Partial Orderings for Hesitant Fuzzy Sets

L. Garmendia, R. González del Campo
Ingeniería del Software e Inteligencia Artificial
Facultad de Matemáticas
Universidad Complutense de Madrid
Juan del Rosal, Ciudad Universitaria
28040 Madrid
Spain
lgarmend@fdi.ucm.es, rgonzale@estad.ucm.es

J. Recasens
Secció Matemàtiques i Informàtica
ETS Arquitectura del Vallès
Universitat Politècnica de Catalunya
Pere Serra 1-15
08190 Sant Cugat del Vallès
Spain
j.recasens@upc.edu

Abstract

New partial orderings $\leq_o$, $\leq_p$ and $\leq_{\mathbb{H}}$ are defined, studied and compared on the set $\mathbb{H}$ of finite subsets of the unit interval with special emphasis on the last one. Since comparing two sets of the same cardinality is a simple issue, the idea for comparing two sets A and B of different cardinalities n and m respectively using $\leq_{\mathbb{H}}$ is repeating their elements in order to obtain two series with the same length. If $\mathrm{lcm}(n, m)$ is the least common multiple of n and m we can repeat every element of A $\mathrm{lcm}(n, m)/m$ times and every element of B $\mathrm{lcm}(n, m)/n$ times to obtain such series and compare them (Definition 2.2).

$(\mathbb{H}, \leq_{\mathbb{H}})$ is a bounded partially ordered set but not a lattice. Nevertheless, it will be shown that some interesting subsets of $(\mathbb{H}, \leq_{\mathbb{H}})$ have a lattice structure. Moreover in the set $\mathbb{B}$ of finite bags or multisets (i.e. allowing repetition of objects) of the unit interval a preorder $\leq_{\mathbb{B}}$ can be defined in a similar way as $\leq_{\mathbb{H}}$ in $\mathbb{H}$ and considering the quotient set $\overline{\mathbb{B}} = \mathbb{B}/\sim$ of $\mathbb{B}$ by the equivalence relation $\sim$ defined by $A \sim B$ when $A \leq_{\mathbb{B}} B$ and $B \leq_{\mathbb{B}} A$, $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$ is a lattice and $(\mathbb{H}, \leq_{\mathbb{H}})$ can be naturally embedded into it.

Keywords: hesitant fuzzy sets, finite subsets of the unit interval, partial ordering, t-norm, fuzzy conjunction.

1 Introduction

Hesitant fuzzy sets generalize the concept of fuzzy set introduced by Zadeh [15] in the sense that they allow us the possibility of assigning more than a value of the unit interval to an object of a universe of discourse. They are a useful tool when there is doubt or *hesitation* in the process of assigning numerical values to the objects. An illustrative example is in the evaluation of a service by a client. If the client is asked to choose between a scale, say, from 0 to 5 (from less to more satisfied) he or she could hesitate to assigning an excellent mark (5) or a very good one (4). In this case, if he or she is allowed to choose more than one option, he or she will decide for $\{4, 5\}$. In another example, if ranking political views (say from 0 to 10), the concept *extremist* could be described as $\{0, 10\}$. More information and results on hesitant fuzzy sets can be found in [1], [5], [6], [7], [9], [10], [11], [12], [13].

A very important issue is how to compare hesitant fuzzy subsets, which leads to the question of comparing finite subsets of the unit interval (i.e, defining a partial ordering on the set $\mathbb{H}$ of finite subsets of $[0, 1]$). There is a very natural way to compare subsets of the same cardinality; namely, if $A = \{a_1, a_2, ..., a_n\}$ with $a_1 < a_2 < ... < a_n$ and $B = \{b_1, b_2, ..., b_n\}$ with $b_1 < b_2 < ... < b_n$, the pointwise comparison seems adequate ($A \leq_{\mathbb{H}} B$ if and only if $a_i \leq b_i$ for all $i = 1, 2, ..., n$). The problem arises when trying to compare subsets $A = \{a_1, a_2, ..., a_n\}$ with $a_1 < a_2 < ... < a_n$ and $B = \{b_1, b_2, ..., b_m\}$ with $b_1 < b_2 < ... < b_m$ of different cardinalities ($n < m$). In [1] different methods are proposed that basically select n elements of B or add $m - n$ elements to A. Also, in order to calculate distances between finite subsets of $[0, 1]$, [13] proposes a pessimistic and an optimistic way. The pessimistic one consisting in adding $m - n$ copies of the smallest element a_1 of A and then calculating the distance between the vector $(\overbrace{a_1, ..., a_1}^{m-n \text{ times}}, a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_m)$. The optimistic way adds $m-n$ copies of a_n to A and computes the distance between the vector $((a_1, a_2, ..., a_n, \overbrace{a_n, ..., a_n}^{m-n \text{ times}}))$ and $(b_1, b_2, ..., b_m)$. This two ways inspire the definition of an optimistic and pessimistic way to compare finite subsets of $[0, 1]$ in Section 2 (see also [1]). Nevertheless, these ways seem acceptable in some occasions but can be too radical in others. In Section 2 a more balanced ordering $\leq_{\mathbb{H}}$ is defined and discussed.

With this new ordering $\mathbb{H}$ does not have the lattice structure. This is proved in Section 3 where it is also shown that some interesting subsets of $\mathbb{H}$ do satisfy the lattice conditions.

One reason why $(\mathbb{H}, \leq_{\mathbb{H}})$ does not have the lattice structure is because it has "too few" elements. If we consider not only finite subsets but also bags (i.e. if we allow repetitions of objects), then we obtain a lattice $(\overline{\mathbb{B}}, \leq_{\overline{\mathbb{B}}})$ in which $(\mathbb{H}, \leq_{\mathbb{H}})$ can be naturally embedded (Section 4).[1]

$(\mathbb{H}, \leq_{\mathbb{H}})$ and $(\overline{\mathbb{B}}, \leq_{\overline{\mathbb{B}}})$ are bounded partially ordered sets and hence there is the possibility of defining t-norms on them (Section 5).

The paper ends with a section of concluding remarks.

