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NATURAL FREQUENCIES FOR LAMINATED RECTANGULAR PLATES WITH EXTENSION-BENDING OR EXTENSION-TWISTING AND SHEARING-BENDING COUPLING

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PRESENTATION CONTENTS

- **Background** on some applications of mechanically coupled laminates**.**
- **Characterization** of laminated composite materials.
- **Derivation** of Extension-Twisting (and Shearing-Bending) coupled laminates.
- **Form of stacking sequences** - standard angle-ply and cross-ply sub-sequences; combinations which are contrary to the previously assumed form for this class of laminate, i.e., anti-symmetric angle-ply laminates.
- **Dimensionless parameters** - from which the extensional, coupling and bending stiffness terms are readily calculated for any fiber/matrix system.
- **Lamination parameters** (Ply orientation dependent dimensionless parameters) are shown graphically to illustrate the extent of the design space with up to 21 plies.
- **Hygro-Thermally Curvature Stable laminates** - a special sub-group that can be manufactured flat under a standard elevated temperature curing process.
- **Bounds on the natural frequency** are assessed using a closed form solution, applicable to all the laminate groups presented.

BACKGROUND

Exotic mechanical coupling responses are not present in conventional materials, such as metals, and therefore represent an important and significant enabling technology.

However ….. many mechanically coupled laminate designs lead to undesirable warping distortions as a result of the high temperature curing process.

One family of coupled laminate, with immunity to thermal warping, has been used extensively for application to tilt-rotor blade design.

Extension-Twisting coupling, at the structural or blade level, is used to develop optimized twist distribution along the blade for both hover and forward flight: a change in rotor speed, and the resulting centrifugal forces, provides the required twist differential between the two flight regimes.

This behaviour can be achieved from laminate level Extension-Shearing coupling through off-axis alignment of a *fully uncoupled* laminate.

CHARACTERIZATION

Laminated composite plates are typically characterized in terms of their response to mechanical or thermal loading, which is generally associated with a description of the coupling behaviour, described by the **ABD** relation:

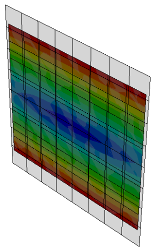
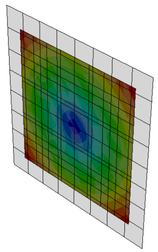
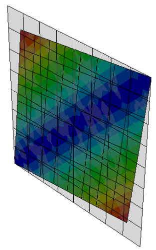
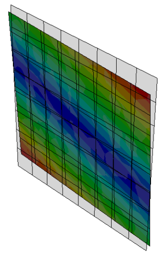
$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ \text{Sym.} & & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ & B_{22} & B_{26} \\ \text{Sym.} & & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ & B_{22} & B_{26} \\ \text{Sym.} & & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ \text{Sym.} & & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Couplings generally exist between:

- in-plane and out-of-plane actions, when $B_{ij} \neq 0$, (*5 distinct forms for matrix* **B***!*)
- Shearing and Extension, when $A_{16} = A_{26} \neq 0$, and
- Bending and Twisting, when $D_{16} = D_{26} \neq 0$.

Unrestrained thermal (contraction) response of square, initially flat, composite laminates, **B** = **0**:

Uncoupled in Extension		Extension-Shearing		
Uncoupled in Bending	**Bending-Twisting**	**Bending-Twisting**	**Uncoupled in Bending**	
$[+/-_2/\bigcirc/+_2/-]_T$  *Simple laminate*	$[+/-/-/+]_T$  *B-T*	$[+/+]_T$  *E-S;B-T*	$[\pm/\bigcirc/-/\bigcirc/-_3/\bigcirc/-_3/\bigcirc/+]_T$  *E-S*	**Uncoupled (B = 0)**

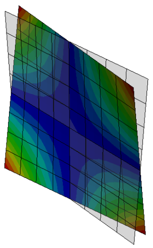
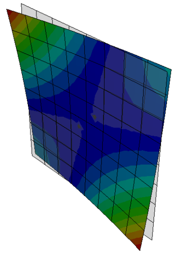
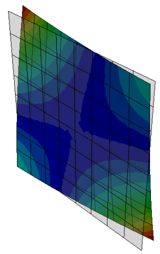
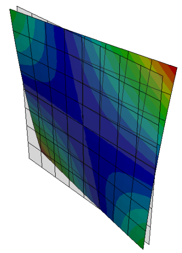
Off-axis alignment of uncoupled (*Simple*) laminates leads to significant Bending-Twisting coupling at the laminate level, and to detrimental effects on the compression buckling strength of the blade[1]; which represents an important static design constraint.

[1] York, C. B. (2012). Extension-Twisting coupled laminates for aero-elastic compliant blade design. Proc. 53rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, Paper No. AIAA-2012-1457, Honolulu, Hawaii.

By contrast, laminate level Extension-Twisting coupling can also be achieved with immunity to the thermal warping distortions, but special tailoring strategies must be employed.

Note: the vast majority of configurations in this class require specially curved tooling in order to possess the required shape after high temperature curing!

