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Title: **Analysis of measurement and simulation errors in structural system identification by observability techniques.**

Authors:

Jun Lei, José Antonio Lozano-Galant, María Nogal , **Dong Xu**(Corresponding Author), José Turmo

Corresponding author:

Dong Xu,
Prof., Ph.D.
Department of Bridge Engineering at Tongji University
1239, Siping Road, Shanghai, China
Post: 200092
Email: xu_dong@tongji.edu.cn
Tel: 021-65980953
Fax: 021-65980953

Analysis of measurement and simulation errors in structural system identification by observability techniques.

Summary

During the process of structural system identification, it is unavoidable to introduce errors in measurement and errors in the identification technique. This paper analyzes the effects of these errors in structural system identification based on observability techniques. To illustrate the symbolic approach of this method a simply supported beam is analyzed step-by-step. This analysis provides, for the very first time in the literature, the parametric equations of the estimated parameters. The effects of several factors, such as errors in a particular measurement or in the whole measurement set, load location, location of the measurement or sign of the errors, on the accuracy of the identification results are also investigated. It is found that error in a particular measurement increases the errors of individual estimations and this effect can be significantly mitigated by introducing random errors in the whole measurement set. The propagation of simulation errors when using observability techniques is illustrated by two structures with different measurement sets and loading cases. A fluctuation of the observed parameters around the real values is proved to be a characteristic of this method. Also, it is suggested that a sufficient combination of different load cases should be utilized to avoid the inaccurate estimation at the location of low curvature zones.

Keyword: structural system identification; stiffness method; observability technique; measurement error; simulation error; observability flow

1. Introduction

Structural System Identification (SSI) methods enable the estimation of stiffnesses and/or masses of actual structures from their monitored data. A wide number of SSI methods have been presented in the literature. In fact, the state of the art of these method have been reviewed in a number of works [1,2]. According to most of these works, system identification methods can be classified as parametric [3–6] and non-parametric (genetic algorithms [7–9], evolutionary strategy [10–13], neural networks [14,15] or least-squares estimation [16–18]).

The major difference between these two methods refers to the equations that link the input and output data, as only in the parametric methods those have a physical meaning. For this reason, parametric methods might be preferred over non-parametric ones.

A major concern for the structural system identification in actual structures refers to the sensitivity of the SSI method to errors. Sanayei et al. [19] summarized the different errors that influence the accuracy of these methods as follows: (1) Measurement errors: Independent of the measurement device, error free measurements cannot be obtained in any actual nondestructive test. In this way, when these measurements are introduced into the SSI technique, deviations in the estimates appear. These unbiased errors can be reduced by technological developments but cannot be avoided. (2) Errors in the parameter estimation technique: Every SSI method is characterized by its characteristic simulation error. This error appears even when noise-free measurements are considered as it depends on the technique formulation. Examples of this error refer to the hypotheses of iterative or optimization processes used in the identification method or the loss of numerical accuracy in computer calculation. However, for the very first time in the literature, the explicit analytical solutions of these estimated parameters can be derived from the observability method in a symbolic way. Hence, those errors in parameter estimation might be avoided if noise-free data were used. (3) Modeling errors: These errors are due to uncertainties in the parameters of the simplified Finite Element Model. Some examples of this error refer to the inaccuracy in material properties, the existence of elements which stiffness was not accounted for, or errors in the boundary conditions.

Significant research has been carried out to study the impact of the different errors on parametric methods. Saneyei and Saletnik [20] proposed an error sensitivity analysis to evaluate the effect of noise in measurements. Saneyei et al. [21] compared the results of different error functions to evaluate the errors in the parameter estimation technique in a small scale model. Saneyei et al. [19] studied the effects of modeling errors in frame structures with elastic supports. Yuen and Katafygiotis [22] studied the effects of noisy measurements in structural system identification. Caddemi and Greco [23] studied the influence of instrumental errors on the static identification of damage parameters for elastic beams. Zhang et al. [24] used intervals analyses to limit the values for the identified parameters under the effect of modeling errors. Wang [25] studied the effects of flexible joints and boundary conditions for model updating. Sanayei et al. [4] presented an error sensitivity analysis to study each parameter based on the load cases and measurement locations of the nondestructive tests.

Lozano-Galant et al. [26,27] proposed the observability method [28] for structural system identification from static tests. This parametric technique analyzes the stiffness matrix method as a monomial-ratio system of equations and enables the mathematical identification of element stiffnesses of the whole structure or of a portion of it using a subset of deflection and/or rotation measurements. In all these works, noise-free measurements were considered. Nevertheless, this assumption is far from reality as the data of actual nondestructive tests is always subjected to errors in measurement devices. In order to fill this gap, this paper analyzes the effects of measurement errors in structural identification by observability techniques. The simulation errors inherent to this identification method are also studied in detail.

This article is organized as follows. In Section 2, the application of observability techniques to structural system identification is presented. In Section 3, a simply supported beam is analyzed to illustrate the different errors appearing in the observability technique. In Section 4 the measurement error is analyzed in an illustrative structure. Next, in Section 5 two structures are studied to illustrate the errors inherent to the observability technique. Finally, some conclusions are drawn in Section 6.

2. Structural System Identification by observability techniques

Prior to the application of observability techniques, a FEM of the structure should be established based on the topology of the structure to be identified, which is a common preliminary step in many identification methods [29–31]. With this FEM and the stiffness matrix method, the equilibrium equations together with strength of materials theory might be written in terms of nodal displacements and nodal forces as presented in Equation 1.

$$[K] \cdot \{\delta\} = \{f\}, \tag{1}$$

in which [K] is the stiffness matrix of the structure, {δ}, is a vector of nodal displacements and {f} is a vector of nodal forces. For 2D analysis, Matrix [K] includes the geometrical and mechanical properties of the beam elements of the structure, such as length, L_j, shear modulus, G_j, Young's modulus, E_j, area, A_j, inertia, I_j, and torsional stiffness, J_j, associated with the j-element.

When the SSI is introduced in the stiffness matrix method, the matrix [K] is partially unknown. Usually, L_j is assumed known while the stiffnesses are traditionally assumed as unknown. The determination of the unknown parameters in [K] leads to a nonlinear problem as these parameters are multiplied by the displacements of the nodes (in 2D, horizontal and vertical deflection and rotation associated with the k-node u_k, v_k and w_k, respectively). This implies that non-linear products of variables, such as $E_j A_j u_k$, $E_j A_j v_k$, $E_j I_j u_k$, $E_j I_j v_k$ and $E_j I_j w_k$, might appear, leading to a polynomial system of equations. Before further discussion, one thing should be kept in mind is that the major interest in structural identification is to assess the structural behavior, e.g. axial stiffnesses, EA, or flexural stiffnesses, EI. In order to reduce the number of parameter, these stiffnesses are, respectively, assimilated into areas, A, and inertias, I, by setting the modulus as a assumed value, e.g. unity or typical values from handbooks. When the identification by observability is completed, the axial stiffnesses and the flexural stiffnesses, respectively, can be recovered by the multiplication of the predefined modulus and the estimated area, $\hat{A}$, and the estimated inertia, $\hat{I}$. This strategy is also followed in [32,33].

