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Integral Equations and Operator Theory

Positive Isometric Averaging Operators on $\ell^2(\mathbb{Z}, \mu)$

Santiago Boza and Javier Soria

Abstract. We show that positive isometric averaging operators on the sequence space $\ell^2(\mathbb{Z}, \mu)$ are determined by very subtle arithmetic conditions on μ (even for very simple examples), contrary to what happens in the continuous case $L^2(\mathbb{R}^+)$, where any possible average value is realized by a suitable positive isometry.

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1. Introduction

Isometric properties for averaging operators have been considered in different settings. For example, it is well known that the averaging Hardy operator

$$Sf(x) = \frac{1}{x}\int_0^x f(t)\, dt$$

satisfies that [4]

$$\|(S - I)f\|_{L^2} = \|f\|_{L^2}, \quad f \in L^2(\mathbb{R}^+). \tag{1}$$

Weighted versions of (1) on $L^2(w)$ can be found in [6]. With more generality, the study of necessary and sufficient conditions for an operator T to be an isometry on $L^2(\mathbb{R})$ have been obtained in [1], and a characterization was given in terms of the restriction of T to characteristic functions of intervals. These results were further extended in [3] to integral operators T defined on $L^2(X)$, for a general measure space X, where the isometric condition on T was characterized just by looking at the restriction of T to a certain class of monotone functions. Other estimates for $T = S - I$ on $L^p(w)$, on monotone functions, were studied in [2].

It is also worth noticing that the reason why the estimates are taken with respect to the L^2-norm is motivated by [7,8], where it is proved that, in

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the setting of rearrangement invariant spaces, L^2 is the only space for which there exist nontrivial isometries.

In the discrete case of sequence spaces $\ell^2(\mathbb{Z})$, isometric averaging operators are not so easily obtained. In particular, for the Cesàro operator

$$S^d(\{a_n\}_{n\in\mathbb{N}})(m) = \frac{1}{m}\sum_{j=1}^{m} a_j,$$

it can be proved that the corresponding isometric property (1) does not hold. Motivated by this fact, we are interested in studying the existence of isometries T on $\ell^2(\mathbb{Z})$ which, on the one hand, are positive operators and, on the other, they are also λ-averaging operators (see Definition 1.3). To this end, given a positive operator T on $L^2(X)$, we need to define the action of T on constant functions:

Definition 1.1. *Let $T : L^2(X) \to L^2(X)$ be a positive linear operator. We define*

$$T(\mathbf{1}) = \sup\left\{Tf : f \in L^2(X), 0 \le f \le 1\right\}.$$

Remark 1.2. We observe that if X is a finite measure space, then $\chi_X \in L^2(X)$ and $T(\mathbf{1}) = T(\chi_X)$. In general, under no restrictions on either T or X, $T(\mathbf{1})$ could be identically equal to infinity.

One of our main goals in this work is to describe, for a class of discrete measures μ on $\mathbb{Z}$ (which includes the counting measure), the set of all possible averages for positive isometries. We will see that, even in this particular setting, these averages are determined by very subtle arithmetic conditions (see Theorem 3.1). Analogously, we can also consider the case when μ is a measure in $\mathbb{N}$ (see Proposition 2.8). In Remark 1.5 we will motivate why we are restricting our attention to atomic measures.

To formulate the problem in a appropriate way and to fix the notation, we first give the following definition:

Definition 1.3. *Given a discrete and positive measure $\mu = \{\mu_k\}_{k\in\mathbb{Z}}$ on $\mathbb{Z}$, we will say that a positive operator T, defined for a given sequence $x \in \ell^2(\mathbb{Z},\mu)$ as*

$$T(x)(j) = \sum_{k\in\mathbb{Z}} a_{jk} x_k \mu_k, \quad j \in \mathbb{Z}, \tag{2}$$

is a λ-averaging operator, if it satisfies that

$$T(\mathbf{1}) \equiv \lambda\mathbf{1}. \tag{3}$$

We will denote by $\sigma_A^+(\mu,\mathbb{Z}) = \sigma_A^+(\mu)$ the set of all values $\lambda \ge 0$ for which there exists a positive isometric operator such that (3) holds.

Remark 1.4. A straightforward argument shows that (3) is equivalent to the condition

$$\sum_{k\in\mathbb{Z}} a_{jk}\ \mu_k = \lambda,\ \text{ for all } j \in \mathbb{Z}, \tag{4}$$

which implies that the sequence $\{a_{jk}\}_{k\in\mathbb{Z}}$ is in $\ell^1(\mathbb{Z},\mu)$, for any $j \in \mathbb{Z}$.

Clearly, the identity is a trivial example of an operator satisfying Definition 1.3, with $\lambda = 1$, and hence for all measures μ on $\mathbb{Z}$, $1 \in \sigma_A^+(\mu)$.

Observe also that $\lambda = 0$ would imply that $T \equiv 0$. Therefore, $\sigma_A^+(\mu) \subset (0, \infty)$. It also easily seen that $\sigma_A^+(\mu)$ is closed under multiplication (just by looking at the composition of the corresponding operators).

