The number and degree distribution of spanning trees in the Tower of Hanoi graph

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Abstract

The number of spanning trees of a graph is an important invariant related to topological and dynamic properties of the graph, such as its reliability, communication aspects, synchronization, and so on. However, the practical enumeration of spanning trees and the study of their properties remain a challenge, particularly for large networks. In this paper, we study the number and degree distribution of the spanning trees in the Hanoi graph. We first establish recursion relations between the number of spanning trees and other spanning subgraphs of the Hanoi graph, from which we find an exact analytical expression for the number of spanning trees of the n-disc Hanoi graph. This result allows the calculation of the spanning tree entropy which is then compared with those for other graphs with the same average degree. Then, we introduce a vertex labeling which allows to find, for each vertex of the graph, its degree distribution among all possible spanning trees.

Keywords: Spanning trees, Tower of Hanoi graph, Degree distribution, Fractal geometry

1. Introduction

The problem of finding the number of spanning trees of a finite graph is a relevant and long standing question. It has been considered

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in different areas of mathematics [1], physics [2], and computer science [3], since its introduction by Kirchhoff in 1847 [4]. This graph invariant is a parameter that characterizes the reliability of a network [5, 6, 7] and is related to its optimal synchronization [8] and the study of random walks [9]. It is also of interest in theoretical chemistry, see for example [10]. The number of spanning trees of a graph can be computed, as shown in many basic texts on graph theory [11], from Kirchhoff's matrix-tree theorem [12] and it is given by the product of all nonzero eigenvalues of the Laplacian matrix of the graph. Although this result can be applied to any graph, the calculation of the number of spanning trees from the matrix theorem is analytically and computationally demanding, in particular for large networks. Not surprisingly, recent work has been devoted to finding alternative methods to produce closed-form expressions for the number of spanning trees for particular graphs such as grid graphs [13], lattices [14, 15, 16, 17], the small-world Farey graph [18, 19, 20], the Sierpiński gasket [21, 22], self-similar lattices [23, 24], etc.

Most of the previous work focused on counting spanning trees on various graphs [1]. However, the number of spanning trees is an integrated, coarse characteristic of a graph. Once the number of spanning trees is determined, the next step is to explore and understand the geometrical structure of spanning trees. In this context, it is of great interest to compute the probability distribution of different coordination numbers at a given vertex among all the spanning trees [25], which encodes useful information about the role the vertex plays in the whole network. Due to the computational complexity of the calculation, this geometrical feature of spanning trees has been studied only for very few graphs, such as the $\mathbb{Z}^d$ lattice [26], the square lattice [27], and the Sierpiński graph [28]. It is non-trivial to study this geometrical structure for other graphs.

In this paper, we study the number and structure of spanning trees of the Hanoi graph. This graph, which is also known as the Tower of Hanoi graph [29], comes from the well known Tower of Hanoi puzzle, as the graph is associated to the allowed moves in this puzzle. There exist an abundant literature on the properties of the Hanoi graph, which includes the study of shortest paths, average distance, planarity, Hamiltonian walks, group of symmetries, average eccentricity, to name a few, see [29] and references therein. In [24], Teufl and Wagner obtained the number of spanning trees of different self-similar lattices, including the Hanoi graph. Here, based on the self-similarity of the Hanoi graph, we enumerate its spanning trees and compute for each vertex of the graph its degree distribution among all spanning trees.

2. The Hanoi graph

The Hanoi graph is derived from the Tower of Hanoi puzzle with n discs [29]. We can consider each legal distribution of the n discs on the three peg, a state, as a vertex of the Hanoi graph, and an edge is defined if one state can be transformed into another by moving one disc. If we label the three pegs 0, 1 and 2, any legal distribution of the n discs can be written as the vector/sequence $\alpha_1\alpha_2\ldots\alpha_n$ where α_i $(1 \leq i \leq n)$ gives the location of the $(n+1-i)$th largest disc. We will denote as H_n the Hanoi graph of n discs. Fig. 1 shows H_1, H_2 and H_3.

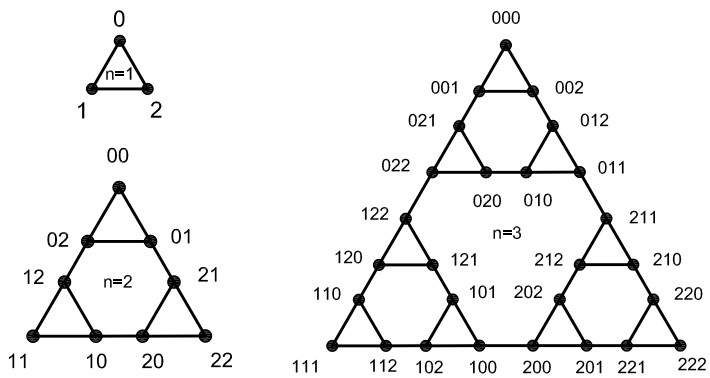

Figure 1: Hanoi graphs H_1, H_2 and H_3.

Note that $H_{n+1}(n \geq 1)$ can be obtained from three copies of H_n joined by three edges, each one connecting a pair of vertices from two different replicas of H_n, as shown in Fig. 2. From the construction rule, we find that the number of vertices or order of H_n is 3^n while the number of edges is $\frac{3}{2}(3^n - 1)$.

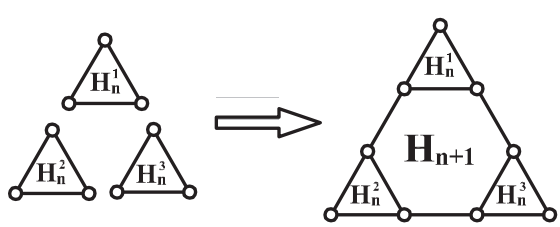

Figure 2: Construction rules for the Hanoi graph. H_{n+1} is obtained by connecting three graphs H_n labeled here by H_n^1, H_n^2 and H_n^3.

In the next section will make use of this recursive construction to find the number of spanning trees of H_n at any iteration step n.

3. The number of spanning trees in H_n

If we denote by V_n and E_n the number of vertices and edges of H_n, then a spanning subgraph of H_n is a graph with the same vertex set as H_n and a number of edges E'_n such that $E'_n < E_n$. A spanning tree of H_n is a spanning subgraph that is a tree and thus $E'_n = V_n - 1$.

In this section we calculate the number of spanning trees of the Hanoi graph H_n. We adapt the decimation method described in [30, 31, 32], which has also been successfully used to find the number of spanning trees of the Sierpiński gasket [22], the Apollonian network [33], and some fractal lattices [16]. This decimation method is in fact the standard renormalization group a pproach [34] in statistical physics, which applies to many enumeration problems on self-similar graphs [35]. We make use of the particular structure of the Hanoi graph to obtain a set of recursive equations for the number of spanning trees and spanning subgraphs, which then can be solved by induction.

Let S_n denote the set of spanning trees of H_n. Let P_n (R_n, T_n) denote the set of spanning subgraphs of H_n, each of which consists of two trees with the outmost vertex $22\ldots2$ ($00\ldots0$, $11\ldots1$) belonging to one tree while the other two outmost vertices being in the second tree. And let L_n denote the set of spanning subgraphs of H_n, each of which contains three trees with every outmost vertex in a different tree. These five types of spanning subgraphs are illustrated schematically in Fig. 3, where we use only the three outmost vertices to represent the graph because the edges joining the subgraphs to which they belong provide all the information needed to obtain the Hanoi graph at the next iteration. Let s_n, p_n, r_n, t_n, and l_n denote the cardinality of sets S_n, P_n, R_n, T_n, and L_n, respectively.

