POINCARE'S PHILOSOPHY OF MATHEMATICS

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POINCARÉ'S PHILOSOPHY OF MATHEMATICS

by

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DECLARATIONS

I, Janet Folina, hereby certify that this thesis which is approximately 65,000 words in length has been written by me, that it is a record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

date 20 May 1986 signed [redacted]

I was admitted as a research student under Ordinance No.12 on 1 October 1983 and as a candidate for the degree of Doctor of Philosophy on 18 January 1984; the higher study for which this is a record was carried out in the University of St. Andrews between October 1983 and May 1986.

date 20 May 1986 signed [redacted]

I hereby certify that the candidate has fulfilled the conditions of the Resolutions and Regulations appropriate to the degree of Doctor of Philosophy of the University of St. Andrews and that she is qualified to submit this thesis in application for that degree.

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Supervisor

ABSTRACT

The primary concern of this thesis is to investigate the explicit philosophy of mathematics in the work of Henri Poincaré. In particular, I argue that there is a well-founded doctrine which grounds both Poincaré's negative thesis, which is based on constructivist sentiments, and his positive thesis, via which he retains a classical conception of the mathematical continuum. The doctrine which does so is one which is founded on the Kantian theory of synthetic apriori intuition. I begin, therefore, by outlining Kant's theory of the synthetic apriori, especially as it applies to mathematics. Then, in the main body of the thesis, I explain how the various central aspects of Poincaré's philosophy of mathematics - e.g., his theory of induction; his theory of the continuum; his views on impredicativity; his theory of meaning - must, in general, be seen as an adaptation of Kant's position. My conclusion is that not only is there a well-founded philosophical core to Poincaré's philosophy, but also that such a core provides a viable alternative in contemporary debates in the philosophy of mathematics. That is, Poincaré's theory, which is secured by his doctrine of apriori intuitions, and which describes a position in between the two extremes of an "anti-realist" strict constructivism and a "realist" axiomatic set theory, may indeed be true.

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For my mother

"... this disinterested pursuit of truth for its own beauty is also wholesome, and can make men better."

(Poincaré, (1908))

"Now the majority of men do not like thinking, and this is perhaps a good thing, since instinct guides them, and very often better than reason would guide a pure intelligence, at least whenever they are pursuing an end that is immediate and always the same. But instinct is routine, and if it were not fertilized by thought, it would advance no further with man than with the bee or the ant. It is necessary, therefore, to think for those who do not like thinking, and as they are many, each one of our thoughts must be as useful in as many circumstances as possible."

(Poincaré, (1908))

"But all this is ancient history. Mr. Russell has realized the danger and is going to reconsider the matter. He is going to change everything, and we must understand clearly that he is preparing not only to introduce new principles which permit of operations formerly prohibited, but also to prohibit operations which he formerly considered legitimate. He is not content with adoring what he once burnt, but he is going to burn what he once adored, which is more serious. He is not adding a new wing to the building, but sapping its foundations."

(Poincaré, (1906))

INTRODUCTION

Jules Henri Poincaré (1854-1912), the Gauss of modern mathematics, was a "universal" mathematician whose contributions were seminal in the development of contemporary pure mathematics, in mathematical physics, and in the philosophical foundations of mathematics. The first two claims, concerning his influence in the technical areas, are uncontestable. Poincaré was the greatest practitioner of mathematics of his time, and he is rightly credited for this. In contrast, his contribution to the philosophy of mathematics is, in general, profoundly underestimated. His thoughts are regarded as idiosyncratic and based upon a misunderstanding of the logicist tradition which he criticised. This interpretation is not, however, entirely unfounded, for at first glance, his writings seem glib, not very deep, and at times based on an indeed polemical reaction to the work of Russell, Zermelo, Peano and Couturat. His papers are often based on speeches he was requested to give to the scientific community, and sometimes, even, to a general audience. This results in writings which are often conversational in tone, the intent being to <u>amuse</u> as well as to <u>inform</u>. Reading them now, therefore, they may on occasion appear glib and even sarcastic. However, it is important not to allow the manner in which he expresses himself to obscure the depth and philosophical import which we may gain from his ideas. Since his philosophical work was always quite clearly secondary to his work <u>in</u> mathematics itself, he never attempted to expound his ideas in a structured, systematic presentation. However, this does not mean that there is no general philosophical foundation which properly characterises the insights he had into the foundations of his work. It just means we must try a little harder to interpret his comments in terms of the whole of his philosophy, and thus to be cautious in what views we attribute to him.

Unfortunately, sufficient care is <u>not</u> always taken when presenting his views; and interpretations which are in my view not entirely fair, are sometimes implied. For instance, in the context of a brief survey of the emergence of the concept of impredicativity, Kneale and Kneale (1962) comment on Poincaré's view that there is a relation between the set-theoretic paradoxes and the attempt to treat infinities as completed wholes. Poincaré <u>did</u> hold such a view; but this was not the full extent of his view. Kneale and Kneale go on to cite a short passage by Poincaré to support their claim.[1] However, this passage, on its own, is misleading; and Poincaré follows a similar passage on the previous page with an explanation of his view: "I must explain myself ..."[2]. Later, they again comment that

> Poincaré suggested that the paradoxes of the theory of sets were due to the fundamental mistake of assuming actually

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1 See Kneale and Kneale, (1962), pp.655-656.

2 Poincaré, (1906b), p.194.

> infinite aggregates. He did not explain in detail why ... 3

In Chapter 4, Section 5, and in Chapter 6, I discuss Poincaré's theory of impredicativity.

Chihara, in his book Ontology and the Vicious-Circle Principle, is fairer to Poincaré, and devotes a chapter to discussing his views in their philosophical context. However, he also seems to have a rather superficial grasp of Poincaré's theory of the relation between the belief in actual infinity and the contradictions.[4] In addition, he concludes that Poincaré is a "nominalist" with regard to mathematics, citing Poincaré's remarks that the continuum is a mere "system of symbols", and that mathematics can "give to the physicist only a convenient language".[5] This, in my view, is a very rash pronouncement. In view of Poincaré's theory of the apriori geometric foundations of the mathematical theory of the continuum (which is the subject of Chapter 5), it seems clear that his remarks on the continuum being only "a system of symbols" is not entirely meant in a straightforward sense. And the "language" the

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3 Kneale and Kneale, (1962), pp.672-673.

4 See p.140. In contrast we may cite Heinzmann, (1985), "Entre Intuition et Analyse" for a very detailed and scholarly account (though not always philosophically deep in its explanation) of the development of the concept of predicativity in Poincaré.

5 Chihara, p.154-155.

mathematician provides for the scientist has - contra-Chihara - not to do with the fact that physical laws are expressed in terms of mathematical symbols and notation. Rather, it is a consequence of the strong Kantian nature of Poincaré's philosophy: mathematics expresses what is necessarily common to all thinking beings; and the best science can do, insofar as discovering true relations, is to discover mathematical relations which survive the inevitable changes in background theory and conventions.[6] There is no "only" in Poincaré's view of the language which mathematics provides for science; the use of the term ("only") not being "misleading" (as Chihara claims), but surely intended as an irony.

Perhaps most surprising of all the (mis)interpretations of Poincaré's philosophy, is a claim made by Parsons[7] that Poincaré is an intuitionist, but not a Kantian, because he seems "quite uninfluenced" by Kant's notion of pure intuition. This is particularly astonishing in view of the calibre, in general, of Parsons' scholarship. I devote Chapter 4 to an argument that Poincaré, indeed, had a very strong theory of "pure intuition", and it is in terms of this theory that we must make sense of his very general claims against logicism and set theory.

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6 See Giedymin, (1982), for a very good account of the Kantian element of Poincaré's "conventionalism".

7 In his, (1964), p.108, note 6.

There is no doubt that it is not easy to make sense of Poincaré's remarks[8], which are often, at first blush, paradoxical and even trite. For example, he appears both to condone and oppose the formalisation of mathematics. He devotes pages to extolling the virtues of the "new" precise methods, e.g., those which are involved in the rigorisation of the concepts of continuity and limit. Yet never far from such praise is a corresponding criticism of formal methods. His fear appears to be that the benefits of exactness were being bought at the cost of purging our mathematical concepts of all intuitive content. Poincaré wants both precision and intuition to be a part of mathematics "proper". From his own experience he knows that "creative intuition" is hardly a formal matter. The relation between our formal characterisations and our intuitive concepts was a tension which he sought to resolve.

There are also prima facie difficulties in coming to grips with his views on set theory. Although he was one of the first mathematicians to employ Cantor's theory of sets, and thus one of the first to reap the benefits of the theory, he explicitly rejected its fundamental theorem in its standard interpretation - that of a proof of the existence of an uncountable set. And there is an apparent outrageous inconsistency in his attitude towards the continuum. Time and

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8 "I know very well there are disappointments, that the thinker does not always find the serenity he should, and even that some scientists have thoroughly bad tempers." (Poincaré, (1908), p.24.)

again he stresses that all infinity is potential, that "there is no actual infinity"; that ineliminably impredicative specifications must be rejected; and so, the greatest cardinal is $\aleph_0$. And yet he wishes to retain a classical notion of continuity and the continuum, as he does not hesitate to employ, in his proofs, variables which range over all the points on the line. Indeed, the notion of continuity is one of the most central to his thinking, and his greatest theoretical achievements in the development of "analysis situs" occurred when he considered what happens if certain parameters are allowed to vary continuously. It seems all the points on the line exist, but there is no cardinal number of all the points on the line.

The diversity and global nature of Poincaré's thinking is depicted in the diverse schools of thought, the foundations to which he contributed. For instance, his theory of meaning - the criterion of verifiability in principle - became foundational in intuitionism. This theory is the subject of Chapter 7. Whereas his theory of impredicativity and vicious circles, the subject of Chapter 6, led to the development of predicative set theory and predicative analysis (as for instance is found in Feferman, (1964), and, more recently, in S. Shapiro, (1985)). Chapter 5 consists in an examination of Poincaré's theory of the continuum; the proper interpretation of the continuum being an open philosophical matter to this day (and, perhaps, for a very long time). Poincaré's contributions also led to a critical reassessment of metamathematics, in general. In Chapter 4 I discuss how, in Poincaré's view, intuitions are epistemologically prior to any significant formal structure; and in Chapter 3 I focus, with a view to the same end, on Poincaré's theory of induction. The question of the apriority of mathematics (and not only its synthetic apriori character) also stands in need of a defence. However, in view of the fact that Poincaré did not himself explicitly address the topic of a modern type of empiricist challenge, I have included a defence of the apriori as an appendix.

My project in this thesis has been to determine whether there is a general philosophical core which underpins Poincaré's scattered, diverse, yet often profoundly insightful remarks. Is there a foundation which makes even his apparently paradoxical views cohere? My answer is an unqualified yes. Poincaré's philosophy is coherent; the fundamental key to an appropriate understanding of his philosophy on the whole is not to underestimate the legacy of Kant in his views. Poincaré adopts Kant's view that mathematics is synthetic apriori. (He adapts it, too, for on his account, contrary to Kant, geometry is not synthetic apriori; in fact it is conventional.) His philosophical position can, on the whole, be described as "neo-Kantian". Chapter 2 is a brief introduction to Poincaré's theory of the synthetic apriori. The main body of this thesis can, in general, be seen as a description of the way in which Poincaré adapts the Kantian thesis, with a view to defending Kant from the "Leibnizian" impulses of the time: i.e., Russell, Zermelo, Peano, etc.. Our first task will thus be to examine Kant's position; to this Chapter 1 is devoted. Throughout, it will be important to bear in mind that from the point of view of one of the very greatest practicioners of classical mathematics since Gauss, one can see Poincaré's philosophical work as possibly being motivated by a desire to steer a middle course between the "Scylla" of triviality and the "Charybdis" of contradiction; in fact, to steer a middle course between strict constructivism and set theory.

CHAPTER ONE

KANT'S PHILOSOPHY OF MATHEMATICS

In order to properly describe and fairly appraise Poincaré's philosophy of mathematics, we must first clarify what he thinks, and why he espouses certain Kantian themes but not others. He explicitly rejects Kant's thesis that Euclidean geometry is synthetic apriori. And he even rejects the more minimal Kantian thesis that the three-dimensionality of space is a synthetic apriori matter. However, he follows Kant in asserting that the theorems of <u>pure</u> mathematics have a synthetic apriori status.

What does Kant mean by saying that mathematics is "synthetic apriori"?[1] First, mathematical truths are not all analytic truths. Both Kant and Poincaré maintain that some mathematical <u>principles</u> are analytic, e.g., "a=a" or, more interestingly, "equals added to (or subtracted from) equals provide equal results".[2] But bona

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1 For instance, "All mathematical judgements, without exception, are synthetic ... [In addition,] mathematical propositions, strictly so called, are always judgements <u>apriori</u>, not empirical, ..." (B p.14) and "... bodies of <u>apriori</u> synthetic knowledge can be derived (Pure mathematics is a brilliant example of such knowledge ...)" (B p.55). (References to Kant's <u>Critique of Pure Reason</u> will be given by prefixing the page number with the appropriate edition, i.e. "A" for 1781 and "B" for 1787.)

2 See B pp.204-205 and Poincaré, (1894a), p.3.

fide mathematical judgements must be synthetic (according to the above reference). And second, despite the non-analytic character of our mathematical judgements, the knowledge gained from them is not empirical - it is not obtained from experience - but is apriori. The invention of the notion of apriori forms of experience enables Kant to conjoin synthetic with apriori, and hence, to maintain this view. What the synthetic apriori consists in and from where it comes will be the main subject of this chapter.

(1) The Basic Distinctions

Preliminary to understanding Kant's philosophy, there are two general claims: one in the theory of meaning, and one in epistemology. The first claim is that there is a well-defined distinction in our language between analytic and synthetic statements (judgements, propositions). Our understanding of the content of our propositions is such that there are two exclusive classes: the analytic and the synthetic. Now this distinction may be explicated (or fail to be explicated) in various ways. Kant himself wavers between the view that these are types of judgements (B p.10), and that these are types of propositions (B p.56). Most likely, he felt the distinction could be applied to propositions as a result of its application to judgements, which seems to be primary. For Kant, the distinction amounts to the existence of a "containment" relation among concepts.

> Either the predicate B belongs to the concept A, as something which is (covertly) contained in this concept A; or B lies outside the concept A, although it does [in virtue of our judgement] indeed stand in connection with it. In the one case I entitle the judgement analytic, in the other synthetic. (B p.10)

Hence analytic truths are those which, in thinking the subject (of the sentence being judged) we cannot help but think the predicate. The predicate, thus, gives us no new information about the subject, for it is already "contained in" the concept of the subject.

There is no doubt that Kant believed that this relation among concepts existed and was well-defined. But this claim, since it presupposes determinacy of meaning - the objectivity of our linguistic conventions - is widely contested in most discussions in modern philosophy of language. Perhaps the boundary between analytic and synthetic is not as straightforward as Kant thought; or perhaps Kant's metaphor of "containment" is not desirable. Whatever way the distinction is made, however - e.g., that the negation of an analytic statement, when appropriate definitions are substituted, produces a contradiction - it is necessary for an understanding of Kant.

The other major distinction which underlies Kant's philosophy is between pure and empirical, or apriori and aposteriori knowledge. Whereas analyticity/syntheticity is a linguistic distinction concerning the content of our statements (or judgements), the apriori/aposteriori distinction is epistemological, and concerns how we can come to know the truth of, and how we can justify making, our assertions. So all statements are either analytic or synthetic; and all knowledge is either apriori (pure) or aposteriori (empirical). Apriori knowledge is that which is knowable without consulting the world: it is "prior to" (in a figurative sense), or independent of, any particular experience or set of experiences. In contrast, aposteriori knowledge requires sense experience, or investigation into the world, before one can have good grounds for accepting it. One must be able to cite evidence (facts - usually other aposteriori pieces of information) in order to justify any knowledge which is aposteriori. The question "How do you know," is never adequately answered by "I just do know" when referring to knowledge which is thought to be aposteriori.

As with the analytic/synthetic distinction, there are problems with the neatness of this dichotomy concerning ways of knowing, as was pointed out by the later Wittgenstein throughout On Certainty.[3] However, again, as with

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3 For instance, that the earth existed 100 years ago seems to be a piece of aposteriori knowledge. But anything we might be tempted to cite as evidence for the statement, itself has no grounding (what is the evidence for the evidence?), unless we presuppose the original statement is true. In order to believe any of the evidence that the world existed 100 years ago (e.g., history books, geological methods, parents), I must already believe it is true that the earth

the analytic/synthetic distinction we must accept the existence of the apriori/aposteriori distinction in order to have any chance at all of acquiring a genuine understanding of Kant's philosophy. Whether or not we possess a clear explication of the concepts, we know the distinction exists because we use it. And, in Wang's words, "To say that analyticity [or any distinction] is not sharp is quite different from saying it is not intelligible".[4]

(2) The Synthetic Apriori

Intuitively, analyticity is usually paired with apriori, syntheticity with aposteriori. We do not need to "look at the world" to know "The bachelor is unmarried" is true; for, since it is analytic that all bachelors are unmarried, it is true in every instance. Justifying it is referring to the language, not to the world or experience; so it is apriori because it is analytic. Conversely, we cannot justify the claim that "There are three people in the room next door" - a synthetic statement which informs us - unless we go and look. We must participate in the extra-linguistic, sensory world - someone must do something

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did not pop into existence ready made with its history books. "Does my telephone call to New York strengthen my conviction that the earth exists," (Wittgenstein, (1969), p.240.) The answer is it cannot: for in order to believe I have successfully phoned New York, I must already believe (implicitly, at least) the earth exists.

4 Wang, (1974), p.278.

to act and/or perceive - in order to be in a position to justify this statement. Hence, it is aposteriori.

Famously, Kant disrupts this tidy dichotomy by fusing synthetic with apriori. Certain statements about the nature of space and time, about substance, and about mathematics are said to be synthetic apriori. They are synthetic apriori because, although they are not analytic (i.e., no containment relation exists between subject and predicate), our knowledge of them does not depend upon sense experience for its justification. So they are apriori and synthetic. An example which Kant gives of a synthetic apriori item of knowledge (a synthetic statement, the truth of which is knowable apriori) is that space has three dimensions. We cannot conclude this inductively from observation alone. For in order for an experience to <u>count</u> as perception of a spatial entity (external to the perceiver), it must be perceived as three-dimensional (B p.38). There is no such thing as amassing evidence for the three-dimensionality of space; for in order to decide whether or not a perception is to count as evidence for or against the hypothesis, we must decide whether or not it is spatial. And the only method we have for deciding this, is to determine whether or not the perception stems from outside the body. But it is impossible to know whether something is external to the body without knowing whether it is three-dimensional. Hence, perception of three-dimensionality is <u>necessary</u> for a perception to count as an external object.

Yet it is not logically absurd to conceive of two-dimensional (or any other dimensional) space, as in plane geometry, or analysis situs. So it is not analytic that space has three dimensions. We can conceive of alternate spaces, we just cannot conceive of ourselves (our bodies) living in (experiencing) them. We cannot imagine what experience of non-three-dimensional space would be like, since any perception we imagine is interpreted or seen from a three-dimensional point of view.

This is an instance of what Kant calls the "form of experience" or, in particular, the "apriori form of perception". Perceptions must have a certain form or character before they can be counted as perceptions, rather than mere imaginations of uninstantiated concepts. Kant argues that there must be some "screening off" faculty; otherwise, for instance, how is it that we are able to distinguish sensations stemming from within the body and mental activity ("inner appearance") from perceptions of an external object ("outer appearance"), where the latter are supposed to be caused by something in the world, independent of the perceiver? (B p.38). The objection that we are not always able to distinguish veridical from non-veridical perception, is answered by pointing out that according to Kant this very distinction (veridical/non-veridical) would be impossible without the apriori form of perception. There must in principle be some difference of which we can be aware, between an experience and the memory of the experience. How the distinction - even if it is not always used accurately - is possible at all, is the question with which Kant is concerned.

The <u>form</u> of all spatial and temporal intuition (or of all possible experience) is apriori - prior to sensations which are the <u>matter</u> of perception. Knowledge of the form of space and time is synthetic apriori; knowledge of particulars in space-time is synthetic aposteriori. Space and time are apriori forms of experience which take in and process the matter of our perceptions, e.g., by providing a structure which imposes an ordering relation on our experiences.

(3) <u>The Synthetic Apriori Instantiated: Geometry</u>

Kant held that geometrical knowledge is synthetic apriori because it consists in synthetic judgements concerning the apriori intuition of space. Since these judgements concern only that which is given apriori - i.e., they do not concern any accidental properties of actual, particular lines, points, triangles, etc. - the knowledge obtained from the judgements is apriori also. The statements concerned are synthetic because they are not analytic; and they are not analytic because they depend on something other than logic plus the containment relation among concepts for their truth or proof. In order to justify a belief that, say, the angles of a triangle add up to 180°, we prove it. And the proof requires an active contribution, which is synthetic in nature.

Geometric proofs have a constructive form: they depend upon the performability in principle of certain constructions. For instance, line segments must be extendable; and we must be able to rotate figures or planes 180^{o} around a straight line. So, since the existence of indefinitely extendable lines is not satisfied in all spaces (e.g., a spatially closed surface with the topology of a sphere[5]), we require intuition to underwrite the existential assumptions, which are in turn necessary for the constructions to count as evidential steps. The constructions will not be evidential unless we have independent grounds for believing they are performable, i.e., that they are satisfiable in the space in question. In this way, the possibility of geometric proofs depends upon the satisfiability of certain constructions. Hence, it depends upon (geometric) space having certain properties, and remaining so over time.

But since, for Kant, space just <u>is</u> what we are in principle able to perceive of the outer world (what we are able to perceive apriori), the proofs depend on nothing other than the apriori form of (experiential) space. Our apriori intuition of space underwrites our proofs because it informs us that the necessary constructions are performable. Hence, that in virtue of which geometry is synthetic (construction, or existential assumptions) is

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5 Friedman discusses this point in his (1985), p.500.

something which is apriori. Geometric theorems are synthetic because of the constructive character of their proofs; yet they are apriori, because the existence of the necessary constructions is guaranteed by apriori information. In referring to properties of space given by the apriori forms of perception, we are going "outside" or "beyond" the geometrical concepts themselves. There is thus no relation of containment between the concepts and these properties. Yet in going beyond the concepts, our constructions do not go <u>into</u> that which can only be given empirically (in an aposteriori way). This is because we only go "beyond" the concepts by looking at what <u>we</u> add apriori to the concepts.

> We are not here concerned with analytic propositions, which can be produced by mere analysis of concepts ..., but with synthetic propositions ... For I must not restrict my attention to what I am actually thinking in my concept of a triangle (this is nothing more than the mere definition); I must pass beyond it to properties which are not contained in this concept, but yet belong to it. (B p.746)

If we restrict our attention to the <u>concept</u> of triangle, we cannot do geometry. We also must employ properties of the embedding space, which inform us that we can bisect angles, extend line segements, etc. For Kant, the properties of the embedding space "belong to" the concept (of triangle) by virtue of certain apriori judgements concerning space and time, by way of the apriori conditions of experience.
The apriori form of experience is imposed on (and hence "belongs to") - not contained in - our concepts, by restricting the ways in which we can "pass beyond" the concepts.

(4) The Synthetic Apriori Instantiated: Arithmetic

Mathematical reasoning is reasoning via "construction", where some sort of new entity is considered or "synthesized" in the process. We have seen how, in the case of geometry, this consideration draws from our apriori spatial intuition; it "goes beyond" the (mere) concepts in this way. How does "construction" fit into arithmetic or algebraic proofs?

Kant was not as clear about arithmetic as he was about geometry. Hence, almost inevitably, when giving an example of a construction, he provides a geometric construction. He does, however, address the subject of arithmetic construction. Computational truths, like "2+3=5", are synthetic because they are statements about constructions in time, the truth of which cannot be known merely by considering the concepts involved (two, three, plus, equals, and five) (B pp.15-16). The containment relation is not satisfied here, because just thinking "two plus three" is not sufficient for thinking "five". In order to arrive at the concept "five", I must actively put the two and the three together by "successive synthesis in time" or counting. In the synthesis involved in the successive counting of units,[6] we find the arithmetic analogue to geometric construction. And since counting both actually takes time, and conceptually requires the apriori intuition of

6 See Körner, (1960), p.29.

time[7], the apriori form or intuition of time is involved at the very basis of arithmetic: in grounding our generating of integers, and hence, in everything we do with the integers. From counting we learn addition, and from addition we proceed to other methods of manipulating the integers.

The apriori intuition of time (the form of all experience) is what enables us to learn the discipline of mathematics and to apply arithmetic and mathematics to the world. This is because the distinguishing characteristic of the integers is their successive nature. The claim is, we could not have learned about numbers without the apriori intuition of time. We could not have acquired the intuition of succession (to ground the concept of succession) were it not a form imposed by our minds upon experience. Furthermore, time is what enables us to apply our mathematics to the world, for it is what guarantees that our perceptual experience will be of a mathematical character (B p.206).

As well as for the integers, the distinguishing characteristic of time is that it is successive:

> Time is nothing but the form of inner sense ... It cannot be a determination of outer appearances; it has to do neither with shape nor position, but with the relation of representations in our inner state. And just because

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7 See Kant,(1770), p.62.

> this inner intuition yields no shape, we endeavour to make up for this want by analogies. We represent the time-sequence by a line progressing to infinity, in which the manifold constitutes a series of one-dimension only; and we can reason from the properties of this line to all the properties of time, with this one exception, that while the parts of the line are simultaneous the parts of time are always successive. (B pp.49-50)

The parts of time are entirely successive, or ordered. Our knowledge of the parts of time occurs via the relations of the representations within this successive framework. We can only know parts of time by aposteriori means, for there is no possible experience of all of time - of the general concept of time, or indefinite succession. Hence the intuition of time in general must be apriori. Because our intuition of time is apriori, the fact that it is successive, or ordered, does not mean it must be composed of discrete parts. The parts of time we perceive do not make up the whole; on the contrary, our memories are carved out of the whole, which must then be intuited apriori.

> Space and time are quanta continua, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such a fashion that this part is itself again a space or a time.... Points and instants are only limits, that is, mere positions which limit space and time.... neither space nor time can be constructed. Such magnitudes may also be called flowing, since the synthesis of productive imagination involved in [producing magnitudes] is a progression in time, and the continuity of time is ordinarily designated by the term flowing or flowing away. (B p.211-212.)

Space and time are continuous; hence we can only know parts of space and time by aposteriori means, as defined by the part enclosed between two end points of a representation. The continuity of time is compared to a "flowing" or the movement of a point - thus, it is clear that Kant's conception of continuity was not equivalent to mere density, and was not unsophisticated for his time. In addition, since we have an apriori intuition of time, we have an apriori intuition of a mathematically sophisticated model.

That by which we express our apriori intuition of time - the continuum - is also that which provides a pictorial model or geometric analogue of all the real numbers. Each real number, like Kant's points and instants, is a limit. Hence it is not absurd to hold that Kant's model of the intuition of time grounds our modern conception of number.

Kant argues that the intuition of time must be apriori; otherwise, we could not have acquired our conception of number. For instance, we could not possibly learn the notion of indefinite succession by aposteriori means alone (by reference only to experience). We could not even learn the mere notion of succession by aposteriori means alone; for there is no possible perception of succession, since succession is merely a relation <u>between</u> perceptions. Hence it must be imposed, not acquired; and our understanding of the corresponding concept must be completely apriori.[8] If the order of our representations was not imposed by the mind, we could not perceive order at all. For without the apriori form of inner perception, our memory would not be ordered - the recollection of our experiences would be haphazard and chaotic, unrelated by position. Hence, we could not acquire the understanding of succession from the perception of order in the memory; for there would not be any order if the relation of succession (total order) was not imposed on it. The apriori understanding of an object in general, via the conditions of thought, ensures that our perceptions are of discrete units or objects. In addition, the apriori intuition of time ensures that recollections of our perceptions are structured as well, by ordering our perceptions as they are inscribed onto the memory.

Hence, our concept of the domain of integers, or of all quantities which are either continuous or successively generated, must be grounded in the apriori intuition of the temporal form. For the very notions we employ to describe our concepts of numerical domains would themselves be devoid of sense in the absence of the temporal form. Concepts like indefinite succession, continuity, etc., would be devoid of sense without the apriori intuition of time, for there is no possible aposteriori experience

8 For the necessity of a possible experience to ground each concept, see next section, this chapter.

(intuition) to ground these concepts. Kant's requirement that there be an intuition (possible experience) corresponding to every meaningful concept, will be discussed in the next section.

(5) The Foundations of the Theory of the Synthetic Apriori

Kant had strongly empiricist elements in his theory of meaning. Associated with every concept there must be an "intuition" (or an instance), and, according to Kant, we must be able to associate an intuition or instance with a concept if we can claim to understand the content or sense of the concept.

> Without sensibility no object would be given to us, without understanding no object would be thought. Thoughts without content are empty, intuitions without concepts are blind. It is, therefore, just as necessary to make our concepts sensible, that is, to add the object to them in intuition, as to make our intuitions intelligible, that is, to bring them under concepts. (B p.75) [9]

Knowledge, then, requires both concept and intuition, or concept and "individual representation"[10] of an instance of the concept. Both aspects are necessary; and each has an apriori form which can be described of them. We have already mentioned Kant's description of the apriori form of intuition, i.e., space and time. The apriori form of understanding is complex in its divisions.[11] The important

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9 See also B p.298.

10 See Hintikka, (1973), e.g., p.44, and pp.207-210.

11 See B p.76 *passim*.

point is that there is more than one type of rule of thought. Kant, here, diverges from the ordinary modern view that the way in which we think is constrained only by the "empty" rules of logic. In contrast, certain of Kant's rules of thought are not "empty" in the same way, but rather, participate in the content of our concepts. First there is "pure general logic" which is completely divorced from anything empirical, and is purely formal, and which may or may not correspond to our notion of formal logic and logical possibility.[12] Second, there is "applied general logic", which concerns the

> rules of the employment of understanding under the subjective empirical conditions ... Applied logic has therefore empirical principles, although it ... refers to the employment of the understanding without regard to difference in the objects. (B p.77)

Insofar as pure general logic is pure, it concerns nothing empirical; insofar as it is general, it is only the mere "form of thought". Insofar as applied general logic is applied, it is empirical because it concerns the way in which we - in fact - think about empirical objects. However, insofar as it, too, is general, it concerns only the way we (in fact) think about empirical objects <u>in general</u>. That is, it too concerns the mere form of thought in that it has nothing to do with any particular differences of

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12 See B p.77 for description.

actual objects. General applied logic, then, is the study of the application of restrictions on our understanding of empirical objects in general.

Kant sees three "stages" to knowledge, each of which is divided up into various sections, depending, among other things, on whether the object or intuition concerned is pure or empirical (apriori or aposteriori). The first stage is perception, which is conditioned by the forms of space and time. The second stage is synthesis, where our perceptions are gathered together in a certain way so as to produce clumps or sets of sense data, each of which is unified into an image. This synthesis is performed by what Kant calls the imagination. The third stage takes place in the understanding; and it is here where we judge (for the understanding is a "faculty of judgement" (B p.94)) which concept corresponds to the clump of sense data (formed by the imagination). We also judge here whether different clumps are instances of the same concept (B p.104). In addition to the unifying power of the imagination then, which both synthesizes (different perceptions of a thing over time) and unifies (perception into one image (A p.120)), the understanding also has a unifying power in that it can unify or judge different images as instances of a single concept. The rules of understanding are necessary in my judgement that, for example, a Granny Smith, a Golden Delicious, a McIntosh, a baked apple, are all instances of the concept "apple". It is via the imagination that I perceive each as one image, one object or unit. And it is via the understanding that I can unify the images under one concept - i.e., that I can know that they are all apples.