Without loss of generality, we are going to assume that μ is an infinite measure not vanishing at any $k \in \mathbb{Z}$. In fact, if μ is finite and T is a positive isometry with $T(\mathbf{1}) \equiv \lambda \mathbf{1}$, then

$$\lambda \left(\sum_{k \in \mathbb{Z}} \mu_k \right)^{1/2} = \|T(\mathbf{1})\|_{\ell^2(\mathbb{Z},\mu)} = \|\chi_{\mathbb{Z}}\|_{\ell^2(\mathbb{Z},\mu)} = \left(\sum_{k \in \mathbb{Z}} \mu_k \right)^{1/2},$$

which implies that $\lambda = 1$ and $\sigma_A^+(\mu) = \{1\}$. For this reason, from now on we will work with a measure $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$, with $0 < \mu_k < \infty$ (we can identify the support of μ with $\mathbb{Z}$) and $\sum_{k \in \mathbb{Z}} \mu_k = \infty$.

Remark 1.5. We observe that the existence of λ-averaging positive isometric operators in $\mathbb{R}^+$ is trivial, since for any $\lambda > 0$, the dilation operator $T_\lambda f(x) = \lambda f(\lambda^2 x)$ satisfies all these properties: T_λ is a positive isometry in $L^2(\mathbb{R}^+)$ and $T_\lambda(\mathbf{1}) \equiv \lambda \mathbf{1}$, since if $f_N(x) = \chi_{(0,N)}(x)$, then

$$T_\lambda(f_N)(x) = \lambda \chi_{(0,N)}(\lambda^2 x) = \lambda \chi_{(0,N/\lambda^2)}(x) \to \lambda \chi_{\mathbb{R}^+}(x), \quad \text{as } N \to \infty.$$

Hence, $\lambda \chi_{\mathbb{R}^+} = \sup_N T_\lambda f_N \leq T_\lambda(\mathbf{1}) \leq \lambda \chi_{\mathbb{R}^+}$.

We see that this argument fails for $\mathbb{Z}$, since nontrivial dilations are never isometries on $\ell^2(\mathbb{Z})$.

In Sect. 2, we will prove some results about the possible values of the set $\sigma_A^+(\mu)$, when $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ is a general discrete positive measure. In particular, Theorem 2.2 is the main tool to translate the problem from an algebraic or matricial formulation into a geometrical property regarding some suitable partitions of the integers. Proposition 2.8 shows that we can transfer, in a canonical way, the averaging values from $\mathbb{Z}$ to $\mathbb{N}$. In Sect. 3 we prove our main result (Theorem 3.1) which characterizes $\sigma_A^+(\mu)$ for all possible measures μ of the form $\mu_k = a\chi_A(k) + b\chi_B(k)$, $a, b > 0$, $A \cap B = \emptyset$ and $A \cup B = \mathbb{Z}$. In particular, $\sigma_A^+(\mu)$ is a countable set contained in $(0, 1]$, which depends on arithmetic properties of a/b and the cardinality of A and B.

In what follows we will use the following notation: for a given subset A of the integers, $|A|$ will denote the cardinality of A, and if $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ is a measure defined on $\mathbb{Z}$, $\mu(A) = \sum_{k \in A} \mu_k$.

2. Results on General Measures on $\mathbb{Z}$

We start by recalling the following result concerning isometries in an arbitrary real Hilbert space H, see [5].

Lemma 2.1. *Let H be a real Hilbert space, let $T : H \longrightarrow H$, and let T^* denote its adjoint operator. Then T is an isometry, i.e., $\|Tx\| = \|x\|$ for any $x \in H$, if and only if $T^*T = I$.*

Journal: 20 Article No.: 2284 TYPESET DISK LE CP Disp.:2016/3/1 Pages: 14

We are now going to apply this result to the discrete case $\ell^2(\mathbb{Z},\mu)$, to obtain the main characterization needed in Sect. 3 to completely describe $\sigma_A^+(\mu)$:

Theorem 2.2. *A necessary and sufficient condition to find a positive and isometric λ-averaging operator T on $\ell^2(\mathbb{Z},\mu)$ is the existence of a partition $\{I_j\}_{j\in\mathbb{Z}}$, of the set of integers $\mathbb{Z}$, for which*

$$\frac{\mu(I_j)}{\mu_j} = \frac{1}{\lambda^2},\ \textit{for all } j\in\mathbb{Z}. \tag{5}$$

Proof. Let T be a positive and isometric λ-averaging operator given as in (2). Then,

$$T^*(x)(j) = \sum_{k\in\mathbb{Z}} a_{kj} x_k \mu_k, \quad x\in\ell^2(\mathbb{Z},\mu).$$

As an application of Lemma 2.1, since T is an isometry then, for every $j\in\mathbb{Z}$,

$$\sum_{l\in\mathbb{Z}} a_{lj}^2\ \mu_l = \frac{1}{\mu_j}, \tag{6}$$

and for $j\neq k$,

$$\sum_{l\in\mathbb{Z}} a_{lj}\ a_{lk}\ \mu_l = 0. \tag{7}$$

Condition (7) implies that, for all $j\in\mathbb{Z}$, there exists a unique k_j such that $a_{jk_j}>0$ which, in combination with condition (4), shows that $a_{jk_j} = \lambda/\mu_{k_j}$. Thus, the matrix satisfies that on each row j, just one element is different from zero, and on each column k, all the nonzero elements take the same value.