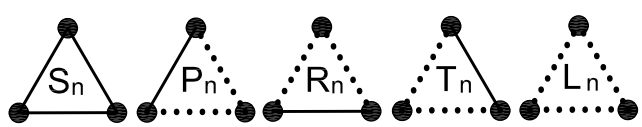

Figure 3: Illustration for the five types of spanning subgraphs derived from H_n. Two outmost vertices joined by a solid line are in one tree while two outmost vertices belong to different trees if they are connected by a dashed line.

Lemma 3.1. *The five classes of subgraphs* S_n, P_n, R_n, T_n *and* L_n *form a complete set because each one can be constructed iteratively from the classes of subgraphs* S_{n-1}, P_{n-1}, R_{n-1}, T_{n-1} *and* L_{n-1}.

We do not prove this Lemma here, since we will enumerate each case. However the proof follows from the fact that H_n can be constructed from three H_{n-1} by joining their outmost vertices and each of the five subgraphs are associated with different ways to produce the spanning trees.

Next we will establish a recursive relationship among the five parameters s_n, p_n, r_n, t_n and l_n. We notice that the equation $p_n = r_n = t_n$ holds as a result of symmetry, thus, in some places of the following text, we will use p_n instead of r_n and t_n.

Lemma 3.2. *For the Hanoi graph* H_n *with* $n \geq 1$,

$$s_{n+1} = 3s_n^3 + 6s_n^2 p_n \,, \tag{1}$$

$$p_{n+1} = s_n^3 + 7s_n^2 p_n + 7s_n p_n^2 + s_n^2 l_n \,, \tag{2}$$

$$l_{n+1} = s_n^3 + 12s_n^2 p_n + 3s_n^2 l_n + 36s_n p_n^2 + 12s_n p_n l_n + 14l_n^3 \,. \tag{3}$$

Proof. This lemma can be proved graphically. Fig. 4 shows a graphical representation of Eq. (1). Fig. 5 provides a case enumeration for p_{n+1}. Fig. 6 and Fig. 7 give the enumeration detail of all configurations that contribute to l_{n+1}. □

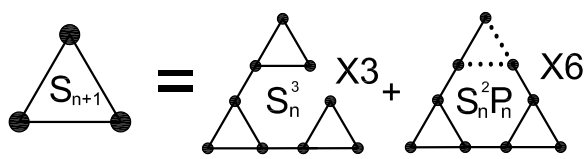

Figure 4: Illustration of the configurations needed to find s_{n+1}.

Lemma 3.3. *For the Hanoi graph* H_n *with* $n \geq 1$, $s_n l_n = 3p_n^2$.

Proof. By induction. For $n = 1$, using the initial conditions $s_1 = 3$, $p_1 = 1$ and $l_1 = 1$, the result is true. Let us assume that for $n = k$, the lemma is true. For $n = k + 1$, using Lemma 3.2, we have that

$$\begin{aligned} s_{k+1}l_{k+1} - 3p_{k+1}^2 &= (3s_k^3 + 6s_k^2 p_k)(s_k^3 + 12s_k^2 p_k + 3s_k^2 l_k + 36s_k p_k^2 + 12s_k p_k l_k \\ &\quad + 14p_k^3) - 3(s_k^3 + 7s_k^2 p_k + 7s_k p_k^2 + s_k^2 l_k)^2 \\ &= 3s_k^2(s_k^2 + 4s_k p_k + 7p_k^2 - s_k l_k)(s_k l_k - 3p_k^2) \,. \end{aligned}$$

By induction hypothesis $s_k l_k - 3p_k^2 = 0$, we obtain the result. □

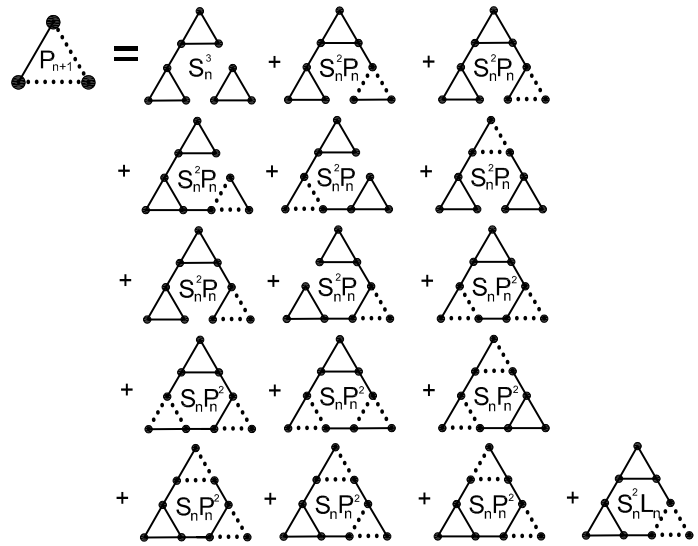

Figure 5: Illustration of the configurations needed to find p_{n+1}.

Lemma 3.4. *For the Hanoi graph H_n with $n \geq 1$, $\frac{s_{n+1}}{s_n^3} = \frac{5^n}{3^{n-1}}$.*

Proof. From Eq. (1), we have

$$\frac{s_{n+1}}{s_n^3} = \frac{3s_n^3 + 6s_n^2 p_n}{s_n^3} = 3 + 6\frac{p_n}{s_n},$$

which can be rewritten as

$$\frac{p_n}{s_n} = \frac{1}{6}\left(\frac{s_{n+1}}{s_n^3} - 3\right).$$

Using Eq. (2) and Lemma 3.3, we obtain

$$\begin{aligned}\frac{p_{n+1}}{s_n^3} &= 1 + 7\frac{p_n}{s_n} + 10\left(\frac{p_n}{s_n}\right)^2 \\ &= 1 + 7\left[\frac{1}{6}\left(\frac{s_{n+1}}{s_n^3} - 3\right)\right] + 10\left[\frac{1}{6}\left(\frac{s_{n+1}}{s_n^3} - 3\right)\right]^2 \\ &= -\frac{s_{n+1}}{2s_n^3} + \frac{5s_{n+1}^2}{18s_n^6},\end{aligned}$$

which leads to

$$\frac{p_{n+1}}{s_{n+1}} = \frac{p_{n+1}}{s_n^3}\frac{s_n^3}{s_{n+1}} = \left(-\frac{s_{n+1}}{2s_n^3} + \frac{5s_{n+1}^2}{18s_n^6}\right)\frac{s_n^3}{s_{n+1}} = -\frac{1}{2} + \frac{5s_{n+1}}{18s_n^3}.$$

According to Eq. (1), we have $s_{n+2} = 3s_{n+1}^3 + 6s_{n+1}^2 p_{n+1}$ and

$$\frac{s_{n+2}}{s_{n+1}^3} = 3 + 6\frac{p_{n+1}}{s_{n+1}} = 3 + 6\left(-\frac{1}{2} + \frac{5s_{n+1}}{18s_n^3}\right) = \frac{5s_{n+1}}{3s_n^3},$$

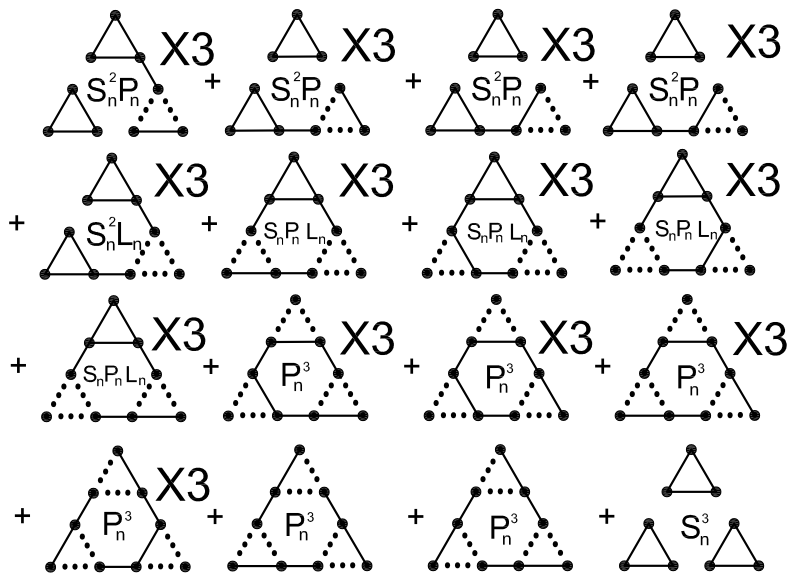

Figure 6: Spanning subgraphs of H_{n+1} that contribute to the term $s_n^3 + 12s_n^2 p_n + 3s_n^2 l_n + 12s_n p_n l_n + 14p_n^3$ of l_{n+1}.

which, together with the initial condition $\frac{s_2}{s_1^3} = 5$ yields

$$\frac{s_{n+1}}{s_n^3} = \frac{5^n}{3^{n-1}} .$$

□

We now give one of the main results of this paper.

