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Topological aspects of Hurewicz tests for the difference hierarchy

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Abstract. We generalize the Baire Category Theorem to the Borel and difference hierarchies, i.e. if Γ is any of the classes Σ^0_ξ, Π^0_ξ, $D_\eta(\Sigma^0_\xi)$ or $\check{D}_\eta(\Sigma^0_\xi)$ we find a representative set $P_\Gamma \in \Gamma$ and a Polish topology τ_Γ such that for every $A \in \check{\Gamma}$ from some assumption on the size of $A \cap P_\Gamma$ we can deduce that $A \setminus P_\Gamma$ is of second category in the topology τ_Γ. This allows us to distinguish the levels of the Borel and difference hierarchies via Baire category. We also present some typical Baire Category Theorem-like applications of the results.

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1. Introduction

If an open set of the Polish space (X, τ) is nonempty it is of second category: this is probably the simplest formulation of the Baire Category Theorem. The purpose of this note is to show that this theorem can be extended to the entire Borel and difference hierarchies: if Γ is any of the classes Σ^0_ξ, Π^0_ξ, $D_\eta(\Sigma^0_\xi)$ or $\check{D}_\eta(\Sigma^0_\xi)$ in a Polish space (X, τ) we define a representative set $P_\Gamma \subseteq X$, $P_\Gamma \in \Gamma$ and a fine Polish topology τ_Γ on X such that if a set $A \subseteq X$, $A \in \check{\Gamma}$ satisfies that $A \cap P_\Gamma$ is of τ_Γ-second category then $A \setminus P_\Gamma$ is also of τ_Γ-second category. This makes possible in particular to distinguish the classes of the Borel and difference hierarchies via Baire category in a suitable topology.

We remark that there is a quite classical extension of the dichotomy expressed by the Baire Category Theorem for open sets. More than a half

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century ago Witold Hurewicz proved the following theorem, also called Hurewicz-dichotomy, about sets failing to be Π^0_2.

Theorem 1.1 (W. Hurewicz). *Let X be a Polish space and $A \subseteq X$ be a coanalytic set. If A is not Π^0_2 there is a continuous injection of the Cantor set into X, $\varphi\colon 2^\omega \to X$ such that $\varphi^{-1}(A)$ is countable dense in 2^ω.*

Again by the Baire Category Theorem in Polish spaces from the weak assumption that a Π^0_2 set is dense we can conclude that it is residual; in particular a countable dense set $H \subseteq 2^\omega$ is never Π^0_2. Thus Theorem 1.1 shows via Baire Category that a not Π^0_2 set is as far from being Π^0_2 as possible on a copy of the Cantor set. This is the reason why the pair $(2^\omega, H)$ is called the *Hurewicz test* for Σ^0_2 sets.

Theorem 1.1 has been strengthened in many successive steps. In some sense the most general existence theorem for Hurewicz tests is the following result (see [3, Theorem 2, p. 27], [2, Corollary 6 and Theorem 7 p. 457] and also [5, Theorem 2, p. 1025]). For the definition of the appearing classes see Section 2; for a class $\Gamma(X) \subseteq 2^X$, $\check{\Gamma}(X) = \{A \subseteq X : X \setminus A \in \Gamma(X)\}$ denotes the *dual class of* Γ.

Theorem 1.2 (A. Louveau, J. Saint Raymond). *Let Γ be a non-selfdual Borel Wadge class. Then there is a zero topological dimensional compact metric space K and a $\Gamma(K)$ subset H of K with the following property: for every Borel set B in a nonempty Suslin space E either $B \in \check{\Gamma}(E)$ or there is a continuous injection $\varphi\colon K \to E$ with $\varphi(H) = B$.*

If $3 \le \xi < \omega_1$ and $1 \le \eta < \omega_1$ or if $\xi = 2$ and $\omega \le \eta < \omega_1$, for the classes $\Gamma = \check{D}_\eta(\Sigma^0_\xi)$ and $\Gamma = D_\eta(\Sigma^0_\xi)$ we can take $K = 2^\omega$ and H can be an arbitrary set in $\Gamma(2^\omega) \setminus \check{\Gamma}(2^\omega)$.

The pair (K, H) is called the *Hurewicz test for* Γ. Observe that Theorem 1.2 gives in particular that a membership of a Borel set in some Γ depends only on zero dimensional sets and can be witnessed via continuous injections $\varphi\colon K \to E$. On the other hand the aspect of dichotomy which was present in the Baire Category Theorem for open sets and in the Hurewicz-dichotomy Theorem for Π^0_2 sets is lacking, e.g. Theorem 1.2 says nothing about how well a set $A \in \Gamma$ can be approximated by sets in $\check{\Gamma}$. Our main result provides a Hurewicz-dichotomy for the classes of the Borel and difference hierarchies.

Theorem 1.3. *Let $0 < \xi, \eta < \omega_1$ be fixed. Then there is a Polish space $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$ homeomorphic to $(2^\omega, \tau_{2^\omega})$, a $\check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ set $P_{\xi,\eta} \subseteq C_{\xi,\eta}$, a nonempty $\Pi^0_\xi(\tau_{C_{\xi,\eta}})$ set $W_{\xi,\eta}(\eta) \subseteq C_{\xi,\eta}$ and Polish topologies $\tau^<_{\xi,\eta}$, $\tau_{\xi,\eta}$ on $C_{\xi,\eta}$ with the following properties:*

1. $\tau^<_{\xi,\eta}$ *refines* $\tau_{C_{\xi,\eta}}$ *and* $\tau_{\xi,\eta}$ *refines* $\tau^<_{\xi,\eta}$*;*
2. $P_{\xi,\eta}$ *and* $C_{\xi,\eta} \setminus P_{\xi,\eta}$ *are both* $\tau_{\xi,\eta}$*-open;*
3. *for every* $D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ *set* $A \subseteq C_{\xi,\eta}$ *and* $\tau^<_{\xi,\eta}$*-open set* $G \subseteq C_{\xi,\eta}$*, if* $G \cap W_{\xi,\eta}(\eta) \neq \emptyset$ *and* $A \cap P_{\xi,\eta}$ *is* $\tau_{\xi,\eta}$*-residual in* $G \cap P_{\xi,\eta}$ *then* $A \setminus P_{\xi,\eta}$ *is of* $\tau_{\xi,\eta}$*-second category in* $G \setminus P_{\xi,\eta}$*.*

Before any further discussion we formulate a corollary of this theorem which is less abundant in notations.

Corollary 1.1. *Let* $0 < \xi, \eta < \omega_1$ *be fixed and let* $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$*,* $P_{\xi,\eta}$ *and* $\tau_{\xi,\eta}$ *be as in Theorem* 1.3*.*

1. *If* $A \subseteq C_{\xi,\eta}$ *is a* $D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ *set and* $P_{\xi,\eta} \subseteq A$ *then* $A \setminus P_{\xi,\eta}$ *is of* $\tau_{\xi,\eta}$*-second category.*
2. *If* $A' \subseteq C_{\xi,\eta}$ *is a* $\check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ *set and* $C_{\xi,\eta} \setminus P_{\xi,\eta} \subseteq A'$ *then* $A' \cap P_{\xi,\eta}$ *is of* $\tau_{\xi,\eta}$*-second category.*

The first statement of Corollary 1.1 is the special $G = C_{\xi,\eta}$ case of Theorem 1.3, while the second follows from Theorem 1.3 applied to $A = C_{\xi,\eta} \setminus A'$ and $G = C_{\xi,\eta}$. This corollary has already the feature of dichotomy we are looking for: it allows to derive from the information that a set A is "badly" structured the conclusion that either it does not contain $P_{\xi,\eta}$ or it is big in category in $C_{\xi,\eta} \setminus P_{\xi,\eta}$, and vice versa. We remark that for the Borel hierarchy a stronger result can be proved: as we will see in Theorem 4.2, for $\eta = 1$ in Theorem 1.3 we can practically conclude that $A \cap P_{\xi,\eta}$ is $\tau_{\xi,\eta}$-residual in $G \cap P_{\xi,\eta}$. The reason why the result we obtain for the difference hierarchy is weaker than what we have for the Borel hierarchy is that $D_\eta(\Sigma^0_\xi)$ $(1 < \eta < \omega_1)$ is not closed under taking countable unions.

Theorem 1.3 gets its real power when combined with Theorem 1.2, as follows.

Corollary 1.2. *Let* (X, τ) *be a nonempty Polish space,* $1 < \xi < \omega_1$*,* $0 < \eta < \omega_1$ *be fixed and let* $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$*,* $P_{\xi,\eta}$ *and* $\tau_{\xi,\eta}$ *be as in Theorem* 1.3*. If a Borel set* $A \subseteq X$ *is*

1. *in* $D_\eta(\Sigma^0_\xi(\tau))$*, then for every continuous injection* $\varphi\colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X, \tau)$*,* $\varphi(P_{\xi,\eta}) \subseteq A$ *implies that* $\varphi^{-1}(A) \setminus P_{\xi,\eta}$ *is of* $\tau_{\xi,\eta}$*-second category;*

1′. *in* $\check{D}_\eta(\Sigma^0_\xi(\tau))$*, then for every continuous injection* $\varphi\colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X, \tau)$*,* $\varphi(C_{\xi,\eta} \setminus P_{\xi,\eta}) \subseteq A$ *implies that* $\varphi^{-1}(A) \cap P_{\xi,\eta}$ *is of* $\tau_{\xi,\eta}$*-second category;*

2. *not in* $D_\eta(\Sigma^0_\xi(\tau))$ *there is a continuous injection* $\varphi\colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X,\tau)$ *such that* $\varphi(P_{\xi,\eta}) = A \cap \varphi(C_{\xi,\eta})$*, hence* $\varphi(P_{\xi,\eta}) \subseteq A$ *but* $\varphi^{-1}(A) \setminus P_{\xi,\eta} = \emptyset$*;*

2′. *not in* $\check{D}_\eta(\Sigma^0_\xi(\tau))$ *there is a continuous injection* $\varphi\colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X,\tau)$ *such that* $\varphi(C_{\xi,\eta} \setminus P_{\xi,\eta}) = A \cap \varphi(C_{\xi,\eta})$*, hence* $\varphi(C_{\xi,\eta} \setminus P_{\xi,\eta}) \subseteq A$ *but* $\varphi^{-1}(A) \cap P_{\xi,\eta} = \emptyset$*.*

In this note we would like to prove only one more corollary, which represents a typical way to apply the preceding results.

Corollary 1.3. *Let* $\aleph_0 < \lambda < 2^{\aleph_0}$ *be a regular cardinal and suppose that in our model the union of* λ *meager sets is meager in Polish spaces (this assumption holds e.g. under Martin's Axiom* $MA(\lambda)$*). Let* (X,τ) *be a nonempty Polish space. Let* $1 < \xi < \omega_1$*,* $0 < \eta < \omega_1$ *be fixed and set* $\Gamma = D_\eta(\Sigma^0_\xi(\tau))$ *or* $\Gamma = \check{D}_\eta(\Sigma^0_\xi(\tau))$*. Suppose that* $A, A_\alpha \subseteq X$ $(\alpha < \lambda)$ *such that* A *is Borel,* $A \notin \check{\Gamma}$*,* $A_\alpha \in \check{\Gamma}$ *and* $A \subseteq A_\alpha$ $(\alpha < \lambda)$*. Then there is a stationary set* $\Lambda \subseteq \lambda$ *such that* $(X \setminus A) \cap \bigcap_{\alpha\in\Lambda} A_\alpha \neq \emptyset$*.*

The paper is structured as follows. In Section 2 we recall some definitions related to the Borel and difference hierarchies and fix some conventions for Polish topologies. In Section 3 we define all the Polish spaces and Hurewicz test sets we will use in the sequel. Since for our problem as well, the $\xi = 1$ case does not fit into the general framework we handle it separately in Section 4.1. In Section 4.2 we prove the theorems concerning the Borel hierarchy, and finally in Section 4.3 we extend the results to the difference hierarchy and prove the corollaries. The results presented here were initiated in [4]. A result of similar flavor for the Borel hierarchy was obtained by S. Solecki [6, Theorem 2.2, p. 526] using effective methods.

2. Preliminaries

Our terminology and notation follow [1]. Let (C, τ_C) denote the Polish space 2^ω with its usual product topology. As usual, $\Pi^0_\xi(\tau)$ and $\Sigma^0_\xi(\tau)$ $(0 < \xi < \omega)$ stand for the ξ^{th} multiplicative and additive Borel class in the Polish space (X,τ), starting with $\Pi^0_1(\tau) =$ closed sets, $\Sigma^0_1(\tau) =$ open sets.

We derive the *difference hierarchy* as follows (see also [1, Section 22.E, p. 175]. Every ordinal η can be uniquely written as $\alpha + n$ where α is limit and $n < \omega$. We call η *even* if n is even and *odd* if n is odd.

Definition 2.1. *Let $0 < \xi, \eta < \omega_1$ and let $(A_\alpha)_{\alpha<\eta}$ be a sequence of subsets of a set X, such that $A_\alpha \subseteq A_{\alpha'}$ $(\alpha \leq \alpha' < \eta)$. Then $D_\eta((A_\alpha)_{\alpha<\eta}) \subseteq X$ is defined by*

$$x \in D_\eta((A_\alpha)_{\alpha<\eta}) \iff x \in \bigcup_{\alpha<\eta} A_\alpha$$

and the least $\alpha < \eta$ with $x \in A_\alpha$ has parity opposite to that of η.

With this operation in the Polish space (X, τ) we set

$$D_\eta\left(\Sigma^0_\xi(\tau)\right) = \left\{D_\eta((A_\alpha)_{\alpha<\eta})\colon A_\alpha \in \Sigma^0_\xi(\tau),\ A_\alpha \subseteq A_{\alpha'}\ (\alpha \leq \alpha' < \eta)\right\}.$$

The definition of the Wadge hierarchy can be found e.g. in [1, Section 21.E, p. 156 and Section 22.B, p. 169]. We will use only the Borel and difference hierarchies in the sequel.

Let ξ, ϑ_i $(i < \omega)$ be ordinals. We write $\vartheta_i \to \xi$ if ξ is successor and $\vartheta_i + 1 = \xi$ $(i < \omega)$ or if ξ is limit, ϑ_i is successor $(i < \omega)$, $\vartheta_i \leq \vartheta_j$ $(i \leq j < \omega)$ and $\sup_{i<\omega} \vartheta_i = \xi$.

For every ordinal $\xi < \omega_1$ we fix once and for all a sequence $(\vartheta_i)_{i<\omega}$ such that $\vartheta_i \to \xi$. To avoid complicated notations, we do not indicate the dependence of the sequence on ξ, it will be always clear which pair of ordinal and sequence is considered.

In this note we will notoriously refine Polish topologies by turning countably many closed sets into open sets. We do this as described in [1], that is the open sets of the ancient topology together with their portion on the members of our collection of closed sets serve as a subbase of the new, finer topology. We will use that the topology obtained in this way is also Polish.

