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FLATNESS CONDITIONS FOR SYSTEMS WITH TWO INPUTS

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Flat systems with 2 inputs are investigated. Our approach is based on invertible differential operators and deformations of structures on diffieties of control systems. Invertible differential operators of size 2×2 are described. The order of a flat output is estimated by the order of the corresponding invertible differential operator linearizing the control system. The minimal order of invertible differential operator linearizing the system is estimated by the order of its deformation.

Keywords: Nonlinear systems, infinite-order prolongations, flat control systems, dynamic feedback linearization, invertible differential operators.

1. INTRODUCTION

Flat control systems were introduced by Fliess, Levine, Martin, and Rouchon [1]. Later, it was shown that many classes of systems commonly used in nonlinear control theory are flat [2]. Besides, control design for flat systems was developed (see [2]). These facts explain the interest in looking for flatness conditions. In the case of one input, flatness conditions are well known (see [2]). In the general case, checking whether a given system is flat still remains up an open problem.

Using geometric methods, Aranda-Bricaire, Moog, and Pomet introduced an infinitesimal Brunovsky form for nonlinear systems and obtained a necessary and sufficient condition for flatness [3]. Namely, flatness of a control system means existence of an invertible linear differential operator of a certain type satisfying some conditions (see Theorem 2 below). In geometry of differential equations [4] the operators of this type are investigated and are called C-differential. A description of invertible linear differential operators was recently obtained in [5].

In this paper, we consider the case of two inputs. Our approach is based on results of [3] and [5]. Here, the description of invertible operators is generalized to the C-differential case. We also show that invertible C-differential 2×2 operators have a simple structure. The main results of the paper follow from this structure.

2. FLAT SYSTEMS

Consider a system of the form

$$\dot{x} = f(t,x,u), \qquad x\in R^n, \quad u\in R^m, \tag{1}$$

where t is the independent variable, the vector $x=(x_1,\dots,x_n)$ is a state, the vector $u=(u_1,\dots,u_m)$ is a control, its coordinates $u_1,\dots,u_m$ are inputs of the system, $f=(f_1,\dots,f_n)$ is a smooth vector function, and $\dot{x}\equiv dx/dt$. Here and throughout the following, smoothness is understood as infinite differentiability. System (1) is said to be *regular* if $\operatorname{rank}\left(\partial f/\partial u\right)=m$ for all considered values of the variables t,x, and u.

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Let k be some nonnegative integer. We treat

$$t, x_1, \ldots, x_n, u_1, \ldots, u_m, \dot{u}_1, \ldots, \dot{u}_m, \ddot{u}_1, \ldots, u_m^{(k)}$$

as independent variables and consider the space with these coordinates. By $O^{(k)}$ denote some domain in this space.

A regular system (1) is said to be *flat* [1] in the domain $O^{(k)}$ if there exist functions

$$y_i = h_i(t, x, u, \dot{u}, \ldots, u^{(k)}), \quad i = 1, \ldots, m, \tag{2}$$

defined in $O^{(k)}$ such that

(i) the variables x and u can be expressed via t, the functions (2), and their time-derivatives up to some finite order;

(ii) any finite set of the functions (2), their time-derivatives, and the function t is functionally independent.

In this case, the set of functions (2) is called a *flat* (or *linearizing*) *output* of system (1).

We shall need the following notions and facts of geometric theory of differential equations [4]. To each system (1) assign the infinite-dimensional space R^{∞} with coordinates

$$(t, x, u^{(0)}, u^{(1)}, \ldots, u^{(l)}, \ldots), \tag{3}$$

where $u^{(l)}$ denotes the vector variable corresponding to the l-th order derivative of u with respect to t. By E^{∞} we denote a subset of the space R^{∞} with coordinates (3) in which system (1) is defined and admissible states and controls lie. The coordinates (3) are called canonical coordinates on E^{∞}. The above-introduced open sets $O^{(l)}$ can be understood as the subsets of E^{∞}. Namely, the first coordinates of any point of this subset are bounded by the conditions defining $O^{(l)}$ and the remaining coordinates are arbitrary. Such subsets of E^{∞} are also denoted by $O^{(l)}$.

