



Non-supersymmetric D-branes with vanishing cylinder amplitudes in asymmetric orbifolds

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| 著者 | Satoh Yuji, Sugawara Yuji, Uetoko Takahiro |
| journal or publication title | Journal of High Energy Physics |
| volume | 2017 |
| number | 8 |
| page range | 82 |
| year | 2017-08 |
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| URL | http://hdl.handle.net/2241/00148318 |

doi: 10.1007/JHEP08(2017)082



Non-supersymmetric D-branes with vanishing cylinder amplitudes in asymmetric orbifolds

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ABSTRACT: We study the type II string vacua with chiral space-time SUSY constructed as asymmetric orbifolds of torus and $K3$ compactifications. Despite the fact that all the D-branes are non-BPS in any chiral SUSY vacua, we show that the relevant non-geometric vacua of asymmetric orbifolds allow rather generally configurations of D-branes which lead to vanishing cylinder amplitudes, implying the bose-fermi cancellation at each mass level of the open string spectrum. After working on simple models of toroidal asymmetric orbifolds, we focus on the asymmetric orbifolds of $T^2 \times \mathcal{M}$, where \mathcal{M} is described by a general $\mathcal{N} = 4$ SCFT with $c = 6$ defined by the Gepner construction for $K3$. Even when the modular invariant partition functions in the bulk remain unchanged, the spectra of such non-BPS D-branes with the bose-fermi cancellation can vary significantly according to the choice of orbifolding.

KEYWORDS: Conformal Field Models in String Theory, D-branes, Superstring Vacua, Superstrings and Heterotic Strings

ARXIV EPRINT: [1704.05262](https://arxiv.org/abs/1704.05262)

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1 Introduction

String theories on the *non-geometric* backgrounds may induce interesting features which are not realized in the standard geometric compactifications. One of the salient aspects of such non-geometric vacua would be the vanishing cosmological constant without unbroken SUSY. This is in contrast to our experiences in ordinary geometric string vacua that the SUSY-violation generically gives rise to cosmological constant at the breaking mass scale (string scale, typically). The attempts of the construction of non-SUSY vacua with vanishing cosmological constant have been initiated by [1–3] based on some non-abelian orbifolds, followed by closely related studies e.g. in [4–9]. More recently, several non-SUSY vacua with this property have been constructed as asymmetric orbifolds [10] by simpler cyclic groups in [11, 12]. Studies of non-SUSY vacua in heterotic string theory have also been presented e.g. in [13–20].

In this paper, we would like to focus on similar interesting aspects of non-BPS D-branes in simple models of non-geometric type II string vacua. Let us first recall that the BPS D-branes are described by the boundary states satisfying the BPS-equation,

$$\left[Q^\alpha + M^\alpha_\beta \tilde{Q}^\beta \right] |B\rangle\rangle = 0, \tag{1.1}$$

where Q^α (\tilde{Q}^β) denotes the left(right)-moving space-time supercharges and M^α_β are some c-number coefficients. Through this paper, we express boundary states by $|\cdots\rangle\rangle$, $\langle\langle\cdots|$.

We then anticipate that the cylinder amplitude of which both ends are attached to the common BPS D-brane, which we call the ‘self-overlap’ in this paper, should vanish,

$$Z_{\text{cyl}}(s) \equiv \langle\langle B|e^{-\pi s H^{(c)}}|B\rangle\rangle = 0. \tag{1.2}$$

Here we identify $s \in \mathbb{R}_{>0}$ as the closed string modulus and $t \equiv 1/s$ as the open string one. Needless to say, this means that we have a precise bose-fermi cancellation at each mass level in the open string spectrum, naturally expected from the BPS property of the D-brane. However, the bose-fermi cancellation (1.2) does not necessarily imply that the boundary state $|B\rangle\rangle$ satisfies the BPS equation (1.1). Indeed, it has been known that, in some superstring vacua, there exist non-BPS configurations of D-branes that however realize the bose-fermi cancellation of open strings [6, 21]. The main purpose of this paper is to demonstrate that *non-geometric* backgrounds of superstring theory rather generally accommodate such non-BPS D-branes with vanishing cylinder amplitudes.

Although we concentrate in this paper mainly on the theoretical aspects from the view points of world-sheet conformal field theory, we would also like to mention a ‘physical’ motivation of this work: since the closed string sector in the bulk is supersymmetric in our setting, the supersymmetry would be broken solely by the effect of the non-BPS D-branes. More concretely, if we have sufficiently generic configurations of the non-BPS D-branes as above, the SUSY-breaking would be brought about by the condensation of the non-BPS ‘D-brane instantons’ (Euclidean D-branes wrapping around internal cycles). In such a case, because the $O(g_s^0)$ -contributions to the cosmological constant, as well as the bulk ones, still vanish due to (1.2), we would be left with a non-perturbatively small cosmological constant induced by the instanton effect, which is exponentially suppressed as long as the string coupling g_s is sufficiently small. Such a possibility in a type II theory has indeed been mentioned in [4] based on the analysis of its heterotic dual. The present work may be a step toward realizing such string vacua with small cosmological constant.

Now, let us make a brief sketch of our basic idea:

- We start with the type II superstring vacua preserving only the *chiral* SUSY, which are straightforwardly constructed by the asymmetric orbifolding by the twist σ that eliminates, say, all the left-moving supercharges Q^α .
- In these vacua, while the cosmological constant in the bulk should vanish due to the existence of unbroken SUSY, *any* D-branes cannot be BPS. In other words, any boundary states cannot satisfy the BPS equation (1.1) due to the lack of Q^α .
- We search for the boundary states realizing nevertheless the vanishing self-overlap (1.2), which are obtained from the BPS D-branes $|B\rangle\rangle_0$ in the untwisted theory by the orbifold projection, $|B\rangle\rangle \propto \mathcal{P}|B\rangle\rangle_0$. The conformal invariance is maintained, since \mathcal{P} commutes with the Virasoro operators.

Of course, in generic chiral SUSY vacua, there are no solutions of the boundary states with the vanishing self-overlaps. However, once the asymmetric twist to preserve the chiral SUSY is given, the self-overlap of the projected D-brane $|B\rangle\rangle$ is likely to be vanishing as

long as it inherits the structure of the bose-fermi cancellation in the bulk torus amplitude. As shown in the following sections, it is indeed possible to find simple models of such asymmetric orbifolds, and thus plenty of boundary states with the vanishing self-overlap. In section 2, we study toroidal models and consider several asymmetric orbifoldings preserving 8 supercharges coming only from the right-mover.

In section 3, which is the main part of this paper, we shall discuss less supersymmetric models constructed as the asymmetric orbifolds of the backgrounds,

$$\mathbb{R}^{3,1} \times T^2 \times \mathcal{M}, \tag{1.3}$$

where \mathcal{M} is described by a general $\mathcal{N} = 4$ superconformal field theory (SCFT) with $\hat{c}(\equiv \frac{c}{3}) = 2$, which geometrically describes compactifications on $K3$ with particular moduli. The relevant asymmetric orbifolds are defined by the twisting,

$$\sigma = (-\mathbf{1}_R)^{\otimes 2} \otimes \sigma_{\mathcal{M}}, \tag{1.4}$$

where $(-\mathbf{1}_R)^{\otimes 2}$ is the chiral reflection on the T^2 -sector ($X^{4,5}$ -directions),

$$(-\mathbf{1}_R)^{\otimes 2} : (X_L^i, X_R^i) \mapsto (X_L^i, -X_R^i), \quad (\psi_L^i, \psi_R^i) \mapsto (\psi_L^i, -\psi_R^i), \quad (i = 4, 5), \tag{1.5}$$

and $\sigma_{\mathcal{M}}$ denotes an involution on the \mathcal{M} -sector, which is allowed to act asymmetrically on the $\mathcal{N} = 4$ superconformal algebra (SCA). As we will clarify later, one obtains in this way the chiral SUSY vacua with the 4-dim. $\mathcal{N} = 1$ SUSY (4 supercharges). We then classify the possible gluing conditions for the boundary states, which are decomposed into the Ishibashi states [40] for each $\mathcal{N} = 4$ unitary irreducible representations (irrep.'s), and examine whether or not their self-overlaps vanish. The spectra of the non-BPS boundary states with this property non-trivially depend on the choice of the twist operator $\sigma_{\mathcal{M}}$, even in the cases when the modular invariant partition functions remain unchanged; different $\sigma_{\mathcal{M}}$'s may lead to the same partition functions in the bulk.