Definition 2.2. *Let (X, τ) be a Polish space, $\mathcal{P} = \{P_i\colon i < \omega\}$ be a countable collection of $\Pi^0_1(\tau)$ sets. Then $\tau[\mathcal{P}]$ denotes that Polish topology refining τ where each P_i $(i < \omega)$ is turned successively into an open set.*

It is easy to see that the resulting finer topology $\tau[\mathcal{P}]$ is independent from the enumeration of $\mathcal{P}$. This will be clear shortly when we fix a base of $\tau[\mathcal{P}]$. We also use the notation $\tau[\mathcal{P}]$ when the countable collection of not necessarily $\Pi^0_1(\tau)$ sets $\mathcal{P}$ can be enumerated on such a way that P_n is $\Pi^0_1(\tau[\{P_i\colon i < n\}])$.

We need a precise notion of basic open sets in our spaces.

Definition 2.3. *Let (X_i, τ_i) $(i \in I)$ be Polish spaces; if a basis $\mathcal{G}_i$ is fixed in the spaces (X_i, τ_i) $(i \in I)$, which are meant to be the basic open sets in (X_i, τ_i), then the* basic open sets of $(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i)$ *are the open sets of the form*

$$\prod_{i\in J} G_i \times \prod_{i\in I\setminus J} X_i,$$

where $J \subseteq I$ is finite and $G_i \in \mathcal{G}_i$ for every $i \in J$.

If the basic open sets $\mathcal{G}$ are fixed in the Polish space (X, τ) and $\tau[\mathcal{P}]$ makes sense for a countable collection $\mathcal{P}$ of subsets of X, then the basic open sets of $\tau[\mathcal{P}]$ *are of the sets of the form $G \cap F_0 \cap \dots \cap F_{n-1}$ or G with $G \in \mathcal{G}$, $F_i \in \mathcal{P}$ $(i < n)$; a basic $\tau[\mathcal{P}]$-open set is said to be* proper *if it is not τ-open.*

Observe that the basic open sets defined on this way form a basis of $\prod_{i \in I} \tau_i$ and $\tau[\mathcal{P}]$, respectively. From now on whenever a Polish space (X, τ) appears we assume that a countable basis comprised of basic τ-open sets is fixed; and this is done with respect to the convention of Definition 2.3 if it is applicable. We take X to be basic τ-open.

The closure of a set $A \subseteq (X, \tau)$ is denoted by $\mathrm{cl}_\tau(A)$. We recall that a $\Pi^0_2(\tau)$ subset G of the Polish space (X, τ) is itself a Polish space with the restricted topology $\tau|_G$ (see e.g. [1, (3.11) Theorem]). In particular, the notions related to category in the topology τ make sense relative to G.

We will have to return to the topologies on the coordinates in product spaces. If (X, σ), (Y, τ) are arbitrary topological spaces and $(\mathcal{X}, \mathcal{S}) = (X \times Y, \sigma \times \tau)$, then we define $\mathrm{Pr}_X(\mathcal{S}) = \sigma$. The projection of product sets in product spaces is defined analogously. If $G_X \subseteq X$ and $G_Y \subseteq Y$, we say that the set of product form $G = G_X \times G_Y \subseteq \mathcal{X}$ is *nontrivial on the X coordinate* if $G_X \neq X$.

3. Spaces, sets and topologies

In this section we define the Polish spaces and the special sets for which a Polish topology featuring the dichotomy we like can be constructed. First we handle the Borel hierarchy.

Definition 3.1. *We set $(C_1, \tau_{C_1}) = (C_2, \tau_{C_2}) = (C, \tau_C)$,*

$$P_1 = \{x \in C_1 \colon x(n) = 1 \ (n < \omega)\}$$

and $\tau_{P_1} = \tau^<_{P_1} = \tau_{C_1}$ on C_1. Let $\{U_{2,n} \colon n < \omega\}$ be an enumeration of the set $\{x \in C_2 \colon \exists n < \omega \ (x(i) = 0 \ (n < i < \omega))\}$ and set

$$P_2 = C_2 \setminus \{U_{2,n} \colon n < \omega\},\ \tau^<_{P_2} = \tau_{C_2},\ \tau_{P_2} = \tau^<_{P_2}[\{U_{2,n} \colon n < \omega\}].$$

Let now $2 < \xi < \omega_1$ and suppose that the spaces $(C_\vartheta, \tau_{C_\vartheta})$, the $\Pi^0_\vartheta(\tau_{C_\vartheta})$ sets P_ϑ and the topologies τ_{P_ϑ}, $\tau^<_{P_\vartheta}$ are defined for every $\vartheta < \xi$. Then with $\vartheta_i \to \xi$ let

$$C_\xi = \prod_{i<\omega} C_{\vartheta_i}, \qquad \tau_{C_\xi} = \prod_{i<\omega} \tau_{C_{\vartheta_i}},$$

$$P_\xi = \{x \in C_\xi \colon x(i,.) \in C_{\vartheta_i} \setminus P_{\vartheta_i}\ (i < \omega)\}, \tag{3.1}$$
$$\tau^<_{P_\xi} = \prod_{i<\omega} \tau_{P_{\vartheta_i}},$$

and let $\tau_{P_\xi} = \tau^<_{P_\xi}[\{U_{\xi,n} \colon n < \omega\}]$ *where*

$$U_{\xi,n} = \prod_{i<n} (C_{\vartheta_i} \setminus P_{\vartheta_i}) \times P_{\vartheta_n} \times \prod_{n<i<\omega} C_{\vartheta_i}$$
$$\subseteq \prod_{i<n} C_{\vartheta_i} \times C_{\vartheta_n} \times \prod_{n<i<\omega} C_{\vartheta_i} = C_\xi\ (n < \omega). \tag{3.2}$$

If ξ is a limit ordinal for every $m < \omega$ we also define

$$C^m_\xi = \prod_{m\le i<\omega} C_{\vartheta_i}, \qquad \tau_{C^m_\xi} = \prod_{m\le i<\omega} \tau_{C_{\vartheta_i}},$$

$$P^m_\xi = \{x \in C^m_\xi \colon x(i,.) \in C_{\vartheta_i} \setminus P_{\vartheta_i}\ (m \le i < \omega)\}, \tag{3.3}$$
$$\tau^<_{P^m_\xi} = \prod_{m\le i<\omega} \tau_{P_{\vartheta_i}}$$

and let $\tau_{P^m_\xi} = \tau^<_{P^m_\xi}[\{U^m_{\xi,n} \colon m \le n < \omega\}]$ *where*

$$U^m_{\xi,n} = \prod_{m\le i<n} (C_{\vartheta_i} \setminus P_{\vartheta_i}) \times P_{\vartheta_n} \times \prod_{n<i<\omega} C_{\vartheta_i}$$
$$\subseteq \prod_{m\le i<n} C_{\vartheta_i} \times C_{\vartheta_n} \times \prod_{n<i<\omega} C_{\vartheta_i} = C_\xi\ (m \le n < \omega). \tag{3.4}$$

If ξ is a limit ordinal and $\vartheta < \xi$ let I be minimal such that $\vartheta \le \vartheta_I$. Set $H_\xi(\vartheta) = \prod_{i<I} (C_{\vartheta_i} \setminus P_{\vartheta_i}) \times \prod_{I\le i<\omega} C_{\vartheta_i}$ and for every $m < \omega$ set $H^m_\xi(\vartheta) = \prod_{m\le i<I} (C_{\vartheta_i} \setminus P_{\vartheta_i}) \times \prod_{I\le i<\omega} C_{\vartheta_i}$ if $m < I$ else let $H^m_\xi(\vartheta) = C^m_\xi$. If ξ is a successor we set $H_\xi(\vartheta) = C_\xi$ $(\vartheta < \xi)$.

We prove a lemma on the relation of P_ξ, $\tau^<_{P_\xi}$ and τ_{P_ξ}.

Lemma 3.1. *Let $0 < \xi < \omega_1$ be fixed.*

1. *(C_ξ, τ_{C_ξ}) is homeomorphic to (C, τ_C).*
2. *P_ξ is a $\Pi^0_\xi(\tau_{C_\xi})$ set.*
3. *P_ξ is a τ_{P_ξ}-nowhere dense $\Pi^0_1(\tau_{P_\xi})$ set.*
4. *For $\xi \ge 2$, $P_\xi \subseteq C_\xi$ is a $\tau^<_{P_\xi}$-dense $\Pi^0_2(\tau^<_{P_\xi})$ hence $\tau^<_{P_\xi}$-residual set.*

5. *For* $\xi \geq 2$, $C_\xi \setminus P_\xi$ *is a* $\tau^<_{P_\xi}$*-dense* $\Sigma^0_2(\tau^<_{P_\xi})$ *set.*

6. *If* G *is basic* τ_{P_ξ}*-open and* $G \cap P_\xi \neq \emptyset$ *then* G *is also basic* $\tau^<_{P_\xi}$*-open.*

7. *If* G *is proper basic* τ_{P_ξ}*-open there is a basic* $\tau^<_{P_\xi}$*-open set* G_0 *and a unique* $n < \omega$ *satisfying* $G = G_0 \cap U_{\xi,n}$.

8. *If* G *is proper basic* τ_{P_ξ}*-open the topologies* $\tau_{P_\xi}|_G$ *and* $\tau^<_{P_\xi}|_G$ *coincide.*

9. *The topologies* $\tau_{P_\xi}|_{P_\xi}$ *and* $\tau^<_{P_\xi}|_{P_\xi}$ *coincide.*

If ξ *is a limit ordinal then* 2–9 *hold for* P^m_ξ *and* $U^m_{\xi,n}$ *instead of* P_ξ *and* $U_{\xi,n}$ $(m \leq n < \omega)$.

Proof. Statement 3.1 follows from the fact that a countable power of (C, τ_C) is homeomorphic to (C, τ_C).

We prove 2 by induction on ξ. For $\xi = 1$ we have that P_1 is a single point, which is clearly $\Pi^0_1(\tau_{C_1})$. For $\xi = 2$, P_2 is the complement of a countable set so it is $\Pi^0_2(\tau_{C_2})$. Let now $\xi \geq 3$ and suppose that P_η is $\Pi^0_\eta(\tau_{C_\eta})$ for every $\eta < \xi$. With $\vartheta_m \to \xi$ we have

$$P_\xi = \bigcap_{m<\omega} \{x \in C_\xi \colon x(m,.) \in C_{\vartheta_m} \setminus P_{\vartheta_m}\}. \tag{3.5}$$

Since τ_{C_ξ} is the product of the topologies $\tau_{C_{\vartheta_m}}$ and P_{ϑ_m} is $\Pi^0_{\vartheta_m}(\tau_{C_{\vartheta_m}})$ by the induction hypothesis, P_ξ is the intersection of sets of additive class lower than ξ, so the statement follows.

We prove 3 and 4 together, by induction on ξ. For $\xi = 1$, P_1 is a single point, which is clearly $\Pi^0_1(\tau_{P_1})$ and τ_{P_1}-nowhere dense. For $\xi = 2$, statements 3 and 4 follow from the fact that $\{U_{2,n} \colon n < \omega\}$ is a τ_{C_2}-dense countable set. Let now $\xi \geq 3$ and suppose that 3 holds for every $\eta < \xi$. We prove 4 for ξ and then 3 for ξ. Let $\vartheta_n \to \xi$. By (3.2) and (3.5), we have

$$P_\xi = C_\xi \setminus \left(\bigcup_{n<\omega} U_{\xi,n} \right). \tag{3.6}$$

By the induction hypothesis P_{ϑ_n} is $\tau_{P_{\vartheta_n}}$-nowhere dense and $\Pi^0_1(\tau_{P_{\vartheta_n}})$ $(n < \omega)$ so since $\tau^<_{P_\xi}$ is the product of the topologies $\tau_{P_{\vartheta_n}}$ $(n < \omega)$, $U_{\xi,n}$ is $\tau^<_{P_\xi}$-nowhere dense $(n < \omega)$. Also, $U_{\xi,n}$ is a finite intersection of $\tau^<_{P_\xi}$-open and $\Pi^0_1(\tau^<_{P_\xi})$ sets, thus it is a $\Sigma^0_2(\tau^<_{P_\xi})$ set $(n < \omega)$. Hence (3.6) shows that P_ξ is $\tau^<_{P_\xi}$-dense and $\Pi^0_2(\tau^<_{P_\xi})$.

Now we prove 3 for ξ. To obtain τ_{P_ξ}, we made open every $U_{\xi,n}$ on the right hand side of (3.6), so P_ξ is $\Pi^0_1(\tau_{P_\xi})$. Using again that P_{ϑ_n} is

$\tau_{P_{\vartheta_n}}$-nowhere dense ($n < \omega$), $\bigcup_{n<\omega} U_{\xi,n}$ meets every $\tau^{<}_{P_\xi}$-open set, hence it is τ_{P_ξ}-dense τ_{P_ξ}-open. So P_ξ is τ_{P_ξ}-nowhere dense.

For 5, we have just observed that $\bigcup_{n<\omega} U_{\xi,n}$ meets every $\tau^{<}_{P_\xi}$-open set, hence it is $\tau^{<}_{P_\xi}$-dense, as stated.

Statements 6 and 7 follow from the fact that $U_{\xi,n}$ ($n < \omega$) are pairwise disjoint and are disjoint from P_ξ. Statements 8 and 9 immediately follow from 7 and 6.

Finally if ξ is a limit ordinal and $m < \omega$ then by taking $\vartheta'_i = \vartheta_{m+i}$ the sequence obtained satisfies $\vartheta'_i \to \xi$ so 2–9 hold for P^m_ξ and $U^m_{\xi,n}$ instead of P_ξ and $U_{\xi,n}$ ($m \le n < \omega$) and the proof is complete. □

Next we point out a property of $H_\xi(\vartheta)$.

Lemma 3.2. *For every* $0 < \xi < \omega_1$, $H_\xi(\vartheta)$ *is a* $\tau^{<}_{P_\xi}$*-dense* $\tau^{<}_{P_\xi}$*-open set in* C_ξ. *We have* $P_\xi \subseteq H_\xi(\vartheta) \subseteq H_\xi(\vartheta')$ ($\vartheta' \le \vartheta < \xi$).

Similarly, $H^m_\xi(\vartheta)$ *is a* $\tau^{<}_{P^m_\xi}$*-dense* $\tau^{<}_{P^m_\xi}$*-open set in* C^m_ξ *and* $P^m_\xi \subseteq H^m_\xi(\vartheta) \subseteq H^m_\xi(\vartheta')$ ($\vartheta' \le \vartheta < \xi$).

Proof. The statements immediately follow from the definition. □

We continue with the definition of the spaces, the sets and the topologies for the difference hierarchy.