On the set E^{∞}, one introduces [4] the structure of a topological space and the usual differential-geometric notions: smooth functions, vector fields, differential forms, etc. In particular, sets of the form $O^{(l)}$ and their arbitrary unions are called *open sets* in E^{∞}. A smooth function on E^{∞} is defined as an infinitely differentiable function depending on a finite (but arbitrary) number of coordinates (3). The algebra of smooth functions on E^{∞} is denoted by $F(E)$. A vector field on E^{∞} is a differentiation of the algebra $F(E)$.

The vector field

$$D = \frac{\partial}{\partial t} + \sum_{j=1}^{n} f_j(t, x, u) \frac{\partial}{\partial x_j} + \sum_{i=1}^{m} \sum_{l=0}^{\infty} u_i^{(l+1)} \frac{\partial}{\partial u_i^{(l)}}$$

on E^{∞} is called the *total derivative with respect to* t on E^{∞}. The Lie derivative of a function $g \in F(E)$ along D coincides with the time-derivative of the function g according to system (1). Phase curves of the vector field D coincide with graphs of solutions of system (1) in E^{∞}. Therefore, the pair (E^{∞}, D) is chosen as a geometric model of system (1). It is called a *diffiety* (or an *infinite prolongation*) of system (1).

Let $\varphi \in F(E)$. By $d_C\varphi$ we denote the 1-form $d\varphi - D(\varphi)dt$ on E^∞, and by $C^1\Lambda(E)$ we denote the $F(E)$-module spanned by the 1-forms $d_C\varphi,\ \varphi \in F(E)$. By H_1 denote the $F(E)$-submodule of the module $C^1\Lambda(E)$ spanned by the 1-forms $d_Cx_1,\ldots,d_Cx_n$. By definition, put $H_{k+1} = \{\omega \in H_k : D\omega \in H_k\}$ for $k > 0$.

The dimension of the space of covectors $\omega_\theta,\ \omega \in H_k$, is finite for arbitrary $k > 0$ and $\theta \in E^\infty$. The dimension of some $F(E)$-submodule H of the module of 1-forms on E^∞ at the point $\theta \in E^\infty$ is defined as the dimension of the space of covectors $\omega_\theta,\ \omega \in H$. A point $\theta \in E^\infty$ is said to be *Brunovský-regular* (or simply *B-regular*) if in a neighborhood of θ system (1) is regular and the dimensions of the modules H_k and $H_k + D(H_k)$ are constant for each $k > 1$.

Theorem 1. [3] In a neighborhood of a B-regular point for system (1) there exist $\rho \geq 0$ functions $\chi_1,\ldots,\chi_\rho$ of $t, x_1,\ldots,x_n$ and m forms $\omega_1,\ldots,\omega_m \in H_1$ such that $\{d_C\chi_1,\ldots,d_C\chi_\rho\} \cup \{D^j(\omega_k) \mid k = 1,\ldots,m,\ j \geq 0\}$ is a basis of the module $C^1\Lambda(E)$.

In the case $\rho = 0$, the set $\{\omega_1,\ldots,\omega_m\}$ is called a *B-basis* or a *linearizing system of differential forms* [3] at the B-regular point. The proof of Theorem 1 gives a procedure for finding a B-basis (see [3]).

A differential operator of the type

$$g_0 + g_1D + g_2D^2 + \ldots + g_kD^k, \quad g_0, g_1,\ldots,g_k \in F(E),$$

is called a C*-differential operator* of order k. A C-differential operator acts on $C^1\Lambda(E)$. A matrix whose entries are C-differential operators determines the operator on the set of columns of differential forms. This operator is called a *matrix* C*-differential operator*. The order of a matrix C-differential operator is the maximal order of its entries.

A matrix C-differential operator Δ is called *invertible* if there exists a matrix C-differential operator ∇ such that the compositions $\nabla \circ \Delta$ and $\Delta \circ \nabla$ are the identity operators. The matrix of an invertible C-differential operator is square. The *size* of an invertible C-differential operator is its matrix size.