Theorem 3.5. *For the Hanoi graph H_n, with $n \geq 1$, the number of spanning trees s_n and spanning subgraphs p_n and l_n is*

$$s_n = 3^{\frac{1}{4}3^n + \frac{1}{2}n - \frac{1}{4}} \cdot 5^{\frac{1}{4}3^n - \frac{1}{2}n - \frac{1}{4}}, \tag{4}$$

$$p_n = \frac{1}{6} \cdot \frac{5^n - 3^n}{5^n} \cdot 3^{\frac{1}{4}3^n - \frac{1}{2}n + \frac{3}{4}} \cdot 5^{\frac{1}{4}3^n + \frac{1}{2}n - \frac{1}{4}}, \tag{5}$$

$$l_n = \frac{1}{4} \cdot (3^n - 5^n)^2 \cdot 3^{\frac{1}{4}3^n - \frac{3}{2}n + \frac{3}{4}} \cdot 5^{\frac{1}{4}3^n - \frac{1}{2}n - \frac{1}{4}} . \tag{6}$$

Proof. From Lemma 3.4, we have $s_{n+1} = \frac{5^n}{3^{n-1}} s_n^3$, which with initial condition $s_1 = 3$ gives Eq. (4).

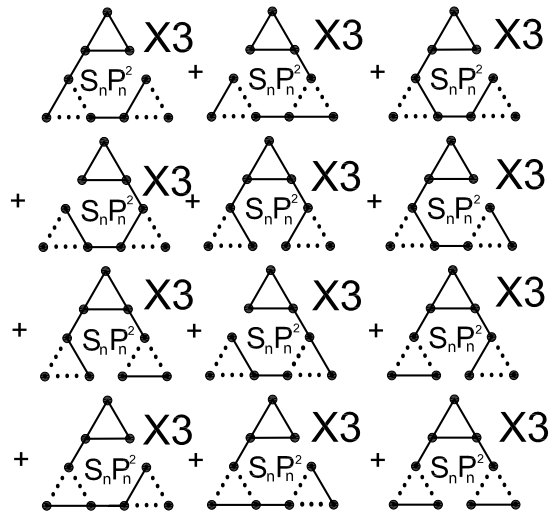

Figure 7: Spanning subgraphs of H_{n+1} that contribute to the term $36 s_n p_n^2$ of l_{n+1}.

From the proof of Lemma 3.4 we know that $p_n = \frac{s_{n+1}-3s_n^3}{6s_n^2}$. Inserting the expressions for s_{n+1} and s_n in Eq. (4) into this formula leads to p_n.

Lemma 3.3 gives $l_n = \frac{3p_n^2}{s_n}$. Using the obtained results for s_n and p_n, we arrive at l_n. $\square$

Note that Eq. (4) was previously obtained [23] by using a different method.

After finding an explicit expression for the number of spanning trees of H_n, we now calculate its spanning tree entropy which is defined as:

$$h = \lim_{V_n \to \infty} \frac{s_n}{V_n} \tag{7}$$

where V_n denotes the number of vertices, see [36].

Thus, for the Hanoi graph we obtain $h = \frac{1}{4}(\ln 3 + \ln 5) \simeq 0.677$.

We can compare this asymptotic value of the entropy of the spanning trees of H_n with those of other graphs with the same average degree. For example, the value for the honeycomb lattice is 0.807 [14] and the 4-8-8 (bathroom tile) and 3-12-12 lattices have entropy values 0.787 and 0.721, respectively [15]. Thus, the asymptotic value for the Hanoi graph is the lowest reported for graphs with average degree 3. This reflects the fact that the number of spanning trees in H_n, although growing exponentially, does so at a lower rate than lattices with the same average degree.

4. The degree distribution for a vertex of the spanning trees

In this section, we compute the probabilities of different coordination numbers at a given vertex on a random spanning tree on the Hanoi graph H_n. We note that, by using similar techniques, it has been possible to obtain more results for the closely related Siperpiński graphs [28, 37]. In the previous calculation, each vertex of H_n corresponds to a state/configuration of all n disks and thus is labeled by an n-tuple $\boldsymbol{\alpha} = \alpha_1\alpha_2\cdots\alpha_n$.

In what follows for the convenience of description, we provide an alternative way of labeling vertices in H_n, by assigning to each vertex a sequence $\boldsymbol{\alpha} = \alpha_1\alpha_2\cdots\alpha_e$, where $1 \le e \le n$ and $\alpha_i \in \{0, 1, 2\}$. The new labeling method is as follows, see Fig. 8. For $n = 1$, H_1 is a triangle, we label the three vertices by 0, 1 and 2. When $n = 2$, H_2 contains three replicas of H_1, denoted by H_1^1, H_1^2, and H_1^3. On the topmost copy H_1^1, we put a prefix 0 on the label of each node in H_1. Similarly, we add a prefix 1 (or 2) to the labeling of vertices on the leftmost (or rightmost) copy H_1^2 (or H_1^3). If a vertex's label ends with several identical digits, we just keep it once. For example, we use 010 to replace 0100. For $n \ge 3$, we label the vertices in H_n by adding prefixes to three replicas of H_{n-1} in the same way, and delete repetitive suffix.

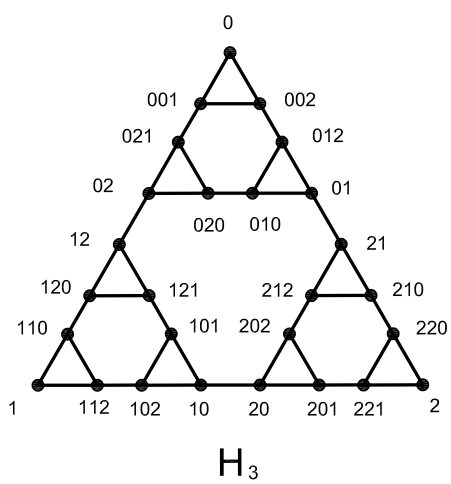

Figure 8: An illustration for a new labeling of vertices in H_3.

In this way, all vertices in H_n are labeled by sequences of three digits 0, 1, and 2, with different length ranging from 1 to n, and each vertex has a unique labeling. For example, for all n, the three outmost vertices of H_n have labels of 0, 1, and 2, while the other six outmost vertices of H_{n-1}^1, H_{n-1}^2, and H_{n-1}^3 forming H_n, each has a label consisting of two digits, which are called connecting vertices hereafter.

After labeling the vertices in H_n, we are now in a position to study the probability distribution of degree for a vertex on all spanning trees. For this purpose, we introduce some quantities.