Unrestrained thermal (contraction) response of square, initially flat, composite laminates, B_{16}, $B_{26} \neq 0$:

<table>
<tr><th colspan="2">Uncoupled in Extension</th><th colspan="2">Extension-Shearing (A_F)</th><th></th></tr>
<tr><th>Uncoupled in Bending</th><th>Bending-Twisting</th><th>Bending-Twisting</th><th>Uncoupled in Bending</th><th></th></tr>
<tr><td>$[+/-]_T$

B-S-T-E</td><td>$[+/-_3/+_2]_T$

B-S;B-T</td><td>$[+/+/-]_T$

E-S;B-S;B-T</td><td>$[\pm/\bigcirc/-_3/\bigcirc/-/\bigcirc/-_3/\pm/\bigcirc/+]_T$

E-S;B-S</td><td>Extension-Twisting, & Shearing-Bending</td></tr>
</table>

How do we obtain these special laminates?

DERIVATION OF STACKING SEQUENCE DATA

- The general rules of symmetry are relaxed!

  Cross plies (0° and 90°) and angle plies (45° and -45°) are <u>not constrained to be symmetric or anti-symmetric</u> about the laminate mid-plane.

- Symbolic notation is adopted *to allow for non-standard ply orientations.*

  All sequences have an angle-ply (+) on one surface (1st ply) of the laminate, but the other surface ply may have equal (+) or opposite (–) orientation or it may indeed be a cross ply (○ or ●) of 0 or 90° orientation.

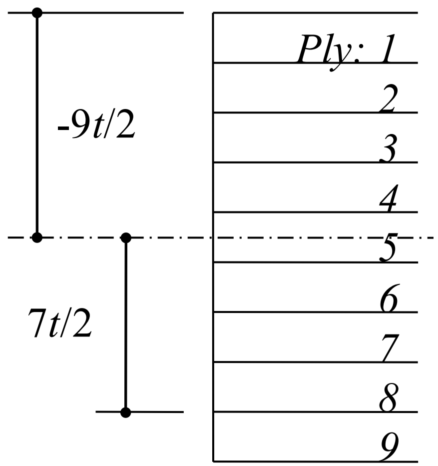
- Non-dimensional parameters are adopted *to allow for any fiber/matrix system*.

DEVELOPMENT OF NON-DIMENSIONAL PARAMETERS

The derivation of non-dimensional coupling stiffness parameters is readily demonstrated for the 9-ply laminate, with non-symmetric stacking sequence $[+/\bigcirc/-/\bigcirc/+_2/-_2/\bigcirc]_T$, where the coupling stiffness terms,

$$B_{ij} = \sum_{k=1}^{n} Q'_{ij,\theta k}(z_k^{\,2} - z_{k\text{-}1}^{\,2})/2$$

$$\begin{aligned}B_{ij} = \{&Q'_{ij+}((-7t/2)^2 - (-9t/2)^2) + Q'_{ij\bigcirc}((-5t/2)^2 - (-7t/2)^2) + Q'_{ij-}((-3t/2)^2 - (-5t/2)^2)\\ &+ Q'_{ij\bigcirc}((-t/2)^2 - (-3t/2)^2) + Q'_{ij+}((t/2)^2 - (-t/2)^2) + Q'_{ij+}((3t/2)^2 - (t/2)^2)\\ &+ Q'_{ij-}((5t/2)^2 - (3t/2)^2) + Q'_{ij-}((7t/2)^2 - (5t/2)^2) + Q'_{ij\bigcirc}((9t/2)^2 - (7t/2)^2)\}/2\end{aligned}$$



where subscripts i,j = 1, 2, 6.
The coupling stiffness contributions from the different ply orientations are:

B_{ij+}	$= 6t^2/2 \times Q'_{ij+}$	$= \chi_+ t^2/4 \times Q'_{ij+}$	$\chi_+ = 12$
B_{ij-}	$= -6t^2/2 \times Q'_{ij-}$	$= \chi_- t^2/4 \times Q'_{ij-}$	$\chi_- = -12$
$B_{ij\bigcirc}$	$= 0t^3/2 \times Q'_{ij\bigcirc}$	$= \chi_\bigcirc t^3/4 \times Q'_{ij\bigcirc}$	$\chi_\bigcirc = 0$

CALCULATION OF THE ABD MATRIX

The calculation procedure for the $[+/\circ/-/\circ/+_2/-_2/\circ]_T$ laminate **ABD** matrix, using these non-dimensional parameters (*derived for each stacking sequence discovered*), is as follows:

$$A_{ij} = \{n_+Q'_{ij+} + n_-Q'_{ij-} + n_\circ Q'_{ij\circ} + n_\bullet Q'_{ij\bullet}\} \times t$$

$$B_{ij} = \{\chi_+Q'_{ij+} + \chi_-Q'_{ij-} + \chi_\circ Q'_{ij\circ} + \chi_\bullet Q'_{ij\bullet}\} \times t^2/4$$

$$D_{ij} = \{\zeta_+Q'_{ij+} + \zeta_-Q'_{ij-} + \zeta_\circ Q'_{ij\circ} + \zeta_\bullet Q'_{ij\bullet}\} \times t^3/12$$

