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To the spectral theory of the Bessel operator on finite interval and half-line

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Abstract. The minimal and maximal operators generated by the Bessel differential expression on the finite interval and a half-line are studied. All non-negative self-adjoint extensions of the minimal operator are described. Also we obtain a description of the domain of the Friedrichs extension of the minimal operator in the framework of extension theory of symmetric operators by applying the technique of boundary triplets and the corresponding Weyl functions, and by using the quadratic form method.

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1. Introduction

The one-dimensional Bessel differential expression was investigated in the classical form

$$\tau_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}, \qquad \nu \in [0,1) \setminus \{1/2\} \tag{1.1}$$

on the half-line $\mathbb{R}_+$ in numerous papers. Here, the parameter $\nu \in [0,\infty) \subset \mathbb{R}$ is the order of the Bessel functions involved. When $\nu = \frac{1}{2}$, it is the regular case. In particular, some results of spectral analysis were investigated in the papers [4, 9–11, 17]. We especially mention papers of Everitt and Kalf [9, 14] the most relevant to our interest. Here, Titchmarsh–Weyl m-coefficient was explicitly computed in $L^2(\mathbb{R}_+)$ using the classical definition. From the Nevanlinna representation of this m-coefficient the spectral function Σ was obtained to describe the spectrum of the associated self-adjoint operator in $L^2(\mathbb{R}_+)$. Additional analysis then yields

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the limit behaviour of the functions in the domain of the Friedrichs extension (see L. Bruneau, J. Dereziński and V. Georgescu [4], Everitt and Kalf [9, 14]) and Krein extension (see [4]).

In this paper we consider Bessel operator (1.1). Under the above restriction ($\nu \in [0,1)$) the endpoint 0 of the equation

$$-y''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} y(x) = \lambda y(x) \tag{1.2}$$

is the singular limit-circle case, with respect to $L^2(\mathbb{R}_+)$ or $L^2(0,b)$, except for the regular case.

We study the minimal and maximal Bessel operators on a finite interval and a half-line. We prove that the domain of the minimal operator $A(\nu,\infty)_{\min}$ associated with τ_ν in $L^2(\mathbb{R}_+)$ is given by

$$\mathrm{dom}(A(\nu,\infty)_{\min}) = H_0^2(\mathbb{R}_+), \tag{1.3}$$

and we prove similar formula for the operator on a finite interval.

We investigate spectral properties of the Bessel operator by applying the technique of boundary triplets and corresponding Weyl functions. This new approach to extension theory of symmetric operators developed during last three decades (see [6, 7, 12] and references in therein).

We construct a boundary triplet for the maximal operator in $L^2(\mathbb{R}_+)$ and $L^2(0,b)$ and compute the corresponding Weyl functions. We determine the domains of (Friedrichs and Krein's) extensions. In addition, all self-adjoint and all nonnegative self-adjoint extensions of the minimal Bessel operator are described. Also we obtained the Weyl functions on half-line as a limit of corresponding Weyl functions of the operator considered in the finite interval. In particular, we obtained other proofs of results of L. Bruneau, J. Dereziński and V. Georgescu [4], Everitt and Kalf [9, 14].

2. Preliminaries

2.1. Boundary triplets and self-adjoint extension

In this section we briefly review the notion of abstract boundary triplets in the extension theory of symmetric operators.

Let A be a closed densely defined symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices

$$n_\pm(A) = \dim\ker(A^* \pm iI) \le \infty.$$

Definition 2.1 ([12]). *A totality* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is called a* boundary triplet *for the adjoint operator* A^* *of* A *if* $\mathcal{H}$ *is an auxiliary Hilbert space and* $\Gamma_0, \Gamma_1 : \mathrm{dom}(A^*) \to \mathcal{H}$ *are linear mappings such that*
(*i*) *the following abstract second Green identity holds*

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}; \tag{2.1}$$

(*ii*) *the mapping* $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ *is surjective.*

First note that a boundary triplet for A^* exists since the deficiency indices of A are assumed to be equal. Moreover, $\mathrm{n}_\pm(A) = \dim(\mathcal{H})$ and $A = A^* \upharpoonright (\ker(\Gamma_0) \cap \ker(\Gamma_1))$ hold. Note also that a boundary triplet for A^* is not unique.

A closed extension $\widetilde{A}$ of A is called *proper* if $A \subseteq \widetilde{A} \subseteq A^*$. Two proper extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of A are called *disjoint* if $\mathrm{dom}(\widetilde{A}_1) \cap \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A)$ and *transversal* if in addition $\mathrm{dom}(\widetilde{A}_1) \dotplus \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A^*)$. The set of all proper extensions of A is denoted by $\mathrm{Ext}\, A$.

With a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* one associates two self-adjoint extensions $A_j := A^* \upharpoonright \ker(\Gamma_j),\ j \in \{0, 1\}$.

Proposition 2.1 ([6,12]). *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*. *Then the mapping*

$$\mathrm{Ext}_A \ni \widetilde{A} := A_\Theta \to \Theta := \Gamma(\mathrm{dom}(\widetilde{A})) = \big\{\{\Gamma_0 f, \Gamma_1 f\} :\ f \in \mathrm{dom}(\widetilde{A})\big\} \tag{2.2}$$

establishes a bijective correspondence between the set of all closed proper extensions Ext_A *of* A *and the set of all closed linear relations* $\widetilde{\mathcal{C}}(\mathcal{H})$ *in* $\mathcal{H}$. *Furthermore, the following assertions hold.*

(*i*) *The equality* $(A_\Theta)^* = A_{\Theta^*}$ *holds for any* $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$.

(*ii*) *The extension* A_Θ *in* (2.2) *is symmetric* (*self-adjoint*) *if and only if* Θ *is symmetric* (*self-adjoint*).

(*iii*) *If, in addition, extensions* A_Θ *and* A_0 *are disjoint, i.e.,* $\mathrm{dom}(A_\Theta) \cap \mathrm{dom}(A_0) = \mathrm{dom}(A)$, *then* (2.2) *takes the form*

$$A_\Theta = A_B = A^* \upharpoonright \ker\big(\Gamma_1 - B\Gamma_0\big), \quad B \in \mathcal{C}(\mathcal{H}). \tag{2.3}$$

2.2. Weyl functions and extension of nonnegative operator

It is known that the classical Weyl–Titchmarsh functions play an important role in the direct and inverse spectral theory of singular Sturm–Liouville operators. In [6] the concept of the classical Weyl–Titchmarsh m-function from the theory of Sturm–Liouville operators was generalized to the case of symmetric operators with equal deficiency indices. The role of abstract Weyl functions in the extension theory is similar to that of the classical Weyl–Titchmarsh m-function in the spectral theory of singular Sturm–Liouville operators.

Definition 2.2 ([6]). *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator valued functions $\gamma : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M : \rho(A_0) \to [\mathcal{H}]$ defined by*

$$\gamma(z) := \big(\Gamma_0 \restriction \mathfrak{N}_z\big)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \qquad z \in \rho(A_0), \quad (2.4)$$

are called the γ-field *and the* Weyl function, *respectively, corresponding to the boundary triplet* Π.

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.4) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the following relations hold (see [6])

$$\gamma(z) = \big(I + (z - \zeta)(A_0 - z)^{-1}\big)\gamma(\zeta), \quad (2.5)$$

$$M(z) - M(\zeta)^* = (z - \overline{\zeta})\gamma(\zeta)^*\gamma(z), \quad (2.6)$$

$$\gamma^*(\overline{z}) = \Gamma_1(A_0 - z)^{-1}, \qquad z,\ \zeta \in \rho(A_0). \quad (2.7)$$

Identity (2.6) yields that $M(\cdot)$ is an $R_{\mathcal{H}}$-function (or *Nevanlinna function*), that is, $M(\cdot)$ is an ($[\mathcal{H}]$-valued) holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ and

$$\operatorname{Im} z \cdot \operatorname{Im} M(z) \geq 0, \qquad M(z^*) = M(\overline{z}), \qquad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.8)$$

Besides, it follows from (2.6) that $M(\cdot)$ satisfies $0 \in \rho(\operatorname{Im} M(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Since A is densely defined, $M(\cdot)$ admits an integral representation (see, for instance, [7])

$$M(z) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) d\Sigma_M(t), \qquad z \in \rho(A_0), \quad (2.9)$$

where $\Sigma_M(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma_M(t) \in [\mathcal{H}]$ and $C_0 = C_0^* \in [\mathcal{H}]$. The integral in (2.9) is understood in the strong sense.

In contrast to spectral measures of self-adjoint operators the measure $\Sigma_M(\cdot)$ is not necessarily orthogonal. However, the measure Σ_M is uniquely determined by the Nevanlinna function $M(\cdot)$. The operator-valued measure Σ_M is called *the spectral measure* of $M(\cdot)$. If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ up to unitary equivalence (see [7]). Due to this fact, spectral properties of A_0 can be expressed in terms of $M(\cdot)$.

Assume that a symmetric operator $A \in \mathcal{C}(\mathfrak{H})$ is nonnegative. Then the set $\mathrm{Ext}_A(0,\infty)$ of its nonnegative self-adjoint extensions is non-empty (see [32]). Moreover, there is a maximal nonnegative extension A_{F} (also called *Friedrichs'* or *hard* extension) and there is a minimal nonnegative extension A_{K} (*Krein's* or *soft* extension) satisfying

$$(A_F + x)^{-1} \le (\widetilde{A} + x)^{-1} \le (A_K + x)^{-1}, \quad x \in (0,\infty), \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty)$$

(for detail we refer the reader to [32]).

The following proposition characterizes the Friedrichs and Krein extensions in terms of the Weyl function.

