

# Energy conservation in the one-phase supercooled Stefan problem

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## Abstract

A one-phase reduction of the one-dimensional two-phase supercooled Stefan problem is developed. The standard reduction, employed by countless authors, does not conserve energy and a recent energy conserving form is valid in the limit of small ratio of solid to liquid conductivity. The present model assumes this ratio to be large and conserves energy for physically realistic parameter values. Results for three one-phase formulations are compared to the two-phase model for parameter values appropriate to supercooled salol (similar values apply to copper and gold) and water. The present model shows excellent agreement with the full two-phase model.

*Keywords:*

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## 1. Introduction

2     When a solid forms from a liquid at the heterogeneous nucleation tem-  
3     perature the freezing process is relatively slow and the liquid molecules have  
4     time to rearrange into a standard crystalline configuration. However, a su-  
5     percooled (or undercooled) liquid is in an unstable state, ready to solidify  
6     rapidly as soon as the opportunity arises. The solidification process may be  
7     so rapid that the liquid molecules have no time to rearrange themselves into  
8     the usual crystal structure and instead form an unorganised or amorphous  
9     solid structure that is reminiscent of the liquid phase. For this reason solids  
10    formed from a supercooled liquid have been referred to as *liquids on pause*

11 [1]. The different molecular arrangement means that such solids may have  
 12 very different properties to the normal solid phase. Amorphous metal alloys,  
 13 formed by supercooling below the glass transition temperature can be twice  
 14 as strong and three times more elastic than steel [1]. Numerous applications  
 15 for materials formed from a supercooled liquid, such as in sport and electronic  
 16 equipment, medical and aerospace, are discussed in the article of Telford [2].

17 The practical importance of solids formed from a supercooled liquid moti-  
 18 vates the need for the theoretical understanding of the associated phase  
 19 change process. Although the two-phase problem is well defined, it may be  
 20 difficult to solve, given that it involves two partial differential equations on  
 21 an *a priori* unknown, moving domain. The associated one-phase problem is  
 22 a significantly less challenging prospect, particularly when dealing with com-  
 23 plex geometries. However it has been shown that the standard one-phase  
 24 reduction does not conserve energy [3]. In this paper we examine the one-  
 25 phase reduction of the one-dimensional Stefan problem. It is shown that  
 26 the energy conserving form of [3] although mathematically correct is not ap-  
 27 propriate for physically realistic problems and so we propose an alternative  
 28 reduction which shows excellent agreement with the full two-phase model.

## 29 2. Mathematical models

One of the most basic formulations of the two-phase supercooled Stefan  
 problem in non-dimensional form may be written

$$\frac{\partial \theta}{\partial t} = \frac{k}{c} \frac{\partial^2 \theta}{\partial x^2}, \quad 0 < x < s(t) \quad \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad s(t) < x < \infty, \quad (1)$$

$$T(s, t) = \theta(s, t) = T_I(t), \quad T|_{x \rightarrow \infty} \rightarrow -1, \quad T(x, 0) = -1, \quad s(0) = 0, \quad (2)$$

$$[\beta - (1 - c)s_t] s_t = \left( k \frac{\partial \theta}{\partial x} - \frac{\partial T}{\partial x} \right) \Big|_{x=s} \quad (3)$$

30 where  $T, \theta$  represent the liquid and solid temperatures,  $k = k_s/k_l$  the ther-  
 31 mal conductivity ratio,  $c$  the specific heat ratio,  $\beta = L_m/(c_l \Delta T)$  the Stefan  
 32 number,  $L_m$  the latent heat and  $\Delta T$  the degree of supercooling. The above  
 33 system describes the phase change process of a supercooled semi-infinite ma-  
 34 terial which solidifies from the boundary  $x = 0$ . The phase change boundary  
 35 is at  $x = s(t)$ , where  $s(0) = 0$ . The variable  $T_I(t)$  represents the temperature