Definition 3.2. *Let* $0 < \xi, \eta < \omega_1$ *be fixed. If* ξ *is a successor ordinal take sequences* $(C_\xi(\alpha), \tau_{C_\xi(\alpha)})$, $P_\xi(\alpha)$, $\tau^{<}_{P_\xi(\alpha)}$ *and* $\tau_{P_\xi(\alpha)}$ ($\alpha < \eta$) *of copies of the Polish space* (C_ξ, τ_{C_ξ}), *the set* P_ξ *and the topologies* $\tau^{<}_{P_\xi}$, τ_{P_ξ} *defined in Definition* 3.1; *and set* $\rho_\xi(\alpha) = 0$ ($\alpha < \eta$). *If* ξ *is a limit ordinal take an injection* $\rho_\eta \colon \eta \to \omega$ *and let* $(C_\xi(\alpha), \tau_{C_\xi(\alpha)}) = (C^{\rho_\eta(\alpha)}_\xi, \tau_{C^{\rho_\eta(\alpha)}_\xi})$, $P_\xi(\alpha) = P^{\rho_\eta(\alpha)}_\xi$, $\tau^{<}_{P_\xi(\alpha)} = \tau^{<}_{P^{\rho_\eta(\alpha)}_\xi}$ *and* $\tau_{P_\xi(\alpha)} = \tau_{P^{\rho_\eta(\alpha)}_\xi}$ ($\alpha < \eta$).

Let

$$C_{\xi,\eta} = \prod_{\alpha<\eta} C_\xi(\alpha),\ \tau_{C_{\xi,\eta}} = \prod_{\alpha<\eta} \tau_{C_\xi(\alpha)},$$

for every $\alpha < \eta$ *set*

$$K_{\xi,\eta}(\alpha) = \bigcup_{\beta\le\alpha} \Big(\Big(\prod_{\gamma<\beta} C_\xi(\gamma) \Big) \times (C_\xi(\beta) \setminus P_\xi(\beta)) \times \prod_{\beta<\gamma<\eta} C_\xi(\gamma) \Big),$$

$$V_{\xi,\eta}(\alpha) = \Big(\prod_{\gamma<\alpha} P_\xi(\gamma) \Big) \times (C_\xi(\alpha) \setminus P_\xi(\alpha)) \times \prod_{\alpha<\gamma<\eta} C_\xi(\gamma),$$

$V_{\xi,\eta}(\eta) = \prod_{\gamma<\eta} P_\xi(\gamma)$ *and for* $0 < \alpha \leq \eta$ *let*

$$W_{\xi,\eta}(\alpha) = \prod_{\gamma<\alpha} P_\xi(\gamma) \times \prod_{\alpha\leq\gamma<\eta} C_\xi(\gamma).$$

Now set

$$Q_{\xi,\eta} = D_\eta\big((K_{\xi,\eta}(\alpha))_{\alpha<\eta}\big) = \bigcup\{V_{\xi,\eta}(\alpha)\colon \alpha \leq \eta,\ \alpha \textit{ is odd} \leftrightarrow \eta \textit{ is even}\},$$

$$P_{\xi,\eta} = C_{\xi,\eta} \setminus Q_{\xi,\eta} = \bigcup\{V_{\xi,\eta}(\alpha)\colon\ \alpha \leq \eta,\ \alpha \textit{ is odd} \leftrightarrow \eta \textit{ is odd}\}$$

and define the topologies

$$\tau^{<}_{\xi,\eta} = \prod_{\alpha<\eta} \tau^{<}_{P_\xi(\alpha)},\ \tau^{<}_{\xi,\eta}(\alpha) = \prod_{\gamma<\alpha} \tau^{<}_{P_\xi(\gamma)} \times \prod_{\alpha\leq\gamma<\eta} \tau_{P_\xi(\gamma)}\ (\alpha \leq \eta),$$

$$\tau_{\xi,\eta} = \tau^{<}_{\xi,\eta}(0)[\{W_{\xi,\eta}(\alpha)\colon 0 < \alpha \leq \eta\}].$$

If ξ *is a limit ordinal and* $\vartheta < \xi$ *set* $H_{\xi,\eta}(\vartheta) = \prod_{\alpha<\eta} H_\xi^{\rho_\eta(\alpha)}(\vartheta)$*. If* ξ *is successor we set* $H_{\xi,\eta}(\vartheta) = C_{\xi,\eta}$

Remark 3.1. The notation introduced above is negligent because for limit ξ we do not indicate the dependence of say $C_\xi(\alpha)$ from the particular η and ρ_η we are working with. Since η and ξ will mostly be fixed this will not cause any confusion. On the other hand ρ_η will vary for the following reason. If ξ is limit and $\eta = \eta_0 + (\eta \setminus \eta_0)$ we identify $C_{\xi,\eta}$ with $C_{\xi,\eta_0} \times C_{\xi,\eta\setminus\eta_0}$ by fixing the injection $\rho_\eta\colon \eta \to \omega$ first and taking $\rho_{\eta_0} = \rho_\eta|_{\eta_0}$, $\rho_{\eta\setminus\eta_0} = \rho_\eta|_{\eta\setminus\eta_0}$. In the sequel this convention applies for the terms of product spaces. We will state in advance where this happens.

We summarize the basic properties of our new sets and topologies. Notice in advance that $V_{\xi,\eta}(\eta) = W_{\xi,\eta}(\eta)$, $\tau^{<}_{\xi,\eta} = \tau^{<}_{\xi,\eta}(\eta)$ $(0 < \eta < \omega_1)$ and that the sets $V_{\xi,\eta}(\alpha)$ $(\alpha \leq \eta)$ are pairwise disjoint.

Lemma 3.3. *Let* $0 < \xi, \eta < \omega_1$ *be fixed.*

1. $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$ *is homeomorphic to* (C, τ_C)*.*

2. $W_{\xi,\eta}(\eta)$ *is a* $\Pi^0_\xi(\tau_{C_{\xi,\eta}})$ *set,* $K_{\xi,\eta}(\alpha)$ *is a* $\Sigma^0_\xi(\tau_{C_{\xi,\eta}})$ *set* $(\alpha < \eta)$ *hence* $Q_{\xi,\eta}$ *is a* $D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ *set and* $P_{\xi,\eta}$ *is a* $\check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ *set.*

3. $Q_{\xi,\eta}$ *and* $P_{\xi,\eta}$ *are* $\tau_{\xi,\eta}$*-open.*

Proof. Statements 1 and 2 follows from Lemma 3.1.1 and Lemma 3.1.2. For 3 we have $V_{\xi,\eta}(\eta) = W_{\xi,\eta}(\eta)$ and

$$V_{\xi,\eta}(\alpha) = W_{\xi,\eta}(\alpha) \setminus \left(\left(\prod_{\gamma<\alpha} C_\xi(\gamma) \right) \times P_\xi(\alpha) \times \prod_{\alpha<\gamma<\eta} C_\xi(\gamma) \right) \ (\alpha < \eta)$$

hence $V_{\xi,\eta}(\alpha)$ is $\tau_{\xi,\eta}$-open $(\alpha \le \eta)$. Since $Q_{\xi,\eta}$ and $P_{\xi,\eta}$ are the unions of some $V_{\xi,\eta}(\alpha)$ $(\alpha \le \eta)$ they are $\tau_{\xi,\eta}$-open, as stated. □

Next we analyze the restriction of the topology $\tau_{\xi,\eta}$ to our special sets.

Lemma 3.4. *Let $0 < \xi, \eta < \omega_1$, $\alpha \le \eta$ be fixed and let (Y, σ) be a nonempty Polish space. Then*

1. *if $G \subseteq C_{\xi,\eta} \times Y$ is basic $\tau^<_{\xi,\eta}(0) \times \sigma$-open and $G \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$ then G is basic $\tau^<_{\xi,\eta}(\gamma) \times \sigma$-open $(\gamma \le \alpha)$;*
2. *if $G \subseteq C_{\xi,\eta} \times Y$ is basic $\tau_{\xi,\eta} \times \sigma$-open and $G \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$ then there is a basic $\tau^<_{\xi,\eta}(\alpha) \times \sigma$-open set G' such that $G \cap (V_{\xi,\eta}(\alpha) \times Y) = G' \cap (V_{\xi,\eta}(\alpha) \times Y)$;*
3. *the topologies $(\tau_{\xi,\eta} \times \sigma)|_{V_{\xi,\eta}(\alpha) \times Y}$ and $(\tau^<_{\xi,\eta}(\gamma) \times \sigma)|_{V_{\xi,\eta}(\alpha) \times Y}$ $(\gamma \le \alpha)$ coincide;*
4. *if $0 < \alpha$, G is basic $\tau^<_{\xi,\eta}(0) \times \sigma$-open and $G \cap (W_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$ then G is basic $\tau^<_{\xi,\eta}(\gamma) \times \sigma$-open $(\gamma \le \alpha)$;*
5. *if G is basic $\tau_{\xi,\eta} \times \sigma$-open and $G \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$ then there is a basic $\tau^<_{\xi,\eta} \times \sigma$-open set G' such that $G' \cap (W_{\xi,\eta}(\eta) \times Y) = G \cap (W_{\xi,\eta}(\eta) \times Y)$;*
6. *the topologies $(\tau_{\xi,\eta} \times \sigma)|_{W_{\xi,\eta}(\eta) \times Y}$ and $(\tau^<_{\xi,\eta}(\gamma) \times \sigma)|_{W_{\xi,\eta}(\eta) \times Y}$ $(\gamma \le \eta)$ coincide.*

Proof. For 1, let G be basic a $\tau^<_{\xi,\eta}(0) \times \sigma$-open set satisfying $G \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$. If for some $\gamma < \alpha$, $\Pr_{C_\xi(\gamma)}(G)$ is proper basic $\tau_{P_\xi(\gamma)}$-open then $\Pr_{C_\xi(\gamma)}(G) \cap P_\xi(\gamma) = \emptyset$ hence $G \cap (V_{\xi,\eta}(\alpha) \times Y) = \emptyset$, which is not the case. So $\Pr_{C_\xi(\gamma)}(G)$ is $\tau^<_{P_\xi(\gamma)}$-open $(\gamma < \alpha)$, thus 1 holds.

For 2, let G be basic $\tau_{\xi,\eta} \times \sigma$-open, say $G = G' \cap (W_{\xi,\eta}(\beta) \times Y)$ where G' is basic $\tau^<_{\xi,\eta}(0)$-open and $\beta \le \eta$, such that $G \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$. Since $V_{\xi,\eta}(\alpha) \subseteq W_{\xi,\eta}(\gamma)$ $(\gamma \le \alpha)$ and $V_{\xi,\eta}(\alpha) \cap W_{\xi,\eta}(\gamma) = \emptyset$ $(\alpha < \gamma \le \eta)$, we have $G \cap (V_{\xi,\eta}(\alpha) \times Y) = G' \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$. Thus by 1 with $\gamma = \alpha$, G' is basic $\tau^<_{\xi,\eta}(\alpha) \times \sigma$-open so 2 holds.

Since $\tau^<_{\xi,\eta}(\gamma)$ $(\gamma \le \alpha)$ is finer than $\tau^<_{\xi,\eta}(\alpha)$ and is coarser than $\tau_{\xi,\eta}$, 2 immediately gives 3.

For 4, observe that $W_{\xi,\eta}(\alpha) = \bigcup_{\alpha\leq\beta\leq\eta} V_{\xi,\eta}(\beta)$. Thus we have $G \cap (V_{\xi,\eta}(\beta) \times Y) \neq \emptyset$ for some $\alpha \leq \beta \leq \eta$. So by 1, G is basic $\tau^<_{\xi,\eta}(\gamma) \times \sigma$-open ($\gamma \leq \beta$), as required.

Finally 5 and 6 are the special $\alpha = \eta$ case of 2 and 3, so the proof is complete. □

To close this section we prove three lemmata, the first specially for $\xi = 1$, the second for $1 < \xi < \omega_1$ and the third for $H_{\xi,\eta}(\vartheta)$.

Lemma 3.5. *Let $0 < \eta < \omega_1$ be fixed and let (Y, σ) be a nonempty Polish space. If H is $\tau^<_{1,\eta} \times \sigma$-open and $H \cap (W_{1,\eta}(\eta) \times Y) \neq \emptyset$ then $H \cap (V_{1,\eta}(\gamma) \times Y) \neq \emptyset$ ($\gamma \leq \eta$).*

Proof. By passing to a subset we can assume that H is basic $\tau^<_{1,\eta} \times \sigma$-open. For $\gamma = \eta$ the statement follows from $V_{1,\eta}(\eta) = W_{1,\eta}(\eta)$ so let $\gamma < \eta$. Observe that $\tau^<_{1,\eta} = \tau_{C_{1,\eta}}$; thus $H \cap (W_{1,\eta}(\eta) \times Y) \neq \emptyset$ implies $P_1(\alpha) \in \mathrm{Pr}_{C_1(\alpha)}(H)$ ($\alpha < \eta$) and $\mathrm{Pr}_Y(H) \neq \emptyset$. Since $\mathrm{Pr}_{C_1(\alpha)}(V_{1,\eta}(\gamma)) = P_1(\alpha)$ ($\alpha < \gamma$) and $\mathrm{Pr}_{C_1(\alpha)}(V_{1,\eta}(\gamma))$ is a $\tau_{C_1(\alpha)}$-dense $\tau_{C_1(\alpha)}$-open set ($\gamma \leq \alpha < \eta$) we conclude $H \cap (V_{1,\eta}(\gamma) \times Y) \neq \emptyset$. □

Lemma 3.6. *Let $1 < \xi, \eta < \omega_1$, $\alpha \leq \gamma \leq \eta$ be fixed. Then*

1. *$V_{\xi,\eta}(\alpha)$ is a $\tau^<_{\xi,\eta}(\gamma)$-dense set;*
2. *$V_{\xi,\eta}(\alpha)$ is a $\tau^<_{\xi,\eta}(\alpha)$-residual set;*
3. *$W_{\xi,\eta}(\alpha)$ is a $\tau^<_{\xi,\eta}(\gamma)$-residual $\Pi^0_2(\tau^<_{\xi,\eta})$ set.*

Proof. By Lemma 3.1.4 and Lemma 3.1.5, both $P_\xi(\beta)$ and $C_\xi(\beta) \setminus P_\xi(\beta)$ are $\tau^<_{P_\xi(\beta)}$-dense sets ($\beta \leq \alpha$), so 1 follows. By Lemma 3.1.4, $P_\xi(\beta)$ is a $\tau^<_{P_\xi(\beta)}$-residual $\Pi^0_2(\tau^<_{P_\xi}(\beta))$ set ($\beta < \alpha$) while by Lemma 3.1.3, $P_\xi(\alpha)$ is a $\tau_{P_\xi(\alpha)}$-nowhere dense, so we have 2 and 3. □

In the final lemma we apply the convention of Remark 3.1 for the first time.