For Extension-Twisting (and Shearing-Bending) coupling, i.e., B_{16}, $B_{26} \neq 0$ (with all other $B_{ij} = 0$):

$\chi_+ = -\chi_-$ and $\chi_\circ = \chi_\bullet = 0$ ✓

This does not imply that cross plies ($\circ$ or $\bullet$) *are absent!*

For uncoupled Extensional stiffness properties, i.e., $A_{16} = A_{26} = 0$:

$n_+ = n_-$ ✓

For uncoupled Bending stiffness properties, $D_{16} = D_{26} = 0$,

$\zeta_+ = \zeta_-$ ✓

NUMBER OF SOLUTIONS

Table 4 – Number of *E-T-S-B* coupled angle-ply laminates for each ply number (n) grouping. Numbers in parentheses indicate non-symmetric stacking sequences.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	1	-	2	-	4	-	7	-	16	-	35(4)	-	84(20)	-	194(70)	-	512(256)	-	1,352(850)	-

Table 5 – Number of *E-T-S-B* coupled laminates for each ply number (n) grouping containing combinations of standard ply orientations: 0°, 90° and/or ±45°.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	1	2	4	8	16	30	63	128	256	536	1,085	2,412	4,774	11,292	21,994	58,502	111,908	329,412	632,996	2,095,268

LAMINATION PARAMETERS

Lamination parameters can be used to reveal the extent of the 2D and 3D design space; these introduce ply angle dependency to the previously derived non-dimensional parameters:

$$\xi_1 = \xi_1^A = \{n_+\cos(2\theta_+) + n_-\cos(2\theta_-) + n_\circ\cos(2\theta_\circ) + n_\bullet\cos(2\theta_\bullet)\}/n$$

$$\xi_2 = \xi_2^A = \{n_+\cos(4\theta_+) + n_-\cos(4\theta_-) + n_\circ\cos(4\theta_\circ) + n_\bullet\cos(4\theta_\bullet)\}/n$$

$$\xi_5 = \xi_1^B = \{\chi_+\cos(2\theta_+) + \chi_-\cos(2\theta_-) + \chi_\circ\cos(2\theta_\circ) + \chi_\bullet\cos(2\theta_\bullet)\}/n^2$$

$$\xi_6 = \xi_2^B = \{\chi_+\cos(4\theta_+) + \chi_-\cos(4\theta_-) + \chi_\circ\cos(4\theta_\circ) + \chi_\bullet\cos(4\theta_\bullet)\}/n^2$$

$$\xi_7 = \xi_3^B = \{\chi_+\sin(2\theta_+) + \chi_-\sin(2\theta_-) + \chi_\circ\sin(2\theta_\circ) + \chi_\bullet\sin(2\theta_\bullet)\}/n^2$$

Note that $\xi_8 = \xi_4^B = 0$ if the angle ply orientations are restricted to ±45, 0 and 90°.

$$\xi_9 = \xi_1^D = \{\zeta_+\cos(2\theta_+) + \zeta_-\cos(2\theta_-) + \zeta_\circ\cos(2\theta_\circ) + \zeta_\bullet\cos(2\theta_\bullet)\}/n^3$$

$$\xi_{10} = \xi_2^D = \{\zeta_+\cos(4\theta_+) + \zeta_-\cos(4\theta_-) + \zeta_\circ\cos(4\theta_\circ) + \zeta_\bullet\cos(4\theta_\bullet)\}/n^3$$

LAMINATION PARAMETERS

$A_{11}=\{U_1+\xi_1 U_2+\xi_2 U_3\}H$

$A_{12}=A_{21}=\{-\xi_2 U_3+U_4\}H$

$A_{16}=A_{61}=\{\xi_3 U_2/2+\xi_4 U_3\}H$

$A_{22}=\{U_1-\xi_1 U_2+\xi_2 U_3\}H$

$A_{26}=A_{62}=\{\xi_3 U_2/2-\xi_4 U_3\}H$

$A_{66}=\{-\xi_2 U_3+U_5\}H$

$B_{11}=\{\xi_5 U_2+\xi_6 U_3\}H^2/4$

$B_{12}=B_{21}=\{-\xi_6 U_3\}H^2/4$

$B_{16}=B_{61}=\{\xi_7 U_2/2+\xi_8 U_3\}H^2/4$

$B_{22}=\{-\xi_5 U_2+\xi_6 U_3\}H^2/4$

$B_{26}=B_{62}=\{\xi_7 U_2/2-\xi_8 U_3\}H^2/4$

$B_{66}=\{-\xi_6 U_3\}H^2/4$

$D_{11}=\{U_1+\xi_9 U_2+\xi_{10} U_3\}H^3/12$

$D_{12}=D_{21}=\{U_4-\xi_{10} U_3\}H^3/12$

$D_{16}=D_{61}=\{\xi_{11} U_2/2+\xi_{12} U_3\}H^3/12$

$D_{22}=\{U_1-\xi_9 U_2+\xi_{10} U_3\}H^3/12$

$D_{26}=D_{62}=\{\xi_{11} U_2/2-\xi_{12} U_3\}H^3/12$

$D_{66}=\{-\xi_{10} U_3+U_5\}H^3/12$

Laminate thickness (H) = number of plies (n) × ply thickness (t);

U_i are laminate invariant (material) properties.

