ROTORDYNAMIC STABILITY—A SIMPLIFIED APPROACH

by
Carroll G. (Chet) Stroh
Consultant
Engineering Service Division
Engineering Department
E. I. du Pont de Nemours & Co., Inc.
Wilmington, Delaware

Carroll G. Stroh is a Consultant with E. I. du Pont de Nemours & Company, Incorporated, located in Wilmington, Delaware. He is assigned to the Mechanical Engineering Consultants of the Engineering Service Division. His consulting service involves turbomachinery and reciprocating compressor analysis, repair, and specification. He has a total of eleven years with Du Pont, during which he spent five years in a regional engineering office at Beaumont, Texas, consulting on turbomachinery problems for Du Pont plants along the Gulf Coast, three years in turbomachinery rotordynamics research at the Experimental Station in Wilmington, Delaware, and three years at his present position.

Prior to joining Du Pont, Mr. Stroh worked four years for the Westinghouse Large Steam Turbine Division in Lester, Pennsylvania, where he was involved with the design of rotors and blades. He earned a B.S. degree, an M.S. degree, and did three years of post graduate work in Mechanical Engineering at the University of Pennsylvania in Philadelphia, Pennsylvania. He also has a B.A. degree in Mathematics from Lebanon Valley College in Annville, Pennsylvania. He is a member of Tau Beta Pi.

ABSTRACT

Straightforward mechanics are used to give a physical understanding of the important parameters involved in rotor dynamic instability. A concise analysis procedure is proposed for assembling available information into a meaningful design stability analysis of between-bearing rotors. The technique combines the destabilizing effects on the rotor into an equivalent single source at the rotor mid-span, and uses an excitation "threshold value" to rate the system stability. Results of applying this design analysis technique to an unstable compressor and the benefit from bearing redesign are presented.

INTRODUCTION

Rotordynamic instability, a self-excitation of the rotor/bearing system, can occur without prior warning and has catastrophic potential. It may take only slight changes in such things as oil temperature, bearing clearance and machine alignment to enable the system to become unstable. Once initiated, vibration can grow to levels that cause machine destruction. Even if a machine starts-up successfully, there is no guarantee for the future. Several documented cases exist of machines that have become unstable after more than 15 years of successful operation. In each case, the plant lost significant production during the time required to return the machine to service. To prevent such unexpected occurrences, an adequate stability factor of safety should be part of the machine design.

Although there are numerous references which list the possible sources for a destablizing force in turbomachinery [1,2], and several mathematical treatises exist which point out important rotor/bearing parameters [3,4,5,6,7], it still can be difficult to comprehend the physical reasons for an unstable rotor. A universally accepted approach to a design analysis for stability also does not exist. Some analysts evaluate the rotor/bearing system alone, while others include their best estimate for all possible destablizing forces within the machine. Most approaches use the criterion that the resultant system damping be equal to or greater than some fixed constant. Analyses of unbalance response and free vibration without destabilizing forces do not supply sufficient information to evaluate rotor stability. The inclusion of destabilizing forces is a definite improvement; however, a need exists for a meaningful stability factor of safety which can be more universally applied.

Two objectives will be addressed herein:

- Clarification of the current theory by using simple models to show the physical effects of important parameters on stability, and
- Presentation of a stability analysis procedure based on an energy approach that approximates all of the destabilizing forces with a single parameter.

The single parameter design analysis approach enables the destabilizing forces and the rotor/bearing system to be rated separately, permits the use of a stability factor of safety with physical meaning and encourages comparison with existing machines. Results of applying this design analysis technique to an unstable compressor and the benefit from bearing redesign are presented.

Approximate analytical solutions are derived in the APPENDIX to enhance the understanding of rotordynamic instability and to facilitate the design stability analysis.

CLARIFICATION OF CURRENT THEORY

Description of Terms

Instability

Imagine a ball balanced on top of an upside-down bowl. If the ball is disturbed, it will roll off the bowl and not return to the starting point. The starting point is a position of instability. Turn the bowl upright and set the ball on the bottom. Disturb the ball and it will return to rest at the starting point, which is now a position of stability. Rotor instability is similar. Disturb an unstable rotor, and linear theory predicts it will whirl with increasing amplitude. Disturb a stable rotor and it will return to its original condition.

Rotor instability might be described as a self-excited vibration in which the exciting force is a result of motion and motion is a result of the exciting force. The destabilizing forces in a rotor instability are proportional to the local shaft displacement, which is why a system resonance is excited—where maximum displacements result from a given energy input. The resonant frequency that is excited (usually close to the first critical) is subsynchronous (below the operating speed), because the damping forces increase with frequency above the operating speed and decrease with frequency below the operating speed. For a given subsynchronous frequency, the damping forces continue to decrease as the operating speed is increased, while the destabilizing forces are continually increasing with the operating speed.

Threshold Speed

Threshold speed is the operating speed at which energy added to the system from destabilizing forces is equal to that removed through damping. Once the threshold speed is exceeded, instability occurs. The threshold speed will usually exceed twice the first bending critical speed of the machine.

Cross-coupled Force

Some well known sources for a destabilizing force are fixed geometry bearings, seals, compressor impellers, shrink fits, gear type couplings, rubs, trapped fluids, and steam whirl. While these sources may use different mechanisms to generate a destabilizing force, the end results are similar. Since steam whirl is the easiest to describe physically, it will be used to show how such a force occurs. In Figure 1, a turbine wheel is shown running eccentric to its peripheral seal, so that more steam leaks by the blades on one side than on the other. The varying amount of steam passing through the blades causes a difference in the tangential (torque) forces applied around the wheel. Summing these forces produces a net force at the center of the wheel which grows in magnitude as the amount of wheel offset is increased. Because the force direction is perpendicular to the wheel offset direction, it is referred to as a cross-coupled force. This force can be represented for small displacements by:

$$\vec{F} = K_{xy} Y \vec{i} + K_{yx} X \vec{j} \tag{1}$$

where K_{xy} and K_{yx} are the cross-coupled stiffnesses.