Theorem 2. [3] Let θ be a B-regular point for system (1), $\rho = 0$, and $\{\omega_1,\ldots,\omega_m\}$ be a B-basis at θ. System (1) is flat at a neighborhood of θ if and only if in a neighborhood of θ there exist m functions $h_1,\ldots,h_m$ on E^∞ and an invertible matrix C-differential operator Δ such that

$$\begin{pmatrix} d_Ch_1 \\ \vdots \\ d_Ch_m \end{pmatrix} = \Delta \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix}.$$

The functions $h_1,\ldots,h_m$ form a flat output of system (1).

Let us set $d_C\alpha = d\alpha - D(\alpha) \wedge dt$ for $\alpha \in \Lambda^*(E)$. Necessary and sufficient flatness conditions based on properties of the operator $[d_C, \Delta^{-1}] \circ \Delta$ are obtained in [6]. Let us recall the basic notations from [6]. A map A of $\Lambda^*(E)$ to $\Lambda^*(E)$ such that

$$A(\Omega) = \alpha_0 \wedge \Omega + \alpha_1 \wedge D\Omega + \ldots + \alpha_k \wedge D^k\Omega, \quad \alpha_i \in C^1\Lambda(E),$$

is called a $C\Lambda$*-operator* of order k and of grading 1 and is denoted by

$$\alpha_0 \wedge 1 + \alpha_1 \wedge D + \ldots + \alpha_k \wedge D^k.$$

A matrix whose entries are $C\Lambda$-operators of order $\leq k$ is called a *matrix* $C\Lambda$*-operator* of order $\leq k$. It is easily shown [6] that $R = [d_C, \Delta^{-1}] \circ \Delta$ is a matrix $C\Lambda$-operator of grading 1 satisfying the conditions

$$(d_C - R)(\omega) = 0, \qquad (d_C - R) \circ (d_C - R) = 0.$$

We say that this $C\Lambda$-operator R is a *deformation of the differential* d_C determined by the flat output $(h_1, \ldots, h_m)$ if Δ is the invertible C-differential operator from Theorem 2.

3. MAIN RESULTS

Let $h = \{h_1, \ldots, h_m\}$ be a flat output of system (1), $\omega = \{\omega_1, \ldots, \omega_m\}$ a B-basis. Consider the $F(E)$-modules

$$G_p = \mathrm{Span}_{F(E)}\{D^j(\omega_i) \mid i = 1, \ldots, m, j = 0, 1, \ldots, p\},$$
$$F_k = \mathrm{Span}_{F(E)}\{d_C(D^s h_i) \mid i = 1, \ldots, m, s = 0, 1, \ldots, k\},$$

where $p \geq 0, k \geq 0$.

An integer l is called the *order of the flat output* h with respect to the B-basis ω, if $F_0 \subset G_l$ but $F_0 \not\subset G_{l-1}$. Similarly, an integer L is called the *order of the B-basis* ω with respect to the flat output h, if $G_0 \subset F_L$ but $G_0 \not\subset F_{L-1}$.

Since the B-basis ω consists of 1-forms from H_1, we see that the order of a flat output of the form (2) is more than the integer k in (2).

We say that a B-regular point $\theta \in E^\infty$ is *d-regular for the pair* (ω, h) if in a neighborhood of θ the dimension of the module $F_k \cap G_p$ is constant for each $p = 0, 1, \ldots, l$ and $k = 0, 1, \ldots, L$, where l and L are the integers such that $F_0 \subset G_l$ and $G_0 \subset F_L$.

Theorem 3. Suppose ω is a B-basis at a B-regular point θ for a flat system with 2 inputs, h is a flat output such that the point θ is d-regular for the pair (ω, h), and k_R is order of the deformation of the differential d_C determined by the flat output h; then

1) there exists a flat output $\tilde{h}$ of order $\leq k_R$ with respect to the B-basis ω;

2) the order of the B-basis ω with respect to the flat output $\tilde{h}$ is not more than k_R.