To solve these equations in a linear-form, system (1) can be rewritten as:

$$[K^*] \cdot \{\delta^*\} = \{f\}, \tag{2}$$

in which the products of variables are located in the modified vector of displacements $\{\delta^*\}$ and the modified stiffness matrix $[K^*]$ is a matrix of coefficients with different dimensions from the initial stiffness matrix [K]. Depending on the known information, the unknown variables of vector $\{\delta^*\}$ may be the non-linear products presented above, as well as other factors of single variables, such as $E_j I_j$, $E_j A_j$, E_j, A_j, I_j or node deflections.

Once the boundary conditions and the applied forces at the nodes during the nondestructive test are introduced, it can be assumed that a subset of increments of deflections δ_1^* of $\{\delta^*\}$ and a subset of forces in nodes f_1 of $\{f\}$ are known and the remaining subset δ_0^* of $\{\delta^*\}$ and f_0 of $\{f\}$ are not. By the static condensation procedure, the system in (2) can be partitioned as follows:

$$[K^*]\{\delta^*\} = \begin{pmatrix} K_{00}^* & K_{01}^* \\ K_{10}^* & K_{11}^* \end{pmatrix} \begin{Bmatrix} \delta_0^* \\ \delta_1^* \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_1 \end{Bmatrix} = \{f\}, \tag{3}$$

where K_{00}^*, K_{01}^*, K_{10}^* and K_{11}^* are partitioned matrices of $[K^*]$and δ_0^*, δ_1^*, f_0 and f_1 are partitioned vectors of $\{\delta^*\}$ and $\{f\}$.

In order to join the unknowns, system (3) can be written in the equivalent form, as:

$$[B]\{z\} = \begin{pmatrix} K_{10}^{*} & 0 \\ K_{00}^{*} & -\mathrm{I} \end{pmatrix} \begin{Bmatrix} \delta_0^{*} \\ \mathrm{f}_0 \end{Bmatrix} = \begin{Bmatrix} \mathrm{f}_1 - K_{11}^{*} \times \delta_1^{*} \\ -K_{01}^{*} \times \delta_1^{*} \end{Bmatrix} = \{D\}, \tag{4}$$

where 0 and I are the null and the identity matrices, respectively. In this system the vector of unknown variables, $\{\mathrm{z}\}$, appears on the left-hand side and the vector of observations, $\{\mathrm{D}\}$, on the right-hand side. Both vectors are related by a coefficient matrix[B]. For the system (4) to have a solution, it is sufficient to calculate the null space [V] of [B] and checking that $[\mathrm{V}][\mathrm{D}] = \{0\}$. Examination of matrix [V] and identification of its null rows leads to identification of the observable variables (subset of variables with a unique solution) of vector $\{\mathrm{z}\}$. The number of required deflections can be optimized by using a recursive process that takes advantage of the connectivity of the beams in the stiffness matrix. This connectivity is included in partitioned matrices of $[K^{*}]$ and therefore, in system (4). In this way, when in the initial observability analysis any deflection, force or structural parameter is observed, this information might help to observe new parameters in the adjacent beam elements through a recursive process. In this analysis, the observed information in the previous step is successively introduced as input data in the observability simulation.

A detailed step by step application of the observability techniques is presented in [26,27]. The readers are recommended to refer to those papers for a more detailed explanation of the peculiarities of the proposed methodology.

The symbolical SSI algorithm presented above fails to address the numerical estimation of the observed parameters. To solve this problem, a numerical development of the observability techniques was presented in [2]. This algorithm combines two approaches: a symbolical and a numerical one. On the one hand, the symbolic approach is used for the observability analysis. This analysis reduces the effects of the unavoidable numerical errors during the computation of the null spaces of the system of equations. On the other hand, the second approach enables the numerical estimation of the observed parameters. This mixed algorithm also includes a recursive process, in which the new observed parameters are successively introduced into the analysis. One concern of this method is that a huge burden is expected in the computation of the null space [V] when confronted with a problem involving a large number of observable variables. However, this method has been applied to some large structures, including a 13-storeys frame building [26] and a cable stayed bridge [27,32]. The main time cost of the algorithm is in the computation of the null space, $[V]$, by symbolical approach whereas the time cost by the numeric approach is negligible. However, the computation of the null space by symbolical approach can be carried out efficiently in Matlab subroutine. In the case of the 13-storeys building, it has been checked that 396 seconds are needed, on a laptop with a 2.4 GHz i7 processor and a 16 GB memory, to get the null space of a matrix [B] with the dimension of 258×462. Note that the number of rows in the matrix [B] is three times as the number of the nodes, which is unchanging, while the number of columns in the matrix [B] equals the number of unknowns. Moreover, the number of unknowns decreases with the recursive steps since part of the unknowns has been observed in preceding steps. Thus, the computation of the null space of the matrix [B] will be accelerated during the recursive steps due to the decrease of the scale of [B]. In addition, if a larger structure of more observable parameters is provided, which could not be handled by this laptop, stronger machines, such as desktops or work stations can be employed.

With regard to the ability of this method, until now, it is only applied in 2D structures simulated by 1D elements with 3 DOFs per node. Conceptually, as a mathematical tool, the observability technique is expected to be able to apply in different formulations of the FEM, including but not limited to 3D structures simulated by 1D elements with 6 DOFs per node or 2D structures simulated by 2D elements with 3DOFs per node [34]. However, more work associated with this part needs to be done in future.

To illustrate the application of this process, a simple structure is analyzed in the following section. This example also serves to point out the errors of the observability technique.