Lamination parameters design spaces:

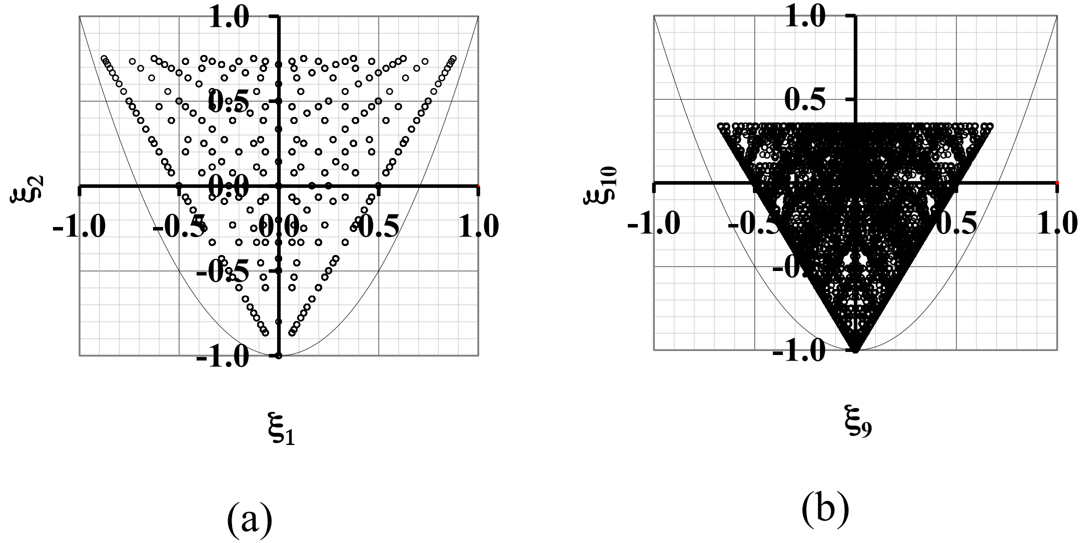


Figure 2 – Lamination parameter design space representing the 42,601 <u>*E-T*</u>-<u>*S-B*</u> laminates with up to (*n* =) 16 plies for standard combinations of ply orientations, i.e., ±45, 0 and 90°, representing (a) extensional stiffness and (b) bending stiffness.