Proposition 2.2 ([6,7]). *Let A be a densely defined nonnegative symmetric operator with finite deficiency indices in $\mathfrak{H}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Let also $M(\cdot)$ be the corresponding Weyl function. Then the following assertions hold.*

(*i*) *Extensions A_0 and A_K are disjoint (A_0 and A_F are disjoint) if and only if*

$$M(0) \in \mathcal{C}(\mathcal{H}) \qquad (M(-\infty) \in \mathcal{C}(\mathcal{H}), \quad \textit{respectively}).$$

Moreover,

$$\begin{aligned} &\mathrm{dom}(A_K) = \mathrm{dom}(A^*) \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0) \\ &(\mathrm{dom}(A_F) = \mathrm{dom}(A^*) \upharpoonright \ker(\Gamma_1 - M(-\infty)\Gamma_0), \quad \textit{respectively}). \end{aligned}$$

(*ii*) *$A_0 = A_K$ ($A_0 = A_F$) if and only if*

$$\begin{aligned} &\lim_{x\uparrow 0}(M(x)f,f) = +\infty, \quad f \in \mathcal{H}\setminus\{0\} \\ &(\lim_{x\downarrow -\infty}(M(x)f,f) = -\infty, \quad f \in \mathcal{H}\setminus\{0\}, \quad \textit{respectively}). \end{aligned}$$

(*iii*) *The set of all non-negative self-adjoint extensions of A admits parametrization* (2.2)*, where Θ satisfies*

$$\Theta - M(0) \ge 0 \qquad (\Theta - M(-\infty) \le 0, \ \textit{respectively}). \tag{2.10}$$

2.3. Bessel functions

The general solution of the equation (1.2) is given by

$$y(x;\lambda) = c_1 x^{1/2} J_\nu(x\sqrt{\lambda}) + c_2 x^{1/2} Y_\nu(x\sqrt{\lambda}), \tag{2.11}$$

where c_1, c_2 are arbitrary constants and J_ν, Y_ν are the Bessel functions of the first and second kind, respectively (see [1, Ch. 9], [32, Appx. 2], [19, p. 284–285]). Recall that the asymptotic behavior of the Bessel functions $J_\nu(t)$ and $J_{-\nu}(t)$ for $t \to 0$ has the form

$$J_\nu(t) = \frac{t^\nu}{2^\nu \Gamma(1+\nu)}[1 + O(t^2)], \quad J_{-\nu}(t) = \frac{2^\nu}{\Gamma(1-\nu)} t^{-\nu}[1 + O(t^2)], \tag{2.12}$$

and the asymptotic behavior of the Bessel functions $Y_\nu(t)$ for $t \to 0$ has the form

$$Y_0(t) = \frac{2}{\pi}\left[\ln\left(\frac{t}{2}\right) + \gamma\right] \cdot [1+O(t^2)], \quad Y_\nu(t) = -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{t}\right)^\nu \cdot [1+O(t^2)], \tag{2.13}$$

where γ is Euler's constant.

Moreover, for $t \to \infty$ we have

$$\begin{cases} J_\nu(t) = \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(|t|^{-\frac{3}{2}}), \\ J_{-\nu}(t) = \sqrt{\frac{2}{\pi t}} \cos\left(t + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(|t|^{-\frac{3}{2}}), \\ Y_\nu(t) = \sqrt{\frac{2}{\pi t}} \sin\left(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(|t|^{-\frac{3}{2}}), \end{cases} \quad t \to \infty. \tag{2.14}$$

We use the following properties of Bessel functions [1, Formula 9.1.28]

$$J_0'(t) = -J_1(t), \qquad Y_0'(t) = -Y_1(t). \tag{2.15}$$

Also recall [32, Appx. 2] that the Bessel function Y_ν of the second kind is given by

$$Y_\nu(t) = \frac{J_\nu(t)\cos \pi\nu - J_{-\nu}(t)}{\sin \pi\nu}, \qquad \nu \neq 0. \tag{2.16}$$

Next, we need formulas (9.1.29) from [1]

$$\begin{aligned} z f_\nu'(z) &= lqz^q f_{\nu-1}(z) + (p - \nu q) f_\nu(z), \\ z f_\nu'(z) &= -lqz^q f_{\nu+1}(z) + (p + \nu q) f_\nu(z), \end{aligned} \tag{2.17}$$

in which $f_\nu(z) = z^p G_\nu(lz^q)$ where $G_\nu(\cdot)$ is one of the Bessel functions $J_\nu(\cdot)$, $Y_\nu(\cdot)$, $H_\nu^{(1)}$, $H_\nu^{(2)}$ or a linear combination, and p, q, l do not depend on ν.

Applying formula (2.17) for $l = 1$, $q = 1/2$, $p = 0$ to the functions $f_\nu = x^{1/2}G_\nu(x\sqrt{z})$ where $G_\nu(\cdot)$ is one of the Bessel functions $J_\nu(\cdot)$, $Y_\nu(\cdot)$, we obtain

$$[f_\nu, x^{1/2+\nu}]_x = \sqrt{z}x^{1/2+\nu}f_{\nu+1}, \qquad [f_{-\nu}, x^{1/2+\nu}]_x = -\sqrt{z}x^{1/2+\nu}f_{-\nu-1}, \tag{2.18}$$

and

$$[f_\nu, x^{1/2-\nu}]_x = -\sqrt{z}x^{1/2-\nu}f_{\nu-1}, \qquad [f_{-\nu}, x^{1/2-\nu}]_x = \sqrt{z}x^{1/2-\nu}f_{-\nu+1}, \tag{2.19}$$

where $[f, g](x) := f(x)\bar{g}'(x) - f'(x)\bar{g}(x)$, for all $x \in \mathbb{R}_+$.

3. Bessel operator $S(\nu; b)$ on the interval

In what follows, we need the following auxiliary lemma.

Lemma 3.1 ([17, p. 318–319]). *Let T_K be the operator in $L^p[0, \infty)$ of the form*

$$T_K : f \mapsto \int_0^\infty K(x,t)f(t)dt, \tag{3.1}$$

and its kernel $K(x,t)$ has a degree of homogeneity -1, i.e. $K(\lambda x, \lambda t) = \lambda^{-1}K(x,t)$, $(\lambda > 0)$. Then the operator T_K is bounded in $L^p[0, \infty)$ and its norm is

$$\|T_K\|_p := \|T_K\|_{L^p \to L^p} = \int_0^\infty |K(1,t)|t^{-1/p}dy. \tag{3.2}$$

Suppose further that $\mathcal{I}$ is the operator of integration, $\mathcal{I}\colon f \mapsto \int_0^x f(t)dt$. Then

$$\mathcal{I}^2 = \int_0^x (x-t)f(t)dt. \tag{3.3}$$

Also assume that $Q : f \mapsto \frac{1}{x^2}f(x)$.

Lemma 3.2. *The operator $Q\mathcal{I}^2$*

$$Q\mathcal{I}^2 : f \mapsto \frac{1}{x^2}\int_0^x (x-t)f(t)dt, \tag{3.4}$$

is bounded in $L^2[0, b]$ for each $b \in (0, \infty]$, and $\|Q\mathcal{I}^2\|_2 = \frac{4}{3}$.

Proof. Let

$$K(x,t) = \begin{cases} \frac{1}{x}\left(1-\frac{t}{x}\right), & t \leqslant x, \\ 0, & t > x. \end{cases} \tag{3.5}$$

Noting that $K(\lambda x, \lambda t) = \lambda^{-1}K(x,t)$ and applying Lemma 3.1 to the operator $T_K = Q\mathcal{I}^2$ we obtain

$$\|Q\mathcal{I}^2\|_2 = \|T_K\|_2 = \int_0^\infty |K(1,t)|t^{-1/2}dt = \int_0^1 (1-t)t^{-1/2}dt = \frac{4}{3}. \tag{3.6}$$

□

Let $H^2[0,a]$ be the Sobolev space on $[0,a]$.

Lemma 3.3. *Let* $\widetilde{H}_0^2[0,1] = \{f \in \widetilde{H}^2[0,1] : f(0) = f'(0) = 0\}$. *If* $f \in \widetilde{H}_0^2[0,1]$ *then the following relations hold:*

$$f(x) = o(x^{3/2}), \qquad f'(x) = o(x^{1/2}). \tag{3.7}$$

Proof. Since $f \in \widetilde{H}_0^2[0,1]$ then $f'(x) = \int\limits_0^x f''(t)dt$. Therefore, by the Cauchy–Bunyakovsky inequality

$$|f'(x)|^2 \leqslant \left(\int_0^x |f''(t)|dt\right)^2 \leqslant x\int_0^x |f''(t)|^2dt = x\cdot o(1) = o(x), \tag{3.8}$$

i.e. $f'(x) = o(x^{1/2})$, which proves the second estimate in (3.7).

Further, since $f \in \tilde{H}_0^2[0,1]$, we get $f(x) = \int_0^x f'(t)dt$. Hence,

$$|f(x)| \leqslant \int_0^x |f'(t)|dt \leqslant \int_0^x o(t^{1/2})dt = o(x^{3/2}) \quad \text{as } x \to 0. \tag{3.9}$$

The first estimate in (3.7) is proved. □

Let D_0^2 be a minimal differential operator of the 2nd order, generated in $L^2[0,a]$ by differential expression $-d^2/dx^2$,

$$\begin{aligned} \operatorname{dom}(D_0^2) &= H_0^2[0,a] \\ &= \{f \in H^2[0,a] : f(0) = f'(0) = f(a) = f'(a) = 0\}. \end{aligned} \tag{3.10}$$

Define by $S(\nu;b) := S(\nu;b)_{\min}$ the minimal operator generated by the differential expression (1.1) in $L^2(0,b)$ $(b<\infty)$.

Proposition 3.1. *Let $S(\nu;b)$ be the minimal Bessel operator generated by* (1.1) *in $L^2(0,b)$, $b<\infty$ for $\nu\in[0,1)$. Then the following assertions hold.*

(*i*) *The deficiency indices of $S(\nu,b)$ are $n_\pm(S(\nu;b))=2$.*

(*ii*) *The domain of the operator $S(\nu;b)$ is given by*

$$\operatorname{dom}(S(\nu;b)) = H_0^2[0,b].$$

(*iii*) $S(\nu;b)_{\max} = S(\nu;b)^*$ *and*

$$\operatorname{dom}(S(\nu;b)^*) = \begin{cases} \widetilde{H}_0^2[0,b] \dotplus \operatorname{span}\{x^{1/2+\nu}, x^{1/2-\nu}\}, & \nu\in(0,1), \\ \widetilde{H}_0^2[0,b] \dotplus \operatorname{span}\{x^{1/2}, x^{1/2}\ln(x)\}, & \nu=0. \end{cases} \tag{3.11}$$

Proof. (*i*)–(*ii*) We denote $\kappa := \nu^2 - \frac{1}{4}$ and note that

$$0\leqslant \nu<1 \Longleftrightarrow -\frac{1}{4}\leqslant\kappa<\frac{3}{4}. \tag{3.12}$$

Then κ admits the representation $\kappa=\frac{3}{4}(1-\varepsilon)$, where $\varepsilon>0$. The function $u\in\widetilde{H}_0^2[0,b]$ admits the integral representation $u(x)=\int_0^x(x-t)u''(t)dt$. Therefore,

$$Qu(x) = \frac{1}{x^2}u(x) = \frac{1}{x^2}\int_0^x (x-t)u''(t)dt = (Q\mathcal{I}^2(D_0^2u))(x). \tag{3.13}$$

By virtue of Lemma 3.2, this yields

$$\begin{aligned}\|Qu\|_{L^2} = \left\|\frac{1}{x^2}u\right\|_{L^2} &= \left\|Q\mathcal{I}^2D_0^2u\right\|_{L^2} \leqslant \|Q\mathcal{I}^2\|_2\cdot\|D_0^2u\|_{L^2} \\ &= \frac{4}{3}\|D_0^2u\|_{L^2}\leqslant\frac{4}{3}\|u\|_{H_0^2[0,b]}. \end{aligned}\tag{3.14}$$

Since κ admits the representation $\kappa=\frac{3}{4}(1-\varepsilon)$ with $\varepsilon>0$, relation (3.14) implies the estimate

$$\begin{aligned}\|\kappa Qu\|_{L^2} = |\kappa|\cdot\|Qu\|_{L^2} &\leqslant \frac{3}{4}(1-\varepsilon)\cdot\frac{4}{3}\|u\|_{H_0^2[0,b]} \\ &= (1-\varepsilon)\|u\|_{H_0^2[0,b]}, \quad u\in H_0^2[0,b]. \end{aligned}\tag{3.15}$$

Estimate (3.15) means that Q is strongly D_0^2-bounded. Therefore, by the Kato–Rellich theorem (see [15]) $n_\pm(S(\nu;b)) = n_\pm(D_0^2) = 2$ and $\operatorname{dom}(S(\nu;b)) = H_0^2[0,b]$.

