On the Spectra of Markov Matrices for Weighted Sierpiński Graphs

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Extended Abstract

Relevant information from networked systems can be obtained by analyzing the spectra of matrices associated to their graph representations. In particular, the eigenvalues and eigenvectors of the Markov matrix and related Laplacian and normalized Laplacian matrices allow the study of structural and dynamical aspects of a network, like its synchronizability and random walks properties.

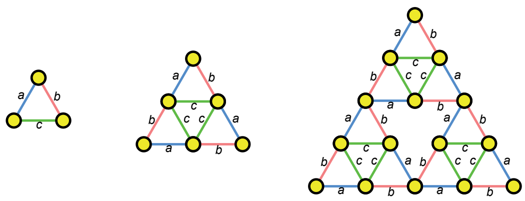
In this study we obtain, in a recursive way, the spectra of Markov matrices of a family of rotationally invariant weighted Sierpiński graphs. From them we find analytic expressions for the weighted count of spanning trees and the random target access time for random walks on this family of weighted graphs.

Construction of W_t. The rotationally invariant weighted Sierpiński graph W_t, $t \geq 0$, is constructed as follows [1]:

For $t = 0$, W_0 is K_3 (a 3-cycle) and its three edges have weights a, b, c.

For $t \geq 1$, W_t is obtained by recurrence by joining three copies of W_{t-1} and identifying two vertices of each copy with one of the vertices of each of the other two copies.

The construction process, as well as the distribution of edge weights (a, b, c), for the rotationally invariant case is shown in the next figure.



The order of W_t is $N_t = (3^{t+1} + 3)/2$ and the total number of edges is $L_t = 3^{t+1}$. At each iteration 3^t new vertices are added to the graph.

Spectrum of the probability transition matrix for random walks on W_t. A_t denotes the adjacency matrix of the weighted graph W_t and has elements $A_t(i,j) = w(i,j)$, where $w(i,j)$ is the weight of edge (i,j). The degree matrix of W_t, denoted by D_t, is a diagonal matrix such that $D_t(i,i) = \sum_j w(i,j)$. The probability transition matrix for random walks on W_t, or Markov matrix, is defined as $M_t = D_t^{-1} A_t$.

When $a = b = c$, W_t degenerates into the unweighted Sierpiński graph S_t. The spectrum of the transition matrix of S_t, denoted $\overline{M}_t$, has been determined elsewhere [2], and the resulting recursive equation for the eigenvalues is $\overline{\lambda}^{(t)} = \overline{\lambda}^{(t+1)} \left(4\overline{\lambda}^{(t+1)} - 3\right)$ where $\overline{\lambda}^{(t+1)} \neq -\frac{1}{4}, \pm\frac{1}{2}$. This equation gives a relationship between the spectra of the transition matrices at steps t and $t+1$, i.e., each eigenvalue of $\overline{M}_{t+1}$, except for the exceptional eigenvalues $\{-\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}\}$, corresponds to an eigenvalue of $\overline{M}_t$. The multiplicities of the exceptional eigenvalues are: $m_{\overline{M}_t}\left(-\frac{1}{2}\right) = \frac{3+3^t}{2}$, $m_{\overline{M}_t}\left(-\frac{1}{4}\right) = \frac{3^{t-1}-1}{2}$), $m_{\overline{M}_t}\left(\frac{1}{2}\right) = 0$, $t > 0$. Thus, in [2] the spectrum of $\overline{M}_t$ is given as $\sigma(\overline{M}_t) = \{1, -\frac{1}{2}\} \bigcup \left(\bigcup_{i=0}^{t-1} Q_{-i}\{\frac{1}{4}\}\right) \bigcup \left(\bigcup_{i=0}^{t-2} Q_{-i}\{-\frac{1}{4}\}\right)$ where $Q_{-i}A$ denotes the preimage of a set A under the i-th composition power of the function $Q(x) = x(4x-3)$.

With a similar technique, we partition the matrix M_t into blocks corresponding to the transition probabilities among old and new vertices, with respect to the last iteration, and study the Schur complement of M_t. In the next theorem, we relate the eigenvalues of $\overline{M}_t$ with those of M_t to find the complete spectrum of M_t. We have also obtained the multiplicities for all the eigenvalues of M_t.