Definition 4.1. *Consider a vertex α in H_n. We define $s_{n,i}(\alpha)$ as the number of spanning trees in which the degree of the node α is i. Then the probability that among all spanning tree the degree of vertex α is i is defined by $S_{n,i}(\alpha) = s_{n,i}(\alpha)/s_n$. Similarly, we define $r_{n,i}(\alpha)$ ($t_{n,i}(\alpha)$, $p_{n,i}(\alpha)$) as the number of spanning subgraphs consisting of two trees such that one outmost vertex 0 (1, 2) is in one tree while the other two outmost vertices 1 and 2 (0 and 2, 0 and 1) are in the other tree, and the degree of α is i. Define the probabilities $R_{n,i}(\alpha) = r_{n,i}(\alpha)/r_n$, $T_{n,i}(\alpha) = t_{n,i}(\alpha)/t_n$ $P_{n,i}(\alpha) = p_{n,i}(\alpha)/p_n$. Finally, we define $l_{n,i}(\alpha)$ as the number of spanning subgraphs containing three trees such that the three outmost vertices 0, 1 and 2 belongs to a different tree, and the degree of α is i. Define the probability $L_{n,i}(\alpha) = l_{n,i}(\alpha)/l_n$.*

In the following text, we will first determine $S_{n,i}(\alpha)$ for the three outmost vertices in in H_n, then we will compute $S_{n,i}(\alpha)$ for the six connecting vertices, and finally we will calculate $S_{n,i}(\alpha)$ for an arbitrary vertex α.

For the three outmost vertices 0, 1, and 2, each has a degree of 2, and thus $s_{n,3}(0) = p_{n,3}(0) = l_{n,3}(0) = 0$. In addition, by symmetry we have $s_{n,i}(0) = s_{n,i}(1) = s_{n,i}(2)$, $p_{n,i}(0) = p_{n,i}(1)$ for $i = 1, 2$, and $l_{n,i}(0) = l_{n,i}(1) = l_{n,i}(2)$ for $i = 0, 1, 2$. Hence, for the outmost vertices, we only need to find $S_{n,i}(0)$ for $i = 1, 2$.

4.1. Determination of $S_{n,i}(0)$ with $i = 1, 2$

For the graph H_n, associated with the Tower of Hanoi puzzle with n disc, we have the following result.

Theorem 4.2. *For the Hanoi graph H_n with $n \geq 1$,*

$$S_{n,1}(0) = \frac{5}{7} - \frac{5}{7}\left(\frac{1}{15}\right)^n, \tag{8}$$

$$S_{n,2}(0) = \frac{2}{7} + \frac{5}{7}\left(\frac{1}{15}\right)^n. \tag{9}$$

$$P_{n,1}(0) = \frac{5}{7} + \frac{9 \cdot 5^n + 5 \cdot 3^n}{7 \cdot 15^n \cdot (5^n - 3^n)}, \tag{10}$$

$$P_{n,2}(0) = \frac{2}{7} - \frac{9 \cdot 5^n + 5 \cdot 3^n}{7 \cdot 15^n \cdot (5^n - 3^n)}, \tag{11}$$

$$P_{n,0}(2) = \frac{5 \cdot (15^n - 1)}{7 \cdot 5^n \cdot (5^n - 3^n)}, \tag{12}$$

$$P_{n,1}(2) = \frac{5}{7} - \frac{12}{7}\left(\frac{1}{15}\right)^n - \frac{3 \cdot (15^n - 1)}{7 \cdot 5^n \cdot (5^n - 3^n)}, \tag{13}$$

$$P_{n,2}(2) = \frac{2}{7} + \frac{12}{7}\left(\frac{1}{15}\right)^n - \frac{2\cdot(15^n-1)}{7\cdot 5^n\cdot(5^n-3^n)}\,, \tag{14}$$

$$L_{n,0}(0) = \frac{10\cdot 3^n}{21\cdot(5^n-3^n)} + \frac{18\cdot 5^n+10\cdot 3^n}{21\cdot 5^n\cdot(5^n-3^n)^2}\,, \tag{15}$$

$$L_{n,1}(0) = \frac{5}{7} + \frac{9}{7}\left(\frac{1}{15}\right)^n - \frac{2\cdot(15^n-1)}{7\cdot 5^n\cdot(5^n-3^n)} - \frac{8\cdot 3^n}{3\cdot 5^n\cdot(5^n-3^n)^2}\,, \tag{16}$$

$$L_{n,2}(0) = \frac{2}{7} - \frac{9}{7}\left(\frac{1}{15}\right)^n - \frac{4\cdot(15^n+6)}{21\cdot 5^n\cdot(5^n-3^n)} + \frac{4\cdot 3^n}{3\cdot 5^n\cdot(5^n-3^n)^2}\,. \tag{17}$$

In order to prove Theorem 4.2 and other main results, we shall first give the following lemma.

Lemma 4.3. *For the Tower of Hanoi graph H_n with $n \geq 1$,*

$$s_{n,1}(0) = \left(\frac{5}{7} - \frac{5}{7}\left(\frac{1}{15}\right)^n\right)\cdot 3^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\,, \tag{18}$$

$$s_{n,2}(0) = \left(\frac{2}{7} + \frac{5}{7}\left(\frac{1}{15}\right)^n\right)\cdot 3^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\,, \tag{19}$$

$$p_{n,1}(0) = \left(\frac{5}{14}\frac{5^n-3^n}{5^n} + \frac{9\cdot 5^n+5\cdot 3^n}{14\cdot 75^n}\right)\cdot 3^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\,, \tag{20}$$

$$p_{n,2}(0) = \left(\frac{1}{7}\frac{5^n-3^n}{5^n} - \frac{9\cdot 5^n+5\cdot 3^n}{14\cdot 75^n}\right)\cdot 3^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\,, \tag{21}$$

$$p_{n,0}(2) = \frac{1}{14}\left(1 - \frac{1}{15^n}\right)\cdot 3^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n-\frac{1}{2}n+\frac{3}{4}}\,, \tag{22}$$

$$p_{n,1}(2) = \frac{5}{14}\left(1 - \frac{8\cdot 3^n}{5\cdot 5^n} - \frac{12}{5\cdot 15^n} + \frac{3}{25^n}\right)\cdot 3^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\,, \tag{23}$$

$$p_{n,2}(2) = \frac{12}{7}\left(1 - \frac{2\cdot 3^n}{5^n} + \frac{6}{15^n} - \frac{5}{25^n}\right)\cdot 3^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\,, \tag{24}$$

$$l_{n,0}(0) = \frac{5}{14}\left(\frac{5^n-3^n}{5^n} + \frac{9}{5\cdot 15^n} + \frac{1}{25^n}\right)\cdot 3^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}\,, \tag{25}$$

$$l_{n,1}(0) = \frac{15}{28}\left(1 - \frac{12\cdot 3^n}{5\cdot 5^n} + \frac{7\cdot 9^n}{5\cdot 25^n} + \frac{9}{5\cdot 15^n} - \frac{16}{5\cdot 25^n}\right)3^{\frac{1}{4}3^n-\frac{3}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{3}{2}n-\frac{1}{4}}\,, \tag{26}$$

$$l_{n,2}(0) = \frac{3}{14}\left(1 - \frac{8\cdot 3^n}{3\cdot 5^n} + \frac{5\cdot 9^n}{3\cdot 25^n} - \frac{9}{2\cdot 15^n} + \frac{5}{25^n} + \frac{25\cdot 3^n}{6\cdot 125^n}\right)3^{\frac{1}{4}3^n-\frac{3}{2}n-\frac{1}{4}}\cdot 5^{\frac{1}{4}3^n+\frac{3}{2}n-\frac{1}{4}}\,. \tag{27}$$