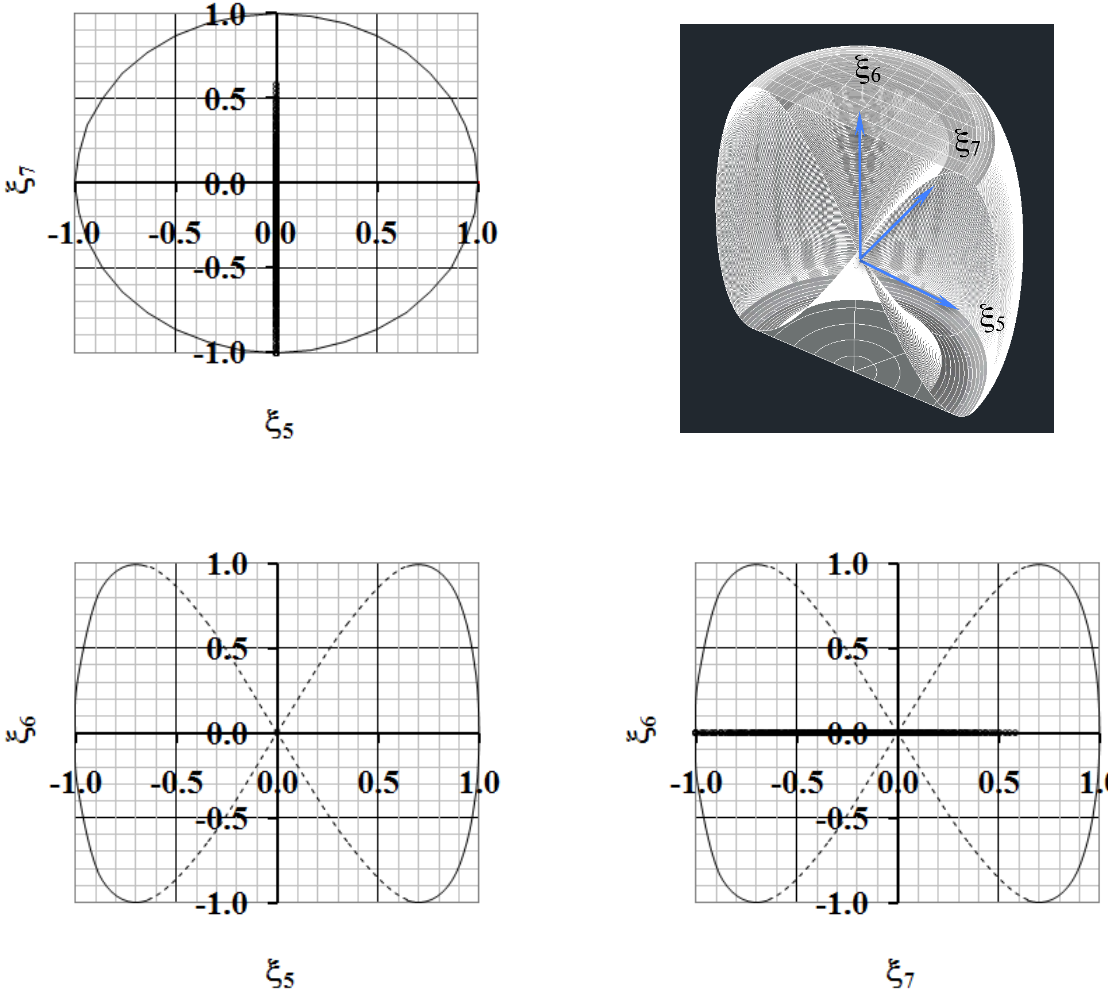


Figure 3 – Three-view illustration of the coupling lamination parameter design space for *E-T-S-B* laminates representing all stacking sequences with up to 21 plies, with standard combinations of ply orientations, i.e., ±45, 0 and 90°.

HYGRO-THERMALLY CURVATURE-STABLE (HTCS) LAMINATES

Here, attention is focused on *E-B-S-T;B-T* coupled extensionally isotropic laminates, which give rise to the *E-T-S-B* coupling behaviour of interest; following off-axis alignment.

Note: *Extensional isotropy implies thermal isotropy* (*but not vice versa!*)*;*

Bending stiffness properties have no influence on HTCS behaviour;

Non-dimensional parameters are unrevealing in the search for HTCS laminates, hence;

Ply-angle-dependent lamination parameters or equivalent A_{ij} *and* B_{ij} *must be interrogated.*

CONDITIONS FOR HTCS BEHAVIOUR

These are dependent on off-axis alignment, β, of the *E-B-S-T* parent laminate (described in first column), which for coupled laminates with extensional isotropy are:

Lamination parameters and stiffness relationships with respect to material axis alignment, β.

$\beta = m\pi/2$	$\beta = \pi/8 + m\pi/2$ (m = 0, 1, 2, 3)	$\beta \neq m\pi/2, \pi/8 + m\pi/2$
$\begin{bmatrix} B_{11} & -B_{11} & 0 \\ -B_{11} & B_{11} & 0 \\ 0 & 0 & -B_{11} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & -B_{16} \\ B_{16} & -B_{16} & 0 \end{bmatrix}$	$\begin{bmatrix} B_{11} & -B_{11} & B_{16} \\ -B_{11} & B_{11} & -B_{16} \\ B_{16} & -B_{16} & -B_{11} \end{bmatrix}$
$\xi_5 = \xi_7 = \xi_8 = 0$	$\xi_5 = \xi_6 = \xi_7 = 0$	$\xi_5 = \xi_7 = 0$
E-B-S-T	*E-T-S-B*	*E-B-E-T-S-B-S-T*

The design space for HTCS laminates, achieved by off-axis alignment, β = π/8, of the *E-B-S-T* parent laminate contains the following number of solutions:

n	8	12	16	20
E-T-S-B	6	20	252	3,076
(*E-T-S-B*;*B-T*)	(6)	(280)	(23,652)	(2,379,722)

NATURAL FREQUENCY ASSESSMENT

Bounds on the natural frequencies are compared for:

angle-ply laminates (with symmetric and non-symmetric configurations);
new laminates containing standard ply orientations and;
HTCS laminates.

All are *E-T*-*S-B* coupled laminates; hence the following closed form solution[2] is applicable:

$$\omega^2 = (\pi^4/\rho)\{T_{33} + (2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2)/(T_{11}T_{22} - T_{12}^2)\}$$

where

$T_{11} = A_{11}(m\pi/a)^2 + A_{66}(n\pi/b)^2$

$T_{12} = (A_{12} + A_{66})(m\pi/a)(n\pi/b)$

$T_{13} = -(3B_{16}(m\pi/a)^2 + B_{26}(n\pi/b)^2)(n\pi/b)$

$T_{22} = A_{22}(n\pi/b)^2 + A_{66}(m\pi/a)^2$

$T_{23} = -(B_{16}(m\pi/a)^2 + 3B_{26}(n\pi/b)^2)(m\pi/a)$

$T_{33} = D_{11}(m\pi/a)^4 + 2(D_{12} + 2D_{66})(m\pi/a)^2(n\pi/b)^2 + D_{22}(n\pi/b)^4$

Results are expressed in terms of the non-dimensional frequency: $\Omega = \omega \dfrac{b^2}{\pi^2}\sqrt{\dfrac{\rho}{D_{\mathrm{Iso}}}}$

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[2] Jones R M. (1999). Mechanics of composite materials, Taylor & Francis, Philadelphia.