Lemma 3.7. *For every $0 < \xi, \eta < \omega_1$, $H_{\xi,\eta}(\vartheta)$ is a $\tau^<_{\xi,\eta}$-dense $\tau^<_{\xi,\eta}$-open set in $C_{\xi,\eta}$. We have $W_{\xi,\eta}(\eta) \subseteq H_{\xi,\eta}(\vartheta) \subseteq H_{\xi,\eta}(\vartheta')$ ($\vartheta' \leq \vartheta < \xi$) and for $\alpha < \eta$, $\tilde{\eta} = \eta \setminus \alpha$, $H_{\xi,\eta}(\vartheta) = H_{\xi,\alpha}(\vartheta) \times H_{\xi,\tilde{\eta}}(\vartheta)$ ($\vartheta < \xi$).*

Proof. The statements immediately follows from Lemma 3.2 and the definition using that $\rho_{\tilde{\eta}} = \rho_\eta|_{\tilde{\eta}}$ and $\rho_\alpha = \rho_\eta|_\alpha$ by the convention of Remark 3.1. □

4. Testing the difference hierarchy

From now on we work to prove that $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$, $W_{\xi,\eta}(\eta)$, $P_{\xi,\eta}$, $\tau^{<}_{\xi,\eta}$ and $\tau_{\xi,\eta}$ of Definition 3.2 have the properties required in Theorem 1.3. We start with the easy observations.

Proposition 4.1. *Let $0 < \xi, \eta < \omega_1$ be fixed. Then $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$ is homeomorphic to $(2^\omega, \tau_{2^\omega})$, $W_{\xi,\eta}(\eta)$ is a $\Pi^0_\xi(\tau_{C_{\xi,\eta}})$ set, $P_{\xi,\eta}$ is a $\check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ set. The topology $\tau_{\xi,\eta}$ is Polish and refines $\tau^{<}_{\xi,\eta}$ which in turn refines $\tau_{C_{\xi,\eta}}$. The sets $P_{\xi,\eta}$ and $C_{\xi,\eta} \setminus P_{\xi,\eta}$ are both $\tau_{\xi,\eta}$-open.*

Proof. The statements follow from Lemma 3.3.1, Lemma 3.3.2, Definition 3.2 and Lemma 3.3.3. □

Thus Theorem 1.3.1 and Theorem 1.3.2 hold. Observe that for $1 < \xi < \omega_1$ if G is basic $\tau^{<}_{\xi,\eta}$-open then $G \cap W_{\xi,\eta}(\eta) \neq \emptyset$ follows from Lemma 3.6.3 with $\alpha = \gamma = \eta$. So this condition in Theorem 1.3.3 is restrictive only for $\xi = 1$. In the sequel we use these properties without further reference. Theorem 1.3.3 will be proved through Theorem 4.1, Corollary 4.1.1 and Theorem 4.3.

4.1. The $\xi = 1$ case

First we prove Theorem 1.3.3 for $\xi = 1$. As we mentioned in the introduction this case does not fit into the general framework. Since the $1 < \xi < \omega_1$ case is complicated enough in itself we treat $\xi = 1$ separately. Also, this is a simple but informative introduction to the techniques we use. In the proof the product structure of Definition 3.2 must be exploited thus we prove Theorem 1.3.3 for $\xi = 1$ in the following more general form. When Y is a single point, we get back Theorem 1.3.3.

Theorem 4.1. *Let $0 < \eta < \omega_1$ be fixed. Let (Y, σ) be a nonempty Polish space and consider a $D_\eta(\Sigma^0_1(\tau_{C_{1,\eta}} \times \sigma))$ set $A \subseteq C_{1,\eta} \times Y$. Suppose that $G \subseteq C_{1,\eta} \times Y$ is $\tau^{<}_{1,\eta} \times \sigma$-open, $G \cap (W_{1,\eta}(\eta) \times Y) \neq \emptyset$ and $A \cap (P_{1,\eta} \times Y)$ is $\tau_{1,\eta} \times \sigma$-residual in $G \cap (P_{1,\eta} \times Y)$. Then $A \cap (Q_{1,\eta} \times Y)$ is of $\tau_{1,\eta} \times \sigma$-second category in $G \cap (Q_{1,\eta} \times Y)$.*

Proof. Notice first that by Definition 3.2, $\tau^{<}_{1,\eta}(0) = \tau^{<}_{1,\eta} = \tau_{C_{1,\eta}}$; so G is basic $\tau_{C_{1,\eta}} \times \sigma$-open. We prove the statement by induction on η. Let first $\eta = 1$; then $A \cap G$ is a nonempty $\Sigma^0_1(\tau_{C_{1,1}} \times \sigma)$ set, $Q_{1,1} = V_{1,1}(0) = C_1(0) \setminus P_1(0)$ and $P_{1,1} = W_{1,1}(1) = P_1(0)$. That is $P_{1,1}$ is one point hence $Q_{1,1}$ is a $\tau_{C_{1,1}}$-dense $\tau_{C_{1,1}}$-open set. Thus $A \cap G \cap (Q_{1,1} \times Y)$ is a nonempty $\tau_{C_{1,1}} \times \sigma$-open set so it is of $\tau_{C_{1,1}} \times \sigma$-second category, i.e. $A \cap (Q_{1,1} \times Y)$ is of $\tau_{C_{1,1}} \times \sigma$-second category in $G \cap (Q_{1,1} \times Y)$. By

Lemma 3.4.3 with $\gamma = 0$, $(\tau_{1,1} \times \sigma)|_{Q_{1,1}\times Y} = \left(\tau_{C_{1,1}} \times \sigma\right)|_{Q_{1,1}\times Y}$ so the statement follows.

Suppose now that $1 < \eta < \omega_1$ and the statement holds for every $\underline{\eta} < \eta$ and nonempty Polish space $(\underline{Y}, \underline{\sigma})$. Let $A = D_\eta((A_\alpha)_{\alpha<\eta})$ with $\Sigma^0_1(\tau_{C_{1,\eta}} \times \sigma)$ sets A_α ($\alpha < \eta$) satisfying $A_\beta \subseteq A_\alpha$ ($\beta \leq \alpha < \eta$). We have that $W_{1,\eta}(\eta)\times Y \subseteq P_{1,\eta}\times Y$ is $\tau_{1,\eta}\times\sigma$-open. Since $G\cap(W_{1,\eta}(\eta) \times Y) \neq \emptyset$, A is $\tau_{1,\eta}\times\sigma$-residual in $G\cap(W_{1,\eta}(\eta) \times Y)$. Thus there is a minimal α such that the parity of α and η are different and $A_\alpha \cap G \cap (W_{1,\eta}(\eta) \times Y)$ is of $\tau_{1,\eta} \times \sigma$-second category. Since A_α and G are $\tau_{C_{1,\eta}} \times \sigma$-open, Lemma 3.5 with $H = A_\alpha \cap G$ gives that $A_\alpha \cap G \cap (V_{1,\eta}(\alpha) \times Y)$ is nonempty hence of $\tau_{1,\eta}\times\sigma$-second category. Now the parity of α and η differ so $V_{1,\eta}(\alpha)\times Y \subseteq Q_{1,\eta} \times Y$. That is if $A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$ is also of $\tau_{1,\eta} \times \sigma$-second category in $G \cap (V_{1,\eta}(\alpha) \times Y)$ then we are done.

Suppose that this is not the case. Then there is a $\beta < \alpha$ and a basic $\tau_{1,\eta} \times \sigma$-open set $G_0 \subseteq A_\alpha \cap G$ such that $G_0 \cap (V_{1,\eta}(\alpha) \times Y) \neq \emptyset$, A_β is $\tau_{1,\eta} \times \sigma$-residual in $G_0 \cap (V_{1,\eta}(\alpha) \times Y)$ and the parity of β and α differ, that is parity of β and η coincide. Since A_α, A_β and G are $\Sigma^0_1(\tau_{C_{1,\eta}} \times \sigma)$ we can assume that G_0 is in fact basic $\tau_{C_{1,\eta}} \times \sigma$-open, $G_0 \subseteq A_\beta \cap A_\alpha \cap G$ and $P_1(\beta) \notin \mathrm{Pr}_{C_1(\beta)}(G_0)$.

Set $\underline{\eta} = \beta$,

$$\underline{Y} = \left(\prod_{\beta\leq\gamma<\eta} C_1(\gamma)\right) \times Y, \qquad \underline{\sigma} = \left(\prod_{\beta\leq\gamma<\eta} \tau_{C_1(\gamma)}\right) \times \sigma,$$

$\underline{A} = D_{\underline{\eta}}((A_\gamma)_{\gamma<\underline{\eta}})$ and $\underline{G} = G_0$. It is clear that $\underline{G}$ is a basic $\tau_{C_{1,\underline{\eta}}}\times\underline{\sigma}$-open set hence $\underline{G}$ is basic $\tau^<_{1,\underline{\eta}} \times \underline{\sigma}$-open, as well. Since $\beta < \alpha$ implies

$$V_{1,\eta}(\alpha) \times Y \subseteq W_{1,\eta}(\beta) \times Y \subseteq W_{1,\underline{\eta}}(\underline{\eta}) \times \underline{Y},$$

we have $\underline{G} \cap (W_{1,\underline{\eta}}(\underline{\eta}) \times \underline{Y}) \neq \emptyset$.

We show that $\underline{A}$ is $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (P_{1,\underline{\eta}} \times \underline{Y})$. For this we have to prove that $\underline{A}$ is $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (V_{1,\underline{\eta}}(\gamma) \times \underline{Y})$ whenever $\gamma \leq \underline{\eta}$ has parity equal to the parity of $\underline{\eta}$. Now $\underline{G} \subseteq G$ implies that A is $\tau_{1,\eta} \times \sigma$-residual in $\underline{G} \cap (P_{1,\eta} \times Y)$, that is A is $\tau_{1,\eta} \times \sigma$-residual in $\underline{G} \cap (V_{1,\eta}(\gamma) \times Y)$ for every $\gamma \leq \eta$ with parity equal to the parity of η. Since the parities of $\underline{\eta}$ and η are equal, $P_1(\underline{\eta}) \notin \mathrm{Pr}_{C_1(\underline{\eta})}(\underline{G})$ implies

$$\underline{G} \cap (V_{1,\underline{\eta}}(\gamma) \times \underline{Y}) = \underline{G} \cap (V_{1,\eta}(\gamma) \times Y) \ \ (\gamma \leq \underline{\eta})$$

and by Lemma 3.4.3,

$$(\tau_{1,\underline{\eta}} \times \underline{\sigma})|_{\underline{G}\cap(V_{1,\underline{\eta}}(\gamma)\times\underline{Y})} = (\tau_{C_{1,\underline{\eta}}} \times \underline{\sigma})|_{\underline{G}\cap(V_{1,\underline{\eta}}(\gamma)\times\underline{Y})} =$$

$$= (\tau_{C_{1,\eta}} \times \sigma)|_{\underline{G}\cap(V_{1,\underline{\eta}}(\gamma)\times\underline{Y})} = (\tau_{C_{1,\eta}} \times \sigma)|_{\underline{G}\cap(V_{1,\eta}(\gamma)\times Y)}$$
$$= (\tau_{1,\eta} \times \sigma)|_{\underline{G}\cap(V_{1,\eta}(\gamma)\times Y)} \quad (\gamma \leq \underline{\eta})$$

we conclude that A is $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (P_{1,\underline{\eta}} \times \underline{Y})$. Now $\underline{G} \subseteq A_\beta$ so A_β is also $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (P_{1,\underline{\eta}} \times \underline{Y})$. Since the parities of β and η are equal this is possible only if $\underline{A} = D_\beta((A_\gamma)_{\gamma<\beta})$ is $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (P_{1,\underline{\eta}} \times \underline{Y})$.

So the induction hypothesis can be applied for $\underline{\eta} < \eta$, the Polish space $(\underline{Y}, \underline{\sigma})$, the $D_{\underline{\eta}}(\Sigma^0_1(\tau_{C_{1,\underline{\eta}}} \times \underline{\sigma}))$ set $\underline{A}$ and the basic $\tau^<_{1,\underline{\eta}} \times \underline{\sigma}$-open set $\underline{G}$ satisfying $\underline{G} \cap (W_{1,\underline{\eta}}(\underline{\eta}) \times \underline{Y}) \neq \emptyset$. We get that $\underline{A}$ is of $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-second category in $\underline{G} \cap (Q_{1,\underline{\eta}} \times \underline{Y})$. Since $\underline{A} = D_\beta((A_\gamma)_{\gamma<\beta}) \subseteq A$, this means that A is of $\tau_{1,\underline{\eta}} \times \underline{\sigma}$-second category in $Q_{1,\underline{\eta}} \times \underline{Y}$. We have

$$V_{1,\underline{\eta}}(\gamma) \times \underline{Y} = V_{1,\eta}(\gamma) \times Y \ (\gamma < \underline{\eta})$$

so by Lemma 3.4.3,

$$(\tau_{1,\underline{\eta}} \times \underline{\sigma})|_{V_{1,\underline{\eta}}(\gamma)\times\underline{Y}} = (\tau_{C_{1,\underline{\eta}}} \times \underline{\sigma})|_{V_{1,\underline{\eta}}(\gamma)\times\underline{Y}}$$
$$= (\tau_{C_{1,\eta}} \times \sigma)|_{V_{1,\underline{\eta}}(\gamma)\times\underline{Y}} = (\tau_{C_{1,\eta}} \times \sigma)|_{V_{1,\eta}(\gamma)\times Y}$$
$$= (\tau_{1,\eta} \times \sigma)|_{V_{1,\eta}(\gamma)\times Y} \quad (\gamma < \underline{\eta}). \quad (4.7)$$

Since β and η have the same parity,

$$Q_{1,\underline{\eta}} \times \underline{Y} = Q_{1,\beta} \times \prod_{\beta\leq\gamma<\eta} C_1(\gamma) \times Y$$
$$= \bigcup\{V_{1,\eta}(\alpha)\colon \alpha < \beta,\ \alpha \text{ is odd} \leftrightarrow \beta \text{ is even}\} \subseteq Q_{1,\eta} \times Y.$$

So A is of $\tau_{1,\eta} \times \sigma$-second category in $Q_{1,\eta} \times Y$, which completes the proof. □

4.2. The $\eta = 1$ case

Similarly to the $\xi = 1$ case, for $1 < \xi < \omega_1$ the proof of Theorem 1.3.3 goes by induction on η. In this section we prove the first step of the inductive argument, namely the $\eta = 1$ case.

Theorem 4.2. *Fix $1 \leq \xi < \omega_1$. Let (Y, σ) be a nonempty Polish space, $G \subseteq C_\xi \times Y$ be a basic $\tau^<_{P_\xi} \times \sigma$-open set.*

1. *If $\xi \geq 2$ and $\vartheta < \xi$, $A \subseteq C_\xi \times Y$ is $\Pi^0_\vartheta(\tau_{C_\xi} \times \sigma)$ and $A \cap (P_\xi \times Y)$ is $\left(\tau_{P_\xi}|_{P_\xi}\right) \times \sigma$-residual in $G \cap (P_\xi \times Y)$ then A is $\tau_{P_\xi} \times \sigma$-residual in $G \cap (H_\xi(\vartheta) \times Y)$.*

2. *If for a set $W \in \Sigma^0_\xi(\tau_{C_\xi} \times \sigma)$, $W \cap (P_\xi \times Y)$ is $(\tau_{P_\xi}|_{P_\xi}) \times \sigma$-residual in $G \cap (P_\xi \times Y)$, then W is $\tau_{P_\xi} \times \sigma$-residual in a $\tau_{P_\xi} \times \sigma$-open set $H \subseteq C_\xi \times Y$ satisfying $G \cap (P_\xi \times Y) \subseteq \mathrm{cl}_{\tau_{P_\xi} \times \sigma}(H \cap (P_\xi \times Y))$.*

The same result holds if (C_ξ, τ_{C_ξ}), P_ξ, τ_{P_ξ}, $\tau^<_{P_\xi}$ and $H_\xi(\vartheta)$ are replaced by $(C^m_\xi, \tau_{C^m_\xi})$, P^m_ξ, $\tau_{P^m_\xi}$, $\tau^<_{P^m_\xi}$ and $H^m_\xi(\vartheta)$ $(m < \omega)$.

