OF THE USE OF THE "ENGLISH SECTOR" IN TRIGONOMETRY: WHAT AMOUNT OF MATHEMATICAL TRAINING WAS NECESSARY IN THE 18TH CENTURY?

Mónica BLANCO
Universitat Politècnica de Catalunya – BarcelonaTech
ESAB, Campus del Baix Llobregat, c/ Esteve Terradas, 8, 08860 Castelldefels (Barcelona), Spain
monica.blanco@upc.edu

Keywords: Edmund Stone, mathematical instruments, sector, solution of triangles

ABSTRACT

In 1723 Edmund Stone published *The construction and principal uses of mathematical instruments*, which was essentially a translation from the French of Bion's *Traité de la construction et des principaux usages des instrumens de mathématique* (1709). As the title of the book indicated, Stone annexed a number of instruments that had been omitted by Bion, in particular, those invented or improved by the English. Hence, after the translation of Book II, on the construction and uses of the "French sector", Stone added a chapter on the "English sector". In the 17th century there had been a number of debates concerning the amount of mathematical training required for the study of mathematical instruments. In the context of the study of mathematical instruments in the 18th century, it is worth exploring the link theory-practice in the books on instruments. The aim of this contribution is to explore the mathematical knowledge involved in the use and applications of the "English sector" in trigonometry in a number of 18th-century books on mathematical instruments.

1 Edmund Stone and the study of mathematical instruments

In 1723 Edmund Stone (1695?-1768) published *The Construction and Principal Uses of Mathematical Instruments*, one of the earliest general manuals on mathematical instruments in English. It was essentially a translation from the French version of Nicolas Bion's *Traité de la construction et des principaux usages des instrumens de mathématique*, first published in 1709.[1] In the Translator's preface, Stone defined mathematics both as a science, with regard to the theory, and as an art, with regard to the practice. Mathematical instruments connected these two sides of mathematics. Since the knowledge of mathematical instruments led to the knowledge of practical mathematics, the study of mathematical instruments could be regarded as one of the most useful branches of knowledge in the world, which, therefore, had to be spread. However useful it might be, Stone lamented the lack of a general treatise like Bion's in English. Although this could partly explain why Stone translated Bion's treatise, his book cannot be said to be merely a translation. As the title of the book indicated, at the end of each book Stone annexed a section on the construction and uses of a number of instruments that had been omitted by Bion, in particular those that were invented or improved by the English.

[1] There is an account of Stone's book and its production in Blanco (2015).

“English instruments” added by Stone. Hence, for instance, after the translation of Book II, “Of the Construction and Uses of the Sector”, Stone added a chapter on the “English sector”.

Stone’s work seems to be addressed to gentlemen, since, as it is stated in the translator’s preface, the study of mathematics “made a part of the education of almost every gentleman” (Stone, 1723, p. v). From the late 16th century onwards, the study of mathematics was an essential element in the education of a gentleman “to maintain his position and engage in the activities traditional to his class” (Turner, 1973, p. 51), such as astronomical navigation, warfare, surveying or trade, among others. In the 17th century there had been a number of debates about the role of mathematical education in the training of gentlemen in mathematical instruments, e.g. the controversy that arose in 1632 between Richard Delamain (1600-1644) and William Oughtred (1574-1660) (Turner, 1973). While Oughtred emphasised the study of mathematics over mathematical instruments, Delamain put the stress on the practice, that is, on the use of instruments. Some fifty years later, John Aubrey (1626-1697) wrote a manuscript on education of gentlemen, which remained unpublished. In his manuscript Aubrey argued that not only a proper mathematical theoretical groundwork was essential in the training of a gentleman, but also the practical concerns had to be taken into consideration. To such and extent, that instruments could be used as pedagogical tools in mathematics education, all the more so since they could be regarded as something new and stimulating in the mathematics classroom. Today we find a similar approach in the works of Bartolini Bussi on how to use ancient instruments in the modern classroom, both as a physical experience and as an alternative way to develop the understanding of specific mathematical concepts (Nagaoka et al, 2000, pp. 343-350). As Greenwald and Thomley (2013) point out, the explicit connection between tools, problems, people and history can boost the teaching and understanding of mathematics. In turn, this connection can be developed further with the integration of traditional instruments with computers in the mathematics classroom.[2]