Damping Force

Damping resists motion and tries to stabilize the system. The viscous and pressure drag of the surrounding fluid on the rotor as it tries to whirl is an example of a velocity dependent damping force and can be written as:

$$\vec{F} = C_{xx} \dot{X} \vec{i} + C_{yy} \dot{Y} \vec{j} \tag{2}$$

Important Parameters

Shaft Stiffness versus Fluid Film Bearing Stiffness

The significance of the bearing to shaft stiffness ratio is explained by considering a single mass supported on a flexible shaft between two identical symmetrical fluid film bearings, as shown in Figure 2. The only source of damping in this model, and the major source in most turbomachinery, comes from relative motion in the bearings. By gradually increasing the bearing stiffness in relation to the shaft stiffness, a greater portion of energy is concentrated into shaft deflection and less into relative motion in the bearings, resulting in diminishing system damping. The stiffer the bearings in relation to the shaft, the smaller the benefit from bearing damping.

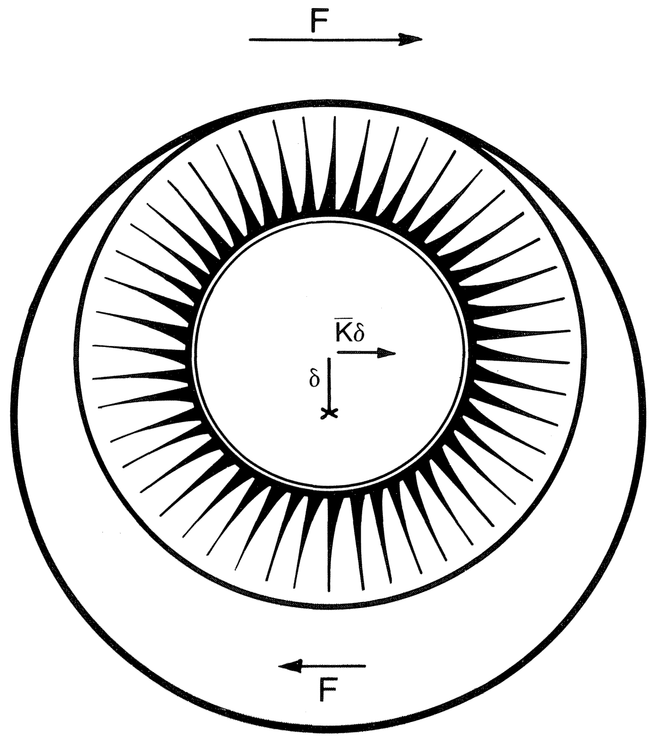

Figure 1. Steam Whirl of a Turbine Wheel.

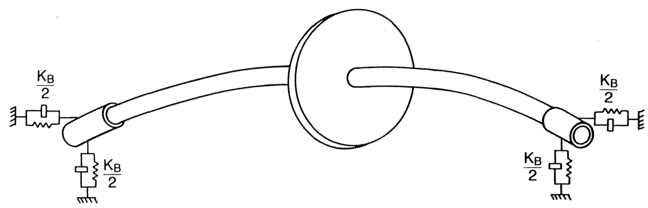

Figure 2. Shaft Stiffness Versus Bearing Stiffness.

Asymmetric Support Stiffness

The benefit of asymmetric support stiffness is explained by referring to an elliptical orbit in the XY plane (Figure 3). The orbit becomes more elliptical as the vertical and horizontal support stiffnesses become more asymmetric. Since work input

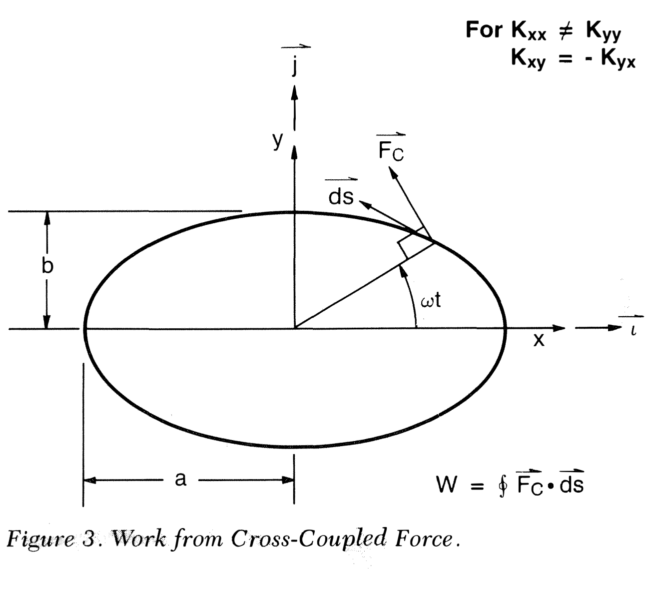

Figure 3. Work from Cross-Coupled Force.

is the integral of force times displacement in the direction of the force, the cross-coupled force puts the maximum amount of work into the system when the orbit is a circle, because force and displacement are always tangent to the orbiting path. When the orbit is an ellipse, the cross-coupled force direction deviates from the direction of displacement and the amount of work put into the orbiting shaft is reduced. For the extreme case of a straight line orbit (Figure 4), the cross-coupled force is always perpendicular to displacement, and there is no work input.

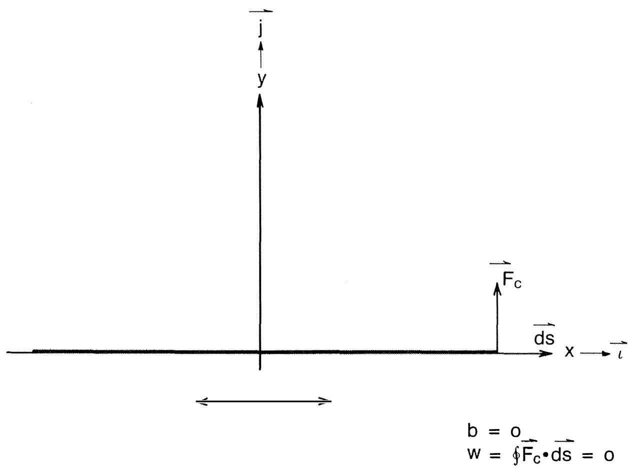

Figure 4. Zero Work from Cross-Coupled Force.