Thus, if we define $I_j = \{l\in\mathbb{Z},\ a_{lj}\neq 0\}$, then it is easy to see that $\{I_j\}_{j\in\mathbb{Z}}$ is a partition of the integers. Finally, condition (6) combined with the equality $a_{lj} = \lambda/\mu_j$, if $l\in I_j$, gives us that for a fixed $j\in\mathbb{Z}$,

$$\frac{1}{\mu_j} = \sum_{l\in I_j} a_{lj}^2\ \mu_l = \sum_{l\in I_j}\frac{\lambda^2}{\mu_j^2}\ \mu_l = \frac{\lambda^2}{\mu_j^2}\sum_{l\in I_j}\mu_l,$$

which is (5).

Conversely, given a partition $\{I_j\}_{j\in\mathbb{Z}}$ and $\lambda>0$ satisfying (5), we define

$$a_{jk} = \begin{cases} \lambda/\mu_k, & \text{if } j\in I_k,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

Finally, if we define T as in (2), T is a positive and isometric λ-averaging operator; i.e., T satisfies (4), (6), and (7). □

Remark 2.3. It is important to observe that the same partition of $\mathbb{Z}$, as a subset of $\mathcal{P}(\mathbb{Z})$, may, or may not, satisfy condition (5), depending on the numeration chosen. For example, if we take

$$I_k = \begin{cases} \{0,1\}, & k=0\\ \{k+1\}, & k>0\\ \{k\}, & k<0,\end{cases}$$

then, there is no measure μ and no $\lambda > 0$ for which (5) holds since, otherwise, for $k = 0$

$$\frac{\mu(I_0)}{\mu_0} = \frac{\mu_0 + \mu_1}{\mu_0} = 1 + \frac{\mu_1}{\mu_0} = \frac{1}{\lambda^2},$$

and, for $k < 0$,

$$\frac{\mu(I_k)}{\mu_k} = \frac{\mu_k}{\mu_k} = 1.$$

Hence, $\lambda = 1$ and $\mu_1 = 0$, which is a contradiction with the fact that $\mu_k > 0$, for all $k \in \mathbb{Z}$.

On the other hand, if we consider the same partition but numbered as follows

$$I_k = \begin{cases} \{0,1\}, & k = 0 \\ \{-k\}, & k > 0 \\ \{-k+1\}, & k < 0, \end{cases}$$

we can arbitrary take μ_0 and μ_1, both strictly positive, such that

$$\frac{\mu(I_0)}{\mu_0} = \frac{\mu_0 + \mu_1}{\mu_0} = 1 + \frac{\mu_1}{\mu_0} := \alpha > 1.$$

For this α, if we define

$$\mu_k = \begin{cases} \alpha^{-2k-1}\mu_1, & k \leq -1 \\ \alpha^{2k-2}\mu_1, & k \geq 2, \end{cases}$$

then, condition (5) holds, with $\lambda = \alpha^{-1/2}$.

Remark 2.4. An example of the matricial representation of an operator T, as in Theorem 2.2, where the sets I_j are intervals, is the following:

$$T \longleftrightarrow \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & \lambda/\mu_k & 0 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & \lambda/\mu_k & 0 & \cdots \\ \cdots & \lambda/\mu_{k-1} & 0 & 0 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \lambda/\mu_{k-1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \lambda/\mu_{k+1} & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & 0 & \lambda/\mu_{k+1} & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

For non necessarily positive isometries, this matricial representation is not longer true. For example, if we consider an isometric averaging convolution operator in $\mathbb{Z}$, $T(a)(j) = (K * a)(j)$, $j \in \mathbb{Z}$, where $K \in \ell^1(\mathbb{Z})$, then Parseval's theorem give us that $|\widehat{K}(\theta)| = 1$, $\theta \in \mathbb{T}$. Since $T\mathbf{1}(j) = \lambda = \widehat{K}(0) > 0$, for every $j \in \mathbb{Z}$, then $\lambda = 1$ (see also [3] for further information). In particular, if we take $\widehat{K}(\theta) = e^{i|\theta|}$, then it can be easily proved that

$$K(j) = \begin{cases} 1/2, & |j| = 1 \\ \dfrac{i}{\pi}\dfrac{1+(-1)^j}{1-j^2}, & |j| \neq 1, \end{cases}$$

and hence, the matrix of the operator T has coefficients $a_{j,k} = K(j-k)$, which do not satisfy condition (8).

We now show that there are some arithmetic restrictions on the measure μ to obtain a nontrivial $\sigma_A^+(\mu)$, together with some general properties of this set.