36 at the phase change interface. If solidification occurs at the heterogeneous  
 37 nucleation temperature we choose  $T_I(t) = 0$ . With supercooling a non-linear  
 38 relation exists between  $T_I$  and  $s_t$  [4, 5]. For small levels of supercooling it is  
 39 standard to choose a linear approximation  $T_I(t) = -s_t$ . This is often referred  
 40 to as a *linear kinetic undercooling model*. For simplicity we will use the linear  
 41 approximation throughout this paper although the methodology translates  
 42 immediately to the non-linear case. The above formulation involves the as-  
 43 sumption that the density change between liquid and solid phases is small  
 44 and so may be neglected compared to other physical changes, such as the  
 45 jump in specific heat. We augment this system with the initial condition  
 46  $\theta(x, 0) = \theta_i$  and a boundary condition  $\theta_x(0, t) = 0$ : for a standard one-phase  
 47 problem these extra conditions are unnecessary but they are required when  
 48 looking for a reduction from a two-phase model. Note, we choose the bound-  
 49 ary condition at  $x = 0$  to match that of [6] and also because it is appropriate  
 50 when working in cylindrical and spherical co-ordinates, but other boundary  
 51 conditions will work in the arguments below.

The standard one-phase Stefan problem is retrieved from the above sys-  
 tem by simply ignoring the  $\theta$  equation and setting  $k = 0$  in the Stefan  
 condition, consequently

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad s < x < \infty \quad (4)$$

$$T(s, t) = -s_t, \quad T|_{x \rightarrow \infty} \rightarrow -1, \quad T(x, 0) = -1, \quad s(0) = 0 \quad (5)$$

$$[\beta - (1 - c)s_t] s_t = -\frac{\partial T}{\partial x} \Big|_{x=s}. \quad (6)$$

52 In fact this is often further reduced by choosing  $c = 1$ . It is well-known  
 53 that if supercooling is neglected, *i.e.*  $T_I(t) = 0$ , and  $c = 1$ , then the well-  
 54 known Neumann solution may be applied to (4)-(6), but this breaks down as  
 55  $\beta \rightarrow 1^+$ . Applying the linear kinetic undercooling temperature  $T_I(t) = -s_t$   
 56 prevents this breakdown and so permits solutions for arbitrary undercooling.

57 Evans and King [3] point out that the above reduction does not conserve  
 58 energy since the limit  $\theta \rightarrow 0$  involves a singular perturbation of the two-  
 59 phase system. Physically the issue is obvious: the reduction is based on  
 60 setting  $\theta$  constant, without the undercooling term the boundary condition  
 61 determines  $\theta = T_I \equiv 0$  and so the (non-dimensional) constant is zero and  
 62 this satisfies the heat equation and boundary condition at  $x = s$  for all time.  
 63 With kinetic undercooling the temperature at  $x = s$  varies with time, so

64  $\theta(s, t)$  is a function of time and the assumption of constant  $\theta$  is no longer  
 65 valid.

66 To determine a consistent one-phase model, Evans and King [3] inves-  
 67 tigate the limit  $k \rightarrow 0$  which is equivalent to neglecting  $\theta$  in the Stefan  
 68 condition. The heat equation in the solid then indicates  $\theta_t \rightarrow 0$  and so  
 69  $\theta \approx \theta(x) = \theta_i$ , (after imposing the initial condition). However, this con-  
 70 tradicts the condition  $\theta(s, t) = -s_t \neq \theta_i$  and so indicates the need for a  
 71 boundary layer. To analyse this boundary layer a new co-ordinate is intro-  
 72 duced,  $x = s(t) - k\hat{x}$  (where  $k \ll 1$ ), which transforms (1b) to

$$s_t \frac{\partial \theta}{\partial \hat{x}} + k \frac{\partial \theta}{\partial t} = \frac{1}{c} \frac{\partial^2 \theta}{\partial \hat{x}^2}. \quad (7)$$

73 Neglecting the small term involving  $k$  allows the equation to be integrated  
 74 and applying  $\theta \rightarrow \theta_i$  as  $\hat{x} \rightarrow \infty$  gives

$$\frac{1}{c} \frac{\partial \theta}{\partial \hat{x}} = s_t (\theta - \theta_i). \quad (8)$$

75 Noting that  $\theta_{\hat{x}} = -k\theta_x$  we may use (8) to replace the solid temperature  
 76 gradient in the Stefan condition (3) and applying  $\theta(s, t) = -s_t$  gives

$$[\beta - s_t - c\theta_i] s_t = -\frac{\partial T}{\partial x}. \quad (9)$$

77 The correct reduction of the two-phase Stefan problem *in the limit*  $k \rightarrow 0$  is  
 78 therefore specified by equations (4)-(5), with the Stefan condition given by  
 79 (9). The properties and behaviour of systems of this form, with appropriate  
 80 modification for different physical situations have been studied for example  
 81 in [7, 8].

