\in H^1_g(\mathbb{Q}_p,V_{g_l h_l}(1)).$$

Via the Block-Kato logarithm, we have a projection of this last space in the dual of $\operatorname{Fil} D(V_{g_l}V_{h_l})$, the three dimensional subspace generated by $\omega_{g_l}\otimes\omega_{h_l}$, $\omega_{g_l}\otimes\eta_{h_l}$ and $\eta_{g_l}\otimes\omega_{h_l}$.

On the other hand, we have that

$$e_{\beta\beta}=\lim_{l\to 1}\eta_{g_l}\otimes\eta_{h_l}.$$

Further,

$$\kappa_p(g_\alpha,h_\alpha)=\lim_{l\to 1}\log(\kappa_p(g_l,h_l))(\eta_{g_l}\otimes\eta_{h_l})=0,$$

since $\eta_{g_l}\otimes\eta_{h_l}$ does not lie in $\operatorname{Fil} D(V_{g_l h_l})$. Observe that the fact of being able to consider the limit comes from [KLZ], where they have proved the existence of global differentials ω and η, that at weight l specializes to ω_l and η_l respectively. □

Assume now for the sake of simplicity that $N_g=N_h=N$. Then, $V_{gh}(N)=V_{gh}$.

Theorem 78. *Assume that $\mathscr{L}_p{}^{g}=\mathscr{L}_p{}^{g_\alpha}(\tilde{g}^*,\tilde{h})\neq 0$. Then, $\kappa_{\alpha\alpha}:=\kappa_p(g_\alpha,h_\alpha)$ and $\kappa_{\alpha\beta}:=\kappa_p(g_\alpha,h_\beta)$ are linearly independent.*

Proof. Recall that $\kappa_{\alpha\alpha}$ and $\kappa_{\alpha\beta}$ can be understood as elements in $\mathrm{Hom}(V_{gh}, U_{gh}^{(p)})$. In particular, if we prove that the action over two different vectors gives rise to a matrix with non-zero determinant we will be done. In particular, setting $\mathscr{L}_p{}^{g_\alpha}(\tilde{g}^*, \tilde{h})(1,1) = \mathscr{L}_p{}^g$, it yields that

$$\begin{pmatrix} \kappa_{\alpha\alpha}(e_{\beta\beta}) & \kappa_{\alpha\alpha}(e_{\beta\alpha}) \\ \kappa_{\alpha\beta}(e_{\beta\beta}) & \kappa_{\alpha\beta}(e_{\beta\alpha}) \end{pmatrix} = \begin{pmatrix} 0 & \mathscr{L}_p{}^g \\ \mathscr{L}_p{}^g & 0 \end{pmatrix},$$

where we have used the previous lemma together with the fact that

$$\kappa_{\alpha\alpha}(e_{\beta\alpha}) = \lim_{l\to 1} \log \kappa(g_l, h_l)(\eta_{g_l} \otimes \omega_{h_l}) = \lim_{l\to 1} \mathrm{AJ}_p(\Delta(g_l, h_l))(\eta_{g_l} \otimes \omega_{h_l}) =$$

$$= \lim_{l\to 1} \mathscr{L}_p{}^{g_\alpha}(g,h)(l,l) = \mathscr{L}_p{}^g,$$

which follows from the formulas of [BDR1]. Since when the rank of vanishing is two $\mathscr{L}_p{}^g \neq 0$, we are done. □