3. Identifying errors in observability techniques

To illustrate the mixed procedure presented above, the simply supported beam presented in Figure 1.A is analyzed. This structure is modeled by a simplified Finite Element Model (FEM) composed of 4 nodes and 3 beam elements. The Young's modulus of all elements is assumed as unknown. Nevertheless, this is not the case of the inertias and the areas, as their values are considered different and unknown for the three different beam elements. To estimate the three unknown flexural stiffnesses of the system (EI_1, EI_2 and EI_3), one rotation (w_1) and two vertical deflections (v_2 and v_3) are measured. In this structure, the application of (4) leads to the following system of equations:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{4\cdot w_1}{L}-\frac{6\cdot v_2}{L^2}\right) & \left(\frac{2}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\left(\frac{1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{12\cdot v_2}{L^3}-\frac{6\cdot w_1}{L^2}\right) & \left(\frac{-6}{L^2}\right) & \left(\frac{12\cdot v_2}{L^3}-\frac{12\cdot v_3}{L^3}\right) & \left(\frac{6}{L^2}\right) & \left(\frac{6}{L^2}\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{2\cdot w_1}{L}-\frac{6\cdot v_2}{L^2}\right) & \left(\frac{4}{L}\right) & \left(\frac{6\cdot v_2}{L^2}-\frac{6\cdot v_3}{L^2}\right) & \left(\frac{4}{L}\right) & \left(\frac{2}{L}\right) & 0 & 0 & 0 & 0 & 0 \\
0 & \left(\frac{-1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{12\cdot v_2}{L^3}-\frac{12\cdot v_3}{L^3}\right) & \left(\frac{-6}{L^2}\right) & \left(\frac{-6}{L^2}\right) & \left(\frac{12\cdot v_2}{L^3}\right) & \left(\frac{6}{L^2}\right) & \left(\frac{6}{L^2}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{6\cdot v_2}{L^2}-\frac{6\cdot v_3}{L^2}\right) & \left(\frac{2}{L}\right) & \left(\frac{4}{L}\right) & \left(\frac{6\cdot v_3}{L^3}\right) & \left(\frac{4}{L}\right) & \left(\frac{2}{L}\right) & 0 & 0 \\
0 & 0 & 0 & \left(\frac{-1}{L}\right) & \left(\frac{1}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{6\cdot v_3}{L^3}\right) & \left(\frac{4}{L}\right) & \left(\frac{2}{L}\right) & 0 & 0 \\
\left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & (-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{6\cdot w_1}{L^2}-\frac{12\cdot v_2}{L^3}\right) & \left(\frac{6}{L^2}\right) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-12\cdot v_3}{L^3}\right) & \left(\frac{-6}{L^2}\right) & \left(\frac{-6}{L^2}\right) & 0 & -1
\end{bmatrix}
\cdot
\begin{Bmatrix}
EA_1u_2 \\ EA_2u_2 \\ EA_2u_3 \\ EA_3u_3 \\ EA_3u_4 \\ H_1 \\ EI_1 \\ EI_1w_2 \\ EI_2 \\ EI_2w_2 \\ EI_2w_3 \\ EI_3 \\ EI_3w_3 \\ EI_3w_4 \\ V_1 \\ V_4
\end{Bmatrix}
=
\begin{Bmatrix}
M_1 \\ H_2 \\ V_2 \\ M_2 \\ H_3 \\ V_3 \\ M_3 \\ H_4 \\ M_4 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\qquad (5)
$$

In this system, the unknown variables {z} include the horizontal reaction, H_1, and vertical reactions, V_1 and V_4, at the boundary, the inertias, EI_1, EI_2 and EI_3, and nonlinear products of coupled areas and inertias, such as EA_1u_2, EA_2u_2, EA_2u_3, EA_3u_3, EA_3u_4, EI_2w_1, EI_2w_3, EI_3w_3 and EI_3w_4. With $\{p_1\}$, being a vector of coefficients, the general solution of system (5) can be expressed in terms of a particular solution $\{z_{p1}\}$ and the null space $[V_1]$ of the matrix of the preceding system as follows:

$$
\{z_1\} = \{z_{p1}\} + [V_1]\cdot\{p_1\} =
\begin{Bmatrix}
EA_1u_2 \\ EA_2u_2 \\ EA_2u_3 \\ EA_3u_3 \\ EA_3u_4 \\ \boldsymbol{H_1} \\ \boldsymbol{EI_1} \\ \boldsymbol{EI_1w_2} \\ EI_2 \\ EI_2w_2 \\ EI_2w_3 \\ EI_3 \\ EI_3w_3 \\ EI_3w_4 \\ \boldsymbol{V_1} \\ \boldsymbol{V_4}
\end{Bmatrix}
= \{z_{p1}\} +
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \left(\frac{-L}{v_2-v_3}\right) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \left(\frac{-L}{v_3}\right) \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\cdot\{p_1\}
\qquad (6)
$$

The analysis of [V1] illustrates the importance of using a symbolic approach. Otherwise, numerical errors with values very close to zero might appear. This might result in reducing the observed parameters. Those variables whose associated rows of [V1] are null indicate that their value has a unique solution (that is to say, that is observable and the particular and general solutions are equal). The variables observed in the first step (H1, EI1, EI1w2, V1 and V4) are highlighted in bold in $\{z_1\}$, (6). Obviously, when the value of EI1 is estimated, w2 can be deduced from EI1w2. The particular solution $\{z_{p1}\}$ of these parameters can be symbolically obtained from system (5) by the left divide, \, in Matlab [35]. Similar functions can be found in other commercial packages, e.g. solve function in both Maple [36] and Mathematica [37]. These functions can be used to provide solutions for symbolic systems of equations.

According to the authors' knowledge, such a type of parametric equations cannot be found in the literature for structural system identification. The obtained parametric equations of the estimates $\widehat{EI}_1$, $\hat{V}_1$ and $\hat{V}_4$ are as follows:

$$\widehat{EI}_1 = \frac{-(L^2 \cdot (8 \cdot M_1 - M_2 - M_3 - M_4 + 2 \cdot L \cdot V_2 + L \cdot V_3)}{\left(18 \cdot (v_2 - L \cdot w_1)\right)} \quad (7)$$

$$\hat{V}_1 = \frac{(M_1 + M_2 + M_3 + M_4 - 2 \cdot L \cdot V_4 - L \cdot V_3)}{3 \cdot L} \quad (8)$$

$$\hat{V}_4 = \frac{(M_1 + M_2 + M_3 + M_4 + L \cdot V_4 + 2 \cdot L \cdot V_3)}{3 \cdot L} \quad (9)$$

in which M_i and V_i are the bending Moments and the Vertical forces (external loads) applied at the *ith* node of the structure during the nondestructive test and L is the length of the beam elements in the model. In these equations, the super index ^ indicates that the value of the estimate is obtained by observability techniques. Obviously, a different equation would be obtained if either the measurement set or the geometry of the structure were changed It should be noted that the parametric equation (7) might lead to unrealistic estimation if the denominator tends to zero or is negative when errors are introduced. This is also discussed in detail in section 4. In order to fill this gap, the researchers are working on an optimization of the measurements which it will be presented in the near future.

The analysis of Equation (5) shows that EI_1 depends on the nodal forces applied at the loading case (M_1 to M_4, V_2 and V_3), the length of the beam elements L, and the measured deflection v_2 and rotation w_1. Both v_2 and w_1 are only found in the denominator of the equation. As the structure is simply supported, V_1 and V_4 can be geometrically determined in terms of the geometry and the forces applied in the loading case. For this reason, these parameters do not depend on the measured deflections.

Once identified the observed parameters, their value can be numerically calculated. To illustrate the results of the method, let's consider a concrete beam of 0.3m height and 0.2m width. The inertia and the Young's modulus are 4.5e-4m^4 and 3.5e7kN/m^2, respectively. The total length (3·L) of the beam is 3m. The loading case is assumed as a concentrated load of -55kN at node 2. This loading case is represented by the following nodal forces: $M_1=M_2=M_3=M_4=V_3=0$ and $V_2=-55$kN. Both the deflections and the rotations obtained throughout the beam for this loading case by FEM program are presented in Figure 1.B and 1.C, respectively for a loading location x=L. In this simulation the shear deformation is neglected.