These three stages are necessary for all knowledge, whether the content concerned is pure or empirical. Hence, there are both apriori and aposteriori intuitions which correspond to different concepts, and there is both an apriori and an aposteriori synthesis (which depends on the type of intuition concerned).

Kant's verificationist theory of meaning is at the root of his development of the synthetic apriori. "All concepts ..., even such as are possible _apriori_, relate to empirical intuitions, that is, to the data for a possible experience." (B p.298). Otherwise, the concept is empty, and has no meaning for us. Since knowledge is thought _plus_ intuition (B p.157), knowing something requires that we pass beyond mere concepts. We can pass beyond mere concepts, or acquire evidence for a synthetic judgement, in two ways. In ordinary (aposteriori) synthetic judgements we refer to sense experience for our evidence. For example, appropriate evidence for "The cat is purring" would include my hearing the cat purr. However, with regard to apriori synthetic judgements we must "pass beyond the mere concepts" in an apriori way, that is, independently of sense experience. We can do this by referring to the apriori conditions, discussed above, for a possible experience; we can inform our concepts by inspecting how they are affected by the conditions for experience (of perception, synthesis and judgement), and this is knowable apriori.

> Synthetic _apriori_ judgements are thus possible when we relate the formal conditions of _apriori_ intuition, the synthesis of imagination, and the necessary unity of this synthesis, in a transcendental apperception [in a unified consciousness (A p.107), i.e., in a single concept], to a possible empirical knowledge in general. We then assert that the conditions of the _possibility of experience_ in general are likewise conditions of the _possibility of the objects of experience_, and that for this reason they have objective validity in a synthetic _apriori_ judgement. (B p.197)

Or, put more starkly: "Time and space, taken together, are the pure forms of all sensible intuition, and so are what make _apriori_ synthetic propositions possible" (B p.56).

(6) _How the Theory Works: Geometry Revisited_

For example, geometry is synthetic apriori because space and time are involved in the construction of figures by guaranteeing the existence and performability of certain constructions. Although we do not draw conclusions about particular figures (hence, our actual sense experience does not inform our proofs, and this is why they are apriori), our employment of figures in proofs does cause us to go "beyond" the mere concepts. There is no relation of containment between the concepts of the figures and the inferences we make via our constructions. The figures or images we employ are arbitrary instances of concepts, like "triangle". Because we can consider an arbitrary instance of the concept, by "drawing it in thought", we satisfy the first condition of knowledge: "that the representation through which the object is thought relates to actual or possible experience" (B p.195). My mental image of a triangle is sufficient to guarantee the meaningfulness of the concept, for any possible experience of an actual triangle will conform to what is given in my mental image. This is because the latter draws only from the conditions for _all_ experience, including experiences of triangles.

Spatial and temporal intuition are employed in my mental image. Space, or "outer sense" guarantees the existence of, or the possibility of constructing, triangles; and time - "inner sense" - is the form of all possible experiences: images and events. The synthesis of the imagination is employed in my mental image in the act of drawing it in thought. And the rules of the understanding are employed when I judge that the mental image is an instance of the concept "triangle".

Hence, via the form of experience, or experience in principle, geometry advances using pure intuitions.

(7) A Precarious Analogy

By "construction" in mathematics Kant intends a non-empirical representation of a concept. Regarding numerical formulas involving small numbers, we construct or synthesize _units_ to determine their truth or falsity. I construct in my mind three units, and then successively add two more units to it, in verifying that "3+2=5" is true. However, construction of abstract units only suffices for a very small part of mathematics. We must also solve equations involving large numbers, for which it is no longer practically possible via "successive synthesis" of units alone. And we must prove general results. Hence, we have more general procedures, the understanding and implementation of which requires construction of _concepts_ rather than units.

> To construct a concept means to exhibit _apriori_ the intuition which corresponds to the concept ... we therefore need a _non-empirical_ intuition. The latter must, as intuition, be a _single_ object, and yet none the less, as the construction of a concept (a universal representation), it must in its representation express universal validity for all possible intuitions which fall under the same concept. Thus I construct a triangle by representing the object which corresponds to this concept either by imagination alone in pure intuition, or in accordance therewith also on paper, in empirical intuition - in both cases completely _apriori_, without having borrowed the pattern from any experience. The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality. For in this empirical intuition we consider only the act whereby we construct the concept, and abstract from the many determinations [for instance the magnitude of the sides and of the angles], which are quite indifferent, as not altering the concept "triangle". (B pp.741-742)

Because we consider an intuition (construction) and not only the concept, we combine the properties which are given by the concept of the object with the properties which it has _qua_ object. And because the intuition is of an _arbitrary_ object, we combine the properties given in the concept with the properties which it has _qua arbitrary_ object. The properties which a construction has _qua arbitrary_ object are those which are present in _every construction_ of the type of object. For example, every construction of a triangle gives us angles which add up to 180°, as can be proved through considering an arbitrary (Euclidean) triangle. Because the point of the representation is to exhibit only the properties which are provided by every construction of the type of object, the intuition of the representation is an intuition which is general, not particular or actual; and hence pure, or apriori, and not empirical. We do not, for instance, measure the angles of the triangle we have constructed for the proof mentioned above. So the proof does not depend on any of the particular, "accidental" properties of the actual figure drawn on paper. If it did, it would not be a proof. Hence, the construction, as we employ it in a proof, is not empirical; and the figure actually drawn is merely a heuristic aid.

> Philosophy confines itself to universal concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept in concreto, though not empirically, but only in an intuition which it presents apriori, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object of the concept thus constructed. (B pp.743-744)

In mathematics we infer things both from what is given in the concept and from what is given in the "construction" of the concept. Because we are constructing (lines, points, triangles), we are "guided throughout [our proofs] by intuition" (B p.745) and by the synthesis of the imagination, so our inferences are synthetic. But since the intuition which guides us is not empirical, the synthesis concerns an apriori image, and our inferences are synthetic apriori.

Conceptual analysis alone is inadequate for mathematics. We must refer to our intuitive understanding of an object or unit in our geomëtrical proofs. But since we refer only to <u>pure</u> intuition, or perception of an <u>arbitrary</u> (ideal, apriori) object, this reference does not impugn the apriority of our inferences. We have seen, in section 6, how the theory of construction of concepts applies to geometry. Space and time contribute to the conclusions we draw even though we only consider apriori, i.e. arbitrary, constructions. The transition from geometry to arithmetic and algebra is important, and requires careful analysis. To this end I will indulge in an extensive quote from Kant.

> But mathematics does not only construct magnitudes (<u>quanta</u>) as in geometry; it also constructs magnitude as such (<u>quantitas</u>), as in algebra. In this it abstracts completely from the properties of the object that is to be thought in terms of such a concept of magnitude. It then chooses a certain notation for all constructions of magnitude as such (numbers), that is, for addition, subtraction, extraction of roots, etc. Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition, in accordance with certain universal rules, all the various operations through which the magnitudes are produced and modified. When, for instance, one magnitude is to be divided by another, their symbols are placed together, in accordance with the sign for division, and similarly in the other processes; and thus in algebra by means of a symbolic construction, just as in geometry by means of an ostensive construction (the geometrical construction of the objects themselves), we succeed in arriving at results which discursive knowledge could never have reached by means of mere concepts. (B p.745)

However, the analogy is contrived. It seems that Kant is trying too hard to draw a tight analogy, where it is not available. The conventions governing the choice of symbols via which we represent the numerals have nothing in any interesting way to do with the synthetic aspect of mathematics. Symbolic conventions are empty conventions. The subject matter of geometry is grounded in the apriori intuition of space (as well as of time). Hence, in representing its concepts we quite naturally employ space: we construct images on paper. However, the subject matter of arithmetic comes only from the apriori intuition of time. So there is no relation between arithmetic construction and space. There is no link between any number concept and the symbolic notation by which we represent it.

Kant locates the parallel in the wrong place. There is a parallel between geometry and the rest of pure mathematics; it just does not lie in the way we <u>represent</u> either our arithmetic or our geometric concepts. Rather, the appropriate parallel lies in our ability in both domains to inform our concepts - of triangle, of number - in an apriori way, via apriori intuition. Mathematical knowledge is only possible because of the existence of our apriori intuitions of space and of time, for without these our exact concepts would not be meaningful.

> All our knowledge relates, finally, to possible intuitions, for it is through them alone that an object is given ... [And] the only intuition that is given <u>apriori</u> is that of the mere form of appearances, space and time. (B pp.747-748)

Both geometrical and number-theoretic statements have a content which is given by the "mere form of appearances". The disanalogy lies in the fact that while geometrical concepts are informed by our apriori intuition of both space and time, our arithmetic and algebraic concepts are informed only by our apriori intuition of time. The content of our number-theoretic statements is independent of spatial intuition. In this way, since the parallel seems to focus on the supposed "construction", it is not apt.[12]

(8) The Key to the Synthetic Aspect of the Science of Number: Induction

The important aspect of construction in mathematics is the way the concepts can be informed, or "added to", by intuition in an apriori way. The performability of geometrical constructions is guaranteed by space and time; the performability of arithmetic constructions is guaranteed only by time. However, mathematical definitions are exact: the commerce of mathematics is exact concepts and ideal objects. We cannot draw ideal objects anyway, so the spatial representation of even geometric concepts - drawings of triangle, lines, etc. - is not related in any important way to the "constructive" aspect of the proofs. (As axiomatic geometry shows, the pictures are dispensable, since

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12 As perhaps, Kant himself realised. See B pp.762-763, where he emphasises the importance of the concepts attached to the symbols; the symbols having a heuristic role.

they are merely heuristic aids to the proofs.) Hence, "construction" of mathematical concepts is a metaphor. The intention of this metaphor is to capture our ability to consider an arbitrary instance - unit, object - of a concept. And, since the concepts involved are (or ought to be) exact, this consideration of an arbitrary instance is mathematically informative, for through it we have access to the "ideal objects" of mathematics.

Considering an arbitrary instance of a concept results in a "construction" which is ideal, because what we are thinking about, or quasi-perceiving "in the mind's eye", is an object which represents all and only the properties of all the elements in a domain. Consideration of an arbitrary element of a domain is an essential part of any general proof (proof of a general result). In geometry, it is essential that our constructions be of arbitrary figures in order to be justified in drawing conclusions about <u>all</u> triangles, or <u>all</u> isosceles triangles. In arithmetic it is an essential part of proof by induction, where we consider an arbitrary element and its successor (or in multiple and transfinite inductions, stages of this pattern). Moreover, induction is necessary for any general proof concerning numbers. Hence, consideration of an arbitrary element of a given domain is a necessary part of any general mathematical proof.

Constructing concepts in mathematics is considering arbitrary instances of concepts. In considering an arbitrary triangle, we employ the apriori intuition of space. Because the "intuition" is non-empirical, the construction is apriori, and the object is arbitrary. For instance, by means of the apriori spatial intuition we know that we can draw a line from one angle to the opposite side of a triangle, which exactly bisects that angle - not approximately, but exactly. "We can draw" only means "there exists" in geometry (we cannot actually bisect an angle); so our apriori spatial intuition vindicates our making certain existential claims about space within a proof.[13]

Likewise, in arithmetic and algebra, considering an arbitrary object of a domain is the key to discovering or understanding general results. For instance, in induction we must consider an arbitrary natural number, *n*.

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13 This coincides with Hintikka's view of the development of the analytic/synthetic distinction (see Hintikka, (1965b), "Are Logical Truths Analytic?"). He claims that prior to Kant, the analytic/synthetic distinction was mainly directed towards geometrical proofs. An analytical "argument" was one in which no constructions were carried out; i.e., "no new lines, points, circles and the like were introduced during the argument" (*ibid*., p.153). If any such new entities were essentially employed in arriving at the conclusion, the argument was considered to be synthetic. There seems to be a direct relation between this view of the distinction and Kant's arguments concerning why mathematical proofs, in contrast with philosophical arguments, are synthetic. Philosophical reasoning is analytic because it depends only on the concepts plus formal logic or the containment relation. Mathematical reasoning is synthetic because it relies upon how the concepts are informed by the rules governing the three stages of knowledge (discussed in section 5): it relies upon construction of concepts.

And this is informative in a way which "goes beyond" the mere concepts, because we employ the apriori intuition of time, since we must consider the relation between n and its successor, n+1. The inference

$$\begin{array}{ll} \text{from} & P(o)\ \&\ P(n) \rightarrow P(n+1) \\ \text{to} & \forall n\ P(n) \end{array}$$

is synthetic, because we must employ our intuition regarding "succession" for the proof of $P(n) \rightarrow P(n+1)$. Since succession is a "mode" of time (B p.219), induction requires the intuition of time. *Pace* Frege, our concepts of P, of number, etc., are inadequate to result in the conclusion $\forall n(Pn)$ via the containment relation alone. Hence, every general (quantified) result, like $\forall x\ \forall y\ (x+y=y+x)$, is synthetic; for its proof requires induction, i.e., consideration of an arbitrary numberical object together with the temporal properties which it has *qua* representational unit. Induction is thus mathematical reasoning "*par excellence*", for it is via induction that we obtain all general, all significant, mathematical results.

Poincaré's explanation of the synthetic apriori character of mathematics relies heavily on his view that induction is synthetic apriori. After a brief introduction to his version of the theory of the synthetic apriori in general, we will begin our explication of Poincaré's philosophy of mathematics, in Chapter 3, with an analysis of his conception that "arithmetic" or "itèrative" intuition is the epistemological source of the principle of induction.

CHAPTER TWO

INTRODUCTION TO POINCARÉ'S THEORY OF THE SYNTHETIC APRIORI

(1) The Synthetic Apriori and Time

(2) The Synthetic Apriori and Space

When Poincaré asserts that mathematics is _synthetic apriori_, he agrees with Kant that true mathematical propositions are not obtainable as theorems of logic alone even when supplemented by appropriate definitions of the non-logical vocabulary involved. Indeed, Poincaré's sense of a synthetic proposition is essentially that of Frege. Whereas Frege might have held that the entire corpus of mathematics (with the possible exception of geometry) is _analytic_, Poincaré held that it is synthetic in precisely Frege's sense; for it is not possible to provide a proof of certain key mathematical claims (even number theoretic ones) without "making use of truths which are not of a general logical nature, but belong to the sphere of some special science"[1]. (This is precisely how Frege characterises a synthetic proposition.) Like Kant, Poincaré maintained that mathematical knowledge is knowledge _apriori_. Where he differs from Kant is in exactly what he takes as synthetic apriori. For Kant the forms of perception of (Euclidean) space and time provide the content of geometrical and number theoretic truths (as well as grounding certain

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1 Frege, (1884), p.4.

general statements concerning substance, cause and effect). In contrast, Poincaré's notion of "synthetic apriori" is that which is

> imposed upon us with such a force that we could not conceive of the contrary proposition, nor could we build upon it a theoretical edifice. 2

The key notion here is "build upon it a theoretical edifice". This is why, contra-Kant, geometrical truths are conventional. It is possible to erect alternative systems of the world (i.e. geometry plus physics), the purely geometric component of which is non-Euclidean. It is this fact, the <u>inter-pretability of experience</u> on the basis of non-Euclidean geometry, which refutes the Kantian explicit claim concerning Euclidean geometry; but it is certainly not the mere existence of consistent non-Euclidean geometries which does so. If Kant had been right, the former fact could not obtain, whereas the latter, the existence of consistent non-Euclidean geometries would still have been possible. We can build alternate theoretical edifices incorporating rival geometrical systems, so no one particular pure geometry has synthetic apriori status.

In contrast Poincaré claims, the principle of mathematical induction is "a true synthetic apriori intuition" because we are unable to imagine a coherent non-standard arithmetic based on the negation of induction, in the same way that we construct a non-Euclidean geometry based on the negation of the parallel postulate.

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2 Poincaré, (1891), p.48.

> Let us next try to get rid of this [the principle of induction], and while rejecting this proposition let us construct a false arithmetic analogous to non-Euclidean geometry. We shall not be able to do it. [3]

Why "we shall not be able to do it" is the subject of the next chapter. Essentially, Poincaré is claiming that arithmetic is so foundational in our conceptual structure that it is a form of understanding. Unlike the geometric case, we can form no concept of an experience which would violate Peano arithmetic. Consequently, if the world does not "measure up" to one of our calculations, we say something went amiss in the observation, or that we did not perform the calculation correctly. We do not call the form or the algorithm of the calculation into question. We correct ourselves or our particular use of an arithmetic identity rather than the identity itself. All this means is that, first, mathematics is not empirical: the world does not determine truth in mathematics, since, for Poincaré, no matter how different a world we imagine, we cannot imagine arithmetic as being any different. And second, mathematics is not conventional, because our actual system is not the simplest (most convenient) out of a selection of possible systems; rather it is the only possible system we can envision. This is because it is grounded in synthetic apriori knowledge.

Whereas Kant saw the synthetic apriori in the metric

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3 Poincaré, (1891), p.49.

properties of space and time, Poincaré locates the synthetic apriori in the weaker structural (or "topological") concepts of "order", continuity and indefinite iterability. I will now discuss the apriori temporal form of experience, or "arithmetic intuition", before turning to the apriori spatial form of experience, or "geometric intuition".

(1) The Synthetic Apriori and Time

One apriori form of intuition which contributes synthetic content to part of mathematics is instantiated in the apriori notion of time. Time is not something external and existing independently of us. Thus our understanding of time is not an empirically acquired concept. Rather, it is an apriori matter. Time is not in the world; we impose time on the world - on our classification and organisation of memory and knowledge.

In addition, what we impose is not chosen, i.e., is not conventional.

> The order in which we arrange conscious phenomena does not admit of any arbitrariness. It is imposed on us and of it we can change nothing. [4]

Ordering our memories and classifying our perceptions is something which we must do in the way in which we do it, because it is something which is imposed by the nature of our minds. The conception of the passage of time is

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4 Poincaré, (1898), p.26

necessary for ordering and classifying our experience, so that we can communicate, understand and remember. Time is manifested as a certain set of relations between memories of events or sensations. But time does not <u>consist in</u> the relations between memories. Rather, the relations between memories, or <u>order</u>, would not exist without the imposition of the form of time upon them.

Poincaré argues that our awareness of time is an apriori form of intuition. If the notion of time was obtained by inductive generalisation from perception of actual events, it could at most consist in an understanding of the order of, and perhaps, the relative "distance" between actual memories, or "filled compartments", of time. However, there is more than this to our conception of time, e.g., the knowledge that our memories could be organised differently - the awareness of the existence of "empty compartments", or possible-but-not-actual memories. Hence, the acquisition of our actual concept of time could not occur by aposteriori means alone. Memories ...

> ... can only be finite in number. On that score, psychologic time should be discontinuous. Whence comes the feeling that between any two instants there are others? We arrange our recollections in time, but we know that there remain empty compartments. How could that be, if time were not a form pre-existent in our mind? How could we know there were empty compartments, if these compartments were revealed to us only by their content? 5

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5 Poincaré, (1898), p.26.

There are many more possible memories than actual memories. Indeed, for Poincaré, our understanding of time is an understanding of an unbounded, dense (total) order; and knowledge of these properties of time could not be empirically acquired.

Before we discuss the structural properties of time, we must first distinguish between time (or our notion or understanding of time) and the temporal form. Time is a construct, an intuitive structure, which satisfies the properties provided as the temporal form. The temporal form is a synthetic apriori intuition, or form of experience, which imposes a notion of linear order (the order type of the rationals, as it turns out) and an understanding of "indefinite iterability". Time, on the other hand is a convenient device for interpreting experience: a construction which satisfies only the impositions of the temporal form. Because the temporal form supplies us with the notion of indefinite iterability, the intuitive structure of time consists in a model for a potentially infinite set. However, in addition to indefinite iterability in the sense in which it produces a potentially infinite set like the natural numbers - i.e., an unbounded domain - there is an indefinite iterability in between the possible compartments, or instants, of time. That is, the temporal form imposes an understanding of indefinite iterability in a dense sense, in the sense in which it produces a domain like the rational numbers. Moreover, time is a structure via which we preserve the order of our memories once they occur; hence it provides a model for a domain in which the order of the elements is fixed. Therefore, time provides a model for an unbounded, dense (total) order.

First, time is a totally ordered domain: "before" and "after" correspond to "less than" and "greater than"; and "simultaneous" corresponds to "equals". Second, time is unbounded. There is no last memory which is fixed in advance; and "indefinite" corresponds to "potentially infinite". Third, time is dense. The infinity of time does not only consist in the fact that there is no last possible compartment of time, no last memory fixed in advance; in addition, the infinity exists between the memories, in virtue of our understanding that between any two instants there is a third. Any procedure describing this understanding or "feeling" must be indefinitely iterable in a way which produces a dense structure.

(2) The Synthetic Apriori and Space

In parallel with the synthetic apriori of time, there is a synthetic apriori element in our notion of space. Time and space are "the frames in which nature seems enclosed"[6] which we impose upon nature. Much of the apriori element of the "frame" of space is not synthetic according to Poincaré. For example, the choice of a geometry is conventional; and

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6 Poincaré, (1905a), p.13.

in additional contrast with Kant, the number of dimensions we attribute to space is not even synthetic apriori for Poincaré. Rather, it is partly a physiological, partly an empirical matter. It is partly determined by the way our sense of muscular movement corresponds with our vision, and obtaining, of objects; and it is partly determined by the way the world is: the nature of physical objects and light, etc. That the three-dimensionality of space is not an apriori condition imposed by the mind is indicated not only by the fact that we can reason on spaces of any given number of dimensions - i.e., that any dimensionality of space is consistent. (Alternative arithmetics are consistent; yet, for Poincaré, standard Peano arithmetic is synthetic apriori.) Rather, the dimensionality of space is not a synthetic apriori matter because we can construct viable empirical theories upon the hypothesis that space is, for instance, four-dimensional. Poincaré admits that spaces of greater than three dimensions are very much harder to work with than three-dimensional spaces. But he maintains that this does not indicate the apriori status of the knowledge of the three-dimensionality of space. If it did, then the same argument could be used to urge the two-dimensionality of space, as an apriori principle, since plane geometry is much easier than the geometry of three dimensions. Although three-dimensional space is most natural for us to imagine living in, a non-three-dimensional space is a viable candidate for an interpretation of our experience. Our visual analogies are merely less direct with four or more coordinates. Hence, according to Poincaré's definition of "synthetic apriori" - as something which forces itself upon us and constrains our thinking, and even our theoretical edifices - the three-dimensionality of space does not qualify. The degree of difficulty of imagining something like four-dimensional space, does not necessarily indicate a mental impossibility.

We attribute three dimensions to perceptual space because it is the description which makes our everyday generalisations (though perhaps not all of our scientific laws or theories) the simplest. Hence, it is a certain set of facts about the exterior world (aposteriori facts) which induces us to think of space in this way, and not apriori conditions imposed by the mind. The reason three-dimensionality may have seemed apriori to Kant is because increasing the number of dimensions beyond three seems unnatural and counterintuitive - it is hard or impossible to "picture". But, the very fact that we can, despite the increase in difficulty, have physical theories concerning spaces of differing dimensions, shows that the three-dimensionality of space is not something which constrains our thinking.

Here we see a consequence of the essential difference between the Poincaréan and the Kantian theory of the synthetic apriori. For Kant, the synthetic apriori is the form of perception, and it is that to which our perceptual pictures must conform. Space is then three-dimensional because he could not picture how living in four-dimensional space would be different perceptually. Whereas for Poincaré, the very fact that we can conceive of some creature (perhaps with a different type of eyes or different muscular sensations, or perhaps with _no_ differences) and some world for which space would be considered to have a different number of dimensions - two, four, five, etc. - reveals the fact that three-dimensionality is not an apriori characteristic of space. This is because Poincaré's requirement is stronger: the significance of the synthetic apriori is something which constrains our ability to _think_ (_and_ to perceive), and not merely our ability to perceive.

There is, however, an aspect of perceptual space which Poincaré considers to be synthetic apriori. This is the _continuity_ of space - the intuition which grounds geometric reasoning on spaces of _any_ number of dimensions. For Poincaré, this is the essence, the _sine qua non_ - of what he calls "spatial intuition". This is Poincaré's "form of spatial perception"; very much reduced from Kant's original position.

To sum up Poincaré's theory of the synthetic apriori, there are two types of apriori intuition, via which we instantiate our concepts in an apriori way. One I shall call Poincaré's "arithmetic" intuition. This is given in the concept of indefinite iterability, and in the order types which are produced by various procedures which are indefinitely iterable. Arithmetic intuition plays an essential part in our conception and characterisation of sets of numbers, e.g., the natural numbers, the rationals, and in induction. The other apriori intuition which plays a major role in the foundations of our mathematical knowledge I shall call Poincaré's "geometric" intuition. This is encapsulated in the concept of continuity or continuous variability. Geometric intuition allows us to understand by the real numbers a (classically) continuous domain; in addition, it is indispensable in disciplines like "analysis situs", or topology, where proofs can require a consideration of parameters which are allowed to vary continuously. Arithmetic intuition, and in particular the synthetic apriori status of the principle of induction, is the subject of Chapter 3, the next chapter. Geometric intuition will be addressed in Chapter 5, the subject of which is Poincaré's theory of the continuum.

CHAPTER THREE

THE ATTACK ON LOGICISM: ARITHMETIC INTUITION AND THE PRINCIPLE OF INDUCTION

According to Poincaré, the concept of indefinite iterability is given by a synthetic apriori "arithmetic" intuition which underlies all mathematical activity. On this view, the principle of induction is not an analytic consequence of the number-concepts, for the definitions of number <u>require</u> induction. Induction is a synthetic apriori principle because it is true of any domain which is a pure instantiation of the synthetic apriori iterative concept. It is important to note that Poincaré seems to equate the natural numbers, the principle of induction, and the concept of indefinite iterability. Though he does not regard them as equivalent, he asserts that they are of equal logical status - no one is logically prior, or more basic, than any other. They are each an apriori manifestation of arithmetic intuition. The concept of indefinite iterability is an apriori form of understanding which expresses arithmetic intuition; the natural numbers are an apriori pure instantiation of the apriori iterative concept; and the principle of induction is known to be true of the natural numbers directly, via arithmetic intuition. The subject of this chapter will be to examine and assess Poincaré's theory of apriori arithmetic intuition.

(1) Analysis of the Principle of Induction

Before Poincaré's time, induction was not a logical tool; although it has been employed by logicians since the development of modern logic after Aristotle. Rather, induction was essentially "mathematical". And though it is now shared with the logicians, it is still a numerical principle, for it hinges on the special successive, or ordered, character of numbers. The principle of mathematical induction can be taken as a second order axiom with the form,

$$\forall P\left[(P(0)\ \&\ \forall n(P(n) \rightarrow P(n+1))) \rightarrow \forall nP(n)\right].$$

For Kant this principle would be synthetic because in considering an element together with its successor, we must employ our intuitions concerning succession. And these intuitions are not "analytic" because our intuitions concerning succession are not present in the concepts of number, of P, but are only present in virtue of the apriori temporal form via which we understand the concepts. It is our apriori intuition of time which allows us to do a proof by induction, because in considering an arbitrary element, n, as a single object or unit within an indefinite collection of successive units, we are considering the temporal attributes of objects. Because we are considering an instantiation of the number concept - albeit an arbitrary instantiation - we must consider n as an object, and hence, as an object in time. And thus we are employing the apriori rules of synthesis of perception, of the imagination, and of understanding.

Just as for Kant, for Poincaré, too, induction is synthetic. However, it is not in virtue of temporal intuition that induction is synthetic. Rather, the principle of induction is a synthetic apriori principle because it is knowable only by virtue of our apriori arithmetic intuition. The principle is a direct consequence of our iterative concept. The important point is not that we must consider the arbitrary n as an object <u>in time</u>. Rather, Poincaré replaces the continuity of time or the temporal intuition with the more minimal concept of iteration. Thus, induction is synthetic because we must consider the arbitrary n as an <u>iterative</u> object, in an indefinite series of objects, the series being defined by an iterative rule of construction. Hence, the significance of induction lies in our special ability to consider n and n+1 (or n and S(n)) as pure instantiations of the iterative concept.

The conditional in the premise of induction, $\forall n\ (P(n) \rightarrow P(n+1))$, is a condition by which we are assured that the property, P, is instantiated in an iterative model. If R is an iterative generating rule, then it defines one element in terms of its predecessor. So that if on the assumption of $P(\alpha)$ - where α is the result of applying R an arbitrary number of times - we can show $P(R(\alpha))$, or $P(\alpha')$, then we know P is true of all the elements generated by R. We know this because α was only an arbitrary instance in the iteration of R. Thus, if P is true of any element generated by R; and if we can show the conditional $P(\alpha) \rightarrow P(R(\alpha))$ to be true; then we know P to be true of all iterations of R.

Essential to the proof of the conditional $\forall n\ (P(n) \rightarrow P(n+1))$, is the arbitrariness of the n. If the proof of $P(n) \rightarrow P(n+1)$ relied upon a particular property of n, then the inference via universal generalisation to $\forall n\ (P(n) \rightarrow P(n+1))$ would fail. It is via our arithmetic intuition that we are able to consider an arbitrary instance of an iteration, and thus that we know the principle of induction to be true. Via the principle of induction we make the leap from

$$P(0),\ P(S(0)),\ P(S(S(0))),\ \ldots,\ P(S^n(0)),\ \ldots$$

to

$$\forall nP(n)$$

(where n, or $S^n(o)$, is the result of applying the rule S (starting with 0) n times), and we can see that these two expressions are equivalent. We can see this, argues Poincaré, because induction, or pure arithmetic intuition, enables us to take the dots in the first expression seriously - in a non-metaphorical way - when the rule, S, is iterative.

(2) The Problem of Induction for the Logicists

The aim of logicism was to disprove Kant's thesis that mathematics has a content which is determined by our apriori intuitions of space and time. The method of showing this was to argue that all true mathematical statements could be proved via logic alone, without depending on any "extra logical" intuitions. At the time of the birth of the modern logicist programmes, i.e., for Frege, Russell, Whitehead, the principle of induction was not regarded as a logical principle. Thus in order to prove any significant mathematical results it was necessary first to prove that induction is true. The logicists needed to show that the principle of induction can be derived as a theorem of a suitably extended logic once the primitives zero, natural number, and immediate predecessor have been defined, together with an equivalent of Frege's axiom 5, the axiom of comprehension. Induction would then be a logical consequence of the logistic definition of number; i.e., that numbers are inductive, would be (for most logicists, though not for Russell, since he considered even logic to be a synthetic matter*) an analytic truth. And if all true mathematical statements could be shown, in a similar way, to be analytic too, then Kant's thesis that the content of mathematical statements concerns space and time would be disproven. Their content would not be space and time, because, since they are analytic truths, they have no (non-logical) content at all. Indeed, it was the stunning achievement

* Russell, (1903), p.434.

of Frege to conceive of such a suitably augmented system, and to show how it could be employed to derive induction as a theorem of logic.

