Iterated line digraphs are asymptotically dense *

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Abstract

We show that the line digraph technique, when iterated, provides dense digraphs, that is, with asymptotically large order for a given diameter (or with small diameter for a given order). This is a well-known result for regular digraphs. In this note we prove that this is also true for non-regular digraphs.

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1 Introduction

To find families of dense digraphs is an important issue in the design of interconnection networks. Dense digraphs are strongly connected digraphs with a relatively large number of vertices with respect to the largest order allowed by their maximum out-degree and diameter. This is related to the degree/diameter problem, that is, to find the largest possible number $N(d,k)$ of vertices in a digraph of maximum out-degree d and diameter k. The directed Moore bound $M(d,k)$, which is an upper bound on the order of such a digraph, is $M(d,k) = \frac{d^{k+1}-1}{d-1}$ if $d \neq 1$, and $M(1,k) = k+1$. The digraphs that attains the directed Moore bound are called Moore digraphs, and they only exist for $k=1$ or $d=1$, that is, the directed cycles on $k+1$ vertices and the complete digraphs on $d+1$ vertices. For more information, see the comprehensive survey by Miller and Širaň [8].

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We recall some basic notation and results. A digraph $G = (V, E)$ consists of a (finite) set $V = V(G)$ of vertices and a set $E = E(G)$ of arcs (directed edges) between vertices of G. If $a = (u, v)$ is an arc from u to v, then vertex u (and arc a) is *adjacent to* vertex v, and vertex v (and arc a) is *adjacent from* v. Let $G^+(v)$ and $G^-(v)$ denote the set of vertices adjacent from and to vertex v, respectively. A digraph G is *d-regular* if $|G^+(v)| = |G^-(v)| = d$ for all $v \in V$.

In the line digraph $L(G)$ of a digraph G, each of its vertices represents an arc of G, that is, $V(L(G)) = \{uv|(u,v) \in E(G)\}$; and vertices uv and wz of $L(G)$ are adjacent if and only if $v = w$, namely, when arc (u, v) is adjacent to arc (w, z) in G. It can be easily seen that every vertex of $L^\ell(G)$ corresponds to a walk $v_0, v_1, \ldots, v_\ell$ of length ℓ in G, where $(v_{i-1,}, v_i) \in E$ for $i = 1, \ldots, k$. Then, if $\boldsymbol{A}$ is the adjacency matrix of G, the uv-entry of the power $\boldsymbol{A}^\ell$, denoted by a^ℓ_{uv}, is the number of ℓ-walks from vertex u to vertex v. Besides, the order N_ℓ of the ℓ-iterated line digraph $L^\ell(G)$ turns out to be $N_\ell = \boldsymbol{j}^\top \boldsymbol{A}^k \boldsymbol{j} = \langle \boldsymbol{j}, \boldsymbol{A}^k \boldsymbol{j} \rangle$, where $\boldsymbol{j}$ stands for the all-1 vector. In particular, if G is a d-regular digraph with n vertices then its iterated line digraph $L^\ell(G)$ is d-regular with $N_\ell = d^\ell N$ vertices.

Recall also that a digraph G is *strongly connected* if there is a (directed) walk between every pair of its vertices. Moreover, it is known that G is strongly connected if and only if its line digraph $L(G)$ is strongly connected. If G is a digraph (different from a directed cycle) with diameter k, then its line digraph $L(G)$ has diameter $k+1$. From this result, it is easy to see that for regular digraphs the iterated line digraph technique provides families of dense digraphs. Two well-known examples of such families are the De Bruijn [2] and Kautz digraphs [6, 7], which can be defined as iterated line digraphs of complete symmetric digraphs with a loop on each vertex, and complete symmetric digraphs, respectively. For both digraphs the number of vertices is $O(d^k)$ for a given degree d and large diameter k. Note that this coincides with the order of the Moore bound. For more details, see Fiol, Yebra and Alegre [4].

In this note, our aim is to show that the line digraph technique gives digraphs with asymptotically optimal diameter (or number of vertices) also for non-regular digraphs. Notice that, in the case of non-regular digraphs, the Moore bound $M(d, k)$ is not tight, since this bound is only attainable for regular digraphs. Then, we give a new Moore bound for a digraph G in terms of the spectral radius (namely, the largest eigenvalue) of its adjacency matrix.