Proof. Based on Figs. 4, 5, 6, and 7, we can establish the following recursive relations

$$s_{n+1,i}(0) = 3s_{n,i}(0)s_n^2 + 2p_{n,i}(0)s_n^2 + 4s_{n,i}(0)s_n p_n \,, \tag{28}$$

$$p_{n+1,i}(0) = s_{n,i}(0)s_n^2 + p_{n,i}(0)s_n^2 + 6s_{n,i}(0)s_n p_n + 4p_{n,i}(0)s_n p_n + 3s_{n,i}(0)p_n^2 + s_{n,i}(0)p_n l_n \,, \tag{29}$$

$$\begin{aligned} p_{n+1,i}(2) = {} & s_{n,i}(0)s_n^2 + 2p_{n,i}(0)s_n^2 + 3p_{n,i}(2)s_n^2 + l_{n,i}(0)s_n^2 + \\ & 2s_{n,i}(0)s_n p_n + 2p_{n,i}(0)s_n p_n + 4p_{n,i}(2)s_n p_n + s_{n,i}(0)p_n^2 \,, \end{aligned} \tag{30}$$

$$\begin{aligned} l_{n+1,i}(0) = {} & s_{n,i}(0)s_n^2 + 2p_{n,i}(0)s_n^2 + 2p_{n,i}(2)s_n^2 + l_{n,i}(0)s_n^2 + \\ & 8s_{n,i}(0)s_n p_n + 12p_{n,i}(0)s_n p_n + 12p_{n,i}(2)s_n p_n + \\ & 4l_{n,i}(0)s_n p_n + 12s_{n,i}(0)p_n^2 + 8p_{n,i}(0)p_n^2 + 6p_{n,i}(2)p_n^2 + \\ & 2s_{n,i}(0)s_n l_n + 2p_{n,i}(0)s_n l_n + 2p_{n,i}(2)s_n l_n + 4s_{n,i}(0)p_n l_n \,. \end{aligned} \tag{31}$$

Using the initial conditions $s_{1,1}(0) = 2$, $s_{1,2}(0) = 1$, $p_{1,1}(0) = 1$, $p_{1,2}(0) = 0$, $p_{1,0}(2) = 1$, $p_{1,1}(2) = p_{1,2}(2) = 0$, $l_{1,0}(0) = 1$, and $l_{1,1}(0) = l_{1,2}(0) = 0$, the above recursive relations are solved to obtain Lemma 4.3. □

From Eqs. (4-6) and Eqs. (18-27), we can prove Theorem 4.2.

4.2. Determination of $S_{n,i}(\alpha)$ with α being connecting vertices

We proceed to calculate $S_{n,i}(\alpha)$ with $i = 1, 2, 3$, where α are the six connecting vertices, the length of whose labels is two. By definition, the six connecting vertices are 01, 10, 02, 20, 12, 21. We obtain that $S_{n,i}(01) = S_{n,i}(02) = S_{n,i}(10) = S_{n,i}(12) = S_{n,i}(20) = S_{n,i}(21)$. Since connecting vertices only exist in H_n for $n \geq 2$, we only need to determine $S_{n+1,1}(01)$ for $n \geq 1$. Thus,

Theorem 4.4. *For the Tower of Hanoi graph H_n and $n \geq 1$,*

$$S_{n+1,1}(01) = \frac{1}{14} 5^{1-2n} \cdot (15^n - 1) \,, \tag{32}$$

$$S_{n+1,2}(01) = \frac{1}{42} \left(30 + 6 \cdot 5^{1-2n} - 3^{2+n} \cdot 5^{-n} - 23 \cdot 5^{-n}\right) \,, \tag{33}$$

$$S_{n+1,3}(01) = \frac{1}{42} \left(12 - 3 \cdot 5^{1-2n} - 2 \cdot 3^{1+n} \cdot 5^{-n} + 23 \cdot 5^{-n}\right) \,, \tag{34}$$

$$P_{n+1,1}(01) = \frac{3 \cdot 5^{1-2n} (5^n - 3^n)(15^n - 1)}{14 (5^{n+1} - 3^{n+1})} \,, \tag{35}$$

$$P_{n+1,2}(01) = \frac{7\left(\frac{9}{5}\right)^n + 5\cdot 3^{1-n} - 2\cdot 3^n - 19\cdot 5^{-n} - 3\cdot 5^{n+1}}{7\cdot 3^{n+1} - 7\cdot 5^{n+1}}\,, \tag{36}$$

$$P_{n+1,3}(01) = \frac{49\cdot 3^{1-n} - 28\cdot 3^{n+1} + 3^n 5^{3-2n} - 56\cdot 5^{1-n} + 42\cdot 5^n + 2\cdot 5^{2-n} 9^n}{14\left(5^{n+1} - 3^{n+1}\right)}, \tag{37}$$

$$P_{n+1,1}(02) = \frac{3\cdot 5^{1-2n}\left(3\cdot 5^n - 3^n\right)\left(15^n - 1\right)}{14\left(5^{n+1} - 3^{n+1}\right)}\,, \tag{38}$$

$$P_{n+1,2}(02) = \frac{19\cdot 3^{1-n} + 19\cdot 3^{n+1} + 2\cdot 3^{n+1} 5^{1-2n} - 113\cdot 5^{-n} - 2\cdot 5^{n+2} - 5^{-n} 9^{n+1}}{14\left(3^{n+1} - 5^{n+1}\right)}\,, \tag{39}$$

$$P_{n+1,3}(02) = \frac{75^{-n}\left(106\cdot 3^n 5^{n+1} - 125\cdot 9^n - 453\cdot 25^n - 2\cdot 5^{n+2} 27^n + 184\cdot 225^n - 86\cdot 375^n\right)}{14\left(3^{n+1} - 5^{n+1}\right)}, \tag{40}$$

$$P_{n+1,1}(20) = \frac{25^{-n}\left(5\cdot 3^n + 21\cdot 5^n + 7\cdot 3^n 5^{2n+1} - 5^{n+1} 9^n\right)}{14\left(5^{n+1} - 3^{n+1}\right)}\,, \tag{41}$$

$$P_{n+1,2}(20) = \frac{55\cdot 3^{-n} - 19\cdot 3^{n+1} + 3^n 5^{1-2n} - 11\cdot 5^{1-n} + 2\cdot 5^{n+2} + 5^{-n} 9^{n+1}}{14\left(5^{n+1} - 3^{n+1}\right)}\,, \tag{42}$$

$$P_{n+1,3}(20) = \frac{75^{-n}\left(95\cdot 9^n - 38\cdot 15^n - 77\cdot 25^n - 14\cdot 3^{n+1} 125^n + 38\cdot 135^n + 52\cdot 225^n\right)}{14\left(3^{n+1} - 5^{n+1}\right)}, \tag{43}$$

$$L_{n+1,1}(01) = \frac{25^{-n}\left(37\cdot 3^n 5^{3n+1} - 25\cdot 9^n + 38\cdot 15^n + 39\cdot 25^n - 2\cdot 3^{2n+1} 25^{n+1} + 5^{n+2} 27^n\right)}{14\left(3^{n+1} - 5^{n+1}\right)^2}\,, \tag{44}$$

$$\begin{aligned} L_{n+1,2}(01) = {} & \frac{75^{-n}\left(-3^{4n+3} 5^n + 27\cdot 5^{3n+1} - 29\cdot 3^{2n+1} 5^{3n+1} + 2\cdot 3^n 5^{4n+3}\right)}{14\left(3^{n+1} - 5^{n+1}\right)^2} \\ & + 75^{-n}\,\frac{20\cdot 27^n + 8\cdot 25^n 27^{n+1} - 13\cdot 45^n - 188\cdot 75^n}{14\left(3^{n+1} - 5^{n+1}\right)^2}\,, \end{aligned} \tag{45}$$

$$\begin{aligned} L_{n+1,3}(01) = {} & \frac{75^{-n}\left(-319\cdot 3^{n+1} 25^n + 62\cdot 3^{3n+1} 25^n + 65\cdot 27^n\right)}{14\left(3^{n+1} - 5^{n+1}\right)^2} \\ & + 75^{-n}\,\frac{199\cdot 45^n + 789\cdot 125^n + 26\cdot 405^n - 562\cdot 1125^n + 254\cdot 1875^n}{14\left(3^{n+1} - 5^{n+1}\right)^2}. \end{aligned} \tag{46}$$