The numeric values of the estimated $\widehat{EI}_1$, $\widehat{w}_2$, $\hat{V}_1$, $\hat{V}_4$ obtained by parametric equations are summarized in the first recursive step of Table 1. This table also includes the ratio of deviation between estimated and actual values. As showed in this table, the maximum deviation 0.017% in $\widehat{EI}_1$, which is due to the round-off error, is negligible.

After introducing the parameters observed in the first recursive step, the system (5) can be rearranged as presented in system (10). This analysis corresponds with the second recursive step. It is worth noticing that in this system the previously identified parameters (V_1, V_4, EI_1 and w_2) are moved from {z} to [B] and {D}.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \left(\frac{1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{12\cdot v_2}{L^3}-\frac{12\cdot v_3}{L^3}+\frac{6\cdot w_2}{L^2}\right) & \left(\frac{6}{L^2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{6\cdot v_2}{L^2}-\frac{6\cdot v_3}{L^2}+\frac{4\cdot w_2}{L}\right) & \left(\frac{2\cdot E}{L}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{-1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{12\cdot v_3\cdot E}{L^3}-\frac{12\cdot v_2\cdot E}{L^3}-\frac{6\cdot w_2}{L^2}\right) & \left(\frac{-6}{L^2}\right) & \left(\frac{12\cdot v_3}{L^3}\right) & \left(\frac{6}{L^2}\right) & \left(\frac{6}{L^2}\right) \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{6\cdot v_2\cdot E}{L^2}-\frac{6\cdot v_3\cdot E}{L^2}+\frac{2\cdot w_2}{L}\right) & \left(\frac{4}{L}\right) & \left(\frac{6\cdot v_3}{L^2}\right) & \left(\frac{4}{L}\right) & \left(\frac{2}{L}\right) \\ 0 & 0 & 0 & \left(\frac{-1}{L}\right) & \left(\frac{1}{L}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{6\cdot v_3}{L^2}\right) & \left(\frac{2}{L}\right) & \left(\frac{4}{L}\right) \\ \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-12\cdot v_3}{L^3}\right) & \left(\frac{-6}{L^2}\right) & \left(\frac{-6}{L^2}\right) \end{bmatrix} \cdot \begin{Bmatrix} EA_2u_2 \\ EA_2u_3 \\ EA_3u_3 \\ EA_3u_4 \\ EEI_2 \\ EI_2w_2 \\ EI_2w_3 \\ EI_3 \\ EI_3w_3 \\ EI_3w_4 \end{Bmatrix} = \begin{Bmatrix} M_1+\left(\frac{6\cdot EI_1\cdot v_2}{L^2}\right)-\left(\frac{4\cdot EI_1\cdot w_1}{L}\right)-\left(\frac{2\cdot EI_1\cdot w_2}{L}\right) \\ H_2 \\ V_2-\left(\frac{12\cdot EI_1\cdot v_2}{L^3}\right)+\left(\frac{6\cdot EI_1\cdot w_1}{L^2}\right)+\left(\frac{6\cdot EI_1\cdot w_2}{L^2}\right) \\ M_2+\left(\frac{6\cdot EI_1\cdot v_2}{L^2}\right)-\left(\frac{2\cdot EI_1\cdot w_2}{L}\right)-\left(\frac{4\cdot EI_1\cdot w_2}{L}\right) \\ H_3 \\ V_3 \\ M_3 \\ H_4 \\ M_4 \\ H_1 \\ V_1+\left(\frac{12\cdot EI_1\cdot v_2}{L^3}\right)-\left(\frac{6\cdot EI_1\cdot w_1}{L^2}\right)-\left(\frac{6\cdot EI_1\cdot w_2}{L^2}\right) \\ V_4 \end{Bmatrix} \quad (10)$$

With $[V_2]$ being the null space of the matrix [B] in system (10), $\{p_2\}$ being a vector of coefficients, and $\{z_{p2}\}$ being the particular solution of the system, the general solution $\{z_2\}$ of the second recursive step can be expressed as follows:

$$\{z_2\}=\{z_{p2}\}+[V_2]\cdot\{p_2\}=\begin{Bmatrix} EA_1u_2 \\ EA_2u_2 \\ EA_2u_3 \\ EA_3u_3 \\ EA_3u_4 \\ \boldsymbol{EI_2} \\ \boldsymbol{EI_2w_3} \\ EI_3 \\ EI_3w_3 \\ EI_3w_4 \end{Bmatrix}=\{z_{p2}\}+\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{-L}{v_3}\right) \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\cdot\{p_2\} \quad (11)$$

The analysis of [V_2] shows that the only observed parameters are EI_2 and EI_2w_3. From this information the calculation of w_3 is a straightforward task. The observed parameters are highlighted in bold in {z_2}, (11). The parametric equation of EI_2 is presented in Equation (12). This equation shows how EI_2 depends on the values of EI_1 and w_2 estimated in the preceding recursive step. The numerical values of EI_2 and w_3 are summarized in the second recursive step of Table 1. As showed in this table, the deviation between the actual value of EI_2 and the estimated one $\widehat{EI}_2$ (-0.014%) is negligible.

$$\widehat{EI}_2 = \frac{-(\mathrm{L}^2 \cdot \mathrm{M}_3 - 2 \cdot \mathrm{L}^2 \cdot \mathrm{M}_2 - 12 \cdot \mathrm{I}_1 \cdot \mathrm{v}_2 + \mathrm{L}^2 \cdot \mathrm{M}_4 + \mathrm{L}^3 \cdot \mathrm{V}_4 + 4 \cdot \mathrm{I}_1 \cdot \mathrm{L} \cdot \mathrm{w}_1 + 8 \cdot \mathrm{L} \cdot \mathrm{w}_2)}{(6 \cdot (v_2 - v_3 + \mathrm{L} \cdot \mathrm{w}_2))} \qquad (12)$$