2 Main result

In our proofs, we use some results from the Perron-Frobenius theorem (see for example Godsil [5]). That is:

Theorem 2.1. [[5], Perron-Frobenius theorem] *Suppose that $\boldsymbol{M}$ is an irreducible non-negative $n \times n$ matrix, that is, $\boldsymbol{M}^k > \boldsymbol{O}$ (the all-0 matrix) for some k. Then,*

$(P1)$ *The spectral radius $\rho(\boldsymbol{M})$ is a positive real number, and it is a simple eigenvalue of $\boldsymbol{M}$, whose corresponding eigenvector can be taken to be positive.*

$(P2)$ *If $\boldsymbol{N}$ is a non-negative $n \times n$ matrix such that $\boldsymbol{N} \le \boldsymbol{M}$, then $\rho(\boldsymbol{N}) \le \rho(\boldsymbol{M})$, with equality if and only in $\boldsymbol{N} = \boldsymbol{M}$.*

2.1 A general upper bound for the order of a digraph

Let $G = (V, E)$ be a (not necessarily regular) strongly connected digraph with $N = |V|$ vertices, $|E|$ arcs, adjacency matrix $\boldsymbol{A}$, and diameter k. Since there exists a walk of length at most k between any pair of vertices, the monic polynomial $p(x) = x^k + x^{k-1} + \cdots + 1$ satisfies

$$p(\boldsymbol{A}) = \boldsymbol{A}^k + \boldsymbol{A}^{k-1} + \cdots + \boldsymbol{I} \ge \boldsymbol{J}, \tag{1}$$

where $\boldsymbol{J}$ is the all-1 $N \times N$ matrix. Let $\lambda_0 = \rho(\boldsymbol{A})$ be the spectral radius of G. Since $\rho(p(\boldsymbol{A})) = p(\lambda_0)$ and $\rho(\boldsymbol{J}) = N$, property $(P2)$ gives

$$N \le M(\lambda_0, k) = p(\lambda_0) = \lambda_0^k + \lambda_0^{k-1} + \cdots + 1 = \frac{\lambda_0^{k+1} - 1}{\lambda_0 - 1}, \tag{2}$$

where $M(\lambda_0, k)$ is the Moore-like bound for a digraph with eigenvalue $\lambda_0 \ne 1$ and diameter k. If $\lambda_0 = 1$, then $N = M(1, k) = k + 1$, and G is a directed cycle. In general, notice that if the digraph is d-regular, then $\lambda_0 = d$ and $M(\lambda_0, k)$ coincides with the known bound $M(d, k)$. Note that $M(\lambda_0, k)$ is of the order of λ_0^k. From (2), we also have

$$k \ge k(\lambda_0, N) = \lceil \log_{\lambda_0}((\lambda_0 - 1)N + 1) \rceil - 1,$$

where $k(\lambda_0, N)$ represents the minimum diameter that a digraph G can have given eigenvalue λ_0 and order N.

Alternatively, assuming that λ_0 has eigenvector $\boldsymbol{v}$ which, by property $(P1)$, can be normalized in such a way that its minimum component, say v_1, equals 1, we can write

$$N\boldsymbol{j} = \boldsymbol{J}\boldsymbol{j} \le p(\boldsymbol{A})\boldsymbol{j} \le p(\boldsymbol{A})\boldsymbol{v} = p(\lambda_0)\boldsymbol{v},$$

where $\boldsymbol{j}$ is the all-1 vector. In particular, considering the first component, we get again (2).

In order to compare the bound (2) with the standard Moore bound for digraphs, Figure 1 shows a digraph on 12 vertices, with maximum out-degree 3, diameter 3, and spectral radius $\lambda_0 = 1 + \sqrt{2}$. Thus, the standard Moore

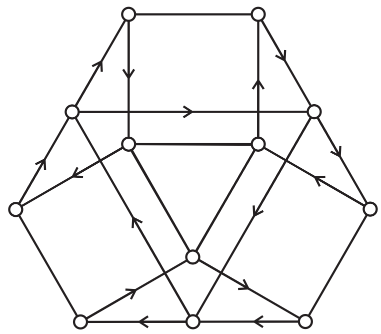

Figure 1: A digraph with maximum out-degree 3 and $\lambda_0 = 1+\sqrt{2}$. The non-directed edges represent two opposite arcs.

bound is $M(3,3) = 1+3+3^2+3^3 = 40$. In contrast, (2) yields the much better value $N \le 1+\lambda_0+\lambda_0^2+\lambda_0^3 = 12+8\sqrt{2} \approx 23.31$.

Since Moore digraphs, with order attaining $M(d,k)$, only exist for $d=1$ or $k=1$, we could ask whether there exist other digraphs attaining the 'spectral Moore bound' $M(\lambda_0,k)$ given in (2). The following lemma answers the question in the negative.

Lemma 2.2. *The only digraphs attaining the bound $M(\lambda_0,k)$ are the (regular) Moore digraphs with $d=1$ (directed cycles) or $k=1$ (complete symmetric digraphs).*

Proof. From property $(P2)$, we see that equality in (2), $N = M(\lambda_0,k)$ holds if and only if $p(\boldsymbol{A}) = \boldsymbol{J}$. Thus, G is a Moore digraph, regular with degree $d=\lambda_0$, and eigenvector $\boldsymbol{v}=\boldsymbol{j}$. □