In the context of the study of mathematical instruments in the 18th century, it is worth exploring the link theory-practice in books on instruments. Was it necessary a sound mathematical groundwork in order to learn how to use an instrument? Or could it be enough to learn a simple set of mechanical rules? Taking inspiration from the debates described by Turner (1973), the aim of this contribution is to explore the mathematical knowledge involved in the use and applications of the “English sector” in trigonometry through a number of 18th-century books on mathematical instruments. To have a more comprehensive perspective, it is also worth having a brief look at how contemporary works on trigonometry presented the solution for the same kind of problems.

2 The ‘English sector’

The sector was a mathematical instrument made of two legs of equal length joined to each other by a hinge, with scales on its sides.[3] It turned out to be useful for solving problems in proportion. Figure 1 shows how the instrument worked with the help of a pair of compasses.

[2] See for instance Isoda’s work in Nagaoka et al (2000, pp. 351-358).

[3] On the history of the sector, see Frémontier-Murphy (2013).

From Euclid's *Elements*, book VI, proposition 4, it is clear that, as a is to x, so is b to y. The lines drawn upon the sides of the sector (e.g. a, b) are called lateral. Parallel lines run from one leg of the sector to the other, in equal divisions from the center (e.g. x, y).

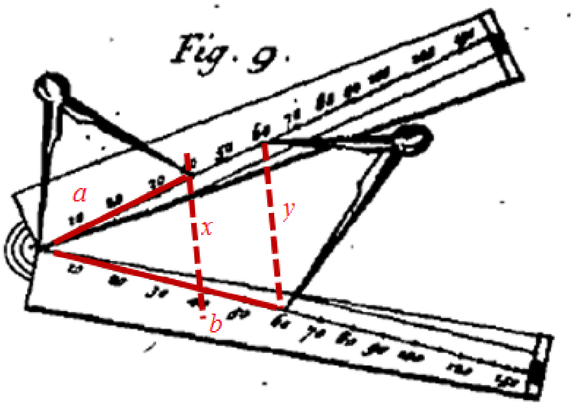


Figure 1. General use of the sector, in Stone (1723), plate VII, Fig. 9 [digitized by the Internet Archive, 2012, with funding from Gordon Bell, available at http://archive.org/details/constructionprin00bion]. Red lines added by the author.

In the second book Bion dealt with "De la construction et des usages du compas de proportion", which Stone translated as the "French sector".[4] At the end of this book, Stone added the section "Of the construction and uses of the English sector". The main differences between the "English sector" and the "French sector" were the nature and the number of lines on each instrument (Figure 2). Upon the faces of Bion's model, there were usually drawn six lines or scales: line of equal parts, line of planes, line of polygons, line of chords, line of solids and line of metals. When it comes to the "English sector", besides the principal lines (equal parts, chords, sines, tangents, secants and polygons), there were also the lines of artificial sines and artificial tangents, Gunter's line of numbers, and a line of inches. The artificial lines represented the logarithms of numbers, sines and tangents and were to be used as on Gunter's scale. In fact, in 1624 Edmund Gunter had produced a book on the sector and the cross staff, wherein he described a sector with up to twelve lines, including the trigonometric lines. At the end of this work, Gunter provided his tables of artificial sines and tangents. While in France Bion's model would be used until the end of the 18th century, sectors with trigonometric lines and artificial lines became very popular in England (Frémontier-Murphy, 2013).