(*iii*) Since
$$\tau_\nu^{1/2\pm\nu} = 0,$$
where the equality is understood in the meaning of the theory of distributions, and $x^{1/2\pm\nu} \in L^2(0,b)$, then
$$\{x^{1/2+\nu}, x^{1/2-\nu}\} \subset \mathrm{dom}(S(\nu;b)_{\max}) = \mathrm{dom}(S(\nu;b)^*),$$
and $\ker(S(\nu;b)^*) = \{x^{1/2+\nu}, x^{1/2-\nu}\} \subset L^2(0,b)$. In addition, it is clear that $\widetilde{H}_0^2[0,b] \subset \mathrm{dom}(S(\nu;b)^*)$ and $\dim(\widetilde{H}_0^2[0,b])/\mathrm{dom}(S(\nu;b))) = 2$. On the other hand, since $n_\pm(S(\nu;b)) = 2$, we have $\dim(\mathrm{dom}(S(\nu;b)^*)/\mathrm{dom}(S(\nu;b))) = 2n_\pm(S(\nu;b)) = 4$ by the first Neumann formula. Therefore, formula (3.11) is valid. □

Consider the quadratic form $\mathfrak{s}'(\nu;b)$ associated with the operator $S(\nu;b)$,
$$\mathfrak{s}'(\nu;b)[u] := (S(\nu;b)u,u), \qquad \mathrm{dom}(\mathfrak{s}'(\nu;b)) = \mathrm{dom}(S(\nu;b)) = H_0^2[0,b]. \tag{3.16}$$
It is clear that $S(1/2;b) = -D_0^2$.

Proposition 3.2. (*i*) *Let $\nu \in [0,1)$. The closure $\mathfrak{s}(\nu;b)$ of the quadratic form $\mathfrak{s}'(\nu;b)$ is given by*
$$\mathfrak{s}(\nu;b) = \mathfrak{s}(1/2;b) + \kappa\mathfrak{q}, \qquad \mathrm{dom}(\mathfrak{s}(\nu;b)) = H_0^1[0,b], \tag{3.17}$$
where
$$\mathfrak{s}(1/2;b)[u] = \int_0^b |u'(x)|^2 dx, \qquad \mathfrak{q}[u] = \int_0^b \frac{|u(x)|^2}{x^2} dx. \tag{3.18}$$

(*ii*) *The domain of the Friedrichs extension $S_F(\nu;b)$ of the operator $S(\nu;b)$ for $\nu \in [0,1)$ takes the form*
$$\mathrm{dom}(S_F(\nu;b)) = \begin{cases} H_0^2[0,b] \dotplus \mathrm{span}\{x^{1/2+\nu}(x-b), x^2(x-b)\}, & \nu \in (0,1), \\ H_0^2[0,b] \dotplus \mathrm{span}\{x^{1/2}(x-b), x^{1/2}\ln x(x-b)\}, & \nu = 0. \end{cases} \tag{3.19}$$

Proof. (*i*) By Hardy's inequality
$$\mathfrak{s}(\nu;b)[u] = \|u'(t)\|^2_{L^2(0,b)} + (\nu^2 - 1/4)\int_0^b \frac{|u(t)|^2}{t^2} dt$$

$$\leqslant \|u'(t)\|^2_{L^2(0,b)}(1+|4\nu^2-1|), \qquad u\in H^1_0[0,b]. \quad (3.20)$$

Thus $H^1_0[0,b]\subset \operatorname{dom}(\mathfrak{s}(\nu;b))$.

We prove the converse inequality. Suppose first that $\nu\in[1/2,1)$. Then

$$\mathfrak{s}(\nu;b)[u]=\|u'(t)\|^2_{L^2(0,b)}+(\nu^2-1/4)\int_0^b\frac{|u(t)|^2}{t^2}dt\geqslant \|u'(t)\|^2_{L^2(0,b)} \quad (3.21)$$

for $u\in H^1_0[0,b]$.

If $\nu\in(0,1/2)$, then for $u\in H^1_0[0,b]$

$$\mathfrak{s}(\nu;b)[u]=\|u'(t)\|^2_{L^2(0,b)}-(1/4-\nu^2)\int_0^b\frac{|u(t)|^2}{t^2}dt$$
$$\geqslant \|u'(t)\|^2_{L^2(0,b)}+(4\nu^2-1)\|u'(t)\|^2_{L^2(0,b)}=4\nu^2\|u'(t)\|^2_{L^2(0,b)}. \quad (3.22)$$

So on $H^1_0[0,b]$ the energy norm of $S(\nu;b)$ is equivalent to the norm of space $H^1_0[0,b]$. Since $H^2_0[0,b]=\operatorname{dom}(S(\nu;b))$ is dense in the energy space of the operator $S(\nu;b)$, then $\operatorname{dom}(\mathfrak{s}(\nu;b))$ and $H^1_0[0,b]$ coincide algebraically and topologically.

(ii) We note that $H^2_0[0,b]\subset H^1_0[0,b]$. If $u(x)=x^{1/2+\nu}(x-b)$ then $u'\in L^2(0,b)$, but $u(\cdot)\notin \operatorname{dom}(S(\nu;b))=H^2_0[0,b]$. By the construction of the Friedrichs extension and the equalities (3.11), we obtain

$$\operatorname{dom}(S_F(\nu;b))=\operatorname{dom}(S(\nu;b)^*)\cap\operatorname{dom}(\mathfrak{s}(\nu;b))=\operatorname{dom}(S(\nu;b)^*)$$
$$\cap H^1_0[0,b]=H^2_0[0,b]\dotplus\operatorname{span}\{x^{1/2+\nu}(x-b),x^2(x-b)\}.$$

The case $\nu=0$ is considered similarly. □

The case $\nu\in[0,1/\sqrt{2})$ in Proposition (3.2) can be treated by means of KLMN-theorem. Indeed, since $\nu\in[0,1/\sqrt{2})$ then $\kappa<1/4$. Therefore, applying Hardy inequality one gets

$$|\kappa\mathfrak{q}[u]|=\int_0^b\frac{|\kappa|}{x^2}|u|^2dx\leqslant 4|\kappa|\int_0^b|u'|^2dx\leqslant(1-\varepsilon)\mathfrak{t}_{D^2_0}[u], \quad u\in H^1_0[0,b]. \quad (3.23)$$

Hence, the form $\kappa\mathfrak{q}$ is strongly $\mathfrak{t}_{D^2_0}$-bounded. Therefore, KLMN-theorem [15] yields $\operatorname{dom}(\mathfrak{s}(\nu;b))=\operatorname{dom}(\mathfrak{t}_{D^2_0})=H^1_0[0,b]$.

4. Bessel operator $A(\nu;b)$ on the interval

Here consider the Bessel operator $A(\nu;b)$ generated by the expression (1.1) in $L^2(0,b)$, $b<\infty$ with the domain

$$\operatorname{dom}(A(\nu;b)) = \{f\in H^2[0,b] : f(0)=f'(0)=f(b)=0\},\quad \nu\in[0,1). \tag{4.1}$$

Proposition 4.1. *(i) The operator $A(\nu;b)$ has equal deficiency indices $n_\pm(A(\nu;b))=1$;*

(ii) and $\operatorname{dom}(A(\nu;b)^*) = \{f\in H^2[0,b] : f(b)=0\}$.

Proof. It is easily seen that $S(\nu;b)\subset A(\nu;b)\subset S(\nu;b)^*$ and

$$\dim(\operatorname{dom}(A(\nu;b))/\operatorname{dom}(S(\nu;b)))=1.$$

But by Proposition 3.1 $n_\pm(S(\nu;b))=2$. Hence, by the second Neumann formula implies $n_\pm(A(\nu;b))=1$. □

Proposition 4.2. *Let $A(\nu;b)$ be the Bessel operator generated by the expression* (1.1) *in $L^2(0,b)$, $b<\infty$ for $\nu\in[0,1)$ with the domain* (4.1). *Then*

(i) Boundary triplet of the operator $A(\nu;b)^*$ can be selected in the form of

$$\mathcal{H}=\mathbb{C},\quad \Gamma_0^{\nu;b}f=[f,x^{\frac12+\nu}]_0,$$
$$\Gamma_1^{\nu;b}f=\begin{cases}-(2\nu)^{-1}[f,x^{\frac12-\nu}]_0, & \nu\in(0,1),\\ [f,x^{\frac12}\ln x]_0, & \nu=0.\end{cases} \tag{4.2}$$

(ii) The corresponding Weyl function $M_{\nu;b}(\cdot)$ is

$$M_{\nu;b}(z)=\begin{cases}-\frac{\Gamma(1-\nu)}{2\nu 4^\nu\Gamma(1+\nu)}\cdot\frac{J_{-\nu}(b\sqrt{z})}{J_\nu(b\sqrt{z})}\cdot z^\nu, & \nu\in(0,1),\\ -\ln\frac{\sqrt{z}}{2}+\frac{\pi}{2}\frac{Y_0(b\sqrt{z})}{J_0(b\sqrt{z})}-\gamma, & \nu=0,\end{cases} \tag{4.3}$$

where γ is Euler's constant.