The damping force and displacement shown in Figure 5 are of opposite sign, but everywhere tangent to the path, whether circular or elliptical in shape. Therefore, as the whirl orbit becomes more and more elliptical, the work input from a cross-coupled force decreases toward zero, whereas the work taken away by damping does not. A ratio of work input to work dissipated versus the orbit minor/major amplitude ratio is plotted in Figure 6. The system used in this example is unstable when the orbit is a circle (ratio of $b/a = 1$ and the work ratio is >1). Stability is achieved by increasing the ellipticity of the orbit (decreasing ratio of b/a) until the work ratio becomes <1.

Note: In light of the above, it becomes apparent that a high preload on tilt-pad bearings is undesirable for rotor stability. Bearing preload increases the bearing stiffness and induces symmetry of the bearing stiffness coefficients [8]. This, however, may conflict with expected results for unbalance response,

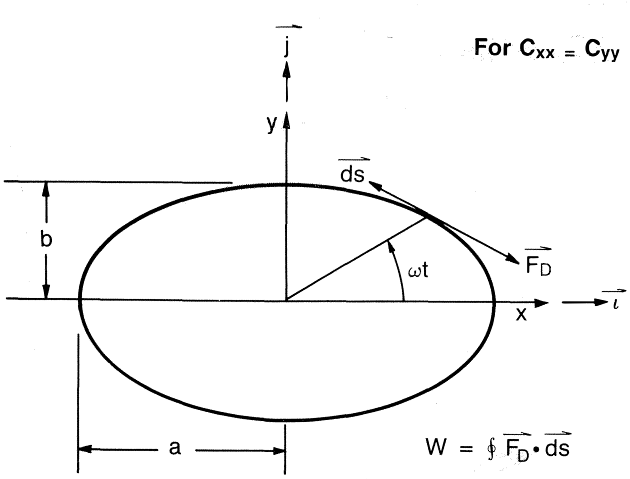

Figure 5. Work from Damping Force.

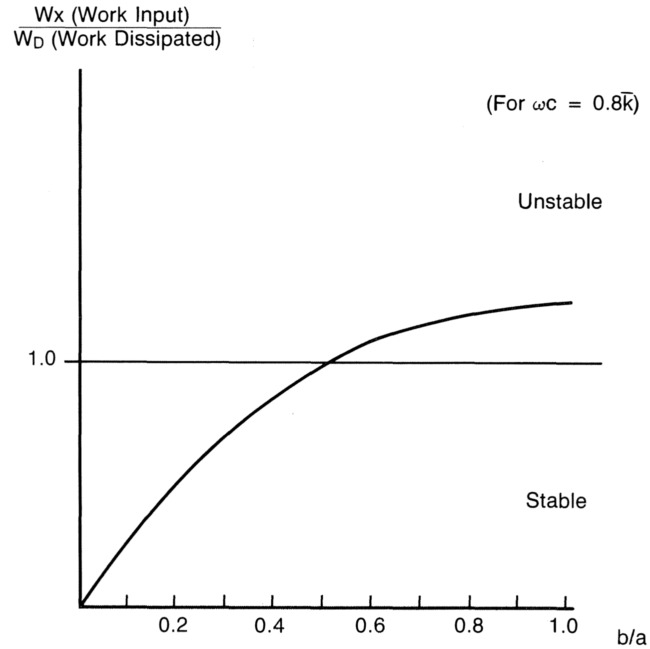

Figure 6. Work Ratio Versus Orbit Minor/Major Amplitude Ratio.

where a stiff bearing sometimes means lower vibration levels for the relative motion between the rotor and the bearing. This is one more compromise the designer must consider.

SINGLE PARAMETER STABILITY ANALYSIS FOR BETWEEN BEARING ROTORS

Definition of Terms

$\bar{K}_{eq}$—Energy Equivalent Mid-Span Cross Coupled Stiffness

The work per revolution of an orbit performed by the cross-coupled force and damping force at each station of the rotor model in Figure 7, and summed over all stations, can be written as [9]:

$$\mathrm{Work}_A = \sum_i \pi \Big\{ (K_{xy}^{(i)} - K_{yx}^{(i)})\, a_i b_i - \omega_D C_{xx}^{(i)} [a_i^2 \cos^2 \beta_i + b_i^2 \sin^2 \beta_i] - \omega_D\, C_{yy}^{(i)} [b_i^2 \cos^2 \beta_i + a_i^2 \sin^2 \beta_i] \Big\} \quad (3)$$

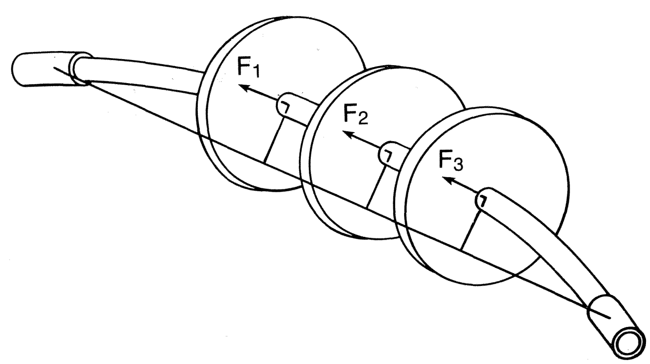

Figure 7. Destabilizing Forces on Rotor Model.