82 Heat conduction occurs on the microscopic scale due to the transfer of  
 83 kinetic energy from hot, rapidly vibrating atoms or molecules to their cooler,  
 84 more slowly vibrating neighbours. In solids the close, fixed arrangement of  
 85 atoms means that conduction is more efficient than in fluids, which have a  
 86 larger distance between atoms. Consequently, in general, the conductivity  
 87 of a solid is greater than that of its corresponding liquid phase, for example  
 88 with water and ice  $k = k_s/k_l \approx 4$ , for solid and molten gold  $k \approx 3$ . Hence  
 89 the limit  $k \rightarrow 0$  has limited applicability and for practical Stefan problems  
 90 it would seem more appropriate to study the large  $k$  limit.

91 Now we let  $k \rightarrow \infty$  and the heat equation (1b) reduces to  $\theta_{xx} \approx 0$ ,  
 92 so to leading order  $\theta = c_0(t) + c_1(t)x = -s_t$  (after applying the boundary

93 conditions). So far this seems a reasonable result, large  $k$  indicates heat  
 94 travels rapidly through the solid (compared to the travel time in the liquid)  
 95 which then equilibrates to the boundary temperature almost instantaneously.  
 96 However, in the Stefan condition we have the term  $k\theta_x$ , which is zero to  
 97 leading order (since  $\theta = -s_t(t)$ ), but since the coefficient  $k$  is large it is  
 98 possible that the first order term plays an important role. If we write  $\theta =$   
 99  $\theta_0 + (1/k)\theta_1 + \mathcal{O}(1/k^2)$  then the leading and first order heat equations are

$$\frac{\partial^2 \theta_0}{\partial x^2} = 0, \quad c \frac{\partial \theta_0}{\partial t} = \frac{\partial^2 \theta_1}{\partial x^2}. \quad (10)$$

100 The appropriate temperatures are  $\theta_0 = -s_t$  and  $\theta_1 = -cs_{tt}(x^2 - s^2)/2$ . The  
 101 Stefan condition becomes

$$[\beta - (1 - c)s_t] s_t = k \left( \frac{\partial \theta_0}{\partial x} + \frac{1}{k} \frac{\partial \theta_1}{\partial x} + \mathcal{O}(1/k^2) \right) \Big|_{x=s} - \frac{\partial T}{\partial x} \Big|_{x=s}. \quad (11)$$

102 Substituting for  $\theta_1$  in (11) we find that the one-phase Stefan problem in the  
 103 limit of large  $k$  is then specified by equations (4)-(5) and the Stefan condition

$$css_{tt} + [\beta - (1 - c)s_t] s_t = - \frac{\partial T}{\partial x} \Big|_{x=s}. \quad (12)$$

104 The inclusion of the derivative  $s_{tt}$  requires an extra initial condition. In  
 105 the absence of supercooling,  $T_I = 0$ , hence  $T(s, t) = T_I$  indicates  $T(0, 0) = 0$ .  
 106 For  $x > 0$  we have  $T(x, 0) = -1$ , hence the temperature gradient

$$T_x(x, 0)|_{x \rightarrow 0} = \lim_{h \rightarrow 0} \frac{T(h, 0) - T(0, 0)}{h} = \lim_{h \rightarrow 0} \left( \frac{-1 - 0}{h} \right) = -\infty. \quad (13)$$

107 In the one phase problem the front velocity is a function of the temperature  
 108 gradient with the result that without kinetic undercooling the above initial  
 109 infinite gradient indicates  $s_t(0) = \infty$ . This may be seen, for example, in  
 110 the well-known Neumann solution where  $s_t \sim 1/\sqrt{t}$ . The singularity is an  
 111 obvious consequence of the unphysical nature of the boundary condition:  
 112 choosing  $T = -1$  for all  $x > 0$  and  $T = 0$  at a single point  $x = 0$  is not  
 113 consistent with an equation based on continuum theory. Kinetic undercooling  
 114 provides a mechanism for removing the unphysical behaviour. The only way  
 115 to avoid the singularity is if  $T(0, 0) = \lim_{h \rightarrow 0} (T(h, 0) + \mathcal{O}(h)) = \lim_{h \rightarrow 0} (-1 +$   
 116  $\mathcal{O}(h)) = -1$ . In physical terms we may think of an undercooled melt at