4. INVERTIBLE DIFFERENTIAL OPERATORS

To prove the main result, we need a description of invertible C-differential operators.

Denote by G_p the set of all C-differential operators of order $\le p$ whose matrices are of size $1\times m$. Clearly, G_p is the $F(E)$-module under the multiplication defined by the equality

$$(f\nabla)(\Omega) = f\nabla(\Omega), \quad f\in F(E), \nabla\in G_p,$$

where Ω is an m-column of 1-forms.

Let Δ be an invertible C-differential operator. Consider the $F(E)$-modules

$$F_k = \{\nabla\circ\Delta \,|\, \nabla\in G_k\}, \quad k\ge 0.$$

If Δ is the invertible C-differential operator from Theorem 2, then

$$G_p = \{\nabla(\omega) \,|\, \nabla\in G_p\}, \quad F_k = \{\nabla(\omega) \,|\, \nabla\in F_k\},$$

where ω is the m-column of 1-forms $\omega_1,\dots,\omega_m$ forming a B-basis.

Each invertible C-differential operator determines the sequence of nonnegative integers $d_{k,p} = \dim(F_k\cap G_p)$ for $p,k\ge 0$. Further, we will classify invertible C-differential operators by the sequences $\{d_{k,p}\}$.

A point $\theta\in E^\infty$ is said to be *d-regular for an invertible C-differential operator* Δ, if in a neighborhood of θ the dimension of the module $F_k\cap G_p$ is constant for each $p=0,1,\dots,l$ and $k=0,1,\dots,L$, where $l=\operatorname{ord}\Delta$, $L=\operatorname{ord}\Delta^{-1}$.

The sequence $\{d_{k,p}\}$ uniquely determines the sequence of integers

$$\rho_{k,p} = \varkappa_{k,p} - \varkappa_{k-1,p-1}, \quad k,p\ge 0, \tag{4}$$

Where

$$\varkappa_{k,p} = d_{k,p} - d_{k-1,p} - d_{k,p-1} + d_{k-1,p-1}$$

(put $d_{k,p} = \varkappa_{k,p} = 0$, if $k<0$ or $p<0$). Conversely, the sequence $\{\rho_{k,p}\}$ uniquely determines the sequence $\{d_{k,p}\}$.

Note that $\{d_{k,p}\}$ is an increasing sequence for large values of k or p, whereas the sequence $\{\rho_{k,p}\}$ has only a finite number of non-zero values (see below).

To describe the sequences of integers (4), we consider a finite set of squares in the first quarter of the (k,p)-plane. Let corners of the squares have integer coordinates, and the sides of the squares be parallel to the coordinate axes. Let us assign the integer -1 to the top right corner of each square, and the integer 1 to the lower left corner. Note that the squares can intersect and even coincide. If there are points that are the corners of several squares, then their integers are added together. The zero value is assigned to all other integer points. We get the sequence of integers $\tilde\rho_{k,p}$, $k\ge 0$, $p\ge 0$. Suppose there exists a sequence of nonnegative integers $a_{k,p}$ such that the sequence $\{\rho_{k,p} = \tilde\rho_{k,p} + a_{k,p}\}$ satisfies the following conditions:

$$\sum_{i=0}^{\infty} \rho_{i,p} = 0, \qquad \sum_{j=0}^{\infty} \rho_{k,j} = 0, \tag{5}$$

$$\sum_{i=0}^{\infty} \rho_{i,0} = m, \qquad \sum_{j=0}^{\infty} \rho_{0,j} = m, \tag{6}$$

$$\sum_{i=0}^{k-1} \rho_{i,p} \geq z_{k,p}, \qquad \sum_{j=0}^{p-1} \rho_{k,j} \geq z_{k,p}, \tag{7}$$

where k, p are arbitrary positive integers, m is some positive integer, $z_{k,p}$ is the number of the squares whose the top right corners are situated at the point (k, p).