2 Ordering Subsets of the Unit Interval

In this section a partial ordering $\leq_{\mathbb{H}}$ is defined on the set $\mathbb{H}$ of finite subsets of the unit interval which is more balanced than the optimistic and pessimistic orderings introduced in the previous section. This definition is discussed and interpreted for sets with low cardinality and compared with the optimistic and pessimistic orderings.

Since comparing two sets of the same cardinality is a simple issue, the idea for comparing two sets A and B of different cardinalities n and m respectively is repeating their elements in order to obtain two series with the same length. Considering the least common multiple ($\text{lcm}(n, m)$) of n and m we can repeat every element of A $\text{lcm}(n, m)/m$ times and every element of B $\text{lcm}(n, m)/n$ times to obtain such series and compare them. The formal definitions and details follow below.

[1]Please note that in general the maps $h : X \to \overline{\mathbb{B}}$ would not be hesitant fuzzy subsets of X since in the definition of fuzzy subset the image of every element of X must be a *subset* of $[0, 1]$.

In order to simplify notations, when giving a finite subset $A = \{a_1, a_2, ..., a_n\}$ of the unit interval we will assume throughout the paper that the elements of A are written in increasing order, i.e., $a_1 < a_2 < ... < a_n$.

Definition 2.1. *Let $A = \{a_1, a_2, ..., a_n\}$ be a finite subset of the unit interval and $r \in \mathbb{N}$. $A_r \in [0,1]^{rn}$ is the vector of rn coordinates defined as*

$$A_r = (\overbrace{a_1, ..., a_1}^{r \text{ times}}, \overbrace{a_2, ..., a_2}^{r \text{ times}}, ..., \overbrace{a_n, ..., a_n}^{r \text{ times}})$$

Definition 2.2. *Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ be two finite subsets of the unit interval and $\mathrm{lcm}(n, m)$ the least common multiple of n and m. Rewriting $A_{\frac{\mathrm{lcm}(n,m)}{m}} = (c_1, c_2, ..., c_{\mathrm{lcm}(n,m)})$ and $B_{\frac{\mathrm{lcm}(n,m)}{n}} = (d_1, d_2, ..., d_{\mathrm{lcm}(n,m)})$,*

$$A \leq_{\mathbb{H}} B \text{ if and only if } c_i \leq d_i \text{ for all } i = 1, 2, ..., \mathrm{lcm}(n, m).$$

Example 2.3. *Consider $A = \{0.2, 0.4\}$ and $B = \{0.3, 0.5, 0.8\}$. The least common multiple of 2 and 3 is 6,*

$$\begin{aligned} A_3 &= (0.2, 0.2, 0.2, 0.4, 0.4, 0.4) \text{ and} \\ B_2 &= (0.3, 0.3, 0.5, 0.5, 0.8, 0.8). \end{aligned}$$

$A \leq_{\mathbb{H}} B$ because $0.2 \leq 0.3$, $0.2 \leq 0.5$, $0.4 \leq 0.5$ and $0.4 \leq 0.8$.

$\leq_{\mathbb{H}}$ is a partial ordering on the set $\mathbb{H}$ of finite subsets of the unit interval. To prove this fact we need the following obvious result.

Lemma 2.4. *Let $A = \{a_1, , ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ be two finite subsets of the unit interval and $r \in \mathbb{N}$. $A \leq_{\mathbb{H}} B$ if and only if every coordinate of $A_{r\frac{\mathrm{lcm}(n,m)}{m}}$ is smaller than or equal to the corresponding coordinate of $B_{r\frac{\mathrm{lcm}(n,m)}{n}}$.*

Example 2.5. *Following the Example 2.3, taking $r = 2$,*

$$\begin{aligned} A_6 &= (0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.4, 0.4, 0.4, 0.4, 0.4, 0.4) \text{ and} \\ B_4 &= (0.3, 0.3, 0.3, 0.3, 0.5, 0.5, 0.5, 0.5, 0.8, 0.8, 0.8, 0.8). \end{aligned}$$

Proposition 2.6. *$\leq_{\mathbb{H}}$ is a partial ordering on the set $\mathbb{H}$ of finite subsets of the unit interval.*

Proof.

- Clearly $A \leq_{\mathbb{H}} A$ for every finite subset $A \in \mathbb{H}$.
- Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ be finite subsets of the unit interval such that $A \leq_{\mathbb{H}} B$ and $B \leq_{\mathbb{H}} A$.

  If A and B have the same cardinality (i.e.: if $m = n$), then clearly $A = B$.

  But if $A \leq_{\mathbb{H}} B$ and $B \leq_{\mathbb{H}} A$, then m and n must coincide. Otherwise, without loss of generality we could assume $n < m$. In $A_{\frac{\mathrm{lcm}(n,m)}{m}}$ and $B_{\frac{\mathrm{lcm}(n,m)}{n}}$ a_1 must be compared with b_1 and b_2. But, since $b_1 < b_2$, it can not happen $a_1 \leq b_1$, $a_1 \leq b_2$ and $b_2 \leq a_1$ simultaneously.
- Transitivity.

  Let $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_m\}$ and $C = \{c_1, c_2, ..., c_p\}$ be finite subsets of the unit interval such that $A \leq_{\mathbb{H}} B$ and $B \leq_{\mathbb{H}} C$. We must prove that $A \leq_{\mathbb{H}} C$. For this, consider the less common multiple $\mathrm{lcm}(n, m, p)$ of n, m and p and compare the vectors $A_{\frac{\mathrm{lcm}(n,m,p)}{n}}$, $B_{\frac{\mathrm{lcm}(n,m,p)}{m}}$ and $C_{\frac{\mathrm{lcm}(n,m,p)}{p}}$. Thanks to Lemma 2.4, each coordinate of $A_{\frac{\mathrm{lcm}(n,m,p)}{n}}$ is smaller than or equal to the corresponding coordinate of $B_{\frac{\mathrm{lcm}(n,m,p)}{m}}$ which in turn is smaller than or equal to the corresponding coordinate of $C_{\frac{\mathrm{lcm}(n,m,p)}{p}}$. By Lemma 2.4, $A \leq_{\mathbb{H}} C$.