Other non-dimensional expressions[3] are also used: $\Omega' = \omega \frac{a^2}{H}\sqrt{\frac{\rho}{E_2}}$; instead of $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho}{D_{\text{Iso}}}}$

Fundamental natural frequency for the laminate [45°/-45°/45°/-45°]$_T$,

a/b	Ω'	Ω'	Ω	$\Omega_{\text{Upper bound}}$	$\Omega_{\text{Lower bound}}$
1	23.53	23.46	2.0801	2.0801	1.2941
2	53.74	53.65	1.1878	1.1878	0.7477
3	98.87	98.78	0.9713	0.9713	0.6192

2nd natural frequency for the laminate [45°/-45°/45°/-45°]$_T$,

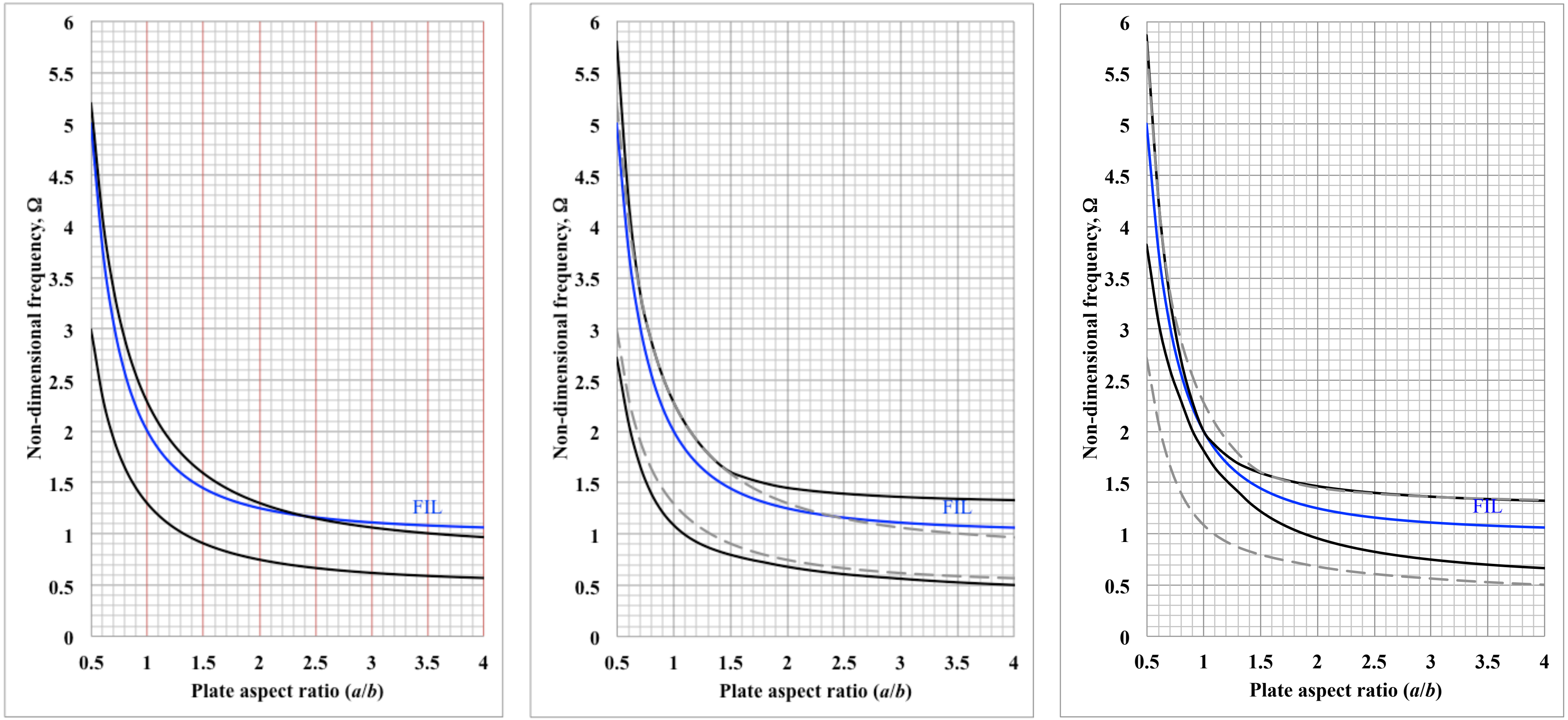
a/b	Ω'	Ω'	Ω	$\Omega_{\text{Upper bound}}$	$\Omega_{\text{Lower bound}}$
1	53.74	53.60	4.7514	4.7514	2.9910
2	94.11	93.89	2.0801	2.0801	1.2941
3	147.65	147.38	1.4504	1.4504	0.9062

3rd natural frequency for the laminate [45°/-45°/45°/-45°]$_T$,

a/b	Ω'	Ω'	Ω	$\Omega_{\text{Upper bound}}$	$\Omega_{\text{Lower bound}}$
1	94.11	53.60	4.7514	4.7514	2.9910
2	147.65	147.31	3.2634	3.2634	2.0390
3	211.75	211.31	2.0801	2.0801	1.2941