Proof. (*i*) Let $f,g\in\operatorname{dom}(A(\nu;b)^*)$. Integrating by parts, we obtain

$$(A(\nu;b)^*f,g)-(f,A(\nu;b)^*g)$$
$$=\lim_{\varepsilon\to0}\Bigg[\int_\varepsilon^b\left(-f''(x)\overline{g(x)}+\frac{\nu^2-\frac14}{x^2}f(x)\right)\overline{g(x)}dx$$

$$- \int_{\varepsilon}^{b} f(x) \left(\overline{-g''(x)} + \frac{\nu^2 - \frac{1}{4}}{x^2} \overline{g(x)} \right) dx \Bigg] \\ = \lim_{\varepsilon \to 0} \left\{ -f(\varepsilon)\overline{g'(\varepsilon)} + f'(\varepsilon)\overline{g(\varepsilon)} \right\}.$$

On the other hand it is easily seen that

$$\begin{aligned}
&(\Gamma_1^{\nu;b} f, \Gamma_0^{\nu;b} g) - (\Gamma_0^{\nu;b} f, \Gamma_1^{\nu;b} g) \\
&\quad = \frac{1}{2\nu} \lim_{x\to 0} \left[\left(\left(\frac{1}{2} + \nu \right) x^{\nu - \frac{1}{2}} f(x) - x^{\frac{1}{2}+\nu} f'(x) \right) \right. \\
&\qquad \times \left(\left(\frac{1}{2} - \nu \right) x^{-\frac{1}{2}-\nu} \overline{g(x)} - x^{\frac{1}{2}-\nu} \overline{g'(x)} \right) \\
&\qquad - \left(\left(\frac{1}{2} - \nu \right) x^{-\nu-\frac{1}{2}} f(x) - x^{\frac{1}{2}-\nu} f'(x) \right) \\
&\qquad \left. \times \left(\left(\frac{1}{2} + \nu \right) x^{-\frac{1}{2}+\nu} \overline{g(x)} - x^{\frac{1}{2}+\nu} \overline{g'(x)} \right) \right] \\
&\qquad = \frac{1}{2\nu} \lim_{x\to 0} 2\nu (f'(x)\overline{g(x)} - f(x)\overline{g'(x)}) \\
&\qquad\qquad = \lim_{x\to 0} \{ -f(x)\overline{g'(x)} + f'(x)\overline{g(x)}) \}.
\end{aligned}$$

Comparing this formula with the previous one we obtain the Green's formula

$$(A(\nu;b)^* f, g) - (f, A(\nu;b)^* g) = (\Gamma_1^{\nu;b} f, \Gamma_0^{\nu;b} g) - (\Gamma_0^{\nu;b} f, \Gamma_1^{\nu;b} g).$$

The case $\nu = 0$ is considered similarly.

$(ii.1)$ First we consider the case $\nu \in (0,1)$.

By the asymptotic relations (2.12) $x^{1/2} J_\nu \in L^2(0,b)$ and $x^{1/2} J_{-\nu} \in L^2(0,b)$. Therefore

$$f_z(x) := x^{\frac{1}{2}} \left(J_\nu(x\sqrt{z}) - \frac{J_\nu(b\sqrt{z})}{J_{-\nu}(b\sqrt{z})} J_{-\nu}(x\sqrt{z}) \right) \in L^2(0,b). \tag{4.4}$$

It is easily seen that $f_z(b) = 0$, and hence $f_z \in \operatorname{dom}(A(\nu;b)^*)$ and $(A(\nu;b)^* - z) f_z = 0$. In other words, deficient space $\mathfrak{N}_z(A(\nu;b))$ of the operator $A(\nu;b)$ generated by the vector f_z.

Using the asymptotic behavior of the Bessel functions (2.12) and formula (2.18) we obtain

$$\left[x^{1/2} J_\nu(x\sqrt{z}), x^{1/2+\nu} \right]_0 = \lim_{x\to 0} \left[z^{1/2} x^{1+\nu} J_{\nu+1}(x\sqrt{z}) \right]$$

$$= \lim_{x\to 0}\left[\frac{z^{1+\nu/2}x^{2(1+\nu)}}{2^{1+\nu}\Gamma(2+\nu)}(1+O(x^2z))\right] = 0,$$

$$\begin{aligned}\left[x^{1/2}J_{-\nu}(x\sqrt{z}), x^{1/2+\nu}\right]_0 &= \lim_{x\to 0}\left[-z^{1/2}x^{1+\nu}J_{-\nu-1}(x\sqrt{z})\right]\\ &= \lim_{x\to 0}\left[-\frac{z^{-\nu/2}2^{1+\nu}}{\Gamma(-\nu)}(1+O(x^2z))\right] = -\frac{z^{-\nu/2}2^{1+\nu}}{\Gamma(-\nu)}.\end{aligned}\qquad(4.5)$$

Similarly, using the asymptotic behavior of the Bessel functions (2.12) and formula (2.19) we obtain

$$\begin{aligned}\left[x^{1/2}J_{\nu}(x\sqrt{z}), x^{1/2-\nu}\right]_0 &= -\lim_{x\to 0}\left[z^{1/2}x^{1-\nu}J_{\nu-1}(x\sqrt{z})\right]\\ &= -\lim_{x\to 0}\left[\frac{z^{\nu/2}}{2^{\nu-1}\Gamma(\nu)}(1+O(x^2z))\right] = -\frac{z^{\nu/2}}{2^{\nu-1}\Gamma(\nu)},\end{aligned}\qquad(4.6)$$

$$\begin{aligned}\left[x^{1/2}J_{-\nu}(x\sqrt{z}), x^{1/2-\nu}\right]_0 &= -\lim_{x\to 0}\left[z^{1/2}x^{1-\nu}J_{-(\nu-1)}(x\sqrt{z})\right]\\ &= -\lim_{x\to 0}\left[\frac{z^{-\nu/2+1}2^{\nu-1}x^{2(1-\nu)}}{\Gamma(2-\nu)}(1+O(x^2z))\right] = 0.\end{aligned}$$

From the formula (4.2), (4.4) and (4.5), (4.6) we arrive to the relation

$$\Gamma_0^{\nu;b}f_z = \frac{2^{1+\nu}}{\Gamma(-\nu)}\cdot\frac{J_\nu(b\sqrt{z})}{J_{-\nu}(b\sqrt{z})}\cdot z^{-\frac{\nu}{2}};\qquad \Gamma_1^{\nu;b}f_z = \frac{1}{\nu 2^\nu\Gamma(\nu)}\cdot z^{\frac{\nu}{2}}.\qquad(4.7)$$

Hence, by (4.7) and Definition 2.2 follows fist part of the formula (4.3).

($ii.2$) The case $\nu = 0$.

By the asymptotic relations (2.12) and (2.13) $x^{1/2}J_0 \in L^2(0,b)$ and $x^{1/2}Y_0 \in L^2(0,b)$. Therefore

$$f_z(x) := x^{\frac{1}{2}}\left(J_0(x\sqrt{z}) - \frac{Y_0(b\sqrt{z})}{Y_0(b\sqrt{z})}J_0(x\sqrt{z})\right) \in L^2(0,b).\qquad(4.8)$$

It is easily seen that $f_z(b) = 0$, and hence $f_z \in \operatorname{dom}(A(0;b)^*)$ and $(A(0;b)^* - z)f_z = 0$. In other words, deficiency space $\mathfrak{N}_z(A(0;b))$ of the operator $A(0;b)$ generated by the vector f_z.

Using the asymptotic behavior of the Bessel functions (2.12) and formula (2.18) we obtain

$$\left[x^{1/2}J_0(x\sqrt{z}), x^{1/2}\right]_0 = \lim_{x\to 0}\left[xz^{1/2}J_1(x\sqrt{z})\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2 z}{2}(1+O(x^2 z))\right] = 0, \quad (4.9)$$

$$\left[x^{1/2}Y_0(x\sqrt{z}), x^{1/2+\nu}\right]_0 = \lim_{x\to 0}\left[xz^{1/2}Y_1(x\sqrt{z})\right]$$
$$= \lim_{x\to 0}\left[-x\sqrt{z}\cdot\frac{2}{\pi\cdot x\sqrt{z}}(1+O(x^2 z))\right] = -\frac{2}{\pi}.$$

Similarly, using the asymptotic behavior of the Bessel functions (2.12), (2.13) and formula (2.15) we obtain

$$\left[x^{1/2}J_0(x\sqrt{z}), x^{1/2}\ln(x)\right]_0 = \lim_{x\to 0}\left[J_0(x\sqrt{z}) + x\ln(x)\cdot\sqrt{z}J_1(x\sqrt{z})\right]$$
$$= \lim_{x\to 0}\left[\left(1+\frac{x^2\ln(x)}{2}z\right)(1+O(x^2 z))\right] = 1, \quad (4.10)$$

$$\left[x^{1/2}Y_0(x\sqrt{z}), x^{1/2}\ln(x)\right]_0 = \lim_{x\to 0}\left[Y_0(x\sqrt{z}) + x\ln(x)\cdot\sqrt{z}Y_1(x\sqrt{z})\right]$$
$$= \lim_{x\to 0}\left[\frac{2}{\pi}\left[\ln\left(\frac{x\sqrt{z}}{2}\right)+\gamma\right]\frac{2}{\pi}\ln(x)(1+O(x^2 z))\right]$$
$$= \frac{2}{\pi}\left[\ln\left(\frac{\sqrt{z}}{2}\right)+\gamma\right].$$

From the formula (4.2), (4.8) and (4.9), (4.10) we arrive to the relation

$$\Gamma_0^{0;b} f_z = \frac{2}{\pi}\cdot\frac{J_0(b\sqrt{z})}{Y_0(b\sqrt{z})}; \qquad \Gamma_1^{0;b} f_z = 1 - \frac{2}{\pi}\cdot\frac{J_0(b\sqrt{z})}{Y_0(b\sqrt{z})}\left[\ln\left(\frac{\sqrt{z}}{2}\right)+\gamma\right]. \quad (4.11)$$

Hence, by (4.11) and Definition 2.2 follows the second part of the formula (4.3). □

Proposition 4.3. *Assume $\nu \in [0,1)$. Let $\Pi_{\nu;b} = \{\mathcal{H}, \Gamma_0^{\nu;b}, \Gamma_1^{\nu;b}\}$ be a boundary triplet of the form (4.2) for the operator $A(\nu;b)^*$ and $M_{\nu;b}(\cdot)$ is the corresponding Weyl function. Then*

(i) *The domain of the Friedrichs extension $A_F(\nu;b)$ of the operator $A(\nu;b)$ has the form*

$$\operatorname{dom}(A_F(\nu;b)) = \ker(\Gamma_0^{\nu;b}) = \left\{f\in\operatorname{dom}(A(\nu;b)^*) : [f, x^{\frac{1}{2}+\nu}]_0 = 0\right\}. \quad (4.12)$$

(*ii*) *The domain of the Krein extension* $A_K(\nu;b)$ *of the operator* $A(\nu;b)$ *has the form*

$$\operatorname{dom}(A_K(\nu;b)) = \begin{cases} \left\{f\in\operatorname{dom}(A(\nu;b)^*) : (2\nu)^{-1}[f, \tfrac{x^{1/2+\nu}}{b^{2\nu}} - x^{1/2-\nu}]_0 = 0\right\}, \\ \hfill \nu\in(0,1), \\ \left\{f\in\operatorname{dom}(A(0;b)^*) : [f, x^{1/2}\ln(x)]_0 = \ln(b)[f,x^{1/2}]_0\right\}, \\ \hfill \nu = 0. \end{cases} \tag{4.13}$$

Proof. (*i*) First we consider the case $\nu\in(0,1)$.