□

Let us see some examples with sets of low cardinality to show the behaviour of this ordering.

Example 2.7.

1. *If $A = \{a\}$ and $B = \{b\}$, then $A \leq_{\mathbb{H}} B$ if and only if $a \leq b$.*
2. *If $A = \{a\}$ and $B \in \mathbb{H}$, then $A \leq_{\mathbb{H}} B$ if and only if $a \leq \min\{B\}$ and $B \leq_{\mathbb{H}} A$ if and only if $\max\{B\} \leq a$.*
3. *If $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, then $A \leq_{\mathbb{H}} B$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$.*

4. *If* $A = \{a_1, a_2, a_3\}$ *and* $B = \{b_1, b_2, b_3\}$*, then* $A \leq_{\mathbb{H}} B$ *if and only if* $a_1 \leq b_1$*,* $a_2 \leq b_2$ *and* $a_3 \leq b_3$*.*

5. *If* $A = \{a_1, a_2\}$ *and* $B = \{b_1, b_2, b_3\}$*, then* $A \leq_{\mathbb{H}} B$ *if and only if* $a_1 \leq b_1$ *and* $a_2 \leq b_2$ *and* $B \leq_{\mathbb{H}} A$ *if and only if* $b_2 \leq a_1$ *and* $b_3 \leq a_2$*.*

Following the pessimistic and optimistic ways to count elements of a set used in [13] a couple of partial orderings on the set $\mathbb{H}$ of finite subsets of the unit interval can be proposed.

Definition 2.8. *Let* $A = \{a_1, a_2, ..., a_n\}$ *and* $B = \{b_1, b_2, ..., b_m\}$ *be two finite subsets of the unit interval with* $n \leq_{\mathbb{H}} m$*.*

- *Optimistic ordering:*

  $A \leq_o B$ *if and only if every coordinate of the vector* $(a_1, a_2, ..., \overbrace{a_n, ..., a_n}^{m-n+1 \text{ times}})$ *is smaller than or equal to the corresponding coordinate of the vector* $(b_1, b_2, ..., b_m)$*.*

- *Pessimistic ordering:*

  $A \leq_p B$ *if and only if every coordinate of the vector* $(\overbrace{a_1, ..., a_1}^{m-n+1 \text{ times}}, a_2, ..., a_n)$ *is smaller than or equal to the corresponding coordinate of the vector* $(b_1, b_2, ..., b_m)$*.*

Let us compare the three partial orderings $\leq_{\mathbb{H}}$, $\leq_o$ and $\leq_p$ in a simple example:

Example 2.9. *Consider the sets* $A = \{a_1, a_2, a_3, a_4\}$ *and* $B = \{b_1, b_2\}$*.*

Pessimistic

$$\begin{aligned} A \leq_p B \quad &\text{if and only if} \quad a_3 \leq b_1 \text{ and } a_4 \leq b_2 \\ B \leq_p A \quad &\text{if and only if} \quad b_1 \leq a_1 \text{ and } b_2 \leq a_4. \end{aligned}$$

Optimistic

$$\begin{aligned} A \leq_o B \quad &\text{if and only if} \quad a_1 \leq b_1 \text{ and } a_4 \leq b_2 \\ B \leq_o A \quad &\text{if and only if} \quad b_1 \leq a_1 \text{ and } b_2 \leq a_4. \end{aligned}$$

$\leq_{\mathbb{H}}$

$$\begin{aligned} A \leq_{\mathbb{H}} B \quad &\text{if and only if} \quad a_2 \leq b_1 \text{ and } a_4 \leq b_2 \\ B \leq_{\mathbb{H}} A \quad &\text{if and only if} \quad b_1 \leq a_1 \text{ and } b_2 \leq a_3. \end{aligned}$$

3 Lattices Related to $(\mathbb{H}, \leq_{\mathbb{H}})$

The partial ordering $\leq_{\mathbb{H}}$ defined on $\mathbb{H}$ seems adequate for comparing finite subsets of the unit interval but it turns out that $(\mathbb{H}, \leq_{\mathbb{H}})$ is not a lattice (Proposition 3.6).

If we restrict ourselves to particular subsets of $\mathbb{H}$, then we will obtain lattices. In particular, it is interesting the case when all the cardinalities of the subsets are powers of a given natural number.

Proposition 3.1. *$\{0\}$ and $\{1\}$ are the lower and upper bounds of $(\mathbb{H}, \leq_{\mathbb{H}})$ respectively.*

Proof. Trivial. □

In order to prove that $(\mathbb{H}, \leq_{\mathbb{H}})$ is not a lattice we will present a couple of elements (namely $A = \{0.2, 0.4\}$ and $B = \{0.1, 0.3, 0.6\}$) of $\mathbb{H}$ with two non-comparable upper bounds. (Dually we could also find two non-comparable lower bounds.)

Lemma 3.2. *$M = \{0.2, 0.4, 0.6\}$ and $N = \{0.3, 0.6\}$ are greater than both $A = \{0.2, 0.4\}$ and $B = \{0.1, 0.3, 0.6\}$. Moreover M and N are non-comparable (i.e.: neither $M \leq_{\mathbb{H}} N$ nor $N \leq_{\mathbb{H}} M$).*

Proof. Straightforward. □

Lemma 3.3. *$N = \{0.3, 0.6\}$ is the smallest two-element set greater than or equal to both A and B of Lemma 3.2.*

Proof. Let $P = \{a_1, a_2\}$. $A \leq_{\mathbb{H}} P \leq_{\mathbb{H}} N$ if and only if $0.2 \leq a_1 \leq 0.3$ and $0.4 \leq a_2 \leq 0.6$. If moreover P should be greater than or equal to B, then we should have to compare the vectors $(0.1, 0.1, 0.3, 0.3, 0.6, 0.6)$ and $(a_1, a_1, a_1, a_2, a_2, a_2)$ which gives $a_1 \geq 0.3$ and $a_2 \geq 0.6$, which means $P = N$. □

Lemma 3.4. *$M = \{0.2, 0.4, 0.6\}$ is the smallest three-element set greater than or equal to both A and B of Lemma 3.2.*