2 Toroidal asymmetric orbifolds

In this section we shall focus on the simpler cases, namely, the asymmetric orbifolds of tori realizing the chiral SUSY vacua of type II string, in order to show how the strategy outlined above is implemented. The discussion is straightforwardly extended to the case of $K3$ in the next section, though it is technically a little more involved.

2.1 Asymmetric orbifold $T^4[D_4]/[(-1)^{F_L} \otimes (-\mathbf{1}_R)^{\otimes 4}]$

Let us first consider the asymmetric orbifold of the 4-dim. torus T^4 , which would be the simplest model that has the desired properties. We assume the torus is along the $X^{6,\dots,9}$ -directions and at the symmetry enhancement point with $\widehat{SO}(8)_1$. We thus denote it as $T^4[D_4]$, the corresponding partition function of which reads

$$Z^{T^4[D_4]}(\tau, \bar{\tau}) = \frac{1}{2} \left\{ \left| \frac{\theta_3}{\eta} \right|^8 + \left| \frac{\theta_4}{\eta} \right|^8 + \left| \frac{\theta_2}{\eta} \right|^8 \right\}. \tag{2.1}$$

The orbifold group is generated by a single element

$$\sigma \equiv (-1)^{FL} \otimes (-\mathbf{1}_R)^{\otimes 4}, \tag{2.2}$$

which acts as the chiral reflection on the right-mover, $X_R^i \rightarrow -X_R^i$, $\psi_R^i \rightarrow -\psi_R^i$ ($i = 6, \dots, 9$), accompanied by the twisting of the space-time fermion number $(-1)^{FL}$ on the left-moving fermions, that is, the sign-flip of arbitrary states in the left-moving R(amond)-sector. Closely related asymmetric orbifolds adopting slightly different setting have been analyzed in the bulk [11, 12] for non-supersymmetric string vacua with vanishing cosmological constant. The analysis below follows these references.

We simply assume that σ^2 acts on the untwisted Hilbert space as an involution for the free bosons X_R^i , whereas we naturally have two possibilities on the fermionic sector; (i) $\sigma^2 = \mathbf{1}$, (ii) $\sigma^2 = (-1)^{FR}$, depending on the definition of the Ramond vacua or the way of bosonization to introduce the spin fields (see also section 3.1). Here, the operator $(-1)^{FR}$ just acts as the sign flip on any states in the right-moving R-sector. We separately examine these two cases:

(i) $\sigma^2 = \mathbf{1}$ (on the untwisted Hilbert space). In this case, the modular invariant is written as

$$Z(\tau, \bar{\tau}) = Z_{\text{bosonic}}^{6d}(\tau, \bar{\tau}) \frac{1}{4} \sum_{a,b \in \mathbb{Z}_4} Z_{(a,b)}^{T^4[D_4]}(\tau, \bar{\tau}) h_{(a,b)}(\tau) \overline{f_{(a,b)}(\tau)},$$

$$Z_{(a,b)}^{T^4[D_4]}(\tau, \bar{\tau}) := \begin{cases} Z^{T^4[D_4]}(\tau, \bar{\tau}) & (a, b \in 2\mathbb{Z}), \\ \epsilon_{(a,b)}^{[4]} \chi_{(a,b)}^{D_4}(\tau) \left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^4 & (a \in 2\mathbb{Z} + 1, \text{ or } b \in 2\mathbb{Z} + 1). \end{cases} \tag{2.3}$$

where $Z_{\text{bosonic}}^{6d}(\tau, \bar{\tau})$ denotes the partition function of the bosonic sector of uncompactified space-time $\mathbb{R}^{5,1}$. The building blocks $\chi_{(a,b)}^{D_4}$, $\overline{\left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^4}$, $h_{(a,b)}$ and $\overline{f_{(a,b)}}$ are evaluated for the sectors of $X_L^{6,\dots,9}$, $X_R^{6,\dots,9}$, the left-moving fermions, and the right-moving fermions, respectively, where the subscript (a, b) labels the sectors with the spatial and temporal twists by σ given in (2.2). They are obtained first for the (0,1) sector with one temporal twist, and then for other sectors by the modular transformation. Their explicit forms are summarized in appendix A. (See (A.7), (A.12), (A.14), (A.17).) The phase factor $\epsilon_{(a,b)}^{[r]}$ is defined in [22], and explicitly written as

$$\epsilon_{(a,b)}^{[r]} := e^{\frac{i\pi}{8} r (-1)^{ab}} \left(\kappa_{(a,b)} \right)^r, \quad (a \in 2\mathbb{Z} + 1 \text{ or } b \in 2\mathbb{Z} + 1), \tag{2.4}$$

with

$$\kappa_{(a,b)} := \begin{cases} -1 & a \equiv 3, 5 \pmod{8}, \quad b \in 2\mathbb{Z} + 1, \\ 1 & \text{otherwise.} \end{cases} \tag{2.5}$$

It is quite useful to note that the combination $\epsilon_{(a,b)}^{[r]} \chi_{(a,b)}^{X_r}(\tau) \overline{\left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^r}$ (or $\epsilon_{(a,b)}^{[-r]} \left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^r \overline{\chi_{(a,b)}^{X_r}(\tau)}$) is organized so as to be modular covariant with respect to (a, b) , where X_r denotes the suitable Lie algebra lattice of rank r presented in [22]. Namely, any modular transformation defined by $A \in \text{SL}(2; \mathbb{Z})$ acts simply on the subscript (a, b) as

$(a, b) \mapsto (a, b)A$. We note that this is an order 4 orbifold due to the existence of the phase factor (2.4) despite $\sigma^2 = \mathbf{1}|_{\text{untwisted}}$, which would be a typical feature in asymmetric orbifolds.

The right-mover preserves 1/2 space-time SUSY, whereas the left-moving space-time SUSY is completely broken. In fact, it is obvious that $\sigma \equiv (-1)^{F_L} \otimes (-\mathbf{1}_R)^{\otimes 4}$ cannot preserve any left-moving supercharges in the even a sector, which are essentially those in the unorbifolded theory. Furthermore, if we had a left-moving supercharges belonging to the sector $a = 1$, we should obtain the equality of the partition functions

$$Z_{a=0}^{(\text{NS}, \text{NS})}(\tau, \bar{\tau}) = -Z_{a=1}^{(\text{R}, \text{NS})}(\tau, \bar{\tau}). \tag{2.6}$$

However, it is easy to see that this is not the case, when observing the explicit forms of relevant partition functions. We can similarly show the absence of supercharges in the $a = -1$ sector. On the other hand, half of untwisted supercharges in the right-mover are σ -invariant, as in the familiar supersymmetric orbifold $T^4 / [(-\mathbf{1}_L)^{\otimes 4} \otimes (-\mathbf{1}_R)^{\otimes 4}]$.

Now, let us move on to the discussion on the non-BPS D-branes. As already pointed out, no D-brane can preserve space-time SUSY. Nevertheless, rather general ‘bulk-type branes’ lead to the vanishing self-overlap.¹ In fact, consider the bulk-type brane written as an orbifold projection,

$$|B\rangle\rangle = \sqrt{2}\mathcal{P}|B\rangle\rangle_0, \tag{2.7}$$

where $|B\rangle\rangle_0$ stands for the GSO-projected boundary state describing any BPS D-brane in the unorbifolded theory on $\mathbb{R}^{5,1} \times T^4[D_4]$, and $\mathcal{P} = \frac{1}{2}(1 + \sigma)$ is the projection operator onto the invariant sector under the twist. As described in the introduction, \mathcal{P} commutes with the Virasoro operators and maintains the conformal invariance. The overall normalization factor $\sqrt{2}$ has been determined by the Cardy condition. By definition, we have ${}_0\langle\langle B|e^{-\pi sH^{(c)}}|B\rangle\rangle_0 = 0$, since $|B\rangle\rangle_0$ is BPS. Moreover, explicit computation gives

$${}_0\langle\langle B|\sigma e^{-\pi sH^{(c)}}|B\rangle\rangle_0 \equiv {}_0\langle\langle B| [(-1)^{F_L} \otimes (-\mathbf{1}_R)^{\otimes 4}] e^{-\pi sH^{(c)}}|B\rangle\rangle_0 \propto f_{(0,1)}(is) \equiv 0. \tag{2.8}$$

Again $f_{(0,1)}(is)$ is defined in (A.14). We thus obtain

$$\langle\langle B|e^{-\pi sH^{(c)}}|B\rangle\rangle = {}_0\langle\langle B|e^{-\pi sH^{(c)}}|B\rangle\rangle_0 + {}_0\langle\langle B|\sigma e^{-\pi sH^{(c)}}|B\rangle\rangle_0 = 0. \tag{2.9}$$

Although the left- and right-movers are correlated in the boundary states due to the conformal invariance, the twist thereon still leads to the same function $f_{(0,1)}$ as in the bulk, which is regarded as a remnant of the bulk computation. In this way, we have successfully shown that the present string vacuum possesses the desired property to have the non-BPS D-branes with vanishing self-overlaps.