A set of squares satisfying the above mentioned conditions is called a *d-scheme of squares* (or simply a *d-scheme*). The corresponding sequence $\{ \rho_{k,p} \}$ is called the *m-table of the d-scheme of squares*.

Note that a d-scheme of squares can define more than one m-table. Besides, if we add an arbitrary positive integer a to the value $\rho_{0,0}$, then conditions (5) and (7) hold true, and the integer m in (6) increases by a. Thus we can go from an m-table to an $(m+a)$-table.

The following theorem generalizes a result of [5] to C-differential operators.

Theorem 4. a) For any invertible C-differential operator in a neighborhood of a d-regular point the sequence of integers (4) coincides with an m–table of some d-scheme.

b) If a d-scheme of squares has an m–table, then there are an invertible C-differential operator such that its sequence of integers (4) coincides with the given m-table.

The proof of Theorem 4 is similar to the proof of the corresponding theorem of [5] and is based on the algebraic theory of chain complexes and their spectral sequences.

Note that an invertible C-differential operator is not uniquely determined by its d-scheme of squares. Let us show how to construct an invertible C-differential operator for a given d-scheme and what structures still should be given for this construction. Suppose we have a d-scheme of squares and a sequence of nonnegative integers $a_{k,p}$ such that the sequence $\{ \rho_{k,p} = \tilde{\rho}_{k,p} + a_{k,p} \}$ satisfies conditions (5)–(7). By Z denote the set of all upper right corners of squares of the d-scheme, and by $Z_{k,p}$ the set of all elements of Z with coordinates (k, p). Besides, consider the set B of elements of two types. The elements of the first type are the lower left corners of the squares. For any $k, p \geq 0$ there are $a_{k,p}$ elements of the second type with coordinates (k, p). There are no other elements in B. It follows from equations (5)–(6) that the sum of the integers $a_{k,p}$ is equal to m. So the set B has m elements of the second type. Denote by $B_{k,p}$ the set of all elements of B with coordinates (k, p).

To each element $b \in B_{k,p}$ assign a C-differential operator $\nabla_b \in G_p$ satisfying the following conditions. The operators $\nabla_b, b \in B_{k,0}, k \geq 0,$ are generators of the module G_0. Each element $z \in Z_{k,p}$ determines a relation of the form

$$\sum_{i=1}^{k} \sum_{j=1}^{p} \sum_{\beta \in B_{i,j}} \square_{z,\beta} \circ \nabla_{\beta} = 0, \tag{8}$$

where $\square_{z,\beta}$ is a scalar C-differential operator of order $\leq \min\{k-i, p-j\}$ satisfying the following properties:

(1) for any $p > 0$ using (8) for $z \in Z_{k,p}, k > 0$, the operators $\nabla_b, b \in B_{k,p}, k > 0$, can be expressed as

$$\nabla_b = \sum_{l \geq 0} \sum_{j=0}^{p-1} \sum_{\beta \in B_{l,j}} A_{b,\beta} \circ \nabla_\beta, \tag{9}$$

where $A_{b,\beta}$ are scalar C-differential operators;

(2) for any $k > 0$ using (8) for $z \in Z_{k,p}, p > 0$, the operators $\nabla_b, b \in B_{k,p}, p > 0$, can be expressed as

$$\nabla_b = \sum_{l \geq 0} \sum_{j=0}^{k-1} \sum_{\beta \in B_{j,l}} B_{b,\beta} \circ \nabla_\beta, \tag{10}$$

where $B_{b,\beta}$ are scalar C-differential operators.

Further, both coordinates of any element of Z are positive. Therefore, the integer $\rho_{k,0}$ coincides with the number of elements in $B_{k,0}$. From the first equality in (6) it follows that the set B contains m elements whose the second coordinate is zero. Let $b_1, \ldots, b_m$ denote these elements. Similarly, the set B contains m elements whose the first coordinate is zero. These elements are denoted by $b_1^1, \ldots, b_m^1$. From (9) we get

$$\nabla_{b_i^1} = \sum_{j=1}^{m} \Delta_{ij} \circ \nabla_{b_j}, \qquad i = 1, \ldots, m.$$

The matrix operator $\Delta = (\Delta_{ij})$ is an invertible C-differential operator such that its sequence of integers (4) coincides with the given m-table. Similarly, from (10) we have

$$\nabla_{b_j} = \sum_{i=1}^{m} \Delta_{ji}^{-1} \circ \nabla_{b_i^1}, \qquad j = 1, \ldots, m.$$

The matrix operator $\Delta^{-1} = (\Delta_{ji}^{-1})$ is the inversion of Δ.