The problem with the augmented system, however (in addition to the unfortunately too strong, and contradictory, nature of Frege's axiom 5), is that in order to even set up a system which is powerful enough for arithmetic, arithmetic principles must be invoked. Poincaré objects, for instance, to considering a typical formulation of the Peano Axioms as defining the concept of number by postulates. The logicist must be able to set up some system which is roughly equivalent to the Peano axioms. Poincaré gives the following formulation of the Peano axioms, and considers their efficacy in defining the concept of natural number.

1. Zero is a whole number.
2. Zero is not the successor of any whole number.
3. The successor of a whole number is another whole number; to which it would be convenient to add: each whole number has a successor.
4. Two whole numbers are equal if their successors are equal.
5. If <u>s</u> is a class which contains 0, and which, if it contains the whole number <u>x</u>, then it contains the successor of <u>x</u>, then it contains all the whole numbers.

This fifth axiom is the principle of complete induction.[1]

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1 Poincaré, (1905b), p.833.

However, because of the iterative nature of the axioms, they cannot be proved consistent without induction. But since induction is one of the axioms, any consistency proof would be circular. Iterative or recursive definitions specify one entity in terms of the previous construction (or iteration). In addition, there is usually no reason to stop the constructing process. The domain thus generated by an iterative rule is, in general, indefinite or potentially infinite. So we cannot directly verify that the axioms are consistent, i.e., provide a finite model of the axioms, for they imply an infinite number of arithmetical propositions.

Showing a definition or domain is consistent involves one of two approaches. Either we show it has a model in the system - an instance in which it is true; or, we prove that no contradiction follows from any propositions implied by the definition together with the totality of propositions of the system to which the definition is being added. When a structure is finite one can always (at least in principle) check or verify that each case of a definition is consistent with the previous structure. However, this is trivial and is not the situation here. In particular, one cannot verify (by checking instances) that an inductive domain is consistent, for the axiom of induction (5) concerns any class of natural numbers, s, and the collection of all such classes is infinite.

Hence, the difficulty with arithmetical induction is that it is, in a sense, "doubly" infinite. First, the range of the implicit class quantifier in 5 is infinite. Indeed, from a purely Platonistic, or extensional set-theoretic, point of view it is uncountably infinite, for its range is: all the subsets of the natural numbers, i.e., the Power set of $\mathbb{N}$. Because the possible instantiations of induction are not finite, we cannot verify by checking each possible case of induction (each possible property of natural numbers) to ensure its consistency. However, the other method of proving the consistency of a set of postulates - exhibiting a model - is also not available to us. For second, in addition to the infinity of the class-variable, the x-variable, too, has an infinite range of possible arguments. Given a specific class, s, the range of x is infinite. Hence, no model of the axioms is finite.

> The demonstration cannot be made by example. We cannot select a portion of the whole numbers - for instance, the three first - and demonstrate that they satisfy the definition.
>
> If I take the series 0, 1, 2, I can readily see that it satisfies axioms I, II, IV and V; but in order that it should satisfy axiom III, it is further necessary that 3 should be a whole number, and consequently that the series 0, 1, 2, 3 should satisfy the axioms. We could verify that it satisfies axioms I, II, IV and V, but axiom III requires besides that 4 should be a whole number, and that the series 0, 1, 2, 3, 4 should satisfy the axioms, and so on indefinitely.
>
> It is therefore, impossible to demonstrate the axioms for some whole numbers without demonstrating them for all.... [2]

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2 Poincaré, (1905b), p.833; quoted from Poincaré (1908), pp.165-166.

We cannot show the consistency of a definition of which an induction axiom, or an iterative constructing rule, is a part, when the domain concerned is infinite, without presupposing the truth of some principle equivalent to the principle of induction. We cannot show directly, in a finite model, that an instance of the Peano axioms is consistent; for with each finite sample, (0, 1, 2), we are "bundled along" by the recursive character of axiom III to a larger sample, (0, 1, 2, 3). Hence, the recursive character of the axioms, coupled with the fact that there is no reason to stop the iteration at a particular, finite n, produces a domain which is inaccessible to a formal consistency proof which excludes an inductive principle.

Poincaré's argument is not merely to object to the lack of a non-circular consistency proof for the logistic structure; nor is it trivialised by Gödel's incompleteness results (which came after Poincaré's death). He has a deeper point to offer, concerning the procedure, in general, of setting up any formal system. First, the logicists wanted to set up a system adequate for arithmetic; so the system had to have the capacity for expressing and implying an infinite number of arithmetical statements. Thus the characterisation, of what an arithmetic formula is, of what a number is, etc., had to be recursive. But then a prior understanding of the principle of induction is required, not merely in setting up the formal system, but in understanding why, for instance, Frege's derivation constitutes a proof of the fifth Peano axiom. The logicist is thus caught in a circle.

The circle is even more serious, however. It is not merely that in order to set up a system adequate for arithmetic - for statements about an infinite domain of objects, the numbers - one requires a prior understanding of induction. In addition, any non-trivial formal system at all requires an inductive intuition for its characterisation. In order to be able to understand the recursive definition of a well-formed formula, of something as simple and basic as this, an inductive or arithmetic intuition must be presupposed. Thus, Poincaré's point is not a mere technical one, but a very bold philosophical claim: <u>induction is epistemologically prior even to logic</u>. Induction is necessary, not only for understanding formal systems, but also for understanding logical structure in general. The logistic arithmetic formal system - astonishing achievement that it is - cannot be regarded as exhibiting the analytic or logical character of our knowledge of induction; because understanding logic itself - understanding the logical methods used to derive the system - presupposes a prior understanding of the principle of induction.

(3) <u>Some Attempts to Avoid the Circle</u>

One may indeed question the importance of epistemological priority, or the necessity of consistency proofs. The logicist, however, has no choice, since his project is to preserve the degree of certainty found in logic. All definitions must then be formal, and may presuppose the existence of no extra-logical intuitions or knowledge. And these definitions require a consistency proof, this being no trivial requirement. Definitions and postulates are not free. There is no way to ensure the acceptability of a definition other than by proving that it is a conservative extension of the system, i.e., is consistent. The problem with a definition which involves (or is strong enough to imply) induction, is that we cannot justify it (show we understand it) without using inductive means. So unless we have a prior grasp of the principle of induction and its consistency, such a proof establishes nothing.

Poincaré attacks logicism by exploiting this fact. Although his real quarrel is epistemological in nature, he argues for his position via a formal point concerning consistency proofs, or logical priority. (He had to do it this way because, whereas for him logic and epistemology are inseparable, for the logicist the true logical nature of grounding relations in a theory are independent of the epistemological question of the order in which we come to know aspects of a theory. Epistemological points would have been glossed over by the logicist; Poincaré saw that the only access for making his point, then, was in terms of a formal point, on the logicist's own ground.) The logicist project was not to assume the existence of any facts outside of logic: to define all arithmetic concepts and relations in terms of logical concepts and relations alone; and, in addition, to prove all the postulates of arithmetic as theorems of the augmented logic. Thus, when proving things about the new system - that its definitions are acceptable, consistent - the logicist can use only previously accepted logical principles and rules.

If showing the consistency of a new definition involves a non-logical principle, then it seems very dubious to consider such a new definition as presupposing only logical concepts. So that, in particular, if the intent of the logicist is to define the arithmetic concepts, metatheoretic proofs about the definitions can only be relative to a system which does not presuppose the arithmetic principles and concepts. One cannot employ, in the justification of a structure in question, one of the principles newly defined _in_ the structure. Otherwise, the justification is circular and no fact has logical priority. The logicist's circle lies in the fact that he cannot use induction until the property of being a natural number is justified; but he cannot justify this property without using induction.

The logicist may here attempt to avoid the circle in another way. He may concede that any consistency proof for the arithmetic concepts requires the principle of induction; but he might claim that the metatheoretic principle is not _the same_ as the object-theory principle;

and hence that the circularity is only apparent. No doubt the structure of the two principles is similar, but the subject matter in each is distinct.

Neither Poincaré nor Russell made a consistent employment of the use/mention distinction, which could perhaps be cited to argue that there are two different principles at work here: one (mathematical induction) concerns objects, and the other (metatheoretic induction) concerns propositions about the objects. However, this would only be fruitful for the logicist if he maintained that the second employment of induction, in the metatheory, is a logical principle. To use a metatheoretic version of a principle implies that the principle is acceptable on independent grounds - grounds which are independent of the particular project for which it is being invoked. So, for instance, the use of a metatheoretic Modus Ponendo Ponens (MPP) type of rule in the proof of the soundness of a logic which includes MPP as a rule in the object language, is circular, but acceptable. For the soundness proof is not intended to be a suasive argument for accepting the MPP rule. So long as we are not calling the validity of MPP reasoning into question, the circularity is thought to be acceptable (and exists in all Tarski-type semantics); for the justification is directed not towards a single use of MPP, which we have accepted on independent linguistic grounds, but more towards investigating the soundness of indefinite iterations, or the arbitrary use, of MPP.

The contrast with MPP is that the subject of the principle of induction _is_ indefinite iterations and arbitrary instances. This is why Poincaré objects that using induction in a metatheoretic proof of a formal system in which arithmetic induction is proved is non-acceptably circular: because he feels the principles are really the same. No doubt there is a difference in the overt subject matter of the two uses; but, for Poincaré, they are merely two different applications of the same principle.

> The possible applications of the principle of induction are innumerable. Take, for instance, one ... in which it is sought to establish that a collection of axioms cannot lead to a contradiction ...
>
> When we have completed the _n_th syllogism, we see that we can form still another, which will be the (_n_+1)th: thus the number _n_ serves for counting a series of successive operations; it is a number that can be obtained by successive additions ... Thus, then, the way we have been brought to consider this number _n_ involves a definition of the finite whole number.... 3

Induction is a numerical principle, and any application of it involves reference to counting, or to the successive ordering of objects so that they can be, in effect, counted. No matter what the subject matter - whether we count numbers or propositions - the domain must be iterative, orderable, so that it possesses numerical properties.

3 Poincaré, (1906a), pp.172-173.

And induction always refers to the <u>numerical</u> properties of the subject matter - in metatheoretic proofs, for example, to the <u>number</u> of steps in an arbitrary proof; or to the <u>degree</u> of complexity of an arbitrary wff and its successor (the next most complex iteration of a formation rule) - in conjunction with the properties which the objects have in virtue of their overt subject matter (in virtue of properties which the objects have <u>qua</u> steps in a proof, or <u>qua</u> well-formed-formulae). Induction on the number of steps in a proof depends upon our being able to consider an arbitrary step of a proof - and thus upon our being able to consider the steps of a proof in an ordered, successive way. And induction on the degree of complexity of a wff depends upon our being able to define "degree" in a successive numerical way. The principles are epistemologically equivalent.

So, for instance, when we do an induction on the complexity of a wff, we can assert no more than that we have good <u>arithmetical</u> reasons for supposing that: if after n steps no contradiction has occurred; and if, by following the procedures as specified, we can prove that after n+1 steps no contradiction will ensue; then, by induction on the complexity, or <u>number</u> of logical constants in the wffs, we can infer that the collection of all wffs which are constructed in accordance with the procedures specified, will be consistent.

An understanding of recursion schemas is necessary for the building of any infinite formal system. Thus, an understanding of some form of the principle of induction is likewise necessary, if we are to understand by the definition schemas that they determine an infinite domain. It is in this sense that the circle is unavoidable for the logicist: an arithmetic principle, the principle of induction, is prior even to logic, for it is necessary for understanding the logical methods employed in setting up and deriving truths about an infinite structure.

(4) The Second Order Principle

Let us consider the thesis of the circularity inherent in any definition of the numbers from another standpoint. Presumably what the logicist really needs is the second order principle of induction,

$$\forall P[P(o) \ \& \ \forall x(P(x) \rightarrow P(x+1)) \longrightarrow \forall x \ P(x)],$$

as a theorem of logic. The reason he needs this is that his thesis is that arithmetic truth as we know it is really logical truth. And thus, he will want his system to denote the standard interpretation of arithmetic. That is, his system ought to exclude the non-standard, i.e., the non-arithmetic, interpretations. Since the first order principle seems to admit too many interpretations, the logicist requires the second order principle in order to capture all and only arithmetic truths.

However, asserting the second order principle means, for the logicist, that some conception of an arbitrary property of the natural numbers must be obtainable from logic alone. In other words, because the "$\forall P$" ranges over all possible properties of the natural numbers, the logicist must find, on purely logical grounds, a characterisation of an arbitrary subset of an infinite set. Perhaps it is possible to extend the notion of logic so as to include quantification over properties as well as objects.[4] However, such an extension is generally taken to imply an extension of the notion of logic into the transfinite: into the theory of types and order, or set theory. And while this may be possible, and even reasonable on certain grounds, there is an unavoidable circularity in so doing. We will have used logic to fix the concept of an arbitrary subset, but we will have, in essence, used set theory to "correct" or extend logic.[5] The comprehension principle might, as Frege thought, be a logical principle, but alas it is a logical falsehood.

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4 See Wright, (1983), especially Section xvii (Chapter 4), for arguments in favour of such an extension.

5 Scott, (1985), pp.vii-viii, makes a parallel point in the following quote, where he is discussing the intuitive arguments for the set theoretic axioms:

> When we come to the Axiom of Choice, we begin to waver: it might be argued that it is implicit in the concept of the totally <u>arbitrary</u> set. On the other hand, there could be other notions of what it means to <u>determine</u> a set for which it would fail; thus, the act of assuming it is indeed axiomatic: it is 'self-evident' but not just a matter of logic. But then, perhaps it <u>is</u> a matter of logic after all, because the <u>finite</u> version is provable. In other words, first order logic is strong enough for some conclusions, but it is in general too weak: we ought perhaps to allow 'infinitary'

As Poincaré well knew, the problem is one of the characterisation of the class of well-founded properties (or extensions). But such a characterisation must be drawn from elsewhere (other than "logic"), if the circularity is to be avoided. For Poincaré, the source is the form of intuition - well-founded, set-theoretic properties must have a content which is instantiable in arithmetic intuition - and the status of our knowledge is synthetic apriori.

(5) Non-Inductive Arithmetic

Poincaré wishes to do more than point out the circle in the various forms of logicism. He has a positive thesis as well. For him, induction is true of the world because the principle of induction is true, and not just true in a model. We have no choice but to consider the numbers as inductive, because the principle of induction cannot coherently be rejected.

> Let us next try to get rid of this [the principle of induction], and while rejecting this proposition let us construct a false arithmetic analogous to non-Euclidean geometry. We shall not be able to do it. 6

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inferences also. And at this point we begin to wonder what is meant by logic. It would seem rather circular if in making set theory precise, we had to use set theory in order to make logic precise.

6 Poincaré, (1891), p.49.

However, this is simply false. We can do it. Peano Axioms (PA) 1 through 4 do not entail 5, the principle of induction; so by the completeness theorem there exist models of formal systems which satisfy PA 1 through 4 plus some form of the negation of axiom 5. So Poincaré was wrong: there is a parallel between axiomatic (Euclidean) geometry and Peano arithmetic. The parallel postulate being independent of Euclid's other axioms, we can construct non-Euclidean geometries; and the induction postulate being independent of Peano's other axioms for arithmetic, we can construct a consistent non-Peanian (or non-inductive, or "false") arithmetic based on some form of the negation of induction.

Geoffrey Hunter[7] points out this formal parallel in an argument to vindicate Kant's view of the synthetic apriori character of Euclidean geometry. Since, he argues, alternative arithmetics are conceptually absurd, it is coherent for Kant to have maintained that non-Euclidean geometrics are, likewise, conceptually absurd or impossible, because mere consistency obviously does not prove "real possibility".

However, the parallel Hunter points out may be employed to argue the opposite thesis. That is, since alternative geometrics are a real possibility, the formal parallel between Peano arithmetic and Euclidean geometry is evidence

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7 Hunter, (1980).

for the real possibility of alternative (non-Peanian, non-inductive) arithmetics. The question turns on just what is meant by "real possibility". For both Kant and Poincaré, mere consistency is not equivalent to real possibility, as Hunter emphasises. For Poincaré, however, the analogy between arithmetic and geometry breaks down at precisely this point.

The breakdown in the analogy between arithmetic and geometry can even be shown formally. Whereas the consistency proofs for non-Euclidean geometries occur in arithmetic strictly independent of any one particular geometry, we must use arithmetic in the consistency proof for arithmetic, standard or non-standard. That is, induction must be assumed elsewhere in our formal theories, in order to show that the negation of induction produces a consistent formal structure. Whereas, in contrast, we do not employ the parallel postulate in proving that a non-Euclidean geometry is consistent. Induction cannot be given up <u>in the same way</u> as the parallel postulate can be given up; it is part of the scaffolding of our formal thinking. Thus, the analogy between non-inductive arithmetics and non-Euclidean geometries is superficial.

The analogy breaks down informally too. "Real possibility" means, for Poincaré, that a coherent account of experience is possible. Non-Euclidean geometry is really possible because a different metric geometry, Riemannian say, is a real candidate for the interpretation of our experience. A non-Euclidean world might be different from our world, but there is nothing in the nature of coherent experience which rules out the possibility of experience of a non-Euclidean metric character.[8] The reason this is so, for Poincaré, is that non-Euclidean geometries - the different, really possible, metric geometries - are, still, continuous: they still accord with apriori geometric intuition. And this is why there is nothing in the nature of coherent experience which rules them out: because they still have a content which is instantiable in our apriori form of understanding.

The contrast with non-standard arithmetics then, lies, for Poincaré, in the fact that they are transgressed by apriori arithmetic intuition; they do not accord with our apriori form of experience. We cannot provide an account of experience of a non-inductive arithmetic world, because non-inductive arithmetic is ruled out by the nature of coherent experience. A non-inductive world is ruled out, because it transgresses apriori arithmetic intuition, which is one of the (apriori) factors which <u>determine</u> the nature of coherent experience. To be sure, non-standard arithmetic is consistent; and we can easily construct a model which shows this: simply by adding a non-inductive

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8 Helmholtz originally argues this point. See, especially, "On the Origin and Significance of the Axioms of Geometry", (1870), and "On the Facts Underlying Geometry", (1868).

(non-successive) number to the set $\mathbb{N}$, of natural numbers. However, since mere consistency does not prove "real possibility", Poincaré can maintain that while arithmetic and geometry are parallel with respect to the formal consistency of their non-standard theories, the parallel does not extend to the real possibility of their non-standard theories. And the reason this is so is also the reason why, for him, metric geometry is not a synthetic apriori matter: because no one, particular metric geometry is fixed by our apriori geometric intuition (of continuity). Whereas, in contrast, apriori arithmetic intuition forces or imposes the standard (inductive) interpretation of arithmetic, thereby ruling out the possibility of a coherent "non-inductive" (mathematical) experience.

(6) The Synthetic Apriori Nature of Arithmetic Intuition

For Poincaré, a synthetic apriori arithmetic intuition must exist; otherwise, we have no access to the cluster of concepts: indefinite iteration, (potential) infinity, "and so on", "etc.", the principle of induction. The world does not provide us with an indefinite iteration; indeed the world falsifies the actual (aposteriori) instantiation of the concept: i.e., we die.[9] So we do not

9 See Poincaré, (1893), p.22; see also the discussion on the apriori nature of the concept of continuity in Chapter 5.

acquire the concept of indefinite iteration by ostension, for, being finite, we cannot perceive an indefinite iteration.

Further, we cannot formally exhibit the concept, either. Any formal explication of a concept which denotes an iterative process, will necessarily involve a procedure which is itself iterative. That is, in order to understand the explication in the appropriate, "intended" way, we must already possess the concept which is being explicated.

> The idea of infinite divisibility or denseness is not capturable by a formula or sentence, but only by an intuitive procedure that is itself dense in the appropriate respect. 10

An intuitive understanding of indefinite iteration is necessary for the intended interpretation of quantified formulae, such as "$\forall x \exists y$ ...". That is, a prior intuitive understanding of the numbers is necessary for the proper interpretation of the logical formulae, in quantified logic, and by which the logicist defines the numbers. It is in this sense that our arithmetical understanding is prior to, and indeed, foundational for, our understanding of quantified logic - and hence for any definitions which involve such quantifiers, as well as for metatheoretical proofs. We cannot <u>acquire</u> the relevant concepts (indefinite

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10 Friedman, (1985), p.469.

iteration, succession, etc.); and yet they are necessary for experience as we know it.

Hence, the very possibility of the natural numbers - the very existence of the concept of "indefinitely" which is inherent in the notions of "+1", of "and so on", of "..." - shows that such an understanding must be apriori. This understanding is not what is extracted from our concepts (of "+1" etc.). That "there is no reason for stopping" is not part of the concept of the successor function. Rather, understanding by certain collecting rules that they determine infinite sets is something we put into our concepts. It is a fact about us, and hence, it is a fact which conditions what we count as understanding certain rules. The "true" understanding comes from the form of our "pictures" - that which shapes our concepts, and to which our concepts necessarily conform by an "active synthesis" (in Kantian language) of the mind. That is, that such an understanding of certain rules is the true or proper one, is a fact about our interpretation of concepts, and not about the content of the concepts themselves.

And it is in this way, moreover, that induction - though true of the numbers - is not part of the number-concept; and thus, is not analytic. Rather, that numbers are inductive is our most natural interpretation - indeed, the only "true" interpretation - of the concept of number, because the numbers and the principle of induction are both a pure manifestation of apriori arithmetic intuition. Knowledge of induction is apriori, because it

> ... is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act when the act is once possible. The mind has a direct intuition of this power, and experiment can only be for it an opportunity of using it, and thereby becoming conscious of it. [11]

And it is because our ("most natural") interpretation of many other concepts and theories is a consequence of the same form of experience as that which imposes induction, that rejecting arithmetic induction is not a coherent possibility, with respect to an indefinite portion of our scientific and linguistic (or logical) practice. Because arithmetic intuition is a form of experience, it has consequences in domains which are not strictly mathematical. The principle of induction is thus a necessary aspect of the numbers, for induction expresses the apriori form of experience in general, via which we determine what is to count as a coherent experience, of which the standard arithmetic numbers are a pure model.

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11 Poincaré, (1894a), p.13.

CHAPTER FOUR

POINCARÉ'S THEORY OF INTUITIONS

(1) Poincaré's Conception of Logic: is it a Mere Misconception?

(2) Russell's Logicism Does Not Refute Kant

(3) Intuitions and Poincaré's Theory of "Glossing Over"

(4) Intuitions and Poincaré's Theory of Definitions

(5) Set Theory and Intuitions

Poincaré is often called a "pre-intuitionist" or a "semi-intuitionist" in virtue of the fact that he contributed to the foundations of the school of thought called "Intuitionism". He certainly wielded a very strong theory of intuitions in his arguments against both the logicist and the Platonistic aspects of set theory. And yet, at times, he advocated non-constructive methods, even on undecidable domains, a practice which the intuitionist would, in general, reject: hence he is only "semi"-intuitionistic. My aim in this chapter is to show, via an exploration of his use of the term "intuition", how Poincaré's theory of intuitions is a very strong expression of the extent to which he was influenced by Kant, and the extent to which his philosophical writings were largely devoted to defending Kant's thesis of the synthetic apriori character of mathematics against the new challenges of logicism and set theory. I begin with an examination of Poincaré's conception of logic.

(1) Poincaré's Conception of Logic: is it a Mere Misconception?

One could use various remarks Poincaré made about the nature of logic to argue that his conception of logic is a misconception, and thus that his arguments against logicism ought to be dismissed. For the same reason that Russell dismisses Kant's theory concerning the spatio-temporal foundation of mathematical knowledge - i.e., due to his ignorance of modern logic[1] - Poincaré's theory can likewise be dismissed. The four following remarks are arguably simply four mistakes. First, Poincaré claims that the success of logicism would entail the emptiness of mathematics.

> The very possibility of mathematical science seems an insoluble contradiction ... If ... all the propositions which it enunciates may be derived in order by the rules of formal logic, how is it that mathematics is not reduced to a giant tautology? 2

Second, even if the logicist admitted that he needed some axioms or first principles, this still trivialises mathematics; for given the axioms,

> it seems that a sufficiently powerful mind could with a single glance perceive all its truths. 3

Third, his idea of what logic is, would exclude from logic a certain type of definition which is a necessary part of mathematics. For Poincaré, defining "by recurrence" is peculiarly non-logical in character.

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1 Russell, (1903), p.4.

2 Poincaré, (1894), p.1.

3 Poincaré, (1894), p.3.

> It is of a particular nature which distinguishes it even at this stage from the purely logical definition; the equality [x + a = (x + (a - 1) + 1)], in fact, contains an infinite number of definitions, each having only one meaning when we know the meaning of its predecessor. 4

Fourth, and last, Poincaré seems to think that since the theorems of a formal mathematical theory form a recursively enumerable set, mathematics would thereby be trivialised by a strict formalisation in the logicist spirit.

> What strikes us first of all in the new mathematics is its purely formal character ... in order to demonstrate a theorem, it is not necessary or even useful to know what it means ... we might imagine a machine where we should put in axioms at one end and take out theorems at the other ... It is no more necessary for the mathematician than it is for these machines to know what he is doing. 5

Let us consider the third point, about the "recursive" character of mathematical definitions, first. The logicist may object to Poincaré's remarks, that disallowing recursive definitions from logic would be to <u>beg the question</u> against logicism. There is nothing contrary to "logic" in such definitions; and the fact that Poincaré assumed that these specifications "transgress" the boundaries of "logic proper", just reveals how limited his account of logic was. The second and the fourth remarks go together. They each seem to involve the mistaken view that logic is trivial.

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4 Poincaré, (1894), p.7.

5 Poincaré, (1905b), p.147.

Poincaré is often saddled with the following argument: logic is decidable; if mathematics is reducible to logic, then mathematics would be decidable; but mathematics is not decidable; hence, mathematics is not reducible to logic. This is a correct form of argument, but the first premise is false, for as is well known, even first-order logic is not decidable. However, this is certainly not Poincaré's argument. He never mentions "decidability" - his comments concern the mechanical nature of deduction. What he is really objecting to is much deeper and more global than a mere objection to logicism, that the essence of mathematics is the systematic deduction of consequences of postulates. It is essentially a matter of emphasis. Poincaré is interested in <u>why</u> some postulates are accepted and not others. What concerns him again and again is mathematical insight, creative mathematics, and not simply the generation of results on the basis of postulates. This is central to Poincaré's thought and should not be lost sight of.

Finally, regarding the first point, if logicism (i.e., Frege's logicism) had been successful, then mathematics <u>would</u>, in a sense, be empty, for there would be no synthetic, no "extralogical", content in our mathematical statements. Theorems would be analytic truths because they would be provable via logic plus stipulated conventions alone.[6]

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6 This is the classic form of logicism, so pellucidly presented by C.G. Hempel in his famous paper, "On the Nature of Mathematical Truth"; Hempel, (1945).

However, it is extremely important to note that the "emptiness" of the logicist reconstruction of mathematics of course depends upon one's theory of the relation between logic and the analytic/synthetic distinction. Now, the Russellian thesis of logicism began with a theory of the syntheticity of symbolic logic - so the success of his programme would not have indicated the analytic character of mathematical truth. Rather, surprisingly, it would have indicated its synthetic apriori character.[7]

Regardless of such labels, however, Russell believed he had refuted Kant's thesis concerning the spatio-temporal character of the foundation of mathematical knowledge.

> The proof that all pure mathematics, including geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy ... The whole doctrine of apriori intuitions, by which Kant explained the possibility of pure mathematics, is wholly inapplicable to mathematics in its present form. 8

And in Chapter 1 of The Principles of Mathematics, Russell makes clear that one of the aims of his book is a refutation of Kant.

> There was, until very lately, a special difficulty in the principles of mathematics. It seemed plain that mathematics consisted

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7 "Kant ... rightly perceived that [the propositions] of mathematics are synthetic. It has since appeared that logic is just as synthetic as all other kinds of truth; but this is a purely philosophical question, which I shall here pass by." (Russell, (1903), see p.434).

8 Russell, (1901), p.96.

> of deductions, and yet the orthodox accounts of deduction were largely or wholly inapplicable to [insufficient for] existing mathematics ... In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, <u>i.e.</u>, the apriori knowledge of space and time. Thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable of a final and irrevocable refutation ... The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age ... [9]

Now we may pose the following question: is it right to dismiss Poincaré's arguments against logicism for the same reason that Russell dismisses Kant's arguments against Leibniz, i.e., on the basis of his (Kant's) ignorance of modern logic?

(2) <u>Russell's Logicism Does Not Refute Kant</u>

Actually, Russell never refuted Kant. It was his view that the truths of mathematics are nothing over and above the truths of symbolic logic. Mathematical truth - just like logical truth - is not mind-dependent; it does not concern our knowledge of space and time; it is simply true.[10]

> Throughout logic and mathematics, the existence of the human or any other mind is totally irrelevant; ... the subject

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9 Russell, (1903), Section 4.

10 Hylton, (1986), p.9.

> matter of logic ... would be equally true if there were no mental processes. It is true that, in that case, we should not know logic; but our knowledge must not be confounded with the truth we know ... a truth and the knowledge of it are as distinct as an apple and the eating of it. 11

Russell rejects epistemological issues as irrelevant. He takes the truths of logic - though perhaps "synthetic" - as independent of space and time. Thus, showing mathematics to be reducible to logic would be to show that mathematics, too, is independent of space and time, thus refuting Kant.

The enrichment of logic which is necessary in order to obtain mathematics is a logic which is as powerful as set theory. For Russell this is the system of Principia Mathematica (PM). This augmented system, however, requires an axiom of infinity and principles which are at least as powerful as the set-theoretic power set axiom and the axiom of choice. Thus, Russell's argument must be that these axioms are truths which are independent of our knowledge of space and time in order to be justified in considering them as part of logic. So the issue now hinges on how far these additional postulates can be regarded as independent of space and time.

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11 Russell, (1904), p.259.

Let us take, for example, the axiom of infinity. Kant argues in the antinomies that certain concepts lead to contradictions, and one of these is the concept of a completed infinity (the famous first antinomy). Russell's argument against Kant's first antinomy consists in noting that (though) an

> enumeration of an infinite series is practically impossible. But the series may be none the less perfectly definable, as the class of terms having a specified relation to a specified term. It then remains a question, as with all classes, whether the class is finite or infinite; and in the latter alternative, as we saw in Part V, that there is nothing self-contradictory. 12

Russell construes Kant as asserting the self-contradictory nature of the <u>concept</u> of a completed infinity. This is, however, a misconstrual. Kant explicitly admits that the concept of a completed or actual infinity is consistent (is possible in the intellect). What he denies is that the completability of the infinite is anything like a logical principle. For it to be a logical principle, it would at least have to be true of our world that it contained a completed infinity. But for Kant this is precisely what is untrue; for the notion of a completed infinity results in a contradiction when it is applied to the totality of our experience.

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12 Russell, (1903), p.435.

> Since <u>unrepresentable</u> and <u>impossible</u> are commonly treated as having the same meaning, the concepts both of the <u>continuous</u> and of the <u>infinite</u> come to be rejected by large numbers of people. For indeed, <u>according to the laws of intuitive cognition</u>, any representation of these concepts is absolutely impossible. 13

However, this is just to say that there is a

> lack of accord between the <u>sensitive</u> faculty and the <u>intellectual</u> faculty ... <u>the abstract ideas which the mind entertains when they have been received from the intellect very often cannot be followed up in the concrete and converted into intuitions</u> ... And this <u>subjective</u> resistance is, as frequently, no true indication of any <u>objective</u> inconsistency ... 14

The objection to notions like "actual infinity" is not that they are logically impossible, since there is no internal inconsistency in the concept of the actually infinite. Rather, Kant's argument is that certain concepts cannot hold without contradiction when applied to the totality of experience.