Proof. Based on Fig. 4, we have the recursion relations for the connecting vertex 01:

$$s_{n+1,1}(01) = s_{n,1}(1) s_n^2 + p_{n,0}(2) s_n^2\,, \tag{47}$$

$$s_{n+1,2}(01) = s_{n,2}(1)s_n^2 + 2s_{n,1}(1)s_n^2 + 4s_{n,1}(1)s_n p_n + p_{n,1}(1)s_n^2 + p_{n,1}(2)s_n^2 \,, \quad (48)$$

$$s_{n+1,3}(01) = 2s_{n,2}(1)s_n^2 + 4s_{n,2}(1)s_n p_n + p_{n,2}(1)s_n^2 + p_{n,2}(2)s_n^2 \,. \quad (49)$$

Since the quantities on the right-hand side of Eqs. (47-49) have been explicitly determined, according to the relation $S_{n+1,i}(01) = s_{n+1,i}(01)/s_{n+1}$, we obtain Eqs. (32-34).

Analogously, we find $P_{n+1,i}(\alpha)$ and $L_{n+1,i}(\alpha)$, when α are connecting vertices. It is obvious that $P_{n,0}(\alpha) = L_{n,0}(\alpha) = 0$. Note that $l_{n,i}(01) = l_{n,i}(02) = l_{n,i}(10) = l_{n,i}(12) = l_{n,i}(20) = l_{n,i}(21)$, $p_{n,i}(01) = p_{n,i}(10)$, $p_{n,i}(02) = p_{n,i}(12)$, $p_{n,i}(20) = p_{n,i}(21)$. Using Figs. 5, 6, and 7, we can establish the recursive relations

$$p_{n+1,1}(01) = s_{n,1}(1)s_n p_n + p_{n,0}(2)s_n p_n \,, \quad (50)$$

$$\begin{aligned} p_{n+1,2}(01) = {} & s_{n,2}(1)s_n p_n + s_{n,1}(1)s_n^2 + 5s_{n,1}(1)s_n p_n + s_{n,1}(1)s_n l_n + \\ & 3s_{n,1}p_n^2 + p_{n,1}(1)s_n^2 + 3p_{n,1}(1)s_n p_n + p_{n,1}(2)s_n p_n \,, \end{aligned} \quad (51)$$

$$\begin{aligned} p_{n+1,3}(01) = {} & s_{n,2}(1)s_n^2 + 5s_{n,2}(1)s_n p_n + s_{n,2}(1)s_n l_n + 3s_{n,2}(1)p_n^2 \\ & p_{n,2}(1)s_n^2 + 3p_{n,2}(1)s_n p_n + p_{n,2}(2)s_n p_n \,, \end{aligned} \quad (52)$$

$$p_{n+1,1}(02) = s_{n,1}(1)s_n^2 + 3s_{n,1}(1)s_n p_n + p_{n,0}(2)s_n^2 + 3p_{n,0}(2)s_n p_n \,, \quad (53)$$

$$\begin{aligned} p_{n+1,2}(02) = {} & s_{n,2}(1)s_n^2 + 3s_{n,2}(1)s_n p_n + 3s_{n,1}(1)s_n p_n + s_{n,1}(1)s_n l_n + \\ & 3s_{n,1}(1)p_n^2 + p_{n,1}(2)s_n^2 + 3p_{n,1}(2)s_n p_n + p_{n,1}(1)s_n p_n \,, \end{aligned} \quad (54)$$

$$\begin{aligned} p_{n+1,3}(02) = {} & 3s_{n,2}(1)s_n p_n + s_{n,2}(1)s_n l_n + 3s_{n,2}(1)p_n^2 + p_{n,2}(2)s_n^2 + \\ & 3p_{n,2}(2)s_n p_n + p_{n,2}(1)s_n p_n \,, \end{aligned} \quad (55)$$

$$p_{n+1,1}(20) = s_{n,1}(1)s_n^2 + s_{n,1}(1)s_n p_n + 2p_{n,1}(1)s_n^2 + p_{n,0}(2)s_n^2 + p_{n,0}(2)s_n p_n \,, \quad (56)$$

$$\begin{aligned} p_{n+1,2}(20) = {} & s_{n,2}(1)s_n^2 + s_{n,2}(1)s_n p_n + 2p_{n,2}(1)s_n^2 + s_{n,1}(1)s_n p_n + \\ & s_{n,1}(1)p_n^2 + 2p_{n,1}(1)s_n^2 + 5p_{n,1}(1)s_n p_n + p_{n,1}(2)s_n^2 + \\ & p_{n,1}(2)s_n p_n + l_{n,1}(1)s_n^2 \,, \end{aligned} \quad (57)$$

$$\begin{aligned} p_{n+1,3}(20) = {} & s_{n,2}(1)s_n p_n + s_{n,2}(1)p_n^2 + 2p_{n,2}(1)s_n^2 + 5p_{n,2}(1)s_n p_n + \\ & p_{n,2}(2)s_n^2 + p_{n,2}(2)s_n p_n + l_{n,2}(1)s_n^2 \,, \end{aligned} \quad (58)$$

$$\begin{aligned} l_{n+1,1}(01) = {} & s_{n,1}(1)s_n^2 + 6s_{n,1}(1)s_n p_n + s_{n,1}(1)s_n l_n + 4s_{n,1}(1)p_n^2 + \\ & 2p_{n,1}(1)s_n^2 + 8p_{n,1}s_n p_n + p_{n,0}(2)s_n^2 + 6p_{n,0}(2)s_n p_n + \\ & p_{n,0}(2)s_n l_n + 4p_{n,0}(2)p_n^2 \,, \end{aligned} \quad (59)$$

$$
\begin{aligned}
l_{n+1,2}(01) = {} & s_{n,2}(1)s_n^2 + 6s_{n,2}(1)s_n p_n + s_{n,2}(1)s_n l_n + 4s_{n,2}(1)p_n^2 + \\
& 2p_{n,2}(1)s_n^2 + 8p_{n,2}(1)s_n p_n + 2s_{n,1}(1)s_n p_n + s_{n,1}(1)s_n l_n + \\
& 8s_{n,1}(1)p_n^2 + 4s_{n,1}(1)p_n l_n + p_{n,1}(1)s_n^2 + 10p_{n,1}(1)s_n p_n + \\
& 3p_{n,1}(1)s_n l_n + 10p_{n,1}(1)p_n^2 + p_{n,1}(2)s_n^2 + 6p_{n,1(2)}s_n p_n + \\
& p_{n,1}(2)s_n l_n + 4p_{n,1}(2)p_n^2 + l_{n,1}(1)s_n^2 + 4l_{n,1}(1)s_n p_n \,, \qquad (60)
\end{aligned}
$$

$$
\begin{aligned}
l_{n+1,3}(01) = {} & 2s_{n,2}(1)s_n p_n + s_{n,2}(1)s_n l_n + 8s_{n,2}(1)p_n^2 + 4s_{n,2}(1)p_n l_n + \\
& p_{n,2}(1)s_n^2 + 10p_{n,2}(1)s_n p_n + 3p_{n,2}(1)s_n l_n + 10p_{n,2}(1)p_n^2 + \\
& p_{n,2}(2)s_n^2 + 6p_{n,2}(2)s_n p_n + p_{n,2}(2)s_n l_n + 4p_{n,2}(2)p_n^2 + \\
& l_{n,2}(1)s_n^2 + 4l_{n,2}(1)s_n p_n \,. \qquad (61)
\end{aligned}
$$