117 temperature  $T = -1$  everywhere when some infinitesimally small amount  
 118 of energy is input at the boundary resulting in  $T(0, 0) = -1 + \mathcal{O}(h)$ : this  
 119 is sufficient to set off the solidification process (and it is well-known that  
 120 ‘working with undercooled liquids is a bit like juggling mousetraps: they’re  
 121 prone to suddenly “snap” and ruin the trick’ [1]). Since  $T_I(0) = T(0, 0) =$   
 122  $-1$  we find that in the case of linear undercooling the additional boundary  
 123 condition required to close the Stefan problem is

$$s_t(0) = -T_I(0) = 1. \quad (14)$$

124 This argument also helps us with the one-phase formulation of equation (9)  
 125 which requires an initial solid temperature,  $\theta_i$  (despite the solid phase not  
 126 entering the one-phase problem). Since the initial ‘kick’ to start solidification  
 127 may be infinitesimal, and for  $t$  sufficiently close to zero an infinitesimally  
 128 small amount of latent heat has been released, the only physically sensible  
 129 value for the solid temperature is  $\theta_i = -1$ . These initial conditions on  $\theta_i$  and  
 130  $s_t$  are obtained more formally through a short time asymptotic analysis in  
 131 [9].

### 132 3. Energy conservation

133 The non-dimensional thermal energy in the two-phase system is given by

$$E = \int_0^s c \theta dx + \int_s^\infty T dx. \quad (15)$$

134 During the phase change the molecular rearrangement also releases (or uses)  
 135 energy, namely the latent heat. So the rate of change of thermal energy,  $E_t$ ,  
 136 must balance the rate at which energy is produced by the phase change,  $\beta s_t$ .  
 137 Differentiating the above equation we find

$$\frac{dE}{dt} = \int_0^s c \frac{\partial \theta}{\partial t} dx + c \theta(s, t) \frac{ds}{dt} + \int_s^\infty \frac{\partial T}{\partial t} dx - T(s, t) \frac{ds}{dt}. \quad (16)$$

138 The heat equations in (1) allow the time derivatives to be replaced with  
 139  $x$  derivatives in the integrals, which may then be evaluated immediately.  
 140 Noting that  $\theta(s, t) = T(s, t) = -s_t$  then (16) becomes

$$\frac{dE}{dt} = k \frac{\partial \theta}{\partial x} \Big|_{x=0} + \left( k \frac{\partial \theta}{\partial x} - \frac{\partial T}{\partial x} \right) \Big|_{x=s} + \frac{\partial T}{\partial x} \Big|_{x=\infty} + (1 - c) \left( \frac{ds}{dt} \right)^2. \quad (17)$$

141 The temperature gradients at  $x = s$  may be removed via the Stefan condition  
 142 (3). The insulated boundary condition of the current study requires  $\theta_x(0, t) =$   
 143  $0$ , and as  $x \rightarrow \infty$  the gradient  $T_x \rightarrow 0$ , so we are left with

$$\frac{dE}{dt} = \left[ \beta - (1 - c) \frac{ds}{dt} \right] \frac{ds}{dt} + (1 - c) \left( \frac{ds}{dt} \right)^2 = \beta \frac{ds}{dt}. \quad (18)$$

144 So the rate of change of thermal energy balances the latent heat release and  
 145 the two-phase formulation conserves energy. Note, the argument follows in  
 146 the same way for different boundary conditions, for example if we choose a  
 147 constant flux  $k\theta_x(0, t) = q$  then the rate of change of thermal energy balances  
 148 the latent heat release plus the heat input at the boundary.