The advantage of the "English sector" above the common scales, or rules, was that all its lines could be fitted to any radius, not exceeding the length of the legs. In Section II, "Of the

[4] See, for instance, Stone (1723, p. 66).

general use of the line of chords, sines, tangents, and secants, on the sector”, Stone illustrated this fact:

> Suppose the chord, sine, or tangent of 10 degrees, to a radius of three inches, is required. Take that three inches, and make it a parallel between 60 and 60 on the line of chords, then, as I have already said, the same extent will reach from 45 to 45, on the line of tangents, also on the other side of the sector, the same distance of three inches will reach from 90 to 90 on the line of sines so that if the lines of chords be set to any radius, the lines of sines and tangents are also set to the same. Now the sector being thus opened, if you take the parallel distance between 10 and 10 on the line of chords, it will give the chord of 10 degrees. Also if you take the parallel distance on the line of sines between 10 and 10, you will have the sine of 10 degrees. Lastly, if you take the parallel extent on the line of tangents, between 10 and 10, it will give you the tangent of 10 degrees. (Stone, 1723, pp. 67-68)

The sector was soon regarded as a universal instrument, “generally useful in all the practical parts of the mathematicks, and particularly contrived for navigation, surveying, astronomy, dialling, projection of the sphere, &c.” (Harris, 1704-1710, entry *SECTOR*). Therefore, the sector could be employed to solve trigonometric problems, such as the solution of oblique triangles.

As mentioned above, Turner (1973) gives an account of the discussions about how necessary the teaching of a proper mathematical groundwork was for the education of gentlemen, in the context of the study of mathematical instruments in the 17th century. It is true that Stone (1723) was intended for gentlemen. Yet, other works dealing with the “English sector” also addressed those interested in practical mathematics (e.g. Gunter, 1624; Cunn, 1729; Webster, 1739), architects, engineers and artificers (e.g. Robertson, 1747) and learners in general (e.g. Rea, 1717). Widening the audience to all kind of readers and getting into the 18th century, did works on the “English sector” include the principles of trigonometry? And if so, how was the doctrine of triangles taught? Since this contribution is a work-in-progress, next I will show how a few works on the sector presented the solution of triangles.

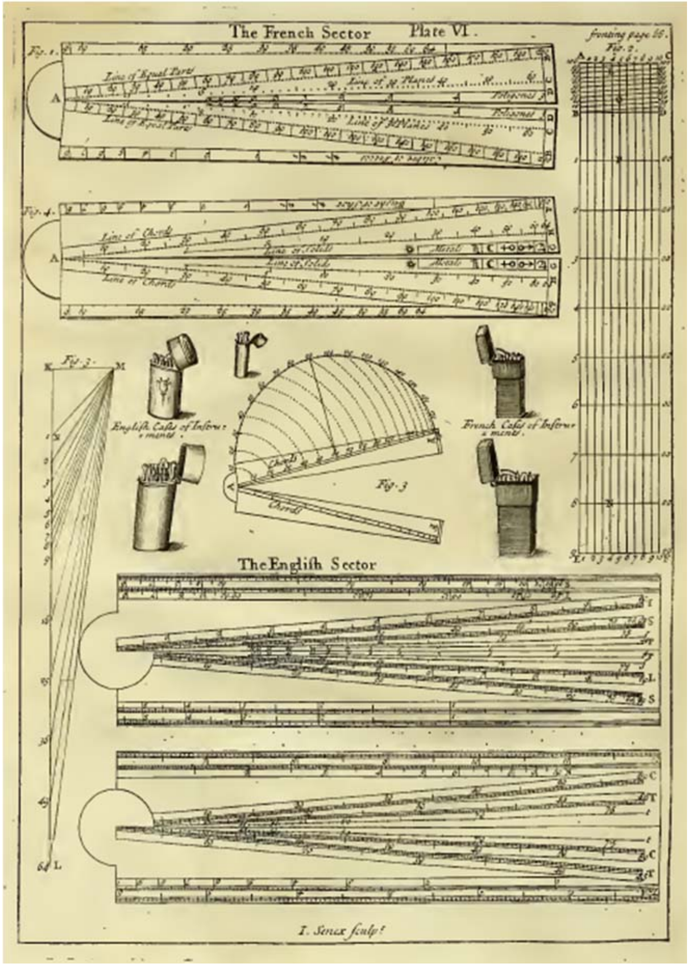


Figure 2. The French sector and the English sector, in Stone (1723), plate VI [digitized by the Internet Archive, 2012, with funding from Gordon Bell, available at http://archive.org/details/constructionprin00bion]