From Theorem 3.5 and Lemma 4.3, we obtain the exact expressions for $p_{n+1,i}(01)$, $p_{n+1,i}(02)$, $p_{n+1,i}(20)$, $l_{n+1,i}(01)$, and thus for $P_{n+1,i}(01)$, $P_{n+1,i}(02)$, $P_{n+1,i}(20)$, $L_{n+1,i}(01)$. $\square$

4.3. Determination of $S_{n,i}(\boldsymbol{\alpha})$ for an arbitrary vertex $\boldsymbol{\alpha}$

We finally calculate $S_{n,i}(\boldsymbol{\alpha})$ for an arbitrary vertex $\boldsymbol{\alpha}$. Note that a vertex $\boldsymbol{\alpha}$ in H_n has a label $\gamma_1\gamma_2\cdots\gamma_p$ with length p, where $1 \le p \le n$ and $\gamma_z \in \{0,1,2\}$ for $1 \le z \le p$. In the preceding subsections we have determined the degree distribution among all spanning trees for the three outmost vertices corresponding to the case $p = 1$ and the six connecting vertices associated with the case $p = 2$. Next, we will show that for any vertex $\gamma_1\gamma_2\cdots\gamma_p\gamma_{p+1}$ in H_{n+1} with label length $p+1$, $S_{n+1,i}(\gamma_1\gamma_2\cdots\gamma_p\gamma_{p+1})$ is obtained from some related quantities for the vertex $\gamma_1\gamma_3\cdots\gamma_p\gamma_{p+1}$ in H_n with label length p.

Let γ be a sequence of $\{0,1,2\}$, and let $|\gamma|$ be the length of γ satisfying $0 \le |\gamma| \le n-1$. Then, all vertices in H_n have the label form 0γ, 1γ, or 2γ, while all vertices in H_{n+1} have the label form $0k\gamma$, $1k\gamma$, or $2k\gamma$ with $k \in \{0,1,2\}$, corresponding to vertices in H_n^1, H_n^2, and H_n^3 that form H_{n+1}. Below we are only concerned with $S_{n+1,i}(0k\gamma)$, since $S_{n+1,i}(1k\gamma)$ and $S_{n+1,i}(2k\gamma)$ can be easily obtained from $S_{n+1,i}(0k\gamma)$ by symmetry.

For an arbitrary vertex $\boldsymbol{\alpha}$ in H_n, we define the following row vector, which contains all quantities we are interested in:

$$
\boldsymbol{M}_{n,i}(\boldsymbol{\alpha}) = \begin{bmatrix} S_{n,i}(\boldsymbol{\alpha}) & P_{n,i}(\boldsymbol{\alpha}) & T_{n,i}(\boldsymbol{\alpha}) & R_{n,i}(\boldsymbol{\alpha}) & L_{n,i}(\boldsymbol{\alpha}) \end{bmatrix},
$$

where $n \ge 1$. Then, our task is reduced to evaluating $\boldsymbol{M}_{n+1,i}(0k\gamma)$.

Before giving our main result for this subsection, we introduce some matrices. Let $\boldsymbol{E}_0$ be the 5×5 identity matrix, and let $\boldsymbol{E}_1$ ($\boldsymbol{E}_2$) be an elementary matrix obtained by interchanging the third (second) column

and the fourth column of $\boldsymbol{E}_0$. In other words,

$$\boldsymbol{E}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover, for any non-negative integer n, the matrix $\boldsymbol{C}_n$ is defined as

$$\boldsymbol{C}_n = \begin{bmatrix} \frac{2\cdot 5^n+3^n}{3\cdot 5^n} & \frac{3\cdot 25^n-9^n}{5^n(5^{n+1}-3^{n+1})} & \frac{3\cdot 25^n-9^n}{5^n(5^{n+1}-3^{n+1})} & \frac{(5^n+3^n)^2}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{6\cdot 25^n-2\cdot 9^n}{(5^{n+1}-3^{n+1})^2} \\ \frac{5^n-3^n}{6\cdot 5^n} & \frac{9^n-4\cdot 15^n+3\cdot 25^n}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{(5^n-3^n)^2}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{25^n-9^n}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{7\cdot 125^n+45^n+27^n-3^{n+2}\cdot 25^n}{2\cdot 5^n(5^{n+1}-3^{n+1})^2} \\ \frac{5^n-3^n}{6\cdot 5^n} & \frac{(5^n-3^n)^2}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{9^n-4\cdot 15^n+3\cdot 25^n}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{25^n-9^n}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{7\cdot 125^n+45^n+27^n-3^{n+2}\cdot 25^n}{2\cdot 5^n(5^{n+1}-3^{n+1})^2} \\ 0 & 0 & 0 & \frac{2\cdot 25^n-9^n-15^n}{5^n(5^{n+1}-3^{n+1})} & \frac{2\cdot(5^n-3^n)(3\cdot 25^n-9^n)}{5^n(5^{n+1}-3^{n+1})^2} \\ 0 & 0 & 0 & \frac{3\cdot(5^n-3^n)^2}{2\cdot 5^n(5^{n+1}-3^{n+1})} & \frac{3\cdot(2\cdot 5^n-3^n)(5^n-3^n)^2}{5^n(5^{n+1}-3^{n+1})^2} \end{bmatrix}.$$

Then, by alternatively computing $\boldsymbol{M}_{n+1,i}(00\gamma)$, $\boldsymbol{M}_{n+1,i}(01\gamma)$ and $\boldsymbol{M}_{n+1,i}(02\gamma)$, we obtain $\boldsymbol{M}_{n+1,i}(0k\gamma)$, as the following lemma states.

Lemma 4.5. *For the Tower of Hanoi graph H_n and $n > 2$,*

$$\boldsymbol{M}_{n+1,i}(0k\gamma) = \boldsymbol{M}_{n,i}(0\gamma)\boldsymbol{E}_k\boldsymbol{C}_n \tag{62}$$

holds for all $i = 1, 2, 3$ and $k = 0, 1, 2$.

Proof. We first prove the case $k = 0$.

For this case, based on Figs. 4, 5, 6, and 7, we can establish the following relations:

$$s_{n+1,i}(00\gamma) = 3s_{n,i}(0\gamma)s_n^2 + p_{n,i}(0\gamma)s_n^2 + t_{n,i}(0\gamma)s_n^2 + 4s_{n,i}(0\gamma)s_n p_n\,,$$

$$\begin{aligned} p_{n+1,i}(00\gamma) = {} & s_{n,i}(0\gamma)s_n^2 + p_{n,i}(0\gamma)s_n^2 + 6s_{n,i}(0\gamma)s_n p_n + 3p_{n,i}(0\gamma)s_n p_n + \\ & t_{n,i}(0\gamma)s_n p_n + 3s_{n,i}(0\gamma)p_n^2 + s_{n,i}(0\gamma)s_n l_n\,, \end{aligned}$$

$$\begin{aligned} t_{n+1,i}(00\gamma) = {} & s_{n,i}(0\gamma)s_n^2 + t_{n,i}(0\gamma)s_n^2 + 6s_{n,i}(0\gamma)s_n p_n + p_{n,i}(0\gamma)s_n p_n + \\ & 3t_{n,i}(0\gamma)s_n p_n + 3s_{n,i}(0\gamma)p_n^2 + s_{n,i}(0\gamma)s_n l_n\,, \end{aligned}$$

$$\begin{aligned} r_{n+1,i}(00\gamma) = {} & s_{n,i}(0\gamma)s_n^2 + p_{n,i}(0\gamma)s_n^2 + 3r_{n,i}(0\gamma)s_n^2 + t_{n,i}(0\gamma)s_n^2 + \\ & l_{n,i}(0\gamma)s_n^2 + 2s_{n,i}(0\gamma)s_n p_n + p_{n,i}(0\gamma)s_n p_n + \\ & 4r_{n,i}(0\gamma)s_n p_n + t_{n,i}(0\gamma)s_n p_n + s_{n,i}(0\gamma)p_n^2\,, \end{aligned}$$