149 The energy balance for the standard one-phase problem specified by equa-  
 150 tions (4)-(5) can be obtained from the above argument by neglecting all  $\theta$   
 151 terms in (16) (or equivalently setting  $c = k = 0$  in (17)) and applying the  
 152 Stefan condition (6) to replace  $T_x(s, t)$

$$\frac{dE}{dt} = \left[ \beta - (1 - c) \frac{ds}{dt} \right] \frac{ds}{dt} + \left( \frac{ds}{dt} \right)^2 \neq \beta \frac{ds}{dt}. \quad (19)$$

153 This demonstrates that energy is not conserved in this formulation. The  
 154 equivalent expression in limit  $k \rightarrow 0$  is obtained by replacing  $\theta_x(s, t)$  via (8)  
 155 and applying the Stefan condition (9) to replace  $T_x(s, t)$  to equation (17) to  
 156 obtain

$$\frac{dE}{dt} = c \frac{ds}{dt} \left( \frac{ds}{dt} + \theta_i \right) + \left[ \beta - \frac{ds}{dt} - c\theta_i \right] \frac{ds}{dt} + (1 - c) \left( \frac{ds}{dt} \right)^2 = \beta \frac{ds}{dt}. \quad (20)$$

157 Finally the one-phase limit with  $k \rightarrow \infty$  is determined using the definition  
 158 of  $\theta_1$  to give  $k\theta_x(s, t) = -css_{tt}$  and  $T_x(s, t)$  comes from the Stefan condition  
 159 (12) to give

$$\frac{dE}{dt} = -cs \frac{d^2s}{dt^2} + \left( cs \frac{d^2s}{dt^2} + \left[ \beta - (1 - c) \frac{ds}{dt} \right] \frac{ds}{dt} \right) + (1 - c) \left( \frac{ds}{dt} \right)^2 = \beta \frac{ds}{dt}. \quad (21)$$

160 Hence the large and small  $k$  formulations also conserve energy.

#### 161 4. Comparison of results

162 We now present two sets of results for the solidification of salol and water.  
 163 The results were computed numerically using the boundary immobilisation

164 method and Keller box finite difference technique used in [10, 11]. The  
 165  $k \rightarrow 0$  result was rather unexpected so the computations were verified using  
 166 an accurate heat balance method, as described in [12, 13]. This provided  
 167 solutions typically within 0.5% of the numerics. As discussed above, the  
 168  $k \rightarrow 0$  formulation requires a value for the solid temperature  $\theta_i$ . At the end  
 169 of §2 we demonstrated that  $\theta_i = -1$ . We also tried  $\theta_i = 0$  but this did not  
 170 improve the correspondence.

171 In Figure 1 we compare the position of the phase change front for the  
 172 three one-phase formulations against the two-phase solution using parameter  
 173 values appropriate for salol and with two values of  $\beta$ . Salol was chosen  
 174 since it was the material with the lowest value of  $k \approx 1.4$  for which we had  
 175 all the necessary data, see [4]. The values of  $\beta$  correspond to dimensional  
 176 temperatures of 234.8, 272.4K (the heterogeneous phase change temperature  
 177  $T_m \approx 314.7\text{K}$ ), the value of  $c = c_s/c_l = 0.73$ . The solid line in the figure  
 178 represents the two-phase model, the dotted line the standard one-phase model  
 179 of equations (4)-(6), this is bounded by the two limiting cases which conserve  
 180 energy using the Stefan conditions (9) for  $k \rightarrow 0$  (dot-dash line) and (12) for  
 181  $k \rightarrow \infty$  (dashed line). Even in this case, where  $k$  is relatively small we find  
 182 that the large  $k$  solution is extremely close to the two-phase model while the  
 183 limit  $k \rightarrow 0$  shows an approximately 40% difference to the two-phase solution.  
 184 It is also surprising that this latter energy conserving form is further from  
 185 the two-phase solution than the form that does not conserve energy. The  
 186 two sets of plots are for small values of  $\beta$  (in particular we wished to show  
 187 results with  $\beta < 1$  and  $\beta > 1$ ). In the limit of large  $\beta$  the curves all coincide  
 188 but for the  $k \rightarrow 0$  case the convergence is slow: for  $\beta = 40$  the  $k \rightarrow \infty$  result  
 189 is within 0.005% of the 2 phase result, the  $k \rightarrow 0$  solution is within 1.8%.