Finally, if P, Q is invertible C-differential operators of order 0, i.e., invertible matrices of functions, then the sequences $\{ \rho_{k,p} \}$ for the invertible C-differential operators Δ and $P \circ \Delta \circ Q$ coincide. It can be proved that any invertible C-differential operator can be constructed in such a way. Thus an invertible C-differential operator is uniquely determined by its d-scheme, scalar C-differential operators $\square_{z,\beta}$ from (8), and matrices P, Q. However this operator does not uniquely determine operators $\square_{z,\beta}$ and matrices P, Q.

Theorem 5. If for a d-scheme of squares there exists a 2-table, then the d-scheme has the form specified on Fig. 1, where s is the number of squares ($s \geq 0$), d_l is the side length of l-th square ($d_l > 0$), $(\sum_{i=l}^{s} d_i, \sum_{i=1}^{l} d_i)$ are the coordinates of the element $z_l \in Z$, $(\sum_{i=l+1}^{s} d_i, \sum_{i=1}^{l-1} d_i)$ are the coordinates of the element $b_l \in B, l = 1, \ldots, s$, $(\sum_{i=1}^{s} d_i, 0)$

are the coordinates of the element $b_0 \in B$, $(0, \sum_{i=1}^{s} d_i)$ are the coordinates of the element $b_{s+1} \in B$.

Theorem 5 follows from the definition of 2-tables of the d-schemes.

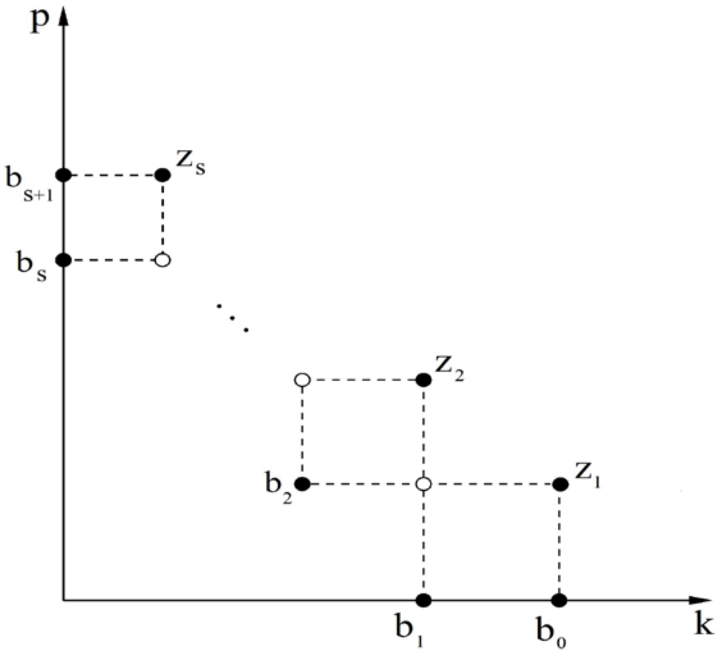

Fig. 1. d-Scheme with 2-table

5. PROOF OF THEOREM 3

Consider a flat system with 2 inputs. Let $\omega = \{ \omega_1, \omega_2 \}$ be a B-basis of the system, $h = \{h_1, h_2 \}$ a flat output. Denote by Δ the invertible C-differential operator from Theorem 2. From Theorem 4 it follows that the sequence $\{ \rho_{k,p} \}$ for the operator Δ^{-1} coincides with a 2–table of some d-scheme. From Theorem 5 it follows that this d-scheme has the form specified on Fig. 1. Relation (8) for the element $z_i \in Z$ of this d-scheme has the form