Proof. Let $P = \{a_1, a_2, a_3\}$. $B \leq_{\mathbb{H}} P \leq_{\mathbb{H}} M$ if and only if $0.1 \leq a_1 \leq 0.2$, $0.3 \leq a_2 \leq 0.4$ and $0.6 \leq a_3 \leq 0.6$. If moreover P should be greater than or equal to A, then we should compare the vectors $(0.2, 0.2, 0.2, 0.4, 0.4, 0.4)$ and $(a_1, a_1, a_2, a_2, a_3, a_3)$ which gives $a_1 \geq 0.2$, $a_2 \geq 0.4$ and $a_3 \geq 0.4$, which means $P = M$. □

Lemma 3.5. *There is no joint of the sets A and B of Lemma 3.2.*

Proof. Let us suppose that $P = \{a_1, a_2, ..., a_n\}$ is such that $A \leq_{\mathbb{H}} P$, $B \leq_{\mathbb{H}} P$, $P \leq_{\mathbb{H}} M$ and $P \leq_{\mathbb{H}} N$. Consider the vectors

$$\begin{aligned}
A_{3n} &= (\overbrace{0.2, ..., 0.2}^{3n}, \overbrace{0.4, ..., 0.4}^{3n}) \\
B_{2n} &= (\overbrace{0.1, ..., 0.1}^{2n}, \overbrace{0.3, ..., 0.3}^{2n}, \overbrace{0.6, ..., 0.6}^{2n}) \\
P_6 &= (\overbrace{a_1, ..., a_1}^{6}, \overbrace{a_2, ..., a_2}^{6}, ..., \overbrace{a_n, ..., a_n}^{6}) \\
N_{3n} &= (\overbrace{0.3, ..., 0.3}^{3n}, \overbrace{0.6, ..., 0.6}^{3n}) \\
M_{2n} &= (\overbrace{0.2, ..., 0.2}^{2n}, \overbrace{0.4, ..., 0.4}^{2n}, \overbrace{0.6, ..., 0.6}^{2n})
\end{aligned}$$

If $n > 3$, then, since $A \leq_{\mathbb{H}} P \leq_{\mathbb{H}} N$ we would have $a_1 = 0.2$ and $a_2 = 0.2$. But $a_1 < a_2$. Then n must be 2 or 3 and thanks to the previous lemmas, if $n = 2$ we have $P = N$ and if $n = 3$ we have $P = M$. □

As a corollary we get the following proposition.

Proposition 3.6. $(\mathbb{H}, \leq_{\mathbb{H}})$ *is not a joint semi-lattice.*

Corollary 3.7. $(\mathbb{H}, \leq_{\mathbb{H}})$ *is not a lattice.*

Nevertheless, if the cardinalities m and n of A and B are one multiple of the other one, than there exist meet and joint of these sets.

Proposition 3.8. *Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ $(m \geq n)$ be finite subsets of the unit interval with cardinalities n and m respectively and with $m = kn$ for some $k \in \mathbb{N}$. Then there exists the meet and the joint of A and B.*

Proof.
Consider

$$\begin{aligned}
A_k &= (\overbrace{a_1, ..., a_1}^{k}, \overbrace{a_2, ..., a_2}^{k}, ..., \overbrace{a_n, ..., a_n}^{k}) = (c_1, ..., c_k, c_{k+1}, ..., c_{2k}, ..., c_{(n-1)k+1}, ..., c_{nk}) \\
B_1 &= (b_1, b_2, ..., b_m).
\end{aligned}$$

- If $B \leq_{\mathbb{H}} A$, then $A \wedge B = B$ and $A \vee B = A$.
- If $B \nleq A$, then there exists $i \in \{1, 2, ..., n\}$ such that $b_i > c_{in}$.
  For every $i = 1, 2, ..., n$ let

$$\begin{aligned} d_i &= \max(b_i, c_{(i-1)k+1}, ..., c_{ik}) \\ e_i &= \min(b_i, c_{(i-1)k+1}, ..., c_{ik}) \end{aligned}$$

  and

$$\begin{aligned} D &= \{d_1, d_2, ...d_n\} \\ E &= \{e_1, e_2, ...e_n\}. \end{aligned}$$

  Then $D = A \vee B$ and $E = A \wedge B$.

$\square$

As a corollary we obtain the following result.

Proposition 3.9. *Let $R = r_1, r_2, ..., r_n, ...$ be a sequence of natural numbers with r_{i+1} a multiple of r_i for every $i \geq 0$, $\mathbb{H}_{r_i}$ the set of finite subsets of the unit interval of cardinality r_i and $\mathbb{H}_R = \bigcup_{i \geq 1}^{\infty} \mathbb{H}_{r_i}$. Then $(\mathbb{H}_R, \leq_{\mathbb{H}})$ is a lattice.*

Considering constant sequences $R = r, r, ..., r, ...$ we obtain the following result.

Corollary 3.10. *$(\mathbb{H}_r, \leq_{\mathbb{H}})$ is a lattice for all $r \in \mathbb{N}$.*

Also,

Corollary 3.11. *Given $r \in \mathbb{N}$ for $R = r, r^2, r^3, ..., r^n, ...$ $(\mathbb{H}_R, \leq_{\mathbb{H}})$ is a lattice.*

4 Bags. Embedding $(\mathbb{H}, \leq_{\mathbb{H}})$ into a Lattice

A natural embedding of $(\mathbb{H}, \leq_{\mathbb{H}})$ into a lattice can be given explicitly. The idea is to allow sets with some elements repeated, the so called bags or multisets [14]. A finite bag of the unit interval can be represented as a vector $\vec{v} = (a_1, a_2, ..., a_n)$ of $[0, 1]^n$ and we will always assume that $a_1 \leq a_2 \leq ... \leq a_n$. The set of finite bags of $[0, 1]$ will be denoted by $\mathbb{B}$.

The following two definitions are similar to Definitions 2.1 and 2.2.