Theorem 1 *Any eigenvalue* $\overline{\lambda}^{(t-1)}$ *of* $\overline{M}_{t-1}$*, is related to several eigenvalues of* M_t*, denoted* $\{\lambda_i^{(t)}\}$*, and they are the preimage of* $\overline{\lambda}^{(t-1)}$ *under the function* R *given by*

$$R(z) = \frac{(a+b)z(sz-2c)(sz+c)-(a^2+b^2)sz+c(a-b)^2}{2(a^2+b^2)c+2absz}$$

where $s = a+b+2c$ *and* $z \notin \{-\frac{c}{s}, \frac{2c}{s}, -\frac{(a^2+b^2)c^2}{abs}\}$*, the exceptional eigenvalues of* M_t*.*

Weighted count of the spanning trees of W_t*.* A spanning tree of W_t is a subgraph that includes all the vertices of W_t and is a tree. Let $\Upsilon(W_t)$ denote the set of spanning trees of W_t. For $\mathcal{T} \in \Upsilon(W_t)$, we define its weight $w(\mathcal{T})$ as $\prod_{e\in\mathcal{T}} w_e$, where w_e denotes the weight of edge e. Let $\tau(W_t) = \sum_{\mathcal{T}\in\Upsilon(W_t)} w(\mathcal{T})$ denote the weighted count of the spanning trees of W_t. From [3], $\tau(W_t) = (\prod_{k=1}^{N_t-1} \gamma_k^{(t)} \prod_{i=1}^{N_t} d_i^{(t)})/\sum_{i=1}^{N_t} d_i^{(t)}$, where $0 = \gamma_0^{(t)} < \gamma_1^{(t)} \leq \cdots \leq \gamma_{N_t-1}^{(t)}$ are the eigenvalues of $\mathcal{L}_t$, the normalized Laplacian matrix of W_t, and $d_i^{(t)} = D_t(i,i)$. Obviously, the eigenvalue spectrum of the matrix $D_t^{\frac{1}{2}} M_t D_t^{-\frac{1}{2}} = (I - \mathcal{L}_t)$ is the same as the spectrum of M_t. Thus, for each k, $(1-\gamma_k^{(t)})$ is an eigenvalue of M_t. We find the following result:

Theorem 2 *The number of spanning trees of* W_t*,* $t > 0$*, is*

$$2^{\frac{3^{t-1}-1}{2}} 3^{\frac{3^t+2t-1}{4}} 5^{\frac{3^{t-1}-2t+1}{4}} (a+b)^{3^{t-1}} (ab+ac+bc)^{\frac{3^t+1}{2}} (a+b+3c)^{\frac{3^{t-1}-1}{2}}.$$

This result coincides with the the values obtained from generating functions by D'Angeli and Donno [1], and verifies the correctness of our computation of the spectrum of M_t.

Random target access time for random walks on W_t*.* Let $\pi = (\pi_1, \pi_2, \cdots, \pi_{N_t})$ denote the stationary distribution for random walks on W_t, which is an eigenvector of M_t associated to the eigenvalue 1. Let $H_{ij}(t)$ represent the mean first passage time from vertex i to vertex j. The random target access time for random walks on W_t, denoted by H_t, is defined as the expected time for a walker starting from vertex i to reach for the first time a target vertex j, selected stochastically according to the stationary distribution. Thus, $H_t = \sum_{j=1}^{N_t} \pi_j H_{ij}(t)$. The random target access time, which reflects the structure of the the entire graph, is independent of the choice of the starting vertex. It has been proved [5] that H_t can be expressed in terms of the nonzero eigenvalues of $\mathcal{L}_t$, given as $H_t = \sum_{i=1}^{N_t-1} \frac{1}{\gamma_i^{(t)}}$. We find the following result:

Theorem 3 *The random target access time* H_t *for random walks on* W_t *is*

$$\frac{s^2-c^2}{ab+ac+bc}\left(\frac{14\cdot 5^{t-2}}{3} - \frac{3^{t-1}}{5} + \frac{3}{5}\right) + \frac{s(3^{t-1}-1)}{2(s+c)} - \frac{s(3+3^{t-1})}{2(a+b)} + \frac{abs(3^t+1)}{2(ab+ac+bc)(a+b)} + \frac{2}{3},$$

where $s = a+b+2c$ *and* $t > 1$*.*

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