Now, Russell's concern is with logical truth; but for Russell, logical truth did not have its modern interpretation of "true in all possible worlds". It carried, rather, the sense of simply true. In particular, then,

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13 Kant, (1770), p.48.

14 Kant, (1770), p.49.

the axiom of infinity must be a truth about our world, i.e., it must be true. So before Russell can claim to have "refuted" Kant he must argue that the axioms of the augmented system PM are true. Thus, for example, he owes Kant an argument that the notion of a completed infinity is satisfiable in our world. But he does not supply us with such an argument. His remark concerning the "mere medical impossibility" of our completing the infinite[15] simply begs the question against Kant.[16]

Now, there do exist arguments concerning the status of our knowledge of the infinite which might be employed to support Russell. For example, Harold Hodes (1984) argues that the concept of infinity is available in logic, in modal logic. So that even if our actual world is finite, mathematical statements may still be true in some richer or bigger world which is accessible to our world. There is always a world with more objects in it than our world. The truths of statements about infinite domains are, thus, truths of a suitable modal logic; and our understanding of such truths is to be explained in terms of possible world semantics. And Nicolas Goodman

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15 See Dummett, (1977), pp.59-60.

16 "It is a most certain empirical fact that the mind is not capable of endlessly repeating the same act. Even apart from the fact that man is mortal, he is doomed to intervals of sleep ..." (Russell, (1904), p.259.)

(1984) also argues that truths about infinity are available in modal logic, where on his account, the accessibility relation is one of "knowability in principle" - the totality of true mathematical statements being those which are knowable (or provable) by an "ideal mathematician". Like the extendability in the heuristic of richer worlds, the capacities of the ideal mathematician are always extendable: he can always perform more acts, compute faster, he never sleeps, etc.

Both of these accounts supply a notion of potential infinity. But this is insufficient for Russell's purposes, for his theory requires a notion of actual infinity. Moreover, while Poincaré and Kant argue against a belief in actual infinity, neither objects to a notion of potential infinity as satisfiable in the totality of our experience. What they would object to, on the other hand, is the assumption that the semantic notions in the above, modern accounts - the notions of increasingly richer worlds and knowable in principle - are self-explanatory. It would be their view that in order to understand these semantical notions we need to already possess a notion of potential infinity. That is, the notion of increasingly richer worlds only supplies a model for the mathematical concept of potential infinity, in virtue of the apriori iterative intuition. For, in Poincaré's view, it is only in virtue of this intuition that we interpret such iterative concepts in the standard way. But the iterative intuition directly supplies us with an understanding of potential infinity. So any explication of infinity via modal logic does not provide an epistemological source of <u>even potential</u> infinity, which can be seen as purely logical in nature, for an understanding <u>of</u> the required logical conception (the box, "$\Box$", plus the heuristics designed to explain this symbol), requires "extralogical" intuitions.

(3) <u>Intuitions and Poincaré's Theory of "Glossing Over"</u>

Poincaré argues not only against the logical status of the existential <u>axioms</u> which are necessary to obtain mathematics from logic. He also objects to the status of the <u>machinery</u> of the augmented system. That is, he agrees to the "arithmetisation" of mathematics insofar as this is possible (see Chapter 5, below); but he does not agree to the reducibility of arithmetic concepts to logic. We must not imagine that merely by revealing that certain concepts and methods <u>are</u> reducible to logic, we have thereby shown that they are essentially, and were all along, purely logical concepts and purely logical methods. Poincaré points out that mere reducibility to logic would not - even if it were successful - reveal the epistemological character and epistemological source of the so-reduced concepts.

> Even if they [the logicists] had been entirely successful, would the Kantians be finally condemned to silence? ...

> Even admitting that it has been established ... the philosopher would still retain the right to seek the origin of these conventions [or postulates], and to ask why they were judged preferable to the contrary conventions. 17

The source of our understanding of both the concepts, reduced and unreduced, and of the methods employed in the augmented logicist system lies, according to this argument, in the form of intuition. In this section we shall explore why Poincaré holds the view that the epistemological source of both logical and mathematical knowledge is (apriori) spatio-temporal intuition.

Poincaré's theory is a version of Kant's thesis that it is a consequence of the kind of beings that we are - i.e., finite - that without an active faculty which lays down the form of experience, we could not think (in the way that we do) or communicate (to the extent that we can). This is the epistemological root of our mathematical abilities.

> Does the harmony the human intelligence thinks it discovers in nature exist outside of this intelligence? No, beyond doubt a reality completely independent of the mind which conceives it, sees or feels it, is an impossibility. A world as exterior as that, even if it existed, would for us be forever inaccessible. But what we call objective reality is, in the last analysis, what is common to many thinking beings, and could be common

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17 Poincaré, (1905b), p.148.

> to all; this common part, we shall see, can only be the harmony expressed by mathematical laws ... [18]

What is "common to many thinking beings, and could be common to all", what is objective, is the harmony established by mathematical laws and relations, because these stem only from our apriori form of perception. Mathematical knowledge is "common to many thinking beings", but not to all, because only some thinking beings employ the apriori form of perception to this extent. And mathematical knowledge "could be common to all" thinking beings, because all thinking beings have the potential to do mathematics. Hence, it seems that Poincaré must say that the apriori form of perception which enables us to do mathematics is a necessary aspect - a defining condition - of any finite thinking being. (I insert "finite" here to distinguish "thinking being" from a god-like being with infinite powers of surveillance, concentration, etc., which is obviously not Poincaré's interest.)

Let us explore this notion further. The claim is that it is a consequence of the fact that we require an apriori synthesising faculty to make sense of perceptual experience, that we can do mathematics. And thus "our need of thinking in images"[19] is common to both our thinking about empirical objects and our thinking about mathematical objects. In

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18 Poincaré, (1905a/1946), p.209.

19 Poincaré, (1889/1908), p.131.

both domains our actual data can never be "complete", can never exhaust all the possible data. A finite being cannot perceive all possible aspects of an empirical object; and thus, he can never obtain the (extensional) "thing-in-itself", the "noumenon". Analogously, a finite being cannot construct an (extensional) infinite domain of mathematical objects, whereas there is no end to the number of possible iterations of certain rules. And yet we group together our perceptions, and we understand by this group a unified whole, an object. Moreover, we all do this in roughly the same way, so that experience is ordered and common. Analogously, we are able to "construct" potentially infinite mathematical sets and perceive them as determinate objects. And we see them as determinate potential infinities because we understand by the rule of construction that there is no end to the number of possible iterations of the rule.

Apriori intuition - the form of experience - is that via which, despite the inevitable incomplete character of experience, we understand by our experience an experience of a completed object, we understand by a rule that it characterises an infinite, yet determinate, collection. Apriori intuition can thus be regarded as a "glossing over" faculty, whereby we "gloss over" the incomplete character of both empirical and non-empirical (mathematical and linguistic) experience. It is a procedure whereby we <u>ignore</u> all the elements which could be generated by a rule, and we disregard or "smooth out" the disparate character of perception:

> it is therefore necessary that by an active operation of the mind we agree to consider two states of consciousness as identical by disregarding their differences. 20

And we all do this in (roughly) the same way. It is in virtue of the fact that we "complete our pictures" and satisfy "our need of thinking in images" in a systematic, uniform way that we are able to communicate and that we possess a mathematical concept of set.[21] For both Poincaré and Kant the glossing over faculty, or the apriori form of experience, is what is common to all finite thinking beings. It thus potentially provides an explanation of why we can communicate, why our concepts "overlap" and meaning can be determinate, even though our actual perceptual data is always incomplete in different ways: because qua finite thinking beings we all gloss over the data and complete our "pictures", or attribute determinacy to our concepts, in roughly the same way.

Even something as straightforward as the understanding which is acquired in learning to apply the rules of

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20 Poincaré, (1912b), p.31 (his emphasis).

21 Hallett, (1983) uses the notion of "smoothing out" instead of "glossing over" to explain the same aspect of Kant's theory, and to apply it to Gödel's understanding of the set concept. The concept of unity and that "of enduring object is used to smooth out an otherwise diverse and complicated manifold of representations", (p.19) and it is that which supplies our mathematical concept of set. (p.24)

elementary formal logic requires intuition, on Poincaré's account. In order to understand the abstract characterisation of a rule we must understand an arbitrary instance of it. And this is an intuitive, structural ("pictorial") understanding. Applying a rule requires that we see that the application possesses the same essential "structural" properties ("shape") as the arbitrary instance given in the schematic characterisation of the rule. The aspects which are structural are those properties which an arbitrary instance possesses. Apriori intuition supplies a uniform way to understand what these are.

For example, to see that

$$(P \vee Q) \;\&\; ((R \& S) \vee T) \vdash ((P \vee Q) \;\&\; (R \& S)) \vee ((P \vee Q) \;\&\; T)$$

is true, it is sufficient to see that it is an instance of the law of Distributivity:[22]

$$A \;\&\; (B \vee C) \dashv\vdash (A \& B) \vee (A \& C).$$

And to see that the first is an instance of the second, we must be able to "gloss over" the inessential differences, and see that the two formulae have the same overall structure: that "PvQ" can be taken as the "A"; that "R&S" can be taken as the "B"; and that "T" can be taken as the "C". Understanding the rules of formal logic is a pictorial

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22 Lemmon, (1971), pp.62-63.

understanding, insofar as doing a proof is picturing and assimilating various instances of a class of structures. We make certain strings of symbols "look" the same by disregarding or glossing over the actual symbols used, which are inessential to its being an instantiation of an arbitrary instance.

For Poincaré, "glossing over" or "smoothing out", which is imposed apriori by our forms of experience and understanding of an object, supplies organisation (via a concept of enduring object and unified whole) to our experience of the world. In addition, it supplies both an understanding of the mathematical concept of set, and an understanding of the notion of an arbitrary object. This results in two mathematical theses. One is that only potential infinities can be "constructed" sets (the one exception being the "set" of real numbers which is a primitive and immediate object of intuition, supplied by our apriori "geometric" understanding of continuity). Collections must be finite or rule-generated; otherwise, without a rule, there is nothing for apriori intuition to gloss over. (This is where Poincaré's notion of glossing over is different from Hallett's account (1983) of Gödel, who, he claims, has access to the full, classical, set-theoretic universe in virtue of some such smoothing out faculty.) The incompletability of the infinite is thus a direct result of the in principle incompletability of any experience of an object. For Poincaré and Kant, the potentially infinite thus describes the limits of our understanding, for it is a consequence of the theory of the synthetic apriori applied to understanding.

The other thesis which is a consequence of Poincaré's theory of the synthetic apriori is that whenever a notion of arbitrary object intrudes, or is necessitated, so does apriori intuition. A "glossing over" is necessary in order to employ our notion of arbitrary instantiation; and this glossing over faculty is not part of logic. It is not characterisable by logic, and, indeed, it is presupposed by logic. Thus, in a deep sense, an understanding of logic presupposes an understanding of arithmetic (as was shown in Chapter 3, above). But also, in a more explicit sense, quantification theory - since it directly and explicitly employs a notion of arbitrary instance (in the rule of universal generalisation) - requires apriori intuition in its glossing over capacity. Therefore, even if Russell's logic is accepted as logic proper, employing this machinery or method requires apriori intuition, according to Poincaré. Thus any "reduction" of mathematics to such a logic does not show that Kant was wrong - that mathematics is independent of spatio-temporal intuition - for this machinery is founded on apriori spatio-temporal intuition.

(4) Intuitions and Poincaré's Theory of Definitions

Poincaré was in favour of the formalisation of mathematics; [23]

23 "Intuition cannot give us exactness, nor even certainty, and this has been recognised more and more." (Poincaré, (1889/1908), p.123.)

yet he felt that intuition ought not to be entirely banished from mathematics; since, on his account, intuitions are the epistemological source of even the formal, precise concepts and methods. In addition, it was his view that in order to apply mathematical results we require rough, intuitive "pictures", in order to be able to compare the mathematical objects in our theorems and definitions with the empirical objects of our scientific problems. Being an applied (as well as a "pure") mathematician himself, Poincaré felt strongly about the necessary usefulness of mathematics:

> The eternal contemplation of its own navel is not the sole object of the science. It touches nature, and one day or other it will come into contact with it. Then it will be necessary to shake off purely verbal definitions and no longer to content ourselves with words. 24

The logicist divorces the truth of mathematics from the forms of experience; but in so doing, he also divorces the content of mathematics from our experience of the world. The formal objects of logicist mathematics are unable to bridge the gap between symbol and reality; thus the intuitive, pictorial notions which are epistemologically at the foundation of the formal concepts and methods are also pragmatically indispensable for the employment of mathematics.

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24 Poincaré, (1906b), p.183.

> It is through [intuition] that the mathematical world remains in touch with the real world, and even if pure mathematics could do without it, we should still have to have recourse to it to fill up the gulf that separates the symbol from reality. 25

Poincaré's theory of mathematical definitions is the result of a compromise between his desire for certainty, and his theory of both the practical and the theoretical need "of thinking in images", for intuitions. Thus, for example, on his account there must be two parts to any definition of a set. The first part is that via which we distinguish all the objects which have a certain property; this is what is ordinarily thought of as the "defining condition" of the sets. In addition, there must be a second part to any well-founded mathematical definition, where an account of the nature of (and relations between) the particular members of the set is provided. In order to be said to understand a definition, in addition to the formal set-membership condition, ϕ, "it is necessary to understand the set of particular objects which satisfy the [first part of the] definition".[26] We must have an idea of the sorts of objects for which ϕ is true.

> The first part of the definition, common to all the elements of the set, will teach us to distinguish them from the elements which are alien to this set; this will be the definition of the set; the second part will teach us to distinguish the different elements

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25 Poincaré, (1889/1908), pp.128-129. (My emphasis.)

26 Poincaré, (1912a), p.69.

> of the set from one another ... [The second part is necessary;] otherwise the object would be inconceivable and the proposition [about an arbitrary object of the set] would have no meaning. 27

The reason the second part of a definition is essential for Poincaré, has to do with his "constructivism" or non-Platonism. There is no universe of sets given prior to our mathematical specifications; so the nature of the objects which satisfy any defining condition ϕ, must be explicitly given by us. In the empirical case, the second part of a definition is, in general, not necessary. Facts about the particular objects which are called "birds" - such as similarities and differences between different species, i.e., the relations between the elements - can be revealed gradually. The definition of "bird" carves out a piece of the independently given aposteriori world, and investigation into the world discloses certain contingent facts about the objects which we have picked out by the definition. However, in mathematics the situation is different. There is no world independent of our postulates and domains which we can investigate by other (i.e., causal) means. So our mathematical definitions must either carve out a piece of a previously given mathematical domain (in a way which parallels definitions of things in the world), or we must <u>invent</u> a new domain. But if we are doing the latter, we must explicitly state the nature of the particular

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27 Poincaré, (1909b), p.61.

elements in the domain. Otherwise, our understanding of such abstract "objects" will not sustain an intuitive comparison with the perceptual objects of experience. For Poincaré, the requirement of this second part of the definition precludes the acceptability both of arbitrary infinite sets and of ineliminably impredicative specifications. Arbitrary infinite collections have no particular characteristics as objects; and we cannot understand the particular nature and relations between all the objects of an ineliminably impredicatively specified collection, because the structure of such a set is not fixed - the order of its elements is always disruptible.[28] Thus, Poincaré's theory of definitions prohibits ineliminably impredicative specifications and arbitrary infinite collections from determining bona fide mathematical objects. Both of these notions feature in the classical set theoretic interpretation of the continuum (e.g., in the power set operation applied to an infinite set).

I will now (in the next section of this chapter) examine one very general argument against axiomatic set theory in the light of Poincaré's theory of intuitions. In Chapters 5 and 6, specific complaints regarding the set-theoretic construction (or non-construction) of the continuum will

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28 See, for example, Poincaré, (1909b), pp.46-48. Poincaré's theory of impredicativity is the subject of Chapter 6.

be dealt with in more detail.

(5) Set Theory and Intuitions

Set theory developed as an extension of the logicist programme; and Poincaré's general arguments against set theory are an extension to his general arguments against logicism. The logicist wishes to bypass intuition in an effort to provide a sceptic-proof foundation for mathematics. His aim is to exhibit a purely logical basis for mathematics, which, with appropriate definitions, could produce all known and all knowable mathematical results. When this proved to be impossible without the aid of some non-logical axioms (e.g., Infinity, Reducibility, Choice), the programme changed to "set theory". And the new aim was to provide a set-theoretic foundation for all mathematics which employed as few non-logical axioms as possible. Poincaré's objection to this programme is that it is impossible to capture our mathematical intuitions by writing down axioms; what we can construct via apriori intuition, e.g., a line with no gaps, cannot be constructed via set theory, via logic and axioms alone. For when we try to write down axioms which produce sets via which we can mirror all results that our mathematical intuition formerly produced, axioms necessitating the acceptance of impredicatively defined sets are employed. And this violates what it _means_ to "construct". Either the axioms are false in intuition, since they validate impredicative specifications; or, by limiting our domains and proofs to those which are "constructive" (e.g., provable in Intuitionist analysis), the result is a gap between what we could formerly do - i.e., classical analysis - and what we can do via construction of sets alone. Hence, neither classical set theory nor Intuitionistic set theory is a faithful characterisation of pre-formalised mathematics. Both violate our mathematical intuition, according to Poincaré.

Now Russell considered this to be a stilted view of the formalisation of mathematics.

> The object is not to banish "intuition", but to test and systematise its employment, to eliminate the errors to which its ungoverned use gives rise, and to discover general laws from which, by deduction, we can obtain true results never contradicted, and in crucial cases confirmed, by intuition. 29

However, on Poincaré's view, set theory does "banish" intuition, for it contradicts it. The axiom of infinity plus the power set axiom entails the acceptance of sets which cannot be specified in a predicative way, or, indeed, at all: e.g., in the classical acceptance of both arbitrary and uncountable infinities. In Poincaré's view, such "sets" directly violate our intuitive conception of "the set of ..." operation.

The formalisation of mathematics found in modern set

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29 Russell, (1906b), p.194.

theories, rather than providing a foundation for the existing mathematical intuition, had to revise what was considered to be a mathematical intuition, a well-founded property, in order to recapture that which was formerly provided (albeit imprecisely) by intuitive mathematics, while at the same time avoiding contradictions. However, the new objects of mathematical intuition - i.e., arbitrary infinite collections and ineliminably impredicatively specified sets - rather than merely extending our notion of mathematical object, constituted (in Poincaré's view) a violation of our prior concept of mathematical object. The new objects gave rise to formally undecidable statements, or postulates which are independent of the theory itself, e.g., the Continuum Hypothesis; and they are only modelled in formal structures which presuppose their acceptability.

Now, what can never be verified,[30] and hence what is meaningless for Poincaré, is any statement concerning all the elements of an arbitrary infinite collection, or any statement which refers to an ineliminably impredicative specification. We can never, even in principle, construct either object in intuition - our glossing over faculty does not address such "objects". There is no extension to our capacities via which we can imagine a determinate verification

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30 See Chapter 7.

of a statement about such an entity. So they are disallowed from "meaningful" mathematics. Poincaré often simultaneously criticises the global acceptance of impredicativity and the belief in actual infinity[31] because they both violate his theory of meaning (i.e., his theory of the synthetic apriori) for the same reason. They both purport to specify objects (sets) which are inconceivable to us <u>as</u> determinate objects.

The problem is not merely that of the counterintuitive objects of the classical set-theoretic universe. In addition, by rejecting intuitive constraints on the notion of possible mathematical object, the set-theoretic characterisation of the mathematical universe leaves itself without a foundation. Axiomatic set theory, on Poincaré's view, makes our intuitions <u>secondary</u> to the formal rules concerning the employment of the axioms. This results in two general objections, both of which stem from epistemological or foundational concerns. One is that certain intuitions cannot be formalised - they "can be felt but not expressed"[32] - so no formal expression will be "reducible" in the intended way.[33] The second general objection is that if our intuitions are secondary to the formal postulates, then what constrains the acceptance

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31 Leading to unfortunate misinterpretations of his actual views when <u>why</u> he does so is disregarded, e.g., in Chihara, (1973), p.140, and in Kneale and Kneale, (1962), pp.672-673.

32 Poincaré, (1905b), p.149.

33 This is discussed in detail in Chapters 3 and 5.

of formal postulates? The choice and interpretation of axioms, in the absence of intuitive considerations, seems arbitrary, this view being confirmed by the discovery of the set-theoretic paradoxes. Consistency is necessary, so a change in the axioms was essential; but how do we change them without the alteration being merely ad hoc? Goodman, echoing Poincaré's objection that modern axiomatic set theory is independent of our intuitions concerning the concepts of set and mathematical object, complains: "If mathematics is set theory, which set theory is it?"[34]

In Poincaré's view, the only choice between triviality (of a strict constructivism) and contradiction (of set theory) was one which relied upon the theory of synthetic apriori intuitions. I now turn, in Chapter 5, to a discussion of how he implements this view in his arguments against the set theoretic "construction" of the classical continuum.

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34 Goodman, (1984), p.22.

CHAPTER FIVE

POINCARÉ'S THEORY OF THE CONTINUUM

(1) Epistemology and the Characterisation Problem

(2) Sets as Contained Collections

(3) The Limits of the Arithmetisation of the Continuum

(4) The Crucial Importance of Cantor's Result for Poincaré's Theory of the Continuum

Poincaré's theory of the continuum is that it is a domain of continuous variation, the understanding of which is provided by geometric intuition. The continuum must be an intuitive geometric structure, because, while we possess an understanding of the concept of mathematical continuity, the continuum cannot be regarded as a set. The set-theoretic characterisation of the classical continuum is precluded by Poincaré's theory of meaning, and consequently by his theory of infinity, whereby only potentially infinite (or countable) sets are coherent. Yet his conception of the continuum is not that of the intuitionist; nor does his theory involve any of the specific Brouwerian methods[1] for generating the intuitionist, i.e., α countable, continuum. Rather, Poincaré's theory of the continuum lies much closer to the classical Cantorian conception, for he regards the continuum as a structure which cannot be exhausted by any countable sequence - a structure which is, nevertheless, determinate.

(1) Epistemology and the Characterisation Problem

Poincaré's objections to the set theoretic characterisation of the continuum stem from epistemological concerns.

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1 See Brouwer, (1913).

His arguments concerning the continuum and geometric intuition are in this way parallel to his arguments concerning induction and arithmetic intuition. He wishes to establish their synthetic apriori nature, thus defending Kant's thesis.

Just as the arithmetic intuition of "indefinite iterability" cannot be a feature of the aposteriori world (the argument for the apriority of arithmetic intuition), neither can the geometric intuition of continuity be a feature we directly experience. Rather, in each case, apriori intuition must be imposed upon experience - must be a feature of our interpretation of experience - in order to maintain the coherency of experience. Thus, Poincaré claims, even our experience of the physical continuum requires geometric intuition in order to "gloss over" the incoherent - indeed, contradictory[2] - situation provided by direct experience alone, which is a consequence

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2 What Poincaré actually says is the following:

> The continua which we have just considered are mathematical continua; each of their points is an individual thing absolutely distinct from the others and, moreover, absolutely indivisible. The continua directly revealed by our senses and which I have called physical continua are altogether different ... It is possible to tell the difference between a 10-gram weight and a 12-gram weight at a guess; it would not be possible to tell an 11-gram weight from either a 10-gram or a 12-gram weight ... in order to construct a physical continuum, it is essential from what has been said before that two of their elements can, in certain cases, be considered as indistinguishable ... It is therefore necessary that by an active operation of the mind we agree to consider two states of consciousness as identical by disregarding their differences. (1912b),pp.30-31. (My underlining of "active operation of the mind".)

of the inevitably limited or incomplete nature of our sense data. And since even our conception of the _physical_ continuum requires apriori intuition to maintain the coherency of experience, we could not have arrived at the notion of mathematical continuity from our experience of physical continuity, independent of apriori intuition.

The objection to the standard account of the continuum obtained, say, as the set of all subsets of the natural numbers, is epistemological. For Poincaré, interestingly, we _can_ characterise the continuum. He appreciates the _mathematical achievement_ represented by the "arithmetisation" of analysis. What he is denying is that this achievement of the rigorisation of analysis has the significance that the logicist claims it has: that it _shows_ that mathematics is independent of extralogical intuitions. That is, Poincaré's claim is that _if we did not already possess_ an apriori understanding of continuity, the set-theoretic characterisation could not _produce_ such a concept in our minds.

> A continuum of _n_ dimensions is a set of _n_ coordinates, that is, a set of _n_ quantities capable of varying _independently_ one from the other and of assuming all real values which satisfy certain inequalities. This definition, flawless from the point of view of mathematics*, nevertheless could not be entirely satisfactory to us ... What it does not reveal is the profound reason for which these materials have been assembled in this fashion rather than in another. I do not mean that this 'arithmetization' of mathematics is undesirable; I say that it is not everything. 3

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3 Poincaré, (1912b), pp.28-29.

* This account of the n-dimensional continuum is not adequate,

The definition of an n-dimensional continuum is inadequate as an epistemological source for the concept of continuity. This characterisation presupposes an understanding of "all the real values of an interval"; but this presupposes an understanding of a continuous domain. Though, perhaps, mathematically acceptable, Poincaré's point is that the characterisation is not epistemologically satisfying or revealing.

Poincaré's argument, however, is general. It is not merely that <u>this</u> characterisation is unacceptable, but that <u>no</u> characterisation could, in principle, succeed in capturing, in an exhaustive way, the notion of a continuous set of real numbers. Given Poincaré's theory of meaning, a continuum could be a set-theoretic object only if we could characterise it (say, like the natural number sequence) by an algorithm for generating its members. Every infinite collection requires a rule for generating its members. There is the rule "+1" for generating sequentially the natural numbers; and there is a method for generating sequentially the rationals. These are both countable sets, the members of which are generated sequentially by a finitely specifiable procedure; so they are acceptable, well-founded collections according to Poincaré.

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as indeed Poincaré's own foundational work on topological invariance shows. Poincare (1912b) constitutes an expository article on the concept of dimension; see Brouwer, (1913), for a fundamental extension of Poincaré's idea.

However, it is not possible for a rule or algorithm to generate (sequentially) an uncountable domain. And since the domain of real numbers cannot be generated by a rule (this is Poincaré's interpretation of Cantor's result, discussed below), Poincaré concludes that the domain of real numbers cannot be regarded as a set. It is impossible to characterise as a set the classical domain of real numbers, in a way which meets the requirements of his theory of meaning.

The set-theoretician may now wonder whether this requirement does not merely beg the question against the classical set-theoretic characterisation of the continuum. Are there any independent arguments for accepting Poincaré's requirements? We will now take a small detour into his theory of definitions of sets, in order to expand on the underlying reasons for rejecting the well-foundedness of uncountable collections.

(2) Sets as Contained Collections

For Poincaré, a set is like a container which has just enough places for the objects which occupy it. In everyday life these containers are finite: the logic of sets is the logic of definite, contained collections. For example, "There were less than 100 students at the lecture" can be analysed as: "Of the finite and determinate set 'all the students at the lecture', its cardinal number is less than 100." We talk of sets or collections all the time, but these are usually finite and determinate sets. Now, when the containers are not finite, in order to reason coherently about the elements which are inside, we must have an exact understanding of how the "container" expands, or how the elements take up their places inside it. Since Poincaré claims that "infinite" only means "unending", or "a collection which never stops growing", in order to understand an infinite set, we must have an exact understanding of how it grows. If a set is not "contained" in a strict, finitistic, sense, it must at least be restricted and deterministic in its growth. This is what we understand when we understand an infinite set via its generating rule: the way in which its growth is restricted or defined.

Poincaré's objection, then, to considering the continuum as a well-defined collection, as a set, is that because we have no exact understanding of a generating rule for an arbitrary irrational number, we have no exact understanding of the container which houses them. We do not have a constructively acceptable procedure for determining how this container would have to grow, or expand, in order to capture all the real numbers, say, between 0 and 1. Hence, such a container is not determinate; hence, such a container is not a set.

The reason we have no exact understanding of how the container which holds, say, the open interval (0,1) grows, is because in order to obtain each of the elements included in this classical real interval, we need to accept the existence of impredicatively characterised reals or arbitrary infinite sets. Impredicative definitions are sentences which pick out an entity by referring to a collection of which it is a member: e.g., "tallest man in the room", "least upper bound", etc. The problem with impredicative definitions occurs when 1) the entity being defined is necessary for the completion, or for the complete characterisation, of a collection; and 2) the entity cannot be defined in any way - is not accessible via any route - other than via this very same <u>completed</u> collection, of which it is supposedly an element. In this way, so-called "vicious circles" are generated. We do not have the object (we cannot pick it out) until the collection is completed (since it cannot be characterised other than <u>via</u> the completed collection); and yet we cannot complete the collection without the object. (Note that vicious circles are not produced when an element can be defined by reference to the partially completed set of which it is a member, e.g., any natural number. It is only when the collection must be <u>completed</u> before one defines the entity (as in "class of all classes"), that vicious circles can be produced, for instance, by the question of whether or not the "completed" collection belongs <u>to</u> the collection, to itself.) Now, according to Poincaré, vicious circles are to be excluded from mathematics. So the maxim is there is no legitimate mathematical object whose characterisation is ineliminably impredicative. (Poincaré's theory of impredicativity is the subject of Chapter 6, below.) Let us suppose this is correct; impredicative definitions do not characterise mathematical objects. Where does this leave us with regard to the continuum? Is it true that in order to accept the classical account of the real numbers, we must accept that impredicative specifications do characterise legitimate mathematical objects?

The matter is not absolutely clear. It would seem that in order to prove the absolute indenumerability of a domain, the impredicativity is ineliminable. As Wang says,

> It is no accident that all proofs of absolute indenumerability use impredicative sets. Indeed, it is certainly intuitively plausible that all predicative sets are denumerable. 4

In order to avoid the problem of impredicativity, we must avoid the question of proving absolute indenumerability.

However, Poincaré would object even to the "intuitive" characterisation of an uncountable set, as the power set of a countable set. The classical set of real numbers is often characterised by the power set axiom applied to the countable and determinate set $\mathbb{N}$, of natural numbers.

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4 Wang, (1954), p.246.

The outcome is the set of all subsets of $\mathbb{N}$, the understanding of which presupposes the notion of an *arbitrary* subset of $\mathbb{N}$. $\mathbb{N}$, however, being infinite, the set of all subsets of $\mathbb{N}$ includes all the *infinite* subsets of $\mathbb{N}$, thus presupposing the notion of an *arbitrary infinite* subset of $\mathbb{N}$. However, for Poincaré, there is no such thing as an arbitrary infinite set. Infinite collections which are not rule-generated are incoherent; the *very notion of an arbitrary infinite set is contradictory*, for it is unfaithful both to the notion of the *infinite* and to the notion of a *set* or collection.

Sets are objects which are *formed* by collecting the members together, and putting them into one "container". Finite sets are objects which are actually formable: we can list their elements between two brackets: {a,b,c,...n}. Infinite sets are objects for which we have a rule. The rule governs the generation of elements, and thus the forming of the set.