3 Of the use of the 'English sector' in trigonometry

In Section III Stone illustrated the use of the sector in trigonometry with the solution of several cases of plane triangles, both right-angled (uses I-VI) and oblique-angled (uses VII-IX), and spherical triangles (uses X-XI). The doctrine of triangles was involved in a number of practical applications. For instance, in surveying:

> How to find the distance of a fort, or walls of a city, or castle, that you dare not approach for fear of gun-shot; or the breadth of a river or water, that you cannot pass, or measure over it; made by two stations. (Sturmy, 1684, p. 20)

and in navigation:

> Two ships, one sails N. 36º 00’ E. 50 miles, the other sails S. 50º 00’ E. 70 miles; I demand their bearing and distance? Jones (1705, p. 28)

In Use VII Stone studied the case of an oblique triangle, when two angles and one side were known, and solved it with the help of the “English sector”:

> Use VII: *The angles* CAB*, and* ACB*, being given, and the side* CB*: to find the base* AB.
>
> Take the given side CB, and turn it into the parallel sine of its opposite angle CAB, and the parallel sine of the angle ACB, will be the length of the base AB. (Stone, 1723, p. 69)

Despite giving no further explanation, it is evident that here Stone was using the following trigonometric result: in all plane triangles, the sides are in proportion one to another as the sines of the angles opposite to these sides, that is to say, what today we call the law of sines.[5] Stone simply showed the mechanical process, the instrumental way of solving the problem by means of the sector. We find a similar approach in Gunter (1624):

> *To find a side by having the angles and one of the other sides given.*
>
> Take the side given, and turn it into the parallel sine of his opposite angle; so the parallel sines of the other angle shall be the opposite sides required. (Gunter, 1624, p. 70)

and later in Rea (1717) and in Martin (1745), though not as concise:

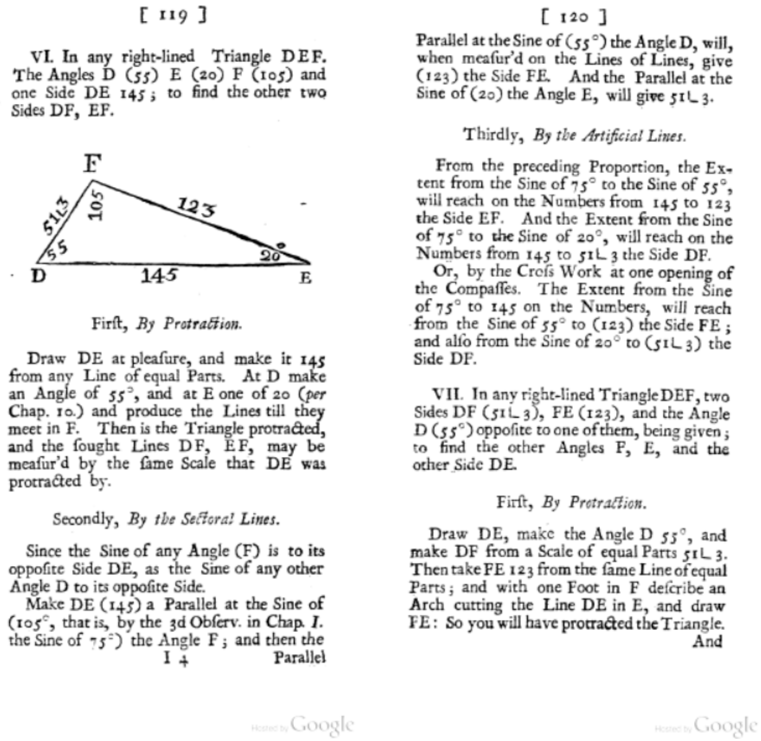
> In the oblique right lin’d triangle ABC, there is given the angle ACB 115 deg. 24, the angle BAC 28 deg. 30, with the length of the side AC 75 lea. to find the quantity of the angle ABC, and the length of the two sides AB and BC. (…)
>
> From the center of the line of lines take the length of the given side AC 75 lea. between your compasses and make it a parallel between the sines of its opposite angle ABC 36 deg. 6, (found by the substraction) the sector being continued at that angle, take the distance between the sines of the angle BAC 28 deg. 30 between your compasses, and measure that distance from the center of the line of lines, and it will be 61 lea. the length of its opposite side BC, and the distance taken between 64 deg. 36 (the complement of the angle ACB 115 deg. 24 to 180 deg.) it will be 115 lea. the length of the side AB, as was required. (Rea, 1717, p. 33)