Proof. First observe that if ξ is a limit ordinal and $m < \omega$ then by taking $\vartheta'_i = \vartheta_{m+i}$ the sequence obtained satisfies $\vartheta'_i \to \xi$ so the statements for $(C^m_\xi, \tau_{C^m_\xi})$, P^m_ξ, $\tau_{P^m_\xi}$, $\tau^<_{P^m_\xi}$ and $H^m_\xi(\vartheta)$ follow from the special $m = 0$ case.

Note also that for $\xi \geq 2$ if $G \subseteq C_\xi \times Y$ is a nonempty basic $\tau^<_{P_\xi} \times \sigma$-open set then $G \cap (P_\xi \times Y) \neq \emptyset$ follows from Lemma 3.1.4.

If we have proved 1 for a $2 \leq \xi < \omega_1$ then 2 is automatic for ξ, as follows. Let $\vartheta_i \to \xi$ and write $W = \bigcup_{i<\omega} A_i$ where A_i is $\Pi^0_{\vartheta_i}(\tau_{C_\xi} \times \sigma)$. Suppose that $W \cap (P_\xi \times Y)$ is $(\tau_{P_\xi}|_{P_\xi}) \times \sigma$-residual in $G \cap (P_\xi \times Y)$. For every $i < \omega$ let H_i denote the maximal $\tau_{P_\xi} \times \sigma$-open set in which A_i is $\tau_{P_\xi} \times \sigma$-residual. Now $P_\xi \subseteq H_\xi(\vartheta)$ $(\vartheta < \xi)$ by Lemma 3.2 so by 1 the $\tau_{P_\xi} \times \sigma$-open set $H = \bigcup_{i<\omega} H_i$ meets $G' \cap (P_\xi \times Y)$ for every basic $\tau^<_{P_\xi} \times \sigma$-open set G' intersecting $G \cap (P_\xi \times Y)$, which proves 2.

So we need only to prove 1. We do this by induction on ξ, namely we prove 2 for $\xi = 1$ and then we prove 1 for a fixed $1 < \xi < \omega_1$ by assuming that 2 holds for every $\eta < \xi$. For 2 if $\xi = 1$, $H = W$ can be chosen by the Baire Category Theorem.

Let now $\xi \geq 2$ and suppose that 2 holds for every $\eta < \xi$ and Polish space (Y, σ), no matter how we have fixed $\vartheta_i \to \eta$ for a limit $\eta < \xi$. Consider a $\Pi^0_\vartheta(\tau_{C_\xi} \times \sigma)$ set $A \subseteq C_\xi \times Y$ for a $\vartheta < \xi$ and suppose that $A \cap (P_\xi \times Y)$ is $(\tau_{P_\xi}|_{P_\xi}) \times \sigma$-residual in $G \cap (P_\xi \times Y)$ for the basic $\tau^<_{P_\xi} \times \sigma$-open set G. By Lemma 3.1.9 the topologies $\tau_{P_\xi}|_{P_\xi}$ and $\tau^<_{P_\xi}|_{P_\xi}$ coincide so $A \cap G \cap (P_\xi \times Y)$ is also $\tau^<_{P_\xi}|_{P_\xi} \times \sigma$-residual in $G \cap (P_\xi \times Y)$. But by Lemma 3.1.4, P_ξ is a $\tau^<_{P_\xi}$-residual subset of C_ξ, so

$$A \cap G \text{ is } \tau^<_{P_\xi} \times \sigma\text{-residual in } G. \tag{4.8}$$

Suppose that A is not $\tau_{P_\xi} \times \sigma$-residual in $G \cap (H_\xi(\vartheta) \times Y)$, that is $A \cap \tilde{G}$ is $\tau_{P_\xi} \times \sigma$-meager for some basic $\tau_{P_\xi} \times \sigma$-open set $\tilde{G} \subseteq G \cap (H_\xi(\vartheta) \times Y)$; by passing to a basic $\tau_{P_\xi} \times \sigma$-open subset we can assume that $\tilde{G}$ is not $\tau^<_{P_\xi} \times \sigma$-open. By Lemma 3.1.7, there is a basic $\tau^<_{P_\xi} \times \sigma$-open set G_0 and a unique $J < \omega$ such that $\tilde{G} = G_0 \cap (U_{\xi,J} \times Y)$. By passing to a basic

$\tau^{<}_{P_\xi} \times \sigma$-open subset $\tilde{G}_0 \subseteq G_0$ we can assume that

$$\tilde{G}_0 \cap (U_{\xi,J} \times Y) = \tilde{G}_0 \cap \left(\left(\prod_{i<J} C_{\vartheta_i} \right) \times P_{\vartheta_J} \times \left(\prod_{J+1 \leq i < \omega} C_{\vartheta_i} \right) \times Y \right) \neq \emptyset. \tag{4.9}$$

Note that $\tilde{G} \subseteq G \cap (H_\xi(\vartheta) \times Y)$ implies $\vartheta \leq \vartheta_J$ and $\tilde{G}_0 \subseteq G$. To summarize, we obtained that

$$A \cap \tilde{G}_0 \cap (U_{\xi,J} \times Y) \text{ is } \tau_{P_\xi} \times \sigma\text{-meager in } \tilde{G}_0 \cap (U_{\xi,J} \times Y)\,. \tag{4.10}$$

Set $\underline{\xi} = \vartheta_J$,

$$\underline{Y} = \left(\prod_{i<J} C_{\vartheta_i} \times \prod_{J+1 \leq i < \omega} C_{\vartheta_i} \right) \times Y,$$

$$\underline{\sigma} = \left(\prod_{i<J} \tau_{P_{\vartheta_i}} \times \prod_{J+1 \leq i < \omega} \tau_{P_{\vartheta_i}} \right) \times \sigma$$

$$\underline{W} = (C_\xi \times Y) \setminus A \subseteq C_{\underline{\xi}} \times \underline{Y},$$

$$\underline{G} = \tilde{G}_0 \subseteq C_{\underline{\xi}} \times \underline{Y}.$$

The space $(\underline{Y}, \underline{\sigma})$ is clearly Polish, $\underline{G}$ is a basic $\tau_{P_{\underline{\xi}}} \times \underline{\sigma}$-open subset of $C_{\underline{\xi}} \times \underline{Y}$. Since $\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y}) \neq \emptyset$, Lemma 3.1.6 implies that $\underline{G}$ is basic $\tau^{<}_{P_{\underline{\xi}}} \times \underline{\sigma}$-open. By $\vartheta \leq \underline{\xi}$, $\underline{W}$ is a $\Sigma^0_{\underline{\xi}}(\tau_{C_{\underline{\xi}}} \times \underline{\sigma})$ set. By (4.9),

$$\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y}) = \tilde{G}_0 \cap (U_{\xi,J} \times Y)$$

thus by Lemma 3.1.8, $(\tau^{<}_{P_\xi} \times \sigma)|_{\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y})} = (\tau_{P_\xi} \times \sigma)|_{\tilde{G}_0 \cap (U_{\xi,J} \times Y)}$. So we have

$$(\tau_{P_{\underline{\xi}}} \times \underline{\sigma})|_{\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y})} = (\tau^{<}_{P_\xi} \times \sigma)|_{\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y})} = (\tau_{P_\xi} \times \sigma)|_{\tilde{G}_0 \cap (U_{\xi,J} \times Y)}.$$

Thus by (4.10), $\underline{W}$ is $(\tau_{P_{\underline{\xi}}}|_{P_{\underline{\xi}}}) \times \underline{\sigma}$-residual in $\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y})$. So by the induction hypothesis $\underline{W}$ is $\tau_{P_{\underline{\xi}}} \times \underline{\sigma}$-residual in some $\tau_{P_{\underline{\xi}}} \times \underline{\sigma}$-open set $\underline{H} \subseteq C_{\underline{\xi}} \times \underline{Y}$ such that $\underline{G} \cap (P_{\underline{\xi}} \times \underline{Y}) \subseteq \mathrm{cl}_{\tau_{P_{\underline{\xi}}} \times \underline{\sigma}}(\underline{H} \cap (P_{\underline{\xi}} \times \underline{Y}))$; in particular, $\underline{H} \cap \underline{G} \neq \emptyset$. Let $H = \underline{H} \subseteq C_\xi \times Y$. Since $\tau_{P_{\underline{\xi}}} = \mathrm{Pr}_{C_{\underline{\xi}}}(\tau^{<}_{P_{\underline{\xi}}})$ by definition, we have $H \cap \tilde{G}_0 \neq \emptyset$ and $A \cap H \cap \tilde{G}_0$ is $\tau^{<}_{P_\xi} \times \sigma$-meager in $H \cap \tilde{G}_0 \subseteq G$. This contradicts (4.8) so the proof is complete. □

Theorem 4.2.2 is the key for the proof of the $1 < \xi < \omega_1$, $\eta = 1$ case of Theorem 1.3.3, that we prove in Corollary 4.1.1 in the usual more general form.

Corollary 4.1. *Let $1 < \xi < \omega_1$ be fixed and let (Y, σ) be a nonempty Polish space. Consider a $D_1(\Sigma^0_\xi(\tau_{C_{\xi,1}} \times \sigma))$ set $A \subseteq C_{\xi,1} \times Y$. Let $G \subseteq C_{\xi,1} \times Y$ be a basic $\tau^<_{\xi,1} \times \sigma$-open set.*

1. *If $A \cap (P_{\xi,1} \times Y)$ is $\tau_{\xi,1} \times \sigma$-residual in $G \cap (P_{\xi,1} \times Y)$ then $A \cap (Q_{\xi,1} \times Y)$ is of $\tau_{\xi,1} \times \sigma$-second category in $G \cap (Q_{\xi,1} \times Y)$.*

2. *If $\vartheta < \xi$, A is $\Pi^0_\vartheta(\tau_{C_{\xi,1}} \times \sigma)$ and $A \cap (P_{\xi,1} \times Y)$ is $\tau_{\xi,1} \times \sigma$-residual in $G \cap (P_{\xi,1} \times Y)$ then A is $\tau_{\xi,1} \times \sigma$-residual in $G \cap (H_{\xi,1}(\vartheta) \times Y)$.*

Proof. In advance, observe that by Definition 3.2 we have $D_1(\Sigma^0_\xi(\tau_{C_{\xi,1}})) = \Sigma^0_\xi(\tau_{C_{\xi,1}})$, $(C_{\xi,1}, \tau_{C_{\xi,1}}) = (C^{\rho_1(0)}_\xi, \tau_{C^{\rho_1(0)}_\xi})$, $Q_{\xi,1} = V_{\xi,1}(0) = C^{\rho_1(0)}_\xi \setminus P^{\rho_1(0)}_\xi$, $P_{\xi,1} = V_{\xi,1}(1) = W_{\xi,1}(1) = P^{\rho_1(0)}_\xi$, $\tau^<_{\xi,1} = \tau^<_{P^{\rho_1(0)}_\xi}$, $\tau_{\xi,1} = \tau_{P^{\rho_1(0)}_\xi}[\{P^{\rho_1(0)}_\xi\}]$ and $H_{\xi,1}(\vartheta) = H^{\rho_1(0)}_\xi(\vartheta)$ $(\vartheta < \xi)$.

For 1, if $A \cap (P_{\xi,1} \times Y)$ is $\tau_{\xi,1} \times \sigma$-residual in $G \cap (P_{\xi,1} \times Y)$ then $A \cap (P^{\rho_1(0)}_\xi \times Y)$ is $(\tau_{P^{\rho_1(0)}_\xi}|_{P^{\rho_1(0)}_\xi}) \times \sigma$-residual in $G \cap (P^{\rho_1(0)}_\xi \times Y)$. So by Theorem 4.2.2, A is of $\tau_{P^{\rho_1(0)}_\xi} \times \sigma$-second category in G. But $P^{\rho_1(0)}_\xi$ is $\tau_{P^{\rho_1(0)}_\xi}$-nowhere dense by Lemma 3.1.3 so $G \setminus (P^{\rho_1(0)}_\xi \times Y) \neq \emptyset$ and A is of $\tau_{P^{\rho_1(0)}_\xi} \times \sigma$-second category $G \setminus (P^{\rho_1(0)}_\xi \times Y)$. Now $G \setminus (P^{\rho_1(0)}_\xi \times Y) = G \cap (Q_{\xi,1} \times Y)$ and $\tau_{\xi,1}|_{Q_{\xi,1}} = \tau_{P^{\rho_1(0)}_\xi}|_{Q_{\xi,1}}$ so we obtained that A is of $\tau_{\xi,1} \times \sigma$-second category in $G \cap (Q_{\xi,1} \times Y)$, as stated.

For 2, by repeating the previous argument Theorem 4.2.1 implies that A is $\tau_{P^{\rho_1(0)}_\xi} \times \sigma$-residual in $G \cap (H^{\rho_1(0)}_\xi(\vartheta) \times Y)$. Since $\tau_{\xi,1}|_{P_{\xi,1}} = \tau_{P^{\rho_1(0)}_\xi}|_{P_{\xi,1}}$, $\tau_{\xi,1}|_{Q_{\xi,1}} = \tau_{P^{\rho_1(0)}_\xi}|_{Q_{\xi,1}}$ and A is $\tau_{\xi,1} \times \sigma$-residual in $G \cap (P_{\xi,1} \times Y)$ by assumption we conclude that A is $\tau_{\xi,1} \times \sigma$-residual in $G \cap (H_{\xi,1}(\vartheta) \times Y)$, which completes the proof. ☐

In the following corollary we show that $W_{\xi,\eta}(\eta)$ with the topology $\tau^<_{\xi,\eta}(0)$ exhibits the same feature of dichotomy as P_ξ with τ_{P_ξ}.