> A collection is formed by the successive addition of new members; we can construct new objects [e.g., n+1] by combining old objects [n and 1, using the combiner," +"], then with these new objects construct newer ones, and if the collection is infinite, it is because there is no reason for stopping. 5

Finite collections are formed when the successive addition of new members terminates. Infinite collections are those

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5 Poincaré, (1912a), p.67.

for which the successive addition of new members proceeds according to some rule of construction, such that we can see in the rule that there is no end to the number of possible constructions. That is, such that we can see in the rule that it is indefinitely iterable. If we cannot see the infinity of a set in the indefinite interability of a rule which generates it, we can have no sense of the individuating properties of its elements, of the relations which hold between them. Thus, for Poincaré, we cannot understand the object as a set.[6] In this way, collections must either be finite or specifiable by an iterative rule, in order for the particular elements to be capable of possessing a sense (in intuition).

Sets must be collectable or constructible in intuition. So coherent infinite sets must be rule-governed, so that they can be modelled in arithmetic intuition - the iterative or collecting intuition. For Poincaré, potential, or rule-governed, infinity describes the limits of our arithmetic intuition which alone can provide an access to a sense of the particular elements of an infinite set. The characterisation of the set of reals, via the notion of arbitrary subset of $\mathbb{N}$, is, therefore, irredeemably illicit since it necessitates the notion of an <u>arbitrary infinite</u> subset of $\mathbb{N}$.

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6 See Chapter 4, section 4, above, for discussion on the necessity of understanding <u>two</u> parts of a definition of a set.

(3) The Limits of the Arithmetisation of the Continuum

The arithmetisation of the continuum attributes a concrete existence to all the points of the real line. π is not an empty formal symbol. And yet the set-theoretic characterisation of the continuum involves accepting the propriety of impredicative specifications, or the existence of arbitrary infinite subsets of $\mathbb{N}$. How do we come to consider each point, each real number, as an object the ontological status of which is equal to all the others? How do we come to understand particular irrationals, like π, $\sqrt{2}$, e?

These particular elements have determinate content which is provided by their use in mathematical practice, and insofar as we can construct them geometrically. They play a role in the arithmetisation of geometry, and so we understand them via the operations which produce them. We understand $\sqrt{2}$ as the length of the hypotenuse of a right angle triangle whose other sides are length 1 unit. We understand π as the circumference of a circle whose radius is $\frac{1}{2}$. There is a theoretical need for these numbers stemming from geometrical argument, so they are intrinsic to any account of the continuum.

However, since we understand particular irrationals via geometrical operations which produce them, it is Poincaré's view that our understanding of geometry, our geometric intuition of continuity, must be epistemologically prior to the arithmetic construction of irrationals. Otherwise, his claim is, we could not explain why it is that we attribute the same "concrete" and ontological status to the constructions (irrationals) as to the materials employed in the constructions (the natural numbers), which form the basis of our operations.

> Should we have a notion of these numbers if we did not previously know a matter which we conceive ... as a continuum? 7

Without apriori geometric intuition we should have had no reason for transcending the rationals. This set represents the limits of arithmetic intuition, of indefinite iteration, since it is everywhere dense. It is geometric intuition which suggests that there are numbers which are not members of the dense set $\mathbb{Q}$. It suggests the notion that there are numbers corresponding to <u>every</u> point on the line; that there are magnitudes corresponding <u>exactly</u> to every length.

The Poincaré thesis is that history and epistemology here coincide. The geometric continuum is a line with no gaps; not a domain which has no gaps which we can see, but a domain with no gaps <u>in principle</u>. Hence, when a gap in the rationals is discovered, it is the apriori

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7 Poincaré, (1893), p.21.

geometric intuition which warrants us in filling it, by inventing a new number, and considering this as an entity with the same ontological status as a natural or rational number. If our intuition of continuity were not more primitive than operations on rationals, we would have no more reason for admitting irrationals to the set of real numbers - i.e., defining this new set - than for concluding that we have "discovered" that the arithmetic numbers do not correspond to the points on the geometric line. Instead, in the domain of real analysis, we treat limits and Dedekind cuts - no matter what operations, and infinite sets these cuts involve - as units, just like 257. And we gloss over all the points for which we have no characterisation, and - unlike the integers and rationals - for which we have no finite specification. This "glossing over" enables us to conceptualise and prove theorems about "any real number". For example, for any real number, it is either greater than, less than, or equal to any other real number; i.e., real numbers satisfy the axiom of total order; i.e.:

$$(\forall x)\ (\forall y)\ (x,y \in \mathbb{R}:\ x < y\ \vee\ x=y\ \vee\ x > y).$$

It is our apriori intuition of continuity which grounds this axiom, and which grounds this practice of "glossing over".

Via geometric intuition we "gloss over" the ineliminable impredicativity in any characterisation of a continuous domain, thus preventing such a specification from being hopelessly vague or viciously circular. For instance, if real numbers are defined as the set of all possible Dedekind cuts, then some real numbers will be impredicative; for all Dedekind cuts will include some impredicatively specified cuts. Now, Poincaré has no objection to the way we characterise particular irrationals. For example, algebraic irrationals, like $\sqrt{3}$, are straightforwardly expressed as: the positive root x, which satisfies the equation, $x^2-3=0$. (The other x, of course is $-\sqrt{3}$.) But the algebraic numbers form only a countable subset of Cantor's uncountable set of reals. There can only be a denumerable number of such algebraically characterised numbers. Hence, in order to obtain the full classical set of real numbers, we must postulate the existence of an uncountable number of transcendental - i.e., non-algebraic numbers.

An algebraic number is one which

> satisfies some algebraic equation of the form
> (1) $a_n x^n + a_{n-1} x^{n-1} + \ldots + ax + a = 0$ $(n \geqslant 1,\ a_n \neq 0)$
> where the a_k are integers. 8

A transcendental number is one which cannot be so expressed: it transcends the powers of algebraic methods.

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8 Courant and Robbins, (1941), p.103.

Transcendental numbers are the problematic, i.e., the interesting, objects. For, since there are only a countable number of algebraic numbers, if the set of real numbers is uncountable, there must be an uncountable number of Dedekind cuts which necessarily refer to limits of infinite series, some of which will be impredicatively specified, or which will not be specifiable at all. Can we really understand such a collection; can such a vague, indeterminate collection form a set?

Poincaré's answer is, of course: no. Specific transcendental numbers are acceptable because we can comprehend them as the limit of an ordered series. For instance, "π" and "e" are obtainable, in a sense; for though they are not constructible (algebraically)[9] we can approach or approximate their construction. Since, via our geometric intuition, we know that, for instance, π exists - provided we assume that there are arithmetic numbers corresponding to all geometric lengths or quantities, i.e., that geometry has a model[10] - we can look for an arithmetic specification of π, in terms of operations on rational

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9 Proof for transcendence of e to be found in Courant and Robbins, (1941), pp.297-299.

10 Interestingly, for Poincaré, this assumption leads to a circle (in the absence of the theory of geometric intuition), for in order to model geometry we must presuppose the real number system. Whereas here, we are supposedly "constructing" a real number.

numbers. For example, the following are all characterisations of π as the limit of some infinite ordered sequence:

$$(1)\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n+1}\left(\frac{1}{2n-1}\right) + \ldots$$

$$(2)\quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots$$

$$(3)\quad \frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots$$ 11

But, Poincaré would wonder, how do we know that these specifications taken to the limit ($n \rightarrow \infty$) capture the object π? It is only by relying upon a prior geometric understanding of the continuum, and hence of π - its place in geometry - that π is a determinate object; that our specifications have something to which they must conform, to which they must be faithful. Hence, our geometric intuition is primary, and conditions our construction of arithmetic numbers.

> One sees what a role geometric images play in all this; and this role is justified by the philosophy and history of the science. If arithmetic had remained free from all admixture of geometry, it would have known only the whole number; it is to adapt itself to the needs of geometry that it invented anything else. 12

It is only via geometric intuition - the intuition of continuity - that our arithmetic constructions, especially

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11 Courant and Robbins, (1941), p.300.

12 Poincaré, (1889/1908), quote taken from (1946) volume, p.442.

constructions of transcendental numbers, have a determinate content over and above purely formal manipulations. Without this apriori intuition, providing a model, the constructions would be correct, and pure; but they would be devoid of any non-formal content, for they would not be part of mathematics. In order to be vindicated in believing that our constructions are not arbitrary, we rely upon geometric intuition to provide content and determinacy to our constructions.

However, in addition, there is a vast domain of transcendental numbers for which we have no specification, and hence, no definite understanding. "All the Dedekind cuts" includes all arbitrary infinite sets of numbers. But this notion is not well-founded. "All the Dedekind cuts" is hopelessly vague and indefinite as the characterisation of the <u>construction of a set</u>, for we have no general understanding of the particular elements which it includes. The only reason this characterisation of the reals - in terms of Dedekind cuts - makes sense at all, is because it accords with our apriori intuition of continuity. Our concept of continuity can thus be <u>refined</u> or explicated, but not <u>brought about</u> via the construction of a set, for this concept is required in order to gloss over the vagueness and impredicativity in any characterisation of an absolutely indenumerable domain, so that we can understand by the explication the determinate mathematical continuum.

The determinacy of elements of the set-theoretic construction of the continuum does not extend beyond particular transcendental numbers for which we have specifications (as limits). Our understanding of the nature of the classical continuum cannot, therefore, depend upon the construction of the elements within it. It is not an uncountable domain, a line with no gaps, because we can specify an uncountable number of elements. Rather, we see that it is uncountable precisely because we cannot completely specify or capture it by constructing all its elements, or "all Dedekind cuts". Insofar as we have a notion of the continuum as an uncountable domain, with no gaps in principle, it is a domain which transcends both constructibility and specifiability. Thus, insofar as we can strictly construct certain elements of the continuum, this "set" is countable; and conversely, insofar as we know the continuum is uncountable, we do not know it as a set.

(4) The Crucial Importance of Cantor's Result for Poincaré's Theory of the Continuum

One might expect, given Poincaré's thesis of the incoherence of the notion of "arbitrary subset", that Cantor's foundational theorem in set theory, to the effect that for any set x, card $(\mathbf{P}(x)) >$ card (x), would have no significant content. This expectation might well be reinforced by noting that the proof that there is no map from a set x onto its power set, involves an impredicative specification. Suppose $f: x \to \mathbb{P}(x)$; then the set z, characterised by

$$z = \{u \in x : \exists y \in \mathbb{P}(x) \ \& \ y = f(u) \ \& \ u \notin y,$$

is ineliminably impredicatively defined, since the set z is defined by reference to a totality ($\mathbb{P}(x)$) of which it itself is a member. Indeed, Poincaré did object to the standard interpretation of the theorem, and in particular to its important instance,

$$\overline{\overline{\mathbb{P}(\mathbb{N})}} > \overline{\overline{\mathbb{N}}}.$$

However, in the case of the instance $\overline{\overline{\mathbb{P}(\mathbb{N})}} > \overline{\overline{\mathbb{N}}}$, he reinterpreted the proof of the specific claim, and took the result so interpreted as showing something fundamental about the nature of the continuum, and consequently, something fundamental about the epistemology of the continuum. He says of his account of Cantor's result,

> According to this reckoning, there would be only a single infinite cardinal number possible, the number Aleph-zero. Why, then, do we say the power of the continuum is not the power of the integers? 13

The point is that he takes the theorem in the form that the power of the continuum is not the power of the integers. It is in his account of the proof of Cantor's theorem of the uncountability of the continuum,[14] that he makes

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13 Poincaré, (1912a), p.68.

14 Cantor, (1874).

the most telling point. To see what it is, first consider the set of rational points. This collection has the property that it is dense, so any "natural" conception of the next rational, after a given one, seems to lose all significance. Consequently it might be thought that the collection of rationals, having such radically different properties from the collection of integers, it would be required that the former collection would have to have a distinct treatment from that afforded by the collection of natural numbers. That is, it might be thought that they could not be seen as a countable, ordered sequence generated by the successive systematic application of an iterative rule.

But that this is not so is indeed shown by Cantor's proof of the countability of the collection of rationals. That is, although the rationals have a different structure from the naturals and a quite distinct order type, this fact induces no special epistemological problems concerning our knowledge of the set of rational numbers. Therefore, it might be thought possible to treat the collection of all real numbers in the same way. It might be thought that the operation of "filling in" the gaps in the rationals, though inducing new structural properties, and yet again a distinct order type, nevertheless would still induce from the epistemological viewpoint no especially new difficulties. If every such "filling in" of the rationals could be effected by employing infinite sets of natural numbers (say in the manner of Dedekind or Russell), each of which was a countable collection specified by a general rule for generating the members of the collection (e.g., without appeal to the notion of an arbitrarily specified subset), then nothing especially significant concerning the epistemology of analysis would follow. Poincaré points out, however, that this is quite impossible given Cantor's result. No matter how we attempt to "order" or list the collection of all real numbers, that list cannot exhaust the continuum; the list is always "disruptable", as he says.

> And this is what we mean, according to the pragmatists [i.e., according to his own view], when we say that the power of the continuum is not the power of the integers. We mean that it is impossible to establish between these two sets a law of correspondence which will be free from this sort of disruption; whereas it is possible to do it, for example, when a straight line and a plane are involved. [15]

The point Poincaré makes is entirely well taken for it shows that we cannot hope to approach the continuum from "below", so to speak, without allowing impredicative specifications[16]: but this effectively rules out any (even

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15 Poincaré, (1912a), p.68

16 Poincaré could not accept the Cantorian result in the form which asserts the absolute indenumerability of the continuum, for such a proof is bound to employ impredicative characterisations, e.g., in specifying the set z (p.119, above) (Cf. Wang, (1954), pp.244-245). But this of course does not detract from Poincaré's reinterpretation of the proof, in essence, to the effect that

minimally) constructive theory of the (classical) continuum from "below".

If we want to effect the "construction" of the continuum from below, say by employing the full power set axiom, we will be involved in the epistemologically hopeless notion of "arbitrary subset". Thus Cantor's result (so interpreted), effectively rules out a constructive theory of the continuum, while if we use the full set-theoretic (or logistic) method, we will be involved in an epistemological absurdity. So there is only one avenue left open. The existence of the continuum is guaranteed by nothing less than geometric intuition, and our knowledge of it is just as Kant claimed; knowledge synthetic apriori. Russell's "irrevocable refutation" of Kant is, after all, no refutation at all. If Kant's theory of the synthetic apriori is epistemologically problematic, it is no more so than the inevitable logistic reliance on the collection of all subsets of a given countable set.

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given any law which enumerates sets of positive integers, we can find a set which differs from every one in the enumeration, and so on indefinitely. Thus Cantor's theorem takes on the form of the "disruptability" of all enumerations of the points on the continuum.

CHAPTER SIX

POINCARÉ'S THEORY OF PREDICATIVITY

Understanding Poincaré's theory of impredicativity is essential to a proper understanding of his philosophy in general. Not only are his views considered to be foundational for intuitionism, he is also rightly considered to be a precursor to modern programmes in predicative analysis and predicative set theory. However, just as it is wrong to consider modern intuitionism as a natural extension of the whole of his philosophy, so is it a mistake to consider a predicative set theory as merely a more precise (yet faithful) encapsulation of his general position. Indeed, in view of the formality of these programmes, he would probably have opposed them. Poincaré did not have a theory of predicativity *per se*; rather the concept of impredicativity developed as part of his more general negative theses, of anti-logicism, anti-formalism, and above all, of anti-Platonism. For example, it was convenient for him to argue against "Cantorism", or axiomatic set theory, by exploiting the problem of the set-theoretic paradoxes, a problem which is (for Poincaré) most naturally expressed via the notion of impredicativity. In this way, the issue of predicativity vs. impredicativity became part of a more general debate between Poincaré on the one hand, and Russell, Zermelo, Peano, etc., on the other. It is the purpose of this chapter to analyse the concept of predicativity in terms of the history of its upbringing, and to explain Poincaré's theory of impredicativity and the paradoxes as a feature of his arguments on more foundational issues concerning the meaningfulness of mathematical statements, and concerning the joint issue of the ontology of the mathematical universe.

There are two senses of Platonism, corresponding to the two foundational issues with which we are concerned - meaning and ontology - and we must first note that Poincaré opposes both of them. The better known and less plausible explication is "ontological Platonism", which is a doctrine about a realm of mathematical objects the existence of which is somehow independent of our mathematical activity - of our awareness and access to them. Poincaré opposes this in virtue of his constructivism: mathematical objects do not exist except insofar as we construct or define them - insofar as they are "conceived by the mind". The second explication of Platonism - which is at least _prima facie_ distinct from the first (although it might entail the first, as Poincaré seemed to believe[1]) - is one whereby the question is shifted from the ontological issue of the existence of mathematical objects to the issue (in the theory of meaning) of the objectivity of mathematical truth.

Now, Poincaré certainly does not deny that mathematical

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1 For example, he regarded the acceptance of actual infinites as a _necessary_ aspect of classical set-theoretic "reductionism" (1906b), p.195.

truth is objective truth. He does not dispute the intelligibility of a distinction between meeting our most refined criteria and actually being mathematically true.[2] (Indeed, the intelligibility of such a distinction is arguably a necessary feature of any "knowledge-domain".) Our criteria and concepts are refinable, so those which we possess at any particular time need not capture or exhaust mathematical truth (as is argued in the Appendix). However, what Poincaré denies, and what the Platonist qua Platonist asserts, is that this objectivity or determinacy of truth extends to statements which are in principle unverifiable. The distinction between Poincaré and the Platonist lies in Poincaré's theory of meaning, which requires that all statements be in principle verifiable. (The content of this notion is the subject of Chapter 7, below.) When properly interpreted (in the light of his theory of the synthetic apriori), Poincaré's criterion of in principle verifiability produces a neo-Kantian constructivist account of mathematics. And it is in virtue of this account that impredicative specifications are objectionable.

(1) Analysis of the Concept of Impredicativity

"Impredicative" is an adjective which describes a class of definitions. In particular, it is used in mathematical

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2 Wright, (1980), Chapter 1.

logic to describe a way of specifying sets or mathematical objects. (It is also sometimes misused to describe a class of sets or objects.[3]) Impredicative definitions are (roughly) those which characterise an object (or concept, or property) (i) by referring to a totality of which the object is a member; or (ii) by employing a concept of which the object (concept, property) is an instantiation. For example, the specification, "red is that colour which, of all the colours in the visible spectrum, is at the long-wavelength end", is impredicative, for it defines "red" via a totality (all the colours), or via a concept (colour), of which red is an instantiation.

Impredicative definitions are objectionable to Poincaré only if they are viciously circular; and they are viciously circular only when they provide the sole access we can have to a mathematical (concept, property, or) object. I shall call impredicative characterisations "irreducible" or "ineliminable" when there is no corresponding predicative equivalent available (e.g., by virtue of a prior theory). Ineliminable impredicative specifications must be excluded by the constructivist, for one certainly cannot construct an object via a circular procedure. If in order to construct an object we must first construct a a collection of which the object is a member, then such a construction (i.e., both constructions) is impossible.

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3 For example, Wang, (1954).

The ineliminable impredicativity of the characterisation of certain constructed objects, like "the set of all sets", is taken by the constructivist to be "viciously circular", and thus not a bona fide part of meaningful mathematics.

For Poincaré and Russell it is the absence of any explicit constraints on "the set of ..." operation which, in the context of a realist interpretation of the quantifiers, is to blame for the paradoxes. Such a lack of constraint allows ineliminable impredicative characterisations to be one of "the set of ..." operations. "Naive realism" is generally characterised by the following (Russell's) axiom schema of set existence:

$$\exists y\ \forall x\ (x \in y \leftrightarrow \phi(x)).$$

It is "naive", presumably, because we translate this as, literally, "<u>every</u> property determines a set". And it is "realist" because of the way the quantifiers are interpreted: the range of the "$\forall$" being the whole of the universe. Naive realism is contradictory, then, because the axiom schema, above, is contradictory under the ordinary (logical, logicist, or realist) interpretation of the quantifiers. If <u>every</u> property determines a set, then so does the property (Russell's) of being "non-self-membered"; so we may instantiate the axiom as follows:

$$\exists y\ \forall x\ (x \in y \leftrightarrow -(x \in x)).$$

Let us call the set we are forming "y_0":

$$\forall x\ (x \in y_0 \leftrightarrow -(x \in x)).$$

But because the property ϕ is impredicative - it characterises a set in terms of a property of sets - and because of the unrestricted "$\forall$" quantifier, y_0 is itself a candidate for ϕ-ness:

$$y_0 \in y_0 \leftrightarrow -(y_0 \in y_0).$$

Poincaré and Russell both felt that it is the impredicativity of considering a set for membership of itself which leads to the contradiction above. The attempt is to collect together and put into a set all the sets which are not members of themselves. But if we must really collect all the sets which are not members of themselves, we must also consider y_0, the set of all such sets, it being by definition (in virtue of the axiom) a set. We must decide whether $\phi(y_0)$ in order to obtain all the sets which possess ϕ. But then the contradiction, $y_0 \in y_0 \leftrightarrow -(y_0 \in y_0)$, immediately follows.

Poincaré and Russell saw the same vicious circularity which is a feature of this paradox in many other paradoxes (e.g., Richard's paradox, and the Burali-Forti paradox). They thus formed the opinion that there is a class of such paradoxes, each of which is a consequence of the same viciously circular, or ineliminably impredicative element in the specification of the object concerned. A general characterisation of the guilty class was needed. We will now turn to an examination of how this general characterisation, the concept of impredicativity, emerged in Poincaré and in Russell.

(2) The Emergence of the Concept

The concept of predicativity evolved gradually by way of discussions between Poincaré, Russell, Zermelo, and Peano in the period 1905-1912, the main venue being the Revue de Métaphysique et de Morale. Part of the obscurity which may be associated with this concept and with the "Vicious Circle Principle" (VCP) which concerns it, lies in the fact that the "discussions" out of which the notion arises, were anything but friendly. Thus, the meaning of predicative/impredicative is influenced by its role in the general dispute between the logicists, formalists or "Cantorians", and the anti-logicists, anti-formalists, or "pragmatists". Poincaré lies in the latter camp; while Russell, Zermelo, Peano, and one might add, Couturat, all lie somewhere in the former camp.

2.1 Poincaré's account

Poincaré blames the paradoxes on a Cantorian point of view, and a logicist method, in general. He thus claims in 1905[4] that it is the belief in the actual infinite, or defining the finite in terms of the infinite, which makes the set-theoretical paradoxes arise. That is, the

4 (1905b), pp.143-145.

purely formal treatment of sets, in conjunction with realism about the existence of sets, is the source of the problem. Thus, the existence of the paradoxes indicates the misguided nature of logicism and formal set-theory - that it is based, as Poincaré claimed all along, on mistaken presuppositions concerning the possibility of refuting Kant. It is for this reason that he also claims, in the same section[5], that it is not to formalism or Cantorism that we can turn to seek a solution. For instance, the Burali-Forti paradox arises because the logical apparatus cannot prevent one from considering "the collection of all the ordinal numbers"; whereas on the basis of intuitive considerations, on Poincaré's view, we have no right to so abuse Cantor's notion of set formation[6], and consider such a collection as even possibly well-founded.[7] That is, in the context of a set theory whereby every collection can be well-ordered or associated with an ordinal number - $\forall x \exists \alpha (x \sim \alpha)$, x being a set and α being an ordinal number, such a result being necessary to generate the Burali-Forti paradox - the guilty classification is parallel to something as obviously problematic as "the set of all sets". The point Poincaré is making, is that the problem lies at the very heart of these programmes, in their formalism; so

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5 And later, in a continuation of the same article (1906b), p.180.

6 Which is not that of Russell's naive axiom. See Hallett, (1984), pp.16-17, 33 for Cantor's requirement that our set-determining concepts be "fixed", "orderly" and "definite".

7 Poincaré, (1905b), p.159 and (1906b), p.185.

any revision in the formal rules will not guarantee the exclusion of any new paradoxes.

Poincaré was led to readdress the issue of the solution of the paradoxes, in the third (and final) of his series of articles entitled "Les Mathématiques et la Logique"[8], in response to a paper by Russell ("On Some Difficulties in the Theory of Transfinite Numbers and Order Types"), published in the meantime (March 7, 1906), which centred on this very question.[9] This time Poincaré was somewhat more positive and clear on his views, and did not merely dismiss the problem as being that of the "other camp". He discusses Russell's various proposals - the zig-zag theory, the theory of the limitation of size, and the no-class theory - makes different objections to each, then provides the first direct statement of his positive thesis in a section entitled "The True Solution". (I will continue to trace the fundamentals of Poincaré's notion; then, in sections 2.2 and 3 below I will turn to the solutions of Russell and Zermelo, and explain Poincaré's objections to them.) In brief, Poincaré considered the main heuristic of the zig-zag theory (that of sufficient simplicity of definitions) to be obscure; the limitation of size theory to be ridiculous: a class may "be infinite, but it must not be too infinite"[10]; and the no-class

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8 Poincaré, (1905-1906).

9 Russell, (1906a).

10 Poincaré, (1906b), p.188.

theory to be so drastic as to constitute an open defeat of logicism.

Russell's first characterisation of the predicative/non-predicative notion shows how it essentially arises out of the problem of set-theoretic paradoxes:

> Norms [properties, propositional functions] which do not define classes I propose to call <u>non-predicative</u>; those which do define classes I shall call <u>predicative</u>. 11

Compare with Poincaré's statement which comes just a bit later that year:

> <u>The definitions that must be regarded as non-predicative are those which contain a vicious circle</u>. 12

At this point, however, Poincaré was not attempting to <u>define</u> "impredicativity" (or "non-predicative"). It was his view that the examples <u>show</u> exactly what is meant. Thus he intended the example of Richard's antinomy, and Richard's diagnosis, to be clearly generalisable to the other similar antinomies. Richard's antinomy concerns the collection, E, of all the decimal numbers expressible in a finite number of words. Since the collection is denumerable (because each specification is finite), its elements can be ordered one-one and onto with the natural numbers, each element of E being assigned an $n \in \mathbb{N}$. However, upon

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11 Russell, (1906a), p.141.

12 Poincaré, (1906b), p.190.

supposing that the order has been established we can define a new number, e*, which differs from each e in E in its n^{th} place (by +1). Yet e* is defined, here, in a finite number of words, so it is by hypothesis one of the members of E. Thus, the contradiction:

> $e^* \in E$, and e* differs from every element in E, or $-(e^* \in E)$.

According to Poincaré, Richard correctly analyses the problem, as he states below:

> E is the aggregate of <u>all</u> the numbers that can be defined by a finite number of words, <u>without introducing the notion of the aggregate E itself</u>, otherwise the definition of E would contain a vicious circle, for we cannot define E by the aggregate E itself.
>
> ... the same explanation serves for the other antinomies, as may be easily verified. Thus <u>the definitions that must be regarded as non-predicative are those which contain a vicious circle</u>. The above examples show sufficiently clearly what I mean by this. 13

Poincaré begins in the next section of the paper to explicate his views on what is wrong with viciously circular or non-predicative definitions. Not all properties determine a class, because some properties are incapable of determining the precise boundaries of any class which we might suppose satisfies the properties. But since mathematical classes <u>must</u> have precise "boundaries" - i.e., we must be able to form a determinate conception

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13 Poincaré, (1906b), pp.189-190.

of "the set of ..." entities in a way which parallels (the determinacy of) collections of empirical objects - definitions must not make essential use of inexact concepts.

> A definition which contains a vicious circle defines nothing. It is of no use to say we are sure, whatever be the meaning given to our definition, that there is at least zero which belongs to the class of inductive numbers. It is not a question of knowing whether this class is empty, but whether it can be rigidly delimited. A "non-predicative class" is not an empty class, but a class with <u>uncertain boundaries</u>. 14

A class must be rigid, or "immutable"; and the properties which define a class must fix the class so that it is not "disruptable" or "uncertain". That is, the problem of paradoxical collections is the problem of the well-foundedness or exactness of mathematical properties.[15]

"A definition which contains a vicious circle", however, is not very precise, nor even intuitively clear. In 1909 Poincaré shifts the problem to clarifying the nature of the "classes" which are taken to be denoted by the non-predicative specifications. Here he tries to explain what he means by the "rigidity", or "immutability" of a class, and, of great importance, to relate this <u>effect</u> with the non-predicative nature of the definition.

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14 Poincaré, (1906b), p.191.

15 I will return to the "geographical" metaphors - "boundary", "frontier", "rigidity" - in Section 4, below.

> From this we draw a distinction between two types of classifications applicable to the elements of infinite collections: the predicative classifications, which cannot be disordered by the introduction of new elements; the non-predicative classifications in which the introduction of new elements necessitates constant modification. 16

In both 1906 and 1909 Poincaré claims that there is a relation between the belief in actually infinite collections, e.g., in the Cantorian method of placing "the infinite before the finite"[17], which is "contrary to all healthy psychology"[18] and the viciously circular definitions. This relation is often misconstrued (e.g., by Russell in his (1906b) and Chihara in his (1973), p.140) as a direct causal connection from a belief in actual infinity to the ensuing of paradoxes. However, this is not what Poincaré intends, even in the earliest (1905) paper. What he asserts is that Cantorism, or the acceptance of actual infinities, leads one to employ impredicative specifications where they cannot (according to Poincaré's account) meaningfully be employed, to denote a determinate mathematical object. In particular, Poincaré objects to the unrestricted extensional account of quantification which is validated by the realist (or Cantorian) belief in actual infinity. That is, for Poincaré, it is the mistake of

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16 Poincaré, (1909b), p.47.

17 Poincaré, (1906b), p.195.

18 Poincaré, (1905b), pp.144-145.

the realist to assume that the correct account of the "$\forall$" quantifier, applied to infinite sets, is that of an infinite conjunction. It is the combination of realism about a totality - considering it as there before we generate its elements - plus unrestricted quantification on the totality, which allows the range of a quantifier in the specification of a totality to include the totality itself.

By 1912 Poincaré offers a more precise, more general analysis of the concept of impredicativity. Impredicative definitions are those which use:

> [a] a relation between the object to be defined and all the individual objects of a genus of which the object to be defined is itself supposed to be a member (or [b] of which one supposes to be members objects which themselves can be defined only by the object to be defined). This is what happens if we posit the two following postulates:
>
> [a] X (object to be defined) is related in such and such a way to all members of the genus G.
>
> X is a member of G.
>
> Or else the following three postulates:
>
> [b] X is related in such and such a way to all the members of the genus G.
>
> Y is related in such and such a way to X.
>
> Y is a member of G. 19

Chihara provides a more modern, more precise characterisation of the logical notion of impredicativity, in terms of quantification:

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19 Poincaré, (1912a), p.70.

> A specification of a set A by means of the schema
>
> $$(x)\ (x \in A \leftrightarrow \phi x)$$
>
> is impredicative if the set A, were it to exist, or any set presupposing the existence of A, falls within the range of a bound variable in the specification. 20

If the range of any bound variable in ϕ has A as a member, then the specification via ϕ of A is impredicative. Finally, Poincaré relates the impredicativity of a specification to the presence of a vicious circle, thus distinguishing the two notions:

> To the pragmatists such a definition implies a vicious circle. It is not possible to define X without knowing all the members of the genus G, and consequently without knowing X which is one of them. 21

2.2 Russell's account

Russell's concept of impredicativity is eventually expressed in his Ramified Theory of Types (RT). Like Poincaré's concept, Russell's too evolved gradually out of a desire to isolate and exclude the mistake involved in producing the paradoxes. The Simple Theory of Types, outlined in Appendix B of Russell's volume, The Principle of Mathematics[22], is later rejected by Russell, and the more complicated Ramified Theory put forward, for various

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20 Chihara, (1973), p.5.