However, in *The description, nature and general use of the sector and plain-scale* (1721), a book most likely written by Stone, the author formulated the law of sines verbally before solving the problem instrumentally by means of the sector: “As the sine of the angle ABC is to the length of the base AC, so is the sine of the angle ACB to the length of the side AB sought” (Stone, 1721, p. 30). A few years later, Cunn (1729), in a work published posthumously and carefully revised by Edmund Stone, proceeded similarly, after having solved the problem by protraction (Figure 3). Cunn went to the extent of presenting a third

---

[5] On the history of the law of sines, see van Brummelen (2009).

way of solving the problem, by the artificial lines. It would be worth analysing the use of the artificial lines in the doctrine of plane triangles, all the more so since other books on mathematical instruments, such as Rea (1717), Stone (1723), Webster (1739), Martin (1745) and Robertson (1747), explained how to use these lines in order to solve an oblique triangle. We can also find this alternative method in some books of trigonometry, as in Newton (1658) and Heynes (1701). Yet, a detailed discussion of the use of these lines is beyond the scope of this paper.



[119]

VI. In any right-lined Triangle DEF. The Angles D (55) E (20) F (105) and one Side DE 145; to find the other two Sides DF, EF.

Firſt, *By Protraction.*

Draw DE at pleaſure, and make it 145 from any Line of equal Parts. At D make an Angle of 55°, and at E one of 20 (*per* Chap. 10.) and produce the Lines till they meet in F. Then is the Triangle protracted, and the ſought Lines DF, EF, may be meaſur'd by the ſame Scale that DE was protracted by.

Secondly, *By the Sectoral Lines.*

Since the Sine of any Angle (F) is to its oppoſite Side DE, as the Sine of any other Angle D to its oppoſite Side.

Make DE (145) a Parallel at the Sine of (105°, that is, by the 3d Obſerv. in Chap. *I.* the Sine of 75°) the Angle F; and then the

I 4 Parallel

[120]

Parallel at the Sine of (55°) the Angle D, will, when meaſur'd on the Lines of Lines, give (123) the Side FE. And the Parallel at the Sine of (20) the Angle E, will give 51L3.

Thirdly, *By the Artificial Lines.*

From the preceding Proportion, the Extent from the Sine of 75° to the Sine of 55°, will reach on the Numbers from 145 to 123 the Side EF. And the Extent from the Sine of 75° to the Sine of 20°, will reach on the Numbers from 145 to 51L3 the Side DF.

Or, by the Croſs Work at one opening of the Compaſſes. The Extent from the Sine of 75° to 145 on the Numbers, will reach from the Sine of 55° to (123) the Side FE; and alſo from the Sine of 20° to (51L3) the Side DF.

VII. In any right-lined Triangle DEF, two Sides DF (51L3), FE (123), and the Angle D (55°) oppoſite to one of them, being given; to find the other Angles F, E, and the other Side DE.