We know (by our assumption) that $\mathrm{Hom}_{G_\mathbb{Q}}(V_{12}, U_{gh}^{(p)})$ is a two-dimensional L-vector space, so we can choose a basis (u, v), that can be written as

$$u = u_{\beta\beta}\phi_{\alpha\alpha} + u_{\beta\alpha}\phi_{\alpha\beta} + u_{\alpha\beta}\phi_{\beta\alpha} + u_{\alpha\alpha}\phi_{\beta\beta},$$

$$v = v_{\beta\beta}\phi_{\alpha\alpha} + v_{\beta\alpha}\phi_{\alpha\beta} + v_{\alpha\beta}\phi_{\beta\alpha} + v_{\alpha\alpha}\phi_{\beta\beta},$$

where as before the Frobenius acts on $u_{\alpha\alpha}$ as multiplication by $\alpha_g\alpha_h$ and analogously for the other coordinates.
Recall that

$$R_{g_\alpha}(V_{12}) = \begin{pmatrix} \log_p u_{\beta\beta} & \log_p v_{\beta\beta} \\ \log_p u_{\beta\alpha} & \log_p v_{\beta\alpha} \end{pmatrix},$$

and that the main conjecture of [DLR2] was that, under the hypothesis that $L(V_{gh}, s)$ vanishes to order 2 at $s = 1$, then there exists a choice of test vectors $(\tilde{g}^*, \tilde{h})$ such that

$$\mathscr{L}_p(\tilde{g}^*, \tilde{h}) = \frac{\mathrm{Reg}_{g_\alpha}(V_{12})}{\log_p u_{g_\alpha}},$$

where $\mathrm{Reg}_{g_\alpha}(V_{12}) = \det(R_{g_\alpha}(V_{12}))$.

8.2 Enhanced regulators

We are going to define a certain kind of objects, the so-called enhanced regulators, that will play a crucial role from now on. They will belong to the following spaces:

$$\widetilde{\mathrm{Reg}}(V_{12}) \in \mathrm{Hom}_{G_\mathbb{Q}}(V_{12}, U_{gh}^{(p)}) \otimes \mathrm{Hom}_{G_\mathbb{Q}}(V_{12}, U_{gh}^{(p)}).$$

$$\widetilde{\mathrm{Reg}}_{\alpha,\alpha}(V_{12}) \in (H_p)^{\mathrm{Frob}_p = \beta_g\beta_h} \otimes \mathrm{Hom}_{G_\mathbb{Q}}(V_{12}, U_{gh}^{(p)}).$$

$$\widetilde{\mathrm{Reg}}(V_{gh}) \in (U_{gh}^{(p)} \otimes V_{gh}) \otimes \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}, U_{gh}^{(p)}).$$

$$\widetilde{\mathrm{Reg}}_{\alpha,\alpha}(V_{gh}) \in (H_p)^{\mathrm{Frob}_p = \beta_g\beta_h} \otimes \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}, U_{gh}^{(p)}).$$

Then,

$$\widetilde{\mathrm{Reg}}(V_{12}) = u \otimes v - v \otimes u = u \wedge v$$

where (u,v) is a basis of the two-dimensional vector-space $\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)})$. It is well-defined up to multiplication by $L^\times$.
To define the second one, consider $\log_{\alpha\alpha} : \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}) \to (H_p)^{\mathrm{Frob}_p=\beta_g\beta_h}$ given by

$$\log_{\alpha\alpha}(u) := \log_p(u_{\beta\beta}).$$

This induces a linear map

$$\log_{\alpha\alpha} \otimes 1 : \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}) \otimes \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}) \to (H_p)^{\mathrm{Frob}_p=\beta_g\beta_h} \otimes \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}).$$

We are going to sketch now how to construct another another type of enhanced regulator, namely, $\widetilde{\mathrm{Reg}}(V_{gh})$, that has properties that will interest us more. In particular, we will consider a map

$$\tilde{j}_{gh}^* \otimes \tilde{j}_{gh} : \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}) \otimes \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)}) \to (U_{gh}^{(p)} \otimes V_{gh}) \otimes \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}, U_{gh}^{(p)}).$$