2.2 The iterated line digraphs

Moreover, if G is a digraph (different from a directed cycle) with diameter k and maximum eigenvalue λ_0, then its ℓ-iterated line digraph $L^\ell(G)$ has diameter $k_\ell = k+\ell$ (see Fiol, Yebra, and Alegre [3, 4]), maximum eigenvalue λ_0 (the line digraph technique preserves all the eigenvalues, see Balbuena, Ferrero, Marcote, and Pelayo [1]), and number of vertices

$$N_\ell = \langle \boldsymbol{j}, \boldsymbol{A}^\ell \boldsymbol{j}\rangle \ge \langle \boldsymbol{v}, \boldsymbol{A}^\ell \boldsymbol{v}\rangle = \langle \boldsymbol{v}, \lambda_0^\ell \boldsymbol{v}\rangle = \lambda_0^\ell \|\boldsymbol{v}\|^2,$$

where, now, $\boldsymbol{v}$ is normalized in such a way that its maximum component is 1. Thus, we prove the following two results concerning the number N_ℓ of vertices and the diameter k_ℓ.

Theorem 2.3. *Given a digraph G on N vertices, with diameter k and spectral radius λ_0, let $L^\ell(G)$ be its ℓ-iterated line digraph on N_ℓ vertices, with diameter k_ℓ and spectral radius λ_0.*

(a) *The number N_ℓ of vertices of $L^\ell(G)$ has the same order $O(\lambda_0^\ell)$, for $\ell \to \infty$, as its corresponding Moore bound $M(\lambda_0, N_\ell)$. More precisely,*

$$\lim_{\ell\to\infty} \frac{N_\ell}{M(\lambda_0, k_\ell)} = \frac{\|\boldsymbol{v}\|^2}{\lambda_0^k}.$$

(b) *The diameter k_ℓ of $L^\ell(G)$ has the same order $O(\ell)$, for $\ell \to \infty$, as the diameter $k(\lambda_0, N_\ell)$ of the digraph corresponding to the Moore bound $M(\lambda_0, N_\ell)$. More precisely,*

$$\lim_{\ell\to\infty} \frac{k_\ell}{k(\lambda_0, N_\ell)} = 1.$$

Proof. (a) We compute the ratio $N_\ell/M(\lambda_0, k_\ell)$ when $\ell \to \infty$:

$$\begin{aligned}
&\lim_{\ell\to\infty} \frac{N_\ell}{M(\lambda_0, k_\ell)} = \lim_{\ell\to\infty} \frac{N_\ell}{M(\lambda_0, k+\ell)} \\
&\geq \lim_{\ell\to\infty} \frac{\lambda_0^\ell \|\boldsymbol{v}\|^2}{\lambda_0^{k+\ell} + \lambda_0^{k+\ell-1} + \cdots + 1} = \frac{\|\boldsymbol{v}\|^2}{\lambda_0^k}.
\end{aligned}$$

When $\ell \to \infty$, $N_\ell \geq \frac{\|\boldsymbol{v}\|^2}{\lambda_0^k} M(\lambda_0, k_\ell)$. Besides, $N_\ell \leq M(\lambda_0, k_\ell)$, because $M(\lambda_0, k_\ell)$ is an upper bound for N_ℓ. Then, N_ℓ and $M(\lambda_0, k_\ell)$ have the same order $O(\lambda_0^\ell)$.

(b) We compute the ratio $k_\ell/k(\lambda_0, N_\ell)$ when $\ell \to \infty$:

$$\begin{aligned}
&\lim_{\ell\to\infty} \frac{k_\ell}{k(\lambda_0, N_\ell)} = \lim_{\ell\to\infty} \frac{k+\ell}{\log_{\lambda_0}((\lambda_0 - 1)N_\ell + 1) - 1} \\
&\leq \lim_{\ell\to\infty} \frac{k+\ell}{\log_{\lambda_0}((\lambda_0 - 1)\lambda_0^\ell \|\boldsymbol{v}\|^2 + 1) - 1} \\
&= \lim_{\ell\to\infty} \frac{k+\ell}{\ell + \log_{\lambda_0}((\lambda_0 - 1)\|\boldsymbol{v}\|^2) - 1} = 1
\end{aligned}$$

Reasoning as in (a), when $\ell \to \infty$, we get $k_\ell \leq k(\lambda_0, N_\ell)$. Besides, $k_\ell \geq k(\lambda_0, N_\ell)$, because $k(\lambda_0, N_\ell)$ is a lower bound for k_ℓ. Then, k_ℓ and $k(\lambda_0, N_\ell)$ have the same order $O(\ell)$.

□

For example, the digraph of Figure 1 has spectral radius $\lambda_0 = 1+\sqrt{2}$ with normalized eigenvector $\boldsymbol{v} = \frac{1}{\lambda_0}(1, \lambda_0 - 1, 1, 1, 1, \lambda_0, \lambda_0 - 1, \lambda_0 - 1, 1, \lambda_0, \lambda_0, 1)$, which, for $\ell \to \infty$, gives

$$N_\ell \;\to\; \alpha M(\lambda_0, k_\ell)$$

where $\alpha = \frac{\|\boldsymbol{v}\|^2}{\lambda_0^3} \approx 0.3595$.

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