Definition 4.1. *Given* $\vec{v} = (a_1, a_2, ..., a_n) \in \mathbb{B}$ *and* $r \in \mathbb{N}$, $\vec{v}_r$ *is defined by*

$$\vec{v}_r = (\overbrace{a_1, ..., a_1}^{r \text{ times}}, \overbrace{a_2, ..., a_2}^{r \text{ times}}, ..., \overbrace{a_n, ..., a_n}^{r \text{ times}}).$$

Definition 4.2. *Let* $\vec{u} = (a_1, a_2, ..., a_n)$ *and* $\vec{v} = (b_1, b_2, ..., b_m)$ *be two finite bags of the unit interval and* $\operatorname{lcm}(n, m)$ *the least common multiple of* n *and* m. *Rewriting* $\vec{u}_{\frac{\operatorname{lcm}(n,m)}{m}} = (c_1, c_2, ..., c_{\operatorname{lcm}(n,m)})$ *and* $\vec{v}_{\frac{\operatorname{lcm}(n,m)}{n}} = (d_1, d_2, ..., d_{\operatorname{lcm}(n,m)})$,

$$\vec{u} \leq_{\mathbb{B}} \vec{v} \text{ if and only if } c_i \leq d_i \text{ for all } i = 1, 2, ..., \operatorname{lcm}(n, m).$$

Similar to Proposition 2.6 we can prove the following result.

Proposition 4.3. *The relation* $\leq_{\mathbb{B}}$ *on* $\mathbb{B}$ *is a preorder (i.e., it is irreflexive and transitive).*

The following lemma is straightforward.

Lemma 4.4. *Given two bags* $\vec{u}$ *and* $\vec{v}$ *of* $\mathbb{B}$, $\vec{u} \leq_{\mathbb{B}} \vec{v}$ *and* $\vec{v} \leq_{\mathbb{B}} \vec{u}$ *if and only if there exists* $\vec{w} \in \mathbb{B}$ *and* $r, s \in \mathbb{N}$ *such that* $\vec{u} = \vec{w}_r$ *and* $\vec{v} = \vec{w}_s$.

Example 4.5. *For* $\vec{u} = (0.1, 0.1, 0.1, 0.1, 0.3, 0.3)$ *and* $\vec{v} = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.3, 0.3, 0.3)$ *we have* $\vec{u} \leq_{\mathbb{B}} \vec{v}$ *and* $\vec{v} \leq_{\mathbb{B}} \vec{u}$ *because* $\vec{u} = \vec{w}_2$ *and* $\vec{u} = \vec{w}_3$ *for* $\vec{w} = (0.1, 0.1, 0.3)$.

Definition 4.6. *On* $\mathbb{B}$ *consider the equivalence relation* $\sim$ *defined for all* $\vec{u}, \vec{v} \in \mathbb{B}$ *by*

$$\vec{u} \sim \vec{v} \text{ if and only if there exists } \vec{w} \in \mathbb{B} \text{ and } r, s \in \mathbb{N} \text{ such that } \vec{u} = \vec{w}_r \text{ and } \vec{v} = \vec{w}_s$$

and denote the quotient $\mathbb{B}/\!\sim$ *by* $\overline{\mathbb{B}}$.

The vector of a class with the smallest number of coordinates will be called its canonical representative.

$\leq_{\mathbb{B}}$ is compatible with $\sim$ and we obtain the following result.

Proposition 4.7. $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$ *is a lattice.*

Proof. Let $[\vec{u}]$ and $[\vec{v}]$ be the classes with representatives $\vec{u} = (a_1, a_2, ..., a_n)$ and $\vec{v} = (b_1, b_2, ..., b_m)$ respectively on $\overline{\mathbb{B}}$ and $\operatorname{lcm}(n, m)$ the least common multiple of n and m. Rewriting $\vec{u}_{\frac{\operatorname{lcm}(n,m)}{m}} = (c_1, c_2, ..., c_{\operatorname{lcm}(n,m)})$ and $\vec{v}_{\frac{\operatorname{lcm}(n,m)}{n}} = (d_1, d_2, ..., d_{\operatorname{lcm}(n,m)})$,

$$\begin{aligned} [\vec{u}] \vee [\vec{v}] &= [(\max\{c_1, d_1\}, \max\{c_2, d_2\}, ..., \max\{c_{\operatorname{lcm}(n,m)}, d_{\operatorname{lcm}(n,m)}\})] \\ [\vec{u}] \wedge [\vec{v}] &= [(\min\{c_1, d_1\}, \min\{c_2, d_2\}, ..., \min\{c_{\operatorname{lcm}(n,m)}, d_{\operatorname{lcm}(n,m)}\})]. \end{aligned}$$

□

Proposition 4.8. *The map $i : \mathbb{H} \to \mathbb{B}$ sending every finite set $A = \{a_1, a_2, ..., a_n\}$ of $[0,1]$ to $i(A) = (a_1, a_2, ..., a_n)$ is an order embedding. The images of different finite sets belong to different classes in $\mathbb{B}$ and so the mapping $\bar{i} : \mathbb{H} \to \overline{\mathbb{B}}$ is an embedding. Moreover $i(A)$ is the canonical representative of its class.*

Example 4.9. *In Section 3 we have seen that $A = \{0.2, 0.4\}$ and $B = \{0.1, 0.3, 0.6\}$ do not have neither meet nor joint in $\mathbb{H}$. Nevertheless,*

$$\begin{aligned} \bar{i}(A) \vee \bar{i}(B) &= [(0.2, 0.4)] \vee [(0.1, 0.3, 0.6)] = [(0.2, 0.2, 0.3, 0.4, 0.6, 0.6)] \\ \bar{i}(A) \wedge \bar{i}(B) &= [(0.2, 0.4)] \wedge [(0.1, 0.3, 0.6)] = [(0.1, 0.1, 0.2, 0.3, 0.4, 0.4)]. \end{aligned}$$

5 t-norms on $(\mathbb{H}, \leq_{\mathbb{H}})$ and $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$

t-norms are binary operations on $[0,1]$ used for modelling the "and" connective [4] [8]. Extensions to more general lattices or partially ordered sets are necessary and widely used. For example, if $T_1, T_2, ..., T_n$ are t-norms on $[0,1]$, we can define the t-norm T on $[0,1]^n$ componentwise; namely, $T((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) = (T_1(x_1, y_1), T_2(x_2, y_2), ..., T_n(x_n, y_n))$ [2]. Also in [2] t-norms are defined on bounded partially ordered sets. With this in mind, since $(\mathbb{H}, \leq_{\mathbb{H}})$ and $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$ are bounded partially ordered sets, t-norms can be defined on them. However, the conditions required for a connective to be a t-norm are sometimes too strong and difficult to obtain. Associativity is the most discussed property and is omitted in the definition of more general conjunctors. In this sense, the concept of general conjunctor is introduced in [3].