where the subscript and superscript "i" refer to values at station "i" of the rotor model and the other variables are defined in Figure 8. The work per revolution of an orbit for a single equivalent cross-coupled force at the rotor mid-span in Figure 9 can be written as:

$$\mathrm{Work}_B = 2\pi \bar{K}_{eq}\, a_{MS}\, b_{MS} \tag{4}$$

The equivalent mid-span cross-coupled stiffness is found by equating the above two expressions for work and solving for $\bar{K}_{eq}$:

$$\bar{K}_{eq} = \mathrm{Work}_A / 2\pi\, a_{MS}\, b_{MS} \tag{5}$$

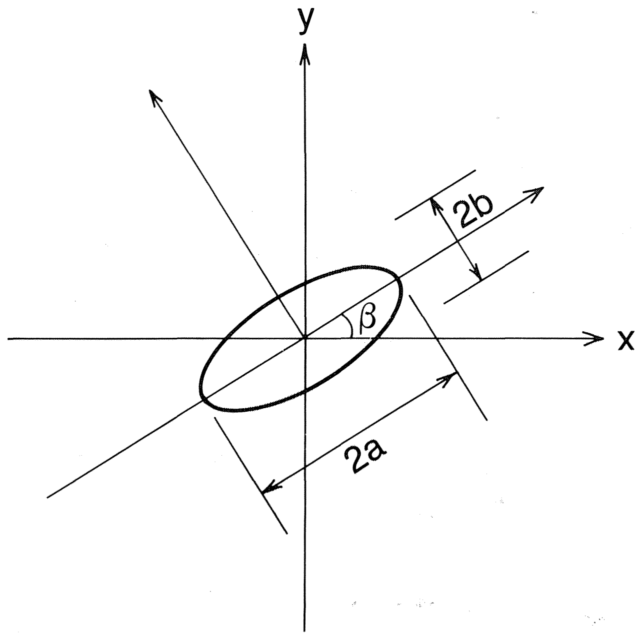

Figure 8. Orbit Nomenclature.

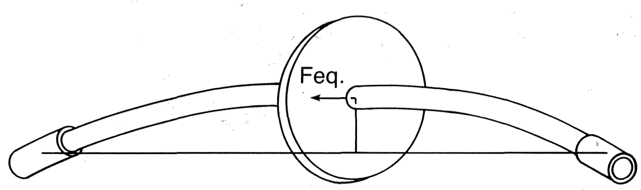

Figure 9. Single Equivalent Cross-Coupled Force.

K_{th}—Threshold Value

The equivalent mid-span cross-coupled stiffness which produces a growth factor of zero (the instability threshold) is labeled $\bar{K}_{th}$. An approximate expression for $\bar{K}_{th}$ is derived in the APPENDIX and is written as:

$$\bar{K}_{th}^2 = [1-(C_o/C_e)^2]\,[K_o^2 + C_e^2\, K_e\, (1 - \frac{C_oK_o}{C_eK_e})/M] \tag{6}$$

The above approach assumes that energy equivalent sources can be moved about on the rotor without significantly changing the resultant mode shape. Although this is not strictly true, the effect has been found to be minor. Using a full rotor model, the threshold value $\bar{K}_{th}$ produced by an excitation source close to the bearing (worst case) was reduced by only six percent when an equivalent source was placed at the mid-span of the rotor. Since a factor of safety of two or better on the threshold value is desired, the error is insignificant.

Procedure

The following procedure may be used as a guide to determining the stability or instability of a rotor and taking corrective actions, if required:

1. Calculate the first critical speed of the rotor on rigid supports.

2. Run a stability program for the rotor/bearing system [10], with no additional destabilizing effects, to obtain the first forward whirl damped natural frequency and the corresponding growth factor.

Whirl can occur in the direction of rotation (forward whirl) or against rotation (backward whirl). Only forward whirl is considered, because subsynchronous backward whirl is rarely excited. All stability runs should be made at the maximum operating speed of the rotor.

3. Use the values from the previous two steps to compute, from Equation (6) the first estimate of $\bar{K}_{th}$ (the equivalent mid-span cross-coupled stiffness threshold value). Equation (6) is for a between bearing rotor whirling in the first "U" mode shape.

4. Run a stability program to improve on the first estimate of $\bar{K}_{th}$. One of the significant advantages of this procedure is that a lot of time and effort is saved, without a significant loss of accuracy, by inputting cross-coupled stiffness (no damping) values at only the rotor mid-span.

5. Compute the equivalent mid-span cross-coupled stiffness $\bar{K}_{eq}$ from Equation (5) by summing the destabilizing work over the rotor length. Use the mode shape produced by the threshold value $\bar{K}_{th}$.

This step requires information on the expected destabilizing cross-coupled stiffness and damping. There is analytical and/or experimental information available on what is presently thought to be the most significant contributors. Further information about this is presented in the *Discussion* section. For existing field instabilities, $\bar{K}_{eq}$ is found by increasing the mid-span cross-coupled stiffness value until the rotor model becomes unstable at the threshold speed observed in the field.

6. Design for a $\bar{K}_{eq} \leq \frac{1}{2}\bar{K}_{th}$ for a minimum factor of safety of two.

By separating the destabilizing forces from the rotor/bearing system, all machines can be reduced to a parameter for the destabilizing forces ($\bar{K}_{eq}$) and a parameter for the rotor/bearing system ($\bar{K}_{th}$). The simplicity of the single parameter approach enables and encourages the user to compare data for each new application with his data base of past machines. Through such comparisons, it can be quickly determined if excessive destabilizing forces are present and if the rotor/bearing system has fully benefited from design support stiffness asymmetry, bearing to shaft stiffness ratio, and system damping.

Discussion

Application to a Real Machine

The centrifugal compressor shown in Figure 10 has six closed face impellers mounted between bearings in a straight through design, oil seals, and a balance piston. The rotor weight is 1620 lbs and the bearing span is 68 in. The rotor rotates at 9500 cpm. Subsynchronous whirl was observed at a frequency of 4200 cpm and prevented the plant from attaining full capacity operation.

The stability analysis results for the unstable compressor with the original rotor/bearing system and one comprised of the same rotor with modified bearings is graphically compared in Figure 11. The bearing configuration is a five pad, tilt-pad design with load on the pad. The lower curve shows this bearing with the original preload of 0.5 and an installed diametral clearance of seven mils. The upper curve shows this bearing with zero preload and an installed diametral clearance of eight mils. As shown, the threshold value $\bar{K}_{th}$ of the rotor/bearing system has improved by a factor of four by only the bearing modifications. This improvement produced a stability factor of safety of four since the original design was unstable at the operating speed. The dashed lines shown on Figure 11 are the results of the following approximate analytical solution.