Corollary 4.2. *Let $1 < \xi, \eta < \omega_1$, (Y, σ) be a nonempty Polish space and let G be a nonempty basic $\tau^<_{\xi,\eta} \times \sigma$-open set.*

1. *If $\vartheta < \xi$, $A \subseteq C_{\xi,\eta} \times Y$ is a $\Pi^0_\vartheta(\tau_{C_{\xi,\eta}} \times \sigma)$ set and $A \cap (W_{\xi,\eta}(\eta) \times Y)$ is $(\tau^<_{\xi,\eta}(0) \times \sigma)|_{W_{\xi,\eta}(\eta) \times Y}$-residual in $G \cap (W_{\xi,\eta}(\eta) \times Y)$ then A is $\tau^<_{\xi,\eta}(0) \times \sigma$-residual in $G \cap (H_{\xi,\eta}(\vartheta) \times Y)$.*

2. *If $B \subseteq C_{\xi,\eta} \times Y$ is a $\Sigma^0_\xi(\tau_{C_{\xi,\eta}} \times \sigma)$ set and $B \cap (W_{\xi,\eta}(\eta) \times Y)$ is of*

$$(\tau^<_{\xi,\eta}(0) \times \sigma)|_{W_{\xi,\eta}(\eta) \times Y}\text{-second category in } G \cap (W_{\xi,\eta}(\eta) \times Y)$$

then B is of $\tau^{<}_{\xi,\eta}(0)\times\sigma$-second category in G.

3. *If $R\subseteq C_{\xi,\eta}\times Y$ is a $\Pi^0_\xi(\tau_{C_{\xi,\eta}}\times\sigma)$ set such that R is $\tau^{<}_{\xi,\eta}(0)\times\sigma$-residual in G then R is $\tau^{<}_{\xi,\eta}\times\sigma$-residual in G.*

Proof. First we show that if $\xi'\le\xi$ and 1 holds for $\Pi^0_\vartheta(\tau_{C_{\xi,\eta}}\times\sigma)$ sets A whenever $\vartheta<\xi'$ then 2 and 3 hold for $\Sigma^0_{\xi'}(\tau_{C_{\xi,\eta}}\times\sigma)$ sets and $\Pi^0_{\xi'}(\tau_{C_{\xi,\eta}}\times\sigma)$ sets, respectively.

Fix $\vartheta_i\to\xi'$ and let $B\subseteq C_{\xi,\eta}\times Y$ be a $\Sigma^0_{\xi'}(\tau_{C_{\xi,\eta}}\times\sigma)$ set such that $B\cap(W_{\xi,\eta}(\eta)\times Y)$ is of $(\tau^{<}_{\xi,\eta}(0)\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-second category in $G\cap(W_{\xi,\eta}(\eta)\times Y)$. Let $B=\bigcup_{i<\omega}B_i$ where B_i is $\Pi^0_{\vartheta_i}(\tau_{C_{\xi,\eta}}\times\sigma)$ $(i<\omega)$. Then for some $i<\omega$ and basic $\tau^{<}_{\xi,\eta}(0)\times\sigma$-open set $G_0\subseteq G$ we have $G_0\cap(W_{\xi,\eta}(\eta)\times Y)\neq\emptyset$ and $B_i\cap(W_{\xi,\eta}(\eta)\times Y)$ is $(\tau^{<}_{\xi,\eta}(0)\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-residual in $G_0\cap(W_{\xi,\eta}(\eta)\times Y)$. By Lemma 3.4.5 there is a basic $\tau^{<}_{\xi,\eta}\times\sigma$-open set G' satisfying $G'\cap(W_{\xi,\eta}(\eta)\times Y)=G_0\cap(W_{\xi,\eta}(\eta)\times Y)$. Since $G_0\subseteq G$ we can assume that $G'\subseteq G$. Then the conditions of 1 are satisfied and we get that B_i is $\tau^{<}_{\xi,\eta}(0)\times\sigma$-residual in $G'\cap(H_{\xi,\eta}(\vartheta_i)\times Y)$. By Lemma 3.7, $H_{\xi,\eta}(\vartheta_i)$ is $\tau^{<}_{\xi,\eta}$-dense $\tau^{<}_{\xi,\eta}$-open, in particular $G'\cap(H_{\xi,\eta}(\vartheta_i)\times Y)\neq\emptyset$. Hence $H_{\xi,\eta}(\vartheta_i)$ is $\tau^{<}_{\xi,\eta}(0)$-open as well and $G'\cap(H_{\xi,\eta}(\vartheta_i)\times Y)\subseteq G$, that is B is of $\tau^{<}_{\xi,\eta}(0)\times\sigma$-second category in G, indeed.

For 3 let $R\subseteq C_{\xi,\eta}\times Y$ be a $\Pi^0_{\xi'}(\tau_{C_{\xi,\eta}}\times\sigma)$ set such that R is $\tau^{<}_{\xi,\eta}(0)\times\sigma$-residual in G. Suppose that R is not $\tau^{<}_{\xi,\eta}\times\sigma$-residual in G, that is for some nonempty basic $\tau^{<}_{\xi,\eta}\times\sigma$-open set $G'\subseteq G$ we have that R is $\tau^{<}_{\xi,\eta}\times\sigma$-meager in G'. By Lemma 3.6.3, $G'\cap(W_{\xi,\eta}(\eta)\times Y)\neq\emptyset$ and $W_{\xi,\eta}(\eta)\times Y$ is $\tau^{<}_{\xi,\eta}\times\sigma$-residual in $C_{\xi,\eta}(\eta)\times Y$. Thus $B=C_{\xi,\eta}\setminus R$ is a $\Sigma^0_{\xi'}(\tau_{C_{\xi,\eta}}\times\sigma)$ set and $B\cap(W_{\xi,\eta}(\eta)\times Y)$ is $(\tau^{<}_{\xi,\eta}\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-residual in the nonempty $G'\cap(W_{\xi,\eta}(\eta)\times Y)$. By Lemma 3.4.6 with $\gamma=0$ and $\gamma=n$ the topologies $(\tau^{<}_{\xi,\eta}\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$ and $(\tau^{<}_{\xi,\eta}(0)\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$ coincide so $B\cap(W_{\xi,\eta}(\eta)\times Y)$ is $(\tau^{<}_{\xi,\eta}(0)\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-residual in $G'\cap(W_{\xi,\eta}(\eta)\times Y)$. Hence by 2, $C_{\xi,\eta}\setminus R=B$ is of $\tau^{<}_{\xi,\eta}(0)\times\sigma$-second category in G', which is a contradiction.

For every fixed $1<\xi<\omega_1$ we prove 1 by induction on ϑ and η: we show 3 for $\xi=1$, $0<\eta<\omega_1$ and we show that if $\vartheta<\xi$ and 3 holds for $\Pi^0_\vartheta(\tau_{C_{\xi,\eta^\star}}\times\sigma)$ sets R whenever $\eta^\star\le\eta$ then 1 holds for $\Pi^0_\vartheta(\tau_{C_{\xi,\eta}}\times\sigma)$ sets A with our η. As we have seen above this will complete the proof.

For $\xi=1$, statement 3 follows from $\tau^{<}_{1,\eta}=\tau^{<}_{1,\eta}(0)$. So let $\vartheta<\xi$ and $A\subseteq C_{\xi,\eta}\times Y$ be a $\Pi^0_\vartheta(\tau_{C_{\xi,\eta}}\times\sigma)$ set such that $A\cap(W_{\xi,\eta}(\eta)\times Y)$ is $(\tau^{<}_{\xi,\eta}(0)\times\sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-residual in $G\cap(W_{\xi,\eta}(\eta)\times Y)$.

Suppose that A is not $\tau^{<}_{\xi,\eta}(0)\times\sigma$-residual in $G\cap(H_{\xi,\eta}(\vartheta)\times Y)$ that is for some $\vartheta^\star<\vartheta$, $\Pi^0_{\vartheta^\star}(\tau_{C_{\xi,\eta}}\times\sigma)$ set $A^\star\subseteq(C_{\xi,\eta}\times Y)\setminus A$ and nonempty basic $\tau^<_{\xi,\eta}(0)\times\sigma$-open set $G'\subseteq G\cap(H_{\xi,\eta}(\vartheta)\times Y)$ we have $A^\star$ is $\tau^<_{\xi,\eta}(0)\times\sigma$-residual in G'. Let $I, I^<\subseteq\eta$ be disjoint finite sets such that

$$G'=\left(\prod_{\alpha\in I}G'(\alpha)\times\prod_{\alpha\in I^<}G'(\alpha)\times\prod_{\alpha\in\eta\setminus(I\cup I^<)}C_\xi(\alpha)\right)\times G'(Y)$$

where $G'(\alpha)$ is proper basic $\tau_{P_\xi(\alpha)}$-open for $\alpha\in I$, $G'(\alpha)$ is basic $\tau^<_{P_\xi(\alpha)}$-open for $\alpha\in I^<$ and $G'(Y)$ is basic σ-open. Let $G^\star$ be the basic $\tau^<_{\xi,\eta}\times\sigma$-open set defined by $G^\star=(\prod_{\alpha<\eta}G^\star(\alpha))\times G^\star(Y)$ where

$$G'(\alpha)=G^\star(\alpha)\quad(\alpha\in I^<),\qquad G^\star(\alpha)=C_\xi(\alpha)\quad(\alpha\in\eta\setminus\big(I\cup I^<\big))$$

and $G^\star(Y)=G'(Y)$, while for $\alpha\in I$ let $G'(\alpha)=G^\star(\alpha)\cap U_{\xi,n(\alpha)}$ with the unique $n(\alpha)<\omega$ of Lemma 3.1.7. Since $G'\subseteq G$, we can assume that $G^\star(\alpha)\subseteq\mathrm{Pr}_{C_\xi(\alpha)}(G)$ $(\alpha\in I)$. Thus $G^\star\subseteq G$ and so by Lemma 3.6.3 A is $\tau^<_{\xi,\eta}\times\sigma$-residual in $G^\star$. Observe that $G'\subseteq H_{\xi,\eta}(\vartheta)\times Y$ implies $\vartheta\leq\vartheta_{n(\alpha)}$ $(\alpha\in I)$ and we have $G'\cap G^\star\cap(H_{\xi,\eta}(\vartheta)\times Y)=G'\neq\emptyset$.

Set

$$Y^\star=\left(\prod_{\alpha\in I}C_\xi(\alpha)\right)\times Y,\quad \sigma^\star=\left(\prod_{\alpha\in I}\tau_{P_\xi(\alpha)}\right)\times\sigma,\quad \tau^\star=\prod_{\alpha\in\eta\setminus I}\tau^<_{P_\xi(\alpha)}.$$

Then G' and $G^\star$ are both nonempty basic $\tau^\star\times\sigma^\star$-open sets. We show that A is $\tau^\star\times\sigma^\star$-residual in $G^\star\cap(H_{\xi,\eta}(\vartheta)\times Y)$ and $A^\star$ is $\tau^\star\times\sigma^\star$-residual in G'; since $G'\cap G^\star\cap(H_{\xi,\eta}(\vartheta)\times Y)\neq\emptyset$ this contradicts $A\cap A^\star=\emptyset$ so the proof will be complete.

First we prove that A is $\tau^\star\times\sigma^\star$-residual in $G^\star\cap(H_{\xi,\eta}(\vartheta)\times Y)$. Let $I=\{\eta_i\colon i<|I|\}$ and for every $i<|I|$ set

$$Y_i=\left(\prod_{\alpha\in\{\eta_j\,:\,j<i\}}C_\xi(\alpha)\times\prod_{\alpha\in\eta\setminus\{\eta_j\,:\,j\leq i\}}C_\xi(\alpha)\right)\times Y,$$

$$\sigma_i=\left(\prod_{\alpha\in\{\eta_j\,:\,j<i\}}\tau_{P_\xi(\alpha)}\times\prod_{\alpha\in\eta\setminus\{\eta_j\,:\,j\leq i\}}\tau^<_{P_\xi(\alpha)}\right)\times\sigma,$$

$$H_i=\Bigg(\prod_{\alpha\in\{\eta_j\,:\,j<i\}}\mathrm{Pr}_{C_\xi(\alpha)}(H_{\xi,\eta}(\vartheta))\times\prod_{\alpha\in\{\eta_j\,:\,i<j<|I|\}}C_\xi(\alpha)\times\prod_{\alpha\in\eta\setminus\{\eta_j\,:\,j<|I|\}}\mathrm{Pr}_{C_\xi(\alpha)}(H_{\xi,\eta}(\vartheta))\Bigg)\times Y,$$

$$G_i = \prod_{\alpha \in \eta \setminus \{\eta_i\}} G^\star(\alpha).$$

Then (Y_i, σ_i) is a Polish space, $G_i \subseteq Y_i$ is a basic σ_i-open set so $G^\star$ is a basic $\tau^<_{P_\xi(\eta_i)} \times \sigma_i$-open set $(i < |I|)$. We show by induction on i that A is $\tau_{P_\xi(\eta_i)} \times \sigma_i$-residual in

$$G^\star \cap (\mathrm{Pr}_{C_\xi(\eta_i)}(H_{\xi,\eta}(\vartheta)) \times H_i) \subseteq C_\xi(\eta_i) \times Y_i = C_{\xi,\eta} \times Y \ (i < |I|).$$

Let first $i = 0$. Since A is $\tau^<_{\xi,\eta} \times \sigma$ residual in $G^\star$ and $\tau^<_{\xi,\eta} \times \sigma = \tau^<_{P_\xi(\eta_0)} \times \sigma_0$, by Lemma 3.1.9 we have that $A \cap (P_\xi(\eta_0) \times Y_0)$ is $(\tau_{P_\xi(\eta_0)}|_{P_\xi(\eta_0)}) \times \sigma_0$-residual in $G^\star \cap (P_\xi(\eta_0) \times Y_0)$. We apply Theorem 4.2.1 with the pair $P_\xi(\eta_0)$, $\tau_{P_\xi(\eta_0)}$, the Polish space (Y_0, σ_0) for the $\Pi^0_\vartheta(\tau_{C_\xi(\eta_0)} \times \sigma_0)$ set A and basic $\tau^<_{P_\xi(\eta_0)} \times \sigma_0$-open set $G^\star$. We get that A is $\tau_{P_\xi(\eta_0)} \times \sigma_0$-residual in $G^\star \cap (H_\xi(\vartheta) \times Y_0)$ if ξ is successor and in $G^\star \cap (H_\xi^{\rho_\eta(\eta_0)}(\vartheta) \times Y_0)$ if ξ is limit. So by Lemma 3.7 A is $\tau_{P_\xi(\eta_0)} \times \sigma_0$-residual in $G^\star \cap (\mathrm{Pr}_{C_\xi(\eta_0)}(H_{\xi,\eta}(\vartheta)) \times H_0)$, as well, which proves the $i = 0$ case.