Using the asymptotic behavior of the Bessel functions (2.14), we obtain

$$\begin{aligned} M_{\nu;b}(-\infty)\cdot\frac{2\nu 4^\nu\Gamma(1+\nu)}{\Gamma(1-\nu)} &= \frac{2\nu 4^\nu\Gamma(1+\nu)}{\Gamma(1-\nu)}\lim_{z\to-\infty} M_{\nu;b}(z) \\ &= -\lim_{z\to-\infty}\cdot\frac{J_{-\nu}(b\sqrt{z})}{J_\nu(b\sqrt{z})}\cdot z^\nu = -\lim_{x\to+\infty}\frac{J_{-\nu}(ib\sqrt{-x})}{J_\nu(ib\sqrt{-x})}\cdot(-x)^\nu \\ &= -\lim_{x\to+\infty}\left[\frac{\cos\left(ib\sqrt{-x}+\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}{\cos\left(ib\sqrt{-x}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\cdot(-x)^\nu\right] \\ &= -\lim_{x\to+\infty}\left[(-x)^\nu\cdot\frac{e^{-i(ib\sqrt{-x}+\frac{\nu\pi}{2}-\frac{\pi}{4})}+o(1)}{e^{-i(ib\sqrt{-x}-\frac{\nu\pi}{2}-\frac{\pi}{4})}+o(1)}\right] \\ &= -e^{-i\nu\pi}\lim_{x\to+\infty}(-x)^\nu = -\frac{e^{i\nu\pi}}{e^{i\nu\pi}}\lim_{x\to+\infty}x^\nu \\ &= -\lim_{x\to+\infty}x^\nu = -\infty. \end{aligned}$$

The case $\nu=0$.

Using the asymptotic behavior of the Bessel functions (2.14), we obtain

$$\begin{aligned} M_{0;b}(-\infty) &= \lim_{z\to-\infty}M_{0;b}(z) = \lim_{z\to-\infty}\left[-\ln\frac{\sqrt{z}}{2}+\frac{\pi}{2}\frac{Y_0(b\sqrt{z})}{J_0(b\sqrt{z})}-\gamma\right] \\ &= \lim_{x\to+\infty}\left[-\ln\frac{i\sqrt{x}}{2}+\frac{\pi}{2}\frac{Y_0(ib\sqrt{x})}{J_0(ib\sqrt{x})}-\gamma\right] \\ &= \lim_{x\to+\infty}\left[-\frac{\pi}{2}i-\ln(\sqrt{x})+\frac{\pi}{2}\cdot\frac{\sin(bi\sqrt{x}-\frac{\pi}{4})}{\cos(bi\sqrt{x}-\frac{\pi}{4})}-\gamma\right] \\ &= \lim_{x\to+\infty}\left[-\frac{\pi}{2}i-\ln(\sqrt{x})+\frac{\pi}{2}\cdot i-\gamma\right] = -\infty \end{aligned}$$

So by the Proposition 2.2 the relation (4.12) is valid.

($ii.1$) First we consider the case $\nu \in (0,1)$.

From (4.3) taking into account the asymptotics of the Bessel function (2.12), we obtain

$$M_{\nu;b}(0) = \lim_{z\to -0} M_{\nu;b}(z) = \lim_{z\to -0}\left[-\frac{\Gamma(1-\nu)}{2\nu 4^{\nu}\Gamma(1+\nu)}\cdot\frac{J_{-\nu}(b\sqrt{z})}{J_{\nu}(b\sqrt{z})}\cdot z^{\nu}\right]$$
$$= -\lim_{z\to -0}\left[\frac{\Gamma(1-\nu)}{2\nu\Gamma(1+\nu)4^{\nu}}\cdot\frac{\Gamma(1+\nu)4^{\nu}}{\Gamma(1-\nu)}\cdot b^{-2\nu}z^{-\nu}z^{\nu}\right] = -\frac{b^{-2\nu}}{2\nu}.$$

The first part of the relation (4.13) follows from the Proposition 2.2.

($ii.2$) The case $\nu = 0$.

$$M_{0;b}(0) = \lim_{z\to -0} M_{0;b}(z) = \lim_{z\to -0}\left[-\ln\frac{\sqrt{z}}{2} + \frac{\pi}{2}\frac{Y_0(b\sqrt{z})}{J_0(b\sqrt{z})} - \gamma\right]$$
$$= \lim_{z\to -0}\left[-\ln\frac{\sqrt{z}}{2} + \frac{\pi}{2}\cdot\frac{2}{\pi}\left(\ln\frac{b\sqrt{z}}{2} + \gamma\right) - \gamma\right] = \ln(b).$$

The second part of the relation (4.13) follows from the Proposition 2.2. □

Remark 4.1. Note that for $\nu \in (0,1)$ the solution $x^{1/2+\nu} \in \mathrm{dom}(A_F(\nu;b))$, while the solution $x^{1/2-\nu} \notin \mathrm{dom}(A_F(\nu;b))$, so $x^{1/2+\nu}$ is the principal solution at 0 (see [8, Def. 11.5]). Similarly, for $\nu = 0$ the solution $x^{1/2}$ is the principal solution at 0, while $x^{1/2}\ln x$ is not.

Proof. Indeed,

$$[x^{1/2+\nu}, x^{1/2-\nu}]_0 = \lim_{x\to 0}\left\{\left(\frac{1}{2}-\nu\right)x^{1/2+\nu}x^{-1/2-\nu} - \left(\frac{1}{2}+\nu\right)x^{1/2-\nu}x^{-1/2+\nu}\right\} = -2\nu \neq 0.$$

Therefore, by Proposition 4.3 $x^{1/2-\nu} \notin \mathrm{dom}(A_F(\nu;b))$.

The case $\nu = 0$ is considered similarly. □

5. The Bessel operator $A(\nu;\infty)$ on the half-line

Here consider the minimal Bessel operator $A(\nu;\infty)$ generated by the expression (1.1) in $L^2(\mathbb{R}_+)$ for $\nu \in [0,1)$.

Proposition 5.1. *Let $A(\nu;\infty)$ be the minimal Bessel operator generated by the expression* (1.1) *in $L^2(\mathbb{R}_+)$ for $\nu \in [0,1)$. Then the following assertions hold.*

(*i*) *The operator* $A(\nu;\infty)$ *has equal deficiency indices* $n_{\pm}(A(\nu;\infty)) = 1$;

(*ii*) *The domain of the operator* $A(\nu;\infty)$ *is given by*

$$\mathrm{dom}(A(\nu;\infty)) = H_0^2(\mathbb{R}_+); \tag{5.1}$$

(*iii*) $A(\nu;\infty)_{\max} = A(\nu;\infty)^*$ *and*

$$\mathrm{dom}(A(\nu;\infty)^*) = \begin{cases} H_0^2(\mathbb{R}_+) \dot{+} \mathrm{span}\{x^{1/2+\nu}\varphi(x), x^{1/2-\nu}\psi(x)\}, & \nu \in (0,1), \\ H_0^2(\mathbb{R}_+) \dot{+} \mathrm{span}\{x^{1/2}\varphi(x), x^{1/2}\ln(x)\psi(x)\}, & \nu = 0, \end{cases} \tag{5.2}$$

where $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$.

Proof. (*i*) See [19, p. 284].

The statements (*ii*)–(*iii*) follow from the proof of the corresponding statements (*ii*)–(*iii*) of Proposition 3.1. □

Remark 5.1. Note that it is proved in [4] that the functions $f \in \mathrm{dom}(A(\nu,\infty))$ satisfy conditions (3.7) while the statement (5.1) was not obtained. This follows from (5.1) and Lemma 3.3.

Next we compute the Weyl function and the corresponding spectral function of the operator $A(\nu,\infty)$ using the boundary triplet technique.

Proposition 5.2. *Let* $A(\nu;\infty)$ *be the Bessel operator generated by the expression* (1.1) *in* $L^2(\mathbb{R}_+)$ *for* $\nu \in [0,1)$ *with the domain* (5.1). *Then*

(*i*) *The boundary triplet of the operator* $A(\nu;\infty)^*$ *can be chosen as*

$$\mathcal{H} = \mathbb{C}, \quad \Gamma_0^{\nu;\infty} f = [f, x^{\frac{1}{2}+\nu}]_0,$$
$$\Gamma_1^{\nu;\infty} f = \begin{cases} -(2\nu)^{-1}[f, x^{\frac{1}{2}-\nu}]_0, & \nu \in (0,1), \\ [f, x^{\frac{1}{2}}\ln x]_0, & \nu = 0; \end{cases} \tag{5.3}$$

(*ii*) *The corresponding Weil function* $M_{\nu;\infty}(\cdot)$ *has the form:*

$$M_{\nu;\infty}(z) = \begin{cases} e^{i(1-\nu)\pi}\frac{\Gamma(1-\nu)}{2\nu 4^\nu \Gamma(1+\nu)} z^\nu, & \nu \in (0,1), \\ -\ln\left(\frac{\sqrt{z}}{2}\right) + \frac{i\pi}{2} - \gamma, & \nu = 0, \end{cases} \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \tag{5.4}$$

where γ *is Euler's constant.*

(*iii*) *Spectral function* $\Sigma_\nu(t)$ *of the operator* $A(\nu;\infty)_0 = A(\nu;\infty)^* \restriction \ker\Gamma_0^{\nu;\infty}$ *is given by*

$$\Sigma_\nu(t) = \begin{cases} C_\nu \frac{t^{\nu+1}}{\nu+1}, & \nu \in (0,1), \\ t/2, & \nu = 0, \end{cases} \tag{5.5}$$

where

$$C_\nu = \frac{1}{2\pi}\frac{\Gamma(1-\nu)\sin((1-\nu)\pi)}{\nu 4^\nu \Gamma(1+\nu)}. \tag{5.6}$$