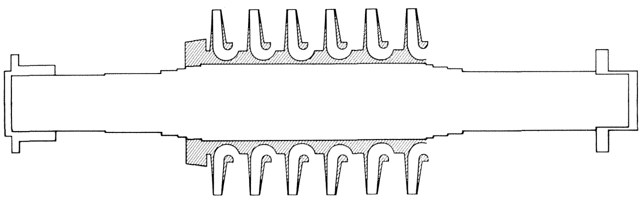

Figure 10. Centrifugal Compressor Rotor.

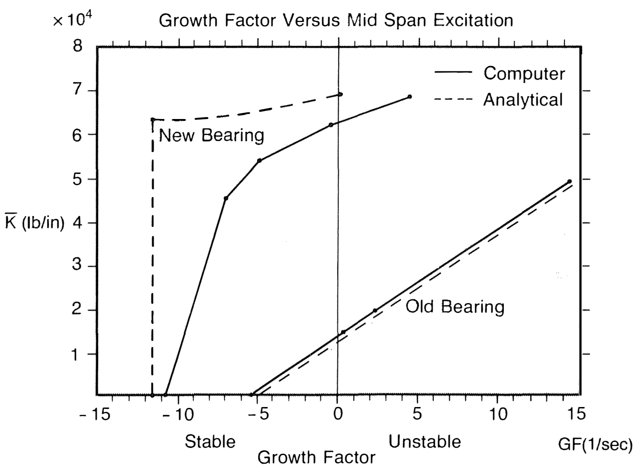

Figure 11. Compressor Stability Analysis Results.

Approximate Analytical Solution

An analytical solution has been derived for:

- A between bearing single mass rotor (Figure 12)
- No gyroscopic effects
- Support by identical bearings with asymmetric stiffness and damping
- Subsynchronous whirl at the first U-mode critical
- A cross-coupled stiffness at the rotor mid-span

The effective mass and effective rotor stiffness are computed from modal properties and are discussed in the APPENDIX. The derivation and solution of the equations of motion are also included in the APPENDIX.

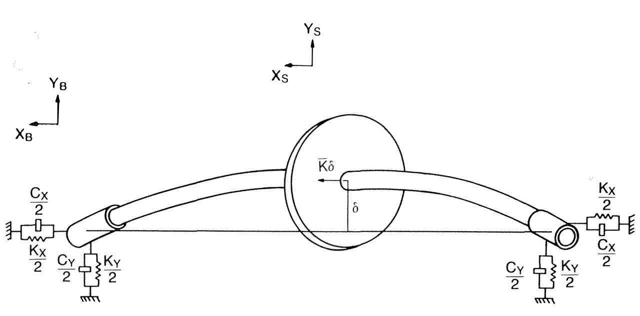

Figure 12. Rotor Model for Analytical Solution.

The resulting characteristic equation for the system eigenvalues has two distinct regions with differing solutions. In the first region, the mid-span cross-coupled stiffness excitation is less than the difference in bearing principal stiffnesses. The analytical solution predicts a constant growth factor from zero excitation up to a transition point, where excitation equals the difference in bearing principal stiffnesses. In the second region, the cross-coupled stiffness excitation exceeds the difference in principal stiffnesses. As excitation increases, the growth factor becomes less negative and the instability threshold is approached.

The result of the analytical solution for the new bearing application plotted in Figure 11 shows the transition point. The calculation is included in the APPENDIX. The excitation has to overcome the bearing stiffness asymmetry before it can move the rotor towards the instability threshold. For the old bearing, there is no transition point. The high preload on the bearing causes the stiffness in the vertical and horizontal direction to be similar, thus the curve moves immediately towards the threshold of instability.

The difference in results for the new bearing (Figure 11) of the computer analysis and the analytical solution is predominately from neglecting the gyroscopic terms in the analytical solution. There is some lesser effect from the actual rotor not having perfect symmetry. These effects combine to smooth out the distinct transition point which occurs with the analytical solution, but the effect of bearing stiffness asymmetry is still quite evident in the computer results. In the region of the instability threshold, the difference in computer and analytical results is small, showing that the analytical solution for $\bar{K}_{th}$ provides a quick and reasonable estimate.

The mode shape described by the analytical solution indicates that between zero cross-coupled stiffness excitation and the transition point, the effect of this excitation alone is insufficient to promote rotor whirl. Once the cross-coupled stiffness excitation exceeds the difference in principal stiffnesses, the rotor begins to whirl in a highly elliptical orbit; and as excitation is further increased, the ellipticity decreases towards a circular orbit.

Field Problems Versus New Design

When a machine exhibits instability in the field, the analyst knows that the destabilizing forces present are sufficient to drive the system unstable. Therefore, when modelling the system, the cross-coupled stiffness is increased until the system becomes unstable at the threshold speed observed in the field. Having thus established $\bar{K}_{eq}$, the analyst can then use this value to evaluate a proposed design [11].

This procedure is in contrast to the case for new designs, where the analyst is completely dependent on the predictive tools available for anticipating destabilizing force levels within the machine [2,12,13]. There are published data on cross-coupled stiffness and damping coefficients for gas labyrinth seals [14], and there are programs in progress to add to this experimental information as well as to develop the analytical tools needed for prediction. There are reasonable analytical tools available to predict expected excitation levels for oil seals [15,18]. These have emerged from the development work previously done on fluid film bearing analysis.