Finally, in the third recursive step all the parameters observed by the first two steps (V_1, V_4, EI_1, $w_{2,}$ EI_2 and w_3) are introduced, and the system of equations (10) is updated to:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \left(\frac{1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{-1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{1}{L}\right) & \left(\frac{-1}{L}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{12 \cdot v_3}{L^3} + \frac{6 \cdot w_3}{L^2}\right) & \left(\frac{6}{L^2}\right) \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{6 \cdot v_3}{L^2} + \frac{4 \cdot w_3}{L}\right) & \left(\frac{2}{L}\right) \\ 0 & 0 & 0 & \left(\frac{-1}{L}\right) & \left(\frac{1}{L}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{6 \cdot v_3}{L^2} + \frac{2 \cdot w_3}{L}\right) & \left(\frac{4}{L}\right) \\ \left(\frac{-1}{L}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{-12 \cdot v_3}{L^3} + \frac{-6 \cdot w_3}{L^2}\right) & \left(\frac{-6}{L^2}\right) \end{bmatrix} \cdot \begin{Bmatrix} EA_2u_2 \\ EA_2u_3 \\ EA_3u_3 \\ EA_3u_4 \\ EI_3 \\ EI_3w_3 \end{Bmatrix} = \begin{Bmatrix} M_1 + \left(\frac{6 \cdot EI_1 \cdot v_2}{L^2}\right) - \left(\frac{4 \cdot EI_1 \cdot w_1}{L}\right) - \left(\frac{2 \cdot EI_1 \cdot w_2}{L}\right) \\ H_2 \\ V_2 - \left(\frac{12 \cdot EI_1 \cdot v_2}{L^3}\right) + \left(\frac{6 \cdot EI_1 \cdot w_1}{L^2}\right) + \left(\frac{6 \cdot EI_1 \cdot w_2}{L^2}\right) - \left(\frac{12 \cdot EI_2 \cdot v_2}{L^3}\right) + \left(\frac{12 \cdot EI_2 \cdot v_3}{L^2}\right) - \left(\frac{6 \cdot EI_2 \cdot w_2}{L^2}\right) - \left(\frac{6 \cdot EI_2 \cdot w_3}{L^2}\right) \\ M_2 + \left(\frac{6 \cdot EI_1 \cdot v_2}{L^2}\right) - \left(\frac{2 \cdot EI_1 \cdot w_2}{L}\right) - \left(\frac{4 \cdot EI_1 \cdot w_2 \cdot E}{L}\right) - \left(\frac{6 \cdot EI_1 \cdot v_2}{L^2}\right) - \left(\frac{4 \cdot EI_2 \cdot w_2}{L}\right) - \left(\frac{2 \cdot EI_2 \cdot w_3}{L}\right) \\ H_3 \\ V_3 + \left(\frac{12 \cdot EI_2 \cdot v_2}{L^3}\right) - \left(\frac{12 \cdot EI_2 \cdot v_3}{L^3}\right) + \left(\frac{6 \cdot EI_2 \cdot w_2}{L^2}\right) + \left(\frac{6 \cdot EI_2 \cdot w_3}{L^2}\right) \\ M_3 - \left(\frac{6 \cdot EI_2 \cdot v_2}{L^2}\right) + \left(\frac{6 \cdot EI_2 \cdot v_3}{L^2}\right) - \left(\frac{2 \cdot EI_2 \cdot w_2}{L}\right) - \left(\frac{4 \cdot EI_2 \cdot w_3}{L}\right) \\ H_4 \\ M_4 \\ H_1 \\ V_1 + \left(\frac{12 \cdot EI_1 \cdot v_2}{L^3}\right) - \left(\frac{6 \cdot EI_1 \cdot w_1}{L^2}\right) - \left(\frac{6 \cdot EI_1 \cdot w_2}{L^2}\right) \\ V_4 \end{Bmatrix} \qquad (13)$$

With [V_3] being the null space of the matrix [B] in system (13), {p_3} being a vector of coefficients, and {z_{p3}} being the particular solution of the system, the general solution {z_3} can be expressed as follows:

$$\{z_3\} = \{z_{p3}\} + [V_3] \cdot \{p_3\} = \begin{Bmatrix} EA_1u_2 \\ AE_2u_2 \\ EA_2u_3 \\ EA_3u_3 \\ EA_3u_4 \\ \boldsymbol{EI_3} \\ \boldsymbol{EI_3w_4} \end{Bmatrix} = \{z_{p3}\} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \{p_3\} \qquad (14)$$

The analysis of matrix [V_3] shows that in this step, EI_3 and EI_3w_4 are observed. From this information w_4 can be directly obtained. These parameters are highlighted in bold in Equation (14). The parametric equation of EI_3 obtained from the particular solution of system (13) is presented in Equation (14). As in the case of EI_2, this equation depends on the values of parameters (such as $\widehat{EI}_2$ and $\widehat{w}_3$) estimated in preceding recursive steps and on measured deflections (v_2 and v_3). The numerical values of $\widehat{EI}_3$ and $\widehat{w}_4$ are presented in the third recursive step of Table 1. This table shows that the deviation between the actual EI_3 and the estimated $\widehat{EI}_3$ (0.02%) is negligible.

$$\widehat{EI}_3 = \frac{-(30 \cdot I_2 \cdot v_2 - 30 \cdot I_2 \cdot v_3 - 3 \cdot L^2 \cdot M_3 + L^3 \cdot V_3 + 12 \cdot I_2 \cdot L \cdot w_2 + 18 \cdot I_2 \cdot L \cdot w_3)}{(6 \cdot (v_3 + L \cdot w_3))} \quad (15)$$

Evidently, axial stiffness of the beam cannot be estimated due to the fact that the axial resistant mechanism was not excited by the external load. However, this did not impede the bending stiffness to be observable, and henceforth, to be estimated.

The analysis of the parametric equations of EI_1, EI_2 and EI_3 shows their dependence to the measurements and therefore, to their errors (measurement errors). These equations also show that the nature of the recursive process tends to increase the errors throughout the analysis (error associated to the simulation method). The sensitivity of the observability techniques to these two kinds of errors is analyzed in the following sections.

Also, a flow chart of the mixed algorithm of structural identification by observability method is provided in Figure 2. All the procedures related with the symbolic approach are enclosed by dashed line whereas the procedures related with the numeric approach are enclosed by dotted line. In *step 0*, input the initial data containing the description of the FEM (nodes, element connectivity, external loads and the unknown set of areas and inertias) and the measurement set. In *substep 1* of step i, absorb the measurements in the matrix $[K^*]$ and collect unknowns in the vector $[\delta^*]$ by static condensation. Next, move the unknowns and the observation, respectively, to the left-hand side and the left-hand side of the system in *substep 2*, by which the system $[B] \cdot \{z\} = \{D\}$ is generated. Then, in *substep 3*, the observability of the unknowns are determined by checking the null row of the symbolic null space, [V], of the matrix [B]. The value of the observed parameters will be evaluated by numeric approach in *substep 4*. And, in *substep 5*, it will be examined first whether the number of the observed parameters, N_i, is zero or the same as the number of N_{i-1} from previous step. If so, the identification process is terminated since no more parameter can be observed. Otherwise, the numeric value of the observed parameters from *substep 5* will be used to update the input and regarded as known parameters to initiate the succeeding recursive step.

4. Measurement errors

This section deals with the role of measurement error in structural system identification by observability techniques. With this aim, two sensitivity analyses of the simply supported beam in Figure 1 are presented. The first simulation analyzes the effects of individual errors in each measurement (deflection or rotation). The deviation in the estimation of $\hat{I}_1$ is also analyzed by means of partial derivatives. Finally, the second sensitivity analysis studies the effect of random errors in all measurements.

- Analysis of errors in single measurement

This section analyzes the sensitivity of parametric equations of I_1, I_2 and I_3 obtained from Equations (7), (12) and (15) to errors in one measurement

The measurement set used here is the same as before, one rotation (w_1) and two deflections (v_2 and v_3). The Young's modulus of the three beam elements is assumed as known (2.5e7kN/m^2).