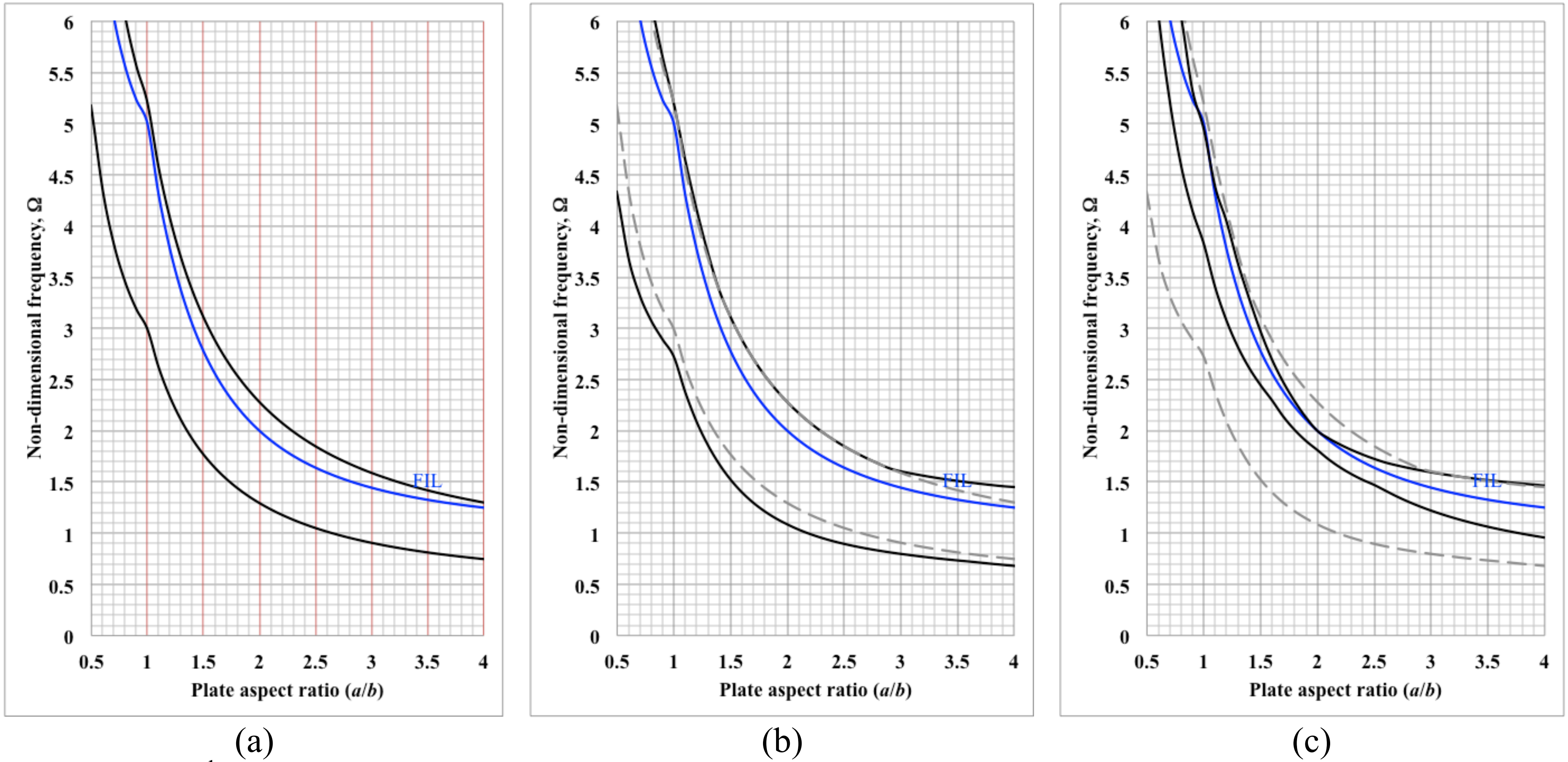
Results have also been validated against NASTRAN FEM analysis.

[3] Khdeir, A. A., (1988). Free vibration of antisymmetric angle-ply laminated plates including various boundary conditions, *Journal of Sound and Vibration*, 122, 377-388.