Reduced Bearing Matrices

A fixed geometry bearing can be adequately represented by a 2×2 matrix for stiffness and a 2×2 matrix for damping. Tilt-pad bearings add a degree of freedom for each pad, which means that an additional row and column must be added to the matrix for each pad. For example, a five pad bearing has a 7×7

matrix array for stiffness and a 7×7 matrix array for damping. These matrices can be reduced, however, to equivalent 2×2 matrices that are frequency dependent. The frequency dependency is slight for highly preloaded bearings, and the 2×2 matrix array which represents the bearing at synchronous speeds is equally valid at subsynchronous speeds [16]. However, the frequency dependency is significant for lightly preloaded bearings. The predicted growth factor for synchronous frequency bearing matrices can be in error by as much as a factor of two over the results of using the correct subsynchronous frequency matrices. When using lightly preloaded bearings, it is important to reduce the bearing matrices at the subsynchronous frequency of interest or to use the full 7×7 matrix in the case of a five pad tilt-pad bearing.

Critical Speeds

The analyst must pay close attention to the movement of critical speeds during design improvements on an unstable machine. He does not want to inadvertently replace one problem with another. On several occasions the redesign of an unstable compressor resulted in the machine running on a heavily damped critical speed. It was concluded, after careful scrutiny, that because of the heavy damping, this was an acceptable design.

Non-Linearities

Once the threshold speed is exceeded, the rotor would whirl with increasing amplitude, if it were not for the non-linearities in the system. Such behavior can come from fluid film bearing characteristics under large displacements, and/or from actual physical contact between rotating and stationary parts when the orbit size exceeds the design running clearances.

CONCLUSION

The design analysis procedure presented offers an efficient way of assembling available information on a machine design into a concise and meaningful stability factor of safety. The simplicity of the single parameter approach encourages machine comparisons which will aid in learning more about expected overall levels of excitation, from repeated application to field stability problems.

NOMENCLATURE

X_s, Y_s	Horizontal and vertical displacements of the rotor at the mid-span location, in
X_B, Y_B	Horizontal and vertical displacements of the rotor at the bearing locations, in
$(\vec{\ })$	Vector quantity
$(\dot{\ })$	Time derivative
δ	Total shaft deflection, in
M	Effective mass of the rotor whirling in the first "U" mode on rigid bearings, lb-sec^2/in
K_s	Effective stiffness of the rotor whirling in the first "U" mode on rigid bearings, lb/in
$\bar{K}$	Cross-coupled stiffness at the rotor mid-span, lb/in
$\bar{K}_{eq}$	Value of $\bar{K}$ for energy equivalent of destabilizing forces, lb/in
$\bar{K}_{th}$	Value of $\bar{K}$ at the threshold of instability, lb/in
K_{xx}, K_{yy}	Horizontal and vertical principal stiffnesses, lb/in
K_{xy}, K_{yx}	Horizontal and vertical cross-coupled stiffnesses, lb/in
C_{xx}, C_{yy}	Horizontal and vertical principal damping, lb-sec/in
C_{xy}, C_{yx}	Horizontal and vertical cross-coupled damping, lb-sec/in
F_c	Cross-coupled force, lb
F_D	Damping force, lb
ω_{rig}	First "U" mode critical speed on rigid bearings, 1/sec
ω	First damped natural frequency for a forward whirl "U" mode at the operating speed, 1/sec
GF	Exponential growth factor, 1/sec
$\vec{i}$, $\vec{j}$	Unit vectors in the horizontal and vertical direction
a, b	Major and minor amplitudes of the elliptical orbit, in
a_{MS}, b_{MS}	Major and minor amplitudes of the elliptical orbit at rotor mid-span, in
β	Angle between major axis and x-axis
"i"	Subscript or superscript denoting mass station in the rotor model

APPENDIX

Approximate Analytical Solution

For a single mass at the mid-span of a massless flexible shaft supported by two identical bearings, and a cross-coupled stiffness at the mass location, the following equations of motion can be written:

$$M\ddot{X}_s + K_s(X_s - X_B) + \bar{K}Y_s = 0$$

$$M\ddot{Y}_s + K_s(Y_s - Y_B) - \bar{K}X_s = 0$$

$$C_{xx}\dot{X}_B + K_{xx}X_B + K_s(X_B - X_s) = 0$$

$$C_{yy}\dot{Y}_B + K_{yy}Y_B + K_s(Y_B - Y_s) = 0 \qquad \text{(A-1)}$$

When close to the threshold of instability, system damping is small and the above equations can be rewritten in terms of only x_s and y_s [17]:

$$M\ddot{X}_s + C_{xeq}\dot{X}_s + K_{xeq}X_s + \bar{K}Y_s = 0$$

$$M\ddot{Y}_s + C_{yeq}\dot{Y}_s + K_{yeq}Y_s - \bar{K}X_s = 0 \qquad \text{(A-2)}$$

where:

$$C_{xeq} = \frac{K_s^2\ C_{xx}}{[K_s + K_{xx}]^2 + \omega^2\ C_{xx}^2}$$

$$C_{yeq} = \frac{K_s^2\ C_{yy}}{[K_s + K_{yy}]^2 + \omega^2\ C_{yy}^2}$$

$$K_{xeq} = \frac{K_s\ [K_{xx}(K_{xx} + K_s) + \omega^2\ C_{xx}^2]}{[K_s + K_{xx}]^2 + \omega^2\ C_{xx}^2}$$

$$K_{yeq} = \frac{K_s\ [K_{yy}(K_{yy} + K_s) + \omega^2\ C_{yy}^2]}{[K_s + K_{yy}]^2 + \omega^2\ C_{yy}^2} \qquad \text{(A-3)}$$

To obtain a solution, the above two equations can be combined into one equation in terms of even and odd coefficients:

$$M\ddot{Z} + C_e\dot{Z} + K_eZ + C_o\dot{\bar{Z}} + K_o\bar{Z} - j\bar{K}Z = 0 \qquad \text{(A-4)}$$

where:

$$Z = X + jY;\ j = \sqrt{-1}$$

$$K_e = \tfrac{1}{2}(K_{xeq} + K_{yeq})$$

$$K_o = \tfrac{1}{2}(K_{xeq} - K_{yeq})$$

$$C_e = \tfrac{1}{2}(C_{xeq} + C_{yeq})$$

$$C_o = \tfrac{1}{2}(C_{xeq} - C_{yeq})$$

Let

$$Z = Ae^{\alpha t} + Be^{\bar{\alpha} t}$$

$$\alpha = GF + j\,\omega$$

$$\bar{\alpha} = GF - j\,\omega \tag{A-5}$$

The resulting characteristic equation is:

$$(M\alpha^2 + C_e\alpha)^2 + 2\,K_e(M\alpha^2 + C_e\alpha) + (K_e^2 + \bar{K}^2) - (C_o\alpha + K_o)^2 = 0 \tag{A-6}$$

Since typically $C_o << C_e$, assume $C_o = 0$:

$$(M\alpha^2 + C_e\alpha) = -K_e \pm \sqrt{K_o^2 - \bar{K}^2} \tag{A-7}$$

This equation has a different solution for each of two distinct regions.