The ratio between each estimated inertia, $\hat{I}_i$, and the actual one, I, with errors from -5% to 5% in v_2, w_1 and v_3 are presented in Figure 3.A. This figure shows how the sensitivity to errors in one of the measurements is increased throughout the recursive process. For example, the deviation between the estimated inertia and the actual one for an error of -5% in v_2 changes from -16.7% in I_1 to 47.1% in I_2, and -45.5% in I_3. In this structure, the deviations in I_1 produced by errors in v_2 correspond with those in I_2 for errors in w_1. Generally, the system is more sensitive to errors in deflections than in rotations. This figure also illustrates the importance of the error sign. In fact, the estimations based on the measured deflections are asymmetric. This asymmetry increases throughout the recursive process and is especially significant in the deflections, v.

In Figure 3.B, the effect of the errors in measurements on the deviation in the estimation throughout the recursive process is presented. This figure also shows that I_1 is not affected by errors in v_3. This is because the parametric equations of these inertias do not depend on the deflection v_3.

The effects of the location of two concentrated loads are analyzed in Figure 4.A and 4.B, respectively, to clarify the influence of the load case on the parametric equations of inertias. Load case one corresponds with a concentrated vertical load V=-55kN, located at an intermediate node (x=L or x=2L). The second load case corresponds with a concentrated bending moment, M=100kN·m, located at the beam edge (x=0 or x=3L). These figures present the deviation between the actual inertia, I, and the $\hat{I}_1$ calculated by parametric equation (7) for different errors in v_2 or w_1 and load locations. It should be highlighted that deviations beyond the range of [0,2] do not have physical meaning and thus they are rejected.

Figure 4 shows that the load case is influential in the accuracy of estimated parameters. In Figure 4.A, the closer the load to the measurements, the smaller the effect of errors. For example, for an error of -5% in v_2, the deviation of $\hat{I}_1$ increases from -16.6% to -25.9% when V is moved from x=L to x=2L. For the same error level in w_1, moving V from x=L to x=2L increases the deviation of $\hat{I}_1$ from 33.3% to 66.7%. Similar conclusion can be drawn when the effect of bending moment M is analyzed. In this case, for an error of -5% in v_2, the deviation in $\hat{I}_1$ increases from -5.8% to -28.6% when M is moved from x=0 to x=3L. For the same error in w_1, the increment is from 12.7% to 81.2%.

In addition, the parametric equations are affected by the location of the measurements. In the observability method, the accuracy of the estimations is highly related to the curvature of the elements where the measurements are performed. Estimates obtained from deflections measured at the low curvatures zone might be more sensitive to errors. For example, in a simply supported beam, the null curvature zones are those adjacent to the support. The influence of the curvature will be discussed in a more extensive way in the simulation error part.

To clarify the effects of curvatures in the accuracy of the estimates, six FEMs, FEM_2, FEM_3, FEM4, FEM_6, FEM_8 and FEM_{12}, with the same length, 3L, but different element numbers were analyzed. The number of elements in these FEMs is indicated by their subscript. In all these models, only the flexural stiffness of the first element, EI_1, is estimated.

In these models, two measurements are considered, the rotation w_1 at the left support and the deflection v_2 of node 2. Note that the location of the measurement v_2 is $\{x = \frac{3L}{2}, L, \frac{3L}{4}, \frac{3L}{8}$ and $\frac{L}{4}\}$ for FEM_2 FEM_3, FEM_4, FEM_6, FEM_8 and FEM_{12}. That is, the measurement v_2 will be located nearer to the null curvature zone in models of more elements.

To analyze the effect of the location of the measurements, the parametric equation of $\widehat{EI}_1$, (7), for FEM_3 is analyzed. Similar equations can be obtained for different FEMs by substituting the length of the different elements in each model. The effect of the errors ranging from -15% to 15% in w_1 and v_2 is obtained by these equations for each FEM is presented in Figure 5. It should be clarified that all these equations are presented as a fraction, in which the numerator indicates information of the load case while the Denominator, D, indicates information of the measurements.

As expected, Figure 5 shows that the denominator of the parametric equation of EI_1, D, depends linearly of the error in measurements w_1 and v_2. In the graph, the closer to the null curvature zone the measurement v_2, the higher the inclination of the denominator line. High inclinations of the lines might lead to estimations with no physical meaning as the errors in measurements lead to denominators close to zero or even negative. It is straightforward that the inertia obtained by this value of the denominator would tend to be infinite or negative. In FEM_2, the threshold error level for w_1 and v_2 to render the denominator null is quite high. Nevertheless, the threshold becomes lower with the decrease of the distance between the support and node 2. Considering the error of w_1, a null denominator is obtained at the following error level: -16.1% (FEM_3), -8.7% (FEM_4), -5.2% (FEM_6), -2.3% (FEM_8) and -1.7% (FEM_{12}). It is suggested to take measurements in the non-null curvature zones to avoid the detrimental effect of the measurement errors on the accuracy of estimations.

- <u>Error by partial derivatives</u>

In previous discussion, estimation of $\widehat{EI}_1$ in FEM_3 depends on errors in v_2 and w_1. With ε being the percentage error in the measurements, the error in $\widehat{EI}_1$, e_1, due to these two parameters can be calculated by the following partial derivatives:

$$e_1 = \sqrt{\left(\frac{\partial \hat{I}_1}{\partial v_2} \cdot \varepsilon v_2\right)^2 + \left(\frac{\partial \hat{I}_1}{\partial w_1} \varepsilon w_1\right)^2} \quad (16)$$

, which can be used to get the deviation in $\widehat{EI}_1$. Using equation (16), the deviation in $\widehat{EI}_1$ against error from -5% to 5% is summarized in Figure 6. It can be seen the estimation of $\widehat{EI}_1$ is quite sensitive to errors in v_2 and w_1. Deviation will be magnified if the signs of the error in v2 and w1 are opposite. And the maximum deviation, 54.3%, is obtained for an error of +5% in v_2 and -5% in w_1.

- <u>Analysis of random errors in all measurements</u>

In practice, measurement errors are inevitable. Furthermore, the actual magnitude of each error is unknown since it depends on a number of parameters including the accuracy of the measurement device. The errors of each measurement are usually assumed to follow a normal distribution. To illustrate the effects of the actual errors, an additional analysis is performed on FEM_3 in Figure 1, in which the inertias of the three elements are assumed as different and unknown. Three different measurement sets were analyzed here. The first of these sets (Set 1) is exclusively composed of nodal rotations, w_1, w_2 and w_3. The second set (Set 2) corresponds with that used in preceding sections, one rotation (w_1) and two deflections (v_2 and v_3). Finally, the third set (Set 3) only includes three deflections v_2, v_3 and v_5. As illustrated in Figure 7, the measurement of (v_5) corresponds with the vertical deflection at one intermediate node located at the first beam element.