Proposition 2.5. *If $\mu = \{\mu_k\}_{k\in\mathbb{Z}}$ is a measure on $\mathbb{Z}$ such that the cardinality of the set $A_\mu = \{k \in \mathbb{Z} : \mu_k \in \mathbb{R}\backslash\mathbb{Q}\}$ is finite and nonempty, then $\sigma_A^+(\mu) = \{1\}$.*

Proof. Due to the compatibility condition (5), we observe that, since there is only a finite number of irrational numbers in the sequence $\{\mu_k\}_{k\in\mathbb{Z}}$, the quotients

$$\mu(I_k)/\mu_k = \alpha$$

must be a constant rational number for all $k \in \mathbb{Z}$. For simplicity in the notation, let us assume that the indices $1 \le k \le N$ correspond to the values μ_k which are irrational. Since $\mu(I_k)$ is irrational, $1 \le k \le N$, and there are only N irrational values for μ then, for each such k, there exists a unique $\sigma(k) \in \{1, \dots, N\}$ such that $\sigma(k) \in I_k$. Clearly, σ is a permutation of the set $\{1, \dots, N\}$ and

$$\mu(I_k) = \mu_{\sigma(k)} + \beta_k = \alpha\mu_k,$$

for some $\beta_k \in \mathbb{Q}$, $1 \le k \le N$.

This system of linear equations can be written as

$$(\mathbb{I}_N + A)\vec{\mu} = \vec{\beta}, \tag{9}$$

where $\vec{\mu} = (\mu_1, \dots, \mu_N)^T$, $\vec{\beta} = (-\beta_1, \dots, -\beta_N)^T$, $\mathbb{I}_N$ denotes the identity matrix of dimension N and A is an $N \times N$ matrix, depending on σ, such that $A^N = (-\alpha)^N \mathbb{I}_N$. Hence, it is enough to study whether $\mathbb{I}_N + A$ is invertible to conclude that $\vec{\mu}$ and $\vec{\beta}$ cannot satisfy (9).

Indeed, due to the properties of A and the Caley–Hamilton theorem, the minimal polynomial of $\mathbb{I}_N + A$ must divide $p(x) = (x-1)^N - (-\alpha)^N$, and hence p is also its characteristic polynomial. From this observation we deduce that $\det(\mathbb{I}_N + A) = (-1)^N p(0) = 1 - \alpha^N$.

Then, $\mathbb{I}_N + A$ is invertible if and only if $\alpha \neq 1$ and hence, solving $\vec{\mu}$ from (9), we get that the components of $\vec{\mu}$ should be rational numbers, which is a contradiction since $\mathbb{I}_N + A$ is a matrix with rational coefficients, and $(\mathbb{I}_N + A)^{-1}\vec{\beta}$ is a vector in $\mathbb{Q}^N$. Thus, necessarily $\alpha = 1$ and in this case, the partition $I_k = \{k\}$ gives the average value $\lambda = 1$. □

Proposition 2.6. *Let $\mu = \{\mu_k\}_{k\in\mathbb{Z}}$ be a measure on $\mathbb{Z}$ satisfying that $\inf_{k\in\mathbb{Z}} \mu_k = m > 0$. Then, $\sigma_A^+(\mu) \subset (0, 1]$.*

Journal: 20 Article No.: 2284 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2016/3/1 Pages: 14

Proof. Let λ be in $\sigma_A^+(\mu)$ and let $\{I_k\}_{k\in\mathbb{Z}}$ be as in Theorem 2.2. For any $\varepsilon > 0$, let us consider the set

$$A_\varepsilon = \{k \in \mathbb{Z} : \ m \le \mu_k < m + \varepsilon\}.$$

Condition (5) implies that, for any $k \in A_\varepsilon$

$$\frac{1}{\lambda^2} = \frac{\mu(I_k)}{\mu_k} = \frac{\mu(I_k \cap A_\varepsilon) + \mu(I_k \cap A_\varepsilon^c)}{\mu_k} > \frac{m|I_k \cap A_\varepsilon| + (m+\varepsilon)|I_k \cap A_\varepsilon^c|}{m+\varepsilon}$$
$$= |I_k \cap A_\varepsilon^c| + \frac{m}{m+\varepsilon}|I_k \cap A_\varepsilon|.$$

Since $\varepsilon > 0$ is arbitrary, and the sets $(I_k \cap A_\varepsilon^c)$ and $(I_k \cap A_\varepsilon)$ cannot be both simultaneously empty, then we obtain that $\lambda \le 1$. ☐

Remark 2.7. The following counterexample shows that this last statement is not true, in general, for discrete measures $\mu = \{\mu_k\}_{k\in\mathbb{Z}}$ whose infimum is equal to zero. To see this, just consider $\mu = \{\lambda_0^{2k}\}_{k\in\mathbb{Z}}$, $\lambda_0 > 1$ and take the partition $I_k = \{k-1\}$, $k \in \mathbb{Z}$. Then,

$$\frac{\mu(I_k)}{\mu_k} = \frac{1}{\lambda_0^2},$$

and hence, $1 < \lambda_0 \in \sigma_A^+(\mu)$.

Observe that this example also shows that, in fact, for every $\lambda > 0$, there exists a measure μ such that $\lambda \in \sigma_A^+(\mu)$.