1. For $K_o^2 \geqslant K^2$: (Stable Region)

$$GF_1 = -C_e/2M$$

(No Whirl)

$$\omega_1^2 = K_e/M - (C_e/2M)^2 - \sqrt{K_o^2 - \bar{I}^2}\,/M \tag{A-8}$$

$$GF_2 = GF_1$$

$$\omega_2^2 = K_e/M - (C_e/2M)^2 + \sqrt{K_o^2 - \bar{K}^2}\,/M \tag{A-9}$$

2. For $K^2 \geqslant K_o^2$:

(Forward Whirl)

$$GF_1 = -C_e/2M + \sqrt{R_1 - R_2}$$

$$\omega_1^2 = R_1 + R_2 \tag{A-10}$$

(Reverse Whirl)

$$GF_2 = -C_e/2M - \sqrt{R_1 - R_2}$$

$$\omega_2^2 = \omega_1^2 \tag{A-11}$$

where

$$R_1 = \tfrac{1}{2}\sqrt{[K_e/M - (C_e/2M)^2]^2 + [\bar{K}^2 - K_o^2]/M^2}$$

$$R_2 = \tfrac{1}{2}\,[K_e/M - (C_e/2M)^2]$$

Removing the assumption of $C_o = 0$ and solving for the cross-coupled stiffness threshold value by setting $GF = 0$ results in:

$$\bar{K}_{th}^2 = [1 - C_o^2/C_e^2]\,[K_o^2 + C_e^2\,K_e\,(1 - \frac{C_oK_o}{C_eK_e})/M] \tag{A-12}$$

The effective mass, M, and rotor stiffness, K_s, are determined from the first undamped rigid bearing bending natural frequency, ω_{rig} [11]. Using the mode shape of this frequency, the effective mass can be found from:

$$M = [\sum_{i=1}^{n} m_i\phi_i^2]\,/\,\phi_{Ms}^2 \tag{A-13}$$

where m_i is the lumped mass at station "i" in the rotor model, ϕ_i is the corresponding deflection, and ϕ_{MS} is the deflection at the rotor mid-span. The effective rotor stiffness can be found from:

$$K_s = M\ \omega_{rig}^2 \tag{A-14}$$

For between bearing rotors, "M" is approximately equal to the total mass of all of the compressor wheels plus ½ the mass of the shaft.

The general expression for a force at a destabilizing source can be written as:

$$F_x = K_{xx}X + K_{xy}Y + C_{xx}\dot{X} + C_{xy}\dot{Y}$$

$$F_y = K_{yy}Y + K_{yx}X + C_{yy}\dot{Y} + C_{yx}\dot{X} \tag{A-15}$$

The general expression for work is found by integrating the force times displacement around an elliptical orbit and summing over the rotor length. The result can be written as:

$$\begin{aligned} \text{Work}_A = \sum_i \pi \Big\{ & (K_{xy}^{(i)} - K_{yx}^{(i)})\ a_ib_i - \omega\ C_{xx}^{(i)}[a_i^2\cos^2\beta_i + b_i^2\sin^2\beta_i] \\ & - \omega\ C_{yy}^{(i)}[b_i^2\cos^2\beta_i + a_i^2\sin^2\beta_i] - \omega\ /2\ (C_{xy}^{(i)} \\ & + C_{yx}^{(i)})[a_i^2 - b_i^2]\ \sin 2\beta_i \Big\} \end{aligned} \tag{A-16}$$

For $\bar{K}_{eq} = K_{xy} = -K_{yx}$ and $C_{xx} = C_{yy} = C_{xy} = C_{yx} = 0$ at the rotor mid-span:

$$\text{Work}_B = 2\pi\bar{K}_{eq}\ a_{MS}\ b_{MS} \tag{A-17}$$

Set $\text{Work}_A = \text{Work}_B$ and solve for $\bar{K}_{eq}$:

$$\bar{K}_{eq} = \text{Work}_A/2\pi\ a_{MS}\ b_{MS} \tag{A-18}$$

Example Calculations

Calculations for the analytical solution plotted as the top curve on Figure 11 are described herein. The tilt-pad bearing has five pads, no preload, load on pad, and an installed diametral clearance of 8 mils. The full 7×7 stiffness and damping matrices were generated at the operating speed of 9500 cpm and reduced to 2×2 matrices for 3400 cpm (first damped frequency from non-excited stability program analysis).