Proof. (*i*) Let $f, g \in \operatorname{dom}(A(\nu;\infty)^*)$. Integrating by parts we obtain

$$\begin{aligned}
&(A(\nu;\infty)^* f, g) - (f, A(\nu;\infty)^* g) \\
&\quad = \lim_{\varepsilon\to 0}\Bigg[\int_\varepsilon^\infty \left(-f''(x)\overline{g(x)} + \frac{\nu^2-\frac14}{x^2} f(x)\right)\overline{g(x)}dx \\
&\qquad - \int_\varepsilon^\infty f(x)\left(\overline{-g''(x)} + \frac{\nu^2-\frac14}{x^2}\overline{g(x)}\right)dx\Bigg] \\
&\qquad\qquad = \lim_{\varepsilon\to 0}\left\{-f(\varepsilon)\overline{g'(\varepsilon)} + f'(\varepsilon)\overline{g(\varepsilon)}\right\}.
\end{aligned}$$

On the other hand it is easily seen that

$$\begin{aligned}
&(\Gamma_1^{\nu;\infty} f, \Gamma_0^{\nu;\infty} g) - (\Gamma_0^{\nu;\infty} f, \Gamma_1^{\nu;\infty} g) \\
&\quad = \frac{1}{2\nu}\lim_{x\to 0}\Bigg[\left(\left(\frac12+\nu\right)x^{\nu-\frac12}f(x) - x^{\frac12+\nu}f'(x)\right) \\
&\qquad \times \left(\left(\frac12-\nu\right)x^{-\frac12-\nu}\overline{g(x)} - x^{\frac12-\nu}\overline{g'(x)}\right) \\
&\qquad - \left(\left(\frac12-\nu\right)x^{-\nu-\frac12}f(x) - x^{\frac12-\nu}f'(x)\right) \\
&\qquad \times \left(\left(\frac12+\nu\right)x^{-\frac12+\nu}\overline{g(x)} - x^{\frac12+\nu}\overline{g'(x)}\right)\Bigg] \\
&\qquad = \frac{1}{2\nu}\lim_{x\to 0} 2\nu(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}) \\
&\qquad\qquad = \lim_{x\to 0}\{-f(x)\overline{g'(x)} + f'(x)\overline{g(x)})\}.
\end{aligned}$$

Comparing this formula with the previous one we obtain the Green's formula

$$(A(\nu;\infty)^* f, g) - (f, A(\nu;\infty)^* g) = (\Gamma_1^{\nu;\infty} f, \Gamma_0^{\nu;\infty} g) - (\Gamma_0^{\nu;\infty} f, \Gamma_1^{\nu;\infty} g).$$

The case $\nu = 0$ is considered similarly.

$(ii.1)$ First we consider the case $\nu \in (0, 1)$.

By the asymptotic relations (2.12) and (2.13) $x^{1/2}J_\nu \in L^2(\mathbb{R}_+)$ and $x^{1/2}Y_\nu \in L^2(\mathbb{R}_+)$. Therefore

$$f_z(x) = x^{\frac{1}{2}} \left\{ J_\nu(x\sqrt{z}) + iY_\nu(x\sqrt{z}) \right\} \in L^2(\mathbb{R}_+). \tag{5.7}$$

It is easily seen that $\lim_{x\to\infty} f_z(x) = 0$, and so $f_z \in \mathrm{dom}(A(\nu;\infty)^*)$ and $(A(\nu;\infty)^* - z)f_z = 0$. In other words, deficiency space $\mathfrak{N}_z(A(\nu;\infty))$ of the operator $A(\nu;\infty)$ generated by the vector f_z.

Using the asymptotic behavior of the Bessel functions (2.12) and formula (2.18) we obtain

$$\begin{aligned}
&\left[x^{1/2}Y_\nu(x\sqrt{z}), x^{1/2+\nu}\right]_0 \\
&\quad= \left[x^{1/2}\frac{J_\nu(x\sqrt{z})\cos(\nu\pi) - J_{-\nu}(x\sqrt{z})}{\sin(\nu\pi)}, x^{1/2+\nu}\right]_0 \\
&\qquad= -\frac{\nu 2^{1+\nu}}{\sin(\nu\pi)\Gamma(1-\nu)} \cdot z^{-\nu/2}.
\end{aligned} \tag{5.8}$$

Similarly, using the asymptotic behavior of the Bessel functions (2.12) and formula (2.19) we obtain

$$\begin{aligned}
&\left[x^{1/2}Y_\nu(x\sqrt{z}), x^{1/2-\nu}\right]_0 \\
&\quad= \left[x^{1/2}\frac{J_\nu(x\sqrt{z})\cos(\nu\pi) - J_{-\nu}(x\sqrt{z})}{\sin(\nu\pi)}, x^{1/2-\nu}\right]_0 \\
&\qquad= -\frac{\nu\cos(\nu\pi)}{\sin(\nu\pi)2^{\nu-1}\Gamma(1+\nu)} \cdot z^{\nu/2}.
\end{aligned} \tag{5.9}$$

From the formula (4.5), (4.6), (5.3), (5.7) and (5.8), (5.9) we arrive to the relation

$$\Gamma_0^{\nu;\infty} f_z = -\frac{i\nu 2^{\nu+1}}{\sin(\nu\pi)\Gamma(1-\nu)} \cdot z^{-\nu/2}; \tag{5.10}$$

$$\Gamma_1^{\nu;\infty} f_z = \left(1 + i\frac{\cos(\nu\pi)}{\sin(\nu\pi)}\right) \cdot \frac{z^{\frac{\nu}{2}}}{2^\nu\Gamma(1+\nu)} = \frac{e^{i\pi(1-\nu)}}{i\sin(\nu\pi)} \cdot \frac{z^{\frac{\nu}{2}}}{2^\nu\Gamma(1+\nu)}. \tag{5.11}$$

Hence, by (5.10), (5.11) and Definition 2.2 follows first part of the formula (5.4).

$(ii.2)$ The case $\nu = 0$.

By the asymptotic relations (2.12) and (2.13) $x^{1/2}J_0 \in L^2(\mathbb{R}_+)$ and $x^{1/2}Y_0 \in L^2(\mathbb{R}_+)$. Therefore

$$f_z(x) = x^{\frac{1}{2}} \left\{ J_0(x\sqrt{z}) + iY_0(x\sqrt{z}) \right\} \in L^2(\mathbb{R}_+). \tag{5.12}$$

It is easily seen that $\lim_{x\to\infty} f_z(x) = 0$, and so $f_z \in \mathrm{dom}(A(0;\infty)^*)$ and $(A(0;\infty)^* - z)f_z = 0$. In other words, deficiency space $\mathfrak{N}_z(A(0;\infty))$ of the operator $A(0;\infty)$ generated by the vector f_z.

From the formula (4.9), (4.10), (5.3), (5.12) we arrive to the relation

$$\Gamma_0^{0;\infty} f_z = -\frac{2}{\pi} i; \tag{5.13}$$

$$\Gamma_1^{0;\infty} f_z = 1 + i \cdot \frac{2}{\pi}\left[\ln\left(\frac{\sqrt{z}}{2}\right) + \gamma\right]. \tag{5.14}$$

Hence, by (5.13), (5.14) and Definition 2.2 follows second part of the formula (5.4).

(*iii*) Since $M_{\nu;\infty}(t+iy)$ is bounded in the rectangle $(0,\infty)\times(0,y_0)$, then its representing measure is absolutely continuous and by Fatou's Theorem for $\nu \in (0,1)$

$$\begin{aligned}\Sigma_\nu'(t) = \frac{1}{\pi}\operatorname{Im} M_{\nu;\infty}(t+i0) &= \frac{1}{\pi}\frac{\Gamma(1-\nu)}{2\nu 4^\nu \Gamma(1+\nu)} \operatorname{Im}\left(e^{i(1-\nu)\pi} t^\nu\right)\\ &= \frac{1}{\pi}\frac{\Gamma(1-\nu)}{2\nu 4^\nu \Gamma(1+\nu)} t^\nu \operatorname{Im}(e^{i(1-\nu)})\\ &= \frac{1}{2\pi}\frac{\Gamma(1-\nu)\sin((1-\nu)\pi)}{\nu 4^\nu \Gamma(1+\nu)} t^\nu = C_\nu t^\nu.\end{aligned}$$

The case $\nu = 0$ is considered similarly. □

Remark 5.2. In addition for $\nu \in (0,1)$ the Weyl function $M_{\nu;\infty}(\cdot)$ admits an integral representation

$$M_{\nu;\infty}(z) = A_\nu + C_\nu \int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) t^\nu dt, \tag{5.15}$$

where the constant C_n is given at (5.6) and

$$A_\nu = \frac{\Gamma(1-\nu)}{2\nu 4^\nu \Gamma(1+\nu)} \cos((1-\nu/2)\pi).$$

Similarly, for $\nu = 0$ the Weyl function $M_{0;\infty}(\cdot)$ admits an integral representation

$$M_{0;\infty}(z) = A_0 + \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) dt, \tag{5.16}$$

where the constant

$$A_0 = -\frac{\pi}{4} - \gamma + \ln 2,$$

γ is Euler's constant.

Remark 5.3. Let us note that for Bessel operators the formula similar to (5.3) has been appeared for $\nu \in (0, 1/2) \cup (1/2, 1)$ in [16, Theorem 2] and for $\nu = 0$ in [16, Theorem 3].

Remark 5.4. Formulas (5.4) and (5.5) were obtained by W.N. Everitt and H. Kalf in [9] using the classical definition of the Weyl function.

Proposition 5.3. *Let* $\Pi_{\nu;\infty} = \{\mathcal{H}, \Gamma_0^{\nu;\infty}, \Gamma_1^{\nu;\infty}\}$ *be a boundary triplet for the operator* $A(\nu;\infty)^*$ *of the form* (5.3) *for* $\nu \in [0, 1)$, $M_{\nu;\infty}(\cdot)$ *is the corresponding Weyl function. Then*

(*i*) *The domain of the Friedrichs extension* $A_F(\nu;\infty)$ *of the operator* $A(\nu;\infty)$ *has the form*

$$\begin{aligned}\operatorname{dom}(A_F(\nu;\infty)) &= \ker(\Gamma_0^{\nu;\infty}) \\ &= \left\{ f \in \operatorname{dom}(A(\nu;\infty)^*) : [f, x^{\frac{1}{2}+\nu}]_0 = 0 \right\}. \end{aligned} \tag{5.17}$$

(*ii*) *The domain of the Krein extension* $A_K(\nu;\infty)$ *of the operator* $A(\nu;\infty)$ *has the form*

$$\begin{aligned}&\operatorname{dom}(A_K(\nu;\infty)) \\ &= \begin{cases} \{f \in \operatorname{dom}(A(\nu;\infty)^*) : [f, x^{\frac{1}{2}-\nu}]_0 = 0\}, & \nu \in (0,1), \\ \{f \in \operatorname{dom}(A(0;\infty)^*) : [f, x^{\frac{1}{2}}]_0 = 0\} = \ker(\Gamma_0^{0;\infty}), & \nu = 0. \end{cases}\end{aligned} \tag{5.18}$$

Proof. To prove these statements we use [6].