As we have already mentioned in the introduction, we can describe the averaging values in $\mathbb{N}$, similarly to the case of the integers, by means of a suitable change of indices:

Proposition 2.8. *There exists a bijection ψ between measures in $\mathbb{Z}$ and measures in $\mathbb{N}$ such that, for every μ in $\mathbb{Z}$*

$$\sigma_A^+(\mu, \mathbb{Z}) = \sigma_A^+(\psi(\mu), \mathbb{N}).$$

Proof. First we notice that considering the bijection $\psi : \mathbb{Z} \longrightarrow \mathbb{N}$ defined by

$$\psi(k) = \begin{cases} 2k+1, & k \ge 0 \\ -2k, & k \le -1, \end{cases}$$

and its corresponding inverse $\phi = \psi^{-1}$, we can obtain, from a positive measure μ defined on $\mathbb{Z}$, a positive measure $\psi(\mu)$ defined on $\mathbb{N}$ by means of

$$\psi(\mu)_j = \mu_{\phi(j)}, \quad j \in \mathbb{N},$$

and conversely.

Similarly, for a given sequence $a = \{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mu)$, we obtain the sequence $\psi(a) := \{a_{\phi(j)}\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}, \psi(\mu))$ and conversely, we have that $\phi(a) := \{a_{\psi(n)}\}_{n\in\mathbb{Z}}$, for $a \in \ell^2(\mathbb{N}, \psi(\mu))$. This is an isometric transformation from $\ell^2(\mathbb{Z}, \mu)$ onto $\ell^2(\mathbb{N}, \psi(\mu))$. Indeed,

$$\sum_{j=1}^{\infty} a_{\phi(j)}^2 \mu_{\phi(j)} = \sum_{k\in\mathbb{Z}} a_k^2 \mu_k.$$

By means of this correspondence, given an operator $T : \ell^2(\mathbb{Z},\mu) \longrightarrow \ell^2(\mathbb{Z},\mu)$ the following operator $\psi(T)$ on $\ell^2(\mathbb{N},\psi(\mu))$ can be defined

$$\psi(T)(a)_j := T(\phi(a))_{\phi(j)}, \ j \in \mathbb{N}.$$

If T is an isometric operator in $\ell^2(\mathbb{Z},\mu)$, $\psi(T)$ is an isometry in $\ell^2(\mathbb{N},\psi(\mu))$, since

$$\|\psi(T)(a)\|^2_{\ell^2(\mathbb{N},\psi(\mu))} = \sum_{j=1}^{\infty} T(\phi(a))^2_{\phi(j)} \ \mu_{\phi(j)} = \sum_{k\in\mathbb{Z}} T(\phi(a))^2_k \ \mu_k$$

$$= \sum_{k\in\mathbb{Z}} \phi(a)^2_k \ \mu_k = \sum_{k\in\mathbb{Z}} a^2_{\psi(k)} \ \mu_k = \sum_{j=1}^{\infty} a^2_j \ \psi(\mu)_j.$$

It is now easy to see that if $\lambda \in \sigma^+_A(\mu,\mathbb{Z})$, with T its associated positive isometry, then $\lambda \in \sigma^+_A(\psi(\mu),\mathbb{N})$, and $\psi(T)$ defined in $\ell^2(\mathbb{N},\psi(\mu))$, is the corresponding positive isometry. In fact, if $T(\mathbf{1}) \equiv \lambda\mathbf{1}$, then $\psi(T)(\mathbf{1}) = T(\phi(\mathbf{1})) = T(\mathbf{1}) \equiv \lambda\mathbf{1}$. The converse embedding $\sigma^+_A(\psi(\mu),\mathbb{N}) \subset \sigma^+_A(\mu,\mathbb{Z})$ is proved similarly. □

3. Case of $\mu_k = a\chi_A(k) + b\chi_B(k)$

In this section we are going to give a complete description of the set $\sigma^+_A(\mu)$ in the case of a positive measure μ defined on $\mathbb{Z}$, taking two possible values. Since the compatibility condition (5) is homogeneous, the set $\sigma^+_A(\mu)$ is invariant under dilations on the measure and, therefore, we can assume without loss of generality that it has the form $\mu_k = r\chi_A(k)+\chi_B(k)$, where A, B is a partition of $\mathbb{Z}$, $r > 0$, and $k \in \mathbb{Z}$. Also, we can assume that B is an infinite set, since otherwise it would suffice to consider the measure $\mu_k = \chi_A(k) + r^{-1}\chi_B(k)$, $k \in \mathbb{Z}$.

Observe that Proposition 2.6 implies that, for these measures, $\sigma^+_A(\mu) \subset (0, 1]$. Moreover, if A is finite, Proposition 2.5 implies that r must be a rational number, and we will prove that, in this case, the set $\sigma^+_A(\mu) \subset \{1/\sqrt{n}\}_{n\in\mathbb{N}}$. However, we are going to see that, in general, the characterization of $\sigma^+_A(\mu)$ is more involved, and it strongly depends on the arithmetic properties of $r > 0$ and $|A|$.

The main tool we are going to use is Theorem 2.2. We will reduce the condition for λ to be in $\sigma^+_A(\mu)$ (or, equivalently, the existence of a nonnegative isometric operator T for which $T(\mathbf{1}) = \lambda$) to finding a suitable partition of $\mathbb{Z}$ satisfying (5).