For this, observe that we have an embedding $j_{gh} : V_{12}^* \to V_{gh}^{*L} \subset V_{gh}^*$ (this comes from the embedding $V_{12} \to V_{gh}$ tensoring with the character). Of course this embedding is completely non-canonical and only defined up to scaling by $\mathbb{Q}_p^\times$.
Now, the canonical dualities on V_{gh} allow us to define $G_{\mathbb{Q}}$-equivariant embeddings $j_{gh}^* : V_{12}^* \to V_{gh}$. Replacing j_{gh} by $\mu \cdot j_{gh}$ for any $\mu \in \mathbb{Q}_p^\times$ replaces j_{gh}^* by $\mu^{-1} \cdot j_{gh}^*$. Hence, the map

$$j_{gh}^* \otimes j_{gh} : V_{12}^* \otimes V_{12}^* \to V_{gh} \otimes V_{gh}^*,$$

is well-defined up to scaling by $L^\times$ and we can now define our desired $\tilde{j}_{gh}^* \otimes \tilde{j}_{gh}$ as the composition of the previous map with the corresponding isomorphisms between $\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, U_{gh}^{(p)})$ and $U_{gh}^{(p)} \otimes V_{12}^*$ (and the analogous ones for the other homomorphism spaces involved).

Definition 55. *The enhanced regulator $\widetilde{\mathrm{Reg}}(V_{gh})$ associated to V_{gh} is*

$$\widetilde{\mathrm{Reg}}(V_{gh}) := (\tilde{j}_{gh}^* \otimes \tilde{j}_{gh})(\widetilde{\mathrm{Reg}}(V_{12})) \in (U_{gh}^{(p)} \otimes V_{gh})^{G_{\mathbb{Q}}} \otimes \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}, U_{gh}^{(p)}).$$

To finish with the definitions, let

$$\mathrm{Log}_p : (U_{gh}^{(p)} \otimes V_{gh})^{G_{\mathbb{Q}}} \to (H_p \otimes V_{gh})^{G_{\mathbb{Q}_p}} = D(V_{gh})$$

be the canonical p-adic logarithm induced from the p-adic logarithm previously defined via the embedding $H \subset H_p$ and let $\mathrm{Log}_{\alpha\alpha}$ be its composition with the functorial projection $D(V_{gh}) \to D(V_{gh}^{\alpha\alpha})$:

$$\mathrm{Log}_{\alpha\alpha} : (U_{gh}^{(p)} \otimes V_{gh})^{G_{\mathbb{Q}}} \to D(V_{gh}^{\alpha\alpha})$$

$$\mathrm{Log}_{\alpha\alpha} := \log_{\alpha\alpha} \otimes e_{\beta\beta}.$$

Recall that $e_{\beta\beta}$ is a basis for $V_{gh}^{\alpha\alpha}$ and that Frobenius acts there as multiplication by $\beta_g\beta_h$. Write

$$\widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{gh}) := (\mathrm{Log}_{\alpha\alpha} \otimes 1)(\widetilde{\mathrm{Reg}}(V_{gh})) = \mathrm{Log}_{\alpha\alpha}(u) \otimes v - \mathrm{Log}_{\alpha\alpha}(v) \otimes u.$$

This regulator $\widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{gh})$ is a canonical invariant well defined up to multiplication by $L^\times$, while $\widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{12})$ depends on the choice of a basis for $V_{gh}^{\alpha\alpha}$. They satisfy

$$\widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{gh}) = \widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{12}) \otimes e_{\beta\beta}.$$

8.3 Some conjectures towards the main result of [DLR2]

We will recall now the construction of the periods ω_{g_α} and ω_{h_α} of [DR2.5] that will play a prominent role in this last part of the article. By the results of Hida, we know that there exists a finite flat extension Λ_g of the Iwasawa algebra anda Hida family $\mathbf{g} \in \Lambda[[q]]$ passing through g_α. In the cuspidal case, it comes with a locally free Λ_g-module $\mathbb{V}_g$ of rank two, affording Hida's ordinary Λ-adic Galois representation. The restriction to $G_{\mathbb{Q}_p}$ gives the filtration

$$0 \to \mathbb{V}_g^+ \to \mathbb{V}_g \to \mathbb{V}_g^- \to 0,$$

being $\mathbb{V}_g^+$ and $\mathbb{V}_g^-$ flat $\Lambda_g[G_{\mathbb{Q}_p}]$-modules locally free of rank one over $\Lambda_{\mathbf{g}}$. Frob_p acts on $\mathbb{V}_g^-$ as multiplication by $a_p(\mathbf{g})$.