190 In Figure 2 we show results for a water-ice system where  $k \approx 4, c \approx 0.49$ .  
 191 This has a significantly lower  $c$  value than salol and a higher  $k$  value. The val-  
 192 ues  $\beta = 0.7, 1.3$  correspond to temperatures 158.9, 211.5 (where  $T_m \approx 273\text{K}$ ),  
 193 see [14]. With the larger  $k$  value we can observe that the two-phase formula-  
 194 tion and the large  $k$  one-phase approximation are almost indistinguishable.  
 195 The  $k \rightarrow 0$  formulation differs by approximately 30% and again the result  
 196 obtained by simply neglecting  $\theta$  is more accurate than this latter energy  
 197 conserving form.

198 In addition to the results shown above we also carried out the same cal-  
 199 culations for molten and solid copper,  $k \approx 2.4, c \approx 0.72$  and gold  $k \approx 3, c \approx$   
 200  $0.79$ . In both cases the value of  $c$  is similar to that of salol and so the cop-  
 201 per results were virtually identical to those of salol, whilst the gold results



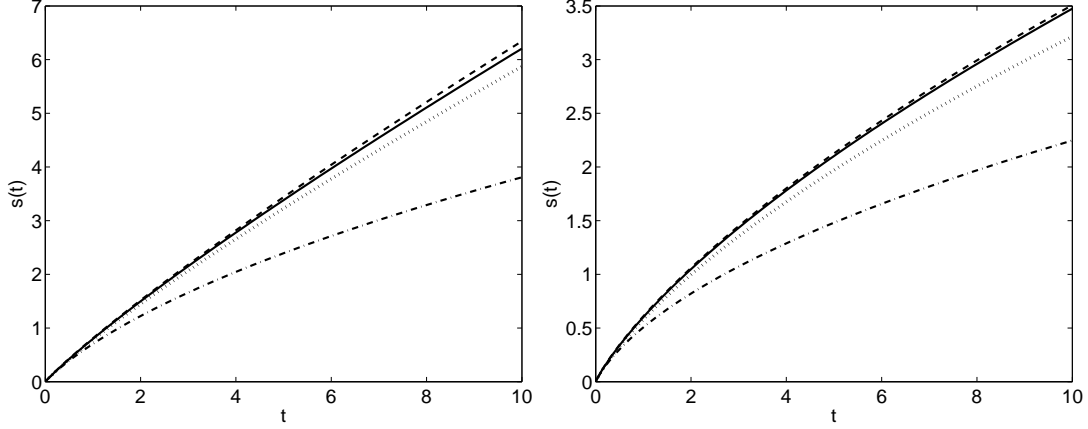


Figure 1: Variation of  $s(t)$  for *salol*,  $k \approx 1.4$ ,  $c = 0.73$  and  $\beta = 0.7, 1.3$

202 showed a very slight decrease in the velocity  $s_t$ .

## 203 5. Conclusions

In summary, our simulations show that the one-phase reduction with large  $k$  can provide an excellent agreement with the two-phase problem for a wide range of physically realistic parameter values and supercooling. The small  $k$  formulation of [3] whilst mathematically correct is highly inaccurate for practical problems and surprisingly significantly less accurate than the non-energy conserving form. Only in the limit of large Stefan number do the solutions coincide (and in this case the supercooled formulation is unnecessary). We therefore propose that an accurate approximation to the two-phase one-dimensional Stefan problem is obtained by the simpler one-phase approximation specified by equations (4)-(5) and the Stefan condition (12). Using standard notation the dimensional form may be written

$$\frac{\partial T}{\partial t} = \frac{k_l}{\rho_l c_l} \frac{\partial^2 T}{\partial x^2}, \quad s(t) < x < \infty \quad (22)$$

$$T(s, t) = T_m - \phi s_t, \quad T|_{x \rightarrow \infty} \rightarrow T_\infty, \quad T(x, 0) = T_\infty \quad (23)$$

$$\rho_l c_s \phi s s_{tt} + \rho_l [L_m - (c_l - c_s) \phi s_t] s_t = -k_l \left. \frac{\partial T}{\partial x} \right|_{x=s}, \quad (24)$$