Theorem 3.1. *Let $\{\mu_k\}_{k\in\mathbb{Z}}$ be a discrete and positive measure on $\mathbb{Z}$ defined by $\mu_k = r\chi_A(k) + \chi_B(k)$, $k \in \mathbb{Z}$, where $r > 0$ and B is an infinite set.*

(i) *If A is finite and $r \notin \mathbb{Q}$, then $\sigma^+_A(\mu) = \{1\}$.*

(ii) *Assume A is finite and $r = p/q \in \mathbb{Q}$, with $p, q \in \mathbb{N}$ and $(p, q) = 1$.*

- *If $|A| < q$, then $\sigma^+_A(\mu) = \left\{\dfrac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}$.*

- *If* $q \le |A|$ *and* $q \mid |A|$, *then*

$$\sigma_A^+(\mu) = \left\{\frac{1}{\sqrt{jq}}\right\}_{j\in\mathbb{N}} \cup \left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}.$$

- *If* $q \le |A|$ *and* $q \nmid |A|$, *then* $\sigma_A^+(\mu) = \left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}$.

(iii) *Assume* A *is infinite.*

- *If* $r \notin \mathbb{Q}$ *satisfies that* $ar^2 + br - c = 0$, *where* $b \in \mathbb{Z}$, $a, c \in \mathbb{N}$ *and* $(a, b, c) = 1$, *then*

$$\sigma_A^+(\mu) = \left\{\frac{1}{\sqrt{j(ar+b)+m}}\right\}_{j\in\mathbb{N},\, m\in\mathbb{N}\cup\{0\}} \cup \left\{\frac{1}{\sqrt{n}}\right\}_{n\in\mathbb{N}}. \tag{10}$$

- *In any other case we have that* $\sigma_A^+(\mu) = \left\{\frac{1}{\sqrt{n}}\right\}_{n\in\mathbb{N}}$.

Proof. The first part is a direct consequence of Proposition 2.5. To prove (ii), we first introduce the following notation: for $\lambda \in \sigma_A^+(\mu)$, we denote by $\alpha = 1/\lambda^2$. Let $\{I_k\}_{k\in\mathbb{Z}}$ be a partition of $\mathbb{Z}$. Then (5) implies that, for $k \in A$,

$$\frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = \alpha, \tag{11}$$

and for $k \in B$,

$$\frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = \alpha. \tag{12}$$

Since A is finite, there is some $k \in B$ such that $I_k \cap A = \emptyset$, and (12) implies that $\alpha = |I_k \cap B| \in \mathbb{N}$. We now consider two possibilities: If $\alpha r \in \mathbb{N}$, then $q \mid \alpha p$ and hence $q \mid \alpha$. On the other hand, if $\alpha r \notin \mathbb{N}$, from (11) we deduce that, for all $k \in A$, $I_k \cap A \neq \emptyset$. Hence, since A is finite, the correspondence between $k \in A$ and the sets I_k, with $k \in A$, must be a one to one mapping, and therefore $|I_k \cap A| = 1$ and $|I_k \cap B| = |I_k| - 1$. Thus, from (11), we have that

$$r|I_k \cap A| + |I_k \cap B| = r + |I_k| - 1 = \alpha r \Rightarrow (\alpha - 1)r \in \mathbb{N} \Rightarrow q \mid (\alpha - 1).$$

So we have obtained that $q \mid \alpha$ or $q \mid (\alpha - 1)$ are necessary conditions if $\lambda \in \sigma_A^+(\mu)$. That is,

$$\sigma_A^+(\mu) \subset \left\{\frac{1}{\sqrt{jq}}\right\}_{j\in\mathbb{N}} \cup \left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}. \tag{13}$$

We will now consider each of the three cases of *(ii)*:

Assume that $|A| < q$. If $q \mid \alpha$, since $\alpha = jq$, for some $j \in \mathbb{N}$, condition (11) implies that, for $k \in A$,

$$p|I_k \cap A| + q|I_k \cap B| = jpq.$$

Thus, $|I_k \cap A|$ is a multiple of q, and therefore $|I_k \cap A| = 0$, for all $k \in A$, since $|I_k \cap A| \le |A| < q$. On the other hand, for $k \in B$, condition (12) implies

$$p|I_k \cap A| + q|I_k \cap B| = jq^2.$$

As before, this equation leads us to the condition $|I_k \cap A| = 0$, for all $k \in B$, which is a contradiction, since $\{I_k\}_k$ is a partition of $\mathbb{Z}$. Therefore, we have proved that $q \mid (\alpha - 1)$ and

$$\sigma_A^+(\mu) \subset \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}. \tag{14}$$

Conversely, if $\alpha = jq + 1$, $j \in \mathbb{N} \cup \{0\}$, let us see that we can find a positive isometric λ-averaging operator, which, by Theorem 2.2, is equivalent to finding a partition satisfying (5). Indeed, we construct $\{I_k\}_{k \in \mathbb{Z}}$ as follows:

If $k \in A$, we take $I_k = \{k\} \cup (I_k \cap B)$, with $|I_k| = pj + 1$, and if $k \in B$, we take $I_k \subset B$ and $|I_k| = \alpha$. It is clear that such a partition of $\mathbb{Z}$ exists, since B is infinite. Finally, let us prove that (11) and (12), equivalently (5), hold:

If $k \in A$,

$$\frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = 1 + \frac{qpj}{p} = 1 + qj = \alpha,$$

and, if $k \in B$,

$$\frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = 0 + \alpha,$$

which finally shows that

$$\left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}} \subset \sigma_A^+(\mu). \tag{15}$$

Therefore, using (14) and (15) we conclude the result.