Let $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Then, there is a Λ_g-adic period $\omega_{\mathbf{g}} \in D(\mathbb{V}_g^-) := (\widehat{\mathbb{Q}_p^{\mathrm{ur}}} \hat{\otimes} \mathbb{V}_g^-)^{G_{\mathbb{Q}_p}}$, corresponding to the normalized Λ-adic eigenform $\mathbf{g}$.

If $y_g : \Lambda_g \to \mathbb{Q}_p$ is the specialization map, we can define

$$\omega_{g_\alpha} := y_g(\omega_{\mathbf{g}}) \in D(V_g^\alpha) := (\mathbb{Q}_p^{\mathrm{ur}} \otimes V_g^\alpha)^{G_{\mathbb{Q}_p}} = (H_p \otimes V_g^\alpha)^{G_{\mathbb{Q}_p}}.$$

In general, there exists a natural Galois-equivariant duality, that specializes to a canonical pairing of $\mathbb{Q}_p$-vector spaces

$$\langle , \rangle : V_g^\beta \times V_g^\alpha \to \mathbb{Q}_p(\chi),$$

which induces a pairing by functoriality

$$\langle , \rangle : D(V_g^\beta) \times D(V_g^\alpha) \to D(\mathbb{Q}_p(\chi)).$$

If the pairing is perfect, we can define a period $\eta_{g_\alpha} \in D(V_g^\beta)$ as the unique element satisfying $\langle \eta_{g_\alpha}, \omega_{g_\alpha} \rangle = \mathfrak{g}(\chi) \otimes 1$, where $\mathfrak{g}\chi(\chi)$ is the usual Gauss sum attached to χ, seen as an element of H_p.

Define p-adic periods

$$\Omega_{g_\alpha} = \Omega_{g_\alpha}(V_g^L) \in H_p^{\mathrm{Frob}_p = \alpha_g^{-1}}, \qquad \Xi_{g_\alpha} = \Xi_{g_\alpha}(V_g^L) \in H_p^{\mathrm{Frob}_p = \beta_g^{-1}}$$

setting

$$\Omega_{g_\alpha} \otimes v_g^\alpha = \omega_{g_\alpha}, \qquad \Xi_{g_\alpha} \otimes v_g^\beta = \eta_{g_\alpha}.$$

These periods depend on the choice of the basis for V_g^L but only up to multiplication by $L^\times$. Further,

$$\Omega_{g_\alpha}(\mu V_g^L) = \mu^{-1} \cdot \Omega_{g_\alpha}(V_g^L), \qquad \Xi_{g_\alpha}(\mu V_g^L) = \mu^{-1} \cdot \Xi_{g_\alpha}(V_g^L),$$

and hence $\mathcal{L}_{g_\alpha} := \Omega_{g_\alpha}/\Xi_{g_\alpha} \in (H_p)^{\mathrm{Frob}_p = \beta_g \alpha_g^{-1}}$ is in $H_p^\times$ and well defined up to multiplication by $L^\times$. This expression is a canonical p-period attached to g_α that can be viewed as a p-adic avatar of the Petersson norm of g.

It seems that these periods are in close relation with the previously defined unit u_{g_α}, as it suggests the following conjecture of [DR2.5]:

Conjecture 13. *The period $\mathcal{L}_{g_\alpha}$ satisfies*

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}) \pmod{L^\times}.$$

Conjecture 14. *Assume that the L-series $L(V_{gh}, s)$ has a double zero at $s = 1$. Then, the generalized Kato class $\kappa(g_\alpha, h_\alpha)$ belongs to $\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}, U_{gh}^{(p)})$ and satisfies the relation*

$$\omega_{g_\alpha}\omega_{h_\alpha} \otimes \kappa(g_\alpha, h_\alpha) \sim_L \widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{gh})$$

in $D(V_{gh}^{\alpha\alpha}) \otimes \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}, U_{gh}^{(p)})$, where $\sim_L$ means equality up to scaling by a non-zero factor in L.