Because of the overall factor in $|B\rangle\rangle$, its coupling to the gravitons (tension) is $\sqrt{2}$ times that in the unorbifolded theory. The coupling to the RR-particles (RR charge) is also multiplied by $\sqrt{2}$. By the modular transformation, the standard open string excitations in the original theory are found to remain in the self-overlap of the unorbifolded part $|B\rangle\rangle_0$. These are common features for all the non-BPS branes with the vanishing self-overlaps treated in this paper.

¹We shall call the boundary states made up only by the untwisted sector as the ‘bulk-type’ to distinguish them from the ‘fractional branes’ that include the contributions from the twisted sectors.

Absence of tachyonic instability. Let us briefly check that no open string tachyons emerge in the cylinder amplitude,

$$Z_{\text{cylinder}}(it) = \langle\langle B|e^{-\pi s H^{(c)}}|B\rangle\rangle, \quad (t \equiv 1/s).$$

In fact, the piece ${}_0\langle\langle B|e^{-\pi s H^{(c)}}|B\rangle\rangle_0$ is just the same as the familiar cylinder amplitude associated to the BPS brane, whereas

$$\begin{aligned} {}_0\langle\langle B|\sigma e^{-\pi s H^{(c)}}|B\rangle\rangle_0 &\propto \left(\sqrt{\frac{2\eta(is)}{\theta_2(is)}}\right)^4 \cdot f_{(0,1)}(is) \equiv \frac{\theta_3(is)^2\theta_4(is)^2}{\eta(is)^4} \cdot f_{(0,1)}(is) \quad (2.10) \\ &= \frac{\theta_3(it)^2\theta_2(it)^2}{\eta(it)^4} \cdot f_{(1,0)}(it) \equiv \frac{\theta_3(it)^4\theta_2(it)^4}{2\eta(it)^8} - \frac{\theta_2(it)^4\theta_3(it)^4}{2\eta(it)^8}. \end{aligned}$$

In the last line, the first and second terms are identified as the NS and R-sector amplitudes in the open string channel, of which leading terms are obviously massless. We then obtain 16 pairs of massless bosonic and fermionic states from the orbifolded part, even though no supercharges in the closed string sector preserve the boundary state $|B\rangle\rangle$. We can similarly show the absence of open string tachyons in the cylinder amplitudes with the bose-fermi cancellation also for other models discussed below.

(ii) $\sigma^2 = (-1)^{F_R}$ (on the untwisted Hilbert space). In this case, σ acts as a \mathbb{Z}_4 -action already on the untwisted sector, and the modular invariant is slightly modified as

$$Z(\tau, \bar{\tau}) = Z_{\text{bosonic}}^{6d}(\tau, \bar{\tau}) \frac{1}{4} \sum_{a,b \in \mathbb{Z}_4} Z_{(a,b)}^{T^4[D_4]}(\tau, \bar{\tau}) h_{(a,b)}(\tau) \overline{\mathbf{f}_{(a,b)}(\tau)}. \quad (2.11)$$

The fermionic chiral block $h_{(a,b)}$ is again given in (A.17), while $\mathbf{f}_{(a,b)}$, given in (A.18), is slightly modified from $f_{(a,b)}$ due to the relation $\sigma^2 = (-1)^{F_R}$. The left-mover has no space-time SUSY as in the first model. At first glance, it seems that the right-moving SUSY is also broken, because all of the supercharges in the untwisted sector are projected out by $(-1)^{F_R}$. However, it is found that (NS,R)-massless states appear in the $a = 2$ twisted sector, suggesting the existence of new 8 supercharges. These states possess the opposite chirality to the case (i), because the orbifolding by $(-1)^{F_R}$ acts like the T-duality transformation (see e.g. [21, 23]). In the end, we indeed obtain a chiral SUSY vacuum. One can check that the partition function vanishes after summing up $a, b \in 2\mathbb{Z}$, although each $\overline{\mathbf{f}_{(a,b)}(\tau)}$ is not necessarily vanishing.

The non-BPS D-branes with vanishing self-overlaps are given by the formula similar to (2.7), but including the contribution from *the $a = 2$ twisted sector*;

$$\begin{aligned} |B\rangle\rangle &= \sqrt{2}\mathcal{P}_4 \left[|B\rangle\rangle_0^{(a=0)} + |B\rangle\rangle_0^{(a=2)} \right] \\ &= \sqrt{2}\mathcal{P}_2 \left[|B\rangle\rangle_0^{(\text{NS}, a=0)} + |B\rangle\rangle_0^{(\text{R}, a=2)} \right] \equiv \sqrt{2}\mathcal{P}_2 |B\rangle\rangle_0^{(\text{opp. BPS})}, \quad (2.12) \end{aligned}$$

where

$$\mathcal{P}_4 \equiv \frac{1}{4} \sum_{n \in \mathbb{Z}_4} \sigma^n, \quad \mathcal{P}_2 \equiv \frac{1}{2} (1 + \sigma), \quad (2.13)$$

and $|B\rangle\rangle_0^{(a=0)}$ is a BPS boundary state in the unorbifolded theory as before. On the other hand, $|B\rangle\rangle_0^{(a=2)}$ is a suitably defined boundary state lying in the $a = 2$ sector, which contains the right-moving Ramond ground states with the opposite chirality as addressed above. (Obviously, we have no solutions of the boundary states in the $a = \pm 1$ sectors.) We return to this point shortly, but here just note σ^2 acts as $(-1)^{F_R-1}$ on the $a = 2$ sector, rather than $(-1)^{F_R}$, which is read off from the expression of $\mathbf{f}_{(a,b)}(\tau)$ in (A.18). The $a = 2$ NSNS-sector is thus projected out by the \mathcal{P}_4 -action, while the $a = 0$ RR-sector drops off. We are then left with the \mathcal{P}_2 -projection of the ‘opposite BPS’ boundary state, which accounts for the second line of (2.12). In other words, if we consider the type IIA (IIB) vacuum of this asymmetric orbifold, $|B\rangle\rangle_0^{(\text{opp. BPS})}$ is regarded as describing a BPS brane in the type IIB (IIA) strings on $\mathbb{R}^{5,1} \times T^4[D_4]$. One could schematically understand these aspects as

$$\begin{aligned} & \text{[IIA (IIB) vacuum on } T^4[D_4]/\sigma \text{ with } \sigma^2 = (-1)^{F_R} \\ & \cong \text{[IIB (IIA) vacuum on } T^4[D_4]/\sigma \text{ with } \sigma^2 = \mathbf{1}. \end{aligned} \tag{2.14}$$

In fact, in the second case (ii), we can resolve the orbifold group as²

$$\mathbb{Z}_4 \text{ generated by } \{\sigma_{\text{case (ii)}}\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ generated by } \{\sigma_{\text{case (i)}}, (-1)^{F_R}\}, \tag{2.15}$$

and by the relation suggested in [21, 23],

$$\text{[IIA (IIB) vacuum]} / (-1)^{F_R} \cong \text{[IIB (IIA) vacuum]}, \tag{2.16}$$

we obtain the above equivalence (2.14). Given this equivalence, a way to construct $|B\rangle\rangle_0^{(a=2)}$ is tracing back the relation in (2.12), as mentioned above. The observation here is used to reduce the number of the cases to be analyzed in the following sections.