Assume now that $q \leq |A|$ and $q \mid |A|$; that is, $|A| = sq$, for some $s \in \mathbb{N}$. Using (13), it suffices to prove that if $q \mid \alpha$ or $q \mid (\alpha - 1)$, then we can find a partition $\{I_k\}_{k \in \mathbb{Z}}$ satisfying both (11) and (12).

If $q \mid \alpha$, then $\alpha = jq$, $j \in \mathbb{N}$. Now, we set $\{I_k\}_{k \in \mathbb{Z}}$ as follows: Choose $k_1, \ldots, k_s \in A$ and take $|I_{k_n} \cap A| = q$, $1 \leq n \leq s$ and $|I_{k_n} \cap B| = p(j-1)$.

For $k \in A \backslash \{k_1, \ldots, k_s\}$, take $I_k \subset B$, with $|I_k| = pj$. If $k \in B$, take $I_k \subset B$, with $|I_k| = \alpha$. Then, if $k \in \{k_1, \ldots, k_s\}$,

$$\frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = q + q(j-1) = qj = \alpha.$$

If $k \in A \backslash \{k_1, \ldots, k_s\}$

$$\frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = 0 + qj = \alpha.$$

If $k \in B$,

$$\frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = 0 + \alpha,$$

which finally shows that

$$\left\{ \frac{1}{\sqrt{jq}} \right\}_{j \in \mathbb{N}} \subset \sigma_A^+(\mu). \tag{16}$$

Still assuming that $q \le |A|$ and $q \mid |A|$, we now consider the case $q \mid (\alpha-1)$; that is, $\alpha = 1 + jq$, for some $j \in \mathbb{N}\cup\{0\}$. If $k \in A$, take $|I_k \cap A| = 1$ and $|I_k \cap B| = pj$. For $k \in B$, take $I_k \subset B$, with $|I_k| = \alpha$. If $k \in A$,

$$\frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = 1 + \frac{qpj}{p} = 1 + qj = \alpha,$$

and, if $k \in B$,

$$\frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = 0 + \alpha,$$

which shows

$$\left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}} \subset \sigma_A^+(\mu). \tag{17}$$

Then, using (13), (16), and (17), we have that

$$\sigma_A^+(\mu) = \left\{\frac{1}{\sqrt{jq}}\right\}_{j\in\mathbb{N}} \cup \left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}.$$

To finish the proof of *(ii)* we now assume that A is finite, $q \le |A|$, and $q \nmid |A|$. If $\{I_k\}_k$ is a partition associated to α, as in Theorem 2.2, using (11) and (12) it is easily seen that q has to divide $|I_k \cap A|$, for every $k \in \mathbb{Z}$, and hence, since $|A| = \sum_k |I_k \cap A|$, then q should also divide $|A|$, which is a contradiction. Thus, from (13) we get

$$\sigma_A^+(\mu) \subset \left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}. \tag{18}$$

Conversely, if $\alpha = jq+1$, we define the same partition $\{I_k\}_{k\in\mathbb{Z}}$ as in the case $|A| < q$: If $k \in A$, we take $I_k = \{k\} \cup (I_k \cap B)$, with $|I_k| = pj+1$, and if $k \in B$, we take $I_k \subset B$ and $|I_k| = \alpha$. Then, as before,

$$\left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}} \subset \sigma_A^+(\mu). \tag{19}$$

Thus, from (18) and (19) we conclude

$$\sigma_A^+(\mu) = \left\{\frac{1}{\sqrt{jq+1}}\right\}_{j\in\mathbb{N}\cup\{0\}}.$$

Finally, we give the proof of *(iii)* and we now assume that both A and B are infinite sets. First, if $\alpha \in \mathbb{N}$, we construct the partition in such a way that $I_k \subset A$, if $k \in A$ and $I_k \subset B$, if $k \in B$, with $|I_k| = \alpha$, for all $k \in \mathbb{Z}$. Using (11) and (12), we see that

$$\{1/\sqrt{n}\}_{n\in\mathbb{N}} \subset \sigma_A^+(\mu). \tag{20}$$

To finish, we will prove the following claim:
$\sigma_A^+(\mu)\backslash\{1/\sqrt{n}\}_{n\in\mathbb{N}} \ne \emptyset$ *if and only if* r *is an irrational number which is the positive root of a polynomial* ax^2+bx-c*, where* $b \in \mathbb{Z}$*,* $a, c \in \mathbb{N}$ *and* $(a,b,c) = 1$.