#1 Bearing:

$K_{xx}/2 = 355000$ lb/in	$K_{yy}/2 = 893000$ lb/in
$C_{xx}/2 = 215$ lb-sec/in	$C_{yy}/2 = 678$ lb-sec/in

#2 Bearing:

$$K_{xx}/2 = 362000 \text{ lb/in} \qquad K_{yy}/2 = 998000 \text{ lb/in}$$

$$C_{xx}/2 = 232 \text{ lb-sec/in} \qquad C_{yy}/2 = 748 \text{ lb-sec/in}$$

Averaging the two bearings together:

$$K_{xx} = 2(359000) \text{ lb/in} \qquad K_{yy} = 2(946000) \text{ lb/in}$$

$$C_{xx} = 2(224) \text{ lb-sec/in} \qquad C_{yy} = 2(713) \text{ lb-sec/in}$$

From a critical speed computer program:

$$\omega_{rig} = 3900 \text{ cpm } (408.4/\text{sec})$$

From non-excited stability computer program:

$$\omega = 3400 \text{ cpm } (356.0/\text{sec})$$

From Equation A-13 and A-14:

$$M = 3.25 \text{ lb-sec}^2/\text{in}$$

$$K_s = 542069 \text{ lb/in}$$

From Equations A-3:

$$\Delta_x = [K_s + K_{xx}]^2 + \omega^2 C_{xx}{}^2 = 1.613 \times 10^{12} \text{ (lb/in)}^2$$

$$\Delta_y = [K_s + K_{yy}]^2 + \omega^2 C_{yy}{}^2 = 6.182 \times 10^{12} \text{ (lb/in)}^2$$

$$K_{xeq} = K_s[K_{xx}(K_{xx} + K_s) + \omega^2 C_{xx}{}^2]/\Delta_x = 312600 \text{ lb/in}$$

$$K_{yeq} = K_s[K_{yy}(K_{yy} + K_s) + \omega^2 C_{yy}{}^2]/\Delta_y = 426410 \text{ lb/in}$$

$$C_{xeq} = K_s{}^2 C_{xx}/\Delta_x = 81.61 \text{ lb-sec/in}$$

$$C_{yeq} = K_s{}^2 C_{yy}/\Delta_y = 67.78 \text{ lb-sec/in}$$

From Equations A-5:

$$K_e = \tfrac{1}{2} (K_{xeq} + K_{yeq}) = 369500 \text{ lb/in}$$

$$K_o = \tfrac{1}{2} (K_{xeq} - K_{yeq}) = -56910 \text{ lb/in}$$

$$C_e = \tfrac{1}{2} (C_{xeq} + C_{yeq}) = 74.70 \text{ lb-sec/in}$$

$$C_o = \tfrac{1}{2} (C_{xeq} - C_{yeq}) = 6.92 \text{ lb-sec/in}$$

From Equation A-12:

$$\bar{K}_{th} = 62230 \text{ lb/in}$$

REFERENCES

1. Enrich, F. and Childs, D., "Self-Excited Vibration in High-Performance Turbomachinery," Mechanical Engineering, pp. 66-79 (May 1984).
2. Kirk, R. G. and Donald, G. H., "Design Criteria For Improved Stability of Centrifugal Compressors," Rotor Dynamical Instability, ASME, AMD *55*, The Applied Mechanics, Bioengineering, and Fluid Engineering Conference, Houston, Texas, pp. 59-71 (June 1983).
3. Gunter, E. J., "Dynamic Stability of Rotor-Bearing Systems," NASA SP-113 (1966).
4. Dimentberg, F. M., *Flexural Vibrations of Rotating Shafts,* London: Butterworths (1961).
5. Lund, J. W., "The Stability of an Elastic Rotor in Journal Bearings with Flexible, Damped Supports," Journal of Applied Mechanics, Transactions of ASME, pp. 911-920 (December 1965).
6. Lund, J. W., "Stability and Damped Critical Speeds of a Flexible Rotor in Fluid-Film Bearings," ASME, Journal of Engineering for Industry, pp. 509-517 (May 1974).
7. Allaire, P. E., "Design of Journal Bearings for High Speed Rotating Machinery," University of Virginia, UVA/643092/MAE81/150, ROMAC Report No. 101 (1979).
8. Nicholas, J. C., Gunter, E. J. and Barrett, L. E., "The Influence of Tilting Pad Bearing Characteristics on the Stability of High-Speed Rotor Bearing Systems," UVA/643092/MAE81/141, University of Virginia, ROMAC Report No. 85, p. 72 (1978).
9. Lund, J. W., "Rotor-Bearing Dynamics Design Technology," Part V: Computer Program Manual for Rotor Response and Stability, Technical Report AFAPL-TR-65-45, Part V, p. 27 (May 1965).
10. Reddi, M. M., "ROTDYN-Flexible Rotor Dynamics Analysis—Volume I," Stability and Time Transient Response Computer Program, The Franklin Institute Research Laboratories, (March 1979).
11. Barrett, L. E., "Journal Bearing Optimization for Aerodynamically Excited Flexible Rotor-Bearing Systems," UVA/643092/MAE82/190, University of Virginia, ROMAC Report No. 168 (1982).
12. Wachel, J. C., "Rotordynamic Instability Field Problems," NASA Conference Publication 2250 – Rotor Dynamic Instability Problems in High-Performance Turbomachinery—1982, pp. 1-7 (May 1982).
13. Alford, J. S., "Protecting Turbomachinery From Self-Excited Rotor Whirl," Journal of Engineering for Power, Transactions of ASME, pp. 333-344 (October 1965).
14. Wyssmann, H. R., Pham, T. C. and Jenny, R. J., "Prediction of Stiffness and Damping Coefficients for Centrifugal Compressor Labyrinth Seals," ASME, Journal of Engineering for Gas Turbine and Power, *106,* pp. 920-926 (October 1984).
15. Schmaus, R. H. and Barrett, L. E., "Static and Dynamic Properties of Finite Length Turbulent Flow Annular Seals," University of Virginia, UVA/643092/MAE81/178, ROMAC Report No. 145 (January 1981).
16. Parsell, J. K., Allaire, P. E., and Barrett, L. E., "Frequency Effects in Tilting Pad Journal Bearing Dynamic Coefficients," University of Virginia (June 1981).
17. Barrett, L. E., Gunter, E. J. and Allaire, P. E., "Determination of Optimum Bearing Damping for Symmetric Rotor Bearing Systems," UVA/643092/MAE81/118, University of Virginia, ROMAC Report No. 47, pp. 2-6 (1976).
18. Kirk, R. G. and Miller, W. H., "The Influence of High Pressure Oil Seals on Turbo-Rotor Stability," ASLE Transactions, *22,* (1), pp. 14-24 (1979).

ACNOWLEDGEMENTS

The author wishes to thank J. C. Swalley and J. M. Perron for their helpful suggestions on the editing of this paper.