Each measurement sets includes three error levels, e={5%,10% and 20%}, which represent a percentage maximum deviation of the actual value of the measured variable. Equation (17) was used to introduce the errors in deflections. The noisy deflection at the i^{th} node, ve_i, is calculated from the error-free deflections, v_i, and the percentage error, e_0, which is the product of the assumed maximum magnitude of the error, e, and a random number, r. The random number r varies between -1.0 and 1.0 according to a truncated normal distribution of null mean and 0.5 standard deviation. A similar equation is used to introduce the errors into the measured rotations we_i.

$$ve_i= v_i+v_i\cdot e_0=v_i+v_i\cdot r\cdot e \qquad (17)$$

Random errors in measurements might lead to estimations with no physical meaning since these noisy measurements should satisfy some geometrical constraints. In FEM_3 from Figure 1, random errors in measurements might result in deformed shapes where the deflection of the node where the load is applied is not the maximum. In each of these analyses, the physical meaning of the deformed shape is analyzed by checking some geometrical restrictions. For this structure, the restrictions assumed are $ve_2>ve_3$ and $we_1<we_2<we_3<we_4$. The vertical deflection and rotation at the intermediate node ve_5 and we_5 are limited by those of the adjacent nodes. If any of these restrictions is not

satisfied a new set of random measurements is obtained until the 200 admissible deformed shapes are obtained.

The ratios between the estimated inertia, $\hat{I}_i$, of the *ith* beam and the actual one, *I*, for different random errors in measurements are presented in Figure 7. As presented in the preceding section, the errors in measurements might lead to estimations with no physical meaning. This lack of meaning comes from those cases where the denominator of the parametric equation is close to zero. This problem can be avoided by adding some physical restrictions to the solutions of the system of equations. For example, in a damaged structure, the estimated inertias cannot be significantly higher than those of the undamaged elements (that is, estimated inertia cannot be twice as big as the original one). In addition, no negative inertias should be considered. In order to fulfill these restrictions, the results in Figure 7 include the average of those analyses where the estimations were bounded by: the 0 and 2 times the original inertia, 0.25 and 1.75 times the original inertia, 0.5 and 1.5 times the original inertia and 0.75 and 1.25 times the original inertia. In this figure, the results are named by the ranges as follows: 0.0-2.0, 0.25-1.75, 0.5-1.5 and 0.75-1.75, respectively. The percentages of analyzed structures satisfying these restrictions are presented in Figure 7.A (Set 1), 6.B (Set 2) and 6.C (Set 3).

From Figure 7, it is deduced that: 1) As expected, the higher the error in measurements, the higher the deviations in estimated inertias. In Set 2, the maximum errors for an error of 5% and a physical restriction of 0.0-2.0 are increased from 4.1% to 26.1% when the maximum random error in measurements is increased to 20%. 2) It is plausible that the smaller the range of allowable estimated inertias, the more accurate the estimations are. For example, in Set 2 with a random error of 20%, changing the allowed range of estimations from 0.0-2.0 to 0.75-1.25 reduces the deviations from 26.1% to 2.2%. 3) The structure is less sensitive to errors in rotations than in deflections. This is appreciable when the results of the different measurement sets are compared. For example, considering a maximum random error of 5% and the physical restriction 0.0-2.0, the maximum errors when only rotations are considered (Set 1 with a deviation of -0.2% in I_3) is significantly lower than the one when w_2 and w_3 are substituted by v_2 and v_3 (Set 2 with a deviation of 3.1% in I_3). These deviations are increased more when only deflections are considered (Set 3 with a deviation of 16.5% in I_2). 4) Deviations in estimations are not increased throughout the recursive process as they fluctuate with the observability flow. In all analyzed sets described in Figure 7, the recursive process is initiated at the first beam element, $\hat{I}_1$. This value is used to estimate $\hat{I}_2$ and then, this new inertia is used to estimate $\hat{I}_3$. As illustrated in the Set 3 for an error of 5%, when $\hat{I}_1$ is underestimated, $\hat{I}_2$ is overestimated to compensate the effect of $\widehat{I_1}$into the system of equations. Conversely, the value of $\hat{I}_3$ is slightly underestimated. This fluctuation in the estimation of inertias is a peculiarity of the observability technique which will be analyzed in detail in the following section.

5. Errors in Parameter Estimation

To clarify the effects of different simulation errors, two examples of increasing complexity are analyzed in this section. On the one hand, the first example corresponds with a cantilever beam. In this example, the errors produced throughout the recursive process are analyzed. To avoid the effect of the curvature, a load case with a uniform curvature distribution is proposed. In addition, to show the effect of the measurement errors, two different measurement precisions are adopted. On the other hand, the second example corresponds with a statically redundant beam. In this structure, the errors produced by the recursive process for a load case that produces a uniform distribution of curvatures are studied first. Finally, to illustrate the effect of the curvature, an additional load case with a non-uniform curvature distribution is simulated.

- <u>Analysis of the recursive process</u>

Assume a cantilever beam with a concentrated bending moment, M=100kN·m at the free end. This load case induces uniform bending moments and curvatures as depicted in Figure 8.A. This curvature enables to focus the analysis on the errors produced by the recursive process. For this load case the maximum deflections (5.11mm) occurs at the beam edge.

The mechanical properties of the structure correspond with those of the structure presented in [33]. The analyzed beam has a length of 30 m. The area and the inertia of the girder are 0.07 m^2 and 0.04 m^4, respectively and Young's of modulus is E = 210 GN/m^2. The simplified FEM of this beam is composed of 31 nodes as presented in Figure 8.A. This assumption leads to a number of 30 elements 1m long. As mentioned in section 2, the flexural stiffnesses can be absorbed in inertias by assuming the Young's modulus as known. Here, these inertias are assumed both different and unknown. As the beam is horizontal, the axial and the flexural mechanisms are uncoupled and can be studied separately. However, only the analysis of the flexural behavior is presented here.

The values of the unknown inertias are estimated by the observability method from two alternative measurement sets derived by the observability trees [32]. The first set is composed of 30 deflections, from v_2 to v_{31}, while the second one includes 29 deflections, from v_3 to v_{31}, and one rotation w_{31}. Each of these measurement sets solves the equations of the stiffness matrix system in a different sequence (or in other words, by a different observability flow). In the first set the solution of the system of equations starts at the clamped node and flows towards the beam edge in 30 steps. The opposite observability flow is obtained by the second measurement set. The observability flows are illustrated in Figures 8.B and 8.C by continuous and dotted arrows, respectively.

Figures 8.B and 8.C, respectively, include the percentage differences between the estimated inertia, $\hat{I}_i$, and the actual one, I, based on different error levels in measurement. Figure 8.B presents the results for error free measurements (with a precision of 1e-9m in v and 1e-9rad), while Figure 8.C presents the results with the measurement errors found in (precision of 1e-5m in v [38] and 1e-5rad in w [39]).