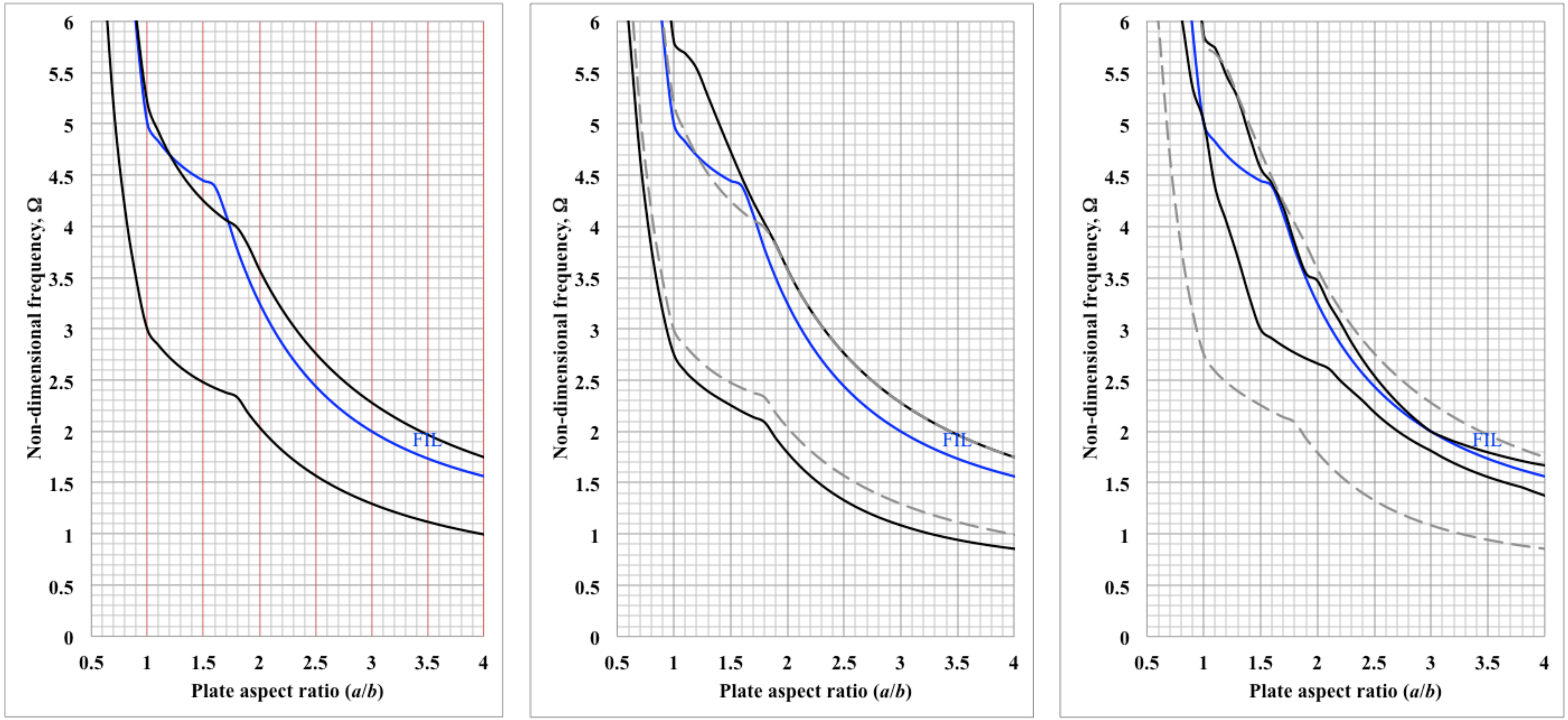


(a) (b) (c)

Bounds on Fundamental Natural Frequency curves for simply supported *E*-*T*-*S*-*B* coupled plates composed of 12-ply laminates with: (a) angle-ply configurations, with ply orientations +45 and -45°, (b) standard configurations with ply orientations +45, -45, 0 and 90° and (c) hygro-thermally curvature-stable configurations with ply orientations +45, -45, 0 and 90° and π/8 off-axis alignment.



Bounds on 2nd Natural Frequency curves for simply supported *E-T-S-B* coupled plates composed of 12-ply laminates with: (a) angle-ply configurations, with ply orientations +45 and -45°, (b) standard configurations with ply orientations +45, -45, 0 and 90° and (c) hygro-thermally curvature-stable configurations with ply orientations +45, -45, 0 and 90° and π/8 off-axis alignment.



(a) (b) (c)

Bounds on 3rd Natural Frequency curves for simply supported *E-T-S-B* coupled plates composed of 12-ply laminates with: (a) angle-ply configurations, with ply orientations +45 and -45°, (b) standard configurations with ply orientations +45, -45, 0 and 90° and (c) hygro-thermally curvature-stable configurations with ply orientations +45, -45, 0 and 90° and π/8 off-axis alignment.

CONCLUSIONS

New Extension-Twisting and Shearing-Bending coupled laminates (*E-T-S-B*), with up to 21 plies, have been derived from combinations of standard ply orientations, i.e., +45, -45, 0 and 90°, and have been shown to exist for all ply number groupings.

The design space for laminates with standard ply orientations has been shown to be vast in comparison to the previously assumed anti-symmetric angle-ply configurations; non-symmetric angle-ply configurations were also identified. The design space is increased yet further with the introduction of HTCS laminates.

Natural frequency results for angle-ply and standard-ply laminates, using a closed form solution, reveal an increase in the bounds for standard ply configurations in comparison to angle-ply laminates; with both anti-symmetric and non-symmetric form.

HTCS laminates , which can be manufactured flat after high temperature curing, are constrained within tighter bounds, about the equivalent isotropic laminate, than for general *E-T-S-B* laminates.

By contrast, HTCS laminates fall on or below the *buckling strength* of the equivalent isotropic laminate. Increasing the mechanical Extension-Twisting coupling behaviour has a negative effect on the compression buckling performance, which is not reflected in the natural frequency response.