$$s(0) = 0, \quad s_t(0) = 1, \quad (25)$$

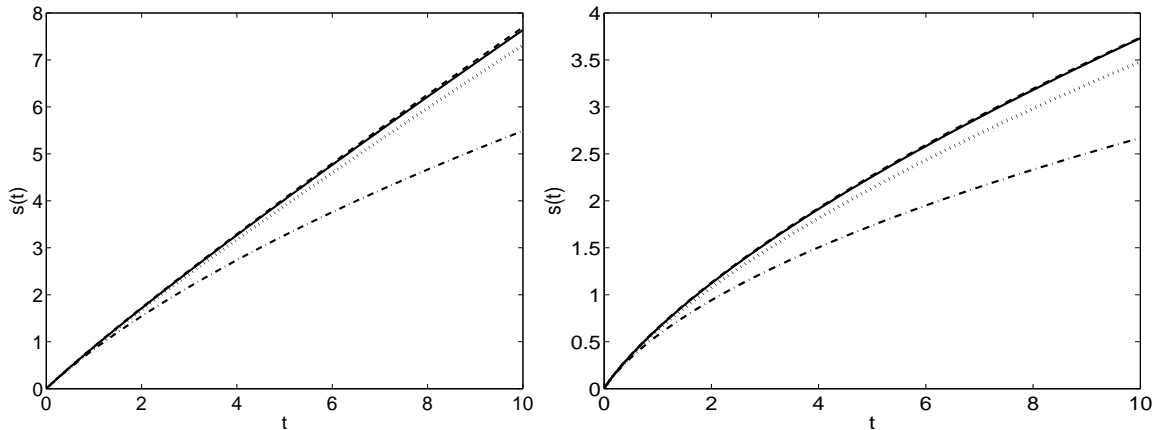


Figure 2: Variation of  $s(t)$  for water,  $k \approx 4$ ,  $c \approx 0.49$  and  $\beta = 0.7, 1.3$

204 where the constant  $\phi$  is the kinetic undercooling coefficient used in the linear  
 205 relation  $T_I(s_t) \approx T_m - \phi s_t$ . Similar reductions can no doubt be obtained for  
 206 related problems and a similar analysis may be easily applied to the nonlinear  
 207 undercooling case.

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## 216 References

- 217 [1] NASA, Nasa science news: Liquids on pause, 2003.  
 218 [http://science.nasa.gov/science-news/science-at-nasa/2003/  
 219 16oct\\_viscosity/](http://science.nasa.gov/science-news/science-at-nasa/2003/16oct_viscosity/), last accessed 12 May 2012.
- 220 [2] M. Telford, *Materials Today* 7 (2004) 36–43.
- 221 [3] J. D. Evans, J. R. King, *Q.J.M.A.M.* 53 (2000) 449–473.

- 222 [4] M. F. Ashby, D. R. H. Jones, *Engineering materials 2: an introduction*  
223 *to microstructures, processing and design*, Butterworth-Heinemann, 3rd  
224 edition, 2006.
- 225 [5] W. L. Chan, R. S. Averback, D. G. Cahill, Y. Ashkenazy, *Phys. Rev.*  
226 *Lett.* 102 (2009) 1230–1234.
- 227 [6] C. Charach, B. Zaltzman, *Phys. Rev. E.* 49 (1994) 4322–4327.
- 228 [7] B. Wu, S. W. McCue, P. Tillman, J. M. Hill, *Appl. Math. Model.* 33  
229 (2009) 2349–2367.
- 230 [8] J. R. King, J. D. Evans, *SIAM J. Appl. Math.* 65(5) (2005) 1677–1707.
- 231 [9] C. Charach, B. Zaltzman, *Phys. Rev. E.* 47 (1993) 1230–1234.
- 232 [10] S. L. Mitchell, M. Vynnycky, *App. Math. Comp.* 215 (2009) 1609–1621.
- 233 [11] S. L. Mitchell, M. Vynnycky, I. G. Gusev, S. S. Sazhin, *App. Math.*  
234 *Comp.* 217 (2011) 9219–9233.
- 235 [12] T. G. Myers, *Int. Commun. Heat Mass Trans.* 36 (2009) 143–147.
- 236 [13] T. G. Myers, *Int. J. Heat Mass Trans.* 53 (2010) 1119–1127.
- 237 [14] P. G. Debenedetti, *J. Phys.: Condens. Matter* 15 (2003) 1669–1726.