Indeed, if $\lambda \in \sigma_A^+(\mu)\setminus\{1/\sqrt{n}\}_{n\in\mathbb{N}}$ and $\alpha = 1/\lambda^2$, equations (11) and (12) imply that

$$\mu(I_k) = r\alpha = rm_k + c_k, \quad \text{for some } m_k, c_k \in \mathbb{N}\cup\{0\}, \; k \in A, \tag{21}$$

$$\mu(I_k) = \alpha = ra_k + l_k, \quad \text{for some } a_k, l_k \in \mathbb{N}\cup\{0\}, \; k \in B. \tag{22}$$

Since $\alpha \notin \mathbb{N}$, then $a_k, c_k \neq 0$ and hence $r \notin \mathbb{Q}$. In fact, if $r = p/q \in \mathbb{Q}$, with $(p,q) = 1$, we would obtain that

$$pq(m_k - l_k) = p^2 a_k - q^2 c_k,$$

and hence, p must divide c_k, which using (21) would imply that $\alpha \in \mathbb{N}$.

Note also that if we combine (21) and (22), we can prove that $a_{k'}r^2 + (l_{k'} - m_k)r - c_k = 0$, for every $k \in A$ and $k' \in B$. Conversely, if $r \notin \mathbb{Q}$ is the positive root of the polynomial $ax^2 + bx - c$, with $a, c \in \mathbb{N}$ and $b \in \mathbb{Z}$, we pick $j \in \mathbb{N}$, write $jb = l - m$, with $l, m \in \mathbb{N}\cup\{0\}$, and define the partition $\{I_k\}_{k\in\mathbb{Z}}$ as follows

$$|I_k \cap A| = m, \quad |I_k \cap B| = jc, \quad \text{if } k \in A,$$

$$|I_k \cap A| = ja, \quad |I_k \cap B| = l, \quad \text{if } k \in B.$$

With this partition we have

$$\frac{\mu(I_k)}{\mu_k} = \begin{cases} \dfrac{mr + jc}{r}, & k \in A, \\ jar + l, & k \in B. \end{cases} \tag{23}$$

The fact that $(mr + jc)/r = jar + l$ shows that (23) satisfies the compatibility condition (5), and this proves the claim, since $jar + l \notin \mathbb{N}$. Moreover, we observe that $\alpha = jar + jb + m$, which gives us that

$$\sigma_A^+(\mu)\setminus\left\{\frac{1}{\sqrt{n}}\right\}_{n\in\mathbb{N}} = \left\{\frac{1}{\sqrt{j(ar+b)+m}}\right\}_{j\in\mathbb{N},\, m\in\mathbb{N}\cup\{0\}}. \tag{24}$$

Finally, if $r \notin \mathbb{Q}$ is the positive root of the polynomial $ax^2 + bx - c$, with $a, c \in \mathbb{N}$ and $b \in \mathbb{Z}$, (20) and (24) prove (10). On the other hand, if r is not as above, the claim and (20) show that $\sigma_A^+(\mu) = \{1/\sqrt{n}\}_{n\in\mathbb{N}}$. □

Example 3.2. We now apply Theorem 3.1 to find $\sigma_A^+(\mu)$, for different sets A and concrete values of $r > 0$:

- If $A = \{0,1,2\}$ and $r = \sqrt{2}$, then $\sigma_A^+(\mu) = \{1\}$.
- If $A = \{0,1,2\}$ and $r = 1/4$, then $\sigma_A^+(\mu) = \left\{1/\sqrt{4j+1}\right\}_{j\in\mathbb{N}\cup\{0\}}$.
- If $A = \{0,1,2\}$ and $r = 1$ (that is, μ is the counting measure in $\mathbb{Z}$), then $\sigma_A^+(\mu) = \left\{1/\sqrt{j}\right\}_{j\in\mathbb{N}}$.
- If $A = \{0,1,2\}$ and $r = 2/3$, then

$$\sigma_A^+(\mu) = \left\{1/\sqrt{3j}\right\}_{j\in\mathbb{N}} \cup \left\{1/\sqrt{3j+1}\right\}_{j\in\mathbb{N}\cup\{0\}}.$$

- If $A = \{0,1,2\}$ and $r = 3/2$, then $\sigma_A^+(\mu) = \left\{1/\sqrt{2j+1}\right\}_{j\in\mathbb{N}\cup\{0\}}$.

- If $A=\mathbb{N}$ and $r=(\sqrt{5}-1)/2$, which is a root of the polynomial x^2+x-1, then

$$\sigma_A^+(\mu)=\left\{\frac{\sqrt{2}}{\sqrt{j\left(\sqrt{5}+1\right)+2m}}\right\}_{j\in\mathbb{N},\,m\in\mathbb{N}\cup\{0\}}\cup\left\{\frac{1}{\sqrt{j}}\right\}_{j\in\mathbb{N}}.$$

- If $A=\mathbb{N}$ and $r=(\sqrt{5}+3)/2$, which is a root of the polynomial x^2-3x+1, then $\sigma_A^+(\mu)=\left\{1/\sqrt{j}\right\}_{j\in\mathbb{N}}$.
- If $A=\mathbb{N}$ and $r=\pi$, then $\sigma_A^+(\mu)=\left\{1/\sqrt{j}\right\}_{j\in\mathbb{N}}$.

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