$$\square_{z_i, b_{i+1}} \circ \nabla_{b_{i+1}} + \square_{z_i, b_i} \circ \nabla_{b_i} + \square_{z_i, b_{i-1}} \circ \nabla_{b_{i-1}} = 0, \tag{11}$$

where $\square_{z_i, b_{i+1}}$ and $\square_{z_i, b_{i-1}}$ are scalar C-differential operator of order 0, i.e., functions, $\square_{z_i, b_i}$ is a scalar C-differential operator of order $\leq d_i$. Denote

$$\nabla_i = \nabla_{b_i}, \quad a_i = \square_{z_i, b_{i+1}}, \quad \square_i = -a_i \square_{z_i, b_i}, \quad c_i = \square_{z_i, b_{i-1}}, \quad i = 1, \ldots, s.$$

From (9) it follows that $a_i \neq 0$. From (10) it follows that $c_i \neq 0$. Therefore, relation (8) can be rewritten as

$$\nabla_{i+1} = \square_i \circ \nabla_i - \alpha_i \nabla_{i-1}, \quad i = 1, \ldots, s, \tag{12}$$

where $\alpha_i = c_i / a_i \neq 0$.

One has

$$\begin{pmatrix} \nabla_{i+1} \\ \nabla_i \end{pmatrix} = \begin{pmatrix} \square_i & -\alpha_i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nabla_i \\ \nabla_{i-1} \end{pmatrix}, \qquad i = 1,\ldots,s.$$

Denote by Δ_i the matrix C-differential operator

$$\begin{pmatrix} \square_i & -\alpha_i \\ 1 & 0 \end{pmatrix}. \tag{13}$$

This operator is invertible and its inversion is

$$\Delta_i^{-1} = \begin{pmatrix} 0 & 1 \\ -\dfrac{1}{\alpha_i} & \dfrac{1}{\alpha_i}\square_i \end{pmatrix}.$$

By construction, $\nabla_0 = (01), \nabla_1 = (10),$ and

$$\begin{pmatrix} \nabla_{s+1} \\ \nabla_s \end{pmatrix} = \Delta_s \circ \ldots \circ \Delta_1$$

is the invertible C-differential operator such that its sequence $\{\rho_{k,p}\}$ coincides with the 2-table. Whence, there exist invertible matrices P, Q of functions such that

$$\Delta^{-1} = P \circ \Delta_s \circ \ldots \circ \Delta_1 \circ Q.$$

Recall that operators $P, Q,$ and $\square_i$ are not uniquely determined. Let us show that we can choose $P, Q,$ and $\square_i$ in such a way that $\alpha_i = 1$ in (13) for $i = 1,\ldots,s.$ Namely, change P to $P \circ O_s^{-1}$, Q to $O_1 \circ Q$, and Δ_i to $O_i \circ \Delta_i \circ O_{i-1}^{-1}$, where

$$O_i = \begin{pmatrix} \gamma_{i+1} & 0 \\ 0 & \gamma_i \end{pmatrix}, \qquad i = 1,\ldots,s,$$

$\gamma_1,\ldots,\gamma_{s+1}$ are some nonzero numbers. Then the operator Δ^{-1} does not change and we have

$$O_i \circ \Delta_i \circ O_{i-1}^{-1} = \begin{pmatrix} \gamma_{i+1}\square_i \circ \dfrac{1}{\gamma_i} & -\alpha_i \dfrac{\gamma_{i+1}}{\gamma_{i-1}} \\ 1 & 0 \end{pmatrix}.$$

Putting $\gamma_0 = 1, \gamma_1 = 1,$ and $\gamma_{i+1} = \frac{\gamma_{i-1}}{\alpha_i}$ for $i = 1,\ldots,s,$ we get the required operators. Below we consider the case $\alpha_i = 1, i = 1,\ldots,s.$

We see that the order of operators Δ and Δ^{-1} is $d_\Sigma = \sum_{i=1}^{s} d_i$. Therefore the order of the flat output h with respect to the B-basis ω and the order of the B-basis ω with respect to the flat output h are d_Σ.