2.2 Asymmetric orbifold $[T^4[D_2 \oplus D_2] \times S^1_R] / [(-1_L)^{\otimes 2} \otimes (-1_R)^{\otimes 4}]$

The point of the construction in the previous subsection is rather general as described in the introduction, and various generalizations would be possible. As an example where the open-string boundary condition is more relevant, we next focus on a case of the 5-dim. torus T^5 along the $X^{5,\dots,9}$ -directions. To be more specific, we begin with the following compactification:

- $X^{6,7,8,9}$ -directions.

We consider

$$T^4[D_2 \oplus D_2] \equiv T^2[D_2] \times S^1[A_1] \times S^1[A_1], \tag{2.17}$$

where $S^1[A_1]$ denotes the circle with the self-dual radius.

- X^5 -direction.

We just consider S^1_R , that is, the circle compactification with an arbitrary radius R .

²If further incorporating a shift operator into the orbifold action, i.e. considering the orbifolding by $\sigma \otimes \mathcal{T}_{2\pi R}$ as in [11], we do not have such a resolution.

Then, we consider the orbifolding by

$$\sigma := (-\mathbf{1}_L)^{\otimes 2}|_{5,6} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{5,7,8,9}, \quad (2.18)$$

where $(-\mathbf{1}_L)^{\otimes 2}|_{5,6}$, for instance, means the chiral reflection acting along the left-movers of $X^{5,6}$ -directions. Based on the twists of this type and related ones, non-SUSY vacua with vanishing cosmological constant have been investigated in [12].

The total modular invariant is given in the form,

$$Z(\tau, \bar{\tau}) = Z_{\text{bosonic}}^{5d}(\tau, \bar{\tau}) \frac{1}{4} \sum_{a,b \in \mathbb{Z}_4} Z_{(a,b)}^{T^4 \times S^1}(\tau, \bar{\tau}) g_{(a,b)}(\tau) \overline{f_{(a,b)}(\tau)}, \quad (2.19)$$

with

$$Z_{(a,b)}^{T^4 \times S^1}(\tau, \bar{\tau}) := \begin{cases} Z^{T^4[D_2 \oplus D_2]}(\tau, \bar{\tau}) Z^{S^1}(\tau, \bar{\tau}) & (a, b \in 2\mathbb{Z}), \\ \epsilon_{(a,b)}^{[2]} \chi_{(a,b)}^{D_2}(\tau) \left| \chi_{(a,b)}^{A_1}(\tau) \right|^2 \left| \tilde{\chi}_{(a,b)}^{A_1}(\tau) \right|^4 \overline{\left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^2} & (a \in 2\mathbb{Z}+1, \text{ or } b \in 2\mathbb{Z}+1). \end{cases} \quad (2.20)$$

Here Z_{bosonic}^{5d} denotes the contribution from the bosonic part of $\mathbb{R}^{4,1}$. In the second line, we have combined $\left| \tilde{\chi}_{(a,b)}^{A_1}(\tau) \right|^2$ from the X^5 -direction, $\epsilon_{(a,b)}^{[-1]} \tilde{\chi}_{(a,b)}^{A_1} \overline{\chi_{(a,b)}^{A_1}}$ from the X^6 -direction, and $\epsilon_{(a,b)}^{[3]} \chi_{(a,b)}^{D_2} \chi_{(a,b)}^{A_1} \overline{\left(\tilde{\chi}_{(a,b)}^{A_1} \right)^3}$ from the $X^{7,8,9}$ -directions. The character functions $\chi_{(a,b)}^{A_1}$, $\tilde{\chi}_{(a,b)}^{A_1}$, $\chi_{(a,b)}^{D_2}$ and the free fermion chiral blocks $f_{(a,b)}$, $g_{(a,b)}$ are summarized in appendix A. As already mentioned, the modular covariance of $Z_{(a,b)}^{T^4 \times S^1}$ is assured due to the phase factor $\epsilon_{(a,b)}^{[*]}$ (2.4).

Again we have various possibilities of the action of σ^2 on the R-sector; (i) $\sigma^2 = \mathbf{1}$, (ii) $\sigma^2 = (-1)^{F_R}$, (iii) $\sigma^2 = (-1)^{F_L}$, (iv) $\sigma^2 = (-1)^{F_L + F_R}$. The modular invariant (2.19) describes the first case (i). The modular invariants for the remaining cases are easy to construct. Namely, we only have to replace the chiral blocks $f_{(a,b)}$, $g_{(a,b)}$ in (2.20) with the ones given in (A.18), (A.19) suitably. However, as mentioned at the last part in the previous subsection, the cases (ii), (iii) reduces to the first case (i) as in (2.14), and the case (iv) corresponds to a non-SUSY vacuum, which is beyond the scope of this work.

Therefore, it is enough to focus on the simplest case (i). We can pick up any BPS boundary states $|B\rangle\rangle_0$ in the unorbifolded theory on $T^4 \times S^1$, and define the non-BPS brane $|B\rangle\rangle$ by the orbifold projection in the same way as (2.7). It is not difficult to show that $|B\rangle\rangle$ has the vanishing self-overlap as long as $|B\rangle\rangle_0$ satisfies the general gluing condition (for the $T^4 \times S^1$ -directions) given by

$$\begin{aligned} \left[\alpha_{L,n}^5 \pm \alpha_{R,-n}^5 \right] |B\rangle\rangle_0 &= 0, & \left[\psi_{L,r}^5 \pm i\psi_{R,-r}^5 \right] |B\rangle\rangle_0 &= 0, \\ \left[\alpha_{L,n}^6 \pm \alpha_{R,-n}^6 \right] |B\rangle\rangle_0 &= 0, & \left[\psi_{L,r}^6 \pm i\psi_{R,-r}^6 \right] |B\rangle\rangle_0 &= 0, \\ \left[\alpha_{L,n}^i + M^i_j \alpha_{R,-n}^j \right] |B\rangle\rangle_0 &= 0, & \left[\psi_{L,r}^i + iM^i_j \psi_{R,-r}^j \right] |B\rangle\rangle_0 &= 0, \end{aligned} \quad (i, j = 7, 8, 9), \quad (2.21)$$

where $\alpha_{L,n}^i, \alpha_{R,n}^i$ and $\psi_{L,r}^i, \psi_{R,r}^i$ denote the bosonic and fermionic oscillators (including the bosonic zero modes), and M_j^i is an arbitrary $SO(3)$ -matrix.³ To show this fact, it is useful to note the relation,

$$\sigma |B\rangle\rangle_0 \equiv (-\mathbf{1}_L)^{\otimes 2}|_{5,6} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{5,7,8,9} |B\rangle\rangle_0 = (-\mathbf{1}_R)^{\otimes 4}|_{6,7,8,9} |B\rangle\rangle_0, \quad (2.22)$$

for any $|B\rangle\rangle_0$ satisfying (2.21). We thus obtain

$${}_0\langle\langle B|\sigma e^{-\pi s H^{(c)}}|B\rangle\rangle_0 \propto f_{(0,1)}(is) = 0, \quad (2.23)$$

similarly to (2.8), leading to the vanishing cylinder amplitude, $\langle\langle B|e^{-\pi s H^{(c)}}|B\rangle\rangle = 0$.

We add a comment: in the model of (2.1) the vanishing self-overlaps have been achieved for arbitrary BPS boundary states $|B\rangle\rangle_0$ in the unorbifolded theory. On the other hand, in the current case, $|B\rangle\rangle_0$ defined by (2.21) is restricted to $(M_j^i) \in SO(3)$ rather than $(M_j^i) \in SO(4)$. If adopting a different orbifold action instead of (2.18), say, $\sigma \equiv (-\mathbf{1}_L)^{\otimes 2}|_{5,7} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{5,6,8,9}$, we still obtain the same modular invariant, yielding the equivalent spectrum of closed string states. Moreover, it is obvious to have an essentially equivalent spectrum of non-BPS branes with the vanishing self-overlaps, in which $(M_j^i) \in SO(3)$ appearing in (2.21) should act on the $X_R^{6,8,9}$ -directions this time. This fact is not surprising, of course. However, in the next section, we will see the examples in which the spectra of the non-BPS branes with vanishing self-overlaps would notably depend on the choice of orbifolding, while the modular invariant partition functions remain unchanged.