(*i*) For $\nu \in (0, 1)$

$$\begin{aligned} M_{\nu;\infty}(-\infty) &= \lim_{z\to-\infty} M_{\nu;\infty}(z) = \lim_{x\to\infty}\left[e^{i(1-\nu)\pi}\frac{\Gamma(1-\nu)}{2\nu 4^\nu\Gamma(1+\nu)}\cdot(-x)^\nu\right] \\ &= -\lim_{x\to\infty}\left[\frac{1}{(-1)^\nu}\cdot\frac{\Gamma(1-\nu)}{2\nu4^\nu\Gamma(1+\nu)}\cdot(-1)^\nu x^\nu\right] = -\infty. \end{aligned}$$

And for $\nu = 0$

$$\begin{aligned} M_{0;\infty}(-\infty) &= \lim_{z\to-\infty} M_{0;\infty}(z) = \lim_{z\to-\infty}\left[\frac{i\pi}{2} - \gamma - \ln\left(\frac{\sqrt{z}}{2}\right)\right] \\ &= \lim_{x\to\infty}\left[\frac{i\pi}{2} - \gamma - \ln\left(i\frac{\sqrt{x}}{2}\right)\right] \\ &= \lim_{x\to\infty}\left[-\gamma - \ln\left(\frac{\sqrt{x}}{2}\right)\right] = -\infty. \end{aligned} \tag{5.19}$$

So by the Proposition 2.2 the relation (5.17) is valid.

(ii.1) First we consider the case $\nu \in (0,1)$.

$$M_{\nu;\infty}(0) = \lim_{z\to -0} M_{\nu;\infty}(z) = -\lim_{z\to -0}\left[e^{i(1-\nu)\pi}\frac{\Gamma(1-\nu)}{2\nu 4^{\nu}\Gamma(1+\nu)}\cdot z^{\nu}\right]$$
$$= -\lim_{z\to -0}\left[\frac{1}{(-1)^{\nu}}\cdot\frac{\Gamma(1-\nu)}{2\nu 4^{\nu}\Gamma(1+\nu)}\cdot z^{\nu}\right] = 0.$$

By the Proposition 2.2 first part of the relation (5.18) is valid.

(ii.2) The case $\nu = 0$.

$$M_{0;\infty}(0) = \lim_{z\to -0} M_{0;\infty}(z) = \lim_{z\to -0}\left[\frac{i\pi}{2} - \gamma - \ln\left(\frac{\sqrt{z}}{2}\right)\right] = +\infty.$$

By the Proposition 2.2 second part of the relation (5.18) is valid. □

Remark 5.5. Formula (5.17) was obtained by W. N. Everitt and H. Kalf in [9] and by L. Bruneau, J. Dereziński and V. Georgescu in [4] using the classical definition of the Friedrichs extension.

Remark 5.6. The Krein extension (5.18) was described in similar form by L. Bruneau, J. Dereziński and V. Georgescu using the property of homogeneity of the operator and its extensions (see [4, Remark 4.20]).

Corollary 5.1. *Let* $\nu \in (0,1)$.

(i) *All self-adjoint extensions of the operator* $A(\nu;\infty)$ *defined in* $L^2(\mathbb{R}_+)$ *are given by*

$$\widetilde{A}(\nu;\infty)_h = A(\nu;\infty)^* \restriction \mathrm{dom}(\widetilde{A}(\nu;\infty)_h), \quad h \in \mathbb{R}\cup\{\infty\};$$

$$\mathrm{dom}(\widetilde{A}(\nu;\infty)_h) = \{f \in \mathrm{dom}(A(\nu;\infty)^*) : [f, x^{\frac{1}{2}-\nu} + 2\nu h x^{\frac{1}{2}+\nu}]_0 = 0\}. \tag{5.20}$$

(ii) *Extension* $\widetilde{A}(\nu;\infty)_h$ *is non-negative,* $\widetilde{A}(\nu;\infty)_h \geqslant 0$ *if and only if*

$$h \geqslant 0.$$

Proof. (i) Using boundary triplet (5.3) one obtains the proof by applying Proposition 2.1(iii).

(ii) From Proposition (5.3) it follows that $M_{\nu;\infty}(0) = 0$ and $A(\nu;\infty)_0$ is the Friedrichs extension, then by virtue of the Proposition 2.2 (ii), the extension $\widetilde{A}(\nu;\infty)_h$ is a non-negative, $\widetilde{A}(\nu;\infty)_h \geqslant 0$ if and only if $h \geqslant M_{\nu;\infty}(0) = 0$. □

Corollary 5.2. *If $\nu = 0$, then the operator $A(0;\infty)$ defined in $L^2(\mathbb{R}_+)$ has a unique non-negative extension*

$$\widetilde{A} = A(0;\infty)^* \upharpoonright \operatorname{dom}(\widetilde{A}),$$
$$\operatorname{dom}(\widetilde{A}) = \{f \in \operatorname{dom}(A(0;\infty)^*) : [f, x^{\frac{1}{2}}]_0 = 0\} = \ker(\Gamma_0^{0;\infty}). \tag{5.21}$$

Proof. According to (5.19) we obtain

$$\lim_{z\downarrow-\infty} M_{0;\infty}(z) = M_{0,\infty}(-\infty) = -\infty.$$

Similarly, (5.19) implies

$$\lim_{z\uparrow 0} M_{0;\infty}(z) = M_{0,\infty}(0) = +\infty.$$

By Proposition 2.2, $A_F(0;\infty) = A_K(0;\infty)$. This completes the proof. □

Proposition 5.4. (*i*) *Let $\nu \in [0,1)$. The closure $\mathfrak{a}(\nu;\infty)$ of the quadratic form $\mathfrak{a}'(\nu;\infty)$ associated with the operator $A(\nu;\infty)$ takes the form*

$$\mathfrak{a}(\nu;\infty)[u] = \mathfrak{a}(1/2;\infty)[u] + \kappa\mathfrak{q}[u], \quad \operatorname{dom}(\mathfrak{a}(\nu;\infty)[u]) = H_0^1(\mathbb{R}_+), \tag{5.22}$$

where

$$\mathfrak{a}(1/2;\infty)[u] = \int_0^\infty |u'(x)|^2 dx, \quad \mathfrak{q}[u] = \int_0^\infty \frac{|u(x)|^2}{x^2} dx. \tag{5.23}$$

(*ii*) *The domain of definition of the Friedrichs extension $A_F(\nu;\infty)$ of the operator $A(\nu;\infty)$ for $\nu \in [0,1)$ takes the form*

$$\operatorname{dom}(A_F(\nu;\infty)) = H_0^2(\mathbb{R}_+) \dotplus \operatorname{span}\{x^{1/2+\nu}\varphi(x)\}, \quad \varphi \in C_0^\infty(\mathbb{R}_+). \tag{5.24}$$

(*iii*) *Let $\nu \in [0,1)$. For the closure $\mathfrak{a}(\nu;\infty)_h$ of the quadratic form $\mathfrak{a}'(\nu;\infty)_h$ associated with the operator $\tilde{A}(\nu;\infty)_h$ the following decomposition is valid*

$$\operatorname{dom}(\mathfrak{a}(\nu;\infty)_h)[u] = H_0^1(\mathbb{R}_+) \dotplus \operatorname{span}\{x^{1/2-\nu}\varphi(x)\}, \quad \varphi \in C_0^\infty(\mathbb{R}_+). \tag{5.25}$$

Proof. (*i*) By Hardy's inequality for $u \in H_0^1(\mathbb{R}_+)$

$$\mathfrak{a}(\nu;\infty)[u] = \|u'(t)\|^2_{L^2(\mathbb{R}_+)} + (\nu^2 - 1/4)\int_0^\infty \frac{|u(t)|^2}{t^2}dt \leqslant \|u'(t)\|^2_{L^2(\mathbb{R}_+)}(1 + |4\nu^2 - 1|). \quad (5.26)$$

Thus $H_0^1(\mathbb{R}_+) \subset \operatorname{dom}(\mathfrak{a}(\nu;\infty))$.

We prove the converse inequality. Suppose first that $\nu \in [1/2, 1)$. Then for $u \in H_0^1(\mathbb{R}_+)$

$$\mathfrak{a}(\nu;\infty)[u] = \|u'(t)\|^2_{L^2(\mathbb{R}_+)} + (\nu^2 - 1/4)\int_0^\infty \frac{|u(t)|^2}{t^2}dt \geqslant \|u'(t)\|^2_{L^2(\mathbb{R}_+)}. \quad (5.27)$$

If $\nu \in [0, 1/2)$, then for $u \in H_0^1(\mathbb{R}_+)$

$$\begin{aligned}\mathfrak{a}(\nu;\infty)[u] &= \|u'(t)\|^2_{L^2(\mathbb{R}_+)} - (1/4 - \nu^2)\int_0^\infty \frac{|u(t)|^2}{t^2}dt \\ &\geqslant \|u'(t)\|^2_{L^2(\mathbb{R}_+)} + (4\nu^2 - 1)\|u'(t)\|^2_{L^2(\mathbb{R}_+)} \\ &= 4\nu^2\|u'(t)\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

So on $H_0^1(\mathbb{R}_+)$ the energy norm of $A(\nu;\infty)$ is equivalent to the norm of space $H_0^1(\mathbb{R}_+)$. Since $H_0^2(\mathbb{R}_+) = \operatorname{dom}(A(\nu;\infty))$ is dense in the energy space of the operator $A(\nu;\infty)$, then $\operatorname{dom}(\mathfrak{a}(\nu;\infty))$ and $H_0^1(\mathbb{R}_+)$ coincide algebraically and topologically.