21 Poincaré, (1912a), p.70.

22 Russell, (1903).

reasons (which, I feel are interrelated; and so I will not attempt to say exactly which doctrines are a consequence of which others, if, indeed, this can be done at all). To be sure, a major reason Russell was not satisfied with the Simple Theory was that it is not global: it only solves the syntactic paradoxes, i.e. those which can be construed in terms of classes. Whereas, with Poincaré, Russell saw the same mistake at the foundation of all the paradoxes. Russell also felt it was necessary for the solution of the paradoxes to be a consequence of a more general theory (of what it makes sense to say), so that the theory could be seen as a logical result, and less arbitrary or ad hoc:

> It is important to observe that the vicious-circle principle is not itself the solution of vicious-circle paradoxes, but merely the result which a theory must yield if it is to afford a solution of them. It is necessary, that is to say, to construct a theory of expressions containing apparent variables which will yield the vicious circle principle as an outcome. It is for this reason that we need a reconstruction of logical first principles, and cannot rest content with the mere fact that the paradoxes are due to vicious circles. 23

And, regarding Poincaré's "intuitive", anti-logicist solution", Russell "quips":

> We may concede that positive errors are less likely to emerge, if we only apply our rules where "intuition" (i.e. common sense) suggests that we may safely do so. But there are some people who would prefer true rules of reasoning ... So long as we only know that a rule holds

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23 Russell, (1906b), p.205.

> in ordinary cases, without knowing what cases are ordinary, our mathematics is in a precarious condition. 24

The theory which excludes the paradoxes must be a result of the principles of logicism, for Russell.

The theory, thus takes the form of a "no-classes" theory, so that it may be construed as a set of logical principles. The thought that classes are fictions is reinforced by two other problems: (i) because the empty set, ϕ, is not a collection; and (ii) because singleton sets, $\{a\}$, are not equivalent to the single elements which are their members ($\{a\} \neq a$). In addition, Russell was perhaps uneasy about the notion of a stratification or layering of objects via logic, which is necessary (in the Simple Theory) if classes are considered as Frege considered them, as objects. There were also problems outside of set theory, Russell's solutions of which were related to his work on solving the paradoxes. For example, the problem of how we can understand sentences which do not successfully refer, and what their precise meanings are, is a problem which Russell solved by turning from an extensional to an intensional point of view. That is, "The King of France is bald"-problem, which led to the theory of definite descriptions, and (then) to the theory of substitutions: the two joint precursors to RT.[25]

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24 Russell, (1906b), p.196.

25 See Russell (1905) and (1906c).

Russell wished to eliminate any assumptions in logic (and in mathematics), and in the linguistic analysis of the meanings of our assertions, about the ontology of the universe. He thus needed to analyse all expressions in terms of propositional functions, and to restrict the ranges of the quantifiers in the expressions of all propositional functions, to those which are meaningful or significant for the concepts employed. (So, in effect, he agrees with Poincaré that at least part of the problem of the paradoxes lies in the unrestricted quantification employed. Whereas, however, Poincaré sees this as the result of a misplaced realism about (certain) totalities, Russell attributes this to logic: to a more general mistake to be found in an imprecise analysis of the workings of our language.) Due to the theory of definite descriptions classes are analysed as propositional functions, and like most nouns (except for "this" and "that") are a mere facon de parler. Impredicativity is to be characterised in terms of propositions and propositional functions; and the VCP is the statement (roughly) that:

> a function is not well-defined unless all its [possible] values are already well-defined. 26
>
> the values of a function cannot contain terms only definable in terms of that function ... ([Otherwise] the values of the function would not be determinate until the function was determinate, whereas we found that the

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26 Russell, (1910), p.39.

> function is not determinate unless its values are previously determinate.) Hence, there must be no such thing as the value for $\phi\hat{x}$ with the argument $\phi\hat{x}$, or with any argument which involves $\phi\hat{x}$ ... In fact "$\phi(\phi\hat{x})$" must be a symbol which does not express anything ... 27

"Arguments" for a propositional function are those expressions which can be substituted for the x in the $\phi\hat{x}$, or are those which fall in the range of any bound variables in ϕ. Thus, for Russell, a predicative definition is a propositional function (or proposition) which accords with RT.

> A predicative function of a variable argument is any function which can be specified without introducing new kinds of variables not necessarily presupposed by the variable which is the argument. 28

In other words, for Russell, the statement of some universally quantified expression for example, presupposes the prior existence of a domain of objects (or expressions) which may or may not be instances of the expression in question (which are possible arguments for the function). And no quantified expression can include itself in its range of possible arguments, or instantiations, because a quantified expression cannot meaningfully presuppose the prior determinacy or existence of itself. The instances which would in general make a specification impredicative

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27 Russell, (1910), p.40.

28 Russell, (1910), p.54.

are thus not possible arguments for the propositional function. They are "values of x with which 'ϕx' is meaningless".[29]

Whatever the merits of Russell's expression of the concept of impredicativity, the full programme (RT) - of which the VCP is a consequence - cannot be accepted. It does not accord with his logicism, for it is not logic, on any augmented account. His rejection of all existential assumptions about the ontology of the universe is bought at the cost of parallel assumptions about the nature of our concepts. These are embodied in the necessary axioms of Infinity - whereby it is possible to have infinitely many arguments in a truth-function - and Reducibility - the assumption that there is always a primitive predicate corresponding to each defined symbol in the hierarchy; an assumption which is, in effect, equivalent to the ontological assumption of the existence of a set of real objects as arguments for every symbol. In addition to the "non-logical" nature of these axioms Poincaré objects that a logicism embodied by a theory like RT is circular. For RT presupposes the theory of ordinals already to be established: it requires functions the specifications of which can refer to all finite orders. And thus, RT is giving up and not expressing logicism, as Russell had intended it to, for

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29 Russell, (1910), p.41.

it shows that a theory of arithmetic is prior to the formalisation of language proposed.

(3) The Objection to Zermelo's Solution

On Zermelo's view, the paradoxes result from allowing the consideration of sets which are simply too "big", i.e., the size of the universe. The totalities which are referred to in the contradictory specifications can be thought of as the size of the universe because they are unextendable (Fraenkel's metaphor). They are closed under diagonalisation; so we do not "get out" of them via certain diagonalising operations, as in the characterisation of e* in Richard's antinomy. Zermelo's solution is to lay down explicit axioms whereby a universe of sets isomorphic to Cantor's transfinite hierarchy can be produced; but where none of the problematic "too big", absolutely infinite sets can be produced.

Poincaré objects, however, that Zermelo's diagnosis of the paradoxes - that it is a matter of the <u>size</u> of the totalities referred to - is simply wrong. On his view - on his interpretation of Zermelo's argument - the argument for a "limitation of size" principle, turns on a mis-assimilation of the two distinct classes of impredicativity.[30]

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30 "La raison invoquée par M. Zermelo ne saurait donc suffire pour justifier l'emploi des définitions 'non prédicatives' car l'assimilation qu'il fait est inexacte." (Poincaré, (1909a), p.119.)

Poincaré admits that there are harmless impredicative specifications; so he agrees that there is no justification for excluding all impredicative characterisations. But the distinction between the harmless and the harmful impredicative specifications, does not lie in the size of any totality which is presupposed (referred to) in the specification; but rather in the issue of whether or not the totality referred to has been previously defined. That is, if our definition (of an object in terms of a collection to which it belongs) is not a construction, is not creative, then the impredicativity is harmless: not viciously circular, and not paradoxical. It is only when both the object to be defined and the totality to which it belongs are being defined (generated, constructed) for the first time, that impredicative specifications must be disallowed.[31]

However, Poincaré's distinction between harmless and harmful impredicative specifications depends on his possessing a sharp distinction between creative and non-creative specifications. And Zermelo, not being (any sort of) a constructivist, does not share such a distinction. On his view, there is no profound difference between creative and non-creative definitions. And thus, the distinction between the two classes of impredicative specifications must lie elsewhere: for him, in the extensional matter of the size of the collections to which the specifications

31 This - Poincaré's diagnosis and solution - will be discussed at length in Section 4, below.

refer. Cantor's idea that every well-founded property determines a set is not really captured by Russell's comprehension principle,

$$\exists y \forall x \ (x \in y \leftrightarrow \phi(x)),$$

nor by Frege's class existence principle,

$$\{x: f(x)\} = \{x: g(x)\} \leftrightarrow \forall x \ (f(x) \leftrightarrow g(x)).$$

Whereas Poincaré accepted Cantor's intuitive comprehension principle - where the <u>well-foundedness</u> of the ϕ (i.e., ϕ must be predicative when the specification is creative) is what obviates the contradictions - Zermelo focuses not on the nature of the ϕ, but on the nature of the range of ϕ. His solution is simple: let us merely stipulate that the range of ϕ is not "too big", by requiring that ϕ determine a subset of some independent set z:

$$\forall z \ \exists y \forall x \ (x \in y \leftrightarrow (x \in z \ \ \& \ \phi(x)).$$ [32]

Thus, we ensure that the set y is small with respect to the set-theoretic universe, for it is always a subset of (and so either the same size or smaller than) some other accepted set z. This, in effect, stipulates that no specifications are "creative" in Poincaré's sense, for we (suppose we) always <u>already have</u> all the objects which might have the property ϕ.

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32 Kunen, (1980), pp.10-11.

Although Zermelo's system - in particular his revised comprehension schema (RCS) - does succeed in blocking the derivation of a contradiction from the impredicative properties of the syntactic paradoxes, Poincaré could not accept it as really solving the paradoxes. The problem with the paradoxical totalities may be that they are too big, and so contradictory; but this "diagnosis" does not address the question of the true origin of the problem. What makes them too big in the first place is the question to which Poincaré demands an answer. For him, the answer is the ineliminable impredicativity, or vicious circularity, of their specifications.

Poincaré sees something common in all the paradoxes - finite/infinite, syntactic/semantic - and he was the first to globalise Richard's diagnosis (of his own paradox).[33] The mere size of a collection cannot be the whole problem, for, as Russell pointed out[34], there are paradoxes which do not even concern a collection (e.g., the liar). Concerning those paradoxes which do involve a reference to a collection, Poincaré correctly sees the problem as one of the acceptance

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33 See Section 2.1, above.

34 Russell, (1906b), p.197.

of totalities with imprecise, unfixed, indeterminate boundaries, frontiers or walls.[35]

> A "non-predicative class" is not an empty class, but a class with <u>uncertain boundaries</u>. 36

To be sure, sets which are "too big" according to Zermelo's account, will have uncertain boundaries - because, in effect, they have no boundaries at all (being the size of the universe). However, the sets which are too big in an intuitive sense do not seem to exhaust all those with uncertain boundaries. So, Zermelo's solution is not intuitively satisfying, for he merely <u>blocks</u> the paradoxes by blocking one symptom of the problem (sets which are too big), rather than <u>solving</u> the paradoxes by looking for their true origin.

In addition, Poincaré objects that the axioms themselves are not "intuitive" - i.e., they do not demand immediate assent in virtue of our concept of set alone. Thus, an argument to the effect that the axioms are true, are faithful to our pre-formal concepts of set and of object, is required. But on Poincaré's view the arguments Zermelo provides (concerning the source of the paradoxes, concerning Russell's comprehension schema) are misleading.

> Mr. Zermelo does not allow himself to consider the set of all the objects which

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35 The nature of such "geographical" metaphors is discussed in Section 4, below.

36 Poincaré, (1906b), p.191. (My emphasis.) (Quoted more fully below, pp.158-159.)

> satisfy a certain condition because it seems to him that this set is never closed; that it will always be possible to introduce new objects [to the range of ϕ]. 37

Poincaré interprets Zermelo's argument for his RCS as pointing out the need for determinate boundaries on the range of ϕ in any set specification. However, the mere stipulation of the existence of the set z on which ϕ is defined does not show that the set y, which the schema asserts exists, is small, or closed in our intuitive sense, because the set z might already be too big.

There are two points here. First, there is no way to tell what the nature of z is. It might very well be the case, given the axioms of infinity and power set, that z is already too big on Poincaré's conception of set (as constrained by our iterative intuition). Thus the fact that the set y is a subset of z does not reassure us that y is "small", does not "contain" y, for y could be the same size as z - i.e., too big.

> Mr. Zermelo has no scruple in speaking of the set of objects which are a part of a certain <u>Menge M</u> and which also satisfy a certain condition. It seems to him one cannot possess a <u>Menge</u> without possessing at the same time <u>all</u> its elements. Among these elements he will choose those which satisfy a certain condition ... without fear of being disturbed by ... unforeseen elements, since he <u>already</u> has all these elements in his hands. By positing beforehand this <u>Menge M</u>, he has erected an

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37 Poincaré, (1909b), p.59.

> enclosing wall which keeps out intruders who could come from without. But he does not query whether there could be intruders from within whom he enclosed inside his walls. If the Menge M possesses an infinite number of elements, this means not that these elements can be conceived of as existing beforehand all at once, but that it is possible for new ones to arise constantly; they will arise inside the wall instead of outside, that is all. 38

That is, for Poincaré, postulating the pre-existence of the set z does not ensure that the specification of y is not in some sense creative, especially if z is infinite - if all the elements of z may not be considered as pre-existing. Furthermore (second), given the independence of the (general) Continuum Hypothesis from ZFC[39], z (or $2^{\aleph_0}$) can be as big as one likes. But this now violates all our intuitions concerning what is closed or determinate.

For Poincaré, what is closed or determinate, or an acceptable set, is not what is merely extendable according to the axioms of ZFC. The mere availability of iterations of the axioms to produce bigger, more extensive sets than in the previous stage of iterations of "the set of ...", is not sufficient to show that for each set thus obtained - at every stage - we only have determinate, acceptable, "small" sets. To show that an arbitrary Zermelo set is extendable, one must indeed presuppose that the axioms

38 Poincaré, (1909b), pp.59-60.

39 Cohen, (1963), highlights the drama of this situation with his "forcing" technique.

(e.g., of separation) are applicable. (For example, in showing that a set, x, is extendable by "diagonalising out" of it, one requires the axiom of separation to form a subset of X.[40]) However, this is not to <u>show</u>, but to <u>presuppose</u>, to <u>proclaim</u> that the Zermelo sets are determinate. That is, crudely, showing that a set is extendable depends on Zermelo's own characterisation of extendability as embodied in his axioms. But establishing the acceptability of (the sets which are produced by) his axioms was the whole point of the argument concerning extendability. The base case of the collection of Zermelo sets would be the first application of his axioms, i.e., $\mathfrak{P}(\mathbb{N})$. But the acceptability of this set is precisely what is in need of justification. For Poincaré the base case is too big; so that showing that iterations of the axioms does not take us out of the domain of such acceptable sets is to show nothing. And even regardless of Poincaré's views on the constraints imposed by our iterative intuition, there seems to be no non-arbitrary way to evaluate the (cardinal) size of this set (the base case); so calling it small is to <u>presume</u> and not to <u>show</u> that Zermelo sets are all small. Thus, Poincaré says:

> A classification was relied upon which was not immutable and which could not be so; the precaution was taken to <u>proclaim</u> it immutable; but this precaution was insufficient. 41

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40 See Hallett, (1984), Chapter 5, especially p.204, for a more detailed argument.

41 Poincaré, (1909b), p.45.

Therefore, not only do Zermelo's axioms violate our pre-formal concept of set; his arguments for his axioms are circular. For there is no way to argue, as Fraenkel attempted (as shown by Hallett), for the thesis that mere extendability captures our intuitions concerning the determinacy of a set, without presupposing that the ability to diagonalise out shows a set to be determinate, i.e. not too big already. For Poincaré, therefore, the Zermelo axioms are insufficient both formally and intuitively.

> This is why Mr. Zermelo's axioms could not be satisfactory to me. Not only do they not seem evident to me, but when I am asked whether they are free from contradictions, I shall not know what to answer ... even though he has closed his sheepfold carefully, I am not sure that he has not set the wolf to mind the sheep. 42

(4) Poincaré's Diagnosis and Solution of the Paradoxes

Poincaré rejected Zermelo's solution of the paradoxes; yet there is something right in Zermelo's diagnosis that Poincaré accepts, and that is the recognition that the mere presence of impredicativity is not sufficient to explain the paradoxes.

4.1 Circles, vicious circles, and two types of definitions

All impredicative definitions are circular, for they define an object in a way which is "self-referential": the

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42 Poincaré, (1909b), p. 60.

definition of the object X either refers directly to X, or it refers to a different mathematical object Y, the existence of Y presupposing the existence of X. However, not all circles are vicious, and not all impredicative definitions are viciously circular or paradoxical. For example, "tallest man in the room" is impredicative, for it picks out a man, the tallest one, via a relation (the relation being "taller than") between the man and all the members of the totality, men in the room. But the object of our definition is a man in the room, so he is a member of the totality via which he is defined. So this definition is impredicative. However, it is by no means viciously circular. It is meaningful, for via it we can pick out, deterministically, the object (the man) which satisfies the specification. Impredicative specifications are common in everyday speech: "Out of all the horses I prefer the bay with the white socks"; "... the chair (out of all the chairs in the room) with the high back"; "the chapter in this thesis which concentrates most on impredicativity". It is because the objects in the totalities of the "everyday" specifications are empirical, because the collections are finite and surveyable, their existence independent of the specifications, that the circularity presents no epistemological problem whatever. Now, the distinction between objects which we need to create or construct, and objects which exist independent of us, or which have already been constructed, is central to Poincaré's philosophy; and it plays a crucial role in his solution of the paradoxes.

Poincaré is some sort of a constructivist (a neo-Kantian sort), and thus he claims that a mathematical "object exists only when it is conceived by the mind".[43] So the specification which makes the object exist (in the mind) must accord with the apriori conditions of the mind (i.e., logic plus the apriori forms of intuition) in order for the specification to be meaningful and succeed in denoting a determinate object.

> X exists only by virtue of its definition, which has meaning only if all the members of G are known beforehand, and X in particular. It would be useless to say ... that it is not a vicious circle to define X by its relation to X ... 44

Impredicative specifications fail to fix a determinate mathematical object because, in the absence of any prior such fixing, it is impossible to construct a (new) mathematical object by virtue of an impredicative definition. Gödel makes the following famous remarks on definitions which are constructive or creative, i.e., definitions which specify objects which are constructed by ourselves:

> In this case there must clearly exist a definition (namely the description of the construction) which does not refer to a totality to which the object defined belongs, because the construction of a

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43 Poincaré, (1912a), p.72.

44 Poincaré, (1912a), p.71.

> thing can certainly not be based on a totality of things to which the thing constructed itself belongs. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described (i.e. uniquely characterised) only by reference to this totality. 45

It is only when the totalities need to be constructed, when the definitions are <u>creating</u> the objects, that impredicative specifications are an issue. It is thus that Russell's famous informal statement of the VCP, which is really two, non-equivalent statements, depends upon a constructive point of view. Russell states that certain sorts of objects, e.g., "propositions, classes, cardinal and ordinal numbers, etc. represent illegitimate totalities, and are therefore capable of giving rise to vicious circle fallacies".[46] On his view the paradoxes, or vicious circles, arise as a result of assuming the existence of such illegitimate totalities, an assumption which is realist, in essence. And the acceptability of Russell's solution requires a non-realist point of view (in keeping with his no-class theory whereby classes are mere fictions).

> The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves <u>all</u> of a collection must not be one of the collection"; or, conversely: "If, provided a certain

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45 Gödel, (1944), p.456.

46 Russell (1910), p.38.

> collection had a total, it would have members only definable in terms of that total, then the said collection has no total. 47

The "involvement" relation in the first statement is imprecise; and in the absence of constructivist constraints, it is unclear why there is anything wrong with one of a collection involving all of the collection. The second statement is better in its emphasis on "only definable", but there is still no argument here for accepting it in the absence of anti-realist presuppositions, i.e., in the absence of Russell's whole theory. We must be constructing the collection which has members only definable (i.e., only constructible) in terms of the collection, in order to explain our rejection of the existence of the totality.

The contrast with realism is the following. If our conception is non-constructive, and sets - like men - exist independent of our specifications, then how we define the sets is independent of the matter of their existence. So, on the face of it at least, impredicativity does not need to be a concern for the Platonist. However, if our conception is constructivist, or non-realist, and sets, unlike empirical objects, exist only insofar as we define or can define them; then, no matter how far we extend the strength of "can" here (thereby strengthening the notion of constructibility), we can never accept the

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47 Russell, (1910), p.37.

existence of a set the construction of which is _only_ characterisable by an impredicative specification. Thus, "all of a collection must not be one of the collection" only when both the element _and_ the collection to which it belongs require each other for constructibility. For then _neither_ will be constructible due to this "vicious circle". Commenting on the paradox of "the smallest integer which cannot be defined by a sentence with fewer than one hundred French words", Poincaré provides an insightful analysis of _why_ the impredicativity here leads to a vicious circle:

> This reasoning rests on a classification of integers into two categories: those which can be defined by a sentence with fewer than one hundred French words and those which cannot be. In asking the question, we proclaim implicitly that this classification is immutable and that we begin our reasoning only after having established it definitively. But that is not possible ... the classification of numbers can be fixed only _after_ the selection of the sentences is completed, and this selection can be completed only _after_ the classification is determined, so that neither the classification nor the selection can _ever_ be terminated. 48

It is the old "chicken and the egg" question, which has no answer, since each object seems to require the other for its existence. This is what the "only definable" in Russell's statement is meant to capture: that the said

48 Poincaré, (1909b), p.46.

collection of elements does not exist relative to some other definition in a prior theory. And it is because it does not exist in virtue of some other fact - e.g., those embraced by Platonistic assumptions - that the postulate is "creative" or constructive, and thus that it must be predicative. (Thus, one of Gödel's criticisms of Russell's RT[49] is that the strongly non-constructive axioms of infinity and reducibility violate the constructive presuppositions necessary for accepting his version of the VCP.)

4.2 Poincaré's conception of set as constructed entity, and his "True Solution"[50]

Poincaré's solution of the paradoxes relies explicitly on his constructivism, to which he remained faithful. Interestingly, he accepted Cantor's "intuitive axioms",[51] and he was in general in favour of increasing precision in mathematics. But the formal definitions cannot replace the intuitive notions; and the axioms are only true in intuitive domains: i.e., domains which are constrained by (his neo-Kantian) constructivist principles.[52] It is thus

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49 Gödel, (1944).

50 The title of a section in his (1906b), p.189, is "The True Solution".

51 Poincaré, (1905b), p.159.

52 "It is true that Cantorism has been useful, but that was when it was applied to a real problem, whose terms were clearly defined, and then it was possible to advance without danger." (Poincaré, (1906b), p.195.)

that his solution of the paradoxes did not require him to alter or add to his principles, for some theory of predicativism is already entailed by his theory of meaning (whereby we must "verify" or prove that any definition of a new entity is consistent), and by, in general, his constructivism. Sets are constructed objects, formed by "the set of ..." operation, with respect to some property or properties of previous well-defined, determinate objects. The existence of a set requires that we construct it by virtue of a definition which collects together its elements, or (when the number of elements is infinite) by virtue of an iterative generating rule. In order for the set to be determinate, the rule must be determinate, the definition must "collect together" the elements in a determinate way. Thus, the nature of the characterisation of an object is relevant to the question of the existence of the object. If the characterisation is viciously circular (if it is impredicative and the totality to which it refers does not exist independently of the specification), then whether or not we know that certain objects are members - e.g., it is determinate that 1, 2 and 3 will be members of the totality E in Richard's paradox - it is impossible to possess a determinate conception of all the members of the totality; for its boundaries are indeterminate.

> A definition which contains a vicious circle defines nothing ... It is not a question of knowing whether this class is empty, but whether it can be rigidly delimited. A "non-predicative class"

> is not an empty class, but a class
> with <u>uncertain boundaries</u>. [53]

Impredicative definitions are unfaithful to our understanding of an object, not because they fail to pick out any members, but because they fail to <u>fix</u> the boundaries of the set of members they purport to pick out.

Objects <u>qua</u> <u>objects</u> have determinate, definite "boundaries". This is just what we mean by an object. If the determinacy is not provided by the empirical world, then in order to consider a concept as determining a well-defined, bona fide object, we must contribute the determinacy ourselves. Since, for Poincaré, mathematical sets are constructed objects, we must ensure that our specifications denote only determinate sets, with precise definite boundaries. According to Poincaré this means we can only consider sets which are predicatively specifiable in a finite number of words. A set will be determinately specified if the nature of <u>all</u> its objects, if the structure of the "container", can be determinately specified. Let us return to the container metaphor of Chapter 5 to explain Poincaré's conception of a closed, determinate set.

A set is like a container or box with just enough room or places to hold all and only the objects which are its members. A set is determinate when all of its spaces have

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53 Poincaré, (1906b), p.191. (My underlining.)

been filled, or when there is a general rule governing the way in which we put or generate the objects in the box. In virtue of arithmetic intuition we can treat potential infinities as completed - as if the container was filled. That is, via our understanding of an iterative rule we can "gloss over" all the elements which have not been generated, or put into the box, "pretend" that they are there, and close the box: put its lid on. Thus, for Poincaré, the determinacy criterion for the acceptability of a set specification means that the set can be completed, either actually or via apriori intuition, and then considered closed because all of the members have been, in general, determined. "The set of ..." operation is in general limited to those rules which can be closed via apriori arithmetic intuition, or enclosed between two brackets: $\{0, 1, 2 \ldots n, \ldots\}$. In addition, he permits a special "set": the set of points on the geometric continuum, the set of real numbers, as a primitive object of intuition - not characterisable by a rule - as an object of apriori geometric intuition. (The finite numbers are primitive and immediate objects of intuition.)

This conception of sets, however, automatically excludes the harmful impredicative specifications from being meaningful. In order to define a new entity by referring to a set, the set must be a determinate object; we must be able to think of it as closed or completely filled "with its lid on", either because it is finite or in virtue of apriori intuition. This is an anti-Platonist or anti-Cantorian (constructive) view, for according to Poincaré, a totality cannot be considered as given independently of its members.[54] So in order to <u>have</u> a totality (in terms of which to specify an object) we must <u>have</u> all its members, and this must be minimally guaranteed in the intuitive, neo-Kantian sense. Thus, if a characterisation, of α, is <u>creative</u>, and if it refers to a totality, X, what α <u>cannot</u> be is one of the elements of X. For, by hypothesis, X is already completely filled, with its lid on, i.e., closed.

> This reasoning rests on a classification ... In asking the question, we proclaim implicitly that this classification is immutable and that we begin our reasoning only after having established it definitively ... [55]

Thus any specification which is creative, and which is in terms of the whole of a totality, must be one whereby we "diagonalise out" of the totality. The only condition under which a specification can be acceptably impredicative is when it is <u>not creating</u> a new member (of the totality), but merely <u>picking out</u> one of the members already (created) in X. In this way, it already being <u>in</u> the box, we do not have to disrupt any elements to put it in. Moreover, on this conception, what can never be acceptable is a set belonging to itself, as in the set of all sets, or the

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54 Heinzmann, (1985), p.60.

55 Poincaré, (1909b), p.46. Quoted more fully above, p.156.

application of "the set of ..." applied to the collection of all ordinals. These impredicative specifications do not arise because one cannot put a box into itself. Impredicative specifications are thus allowed only when they are of the harmless variety, i.e., when they are non-creative. For then the totality can be thought of as completed, the box closed, and the specification in terms of all the members of the totality can only pick out an existing member, and never add a new member to the box.

Therefore, the contrast between creative and non-creative specifications was central for Poincaré, and it is not a distinction merely between empirical and non-empirical domains. Even within constructive (i.e., non-Platonist) mathematics a distinction between creative and non-creative definitions can be sustained. Creative definitions are postulates which introduce a new domain: e.g., the use of the addition of the root of a natural number which has no rational square root (like $\sqrt{2}$, $\sqrt{3}$) to generate the surd field: $a+b\sqrt{2}$ for a, b $\in \mathbb{Q}$; and the addition of $\sqrt{-1}$ to form the field of complex numbers: $a+bi$ for a, b $\in \mathbb{R}$. Non-creative definitions do not introduce or form a new object or domain, so that impredicativity can be tolerated, as in the following examples: the impredicative least upper bound theorem; and the impredicative specification of the neutral element of a group.[56]

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56 Heinzmann, (1985), pp.42-43.

The non-creative specifications will sustain impredicativity without paradoxical consequences because the specification of an element e in terms of all the elements in a collection E (one of which is e) comes after the specification of E (which must be determinate). Thus, we would have independent access to all the members of E, including e, by virtue of the specification of E; so we may pick e out via reference to all the elements of E. It must here be noted that "independent access", for Poincaré, includes not only explicit, prior constructions, but domains which are given primitively, in apriori intuition as well - like $\mathbb{N}$ and $\mathbb{R}$. And it is in virtue of his theory of the synthetic apriori, i.e., arithmetic plus geometric intuition, that Poincaré can accept the least upper bound theorem in the same way that he can accept a specification like $\{x: \mathbb{N}(x) \& x < 3\}$. Neither presents any epistemological problems, because the totalities referred to - $\mathbb{R}$ and $\mathbb{N}$ - "exist" (in intuition) before the impredicative specification.

> Plus généralement, si nous envisageons un ensemble E de nombres réels positifs, par exemple, on peut démontrer que cet ensemble possède une limite inférieure e; cette limite inférieure est définie après l'ensemble E; et il n'y a pas de pétition de principe puisque e ne fait pas en général partie de E. Dans certains cas particuliers, il peut arriver que e fasse partie de E. Dans ces cas particuliers, il n'y a pas non plus de pétition de principe puisque e ne fait pas partie de E en vertu de sa définition, mais par suite d'une démonstration postérieure à la fois à la definition de E et à celle de e. 57

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57 Poincaré, (1909a), p.119.

Impredicative specifications are paradoxical when they refer to totalities which do not (yet) exist as closed, determinate sets. Thus the paradoxical cases arise as a result of a misplaced realism about the existence of such totalities, or about the objectivity of truth of any statements concerning all of the members of such a totality. For example, neither the totality of sets, nor the totality of subsets of $\mathbb{N}$ (for all the subsets would include all the arbitrary infinite subsets), are intuitive mathematical objects for Poincaré. So any "semi-realism" (or realism) about these collections is illicit, and any impredicative specification which presupposes the determinacy of such a collection is unjustifiable. This is Poincaré's objection to the set-theorists: when the totality of real numbers is construed as a totality of arbitrary sets (of natural numbers, or arbitrary Dedekind cuts, for example) the impredicativity in the specification of the least upper bound is no longer acceptable. This is because in the absence of apriori geometric intuition, we no longer have a determinate totality of real numbers which can be thought of as "closed", because every formal characterisation of the classical totality will require the acceptance of ineliminably impredicative specifications. So to attempt to replace geometric intuition with a formal notion like least upper bound (which guarantees the nature of the classical continuum - i.e., that there are no gaps in principle) is illicit, for it is to remove the totality which forms the only foundation we can have for the impredicative specification of least upper bound.