3 Chiral SUSY vacua as asymmetric orbifolds of $T^2 \times K3$

In this section, we shall study less supersymmetric cases with $\mathcal{M} = K3$ in the background (1.3),

$$\mathbb{R}^{3,1} \times T^2[D_2] \times \mathcal{M}. \quad (3.1)$$

The strategy to construct the non-BPS D-branes with vanishing self-overlaps is the same as in the previous section. The discussion is however a little more involved. We thus first summarize the relevant asymmetric orbifolds in the bulk in subsection 3.1. We then concentrate on the examples of the Gepner construction in subsection 3.2. For these subsections, we follow [24] where the modular invariant partition functions of related asymmetric orbifolds ('mirrorfolds') are constructed. (See also [25].) Based on these set-ups, we construct the non-BPS D-branes in subsection 3.3, which is the main part of this section 3.

3.1 Asymmetric orbifolds of $T^2 \times K3$ with chiral SUSY

To begin with, we assume that the \mathcal{M} -sector is described by a general $\mathcal{N} = (4, 4)$ SCFT with $\hat{c} (\equiv \frac{c}{3}) = 2$, not reducing to the toroidal models. We denote the relevant $\mathcal{N} = 4$ SCA [26] by L_n (Virasoro), J_n^α ($\widehat{SU}(2)_1$) with $\alpha = 1, 2, 3$, G_r^a with $a = 0, 1, 2, 3$. Recall that the total R-symmetry is given by $SO(4) \cong SU(2)_c \times SU(2)_f$, in which the inner symmetry

³Since $T^4[D_2 \oplus D_2] = T^4[(A_1)^4]$ holds, the $SO(3) \subset SO(4)$ -rotated gluing condition is well-defined, even though $SO(3)$ is not a part of symmetry on this string vacuum.

(‘color SU(2)’) is generated by the affine currents J^α , whereas the global SU(2)-symmetry (‘flavor SU(2)’) is an outer one. G^0 is a singlet of $SU(2)_{\text{diag}} \subset SU(2)_c \times SU(2)_f$, while G^1, G^2, G^3 compose a triplet of $SU(2)_{\text{diag}}$.

We also assume that $\mathcal{N} = 2$ SCA is embedded into the $\mathcal{N} = 4$ one in the standard fashion by identifying the $\mathcal{N} = 2$ $U(1)_R$ -current as

$$J^{\mathcal{N}=2} = 2J^3, \tag{3.2}$$

and

$$G^\pm = \frac{1}{2} (G^0 \pm iG^3). \tag{3.3}$$

The generators of integral spectral flows $U_{\pm 1}$ are identified with the remaining SU(2)-currents

$$U_{\pm 1} = J^\pm \equiv J^1 \pm iJ^2, \tag{3.4}$$

and the half-spectral flows $U_{\pm 1/2}$ define the Ramond sector.

Let us now consider the asymmetric orbifolding of (3.1) by $\sigma \equiv \sigma_{\mathcal{M}} \otimes (-\mathbf{1}_R)^{\otimes 2}$, where $(-\mathbf{1}_R)^{\otimes 2}$ denotes the chiral reflection along the $T^2[D_2]$ -direction. We first note the action of $(-\mathbf{1}_R)^{\otimes 2}$ on the world-sheet fermions, which we assign to ψ_R^4, ψ_R^5 . We bosonize them as⁴

$$\psi_R^4 + i\psi_R^5 = \sqrt{2}e^{iH_R^{T^2}}, \tag{3.5}$$

and $(-\mathbf{1}_R)^{\otimes 2}$ should act as the shift $H_R^{T^2} \rightarrow H_R^{T^2} + \pi$. It fixes the action of $(-\mathbf{1}_R)^{\otimes 2}$ on the R-sector and we find

$$[(-\mathbf{1}_R)^{\otimes 2}]^2 = (-1)^{F_R}. \tag{3.6}$$

We next consider the \mathcal{M} -sector. We would like to suitably choose the orbifold twisting $\sigma_{\mathcal{M}}$ so as to obtain a 4-dim. vacuum with $\mathcal{N} = (0, 1)$ -chiral SUSY unbroken. Obviously $\sigma_{\mathcal{M}}$ should be an automorphism of both left and right-moving $\mathcal{N} = 4$ SCAs. Furthermore, since working on superstring vacua in the NSR-formalism, $\sigma_{\mathcal{M}}$ should satisfy the following conditions:

- (i) $\sigma_{\mathcal{M}}$ preserves T (energy-momentum tensor) and G^0 , which is necessary for the BRST-invariance.
- (ii) $\sigma_{\mathcal{M}}$ keeps the Ramond sector intact so as to be compatible with $U_{\pm 1/2}$. This means that the automorphism $\sigma_{\mathcal{M}}$ has to satisfy $\sigma_{\mathcal{M}} J^3 \sigma_{\mathcal{M}}^{-1} = J^3$, or $\sigma_{\mathcal{M}} J^3 \sigma_{\mathcal{M}}^{-1} = -J^3$.

The same conditions are required for the right-mover.

⁴If we had adopted the bosonization for the combinations, e.g. $\psi_R^2 + i\psi_R^4$ and $\psi_R^3 + i\psi_R^5$, the L.H.S of (3.6) would have been the identity. However, we shall not consider this possibility here in order to respect the super-Poincare symmetry in $\mathbb{R}^{3,1}$, and (3.5) is the unique choice. We also simply assume that $(-\mathbf{1}_R)^{\otimes 2}$ is involutive on the bosonic coordinates X_R^4, X_R^5 (on the untwisted Hilbert space) in this paper, even though we have more general possibilities if utilizing the fermionization of them as discussed in [12].

Let us now introduce the automorphisms $\sigma_L^{(\alpha)}$ ($\alpha = 1, 2, 3$) of the left-moving $\mathcal{N} = 4$ SCA. They are defined by

$$\begin{aligned} \sigma_L^{(\alpha)} T(z) \sigma_L^{(\alpha)-1} &= T(z), & \sigma_L^{(\alpha)} G^0(z) \sigma_L^{(\alpha)-1} &= G^0(z), \\ \sigma_L^{(\alpha)} J^\alpha(z) \sigma_L^{(\alpha)-1} &= J^\alpha(z), & \sigma_L^{(\alpha)} G^\alpha(z) \sigma_L^{(\alpha)-1} &= G^\alpha(z), \\ \sigma_L^{(\alpha)} J^\beta(z) \sigma_L^{(\alpha)-1} &= -J^\beta(z), \quad (\beta \neq \alpha), & \sigma_L^{(\alpha)} G^\beta(z) \sigma_L^{(\alpha)-1} &= -G^\beta(z), \quad (\beta \neq \alpha), \end{aligned} \quad (3.7)$$

and we assume that they are involutive on the whole Hilbert space; $(\sigma_L^{(\alpha)})^2 = \mathbf{1}_L$ ($\forall \alpha$). We also set $\widehat{\sigma}_L^{(\alpha)} := e^{\frac{i\pi}{2} F_L} \sigma_L^{(\alpha)}$ for convenience. $\widehat{\sigma}_L^{(\alpha)}$ obviously acts on the $\mathcal{N} = 4$ SCA in the same way as (3.7), but it is no longer involutive; $(\widehat{\sigma}_L^{(\alpha)})^2 = (-1)^{F_L}$.

To complete the definition of $\sigma_L^{(\alpha)}$ (and $\widehat{\sigma}_L^{(\alpha)}$), we still have to specify their actions on the Ramond ground states, in other words, on the half-spectral flow operators $U_{\pm 1/2}$. Recalling the simple relation $J^\pm \equiv J^1 \pm iJ^2 = U_{\pm 1} = (U_{\pm 1/2})^2$, we can naturally define

$$\sigma_L^{(1)} U_{\pm 1/2} \sigma_L^{(1)-1} = U_{\mp 1/2}, \quad \sigma_L^{(2)} U_{\pm 1/2} \sigma_L^{(2)-1} = \pm i U_{\mp 1/2}, \quad (3.8)$$

which are surely consistent with $(\sigma_L^{(\alpha)})^2 = \mathbf{1}_L$. We next consider the composition $\sigma_L^{(1)} \sigma_L^{(2)}$. It obviously acts on each $\mathcal{N} = 4$ chiral current in the same way as $\sigma_L^{(3)}$. However, since $\sigma_L^{(1)}$ and $\sigma_L^{(2)}$ are anti-commutative on the R-sector due to (3.8), we find $(\sigma_L^{(1)} \sigma_L^{(2)})^2 = (-1)^{F_L}$. Thus, we should identify

$$\sigma_L^{(1)} \sigma_L^{(2)} = (-1)^{F_L} \sigma_L^{(2)} \sigma_L^{(1)} = \widehat{\sigma}_L^{(3)} \equiv e^{\frac{i\pi}{2} F_L} \sigma_L^{(3)}, \quad (3.9)$$

and $\widehat{\sigma}_L^{(3)}$ acts on the half-spectral flows $U_{\pm 1/2}$ as

$$\widehat{\sigma}_L^{(3)} U_{\pm 1/2} \widehat{\sigma}_L^{(3)-1} = \pm i U_{\pm 1/2}. \quad (3.10)$$

$\sigma_R^{(\alpha)}$ ($\alpha = 1, 2, 3$) for the right-mover are defined in the same way.