Denote

$$\tilde{\Delta} = P \circ \Delta_s \circ \ldots \circ \Delta_2, \quad \hat{\Delta} = \Delta_1 \circ Q.$$

Then

$$\Delta^{-1} = \tilde{\Delta} \circ \hat{\Delta}, \quad \omega = (\tilde{\Delta} \circ \hat{\Delta})(d_C h)$$

and hence

$$R = [d_C, \Delta^{-1}] \circ \Delta = d_C \tilde{\Delta} \circ \tilde{\Delta}^{-1} + \tilde{\Delta} \circ d_C \hat{\Delta} \circ \hat{\Delta}^{-1} \circ \tilde{\Delta}^{-1}. \tag{14}$$

The order of operators $\tilde{\Delta}$ and $\tilde{\Delta}^{-1}$ is $d_\Sigma - d_1$. Therefore, the order of $C\Lambda$-operator $d_C \tilde{\Delta} \circ \tilde{\Delta}^{-1}$ is not more than $2(d_\Sigma - d_1)$.

Let us show that the order of R is determined by the second summand in (14). Let the matrix $C\Lambda$-operator $d_C Q \circ Q^{-1}$ be

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \quad q_{11}, q_{12}, q_{21}, q_{22} \in C^1\Lambda(E).$$

Then an easy calculation shows that

$$d_C \hat{\Delta} \circ \hat{\Delta}^{-1} = \begin{pmatrix} -\square_1 q_{12} + q_{22} & \alpha \\ -q_{12} & q_{11} + q_{12}\square_1 \end{pmatrix},$$

where $\alpha = \square_1 q_{11} - q_{21} + \square_1 q_{12} \square_1 - q_{22} \square_1 + d_C \square_1$.

Let $\square_1 = g_0 + g_1 D + \ldots + g D^{d_1}$ and let us consider 3 cases.

Case 1: $q_{12} \neq 0$. The symbol of the B-basis ω is

$$\tilde{\Delta} \circ \begin{pmatrix} 0 & q_{12} g^2 D^{2d_1} \\ 0 & 0 \end{pmatrix} \circ \tilde{\Delta}^{-1},$$

and $k_\omega = 2d_\Sigma$.

Case 2: $q_{12} = 0, q_0 = g q_{11} - q_{22} g + d_C g \neq 0$. The symbol of ω is

$$\tilde{\Delta} \circ \begin{pmatrix} 0 & q_0 D^{d_1} \\ 0 & 0 \end{pmatrix} \circ \tilde{\Delta}^{-1},$$

and $k_\omega = 2d_\Sigma - d_1$.

Case 3: $g q_{11} - q_{22} g + d_C g = 0$. It can be proved that in this case there exists a flat output $\tilde{h}$ of order $< d_\Sigma$.

Continuing this line of reasoning, we obtain the cases 1, 2 or a flat output of less order. In the cases 1 and 2, $k_R \geq d_\Sigma$. This proves Theorem 3.

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УСЛОВИЕ ПЛОСКОСТИ ДЛЯ СИСТЕМ С ДВУМЯ ВХОДАМИ

В.Н. Четвериков

Изучаются плоские системы с двумя входами. Наш подход основан на обратимых дифференциальных операторах и деформации структур на диффеотопе систем с управлением. Описаны обратимые дифференциальные операторы размерности 2×2 . Введено понятие веса для B-базиса. Если вес B-базиса нулевой, то проверка плоскости тривиальна. Минимальное количество плоских выходов оценивается на основании порядка соответствующего обратимого дифференциального оператора, линеаризующего систему с управлением. Минимальный порядок обратимого дифференциального оператора, линеаризующего систему, оценивается через порядок его деформации.

Ключевые слова: Нелинейные системы, бесконечное продолжение, плоские системы, линеаризация динамической обратной связи, обратимые дифференциальные операторы.