Definition 5.1. *([3]) An operation $C : [0,1]^2 \to [0,1]$ is a fuzzy conjunction if*

- *It is increasing with respect to each variable*
- $C(1,1) = 1$, $C(0,0) = C(0,1) = C(1,0) = 0$.

This definition can be generalized to any bounded partially ordered set.

Definition 5.2. *Let $\mathbb{P} = (P, \leq_P, 0, 1)$ be a bounded partially ordered set. An operation $\mathbb{C} : \mathbb{P}^2 \to \mathbb{P}$ is a fuzzy conjunction if*

- *It is increasing with respect to each variable*

- $\mathbb{C}(1,1) = 1$, $\mathbb{C}(0,0) = \mathbb{C}(0,1) = \mathbb{C}(1,0) = 0$.

Let us recall the definition of a t-norm on a bounded partially ordered set.

Definition 5.3. *A t-norm $\mathbb{T}$ on a bounded partially ordered set $\mathbb{P} = (P, \leq_P, 0, 1)$ is a binary operation on P that for all $x, y, z \in P$ satisfies*

- $\mathbb{T}(x, 1) = x$ *(neutral element)*
- *If $x \leq_P y$, then $\mathbb{T}(x, z) \leq_P \mathbb{T}(y, z)$ (monotonicity)*
- $\mathbb{T}(x, y) = \mathbb{T}(y, x)$ *(commutativity)*
- $\mathbb{T}(x, \mathbb{T}(y, z)) = \mathbb{T}(\mathbb{T}(x, y), z)$ *(associativity).*

From a t-norm T on $[0,1]$ an operation $\mathbb{C}$ can be defined on $\mathbb{H}$.

Definition 5.4. *Let T be a t-norm on $[0,1]$. On $\mathbb{H}$ the binary operation $\mathbb{C}$ is defined in the following way. For two elements $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ of $\mathbb{H}$, writing $A_{\frac{\mathrm{lcm}(n,m)}{m}} = (c_1, c_2, ..., c_{\mathrm{lcm}(n,m)})$ and $B_{\frac{\mathrm{lcm}(n,m)}{n}} = (d_1, d_2, ..., d_{\mathrm{lcm}(n,m)})$,*

$$\mathbb{C}(A, B) = \{T(c_1, d_1), T(c_2, d_2), ..., T(c_{\mathrm{lcm}(n,m)}, d_{\mathrm{lcm}(n,m)})\}.$$

Example 5.5. *If $A = \{0.2, 0.5\}$ and $B = \{0.1, 0.4, 0.5\}$ and T is the Product t-norm, then $A_3 = (0.2, 0.2, 0.2, 0.5, 0.5, 0.5)$, $B_2 = (0.1, 0.1, 0.4, 0.4, 0.5, 0.5)$ and*

$$\mathbb{C}(A, B) = \{0.02, 0.02, 0.08, 0.20, 0.25, 0.25\} = \{0.02, 0.08, 0.20, 0.25\}$$

the last equality obtained by transforming the multiset into a set.

The following lemma is similar to Lemma 2.4 and will be used to prove the monotonicity of $\mathbb{C}$ in Proposition 5.7.

Lemma 5.6. *Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ be two finite subsets of the unit interval and $r \in \mathbb{N}$. Writing $A_{r\frac{\mathrm{lcm}(n,m)}{m}} = (c_1, c_2, ..., c_{r\mathrm{lcm}(n,m)})$ and $B_{r\frac{\mathrm{lcm}(n,m)}{n}} = (d_1, d_2, ..., d_{r\mathrm{lcm}(n,m)})$,*

$$\mathbb{C}(A, B) = \{T(c_1, d_2), T(c_2, d_2), ..., T(c_{r\mathrm{lcm}(n,m)}, d_{r\mathrm{lcm}(n,m)})\}.$$

Proposition 5.7. *Let T be a t-norm on $[0,1]$. Then $\mathbb{C}$ is a commutative fuzzy conjunction on $(\mathbb{H}, \leq_{\mathbb{H}})$.*

Proof.

- Commutativity follows trivially form the commutativity of T.

- Let $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_m\}$ and $C = \{c_1, c_2, ..., c_p\}$ be finite subsets of the unit interval. Writing $A_{\frac{\text{lcm}(m,n,p)}{\text{lcm}(m,p)}} = (d_1, d_2, ..., d_{\text{lcm}(m,n,p)})$, $B_{\frac{\text{lcm}(m,n,p)}{\text{lcm}(n,p)}} = (e_1, e_2, ..., e_{\text{lcm}(m,n,p)})$ and $C_{\frac{\text{lcm}(m,n,p)}{\text{lcm}(m,n)}} = (f_1, f_2, ..., f_{\text{lcm}(m,n,p)})$.
  If $A \leq_{\mathbb{H}} B$, then, thanks to Lemma 2.4, $d_i \leq e_i$ for all $i = 1, 2, ..., \text{lcm}(m, n, p)$ and from this $T(d_i, f_i) \leq T(e_i, f_i)$. Applying Lemma 5.6 we get,

$$\mathbb{C}(A, C) \leq \mathbb{C}(B, C).$$

- The contour conditions $\mathbb{C}(\{1\}, \{1\}) = \{1\}$, $\mathbb{C}(\{0\}, \{0\}) = \mathbb{C}(\{0\}, \{1\}) = \mathbb{C}(\{1\}, \{0\}) = \{0\}$ are trivially satisfied.