Suppose now that the statement holds for some $i < |I| - 1$; we prove it for $i + 1$. Observe that $\tau_{P_\xi(\eta_i)} \times \sigma_i = \tau^<_{P_\xi(\eta_{i+1})} \times \sigma_{i+1}$ and $\mathrm{Pr}_{C_\xi(\eta_i)}(H_{\xi,\eta}(\vartheta)) \times H_i = C_\xi(\eta_{i+1}) \times H_{i+1}$ so by the induction hypothesis we have A is $\tau^<_{P_\xi(\eta_{i+1})} \times \sigma_{i+1}$-residual in $G^\star \cap (C_\xi(\eta_{i+1}) \times H_{i+1})$; hence by Lemma 3.1.9, $A \cap (P_\xi(\eta_{i+1}) \times Y_{i+1})$ is $(\tau_{P_\xi(\eta_{i+1})}|_{P_\xi(\eta_{i+1})}) \times \sigma_{i+1}$-residual in $G^\star \cap (P_\xi(\eta_{i+1}) \times H_{i+1})$. So we can apply Theorem 4.2.1 with the pair $P_\xi(\eta_{i+1})$, $\tau_{P_\xi(\eta_{i+1})}$, the Polish space (Y_{i+1}, σ_{i+1}) for the $\Pi^0_\vartheta(\tau_{C_\xi(\eta_{i+1})} \times \sigma_{i+1})$ set A and basic $\tau^<_{P_\xi(\eta_{i+1})} \times \sigma_{i+1}$-open set $G^\star \cap (C_\xi(\eta_{i+1}) \times H_{i+1})$. We get that A is $\tau_{P_\xi(\eta_{i+1})} \times \sigma_{i+1}$-residual in $G^\star \cap (\mathrm{Pr}_{C_\xi(\eta_{i+1})}(H_{\xi,\eta}(\vartheta)) \times H_{i+1})$, as stated. Since $\tau_{P_\xi(\eta_{|I|-1})} \times \sigma_{|I|-1} = \tau^\star \times \sigma^\star$ and $\mathrm{Pr}_{C_\xi(\eta_{|I|-1})}(H_{\xi,\eta}(\vartheta)) \times H_{|I|-1} = H_{\xi,\eta}(\vartheta)$ we conclude that A is $\tau^\star \times \sigma^\star$-residual in $G^\star \cap (H_{\xi,\eta}(\vartheta) \times Y)$.

It remains to prove that $A^\star$ is $\tau^\star \times \sigma^\star$-residual in G'. Let $C^\star = \prod_{\alpha \in \eta \setminus I} C_\xi(\alpha)$ and $\eta^\star = \eta \setminus I$. Using the convention of Remark 3.1, $(C^\star, \tau^\star) = (C_{\xi,\eta^\star}, \tau^<_{\xi,\eta^\star})$ and $\tau^<_{\xi,\eta^\star}(0) \times \sigma^\star = \tau^<_{\xi,\eta}(0) \times \sigma$. So by the definition of G', $A^\star$ is $\tau^<_{\xi,\eta^\star}(0) \times \sigma^\star$-residual in G'. By our assumption, 3 holds for the $\Pi^0_{\vartheta^\star}(\tau_{C_{\xi,\eta^\star}} \times \sigma^\star)$ set $A^\star \subseteq C_{\xi,\eta^\star} \times Y^\star$ and the nonempty basic $\tau^<_{\xi,\eta^\star} \times \sigma^\star$-open set G', and we get that $A^\star$ is $\tau^<_{\xi,\eta^\star} \times \sigma^\star$-residual hence $\tau^\star \times \sigma^\star$-residual in G'. The proof is complete. □

4.3. The general case

This final section contains the proof of Theorem 1.3.3 for $1 < \xi < \omega_1$. We start with a claim which slightly strengthens Corollary 4.2.2.

Proposition 4.2. *Let $1 < \xi, \eta < \omega_1$, (Y, σ) be a nonempty Polish space and $G \subseteq C_{\xi,\eta} \times Y$ be a nonempty $\tau^<_{\xi,\eta} \times \sigma$-open set. If $A \subseteq C_{\xi,\eta} \times Y$ is $\Sigma^0_\xi(\tau_{C_{\xi,\eta}} \times \sigma)$ and $A \cap (W_{\xi,\eta}(\eta) \times Y)$ is of $(\tau_{\xi,\eta} \times \sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-second category in $G \cap (W_{\xi,\eta}(\eta) \times Y)$ then there is a nonempty basic $\tau^<_{\xi,\eta} \times \sigma$-open set $G_0 \subseteq G$ such that A is $\tau_{\xi,\eta} \times \sigma$-residual in G_0.*

Proof. Before starting the proof, observe that by Lemma 3.6.3, $G \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$ for every nonempty $\tau^<_{\xi,\eta} \times \sigma$-open set G.

Let $\vartheta_i \to \xi$. Since A is $\Sigma^0_\xi(\tau_{C_{\xi,\eta}} \times \sigma)$, there is an $i < \omega$, a $\Pi^0_{\vartheta_i}(\tau_{C_{\xi,\eta}} \times \sigma)$ set $B \subseteq A$ and a nonempty basic $\tau_{\xi,\eta} \times \sigma$-open set $G^\star \subseteq G$ such that $G^\star \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$ and $B \cap (W_{\xi,\eta}(\eta) \times Y)$ is $(\tau_{\xi,\eta} \times \sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-residual in $G^\star \cap (W_{\xi,\eta}(\eta) \times Y)$. By Lemma 3.4.5 and Lemma 3.7 there is a basic $\tau^<_{\xi,\eta} \times \sigma$-open set G_0 for which $G_0 \cap (W_{\xi,\eta}(\eta) \times Y) = G^\star \cap (W_{\xi,\eta}(\eta) \times Y)$ and $G_0 \subseteq H_{\xi,\eta}(\vartheta_i) \times Y$. Then by Lemma 3.4.6 for $\gamma = \eta$, $B \cap (W_{\xi,\eta}(\eta) \times Y)$ is $(\tau^<_{\xi,\eta} \times \sigma)|_{W_{\xi,\eta}(\eta)\times Y}$-residual in $G_0 \cap (W_{\xi,\eta}(\eta) \times Y)$. We show that B is $\tau_{\xi,\eta} \times \sigma$-residual in G_0; then by $B \subseteq A$, G_0 fulfills the requirements.

We have $C_{\xi,\eta} \times Y = \bigcup_{\alpha\le\eta} V_{\xi,\eta}(\alpha) \times Y$ and $V_{\xi,\eta}(\alpha)$ is $\tau_{\xi,\eta} \times \sigma$-open ($\alpha \le \eta$) so we have to prove that B is $\tau_{\xi,\eta} \times \sigma$-residual in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$ ($\alpha \le \eta$).

For $\alpha = \eta$ this follows from $V_{\xi,\eta}(\eta) = W_{\xi,\eta}(\eta)$; so fix some $\alpha < \eta$. Set $\tilde\eta = \eta \setminus \alpha$,

$$\tilde Y = \left(\prod_{\gamma<\alpha} C_\xi(\gamma)\right) \times Y,\ \tilde\sigma = \left(\prod_{\gamma<\alpha} \tau^<_{P_\xi(\gamma)}\right) \times \sigma,$$

$\tilde G = G_0$ and $\tilde B = B$. Then $(\tilde Y, \tilde\sigma)$ is a nonempty Polish space and $\tilde G$ is a nonempty basic $\tau^<_{\xi,\tilde\eta} \times \tilde\sigma$-open set. By Lemma 3.6.3, $W_{\xi,\eta}(\eta) \times Y = W_{\xi,\tilde\eta}(\tilde\eta) \times W_{\xi,\alpha}(\alpha) \times Y$ is $\tau^<_{\xi,\tilde\eta} \times \tilde\sigma$-residual in $C_{\xi,\tilde\eta} \times \tilde Y$ so $\tilde B \cap (W_{\xi,\tilde\eta}(\tilde\eta) \times \tilde Y)$ is $(\tau^<_{\xi,\tilde\eta} \times \tilde\sigma)|_{W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y}$-residual in $\tilde G \cap (W_{\xi,\tilde\eta}(\tilde\eta) \times \tilde Y)$. By Lemma 3.4.6 the topologies

$$(\tau_{\xi,\tilde\eta} \times \tilde\sigma)|_{W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y} \text{ and } (\tau^<_{\xi,\tilde\eta}(\gamma) \times \tilde\sigma)|_{W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y}\ (\gamma \le \tilde\eta)$$

all coincide so $\tilde B \cap (W_{\xi,\tilde\eta}(\tilde\eta) \times \tilde Y)$ is $(\tau^<_{\xi,\tilde\eta}(0) \times \tilde\sigma)|_{W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y}$-residual and

$$(\tau_{\xi,\tilde\eta} \times \tilde\sigma)|_{W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y}\text{-residual in } \tilde G \cap (W_{\xi,\tilde\eta}(\tilde\eta) \times \tilde Y).$$

Thus we can apply Corollary 4.1.2 if $\tilde\eta = 1$ or Corollary 4.2.1 if $1 < \tilde\eta$ for $\tilde B$ and $\tilde G$ in $C_{\xi,\tilde\eta} \times \tilde Y$ and we get that $\tilde B$ is $\tau^<_{\xi,\tilde\eta}(0) \times \tilde\sigma$-residual in $\tilde G \cap (H_{\xi,\tilde\eta}(\vartheta_i) \times \tilde Y)$. Since $G_0 \subseteq H_{\xi,\eta}(\vartheta_i) \times Y$ implies $G_0 \subseteq (H_{\xi,\tilde\eta}(\vartheta_i) \times \tilde Y)$

by Lemma 3.7 and $\tau^<_{\xi,\tilde{\eta}}(0) \times \tilde{\sigma} = \tau^<_{\xi,\eta}(\alpha) \times \sigma$ we get that B is $\tau^<_{\xi,\eta}(\alpha) \times \sigma$-residual in G_0.

By Lemma 3.6.2, $V_{\xi,\eta}(\alpha) \times Y$ is $\tau^<_{\xi,\eta}(\alpha) \times \sigma$-residual in $C_{\xi,\eta} \times Y$ so

$$G_0 \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$$

and B is $(\tau^<_{\xi,\eta}(\alpha) \times \sigma)|_{V_{\xi,\eta}(\alpha)\times Y}$-residual in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$. By Lemma 3.4.3 with $\gamma = \alpha$ the topologies $(\tau_{\xi,\eta} \times \sigma)|_{V_{\xi,\eta}(\alpha)\times Y}$ and $(\tau^<_{\xi,\eta}(\alpha) \times \sigma)|_{V_{\xi,\eta}(\alpha)\times Y}$ coincide. So we obtained that B is $\tau_{\xi,\eta} \times \sigma$-residual in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$. The proof is complete. □

As above, in the proof of the remaining part of Theorem 1.3.3 the product structure of Definition 3.2 must be exploited. So we prove it in the following more general form. When Y is a single point we get back Theorem 1.3.3.

Theorem 4.3. *Let $1 < \xi, \eta < \omega_1$ be fixed. Let (Y, σ) be a nonempty Polish space and consider a $D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}} \times \sigma))$ set $A \subseteq C_{\xi,\eta} \times Y$. If $G \subseteq C_{\xi,\eta} \times Y$ is $\tau^<_{\xi,\eta} \times \sigma$-open such that $A \cap (P_{\xi,\eta} \times Y)$ is $\tau_{\xi,\eta} \times \sigma$-residual in $G \cap (P_{\xi,\eta} \times Y)$ then $A \cap (Q_{\xi,\eta} \times Y)$ is of $\tau_{\xi,\eta} \times \sigma$-second category in $G \cap (Q_{\xi,\eta} \times Y)$.*

Proof. We prove the statement by induction on η. The $\eta = 1$ case is Corollary 4.1.1.

Suppose now that $1 < \eta < \omega_1$ and that the statement holds for every $\underline{\eta} < \eta$ and Polish space $(\underline{Y}, \underline{\sigma})$. Let $A = D_\eta((A_\alpha)_{\alpha<\eta})$ with $\Sigma^0_\xi(\tau_{C_{\xi,\eta}} \times \sigma)$ sets A_α ($\alpha < \eta$) satisfying $A_\beta \subseteq A_\alpha$ ($\beta \leq \alpha < \eta$). We have that $W_{\xi,\eta}(\eta) \times Y \subseteq P_{\xi,\eta} \times Y$ is $\tau_{\xi,\eta} \times \sigma$-open. By Lemma 3.6.3, $G \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$ so A is $\tau_{\xi,\eta} \times \sigma$-residual in $G \cap (W_{\xi,\eta}(\eta) \times Y)$. Thus there is a minimal α such that the parity of α and η are different and for some basic $\tau_{\xi,\eta} \times \sigma$-open set $G^\star \subseteq G$, A_α is of $\tau_{\xi,\eta} \times \sigma$-second category in the nonempty $G^\star \cap (W_{\xi,\eta}(\eta) \times Y)$. Then by Lemma 3.4.5 there is a $\tau^<_{\xi,\eta} \times \sigma$-open set G' such that $G^\star \cap (W_{\xi,\eta}(\eta) \times Y) = G' \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$.

We apply Proposition 4.2 for A_α and G'; we obtain that for some nonempty basic $\tau^<_{\xi,\eta} \times \sigma$-open set $G_0 \subseteq G'$, A_α is $\tau_{\xi,\eta} \times \sigma$-residual in G_0. So in particular, A_α is $\tau_{\xi,\eta} \times \sigma$-residual in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$, which is nonempty by Lemma 3.6.1. But the parity of α and η differ, so $V_{\xi,\eta}(\alpha) \times Y \subseteq Q_{\xi,\eta} \times Y$. That is if $A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$ is also of $\tau_{\xi,\eta} \times \sigma$-second category in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$ then we are done.

Suppose that this is not the case. Then there is a $\beta < \alpha$ and a basic $\tau_{\xi,\eta} \times \sigma$-open set $G_0^\star \subseteq G_0$ such that $G_0^\star \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$ and A_β is $\tau_{\xi,\eta} \times \sigma$-residual in $G_0^\star \cap (V_{\xi,\eta}(\alpha) \times Y)$, moreover the parity of β and α differ, that is the parity of β and η coincide. By Lemma 3.4.2, there is a basic $\tau^<_{\xi,\eta}(\alpha)\times\sigma$-open set G'_0 such that $G^\star_0\cap(V_{\xi,\eta}(\alpha)\times Y)=G'_0\cap(V_{\xi,\eta}(\alpha)\times Y)$; and since G_0 is basic $\tau^<_{\xi,\eta}\times\sigma$-open, we can assume that $G'_0\subseteq G_0$. By Lemma 3.4.3 with $\gamma=\alpha$, A_β is $\tau^<_{\xi,\eta}(\alpha)\times\sigma$-residual in $G'_0\cap(V_{\xi,\eta}(\alpha)\times Y)$. By passing to a subset if necessary, we assume that $\mathrm{Pr}_{C_\xi(\alpha)}(G'_0)$ is proper $\tau_{P_\xi}(\alpha)$-open, that is

$$\mathrm{Pr}_{C_\xi(\alpha)}(G'_0)\subseteq C_\xi(\alpha)\setminus P_\xi(\alpha). \tag{4.11}$$

Set $\tilde\eta=\alpha$,

$$\tilde Y=\Big(\prod_{\alpha\le\gamma<\eta}C_\xi(\gamma)\Big)\times Y,\qquad \tilde\sigma=\Big(\prod_{\alpha\le\gamma<\eta}\tau_{P_\xi(\gamma)}\Big)\times\sigma$$

and $\tilde G=G'_0$. With this setting, using (4.11),

$$\tilde G\cap(W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y)=G'_0\cap(V_{\xi,\eta}(\alpha)\times Y)\neq\emptyset \tag{4.12}$$

and by Lemma 3.4.6,

$$\begin{aligned}(\tau_{\xi,\tilde\eta}\times\tilde\sigma)\,|_{\tilde G\cap\left(W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y\right)}&=(\tau^<_{\xi,\tilde\eta}\times\tilde\sigma)|_{\tilde G\cap\left(W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y\right)}\\ &=(\tau^<_{\xi,\tilde\eta}\times\tilde\sigma)|_{G'_0\cap\left(V_{\xi,\eta}(\alpha)\times Y\right)}=(\tau^<_{\xi,\eta}(\alpha)\times\sigma)|_{G'_0\cap\left(V_{\xi,\eta}(\alpha)\times Y\right)}.\end{aligned} \tag{4.13}$$

By (4.12) and (4.13) we have $\tilde G\cap(W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y)\neq\emptyset$ and A_β is $(\tau_{\xi,\tilde\eta}\times\tilde\sigma)|_{W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y}$-residual in $\tilde G\cap(W_{\xi,\tilde\eta}(\tilde\eta)\times\tilde Y)$. So we can apply Proposition 4.2 in $C_{\xi,\tilde\eta}\times\tilde Y$ for A_β and $\tilde G$. We get that for some nonempty basic $\tau^<_{\xi,\tilde\eta}\times\tilde\sigma$-open set $\tilde G_0\subseteq\tilde G$, A_β is $\tau_{\xi,\tilde\eta}\times\tilde\sigma$-residual in $\tilde G_0$. In particular, $\tilde G_0\cap(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y)\neq\emptyset$ $(\gamma\le\tilde\eta)$ by Lemma 3.6.1 and A_β is $\tau_{\xi,\tilde\eta}\times\tilde\sigma$-residual in $\tilde G_0\cap(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y)$ $(\gamma\le\tilde\eta)$.