We will now prove how under the conjecture relating the canonical period attached to g to the Stark unit u_{g_α}, this conjecture implies the main conjecture of [DLR2].

Proposition 34. *Assume that conjectures* 5.2 *and* 7.1 *are true. Then, conjecture 7.2 implies the main conjecture of [DLR2].*

Proof. Consider the product of periods

$$\eta_{g_\alpha}\omega_{h_\alpha} = (\Xi_{g_\alpha} \otimes e_g^\beta) \cdot (\Omega_{h_\alpha} \otimes e_h^\alpha) = \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \otimes e_{gh}^{\beta\alpha} \in D(V_{gh}^{\beta\alpha}).$$

The pairing we have introduced gives rise to a pairing

$$\langle , \rangle : D(V_{gh}^{\alpha\beta}) \times D(V_{gh}^{\beta\alpha}) \to D(\mathbb{Q}_p) = \mathbb{Q}_p.$$

On the other hand, by the definition of the enhanced regulator, applying now the Log map in the second component we obtain

$$\begin{aligned}\mathrm{Log}_{\alpha\beta}\, \widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{gh}) &= (\log_p u_{\beta\beta} \log_p v_{\beta\alpha} - \log_p v_{\beta\beta} \log_p u_{\beta\alpha}) \otimes e_{\beta\beta} \otimes \phi_{\beta\alpha} \\ &= \mathrm{Reg}_{g_\alpha}(V_{12}) \otimes e_{\beta\beta} \otimes \phi_{\beta\alpha} \pmod{L^\times}.\end{aligned}$$

Hence, we have the following inequality in $D(V_{gh}^{\alpha\alpha})$:

$$\langle \mathrm{Log}_{\alpha\beta}\, \widetilde{\mathrm{Reg}}_{\alpha\alpha}(V_{gh}), \eta_{g_\alpha}\omega_{h_\alpha} \rangle = \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \cdot \mathrm{Reg}_{g_\alpha}(V_{gh}) \otimes e_{\beta\beta} \pmod{L^\times}.$$

We also have the following result from the sections 8.2.10 and 9.7.2 of [KLZ]:

$$\langle \mathrm{Log}_{\alpha\beta}\, \kappa(g_\alpha, h_\alpha), \eta_{g_\alpha}\omega_{h_\alpha} \rangle = \mathscr{L}_p{}^{g_\alpha}(g, h) \pmod{L^\times}.$$

By pairing the value of $\mathrm{Log}_{\alpha\beta}$ at both sides of the displayed identity of the previous conjectures with the class $\eta_{g_\alpha}\omega_{h_\alpha}$, we obtain

$$\omega_{g_\alpha}\omega_{h_\alpha} \otimes \mathscr{L}_p{}^{g_\alpha}(g, h) = \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \cdot \mathrm{Reg}_{g_\alpha}(V_{12}) \otimes e_{\beta\beta} \in D^{\alpha\alpha}) \pmod{L^\times}.$$

Since we know that

$$\omega_{g_\alpha}\omega_{h_\alpha} = \Omega_{g_\alpha}\Omega_{h_\alpha} \cdot e_{\beta\beta} \pmod{L^\times},$$

it follows that

$$\Omega_{g_\alpha}\mathscr{L}_p{}^{g_\alpha}(g, h) = \Xi_{g_\alpha}\, \mathrm{Reg}_{g_\alpha}(V_{12}) \pmod{L^\times},$$

and therefore that

$$\mathscr{L}_p{}^{g_\alpha}(g, h) = \frac{\mathrm{Reg}_{g_\alpha}(V_{12})}{\mathcal{L}_{g_\alpha}} = \frac{\mathrm{Reg}_{g_\alpha}(V_{12})}{\log_p u_{g_\alpha}} \pmod{L^\times}.$$

$\square$

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