Now, let us specify the possible orbifold actions $\sigma \equiv \sigma_{\mathcal{M}} \otimes (-\mathbf{1}_R)^{\otimes 2}$. We again have four possibilities (i) $\sigma^2 = \mathbf{1}$, (ii) $\sigma^2 = (-1)^{F_L}$, (iii) $\sigma^2 = (-1)^{F_R}$, (iv) $\sigma^2 = (-1)^{F_L + F_R}$, as in the previous section. However, all the space-time SUSY are broken in the fourth case, and the second and third cases reduce to the first case by the chirality flip; IIA \longleftrightarrow IIB as mentioned in subsection 2.1. It is thus enough to consider the first case such that σ is involutive, that is, the cases with $\sigma \equiv \sigma_L^{(\alpha)} \otimes \widehat{\sigma}_R^{(\beta)} \otimes (-\mathbf{1}_R)^{\otimes 2}$. We shall especially focus on the following three cases; (1) $\sigma_{\mathcal{M}} \equiv \sigma_L^{(3)} \otimes \widehat{\sigma}_R^{(1)}$, (2) $\sigma_{\mathcal{M}} \equiv \sigma_L^{(3)} \otimes \widehat{\sigma}_R^{(3)}$, (3) $\sigma_{\mathcal{M}} \equiv \sigma_L^{(1)} \otimes \widehat{\sigma}_R^{(1)}$. Of course, we have to examine whether they are actually compatible with the modular invariance. In the next subsection, we explicitly confirm in the case of the Gepner construction that the asymmetric orbifolding by $\sigma \equiv \sigma_{\mathcal{M}} \otimes (-\mathbf{1}_R)^{\otimes 2}$ constructed this way yields superstring vacua with modular invariance. In all the three cases, the space-time SUSY from the left mover is broken to achieve the 4-dim. $\mathcal{N} = (0, 1)$ chiral SUSY.

3.2 Concrete examples: Gepner construction

Let us consider the generic Gepner construction [27, 28] for $K3$, that is, the superconformal system defined by

$$[M_{k_1} \otimes \cdots \otimes M_{k_r}] |_{\mathbb{Z}_N\text{-orbifold}}, \quad \sum_{i=1}^r \frac{k_i}{k_i + 2} = 2, \quad (3.11)$$

where M_k denotes the $\mathcal{N} = 2$ minimal model of level k ($\hat{c} \equiv \frac{c}{3} = \frac{k}{k+2}$), and we set

$$N := \text{L.C.M.}\{k_i + 2; i = 1, \dots, r\}. \quad (3.12)$$

We start with the diagonal modular invariant for simplicity. We have to make the \mathbb{Z}_N -orbifolding that renders the total $U(1)_R$ -charge (in the NS-sector) integral,

$$Q(I) := \sum_{i=1}^r \frac{m_i}{k_i + 2} \in \mathbb{Z}, \quad (3.13)$$

where $I := \{(\ell_1, m_1), \dots, (\ell_r, m_r)\}$ denotes the collective label of the primary state in $M_{k_1} \otimes \cdots \otimes M_{k_r}$ ($0 \leq \ell_i \leq k_i$, $m_i \in \mathbb{Z}_{2(k_i+2)}$, $\ell_i + m_i \in 2\mathbb{Z}$), and the twisted sectors of orbifolding are identified with the ‘spectral flow orbits’ by the actions \mathcal{U}^n ($n \in \mathbb{Z}_N$) with

$$\mathcal{U} : I \equiv \{(\ell_1, m_1), \dots, (\ell_r, m_r)\} \mapsto \mathcal{U}(I) \equiv \{(\ell_1, m_1 - 2), \dots, (\ell_r, m_r - 2)\}. \quad (3.14)$$

See [29] for more detail.

The relevant Hilbert space for the $K3$ sector (before imposing the GSO projection) is schematically expressed as

$$\mathcal{H}_{\text{Gepner}}^{(s, \tilde{s})} = \bigoplus_{n \in \mathbb{Z}_N} \bigoplus_{\substack{I, \tilde{I} \\ Q(I) \in \mathbb{Z}, Q(\tilde{I}) \in \mathbb{Z}}} \left[\delta_{I, \tilde{I}} \mathcal{H}_{\mathcal{U}^n(I), L}^{(s)} \otimes \mathcal{H}_{\tilde{I}, R}^{(\tilde{s})} \right], \quad (s, \tilde{s} = \text{NS}, \text{R}), \quad (3.15)$$

where the Ramond Hilbert space $\mathcal{H}_{I, * }^{(R)}$ is uniquely determined by the half-spectral flow in the standard manner.⁵ Note that the left-right symmetric primary states lie in the $n = 0$ sector, but we also have many asymmetric primary states generated by the spectral flows. As already mentioned, the $\mathcal{N} = 2$ superconformal symmetry with $\hat{c} = 2$ is enhanced to the $\mathcal{N} = 4$ by adding the spectral flow operators, which are identified with the $\widehat{\text{SU}}(2)_1$ currents $J^\pm \equiv J^1 \pm iJ^2$ in the $\mathcal{N} = 4$ SCA [29]. Accordingly, the chiral parts of $\mathcal{H}_{\text{Gepner}}^{(s, \tilde{s})}$ are decomposed into irreducible representations of $\mathcal{N} = 4$ SCA at level 1, that are classified as follows [30, 31]:

- **Massive representations:** $\mathcal{C}_h^{(\text{NS})}, \mathcal{C}_h^{(\text{R})}$.

These are non-degenerate representations whose vacua have conformal weights h . The vacuum of $\mathcal{C}_h^{(\text{NS})}$ belongs to the spin 0 representation of the $\widehat{\text{SU}}(2)_1$ -symmetry. The four-fold degenerate vacua of $\mathcal{C}_h^{(\text{R})}$ generate the representation $2[\text{spin } 0] \oplus [\text{spin } 1/2]$. Unitarity requires $h \geq 0$ for $\mathcal{C}_h^{(\text{NS})}$ and $h \geq \frac{1}{4}$ for $\mathcal{C}_h^{(\text{R})}$. The $1/2$ -spectral flow connects $\mathcal{C}_h^{(\text{NS})}$ with $\mathcal{C}_{h+\frac{1}{4}}^{(\text{R})}$.

⁵Notice however that the label I in $\mathcal{H}_{I, * }^{(R)}$ indicates the quantum numbers in the NS -sector.

- **Massless representations:** $\mathcal{D}_\ell^{(\text{NS})}, \mathcal{D}_\ell^{(\text{R})}$ ($\ell = 0, 1/2$).

These are degenerate representations whose vacua have conformal weights $h = \ell$ for the NS representations $\mathcal{D}_\ell^{(\text{NS})}$, and $h = \frac{1}{4}$ for the Ramond representations $\mathcal{D}_\ell^{(\text{R})}$; they belong to the spin ℓ representation of $\widehat{\text{SU}}(2)_1$. To be more specific, $\mathcal{D}_0^{(\text{NS})}$ (‘graviton rep.’ or ‘identity rep.’) corresponds to the unique vacuum with $h = 0, J_0^3 = 0$, while $\mathcal{D}_{1/2}^{(\text{NS})}$ (‘massless matter rep.’) is generated over doubly degenerated vacua with $h = 1/2, J_0^3 = \pm 1/2$. The Ramond sector $\mathcal{D}_{\frac{1}{2}-\ell}^{(\text{R})}$ is connected with $\mathcal{D}_\ell^{(\text{NS})}$ by the $1/2$ -spectral flow.