Firſt, *By Protraction.*

Draw DE, make the Angle D 55°, and make DF from a Scale of equal Parts 51L3. Then take FE 123 from the ſame Line of equal Parts; and with one Foot in F deſcribe an Arch cutting the Line DE in E, and draw FE: So you will have protracted the Triangle.

And

Figure 3. Cunn (1729) [digitized by Google Books].

Finally, Robertson (1747) expressed the law of sines in a symbolical way, though not fully developed (Figure 4). In treatises on trigonometry, the so-called law of sines was expressed either verbally: “As the side AB is to the side BC. So is the sine of the angle ACB to the sine of the angle BAC” (Pitiscus, 1614, p. 95), or symbolically, as in Newton (1659, p. 63), Heynes (1701, p. 20) and Harris (1703, p. 32):

$$CB \,.\, s.\, CDB :: CD \,.\, s.\, CBD$$

Thus, it seems that books on trigonometry started to use symbols to express the law of sines earlier than books on mathematical instruments.

SOLUTION *of* CASE I.

The Solution of the examples falling under this cafe depend on the proportionality there is between the fides of plane triangles, and the fines of their oppofite angles.

EXAMPLE I. Pl. VI. Fig. 26.

In the triangle ABC : Given AB=56 } equal parts.
AC=64 }
∠B=46° 30′
Required ∠C, ∠ A, & BC.

The proportions are as follow,
As fide AC : fide AB : : fine ∠ B : fine ∠ C.
Then the fum of the angles B and C being taken from 180° will leave the angle A.
And as fine ∠B : fine ∠A : : fide AC : fide CB.

First by the logarithmic fcales.

To find the angle C.

The extent from 64 (=AC) to 56 (=AB) on the fcales of logarithm numbers, will reach from 46° 30′ (=∠B) to 39° 24′, (=∠C) on the fcale of logarithmic fines.

And the fum of 46° 30′ and 39° 24′ is 85° 54′

Then 85° 54′ taken from 180°, leaves 94° 6′ for the angle A.

To

Figure 4. Robertson (1747) [digitized by Google Books].

4 Some final remarks

From this preliminary study, three different ways of teaching how to use the sector for solving triangles can be identified: 1) instrumentally, with no explicit formulation of the law of sines; 2) with a verbal formulation of the law of sines before the instrumental solution; 3) with a symbolic formulation of the law of sines before the instrumental solution. The study of more works dealing with the “English sector” and its applications in trigonometry could contribute to determine a possible pattern in the transition from one way of teaching to another, especially in the first half of the 18th century. It would also be interesting to examine alternative ways of solving triangles by means of the artificial lines and of instruments other than the “English sector” and, consequently, the interaction between the mathematical development (theory) and the instruments (practice). Finally, from a pedagogical point of view it would be worth exploring the use of the sector to solve triangles in the mathematics classroom.

Acknowledgements

I am grateful to the staff of the British Library (especially of the Rare Books & Music Reading Room) and to Tony Simcock (Library and Archives, Museum of the History of Science in Oxford). Thanks to Stephen Johnston for his helpful comments regarding the collection of sectors of the Museum of the History of Science in Oxford. I also thank the Spanish Ministry of Economy and Competitivity for financial support (HAR2013-44643-R).

REFERENCES

Bion, N. (1709). *Traité de la construction et des principaux usages des instrumens de mathématique*. Paris: Chez la Veuve de J. Boudot, J. Collombat et J. Boudot fils.

Blanco, M. (2015). On gardeners, dukes and mathematical instruments. *Bulletin of the Scientific Instrument Society*, *125*, 22-28.

Cunn, S. (1729). *A new treatise of the construction and use of the sector*... Being a work of the late Mr. Samuel Cunn's, Teacher of Mathematicks, &c. Now carefully revised by Edmund Stone. London: Printed for J. Wilcox and T. Heath.