□

The fuzzy conjunctor $\mathbb{C}$ is not a t-norm on $(\mathbb{H}, \leq_{\mathbb{H}})$ because it does not satisfy associativity. Indeed, we can show the following counter-example:

If $A = \{0.7, 0.8\}$, $B = \{0.6, 0.7, 0.9\}$, $C = \{0.3, 0.4, 0.7\}$, and T is the Product t-norm, then

$$\begin{aligned} \mathbb{C}(\mathbb{C}(A, B), C) &= \{0.126, 0.147, 0.196, 0.224, 0.392, 0.504\} \\ \mathbb{C}(A, \mathbb{C}(A, B)) &= \{0.126, 0.196, 0.224, 0.504\}. \end{aligned}$$

Nevertheless, we can consider subsets of $\mathbb{H}$ in which the restriction of $\mathbb{C}$ is a t-norm following the ideas from Section 3.

Proposition 5.8. *Let $R = r_1, r_2, ..., r_n, ...$ be a sequence of natural numbers with r_{i+1} a multiple of r_i for every $i \geq 0$, $\mathbb{H}_{r_i}$ the set of finite subsets of the unit interval of cardinality r_i and $\mathbb{H}_R = \bigcup_{i \geq 1}^{\infty} \mathbb{H}_{r_i}$. Then $\mathbb{C} : \mathbb{H}_R^2 \to \mathbb{H}_R$ is a t-norm on $\mathbb{H}_R$.*

In $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$ we can also derive a binary operation $\mathbb{T}$ from a t-norm on $[0,1]$ in a similar way as in Definition 5.4. In this case, $\mathbb{T}$ is a t-norm.

Definition 5.9. *Let T be a t-norm on $[0,1]$. On $\overline{\mathbb{B}}$ the binary operation $\mathbb{T}$ is defined in the following way: For two elements $[\vec{u}] = [(a_1, a_2, ..., a_n)]$ and $[\vec{v}] = [(b_1, b_2, ..., b_m)]$ of $\overline{\mathbb{B}}$, writing $\vec{u}_{\frac{\mathrm{lcm}(n,m)}{m}} = (c_1, c_2, ..., c_{\mathrm{lcm}(n,m)})$ and $\vec{v}_{\frac{\mathrm{lcm}(n,m)}{n}} = (d_1, d_2, ..., d_{\mathrm{lcm}(n,m)})$,*

$$\mathbb{T}([\vec{u}], [\vec{v}]) = [(T(c_1, d_1), T(c_2, d_2), ..., T(c_{\mathrm{lcm}(n,m)}, d_{\mathrm{lcm}(n,m)}))].$$

Example 5.10. *If $\vec{u} = (0.2, 0.5)$ and $\vec{v} = (0.1, 0.4, 0.5)$ and T is the Product t-norm, then $\vec{u}_3 = (0.2, 0.2, 0.2, 0.5, 0.5, 0.5)$, $\vec{v}_2 = (0.1, 0.1, 0.4, 0.4, 0.5, 0.5)$ and*

$$\mathbb{T}([\vec{u}], [\vec{v}]) = [(0.02, 0.02, 0.08, 0.20, 0.25, 0.25)].$$

Please note the difference between Example 5.10 and Example 5.5. (In Example 5.10 $(0.02, 0.08, 0.20, 0.25)$ is not a representative of $\mathbb{T}([\vec{u}], [\vec{v}])$).

Proposition 5.11. *Let T be a t-norm on $[0,1]$. Then $\mathbb{T}$ is a t-norm on $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$.*

Proof. Let $\vec{u} = (a_1, a_2, ..., a_n)$, $\vec{v} = (b_1, b_2, ..., b_m)$ and $\vec{w} = (c_1, c_2, ..., c_p)$ be finite bags of the unit interval. Writing $\vec{v}_{\frac{\mathrm{lcm}(m,n,p)}{\mathrm{lcm}(m,p)}} = (d_1, d_2, ..., d_{\mathrm{lcm}(m,n,p)})$, $\vec{v}_{\frac{\mathrm{lcm}(m,n,p)}{\mathrm{lcm}(n,p)}} = (e_1, e_2, ..., e_{\mathrm{lcm}(m,n,p)})$ and $\vec{w}_{\frac{\mathrm{lcm}(m,n,p)}{\mathrm{lcm}(m,n)}} = (f_1, f_2, ..., f_{\mathrm{lcm}(m,n,p)})$.

- It is trivial to prove that $\mathbb{T}([\vec{u}], [1]) = [\vec{u}]$.
- Monotonicity: If $\vec{u} \leq_{\mathbb{B}} \vec{v}$, then $d_i \leq e_i$ for all $i = 1, 2, ..., \mathrm{lcm}(m,n,p)$ and from this $T(d_i, f_i) \leq T(e_i, f_i)$. Hence,
$$\mathbb{T}([\vec{u}], [\vec{w}]) \leq \mathbb{T}([\vec{v}], [\vec{w}]).$$
- Commutativity follows trivially form the commutativity of T.
- Associativity:
$$\begin{aligned}
&\mathbb{T}([\vec{u}], \mathbb{T}([\vec{v}], [\vec{w}])) \\
&\quad = \mathbb{T}([(d_1, d_2, ..., d_{\mathrm{lcm}(m,n,p)})], \mathbb{T}([(e_1, e_2, ..., e_{\mathrm{lcm}(m,n,p)})], [(f_1, f_2, ..., f_{\mathrm{lcm}(m,n,p)})])) \\
&\quad = \mathbb{T}([(d_1, d_2, ..., d_{\mathrm{lcm}(m,n,p)})], [(T(e_1, f_1), T(e_2, f_2), ..., T(e_{\mathrm{lcm}(m,n,p)}, f_{\mathrm{lcm}(m,n,p)}))]) \\
&\quad = [(T(d_1, T(e_1, f_1)), T(d_2, T(e_2, f_2)), ..., T(d_{\mathrm{lcm}(m,n,p)}, T(e_{\mathrm{lcm}(m,n,p)}, f_{\mathrm{lcm}(m,n,p)})))] \\
&\quad = [(T(T(d_1, e_1), f_1), T(T(d_2, e_2), f_2), ..., T(T(d_{\mathrm{lcm}(m,n,p)}, e_{\mathrm{lcm}(m,n,p)}), f_{\mathrm{lcm}(m,n,p)})))] \\
&\quad = \mathbb{T}([(T(d_1, e_1), T(d_2, e_2), ..., T(d_{\mathrm{lcm}(m,n,p)}, e_{\mathrm{lcm}(m,n,p)}))], [(f_1, f_2, ..., f_{\mathrm{lcm}(m,n,p)})]) \\
&\quad = \mathbb{T}(\mathbb{T}([\vec{u}], [\vec{v}]), [\vec{w}]).
\end{aligned}$$