The relevant character formulas are summarized in appendix A.

Now, let us construct the asymmetric orbifolds by the involution σ . Since a detailed account of closely related asymmetric orbifolds has been given in [24], based on [25], we here briefly describe the relevant construction.

Since the most non-trivial part $\sigma_{\mathcal{M}}$ has the form $\sigma_L^{(\alpha)} \otimes \widehat{\sigma}_R^{(\beta)}$ ($\alpha, \beta = 1$ or 3), we should specify how the $\mathcal{N} = 4$ involutions $\sigma_L^{(1)}, \sigma_L^{(3)}$ act on the primary states in the Gepner construction. First, we can naturally identify $\sigma_L^{(1)}$ with the $\mathcal{N} = 2$ involution,

$$\sigma_L^{(1)} := \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}, \tag{3.16}$$

where the $\mathcal{N} = 2$ involution $\sigma_L^{\mathcal{N}=2,(i)}$ acts as

$$T^{(i)} \rightarrow T^{(i)}, \quad J^{(i)} \rightarrow -J^{(i)}, \quad G^{\pm(i)} \rightarrow G^{\mp(i)}, \tag{3.17}$$

in each minimal factor M_{k_i} .

On the other hand, $\sigma_L^{(3)}$ acts on the $\mathcal{N} = 4$ SCA as the automorphism (3.7). We still have to define how it acts on the $\mathcal{N} = 4$ primary states $|v\rangle_L$. A simple choice would be given as

$$\sigma_L^{(3)}|v\rangle_L := \begin{cases} \sigma_L^{(1)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = 0), \\ J_{L,0}^+ \sigma_L^{(1)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = |v\rangle_L), \\ -J_{L,0}^- \sigma_L^{(1)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = -|v\rangle_L), \end{cases} \tag{3.18}$$

where $J_L^\pm \equiv J_L^1 \pm iJ_L^2$ are the $\text{SU}(2)$ currents in the $\mathcal{N} = 4$ SCA, which turns out to be compatible with the modular invariance.

By these definitions and the fact that $\sigma_{L(R)}^{(1)}$ and $\sigma_{L(R)}^{(3)}$ induce the equal twisted characters of $\mathcal{N} = 4$ SCA (see appendix B), we find that the torus partition function does not depend on α, β in $\sigma_{\mathcal{M}} \equiv \sigma_L^{(\alpha)} \otimes \widehat{\sigma}_R^{(\beta)}$. The total modular invariant is now written as

$$Z(\tau, \bar{\tau}) := Z_{\text{bosonic}}^{4d}(\tau, \bar{\tau}) \frac{1}{4} \sum_{a,b \in \mathbb{Z}_4} Z_{(a,b)}(\tau, \bar{\tau}). \tag{3.19}$$

As before, Z_{bosonic}^{4d} denotes the contribution from the bosonic part of $\mathbb{R}^{3,1}$, which is related with neither the σ -twisting nor the GSO-projection. Those for the various σ -twisted sectors $Z_{(a,b)}$ ($a, b \in \mathbb{Z}_4$), which are crucial in our arguments, are described in the following way:

- **Even sectors with $a, b \in 2\mathbb{Z}$.**

$$Z_{(a,b)}(\tau, \bar{\tau}) := \frac{1}{4} \sum_{s, \bar{s}} \sum_{\mathcal{I}, \tilde{\mathcal{I}}} N_{\mathcal{I}, \tilde{\mathcal{I}}} F_{\mathcal{I}}^{(s)}(\tau) \overline{F_{\tilde{\mathcal{I}}}^{(\bar{s})}(\tau)} \cdot Z^{T^2[D_2]}(\tau, \bar{\tau}) \cdot \left(\frac{\theta_{[s]}}{\eta} \right)^2 \overline{\left(\frac{\theta_{[\bar{s}]}}{\eta} \right)^2}, \quad (3.20)$$

where $F_{\mathcal{I}}^{(s)}(\tau)$, $F_{\tilde{\mathcal{I}}}^{(\bar{s})}(\tau)$ denote the chiral building blocks with the chiral spin structures $s, \bar{s} = \text{NS}, \widetilde{\text{NS}}, \text{R}, \widetilde{\text{R}}$, in the Gepner model for \mathcal{M} , which are labeled by the spectral flow orbits $\mathcal{I}, \tilde{\mathcal{I}}$. For instance, $F_{\mathcal{I}}^{(\text{NS})}(\tau)$ is explicitly written as

$$F_{\mathcal{I}}^{(\text{NS})}(\tau) = \sum_{\{(\ell_i, m_i)\} \in \mathcal{I}} \prod_{i=1}^r \text{ch}_{\ell_i, m_i}^{(\text{NS})}(\tau),$$

with

$$\mathcal{I} \equiv \{(\ell_1, m_1 - 2n), \dots, (\ell_r, m_r - 2n)\}_{n \in \mathbb{Z}_N},$$

and the $\mathcal{N} = 2$ minimal character $\text{ch}_{\ell_i, m_i}^{(s)}(\tau)$ [32–35]. The chiral blocks for other spin structures are determined by the 1/2-spectral flows and by incorporating the suitable sign factors to impose the GSO condition. (See [29] for more detail.) We also adopted the concise notation $\theta_{[s]}(\tau) := \theta_3(\tau)$, $\theta_4(\tau)$, $\theta_2(\tau)$, $i\theta_1(\tau) (\equiv 0)$ for $s = \text{NS}, \widetilde{\text{NS}}, \text{R}, \widetilde{\text{R}}$ respectively. The modular invariant coefficients $N_{\mathcal{I}, \tilde{\mathcal{I}}}$ are straightforwardly determined due to the Gepner construction, which are independent of a, b , and the overall factor 1/4 originates from the chiral GSO projection.

- **Odd sectors with $a \in 2\mathbb{Z} + 1$ or $b \in 2\mathbb{Z} + 1$.**

$$\begin{aligned} Z_{(a,b)}(\tau, \bar{\tau}) &:= Z_{(a,b)}^{\mathcal{M}}(\tau, \bar{\tau}) \cdot Z_{(a,b)}^{T^2[D_2]}(\tau, \bar{\tau}) \cdot Z_{(a,b)}^f(\tau, \bar{\tau}) \\ &\equiv \sum_{\mathbf{l}, \bar{\mathbf{l}}} N_{\mathbf{l}, \bar{\mathbf{l}}}^{(a,b)} \chi_{\mathbf{l}, (a,b)}(\tau) \overline{\chi_{\bar{\mathbf{l}}, (a,b)}(\tau)} \cdot \epsilon_{(a,b)}^{[2]} \chi_{(a,b)}^{D_2}(\tau) \overline{\left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^2} \\ &\quad \times \epsilon_{(a,b)}^{[2]} \chi_{(a,b)}^{D_2, [-]}(\tau) \frac{1}{2} \left[\overline{\left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^2} - \left(\tilde{\chi}_{(a,b)}^{A_1}(\tau) \right)^2 \right], \end{aligned} \quad (3.21)$$

where we set

$$\chi_{\mathbf{l}, (a,b)}(\tau) := \prod_i \chi_{\ell_i, [a,b]}^{k_i}(\tau), \quad \mathbf{l} \equiv (\ell_1, \dots, \ell_r), \quad (3.22)$$

and $\chi_{\ell_i, [a,b]}^k(\tau)$ denotes the twisted $\mathcal{N} = 2$ character (B.4). Recall that $\epsilon_{(a,b)}^{[r]} \equiv e^{\frac{i\pi}{8} r (-1)^{a,b}}$ and the definitions of the functions $\chi_{(a,b)}^{D_2}(\tau)$, $\chi_{(a,b)}^{D_2, [-]}(\tau)$ and $\tilde{\chi}_{(a,b)}^{A_1}(\tau)$ are summarized in (A.7), (A.8) and (A.12). The 4-dim. $\mathcal{N} = (0, 1)$ chiral SUSY is confirmed from (3.21).