Frémontier-Murphy, C. (2013). A new mathematical vision for an innovative calculating instrument: Galileo's sector. *Bulletin of the Scientific Instrument Society*, *118*, 35-43.

Greenwald, S. J. and Thomley, J. E. (2013). Using the history of mathematics technology to enrich the classroom learning experience. *Electronic Proceedings of the Twenty-fourth Annual International Conference on Technology in Collegiate Mathematics*, paper S071: http://archives.math.utk.edu/ICTCM/i/24/S071.html, accessed 9 May 2016.

Gunter, E. (1624). *The description and use of the sector. The crosse-staffe and other instruments*. London: Printed by W. Jones.

Harris, J. (1703). *The elements of plain trigonometry demonstrated: and its practice and use explained*. London: Printed by E. Midwinter.

Harris, J. (1704, 1710). *Lexicon Technicum: or, An Universal English Dictionary of Arts and Sciences*, 2 vols. London: Printed for D. Brown.

Heynes, S. (1701). *A treatise of trigonometry, plane and spherical, theoretical and practical*. London: Printed by J. Matthews

Jones, W. (1705). *A new epitomy of the whole art of practical navigation : Containing the elements of plain trigonometry, and its application to plain, Mercator's and middle-latitude sailing*. London: Printed by T. Mead.

Martin, B. (1745). *The description and use of a case of mathematical instruments; particularly of all the lines contained on the plain scale, the sector, the Gunter, and the proportional compasses* ... London.

Nagaoka, R., Barrow-Green, J., Bartoloni Bussi, M. G., Isoda, M., van Maanen, J., Michalowicz, K. D., Ponza, M. V., van Brummelen, G. (2000). Non-standard media and other resources. In J. Fauvel and J. van Maanen (eds.), *History in mathematics education: the ICMI study*. Dordrecht: Kluwer, pp. 329-370.

Newton, J. (1658). *Trigonometria Britanica, or, The doctrine of triangles: in two books... the one composed, the other translated from the Latine copie written by Henry Gellibrand.* London : Printed by R. & W. Leybourn.

Newton, J. (1659). *Geometrical trigonometry, or, The explanation of such geometrical problems as are most useful & necessary, either for the construction of the canons of triangles, or for the solution of them*. London.

Pitiscus, B. (1595). *Trigonometry: or the doctrine of triangles*. First written in Latin, by Bartholomew Pitiscus. English Translation: 1614. London: Printed by E. Allde.

Rea, R. (1717). *The sector and plain scale, compared.* London: Printed by J. Cluer.

Robertson, J. (1747). *A treatise of such mathematical instruments, as are usually put into a portable case: containing their various uses in arithmetic, geometry, trigonometry, architecture, surveying, &c.* London: Printed for T. Heath, J. Hodges and J. Fuller.

Stone, E. (1721). *The description, nature and general use, of the sector and plain-scale, briefly and plainly laid down. As also a short account of the uses of the lines of numbers, artificial sines and tangents*. London: Printed for T. Wright.

Stone, E. (1723). *The construction and principal uses of mathematical instruments. Translated from the French of M. Bion, Chief Instrument-Maker to the French King. To which are added, the construction and uses of such instruments as are omitted by M. Bion*. London: Printed for J. Senex and W. Taylor.

Sturmy, S. (1669). *The Mariners Magazine.* 3rd ed: 1684. London: Printed by J. Playford.

Turner, A. J. (1973). Mathematical instruments and the education of gentlemen. *Annals of Science, 30 (1),* 51–88.

Van Brummelen, G. (2009). *The mathematics of the heavens and the earth: the early history of trigonometry*. Princeton, Oxford: Princeton University Press.

Webster, W. (1739). *The description and use of a complete sett or case of pocket-instruments: containing the construction of the several lines laid down on the plain-scale, and sector*. 2nd ed, corrected. London: Printed by A. Betteswroth, C. Hitch and D. Browne.