□

6 Concluding Remarks

We have introduced and studied different orderings on the set $\mathbb{H}$ of finite subsets of the unit interval and on the set $\overline{\mathbb{B}}$. The results of this paper remain valid replacing the unit interval by a bounded totally ordered set.

We have shown that $(\mathbb{H}, \leq_{\mathbb{H}})$ is a bounded partially ordered set with $\{0\}$ and $\{1\}$ the lower and upper bounds respectively but not a lattice. Nevertheless, interesting subsets of if are: among others, the sets $\mathbb{H}_r$ of the unit interval of fixed cardinality r and the sets $\mathbb{H}_R$ of the unit interval with cardinalities $R = r, r^2, ..., r^n$ for a given $r \in \mathbb{N}$. Moreover from the finite bags $\mathbb{B}$ of the unit interval a lattice $(\overline{\mathbb{B}}, \leq_B)$ has been built and $(\mathbb{H}, \leq_{\mathbb{H}})$ embedded into it in a natural way.

Section 5 is a first introduction to t-norms and fuzzy conjunctions on $(\mathbb{H}, \leq_{\mathbb{H}})$ and on $(\overline{\mathbb{B}}, \leq_{\mathbb{B}})$. The topic deserves a much deeper attention. In particular it is important to find reasonable ways to generate t-norms on $\mathbb{H}$ from a t-norm on $[0, 1]$. This will be studied by the authors in forthcoming papers.

Apart from its use for comparing hesitant fuzzy sets, $\mathbb{H}$ can also be useful for comparing different types of fuzzy subsets. For example, interval-valued fuzzy sets, triangular fuzzy numbers $[a, b, c]$ with support the interval $[a, c]$ and core b or trapezoidal numbers $[a, b, c, d]$ with $[a, d]$ its support and $[b, c]$ its core. In this way we can compare different types of fuzzy subsets at the same time, enriching the possibilities of tackling different types of imprecision. For instance, Example 2.7 can have the following interpretation:

1 The usual ordering of the unit interval is preserved.

3 The usual ordering of intervals is preserved.

4 A triangular number $A = [a_1, a_2, a_3]$ is smaller than or equal to a number $B = [b_1, b_2, b_3]$ when the support of A is smaller than or equal to the support of B (these supports considered as intervals) and the core of A is smaller than or equal to the core of B.

5 An interval $A = [a_1, a_2]$ is smaller than or equal to a triangular fuzzy number $B = [b_1, b_2, b_3]$ when the left endpoint of A is smaller than or equal to the left endpoint of the support of B and the right endpoint of A is smaller than or equal to the core of B while B is smaller than or equal to A when the core of B is smaller than or equal to the left

endpoint of A and the right endpoint of the support of B is smaller than or equal to the right endpoint of A. In this sense, A can be seen as a triangular number with unknown core and then $A \leq_{\mathbb{H}} B$ or $B \leq_{\mathbb{H}} A$ if and only if the relation is true for all possibilities of assigning the core to A.

Acknowledgments

We thank the support of the research projects TIN2012-32482 and TIN2015-66471-P.

References

[1] B. Bedregal, R. Reiser, H. Bustince, C. Lopez-Molina, V. Torra. Aggregation Functions for typical hesitant fuzzy elements and the action of automorphisms. Information Sciences, 255 (2014) 82–99.

[2] B. De Baets, R. Mesiar. Triangular norms on product lattices. Fuzzy Sets Systems, 104 (1999) 61-75.

[3] J.C. Fodor, T. Keresztfalvi, Non-standard conjunctions and implications in fuzzy logic, Internat. J. Approx. Reason. 12 (1995) 69-84.

[4] E.P. Klement, R. Mesiar, E. Pap. Triangular norms. Kluwer Academic Publishers, Dordrecht, 2000.

[5] R.M. Rodríguez, L. Martínez, and F. Herrera. Hesitant fuzzy linguistic term sets for decision making. IEEE Transactions on Fuzzy Systems, 20 (2012) 109-119.

[6] R.M. Rodríguez, L. Martínez, and F. Herrera. A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. Information Sciences, 241, (2013) 28-42.

[7] R.M. Rodríguez, L. Martínez, V. Torra, Z.S. Xud, F. Herrera. Hesitant fuzzy sets: State of the art and future directions. International Journal of Intelligent Systems, 29 (2014) 495–524.

[8] B. Schweizer, A. Sklar. Probabilistic Metric Spaces. North-Holland, Amsterdam, 1983.

[9] V. Torra. Hesitant fuzzy sets. International Journal of Intelligent Systems, 25 (2010) 529-539.

[10] V. Torra and Y. Narukawa. On hesitant fuzzy sets and decision. In The 18th IEEE International Conference on Fuzzy Systems (2009) 1378-1382.

[11] C. Wei, N. Zhao, and X. Tang. Operators and comparisons of hesitant fuzzy linguistic term sets. IEEE Transactions on Fuzzy Systems, 22 (2014) 575–585.

[12] M.M. Xia, Z.S. Xu. Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52 (2011) 395–407.

[13] Z.S. Xu, M.M. Xia. Distance and similarity measures for hesitant fuzzy sets. Information Sciences, 181 (2011) 2128–2138.

[14] R:R: Yager. On the theory of bags. International Journal of General Systems, 13 (1986) 23–37.

[15] L. Zadeh. Fuzzy sets. Information and Control, 8 (1965) 338–353.