To solve the system of equations, the recursive process uses information from preceding steps. In this way, the value estimated of a certain rotation or inertia is used in the subsequent steps. It must be emphasized that it is intuitive to think, in the recursive process, that errors will accumulate and propagate, and thus the parameters identified in the final steps will contains significant error. Conversely, this is not the case in the observability techniques. As depicted in Figure 8.B, it is shown that for the first set (continuous blue line) the initial error of -0.01% is increased to 0.04% at the end of the beam. A similar phenomenon can be observed for the second set (dotted red line), where the initial deviation of -0.01% is increased to -0.02% at the proximities of the clamped node. In fact, when an estimated inertia is slightly higher than the actual one (i.e. overestimation), the next estimated inertia tends to be slightly underestimated in order to compensate the overestimation in preceding step. This effect leads to the fluctuation of error. However, this fluctuation might produce even higher errors in some middle steps of the recursive process than the one obtained at the final step. For example, in the first flow, the maximum deviation (0.09% in element 26) is 2.14 times higher than the error obtained at the end of the recursive process. The same effect appears in Figure 8.C. Nevertheless, in this case, because of the error in measurements, higher fluctuations are obtained. For error free measurements, the maximum deviations are observed at I_7 for the first set (v_3 to v_{31} and w_{31}). The obtained estimation at this point represents the 0.55% of I. This value is 46.1% higher than the value obtained at the end of the recursive process (0.38%).

- Analysis of the effects of the curvature

The second structure corresponds with the two-span continuous beam presented in Figure 9.A. This beam has a 60 m length and is evenly divided into 60 elements. The material and mechanical properties are the same as those used in the preceding section. Again, the Young's modulus and the areas are assumed as known whereas the inertias are assumed as different and unknown for each element. This structural system identification problem was presented in [33]. Later, Nogal et al. [2] used this example to illustrate the different simulation errors that might appear in observability techniques. The aim of this example is to extend that study, and to provide a better understanding of the nature and magnitude of the different simulation errors when observability techniques are applied.

To estimate the 60 unknown inertias, two different load cases are studied. The first case includes two concentrated bending moments, M=1000kN·m, at the beam edges and a settlement of 5.4mm at the inner support. This load case induces, as presented in Figure 9.B, a constant bending moments in the structure. The second load case corresponds with a concentrated vertical load V=-100kN applied at node 16 as presented in Figure 9.C, which produces a linear diagram of bending moments with a maximum (500kN·m) at node 16 and a minimum (-250kN·m) at node 31 and null values at the vicinity of node 23.

The measurement set in both load cases is identical and includes 58 deflections (v_1 to v_{30} and v_{32} to v_{60}) and 2 rotations (w_{29} and w_{30}). This measurement set initiates an observability flow at the left hand side of the inner support that is propagated towards both beam edges. The direction of this flow is indicated by the arrows in Figures 8.B and 8.C, respectively. In the first recursive step, three inertias ($\hat{I}_{28}$, $\hat{I}_{29}$ and $\hat{I}_{30}$) are observed. The rest of the inertias are successively estimated after 30 steps. The parameters estimated in the first recursive steps are highlighted in these figures by a circle.

The deviations between the actual inertia, I, and the estimated one, $\hat{I}_i$, in each beam element *i* are summarized in Figures 8.B and 8.C. In these figures, the results obtained by the error free measurements (precision 1e-9m in v and 1-9rad in w) and the state of the art errors (1e-5m in v and 1e-5rad in w) are presented in different colors.

Figure 9.B shows that when a uniform curvature is applied, the errors of the estimations are not increased monotonically throughout the recursive steps. In effect, the deviations from the actual stiffnesses present similar fluctuations to those observed in the cantilever beam. For the error free measurements, the maximum deviation error in the first recursive step (-0.01% in I_{28}) is increased to 0.1% in I_{37} throughout the analysis. In the structures with measurement errors, the fluctuations are slightly more significant since the initial errors (-0.13% in I_{30}) are increased to 1.1% in I_{40}.

Figure 9.C illustrates the importance of the curvature in the identification by the observability. In fact, the maximum errors are obtained in those areas with null curvatures (concretely at x=0, x=27 and x=60m). This effect can be explained by the fact that the bending stiffness is calculated based on the curvature of the beam elements imposed by the load case. As a result, higher errors appear at those locations with low curvatures. As expected, the maximum deviation (1.52%) is found at x=27, which is adjacent to the inflection point of the moment diagram. In this structure, the effects of the magnitude of the curvature are slightly higher than those of the recursive process. To avoid the detrimental effects of the low curvature, adequate load cases are advised for structural system identification by observability techniques.

6. Conclusions

This paper analyzes the effects of two unavoidable sources of errors upon the structural system identification by observability techniques. The first of these sources refers to the measurement errors. To simulate this error, the parametric equations of the estimated inertias were analyzed in detail in a simply supported beam. The analysis of this structure shows that: (1) Estimations in subsequent recursive steps depend on the values estimated in preceding steps. As an academic example it is showed that considering an error in single measurement increases the errors in the estimations throughout the recursive process. This effect is significantly mitigated when errors in all measurements are considered. (2) Parametric equations of the estimated parameters can be obtained. These equations are very useful to study the sensitivity of the estimated parameter. In order to make the estimations less sensitive to the errors, it is recommended to use measurements closer to the load location. The numeric analysis shows that the rotations are less sensitive to errors than the vertical deflections. This parametric approach enables the use of partial derivatives in the error analysis. (3) The loading case is of primary importance. Usually the closer the load location of the concentrated load to the inertia to be estimated the lower the sensitivity of the estimation to measurement errors. This also corresponds to the fact that, for the same loading case, the closer the location of the measurement to the boundary condition, the lower the curvature. (4) The denominator of the parametric equations of the estimated inertia depends, to a large extent, on the measurement errors. Denominators with a value close to zero lead to solutions with no physical meaning. (5) Those estimations based on the measured deflections are asymmetric. Furthermore, the asymmetry in estimates is increased throughout the recursive process. On the other hand, the second analyzed source of error refers to those simulation errors inherent in the observability analysis. To illustrate these effects two structures of growing complexity were analyzed. The simulation of these structures shows that: (1) Fluctuations in the inertias estimated are obtained because of the recursive process. This can be explained by the fact that every time that a certain inertia is underestimated, the next inertia that uses this information will tend to be overestimated to compensate the effect of the preceding one in the system. (2) The curvature of the beam plays an important role in the accuracy of the estimations. In fact, wrong estimations are obtained near points with null curvatures. The effect of the curvature requires an adequate selection of the loading cases for structural system identification by observability techniques.

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Table 1: Numerical estimations of the parameters during the recursive steps and the deviations with the actual values obtained from the parametric equations.

Step 1			**Step 2**			**Step 3**		
Parameter	Estimation	Deviation	Parameter	Estimation	Deviation	Parameter	Estimation	Deviation
$\widehat{EI}_1$ (kN/m^2)	15753.5	0.017%	$\widehat{EI}_2$ (kN/m^2)	15750.0	-0.014%	$\widehat{EI}_3$ (kN/m^2)	15753.6	0.020%
$\widehat{w}_2$ (rad)	-1.1e-3	-0.002%	$\widehat{w}_3$ (rad)	1.3 (rad)	0.002%	$\widehat{w}_4$ (rad)	2.1 (rad)	-0.001%
$\widehat{V}_1$ (kN)	36.7	0.000%						
$\widehat{V}_4$ (kN)	18.3	0.000%						