Furthermore, Poincaré's "solution" can be understood as global if one interprets the semantic paradoxes (as he seemed to, e.g., in his (1909b)) as involving the (misplaced) assumption of a totality of semantic objects. For example, the Grelling paradox of heterologicality can be taken as a paradox in second order logic:

$$(\forall\alpha)\ (\mathrm{Het}\ (\alpha) \leftrightarrow -\alpha(\alpha)).$$

An adjective, α, is heterological if and only if it does not apply to itself. For example, "big" is heterological because "big" is not a big word; but "small" is not heterological (it is autological) because "small" is a small word. The paradox arises under the Platonist assumption that it is a determinate matter in every possible case whether or not an adjective is heterological; that is, under the Platonist assumption of the existence of a determinate totality of adjectives in terms of which we can define "heterological". But "heterological" is an adjective, so it must already be in the "container" of all adjectives so the paradox arises thus:

$$\mathrm{Het}\ (\mathrm{Het}) \leftrightarrow -\ \mathrm{Het}\ (\mathrm{Het}).$$

And the impredicativity of the specification of "heterological" is ineliminable, for "heterological" can only be defined in terms of such a totality, it being a property of adjectives.

Now, Poincaré would object that the (second order) assumption of a container of all adjectives is illicit. No such <u>determinate collection</u> exists. And since it is not a primitive object of apriori intuition, any specification which presupposes the determinacy of such a collection is not meaningful.

Whereas we have a clear grasp of certain (infinite) totalities of finite numbers, as a consequence of our general intuitive grasp of "finite numbers", we have no intuitive grasp of the set-theoretic notion of arbitrary set or collection (nor of the semantic notion of, e.g., arbitrary adjective). The totality of all sets does not exist previously or primitively in intuition - nor can we <u>construct</u> it, or <u>characterise</u> it "from below" via acceptable (exhaustive) operations. Thus, from a neo-Kantian constructivist point of view, any specification, any bound variable in a defining condition, ϕ, which ranges over sets, must range over totalities of sets which are <u>previously defined</u> in an explicit way. This makes available a general notion of the set, and so, an understanding of an arbitrary member, thus enabling the set to be an object of intuition. The set must be an object of intuition in order for statements about the set to be verifiable, as Poincaré's theory of meaning requires. Poincaré's constructivism, therefore, which he explicates in terms of a theory of verifiability in principle, automatically excludes the problematic impredicative characterisations: they are not meaningful.

An object defined by a specification, ϕ, cannot be in the range of any quantifier of its defining condition, nor can it be a member of a totality which is in the range of a quantifier in ϕ, unless it is a member of a totality to which we have independent and prior access - either relative to a prior postulate or theory, or when the totality is a primitive object of apriori intuition. The specification is simply not meaningful otherwise. For Poincaré, the nature of our minds, i.e., the synthetic apriori form of experience, determines "the true solution" of the paradoxes; for it determines what is to count as a meaningful mathematical specification. His theory of meaning is one arena in which he grounds the rejection of the unacceptable cases of impredicativity. Both his theory of meaning and his theory of impredicativity, however, are to be found in the conclusions of a very general constructivism, interpreted - as it must be for Poincaré - in the light of what is guaranteed by apriori intuition. I now turn to an examination of the central notion in Poincaré's theory of meaning: the notion of verifiability in principle.

CHAPTER SEVEN

VERIFIABILITY

(1) Poincaré's Criterion of Meaning

(2) How Poincaré Employs the Notion

(3) A Precise Statement of the General Requirement of Verifiability in Principle

(4) Three Aspects of "Verifiability"

(5) Potential Infinity and the Domain Argument Blocked

(6) Strict Finitism and the Objection to Poincaré's Theory of Verifiability

(7) A Misguided Argument Against the Strict Finitist

(8) Poincaré's Defence of the Notion of Indefinite Iterability

(9) Poincaré's Theory of Verifiability, and a Middle Position Between Intuitionism and Platonism

In Chapters 4 and 6 I showed how Poincaré's predicativist account of mathematics was a consequence of his particular theory of meaning. The central component of his theory of meaning is the notion of verifiability in principle. This notion is, however, far from unproblematic; and we must now enquire as to whether it is coherent. If verifiability in principle - also constructible in principle, provable in principle, decidable in principle - cannot be given a clear sense, then that which is the very cornerstone of Poincaré's philosophy of mathematics is possibly empty, and the whole edifice crumbles. Hence, we must argue that his verificationist theory of meaning, interpreted in the light of his background claims concerning the synthetic apriori, provides a workable account of the foundations of analysis.

As discussed in Chapter 4, Poincaré's notion of understanding - that we must understand in terms of "pictures" - is not verificationist in the classic logical positivist sense. Our concepts do not have to be reducible to ostensive definitions, or to terms which have only empirical content. Our concepts can also irreducibly refer to apriori, non-empirical "pictures": that which we can "represent to ourselves". This aspect of his theory of meaning is directly culled from Kant, and it informs, in particular, the central component of verifiability in principle. I will now take it that the existence of such apriori "pictures" is accepted, so that we can proceed with the present aim: to investigate the notion of verifiability in principle (in the light of the theory of the synthetic apriori), as it functions in Poincaré's theory of meaning.

(1) Poincaré's Criterion of Meaning

There are two distinct thoughts in Poincaré's account of the meaningfulness of mathematical statements. These are related, but they need to be distinguished. First, claimants to mathematical truth must be in principle provable. If a statement is mathematically meaningful, there must be nothing which in principle bars the possibility of proof. This is because Poincaré rejects verification transcendent truth in mathematics. There is no coherent notion of mathematical truth which transcends all (however "informal") methods of proof.

Second, mathematical statements must be "verifiable". By this Poincaré means that a proposed theorem must have an instantiable content. There must either exist a procedure, or computation, which shows that at least one instance of a general claim is true (or which refutes it); or, we must have some conception of what would constitute either a verification of an instance of the claim, or a refutation of it. This second idea is the more philosophically important one, and will play a crucial role in securing both Poincaré's acceptance of potential infinities, and his attack on the existence of uncountable collections. Hence, the important notion in his theory of meaning is verifiability in principle. It is the claim of this chapter that such a conception can arise and withstand criticism only on the basis of the apriori forms of intuition. It is to an examination of this criterion to which I now turn.

(2) How Poincaré Employs the Notion

Poincaré requires of mathematical statements that they be verifiable; otherwise, they are meaningless.

> When a theorem is brought to [my] attention without giving [me] a means of verifying it, [I] see in it only unintelligible verbiage. [1]

> Every mathematical theorem must be capable of verification. When I state this theorem, I assert that all the verifications of it which I shall attempt will succeed; and even if one of these proofs requires efforts which exceed the capability of a man, I assert that, if many generations, one hundred if need be, deem it appropriate to undertake the verification, it will still succeed. The theorem has no other meaning and this is still true if we mention infinite numbers in its statement. [2]

The statement of a theorem just is the statement that

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1 Poincaré, (1912a), p. 66.

2 Poincaré, (1909b), p. 62.

every instance is verifiably true. By verification of a theorem Poincaré means a proof that the mathematical statement is true for at least one instance of the domain to which it refers. Lack of a general proof does not imply that a statement is meaningless; this would be far too strong. Rather, lack of a means of showing a statement is true or false <u>for one instance</u>, implies that it is meaningless. Fermat's last theorem is not yet proved, but it is unquestionably meaningful. And the reason this is so, is that we show by a finite decidable procedure - an arithmetic computation - that an instance verifies the general statement. Though we have no formal proof that there are no *n*'s greater than 2, such that

$$x^n + y^n = z^n, \text{ for } x, y, z, n \text{ integers;}$$

we can show within Peano arithmetic that it is true for an instance. Thus,

$$3^3 + 4^3 \neq 5^3;$$

and this confirms or verifies the theorem, because it instantiates the general claim. For Poincaré, the fact that we can consider an instantiation of the general claim guarantees that the claim has determinate content. That is, the verifiability of a claim shows that it is in principle provable (or refutable in general).

(3) A Precise Statement of the General Requirement of Verifiability in Principle

Let us attempt to generalise Poincaré's notion of "verifiable" in a way which would apply to any sort of statement. We will contrast "verifiability in principle" with "verifiability in practice".

(i) To say that a statement, S, is decidable or verifiable in practice is to say that we can actually position ourselves so as to be in a state of information which enables us to decide (or provide evidence for) the truth-value of S.

(ii) To say that S is decidable or verifiable in principle is to say that it is not impossible to be in a state of information in which we can decide the truth value of S. That is, we can envision indefinite, but finite, extensions of our actual state of affairs or circumstances - such as time available, or powers of memory - such that if these were to obtain then we would be in a state of information which would enable us to decide upon the truth value of S. There must be nothing which in principle bars a decision; only our own actual limitations can be seen as preventing the decision.

The reason verifiability and provability is so important to Poincaré is because there is no such thing as a mathematical fact which is completely independent of actual, or in principle, possible mathematical activity. Meaning must be linked with practice. So - contra classical realism - statements, which are in principle beyond the verification capacities of <u>any</u> finite being, are meaningless because they are <u>in principle</u> undecidable. If a statement cannot be decided or modelled by a finite being no matter how much we extend his or her powers via finite additions, then it has <u>no</u> determinate content.

In order to consider a mathematical claim to be meaningful, we must be able to show, or we must be able to envision showing, that an instance of it is either true or false. But what is it "to be able to show", or "to be able to envision showing"? We must now enquire into the <u>methods allowed</u> for showing that an instance of a statement is true or false, for verifying it. In addition, we require an account of the constraints put on our powers of <u>envisioning</u>: we require an account of the constraints on the <u>in principle</u> aspect of verifiability in principle. A description of the acceptable procedures will clarify the content of the notion of verification, and of verifiable in principle.

(4) Three Aspects of "Verifiability"

What is verifiable in Poincaré's sense is directly determined by three things: (i) the finitude of our human capacities; (ii) our apriori arithmetical intuition; and (iii) our apriori geometrical intuition. These correspond to the ways in which we understand (mentioned above and discussed in Chapter 4): either in terms of a reduction to concepts which we know by acquaintance (parallel to (i) - alone); or in terms of a reduction to concepts which are instantiated in apriori intuition ((i) plus (ii) plus (iii)). The methods allowed, then, for verifying an instance of a general mathematical claim, will be informed by the ways in which we can "picture", or understand. It is not only actual, finitely decidable operations which determine meaningful mathematical structures - which determine our pictures. Our apriori intuitions also provide apriori "pictures"; these extending what is mathematically acceptable, or verifiable, by supplementing the class of permissible operations. Our concepts can be instantiated via constructions, actually carrying out the operation, or via apriori intuition.

Part of the class of what is mathematically meaningful for Poincaré - that which is determined by (i) plus (ii) alone - corresponds to what is intuitionistically acceptable. Namely, those statements which are finitely refutable, or for which we have a constructively acceptable proof, are meaningful for both Poincaré and the intuitionist. The intuitionist accepts the same domains as those which are sanctioned by Poincaré's arithmetic intuition; for he accepts potentially infinite domains, or domains which are "constructible" according to an effective rule. This is the special intuitionistic sense of "constructible". Although there may be no reason for stopping the iteration of a rule - so that we never actually complete the construction - we can treat certain rules as defining determinate sets which are potentially infinite (certain potentially infinite sets are "constructible"), if the rules are sufficiently clear. The rules will, for the intuitionist, be sufficiently clear, if the outcome at every stage of construction of the set is determinate or effective, given the value of the prior stage.[3] The difference between Poincaré and the intuitionist - at this point - is that Poincaré, and not the latter, roots our understanding of the indefinite iterability of a rule, and hence, the potentially infinite nature of certain sets, in our apriori arithmetic intuition (as was discussed in Chapter 2).[4]

However, in addition to (i) and (ii) - in addition to constructive or intuitionistically acceptable proofs or refutations - what is meaningful for Poincaré is determined also by that which the intuitionist would reject: namely, non-constructive procedures, provided the properties of the domains in question are guaranteed by (iii) apriori

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3 See Heyting, (1971), pp.32-34 for the notion of an effective rule, and pp.13-15 concerning the infinity of the natural numbers. See also, Dummett, (1977), e.g., pp.55-65.

4 Poincaré's neo-Kantianism thus provides a foundation for explaining - as against the Wittgensteinian strict finitist - how it is we can have the certainty about a rule that at any stage we will be able to generate the next element, and that this is a determinate matter. See Wittgenstein, (1956); and see below, Sections 6-8, for more detailed discussion on this matter.

(geometric) intuition. This is precisely where Poincaréan mathematics differs in content from intuitionistic mathematics; and why, for example, Poincaré's continuum has such a different character from the intuitionistic one. It is because not only our constructive procedures - sanctioned, for Poincaré, by arithmetic intuition - but also our apriori geometric intuition will determine which domains will be mathematically acceptable, or which domains will be in principle verifiable or "picturable". And the methods which are acceptable for verifying statements will depend upon whether the domain concerned is purportedly arithmetical, or whether it exists only in virtue of our geometric intuition. Now Poincaré is not a revisionist. With regard to effective domains - those which are generated by an effective rule - the intuitionist can agree with Poincaré. However, their theories diverge in that Poincaré accepts the applicability of classical logic to domains which are not generated by an effective rule, i.e., to the mathematical continuum, since our knowledge of this domain is generated by apriori geometric intuition; whereas in the absence of the theory of the synthetic apriori, the intuitionist cannot - according to his own principles - stretch his account of bona fide mathematical domains past the denumerable. This is why Poincaré's theory of the continuum is not intuitionistic.

(5) Potential Infinity and the Domain Argument Blocked

Poincaré's theory of the continuum is not, however, classical either, as was shown in Chapter 5. A statement about an arbitrary real number, or an arbitrary point on the continuum, has content; but its content is not to be explained by reference to the classical, set-theoretic characterisation of the set of all possible subsets of the natural numbers (each subset corresponding to a unique real number). Arbitrary infinite collections are not meaningful domains, so Poincaré's theory of meaning directly prohibits the set-theoretic characterisation of the real line. Thus, statements about the continuum, about continuous variation, are not grounded in the existence of some actual infinity - as Cantor argued - but in virtue of geometric intuition. This is the foundation of Poincaré's rejection of Cantor's domain argument. He (Poincaré) asserts that

> Every mathematical theorem must be capable of verification .... and this is still true if we mention infinite numbers in its statement. But since the verifications can apply only to finite numbers, it follows that every theorem concerning infinite numbers or particularly what are called infinite sets, or transfinite cardinals, or transfinite ordinals, etc., etc., can only be a concise manner of stating propositions about finite numbers. If it is otherwise, this theorem will not be verifiable, and if it is not verifiable, it will be meaningless.
>
> And it follows that there could not be any evident axiom concerning infinite numbers; every property of infinite numbers is nothing more than a translation of a property of finite numbers. It is the latter which could be evident, while it would be necessary to prove the first by comparing it with the latter and by showing that the translation is exact. [5]

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5 Poincaré, (1909b), pp.62-63.

The point being made by Poincaré is thus that since we are finite, every verification must either be composed of a finite number of discrete steps, or it must be grounded in apriori intuition (either arithmetic or geometric). Not even one instance of a theorem can refer ineliminably to an actually infinite set or to an infinite number, for though our theorems cannot fail to refer, the only objects to which they can refer are <u>finite</u>.

Thus, for example, any theorem concerning all real numbers, or even all natural numbers, refers to an infinite domain, as in the commutativity of addition:

$$\forall x\ \forall y\ (x+y=y+x).$$

However, the infinity exists in the number of possible instantiations of the theorem, and not in the objects referred to in a single instantiation. Every instance of this theorem refers only to finite numbers; and so every instance is verifiable (in principle) via a finite deterministic computation. Whereas, the contrast with, e.g., the classical theorem in its full generality - that for any set x, $\overline{\overline{P(x)}} = \overline{\overline{2}}^{x}$ - is that there are <u>instances</u> of this theorem which refer ineliminably to infinite numbers; e.g., as in $\overline{\overline{P(\mathbb{N})}} = 2^{\overline{\overline{\mathbb{N}}}}$. These instances are not even in principle verifiable, for we have no apriori intuitions concerning the unlimited universe of infinite numbers which is sanctioned by classical set theory. For Poincaré, we only have intuitions about the finite numbers, as provided <u>apriori</u> in arithmetic and geometric intuition.

So for Poincaré, our understanding of "limit" was confused and unclear until the notions of "variable" and "going to infinity" were translated into explications in which there was no irreducible reference to infinity or to infinite procedures.

> I shall cite the following theorems as examples: the set of prime numbers is without bound; the series $\Sigma 1/n^2$ is convergent, etc. Each one of these can be translated into equalities or inequalities in which only finite numbers are involved. These theorems partake of infinity not because one of the possible [verifications] itself partakes of infinity but because the possible [verifications] are infinite in number. 6

Weierstrass formulated "limit of a sequence, $\{a_n\} = y$ as n tends to infinity", as:

$$\lim_{n\to\infty} \{a_n\} = y \Leftrightarrow \forall \varepsilon > 0 \; \exists k(\varepsilon) \; (\forall n > k \; |a_n - y| < \varepsilon).$$

In English: no matter how small ε is, you can always get closer to y by taking n large enough. Statements about limits are meaningful insofar as "tending to infinity" can be "translated" into a precise statement which refers only to finite numbers. That is, their meaningfulness depends upon the precise formulation not requiring the existence of actual infinities. So Poincaré's theory is in direct opposition to Cantor's famous domain argument.[7]

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6 Poincaré, (1912a), p.66. (I substitute "verification" for "proof" as it occurs in the 1963 edition; for given the context, "proof" is a misleading translation.)

7 See Hallett, (1984), pp.1-32.

Here we see the clash of the two giants in a dispute which is, at base, realism versus anti-realism. Cantor is unashamedly realist in his conception of "free mathematics", where coherence and the Divine intellect are the only constraints - this leading to the view that there is an actual infinity which corresponds to every potential infinity.[8] And Poincaré is fervently anti-realist in his view that mathematics is not free; the existence of a mathematical object depends on its being conceived by a finite mind; and thus there are no actual infinities.

> And why do the pragmatists refuse to permit objects which could not be defined in a finite number of words? It is because they believe that an object exists only when it is conceived by the mind and that an object could not be conceived by the mind independently of a being capable of thinking. There is indeed idealism in that. And since a rational subject is a man, or something which resembles a man, and consequently is a finite being, infinity can have no other meaning than the possibility of creating as many finite objects as we wish. [9]

Such a dispute, however, can be decided only -if at all - in the context of a detailed investigation into more general foundational issues in the philosophy of language and in epistemology.[10] Poincaré recognises the depth of the issue,

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8 Hallett, (1984), pp.14-25.

9 Poincaré, (1912a), p.72.

10 Such as are founded, for example, in Dummett's arguments concerning the acquisition and manifestation of our concepts in general (See Dummett, (1973).), and crystallised by Wright in, e.g., his (1986), especially section II, pp.18-32)

and in a very poignant passage opines that such a reconciliation is unlikely:

> At all times there have been opposite tendencies in philosophy and it does not seem that these tendencies are on the verge of being reconciled. . .. There is therefore no hope of seeing harmony established between the pragmatists and the Cantorians. Men do not agree because they do not speak the same language, and there are languages which cannot be learned. 11

(6) Strict Finitism and the Objection to Poincaré's Theory of Verifiability

Poincaré's theory of meaning, especially as it is wielded in arguments against Platonism, expresses the view that the conferral of truth or falsity, i.e., the meaning of a statement, must be fundamentally related to the way in which we investigate whether or not it is true. This is why when there is in principle <u>no</u> way to determine whether a statement is true or false (and when there is no apriori intuition corresponding to the domain in question), it <u>has</u> no meaning. Thus, the argument against Platonism is that it accepts the existence of domains which are meaningless, domains which (for Poincaré) cannot be ineliminably involved in meaningful statements. Thus, there is a gap between the meaning or content of a Platonistically acceptable mathematical statement and our

11 Poincaré, (1912a), p.74.

mathematical practice.

Poincaré, however, accepts the existence of potentially infinite collections, and indeed, the set of all real numbers. Are these really meaningful domains? Are the meanings of statements about the domains accessible to us in the appropriate way: by reference to our capacities for verification? The strict finitist employs Poincaré's argument against the Platonist (in the theory of meaning) to argue against Poincaré's own theory. Just as actual infinity transcends all our powers of construction and verifiability, so does potential infinity, for we can never verify a statement about <u>all</u> of (even) a <u>potentially</u> infinite collection. The concepts of both actual and potential infinity are illicit, for acceptance of even potential infinities severs the purported link between practice (i.e., verification) and meaning.[12]

Now, we cannot expect Poincaré to refute the sophisticated strict finitist, for he was not acquainted with any such arguments. Moreover, the latter's position is probably irrefutable, because insular. However, on the basis of Poincaré's <u>own</u> principles, strict finitism is

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12 See, for example, Wittgenstein's remarks (1956), and Wright's systematic treatment of these in his (1980); in addition, see the more localised (to mathematics) strict finitist arguments in Wright, (1982).

simply dismissable, for it is unfaithful to intuitions we, as a matter of fact, have.

> A man, however talkative he may be, will never in his lifetime utter more than a billion words. Consequently, shall we exclude from science the objects whose definition contains one billion and one words? ...
>
> However talkative a man may be, mankind will be still more talkative and, since we do not know how long mankind will last, we cannot limit beforehand the field of its investigations. We merely know that this field will always remain limited; and even though we might be able to determine the date of its disappearance, there are other celestial bodies which could take up the work left unfinished on Earth. The pragmatists, moreover, would have no qualms in imagining a mankind much more talkative than ours, but still retaining something human; they refuse to argue on the hypothesis of some infinitely talkative divinity ... [13]

Potential infinity is meaningful according to Poincaré's notion of verifiability, because of the coherence of the idea of indefinite but finite extensions, to our speed of thought or speech, to the existence of mankind. The understanding of indefinite extendability, then, is what defines the distinction between finite sets and potential infinity, as well as the distinction between potential and actual infinity. For it is through the understanding of this heuristic that we qua finite beings can "construct" potential infinities. The understanding of indefinite

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13 Poincaré, (1912a), pp.66-67.

iterability extends and informs Poincaré's criterion of verifiability in principle, which, in the end, is grounded in the epistemological theory of the synthetic apriori.

(7) A Misguided Argument against the Strict Finitist

One might attempt to argue that the theory of the synthetic apriori is not necessary to establish a stable position in opposition to both the strict finitist and the Platonist by citing the deep distinction between rule-governed and non-rule-governed infinities.

For Poincaré, there are two types of construction (which correspond to the two parts of a definition, discussed above (p.92 *passim*)): (i) the *creative* construction, where we define a new object or domain, and (ii) the *generative* construction, where we generate elements via a given rule which defines the relation between the elements of a domain, e.g., "+1". There is a deep difference between these two types of definition. Each creative definition requires a new act of intuition; hence, there are only ever a finite number of these possible, and no more than a potential infinity of these is coherent. This is because each must be fully defined in a finite number of words. One cannot define a sequence of creative constructions, the successor n' in terms of the prior construct n, for such a definition is not creative in the sense intended. For this reason, there is here no sense of the potentially infinite, for the infinity in potential infinity lies in its characterisability by a rule.

When we are constructing elements according to an accepted generating rule or recursive procedure, each element does not require a separate act of intuition. Because the defining condition allows one element to be defined in terms of another, each element is related to all the others. Thus we can have a general intuitive idea of the domain as a whole - via our intuitive understanding of an arbitrary element of the domain. In this way we can "see at a glance" the whole of the structure of a potentially infinite set. And thus, we can treat certain potentially infinite sets, where an understanding of this sort is possible - i.e., where we can satisfy an inductive axiom - as unified wholes.

This argument falters, however; for the existence of the distinction described above, between types of construction, does not establish the well-foundedness of the classical concept of indefinite iterability. For the strict finitist may allow that the distinction between creative and generative constructions is clear. It is just that he simultaneously denies that the distinction defines two different types of objects. These two different ways of defining do not lead to two essentially different classes of entities. His point is that we are actually limited even in generating elements according to a single accepted procedure, like "+1". Hence our concept of the set produced in this procedure is not a concept of a potentially infinite set. We are mistaken if we think it is so, <u>for the very same reasons as were used in arguing against the Platonist</u>: the meaning of our mathematical notions cannot coherently extend beyond mathematical practice. The meaning of our mathematical concepts is provided by actual mathematical activity; hence, any meaning or content which in principle outruns mathematical activity is incoherent, and the determinacy in content is merely an illusion.

(8) <u>Poincaré's Defence of the Notion of Indefinite Iterability</u>

For Poincaré, we can see in certain generating rules that they determine a potentially infinite collection; and this is no illusion only because of the existence of apriori intuition. This is the whole point of his argument against logicism: that the "and so on", as in "1, 3, 5, 7, and so on", or the dots, as in $\{1, 3, 5, 7, \ldots\}$, can indicate a potential infinity is a fact about the nature of our minds. The infinity is not an illusion because it is universally imposed by the synthetic apriori forms of perception and understanding.

It may at first have seemed paradoxical that Poincaré agrees with the strict finitist that we cannot <u>arrive at</u> the concepts of indefinite iterability or continuity via experience alone; for the procedures we employ to explain or to characterise these concepts are themselves instances of these concepts. That is, the only way to represent these concepts is to provide an interpretation which makes them true. For example,

> The idea of infinite divisibility or denseness is not capturable by a formula or sentence, but only by an intuitive procedure <u>that is itself</u> dense in the appropriate respect ... One simply cannot separate the idea or representation of infinite divisibility from what we would now call a model or realisation of that idea ... 14

However, whereas the strict finitist might for this reason deny that these concepts (or for example, that of the actually infinite) can have the content normally ascribed to them, Poincaré never doubts that the content of these concepts is determinate and standard. He never doubts that we possess certain concepts, their existence to be explained, if at all, via the theory of the synthetic apriori.

Therefore, there is a profound difference between the foundations of the two theories. The difference is manifested in what counts as verifiable, or possible. For Poincaré, apriori intuitions exist; hence they supplement the methods and operations used to delimit the class of acceptable mathematical objects or what is verifiable, by

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14 Friedman, (1985), p.469. (My emphasis.) This conception was discussed at length in Chapter 2.

imposing a certain interpretation on our understanding of our concepts and rules.

Since the strict finitist does not accept the theory of the synthetic apriori, his interpretation of certain concepts will be different. Poincaré's view of what we are able to understand by our concepts is enriched by his theory of the synthetic apriori. Hence his account of legitimate mathematical domains is richer. For Poincaré, then, the strict finitist shuns our true intuitions, and takes the metaphor of "construction" far too literally.

(9) Poincaré's Theory of Verifiability, and a Middle Position Between Intuitionism and Platonism

The usual Platonist argument against the intuitionistic and for the classical iterative conception of the set-theoretic universe,[15] is that just as with potential infinity, which the intuitionist accepts, we can stretch our concept of possibility so that the power set operation - the "construction" of all the subsets of a set - is well-founded at any stage in the hierarchy. That is, our concept of construction can be extended, so that we can understand, say, applying the power set axiom to an infinite set, by analogy with the same axiom applied to finite sets. Our

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15 See Hallett, (1984), pp.214-223.

understanding of the actual, or of the uncountably, infinite, or of arbitrary infinite sets is via analogy with the finite.

However, for Poincaré, no amount of stretching of our concepts, by analogy or otherwise, can succeed. Such "analogies" are lost on finite beings, for the analogy can only be perceived as such, or "grasped", provided we already have a concept of "arbitrary infinite". But this is what the analogy was designed to explain. On the other hand, Poincaré was not an intuitionist, nor even a "pre-intuitionist", as he is sometimes called (e.g., by Brouwer). For he regarded the classical continuum of real numbers as a bona fide mathematical object: it is an object of immediate awareness, a primitive domain given in apriori (geometric) intuition. The distinctive character of the two sorts of apriori intuition (arithmetic and geometric), which supplement Poincaré's criterion of verifiability in principle, necessitates an intermediate position between the intuitionist and the Platonist, i.e., a neo-Kantian position.

The continuum, though it exists, is not an object as the Platonist conceives it. It is an intuitive form or primitive structure, and not a set. The fundamental intuition of continuity will simply not bear further epistemological analysis (at least along logicist or constructivist lines). It thus cannot be treated as a completed collection, or as an object, upon which further operations can be performed. So, for example, on this (Kantian) account, there is simply no sense in the notion of the collection of all subsets of $\mathbb{R}$, or the set of all functions from $\mathbb{R}$ into $\mathbb{R}$. If the essence of the mathematical significance of the notion of set is the idea that, in general, the operation of forming a "set of ..." then creates a new object which can automatically be added to the domain of application of permissible operations (e.g., which is capable of being the argument of a function); then Poincaré is claiming that this conception of set is illusory. For it cannot in general apply to infinite sets. Geometric intuition guarantees the existence of the continuum, but it does not guarantee it as a completed determinate domain, as falling under the above description of set.

To sum up Poincaré's position: though he emphasises the criterion of verifiability of mathematical statements, what is verifiable for him must be informed by his epistemological theory of the synthetic apriori. He is no strict finitist, for what is verifiable is not limited to our finite abilities to perform operations. It also depends upon what we can verify apriori, in arithmetic and geometric intuition. In this way, he is also not an intuitionist, for his notion of in principle verifiable includes non-constructive operations on domains where the operations are guaranteed by apriori intuition.[16] And

16 To take a very simple example, there are domains where Poincaré would accept the assertion of "P V - P" where an intuitionist

thus, the nature of his continuum, since it is guaranteed by geometric intuition, is more "classical" than "intuitionistic". Yet he is not Platonist - even though he employs classical logic in all acceptable domains - for his view of what is acceptable is more limited than that of the Platonist. The constraints provided by the theory of the synthetic apriori disallow, for example, impredicative specifications. Hence, although with the Platonist he accepts the existence of the classical continuum - the determinate domain of all real numbers - in opposition to the Platonist, the continuum cannot be "arithmetised", i.e., it cannot be treated as a collection of set-theoretic entities (infinite sets of natural numbers); so it is not a set. And thus, of particular importance, the continuum is not a set-theoretic object upon which further operations can automatically be performed. The existence of the synthetic apriori, and the sharp distinction between arithmetic and geometric intuition, allows the enrichment of the notion of in principle verifiability to stop at just this point: to determine a position in between the Platonist and the intuitionst.

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would demur. Thus in the decimal expansion of π either a sequence of seven sevens occurs or it does not; this follows from the determinacy of the number π which is a consequence of geometric intuition. More significantly, Poincaré would accept in general, the theorem of the linear order of the reals; for again, on the basis of geometric intuition, it would be assertable - contra-Brouwer - that $\forall\delta\,\forall r\,(\delta < r \vee \delta = r \vee \delta > r)$, for arbitrary "points", δ and r.

APPENDIX

MATHEMATICS AND THE APRIORI

The claim that mathematics is synthetic apriori is most commonly and most famously attacked by focusing on the synthetic aspect; i.e., by arguing against Kant's thesis that mathematics has a subject matter. For example, the logicists (Frege, Russell, etc.) endeavoured to show that Kant was wrong about the content of mathematics by showing that mathematical truths are really analytic truths. Their object was thus to show how any true mathematical statement could be proved using logic plus Kant's containment relation among concepts alone. They would then have succeeded in showing mathematics to be as sceptic-proof as logic, because its foundation would be essentially that of logic. Poincaré's arguments against the coherence and success of both logicism and set theory are discussed throughout this work.[1]

However, of equal importance to the synthetic apriori is a defence of the epistemological claim that mathematical knowledge is knowledge apriori. Modern "empiricists" sometimes extend their empiricism to mathematics by arguing that even our mathematical knowledge is not apriori because

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1 See Chapters 3 and 4 for arguments against logicism; see Chapters 4, 5, 6 for arguments against set theory.

it is not truly independent of experience. On the contrary, coming to know a mathematical statement depends in an important and essential way on the presence of certain experiential factors, such that if they (aposteriori facts) are not present, mathematical knowledge could not be obtained. One such argument, expressed in a sophisticated manner, is found in Philip Kitcher's recent work, The Nature of Mathematical Knowledge.[2] In this chapter I will concentrate on defending mathematical apriorism, i.e., Poincaré's thesis, from Kitcher's very general empiricist attack.