Now $\tilde G_0\subseteq G$ is $\tau_{\xi,\eta}\times\sigma$-open and $V_{\xi,\tilde\eta}(\gamma)\times\tilde Y$ is also $\tau_{\xi,\eta}\times\sigma$-open $(\gamma\le\tilde\eta)$. So A is $\tau_{\xi,\eta}\times\sigma$-residual in $\tilde G_0\cap(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y)$ for every $\gamma\le\tilde\eta$ with parity different from the parity of $\tilde\eta$. We have

$$\tilde G_0\cap(V_{\xi,\eta}(\gamma)\times Y)=\tilde G_0\cap(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y)\neq\emptyset\ (\gamma<\tilde\eta).$$

So by Lemma 3.4.3,

$$\begin{aligned}(\tau_{\xi,\eta}\times\sigma)\,|_{\tilde G_0\cap\left(V_{\xi,\eta}(\gamma)\times Y\right)}&=(\tau^<_{\xi,\eta}(0)\times\sigma)|_{\tilde G_0\cap\left(V_{\xi,\eta}(\gamma)\times Y\right)}\\ &=(\tau^<_{\xi,\tilde\eta}(0)\times\tilde\sigma)|_{\tilde G_0\cap\left(V_{\xi,\eta}(\gamma)\times Y\right)}=(\tau^<_{\xi,\tilde\eta}(0)\times\tilde\sigma)|_{\tilde G_0\cap\left(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y\right)}\\ &=(\tau_{\xi,\tilde\eta}\times\tilde\sigma)\,|_{\tilde G_0\cap\left(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y\right)}\quad(\gamma<\tilde\eta).\end{aligned} \tag{4.14}$$

We get that A is $\tau_{\xi,\tilde\eta}\times\tilde\sigma$-residual in $\tilde G_0\cap(V_{\xi,\tilde\eta}(\gamma)\times\tilde Y)$ for every $\gamma<\tilde\eta$ with parity different from the parity of $\tilde\eta$. Since A_β is also $\tau_{\xi,\tilde\eta}\times\tilde\sigma$-residual in

$\tilde{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})$ for every $\gamma \le \tilde{\eta}$, this is possible only if $D_\beta((A_\gamma)_{\gamma<\beta})$ is $\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma}$-residual in $\tilde{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})$ for every $\gamma < \tilde{\eta}$ with parity different from the parity of $\tilde{\eta}$.

Let $H \subseteq C_{\xi,\eta}$ be a basic $\tau^<_{\xi,\eta}(0)$-open set which is nontrivial only on the $C_\xi(\beta)$ coordinate and $\mathrm{Pr}_{C_\xi(\beta)}(H)$ is proper basic $\tau_{P_\xi(\beta)}$-open, i.e. $\mathrm{Pr}_{C_\xi(\beta)}(H) \subseteq C_\xi(\beta) \setminus P_\xi(\beta)$, and $H \cap \tilde{G}_0 \neq \emptyset$. Set $\underline{\eta} = \beta$,

$$\underline{Y} = \Big(\prod_{\beta \le \gamma < \alpha} C_\xi(\gamma) \Big) \times \tilde{Y}, \qquad \underline{\sigma} = \Big(\prod_{\beta \le \gamma < \alpha} \tau_{P_\xi(\gamma)} \Big) \times \tilde{\sigma},$$

$\underline{A} = D_{\underline{\eta}}((A_\gamma)_{\gamma<\underline{\eta}})$ and $\underline{G} = \tilde{G}_0 \cap H$. Since $\tilde{G}_0$ is basic $\tau^<_{\xi,\tilde{\eta}} \times \tilde{\sigma}$-open, it is $\tau^<_{\xi,\tilde{\eta}}(\beta) \times \tilde{\sigma}$-open and so $\underline{G}$ is basic $\tau^<_{\xi,\underline{\eta}} \times \underline{\sigma}$-open. As above, we have

$$\underline{G}_0 \cap (V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y}) = \underline{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y}) \neq \emptyset \ (\gamma \le \underline{\eta})$$

so by Lemma 3.4.3,

$$\begin{aligned}(\tau_{\xi,\underline{\eta}} \times \underline{\sigma})|_{\underline{G}_0 \cap (V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y})} &= (\tau^<_{\xi,\underline{\eta}}(0) \times \underline{\sigma})|_{\underline{G}_0 \cap (V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y})} \\ = (\tau^<_{\xi,\tilde{\eta}}(0) \times \tilde{\sigma})|_{\underline{G}_0 \cap (V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y})} &= (\tau^<_{\xi,\tilde{\eta}}(0) \times \tilde{\sigma})|_{\underline{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})} \\ &= (\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma})|_{\underline{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})} \quad (\gamma \le \underline{\eta}). \end{aligned} \tag{4.15}$$

Since $\underline{A} = D_{\underline{\eta}}((A_\gamma)_{\gamma<\underline{\eta}})$ is $\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma}$-residual in $\tilde{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})$, we get that $\underline{A}$ is $\tau_{\xi,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y})$ for every $\gamma \le \underline{\eta}$ with parity different from the parity of $\tilde{\eta}$, that is with parity equal to the parity of $\underline{\eta}$. In particular, $\underline{A}$ is $\tau_{\xi,\underline{\eta}} \times \underline{\sigma}$-residual in $\underline{G} \cap (P_{\xi,\underline{\eta}} \times \underline{Y})$. So by the induction hypothesis $\underline{A}$ is of $\tau_{\xi,\underline{\eta}} \times \underline{\sigma}$-second category in $\underline{G} \cap (Q_{\xi,\underline{\eta}} \times \underline{Y})$. Since $\underline{A} = D_\beta((A_\gamma)_{\gamma<\beta}) \subseteq A$, this means that A is of $\tau_{\xi,\underline{\eta}} \times \underline{\sigma}$-second category in $Q_{\xi,\underline{\eta}} \times \underline{Y}$. We have

$$V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y} = V_{\xi,\eta}(\gamma) \times Y \ (\gamma < \underline{\eta})$$

so by Lemma 3.4.3,

$$\begin{aligned}(\tau_{\xi,\underline{\eta}} \times \underline{\sigma})|_{V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y}} &= (\tau^<_{\xi,\underline{\eta}}(0) \times \underline{\sigma})|_{V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y}} \\ = (\tau^<_{\xi,\eta}(0) \times \sigma)|_{V_{\xi,\underline{\eta}}(\gamma) \times \underline{Y}} &= (\tau^<_{\xi,\eta}(0) \times \sigma)|_{V_{\xi,\eta}(\gamma) \times Y} \\ &= (\tau_{\xi,\eta} \times \sigma)|_{V_{\xi,\eta}(\gamma) \times Y} \quad (\gamma < \underline{\eta}). \end{aligned} \tag{4.16}$$

Since β and η have the same parity,

$$Q_{\xi,\underline{\eta}} \times \underline{Y} = Q_{\xi,\beta} \times \prod_{\beta \le \gamma < \eta} C_\xi(\gamma) \times Y$$
$$= \bigcup \{V_{\xi,\eta}(\alpha) \colon \alpha < \beta,\ \alpha \text{ is odd} \leftrightarrow \beta \text{ is even}\} \subseteq Q_{\xi,\eta} \times Y.$$

So A is of $\tau_{\xi,\eta} \times \sigma$-second category in $Q_{\xi,\eta} \times Y$, which completes the proof. □

Before proving Corollary 1.2 and Corollary 1.3 we need to show that Theorem 1.2 can be applied for our pairs $(C_{\xi,\eta}, P_{\xi,\eta})$ and $(C_{\xi,\eta}, Q_{\xi,\eta})$ for $1 < \xi < \omega_1$ and $0 < \eta < \omega_1$.

Lemma 4.1 (A. Louveau, J. Saint Raymond). *Let* $1 < \xi < \omega_1$ *and* $0 < \eta < \omega_1$ *be fixed. Then* $(C_{\xi,\eta}, P_{\xi,\eta})$ *is a Hurewicz test pair for* $\check{D}_\eta(\Sigma^0_\xi)$ *and* $(C_{\xi,\eta}, C_{\xi,\eta} \setminus P_{\xi,\eta})$ *is a Hurewicz test pair for* $D_\eta(\Sigma^0_\xi)$.

Proof. Let first $3 \le \xi < \omega_1$ and $1 \le \eta < \omega_1$ or $\xi = 2$ and $\omega \le \eta < \omega_1$. By Lemma 3.3.1, $(C_{\xi,\eta}, \tau_{C_{\xi,\eta}})$ is homeomorphic to (C, τ_C). By Lemma 3.3.2, $P_{\xi,\eta}$ is $\check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ and $Q_{\xi,\eta}$ is $D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ so it remains to prove that $P_{\xi,\eta} \notin D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$ and $Q_{\xi,\eta} \notin \check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$.

Suppose that $P_{\xi,\eta}$ is $D_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$. By Corollary 1.1.1 with $A = P_{\xi,\eta}$, $P_{\xi,\eta} \setminus P_{\xi,\eta} = \emptyset$ is of $\tau_{\xi,\eta}$-second category, a contradiction. The same argument using Corollary 1.1.2 gives that $Q_{\xi,\eta} \notin \check{D}_\eta(\Sigma^0_\xi(\tau_{C_{\xi,\eta}}))$.

If $\xi = 2$ and $0 < \eta < \omega$ the statement follows from [2, Corollary 9, p. 458] and the special choice of $P_{2,\eta}$. This completes the proof. □

Proof of Corollary 1.2. Let first $A \subseteq X$ be a $D_\eta(\Sigma^0_\xi(\tau))$ set and suppose that the continuous injection $\varphi \colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X, \tau)$ satisfies $\varphi(P_{\xi,\eta}) \subseteq A$. Then $P_{\xi,\eta} \subseteq \varphi^{-1}(A)$ so by Corollary 1.1.1, $\varphi^{-1}(A) \setminus P_{\xi,\eta}$ is of $\tau_{\xi,\eta}$-second category, which proves 1. If $A \subseteq X$ is $\check{D}_\eta(\Sigma^0_\xi(\tau))$ the same argument using Corollary 1.1.2 proves 1′.

For 2 let $A \subseteq X$ be not in $D_\eta(\Sigma^0_\xi(\tau))$. By Lemma 4.1, $(C_{\xi,\eta}, P_{\xi,\eta})$ is a Hurewicz test pair for $\check{D}_\eta(\Sigma^0_\xi)$ so by applying Theorem 1.2 we get a continuous injection $\varphi \colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X, \tau)$ which satisfies $\varphi(P_{\xi,\eta}) = A \cap \varphi(C_{\xi,\eta})$, as stated. The same argument gives 2′ so the proof is complete. □

Proof of Corollary 1.3. We give the proof for $\Gamma = D_\eta(\Sigma^0_\xi)$, the proof for $\Gamma = \check{D}_\eta(\Sigma^0_\xi)$ is the same. By Lemma 4.1 we can apply Theorem 1.2 to have a continuous injection $\varphi \colon (C_{\xi,\eta}, \tau_{C_{\xi,\eta}}) \to (X, \tau)$ satisfying $\varphi(C_{\xi,\eta} \setminus P_{\xi,\eta}) = A \cap \varphi(C_{\xi,\eta})$. Fix a countable base $\mathcal{B}$ in the Polish space $(P_{\xi,\eta}, \tau_{\xi,\eta}|_{P_{\xi,\eta}})$. Set

$$\Lambda_B = \{\alpha < \lambda \colon \varphi^{-1}(A_\alpha) \text{ is } \tau_{\xi,\eta}|_{P_{\xi,\eta}}\text{-residual in } B\} \ (B \in \mathcal{B}).$$

By Corollary 1.1.2, $\varphi^{-1}(A_\alpha)$ is of $\tau_{\xi,\eta}$-second category in $P_{\xi,\eta}$ thus we have $\lambda = \bigcup_{B\in\mathcal{B}} \Lambda_B$. So $\Lambda = \Lambda_B$ is stationary for some $B \in \mathcal{B}$. Since in our model the union of λ meager sets is meager in Polish spaces, $\bigcap_{\alpha\in\Lambda_B} \varphi^{-1}(A_\alpha)$ is $\tau_{\xi,\eta}$-residual in B, in particular $B \cap \bigcap_{\alpha\in\Lambda_B} \varphi^{-1}(A_\alpha) \neq \emptyset$. Now φ is one-to-one hence this implies $(X \setminus A) \cap \bigcap_{\alpha\in\Lambda} A_\alpha \neq \emptyset$, so the proof is complete. □

References

[1] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, 156, Springer-Verlag, 1994.

[2] A. Louveau, J. Saint Raymond, *Borel Classes and Closed Games: Wadge-type and Hurewicz-type Results* // Trans. Amer. Math. Soc., **304** (1987), N 2, 431–467.

[3] A. Louveau, J. Saint-Raymond, *The strength of Borel Wadge determinacy*, Cabal Seminar **81–85**, Lecture Notes in Math., **1333**, Springer, Berlin, 1988, 1–30.

[4] T. Mátrai, *Hurewicz tests: separating and reducing analytic sets the conscious way*, PhD Thesis, Central European University, 2005.

[5] S. Solecki, *Covering analytic sets by families of closed sets* // J. Symbolic Logic **59** (1994), N 3, 1022–1031.

[6] S. Solecki, *Decomposing Borel sets and functions and the structure of Baire class 1 functions* // J. Amer. Math. Soc. **11** (1998), N 3, 521–550.

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