$$
\begin{aligned}
l_{n+1,i}(00\gamma) = {} & s_{n,i}(0\gamma)s_n^2 + p_{n,i}(0\gamma)s_n^2 + 2r_{n,i}(0\gamma)s_n^2 + t_{n,i}(0\gamma)s_n^2 + \\
& l_{n,i}(0\gamma)s_n^2 + 8s_{n,i}(0\gamma)s_n p_n + 6p_{n,i}(0\gamma)s_n p_n + 12r_{n,i}(0\gamma)s_n p_n + \\
& 6t_{n,i}(0\gamma)s_n p_n + 4l_{n,i}(0\gamma)s_n p_n + 12s_{n,i}(0\gamma)p_n^2 + 4p_{n,i}(0\gamma)p_n^2 + \\
& 6r_{n,i}(0\gamma)p_n^2 + 4t_{n,i}(0\gamma)p_n^2 + 2s_{n,i}(0\gamma)s_n l_n + p_{n,i}(0\gamma)s_n l_n + \\
& 2r_{n,i}(0\gamma)s_n l_n + p_{n,i}(0\gamma)s_n l_n + 4p_{n,i}(0\gamma)p_n l_n .
\end{aligned}
$$

By definition of $S_{n,i}(\boldsymbol{\alpha})$, $P_{n,i}(\boldsymbol{\alpha})$, $T_{n,i}(\boldsymbol{\alpha})$, $R_{n,i}(\boldsymbol{\alpha})$, and $L_{n,i}(\boldsymbol{\alpha})$, we have

$$
S_{n+1,i}(00\gamma) = \frac{2\cdot 5^n + 3^n}{3\cdot 5^n} S_{n,i}(0\gamma) + \frac{5^n - 3^n}{6\cdot 5^n} P_{n,i}(0\gamma) + \frac{5^n - 3^n}{6\cdot 5^n} T_{n,i}(0\gamma) ,
$$

$$
\begin{aligned}
P_{n+1,i}(00\gamma) = {} & \frac{3\cdot 25^n - 9^n}{5^n(5^{n+1} - 3^{n+1})} S_{n,i}(0\gamma) + \frac{9^n - 4\cdot 15^n + 3\cdot 25^n}{2\cdot 5^n(5^{n+1} - 3^{n+1})} P_{n,i}(0\gamma) + \\
& \frac{(5^n - 3^n)^2}{2\cdot 5^n(5^{n+1} - 3^{n+1})} T_{n,i}(0\gamma) ,
\end{aligned}
$$

$$
\begin{aligned}
T_{n+1,i}(00\gamma) = {} & \frac{3\cdot 25^n - 9^n}{5^n(5^{n+1} - 3^{n+1})} S_{n,i}(0\gamma) + \frac{(5^n - 3^n)^2}{2\cdot 5^n(5^{n+1} - 3^{n+1})} P_{n,i}(0\gamma) + \\
& \frac{9^n - 4\cdot 15^n + 3\cdot 25^n}{2\cdot 5^n(5^{n+1} - 3^{n+1})} T_{n,i}(0\gamma) ,
\end{aligned}
$$

$$
\begin{aligned}
R_{n+1,i}(00\gamma) = {} & \frac{(5^n - 3^n)^2}{2\cdot 5^n(5^{n+1} - 3^{n+1})} S_{n,i}(0\gamma) + \frac{25^n - 9^n}{2\cdot 5^n(5^{n+1} - 3^{n+1})} P_{n,i}(0\gamma) + \\
& \frac{25^n - 9^n}{2\cdot 5^n(5^{n+1} - 3^{n+1})} T_{n,i}(0\gamma) + \frac{2\cdot 25^n - 9^n - 15^n}{5^n(5^{n+1} - 3^{n+1})} R_{n,i}(0\gamma) + \\
& \frac{3\cdot (5^n - 3^n)^2}{2\cdot 5^n(5^{n+1} - 3^{n+1})} L_{n,i}(0\gamma) ,
\end{aligned}
$$

$$
\begin{aligned}
L_{n+1,i}(00\gamma) = {} & \frac{6\cdot 25^n - 2\cdot 9^n}{(5^{n+1} - 3^{n+1})^2} S_{n,i}(0\gamma) + \\
& \frac{7\cdot 125^n + 45^n + 27^n - 3^{n+2}\cdot 25^n}{2\cdot 5^n(5^{n+1} - 3^{n+1})^2} P_{n,i}(0\gamma) + \\
& \frac{7\cdot 125^n + 45^n + 27^n - 3^{n+2}\cdot 25^n}{2\cdot 5^n(5^{n+1} - 3^{n+1})^2} T_{n,i}(0\gamma) + \\
& \frac{2\cdot (5^n - 3^n)\cdot (3\cdot 25^n - 9^n)}{5^n(5^{n+1} - 3^{n+1})^2} R_{n,i}(0\gamma) + \\
& \frac{3\cdot (2\cdot 5^n - 3^n)(5^n - 3^n)^2}{5^n(5^{n+1} - 3^{n+1})^2} L_{n,i}(0\gamma),
\end{aligned}
$$

which can be rewritten in matrix form as

$$
\boldsymbol{M}_{n+1,i}(00\gamma) = \boldsymbol{M}_{n,i}(0\gamma)\boldsymbol{C}_n = \boldsymbol{M}_{n,i}(0\gamma)\boldsymbol{E}_0\boldsymbol{C}_n . \tag{63}
$$

In this way, we have completed the proof of the case $k = 0$. For the other two cases $k = 1$ and $k = 2$, the proof is completely analogous to the case $k = 0$, we omit the details here. $\square$

The first column of the matrix in Eq. (62) gives $S_{n+1,i}(0k\gamma)$ for any vertex $0k\gamma$, which is recursive expressed in terms of the related quantities for vertex 0γ. Let $\boldsymbol{e}_1$ denote the vector $(1,0,0,0,0)^\top$, from Lemma 4.5, we have the following result.

Theorem 4.6. *For the Tower of Hanoi graph H_n and $n > 2$,*

$$\begin{bmatrix} S_{n+1,i}(00\gamma) \\ S_{n+1,i}(01\gamma) \\ S_{n+1,i}(02\gamma) \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{n,i}(0\gamma)\boldsymbol{E}_0\boldsymbol{C}_n \\ \boldsymbol{M}_{n,i}(0\gamma)\boldsymbol{E}_1\boldsymbol{C}_n \\ \boldsymbol{M}_{n,i}(0\gamma)\boldsymbol{E}_2\boldsymbol{C}_n \end{bmatrix} \times \boldsymbol{e}_1 \tag{64}$$

holds for all $i = 1, 2, 3$.

By symmetry, we can obtain the recursive relations for $S_{n+1,i}(1k\gamma)$ and $S_{n+1,i}(2k\gamma)$. Since for arbitrary n and $|\gamma| = 0$ and 1, the terms on the right-hand side of Eq. (64) have been previously determined, we can repeatedly apply Theorem 4.6 to obtain $S_{n,i}(\boldsymbol{\alpha})$ for any vertex $\boldsymbol{\alpha}$ in H_n.

5. Conclusion

In this paper we have found the number of spanning trees of the Hanoi graph by using a direct combinatorial method, based on its self-similar structure, which allows us to obtain an analytical exact expression for any number of discs. The knowledge of exact number of spanning trees for the Hanoi graph shows that their spanning tree entropy is lower than those in other graphs with the same average degree. Our method could be used to further study in this graph, and other self-similar graphs, their spanning forests, connected spanning subgraphs, vertex or edges coverings. We have used it to provide a recursive solution for the degree probability distribution for any vertex on all spanning tree configurations of the Hanoi graph.

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