(1) The Basic Structure of Kitcher's Argument: A Tenuous Relation Between Certainty and the Apriori

Kitcher argues against mathematical apriorism by pointing out the existence of empirical or experiential factors in mathematical knowledge. Whether or not mathematics seems apriori, underneath it all it is really aposteriori. The structure of mathematical progress and revolutions mirrors that of the physical sciences.[3] And since the nature of knowledge in the physical sciences is the paradigm of aposteriori knowledge, the existence of this parallel is evidence that the two areas of discourse do not involve two essentially different types of knowledge. This claim, though very plausibly argued, does not seem to me correct.

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2 Kitcher, (1983).

3 For a precise development of this view, see Michael Hallett, "Towards a Theory of Mathematical Research Programmes" (I) and (II), British Journal for the Philosophy of Science, (1979).

The reason mathematical knowledge is, for Kitcher, more like scientific knowledge than like some other kind of knowledge, is due to the ever-present possibility in both domains that we might be mistaken. No matter how certain we are about a belief at one time, t, it is always imaginable that we find out at a later time, t + k, that we were wrong at t. Even for the simplest of proofs, or for the solutions of the simplest polynomial equations, it is always possible for the reasoning in the proof or the computation to be wrong. This argument cannot simply be dismissed as a globalised version of Cartesian scepticism and nothing more. The argument is that just like in everyday assertions about the world, the possibility that something might happen to alter our belief about the status of our state of information at time t, is always a coherent possibility. I can always imagine learning something new about the situation at time t - e.g., that I was actually drunk or drugged at the time of computation - so that I come to doubt whether or not any bona fide knowledge was acquired. Both science and mathematics are uncertain disciplines, for what seems to be a similar sort of reason: that we can learn something new or more about the state of affairs at time t which induces doubt about the conclusion then obtained. To summarise, then, Kitcher imagines that mathematical uncertainty arises in exactly the same way as scientific uncertainty: by learning something new about our background experience. Hence, his conclusion is that mathematics could not be apriori.

However, the uncertainty of mathematics indicates to

Kitcher that it is aposteriori, only because he fails to properly distinguish between types of uncertainty. This failing, as we shall see, is rooted in the particular way in which he defines apriori knowledge. Why should certainty be linked in the first place with the apriori? Kitcher, here, plays on the vagueness of our intuitions concerning certainty. He raises the important question: if something is really known apriori how can it not be true? How can we come to doubt whether or not a belief is really knowledge (is really true), if the belief was acquired via apriori means? How can we not be certain about the apriori? The question must be answered by the mathematical apriorist, for Kitcher's point about the lingering coherency of the possibility of doubt is unassailable. And it is not immediately clear what the relation between certainty and the apriori is. What is clear, however, is that it is unreasonable to demand that in order for an item of knowledge to be a candidate for the apriori, it must not be doubtable in any way. For this would be to demand Cartesian certainty of the apriori; i.e., a refutation of Cartesian scepticism would be a prerequisite to mathematical apriorism. And this, it seems, is impossible.

In essence, Kitcher demands too much of mathematical knowledge. His conditions, which an item of knowledge must meet in order to be considered apriori, are much too strong. His arguments against the apriority of mathematical knowledge rely upon his drawing a distinction in a particular way. And the way in which he draws it effectively defines apriori knowledge out of existence.

Mathematical knowledge is thus not apriori because no knowledge is. This is because Kitcher does not take into account the different ways in which uncertainty can be raised. He concedes that some form of experience may be necessary for any sort of knowledge - so that the apriorist is not committed to the thesis that we have innate knowledge. Hence, he intends not to stack the deck against the apriorist, but to allow the minimal experience which may be necessary in order to acquire any concept. So that, though it is usually via experience that we acquire our concepts, our reasoning via these concepts may still be apriori. For instance, though we may come to know what "triangle" means by ostension - by being presented with certain pictures of figures with slightly different shapes and varying in size, but yet with something in common - we can still reason in an apriori way with the concept of "triangle". So that, "The triangle has three sides", may be regarded as apriori. This is an instance of an analytic apriori truth: it is necessarily true given the meanings of the terms. In addition, even if a statement represents a contingent fact about the world, Kitcher wishes not to automatically exclude this class from candidacy for apriori knowledge.

> Our goal is to construe apriori knowledge as knowledge which is independent of experience, and this can be achieved without closing the case against the contingent apriori. 4

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4 Kitcher, (1983), p.24.

Thus, for example, one might consider the statement, "There is something that is conscious" to be an instance of a contingent truth - for it could have been the case that no conscious beings existed - which is knowable apriori - for in merely considering it, one realises it is true; and hence, we do not need to investigate the world to see if it is true. Therefore, on the face of it, Kitcher intends to be very generous and broad-minded in his view of what may or may not count as apriori knowledge.

(2) Kitcher's Definition of "Apriori Warrant", and Some Counterexamples

However, once he starts to pin down his idea of what constitutes the apriori we see that his views are anything but generous to the apriorist. It turns out that, for Kitcher, in order for an item of knowledge to count as apriori it must be indefeasibly certain.

> To generate knowledge independently of experience, apriori warrants must produce warranted true belief in counterfactual situations where experiences are different ... On this account apriori warrants are ultra-reliable; they never lead us astray. [5]

"In counterfactual situations, where experiences are different ..." we must still hold the same (true) belief in order for it to have been produced by an apriori warrant. "Apriori" and "certainty" - absolute certainty - are

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5 Kitcher, (1983), p.24.

necessarily linked, according to Kitcher; where, for him, the possibility of uncertainty implies that the belief could not have been acquired by bona fide apriori means. Rather than generous, this is very strong indeed.

Let us examine his explicit set of conditions.

> (2) X knows apriori that p if and only if X knows that p and X's belief that p was produced by a process which is an apriori warrant for it.
>
> (3) α is an apriori warrant for X's belief that p if α is a process such that, given any life c, sufficient for X for p,
>
> a) some process of the same type could produce in X a belief that p,
> b) if a process of the same type were to produce in X a belief that p, then it would warrant X in believing that p,
> c) if a process of the same type were to produce in X a belief that p, then p. 6

These are not only strong. As it turns out, these conditions actually beg the case against the possibility of any non-trivial apriori knowledge.

Why should uncertainty be incompatible with apriority? For Kitcher it is due to condition (3) b), above.

> If a process of the same type were to produce in X a belief that p, then it would warrant X in believing that p;

and it would do this whatever the background experience. So, for instance, in the case of long formal proofs which replace

6 Kitcher, (1983), p.24.

complex informal proofs, it is reasonable to be uncertain about whether or not there is a mistake in the long formal proof. Indeed, the person who dismisses the possibility of a mistake is unreasonable.[7] If uncertainty is reasonable, then this process - of the same type as the informal proof, but longer and with no steps glossed over - does not warrant belief that p (that the theorem is proved, or true). Hence, the original warrant - the informal proof - is not apriori. This is because (3) b) has been violated: since a process of the same type (i.e., proof) leads to an uncertainty in its warrant for p, due to the background condition of it being longer, the apriority of the original warrant is impugned.

That is damaging enough, but Kitcher does not contest only the apriority of long, complex reasoning as Descartes did. For mathematical proof to count as bestowing apriori knowledge, not only must it be certain, but so must the principles and axioms from which it stems. That is, they must be apriori. And the rules of inference must preserve this status. So, according to him, a mathematical apriorist is committed to something like the following:

(4) There is a class of statements A and a class of rules of inference R such that

a) each member of A is a basic apriori statement,
b) each member of R is an apriority-preserving rule,

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7 Kitcher, (1983), p.42.

> c) each statement of standard mathematics occurs as the last member of a sequence, all of whose members either belong to A or come from previous members in accordance with some rule in R. 8

However, more "counterexamples": since "knowers form a community",[9] then,

> at best correct or reasonable social practice can determine which sequences are proofs. Yet now we must ask what makes the adoption of a theory or system correct or reasonable. 10

Hence, his "social challenge" in a simplified form: if everyone doubts my proof, I will too. Again condition (3) b) is violated. I can envision background circumstances which would cause me to doubt a warrant which is of the same type as the present one; hence the present warrant is not apriori. In addition,

> it is conceivable that we could become reasonably convinced by our own experience that the ingestion of certain substances had enabled us to solve baffling theoretical puzzles and that, during one of these episodes, we had discovered a counterexample to a mathematical axiom ... 11

Even our axioms or principles are in danger of being doubtable, and thus, they are not apriori.

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8 Kitcher, (1983), p.39.

9 Kitcher, (1983), p.14.

10 Kitcher, (1983), p.39.

11 Kitcher, (1983), p.90.

(3) The Problem with Kitcher's Definition

The problem with Kitcher's account of apriori knowledge is that he thinks there is an if-then relation between apriority and certainty. That is, any belief which is produced by an apriori warrant must be certain. If a warrant is apriori, it must be "able to discharge its warranting function, no matter what background of disruptive experience we may have".[12] Kitcher captures this idea primarily with (3) b). It is my opinion that there is a problem with Kitcher's definition and that is, primarily, that his condition (3) b) is too strong.

If the possibility of any mistake must be disallowed in order for knowledge to be apriori, then how can such an item be knowledge at all? If there is no possibility of any sort of mistake whatever, then it seems what is in question is not objective knowledge, but mere subjective awareness. Certain statements, like "I am in pain", "I see red", may come out apriori on Kitcher's account; and these are the avowals: if true, incorrigibly so. Moreover, nothing other than incorrigible truths will be allowed if (3) b) obtains. Yet, according to most apriorists, these will not count as apriori knowledge, because avowals are not items of knowledge which are accessible to everyone, but only to the experiencing subject. Clearly, where and how Kitcher draws the distinction between apriori and

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12 Kitcher, (1983), p.35.

aposteriori knowledge is contentious. And since his arguments against the apriority of mathematical knowledge depend on his particular characterisation of apriori warrant, his argument is invalid. Since the way he makes out the apriori/aposteriori distinction debars <u>any</u> (significant - i.e., not incorrigible) knowledge from being apriori, he begs the question against <u>mathematical</u> knowledge being apriori.

(4) <u>Revising Kitcher's Definition: Two Types of Uncertainty</u>

Kitcher is not entirely wrong in his intuition that there is a relation between apriori knowledge and certainty. Knowledge produced by an apriori warrant <u>is</u> indefeasible, <u>providing the warrant is successful</u> - providing we carry out the operations correctly. In this way, the counterfactual experiences employed, in making out the distinction between apriori and aposteriori warrants, cannot be ones in which we come to doubt whether or not we have proceeded correctly. How accurately we in fact follow a rule should have no bearing on what kind of rule it is. This is essentially where Kitcher's mistake lies: his explication of what kind of procedure an apriori warrant is, implicitly disallows the undeniable fact that we <u>can</u> always misperform an operation. Hence, nothing turns out to be apriori for Kitcher, because he conflates what <u>kind</u> of procedure something is with the question of the correctness of our particular <u>use</u> of a procedure. This is because in the class of possible counterfactual experiences, which must be looked at in order to determine whether or not a particular experience is necessary in producing a belief - and hence, in determining whether or not a belief is produced via an apriori warrant - he allows into this class counterfactual experiences in which we have applied a rule incorrectly. Clearly the apriority of a warrant cannot be decided by holding it accountable for our possible misapplication of such a warrant.

The nature of the set of counterfactual, doubt-inducing experiences, which can count against the possibility of apriori knowledge, should be examined more closely. Kitcher clearly allows anything into this set. Any sort of experience we can imagine which would, if true, cause uncertainty, counts against the apriority of a warrant, according to (3) b). But we will be more discriminating. We will distinguish between two sorts of uncertainty, or two sorts of potential error. (1) The first sort indicates that the warrant for the belief under scrutiny did not fix or force the conclusion: though the evidence-statements which constitute the warrant may still hold, further evidence may lead us to abandon the claim, while simultaneously granting that the warrant we did have for the claim was a good one, indeed the best there could be. If this sort of uncertainty is possible, then the belief is aposteriori. (2) The other type of potential uncertainty is present in both apriori and aposteriori beliefs. This is where the potential error lies not in the world, but somewhere in the warranting processes themselves, for instance, in the way in which the warrant was applied. In this case, when doubting the original claim we are doubting the original warrant: either that it was appropriate to the situation; or that though the type of warrant was appropriate, we doubt our particular application (instantiation) of it. My proposal is that a distinguishing feature of apriori warrants is that only uncertainty of this type (type 2) is possible.

For instance, by making a computation error we are misapplying a rule; and thus we are not successful in obtaining apriori knowledge, not because the warrant was not of a bona fide apriori type, but because we have not obtained any knowledge at all. It is still correct to call the type of warranting procedure - computation - apriori. The point is, just because a warrant is apriori does not mean there is no scope for human error. Yet this is what Kitcher seems to require of apriori warrants, and so this is where his account goes wrong.

If apriori knowledge is to have a chance at being a clear and yet useful distinction - useful in describing some non-empty class - we must be cautious enough to provide an account of apriori warrants which both contrasts with aposteriori warrants, but which has in common with the aposteriori the ability to produce (corrigible) knowledge. The paradigm aposteriori case is perceptual: we draw a conclusion about the world on the basis of perceptual experience - our experience of the world. For example, we see a person in overalls carrying a painting out of a museum room, and we infer on the basis of past experiences of workers in broad daylight, that he is on official relocating business. However, there are two ways in which our inference could be wrong. (1) The man might be a thief. That is, our perceptual faculties may be working properly - the man we seem to see is real, etc. - it is just that the world does not measure up to our hypothesis about his occupation. Or (2) our perceptual faculties may be malfunctioning: there is no man, and perhaps we are not even in a museum. Case 1 is a situation where we may doubt the conclusion or inference without doubting any of the premises or warrants. Our perceptual statements still hold; there really is a man in overalls, etc. And our general rules are still true: it is still true that usually, or in general, men who transport paintings in broad daylight in museums are on official orders. In contrast, case 2 depicts an instance of the type of situation where doubt about the conclusion occurs via doubt about at least one of the warranting premises. I repeat my proposal: what distinguishes aposteriori beliefs or knowledge is that error of type 1 is always possible. Or, to state it conversely, the distinguishing feature of apriori warrants is the <u>impossibility</u> of type 1 doubt or error. Hence, rather than allowing <u>any</u> doubt-inducing counterfactual experiences to count against the apriority of a belief, we ought to restrict the type of counterfactual experience which indicates a belief was produced by an aposteriori warrant to that of type 1. Aposteriori warrants can lead us astray either due to faulty mechanisms, so that what seems to be the case is not really the case (i.e., our perceptions do not mirror the world); or due to the world simply not measuring up to our hypotheses. This second way is where our perceptual evidence is not at fault: there really is a man in overalls ... It is, rather, the judgement made on the basis of that evidence which is at fault.

The inductive character of aposteriori warrants forces the situation where it is always possible to have type 1 error. Why should this be the mark of the aposteriori, rather than a mere distinction between induction and deduction? A distinction between induction and deduction <u>is</u> employed. But this distinction is related to the apriori/ aposteriori distinction, for aposteriori knowledge is generally translated as "knowledge in the light of <u>sense</u> experience".[13] The presence of experience of a certain sort - a sensory sort - is necessary for a warrant to be aposteriori. The relationship between our sense experience and our (aposteriori) hypotheses is evidential or inductive, for the subject of hypotheses which rest on sense experience is the world: that which <u>causes</u> our sense experiences, and which is (for most people, in some sense) independent of our awareness of it. Since our aposteriori hypotheses concern facts the status of which are (on the most ordinary interpretation) independent of our manner of investigating them, there is always a gap between the process of investigation and the discovery of the truth. Hence, it is clear

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13 Peter A. Angeles, (1981). (My emphasis.)

that there are two ways of getting aposteriori knowledge wrong, illustrated by the two cases. So that even if one contests dividing up the counterfactual experiences to coincide with the apriori/aposteriori distinction in the way we have urged, he or she can still admit such a distinction exists.

Furthermore, it is plausible to distinguish between apriori and aposteriori by looking at the _way_ in which uncertainty arises. For it bears out our intuitions concerning the inductive character of aposteriori knowledge - i.e., the presence of type 1 possible error as the mark of aposteriori. And, in addition, it bears out our intuitions concerning the apriori: that mathematics and logic are apriori, and that the root of any mistake here has nothing to do with sense experience, but involves only the warranting processes themselves. To summarise, any warrant can be rejected. Hence, case 2 where the warranting process itself is impugned or doubtable, is no way to distinguish between _types_ of warrant. For this happens with both apriori and aposteriori warrants. Thus, type 1 recalcitrant experience is a plausible candidate for a distinguishing factor.

(5) Further Explication of the Distinction

Let us examine the two sets of cases more closely. If an item of knowledge is apriori, then the belief that it is true must be produced via an apriori warrant. An apriori warrant is one which satisfies Kitcher's conditions, with the added clause "provided we perform the necessary operations in the warrant correctly". Thus, for example (3) b) would be revised to:

> (3) b′) if a process of the same type were to produce in X a belief that p, then it would warrant X in believing that p, provided X carries out all its operations correctly.

Adding this clause excludes all counterfactual experiences which induce doubt about the belief that p by inducing doubt about the warrant for p. We can thus retain a useful notion of apriori which, yet, contrasts with our notion of the aposteriori.

There is no gap between best possible evidence and apriori truth. Having the best possible evidence for believing that p guarantees the truth of p, if p is an apriori belief. For if we really have the best possible evidence, or warrant, then this means that the warrant was carried out properly; and if the warrant was carried out properly, then there is no other place for doubt to arise.[14] The only way we can be mistaken about a possible item of apriori knowledge is via the warrant. Uncertainty here stems only from doubt about the pedigree of the warranting process; e.g., worry

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14 Hence we see questions about the objectivity of apriori truths arise in conjunction with an account of that in which the apriori consists. For if there is no gap between best possible warrant and truth when an item of knowledge is apriori, then it seems such items cannot be regarded as objective in the same sort of way as statements about the empirical world are regarded as objective. This is especially acute for the non-Platonist, for whom following the procedure correctly is what constitutes truth in, for example, mathematics and logic.

about an addition mistake in a long computation; or worry about the clarity of our concepts. If a belief is apriori, the only way we can fail to obtain "knowledge" is by failing to have secured a proper warrant.

In contrast, there is always a gap between best possible evidence and aposteriori truth. Our warrant for an item of aposteriori knowledge may be unassailable, and we can still be mistaken about our belief. That is, doubt about the belief can occur <u>other than</u> via a doubt about our warranting experience. We can have the best possible evidence for believing that <u>p</u>, and yet <u>p</u> may not be true. For instance, the best possible evidence that you are in pain is for me to see you writhing about on the floor. And yet I can still be wrong. You may not be in pain; you may be playing a trick on me. I can have the best possible evidence and still be wrong in my inference, because aposteriori hypotheses require cooperation from the exterior world in order for me to be right. Another example: I can have the best possible evidence for believing that there is a cat on the mat next door: e 1 - there is something that looks like a cat; e 2 - it smells like a cat; e 3 - it is purring; e 4 - it is not the first time I have seen it there; e 5 - there is communal agreement that my neighbour owns a cat (i.e., we have spoken about "his cat"); etc. However, it is possible that what looks, smells, sounds, etc. like a cat is really some sort of sophisticated, smelly automaton. The point is, the conclusion, "There is a cat ...", is doubtable without necessarily doubting any of the items of evidence, any of the e *n*'s, for it. This is because we can add information which is inconsistent with our former conclusion about the existence of the cat, but which is consistent with all the evidence we employed in arriving at the belief. Hence, an aposteriori warrant (a set of aposteriori evidential statements) can never *fix* a particular conclusion. There will always be indefinitely many possible conclusions which - though incompatible with one another - will all be compatible with the warrant. This is why it is not necessary in the case of an aposteriori belief to doubt the warrant when we doubt the conclusion (though, of course, this too is always possible; e.g., an hallucination). Whereas with apriori beliefs, the only avenue to doubt about the conclusion is one which necessitates doubt about (at least part of) the warrant along the way.

Therefore, the proposal is to meet Kitcher's challenge to find a weaker view of apriori warrant which does not "trivialise" the notion. It is possible, because refining his conditions (3) a)-c) turns out to make a big difference in what comes out as apriori. That is, by refining his conditions, apriori knowledge becomes possible.

> The charge that my argument against apriorism presupposes too strong a notion of apriority is relatively easy to rebut ... To abandon it is to abandon the fundamental idea that apriori knowledge is knowledge which is independent of experience. The apriorist would be saying that one can know apriori that *p* in a particular way, even though, given appropriate experiences, one would not be able to know that *p* in the same way. But if alternative experiences could undermine one's knowledge then

> there are features of one's current experience which are relevant to the knowledge, namely those features whose absence would change the current experience into the subversive experience ... To reject condition (3b), the condition of my analysis on which the central arguments have turned, would be to strip apriorism of its distinctive claim.

And footnote 1:

> I would contend that the analysis of apriori knowledge given in Chapter 1 provides the only clear account of the epistemological notion of apriority which is currently available. Hence if someone wants to protest that my analysis stacks the deck against the apriorist, it is incumbent upon him to provide an alternative. Given the arguments ... rehearsed in Chapter 1 ... [either] the distinctive idea of epistemological apriority will have been abandoned, [or] ... apriorism will be vulnerable in just the way I have taken it to be. 15

However, it is here contended that it is unreasonable, and indeed unfaithful to the prior meaning of the term, to analyse apriori knowledge as knowledge which holds despite any counterfactual experience whatever. In particular, it seems acceptable to say that "there are features of one's current experience which are relevant to" apriori knowledge, so long as these features have to do with our warranting procedures. That is, provided the experiences are only necessary in order to support the claim that, for instance, we had carried out certain apriori operations correctly, these features are not of a sensory type, and

15 Kitcher, (1983), pp.88-89.

hence do not impugn the apriority of the warrant. Apriori knowledge is

> knowledge derived from the function of reason without reference to sense experience. Non-empirical knowledge. To know something apriori is to know it prior to experiencing anything like it in the external world. The truth of apriori knowledge (a) is not derived from sense experience, (b) cannot be checked against sense experience, (c) cannot be refuted by any sense experience. 16

Sense experience cannot refute apriori knowledge - the world cannot "step in" to provide doubt-inducing experience in an additive way, as is possible with aposteriori knowledge. However, this is not to say it cannot be refuted at all. My proposal is to distinguish between apriori and aposteriori knowledge by focusing on the non-empirical nature of apriori warrants.

What is distinctive about aposteriori knowledge is that we can be wrong about our conclusion despite the fact that our warrant - even the best possible warrant - still holds. This contrasts with apriori knowledge. The type of counterfactual experience which indicates a warrant is not apriori, is one in which the warrant still holds, and yet we have independent reason to revise the conclusion. On the other hand, if, in order to doubt the conclusion, we must doubt or impugn the warranting process, then such a conclusion is known on apriori grounds. Mathematics turns

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16 Angeles, (1981).

out to be apriori, since it is usually thought that having a proper warrant is what determines truth (or at least is co-extensional with a certain range of what is true) in mathematics. There is no gap here between the propriety or correctness of a warrant and what is, in fact, true.

(6) In Defence of Our New Explication

I will defend the new proposal by testing it against Kitcher's counterexamples, cited above.

1) The example of complex long proofs is straightforward. The uncertainty here stems from an uncertainty about whether or not one has followed the procedures as specified. So doubt is doubt about the warranting process - the actual proving activity. So this example of recalcitrant experience does not fall into the "distinctively aposteriori" category. Hence it does not bear against the apriority of the warranting procedure of mathematical or logical proofs.

2) The example of revolutions in mathematical standards is somewhat more difficult, for it raises a question about the correctness of an apriori warrant itself. Uncertainty about a theorem caused by an imagined change in the standards of correctness makes proofs - a class of purported apriori warrants - seem correct only relative to a community. Whereas, on the ordinary view, if a rule, axiom, or principle is knowable apriori, then it should be knowable in any community in which it is possible to obtain the relevant concepts.

3) The case of the possibility of ingesting magical substances which enable us to see counterexamples to an axiom is equally difficult - albeit a bit more strange - for the same sort of reason. If an axiom is knowable apriori, then it must be true; but if so, how can it be that we can envision even the bare abstract possibility of a counterexample?

One could merely dismiss the objections posed by the latter two examples by saying that in both cases what is occurring is a doubt about a warrant. Standards of correctness and axioms are warranting tools; hence, neither case threatens the apriority of such processes, according to our revised view of the apriori. However, this would be to dodge, rather than to meet, the problem. Case 1 has to do with doubt about the application of a purported apriori warrant; and is answered by referring to our refined conditions (3) a′)-c′). Cases 2 and 3, on the other hand, illustrate a doubt which centres on the warranting process itself, rather than merely on its application. So it is a different kind of case. It will require us to examine part (2) of Kitcher's definition, for it raises questions concerning what it means for a process to be an apriori warrant for a belief.

The objections go: how can a warrant, like an axiom or principle, be apriori if it is ever susceptible to doubt in the way imagined above (either via a possible revolution in rigour (case 2), or via the possible awareness of a counterexample to an axiom, during a drug-induced state (case 3) ? The warrants in cases 2 and 3 depend on the concepts involved: on our interpretation and refinement of those concepts. Thus, for example, the revolution in rigour which accompanied the development of the ε-δ limit concept occurred partly because it was discovered that our concepts of continuity and limit lead to contradictions.

Kitcher's description of the revolution in rigour is misleading. It is not merely that standards change; and so certain warrants which were formerly acceptable are no longer acceptable. This is an incorrect picture, making the warranting process itself seem a matter of mere convention. Rather, given a certain concept of continuity, C (i.e., smoothness); plus certain apriori warranting operations, $\alpha_1, \ldots, \alpha_n$; and the situation where both are embedded in some more general theory, T; we were able to arrive at a belief P: for example, that a continuous function is everywhere differentiable. Later we came to believe not P: that we were mistaken in the view that every continuous function has a tangent at every point. The new ε-δ definition of continuity, for example, allowed for the existence of continuous curves with no unique tangents at some, or indeed all, points. However, this does not mean that we have discovered that our warrant was wrong all along - and hence, that it could not have been apriori, because false. Nor does this indicate a mere change in the standards demanded of one warranting procedure. Rather, the refinement in the concept of continuity (C, smoothness, being replaced by C*, the ε-δ account of the continuity of a real-valued function) entailed that the procedures, $\alpha_1, \ldots, \alpha_n$, which formerly warranted claims employing the concept C, were now abandoned, not because they were discovered to be incorrect after all, but because they were no longer appropriate - they are inappropriate to the new concept C*. Warranting procedures, as envisaged by Kitcher and discussed here, are appropriate only relative to the concepts on which they are based. Our concept of continuity was refined, and the new concept brought with it new warranting procedures appropriate to it. But this in no way impugns the apriority of the warranting procedures appropriate to the earlier concept C. Since progress in mathematics must involve solving new problems, and solving old ones better, it involves either creating new domains, or refining existing concepts. As Poincaré put it,

> it was not long before it was recognized that exactness cannot be established in arguments unless it is first introduced into the definitions. 17

The fact that concepts are refined and revised does not show that the warranting procedures appropriate to those concepts are not apriori.

Case 3 above - where Kitcher attempts again to argue against the apriority of a part of mathematics by referring to a certain way in which a counterexample to an axiom might arise - is another instance where the description of the example is contentious. Kitcher cites the possibility that we can imagine ourselves to be able, only when in a drug-induced state, to see counterexamples to a previously accepted axiom. And he takes this as a sign that axioms could not then be apriori, because we can always imagine

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17 Poincaré, (1889/1908), pp.123-124.

giving them up. So, for instance, let us imagine that drinking a cup of coffee allows me to think just that little bit faster or more clearly so that I can see a counterexample to some axiom which I was thinking of adopting (like the Axiom of Choice). Further, it is only in this drug-induced state - i.e. my caffeine "high" - that I can see and understand this counterexample; although I can remember _that_ it is a coherent counterexample when I am in a caffeine "low" state, it is not clear _why_ until I down at least two cups. Kitcher's point is not merely that axioms are not certain; although this is part of his complaint, since he does link apriority with certainty. Rather, the deeper point is that the uncertainty here is caused by an aposteriori or empirical fact: my drinking a cup of coffee. Hence, he wishes to complain, since the imagination of a certain additional aposteriori fact can induce doubt about an axiom, an axiom is not knowable apriori.

However, Kitcher is glossing over an important distinction here. He is treating the cause of my belief that -_P_, i.e., coffee drinking, as if it is the cause of -_P_. Coffee drinking does not cause or instantiate a possible counterexample to an axiom. Rather, in the example, coffee drinking allows the instantiation, or the "seeing", of the counterexample to occur. Kitcher's description of the example is prejudiced, for he makes it appear that drinking cups of coffee is the warrant for my belief that -_P_. Whereas - it is obvious now - that cannot be the warrant, for it is no warrant at all for -P. If I am asked why I reject the axiom P, the answer could not be merely, "Because I drank a cup of coffee". Hence, it is not an aposteriori fact which impinges on my belief that P; and so this example does not bear against the apriority of our knowledge of mathematical axioms. Although coffee drinking may be episodically related to my rejection of an axiom, it cannot be an explanation of my rejection, but a mere explanation of why, all of a sudden, I can see counter-examples now, where I could not before. That is, the example should be described as another case of further analysing certain concepts employed. The ingestion of chemicals does not cause mathematical facts; the ingestion of chemicals can, however, cause me to see certain mathematical facts. Perhaps drinking coffee enables me to think about a concept in a new, more refined, more fruitful way; or perhaps I can concentrate better, or think of more logical consequences faster, when I am under "the influence". But this is only to say that our ability to perform certain apriori operations (concentrating, deducing) is enhanced (indeed, they can also be tinged, e.g., by alcohol) by certain physical, and hence, aposteriori factors. However, the ability of aposteriori factors to influence our performance of certain warranting processes in no way impugns the apriority of those processes. The ability to be influenced by aposteriori factors does not indicate that the operations we are performing are thereby aposteriori. It does not inform us at all about the epistemological nature of the operations.

Our application of apriori warrants does not need to be indefeasibly certain in order to be able to capture a faithful notion of apriori. Furthermore, our knowledge or acceptance of apriori warrants need not be indefeasible according to the revised account of apriori warrant. There is no direct link between certainty and apriority. Wrong results cause us to reject a use of a rule, and not its apriority. And seeing a counterexample may lead us to reject an axiom not because it is not apriori - that which first induced us to accept the axiom was not an aposteriori warranting procedure - but because it is not true or faithful to our concepts. Neither the possibility of misuse of a warrant, nor the possibility of rejecting a warrant, necessarily informs us that the warrant could not be apriori. Thus it seems to me that none of Kitcher's counterexamples succeed in establishing that pure mathematical knowledge is not knowledge apriori. I shall therefore conclude that Poincaré's thesis of the synthetic apriori character of mathematical truth remains intact with respect to a certain challenge from the modern empiricist.

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