(*ii*) We note that $H_0^2(\mathbb{R}_+) \subset H_0^1(\mathbb{R}_+)$. If $u(x) = x^{1/2+\nu}\varphi(x)$, where $\varphi \in C_0^\infty(\mathbb{R}_+)$ then $u' \in L^2(\mathbb{R}_+)$, but $u(\cdot) \notin \operatorname{dom}(A(\nu;\infty)) = H_0^2(\mathbb{R}_+)$. By the construction of the Friedrichs extension and the equalities (5.2), we obtain

$$\begin{aligned}\operatorname{dom}(A_F(\nu;\infty)) &= \operatorname{dom}(A(\nu;\infty)^*) \cap \operatorname{dom}(\mathfrak{a}(\nu;\infty)[u]) \\ &= \operatorname{dom}(A(\nu;\infty)^*) \cap H_0^1(\mathbb{R}_+) = H_0^2(\mathbb{R}_+) \dotplus \operatorname{span}\{x^{1/2+\nu}\varphi(x)\}.\end{aligned}$$

(*iii*) Let

$$\varphi(x) = \begin{cases} 1, & x \in [0,1], \\ 0, & x \in [2; +\infty). \end{cases} \quad (5.28)$$

Then proof follows from [18, Theorem 1] and from the fact that $x^{1/2+\nu}\varphi(x) \in H_0^1(\mathbb{R}_+)$. □

Corollary 5.3. *Note that the domains of the Friedrichs extensions in* (5.17) *and* (5.24) *are coincide.*

Proof. So $[f, x^{\frac{1}{2}+\nu}]_0 = 0$, it is easy to see that

$$[f, x^{\frac{1}{2}+\nu}]_0 = \lim_{x\to 0}\left(\left(\frac{1}{2}+\nu\right)x^{\nu-\frac{1}{2}}f(x) - x^{\frac{1}{2}+\nu}f'(x)\right) = x^{\frac{1}{2}+\nu}\varphi(x),$$

where $\varphi(x) \in C_0^\infty(\mathbb{R}_+)$. □

6. Connection of the operators $A(\nu; b)$ and $A(\nu;\infty)$

Proposition 6.1. *Let* $\nu \in [0, 1)$. *Consider the operators* $A(\nu; b)$ *and* $A(\nu;\infty)$ *with the domains* (4.1) *and* (5.1) *respectively. Assume* $\Pi_{\nu;b}$ *and* $\Pi_{\nu;\infty}$ *be the boundary triplets defined by relations* (4.2) *and* (5.3) *respectively. Assume also* $M_{\nu;b}(z)$ *and* $M_{\nu;\infty}(z)$ *be the Weyl functions given by the equalities* (4.3) *and* (5.4). *Then the relation*

$$\lim_{b\to+\infty} M_{\nu;b}(z) = M_{\nu;\infty}(z)$$

holds uniformly on compact subsets of $\mathbb{C}_+$.

Proof. First we consider the case $\nu \in (0, 1)$. Since the Bessel functions $J_\nu(t)$ and $J_{-\nu}(t)$ for $t \to \infty$ have the asymptotic behavior (2.14), then

$$\begin{aligned}\lim_{b\to+\infty} M_{\nu;b}(z) &= -\lim_{b\to+\infty}\frac{\Gamma(1-\nu)}{2\nu 4^\nu\Gamma(1+\nu)}\cdot\frac{J_{-\nu}(b\sqrt{z})}{J_\nu(b\sqrt{z})}\cdot z^\nu \\ &= -\lim_{b\to+\infty}\left[\frac{\Gamma(1-\nu)}{2\nu 4^\nu\Gamma(1+\nu)}\cdot\frac{\cos\left(b\sqrt{z}+\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}{\cos\left(b\sqrt{z}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\cdot z^\nu\right] \\ &= -\frac{\Gamma(1-\nu)}{2\nu 4^\nu\Gamma(1+\nu)}\lim_{b\to+\infty}\frac{e^{-i(b\sqrt{z}+\frac{\nu\pi}{2}-\frac{\pi}{4})}}{e^{-i(b\sqrt{z}-\frac{\nu\pi}{2}-\frac{\pi}{4})}}\cdot z^\nu \\ &= e^{i(1-\nu)\pi}\frac{\Gamma(1-\nu)}{2\nu 4^\nu\Gamma(1+\nu)}\cdot z^\nu = M_{\nu;\infty}(z).\end{aligned}$$

The case $\nu = 0$ is treated similarly. Namely

$$\begin{aligned}\lim_{b\to+\infty} M_{0;b}(z) &= \lim_{b\to+\infty}\left[-\ln\frac{\sqrt{z}}{2}+\frac{\pi}{2}\frac{Y_0(b\sqrt{z})}{J_0(b\sqrt{z})}-\gamma\right] \\ &= \lim_{b\to+\infty}\left[-\ln\frac{\sqrt{z}}{2}+\frac{\pi}{2}\cdot\frac{\sin(b\sqrt{z}-\frac{\pi}{4})}{\cos(b\sqrt{z}-\frac{\pi}{4})}-\gamma\right] \\ &= -\ln\frac{\sqrt{z}}{2}+\frac{\pi}{2}\cdot i-\gamma = M_{\nu;\infty}(z).\end{aligned}$$

It is easily seen that convergence in both relations is uniform on compact subsets. □

7. Singular Sturm–Liouville operators of Bessel type

Here we consider in $L^2(\mathbb{R}_+)$ Sturm–Liouville differential expression

$$\tau u := -u'' + qu \tag{7.1}$$

with certain potentials q.

The minimal operator $T_{\min} = T$ associated with (7.1) is the closure of the operator T' of the form

$$T'u := \tau u,\ \operatorname{dom}(T') = \{u : u \in \mathfrak{D},\ u \text{ has the comp. support in } (0,\infty)\}, \tag{7.2}$$

where

$$\mathfrak{D} := \{u : u \in AC_{loc}(\mathbb{R}_+) \cap L^2(\mathbb{R}_+), u' \in AC_{loc}(\mathbb{R}_+), \tau u \in L^2(\mathbb{R}_+)\}. \tag{7.3}$$

T is a densely defined symmetric operator.

The maximal operator associated with (7.1) is

$$T_{\max} = T^* = \tau \restriction \mathfrak{D}. \tag{7.4}$$

The following relations hold

$$T_{\min} = T = \overline{T'} = T^{**} = T^*_{\max}.$$

Corollary 7.1. *Let $q \in L^1_{loc}(\mathbb{R}_+)$ and*

$$q(x) \geq \frac{\beta}{x^2} - \mu, \quad (x \in \mathbb{R}_+) \tag{7.5}$$

for some $\beta > -\frac{1}{4}$ and $\mu \geq 0$. Then:

(i) The closure $\mathfrak{t}_q$ of the quadratic form $\mathfrak{t}'_q$ associated with the operator T is

$$\mathfrak{t}_q[u] = \int_0^\infty |u'(x)|^2 dx + \int_0^\infty q(x) \cdot |u(x)|^2 dx,$$

$$\operatorname{dom}(\mathfrak{t}_q) = \left\{ u \in H^1_0(\mathbb{R}_+) : \int_0^\infty q(x) \cdot |u(x)|^2 dx < \infty \right\} =: H^1_0(\mathbb{R}_+; q). \tag{7.6}$$

(ii) [14] *The domain of the Friedrichs extension T_F of T is*

$$\operatorname{dom}(T_F) = \mathfrak{D} \cap H^1_0(\mathbb{R}_+; q), \tag{7.7}$$

where $\mathfrak{D}$ is given by (7.3) .

Proof. Without loss of generality we can assume that $\mu = 0$. Let $\beta = \nu^2 - \frac{1}{4} > -\frac{1}{4}$. Consider the quadratic form $\mathfrak{t}_q$ associated with the operator T_F. Since $q(x) > \frac{\nu^2 - \frac{1}{4}}{x^2}$, we have

$$\operatorname{dom}(\mathfrak{t}'_q) \subset \operatorname{dom}(\mathfrak{a}(\nu;\infty)) = H^1_0(\mathbb{R}_+), \tag{7.8}$$

where $a(\nu;\infty)$ is given by (5.22).

Further, let $u(\cdot) \in C_0^\infty(\mathbb{R}_+) \subset \operatorname{dom}(T')$. Integrating by parts one obviously has

$$\mathfrak{t}'_q[u] = (Tu, u) = \lim_{x\to\infty}\left[u'(t)u(t)\big|_0^x + \int_0^x |u'(t)|^2 dt + \int_0^x q(t)\cdot|u(t)|^2 dt\right]$$
$$= \int_0^\infty |u'(x)|^2 dx + \int_0^\infty q(x)\cdot|u(x)|^2 dx. \tag{7.9}$$

Taking the closure of these forms with account of (7.8), one arrives at (7.6).

According to the construction of the Friedrichs extension and (7.3)

$$\operatorname{dom}(T_F) = \operatorname{dom}(T^*) \cap \operatorname{dom}(\mathfrak{t}_q) = \mathfrak{D} \cap H^1_0(\mathbb{R}_+; q).$$

The Corollary is proved. □

Corollary 7.2. *Let* $q \in L^1_{loc}(\mathbb{R}_+)$ *and*

$$q(x) \geq -\frac{1}{4x^2} - \mu, \quad (x \in \mathbb{R}_+) \tag{7.10}$$

for some $\mu \geq 0$. *Then*

(*i*) *The closure* $\mathfrak{t}_q$ *of the quadratic form* $\mathfrak{t}'_q$ *associated with the operator* T *takes the form*

$$\mathfrak{t}_q[u] = \int_0^\infty |u'(x)|^2 dx + \int_0^\infty \left|\frac{u(x)}{2x}\right|^2 dx, \tag{7.11}$$

$$\operatorname{dom}(\mathfrak{t}_q) = \{u \in H^1_0(\mathbb{R}_+) : \int_0^\infty \frac{|u(x)|^2}{4x^2} dx < \infty\} =: H^1_0(\mathbb{R}_+; q). \tag{7.12}$$

(*ii*) [14] *The domain of the Friedrichs extension of* T *is*

$$\operatorname{dom}(T_F) = \mathfrak{D} \cap H^1_0(\mathbb{R}_+; q), \tag{7.13}$$

where $\mathfrak{D}$ *is given by* (7.3).

The proof is similarly to Corollary 7.1.

Corollary 7.3. *Another description of the quadratic form* $\mathfrak{t}_q$*, defined by the equation* (7.11)*, was obtained by Kalf in [14]*

$$\mathfrak{t}[u] = \int\limits_0^\infty \left| u'(x) - \frac{u(x)}{2x} \right|^2 dx,$$

$$\mathrm{dom}(\mathfrak{t}_q) = \{u \in H_0^1(\mathbb{R}_+) : \int\limits_0^\infty \frac{|u(x)|^2}{4x^2} dx < \infty\}.$$

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