The coefficients $N_{\mathbf{l}, \bar{\mathbf{l}}}^{(a,b)}$ in the odd sectors are slightly non-trivial. We can determine them in a way parallel to that presented in [24, 25]. We here briefly describe the results, which depend on the spectrum of the level k_i in (3.11) as follows:

(i) **At least one of k_i 's is odd.** In this case, the modular invariant coefficients are very simple,

$$N_{1, \tilde{1}}^{(a,b)} = \prod_{i=1}^r \delta_{\ell_i, \tilde{\ell}_i}. \tag{3.23}$$

(ii) **All k_i 's are even.** In this case, N in (3.12) is even, and we set

$$\begin{aligned} S_1 &:= \left\{ i \in \{1, \dots, r\} ; \frac{N}{k_i + 2} \in 2\mathbb{Z} + 1 \right\}, \\ S_2 &:= \left\{ i \in \{1, \dots, r\} ; \frac{N}{k_i + 2} \in 2\mathbb{Z} \right\}. \end{aligned} \tag{3.24}$$

Then, the relevant coefficients are given by

$$N_{1, \tilde{1}}^{(a,b)} := \begin{cases} \prod_{i \in S_2} \delta_{\ell_i, \tilde{\ell}_i} \left(\prod_{i \in S_1} \delta_{\ell_i, \tilde{\ell}_i} + \prod_{i \in S_1} \delta_{\ell_i, k_i - \tilde{\ell}_i} \right) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \left(1 + (-1)^{\sum_{i \in S_1} \ell_i} \right) \prod_{i=1}^r \delta_{\ell_i, \tilde{\ell}_i} & (a \in 2\mathbb{Z} + 1). \end{cases} \tag{3.25}$$

One can directly confirm that the $Z_{(a,b)}(\tau, \bar{\tau})$ in the odd sectors (3.21) show the suitable modular covariance by using the modular transformation formulas given in (B.7). Note that $\sigma_{L(R)}^{(\alpha)}$ -insertion only provides non-vanishing contributions to the trace over the sectors with $\{(\ell_1, 0), \dots, (\ell_r, 0)\}$ in the spectral flow orbit (of NS-sector). The difference of the two cases (3.23) and (3.25) originates from this fact.

We make a few comments:

- As already mentioned, we are considering the orbifolding by $\sigma = \sigma_L^{(\alpha)} \otimes \widehat{\sigma}_R^{(\beta)} \otimes (-\mathbf{1}_R)^{\otimes 2}$ for various α, β , and obtain the equivalent spectra of closed string states in all these models. However, this fact does not necessarily imply that they are equivalent string vacua. Indeed, it turns out that they have quite different D-branes, as we elucidate in subsection 3.3.
- One finds that the contributions from the $(R, *)$ or $(*, R)$ -sectors do not appear in the building block (3.21) with $a \in 2\mathbb{Z}$ and $b \in 2\mathbb{Z} + 1$. This fact is actually expected so as to achieve the modular invariance. It is not difficult to confirm that this is indeed the case in almost all the Gepner models for $K3$ due to the basic properties of the twisted characters. (See appendix B.) The exception is only the $(4)^3$ type, in which there would exist a non-vanishing (R, NS) contribution⁶ that could spoil the modular invariance. However, by suitably fixing the sign ambiguity of $\sigma_L^{(\alpha)}$ on the Ramond vacua with $Q = 0$, one can avoid this possibility still in the $(4)^3$ -model.

3.3 General construction of boundary states with vanishing self-overlaps

Let us present our main studies. Namely, we investigate how we achieve the vanishing self-overlaps in the current models of asymmetric orbifolds,

$$\mathbb{R}^{3,1} \times [T^2[D_2] \times \mathcal{M}]|_{\sigma\text{-orbifold}}. \tag{3.26}$$

⁶The (R, R) and (NS, R) -contributions trivially vanish due to the fermionic zero-modes in the $\mathbb{R}^{3,1} \times T^2[D_2]$ -directions.

We begin with specifying the boundary conditions in the (unorbifolded) \mathcal{M} -sector that characterize general BPS D-branes.⁷ Naively, any boundary conditions preserving the $\mathcal{N} = 4$ superconformal symmetry with an arbitrary twisting by automorphism may be allowed, which are schematically expressed as in [39] by

$$\left[\mathcal{A}_r^I + g \cdot \tilde{\mathcal{A}}_{-r}^I \right] |B\rangle\rangle = 0. \quad (3.27)$$

Here, $\mathcal{A}_r^I, \tilde{\mathcal{A}}_r^I$ are the chiral currents and g denotes any (inner or outer) automorphism of the $\mathcal{N} = 4$ SCA. However, since we are working on the physical boundary states in the RNS superstrings, we still have to impose the following conditions:

- (i) $|B\rangle\rangle$ preserves G^0 -symmetry without any twisting, which is necessary for the BRST-invariance.
- (ii) $|B\rangle\rangle$ contains the correct components of the RR-sector compatible with the above definition of $U_{\pm 1/2}$. This means that the automorphism g in (3.27) has to satisfy $g \cdot J^3 = J^3$, or $g \cdot J^3 = -J^3$.

Thus, at least generically, the allowed twisting g by the $\mathcal{N} = 4$ automorphism is restricted and we eventually obtain the following two types of gluing conditions:

A-type:

$$\begin{aligned} \left[L_n - \tilde{L}_{-n} \right] |\theta\rangle\rangle_A = 0, & \quad \left[J_n^3 - \tilde{J}_{-n}^3 \right] |\theta\rangle\rangle_A = 0, & (3.28) \\ \left[G_r^0 - i\tilde{G}_{-r}^0 \right] |\theta\rangle\rangle_A = 0, & \quad \left[G_r^3 + i\tilde{G}_{-r}^3 \right] |\theta\rangle\rangle_A = 0, \\ \left[G_r^\alpha - i\hat{R}(\theta)^\alpha_\beta \tilde{G}_{-r}^\beta \right] |\theta\rangle\rangle_A = 0, & \quad \left[J_n^\alpha + \hat{R}(\theta)^\alpha_\beta \tilde{J}_{-n}^\beta \right] |\theta\rangle\rangle_A = 0, \quad (\alpha, \beta = 1, 2). \end{aligned}$$

B-type:

$$\begin{aligned} \left[L_n - \tilde{L}_{-n} \right] |\theta\rangle\rangle_B = 0, & \quad \left[J_n^3 + \tilde{J}_{-n}^3 \right] |\theta\rangle\rangle_B = 0, & (3.29) \\ \left[G_r^\alpha - i\tilde{G}_{-r}^\alpha \right] |\theta\rangle\rangle_B = 0, & \quad (\alpha = 0, 3), \\ \left[G_r^\alpha - iR(\theta)^\alpha_\beta \tilde{G}_{-r}^\beta \right] |\theta\rangle\rangle_B = 0, & \quad \left[J_n^\alpha + R(\theta)^\alpha_\beta \tilde{J}_{-n}^\beta \right] |\theta\rangle\rangle_B = 0, \quad (\alpha, \beta = 1, 2). \end{aligned}$$

In the above, $R(\theta)$ denotes the $SO(2)$ -rotation with the angle parameter θ , and $\hat{R}(\theta) \equiv R(\theta)\sigma_3 \in O(2)$. The relevant Ishibashi states [40] are characterized by the $\mathcal{N} = 4$ irrep. classified in the subsection 3.2 as well as the gluing conditions given above, and should satisfy e.g.

$$\begin{aligned} A \left\langle \left\langle \mathcal{D}_\ell^{(\text{NS})}; \theta \right| e^{-\pi s H^{(c)}} \left| \mathcal{D}_\ell^{(\text{NS})}; \theta \right\rangle \right\rangle_A &= \text{ch}_0^{(\text{NS})}(\ell; is), \\ A \left\langle \left\langle \mathcal{C}_h^{(\text{NS})}; \theta \right| e^{-\pi s H^{(c)}} \left| \mathcal{C}_h^{(\text{NS})}; \theta \right\rangle \right\rangle_A &= \text{ch}^{(\text{NS})}(h; is) \equiv e^{-2\pi(h-\frac{1}{8})} \frac{\theta_3(is)^2}{\eta(is)^3}, \end{aligned} \quad (3.30)$$

⁷In this paper, we shall not work with explicit forms of the boundary states in Gepner model for \mathcal{M} , which should be constructed as tensor products of those for the $\mathcal{N} = 2$ minimal models. See [36] and also e.g. [37, 38] for detail.

where $\text{ch}_0^{(\text{NS})}(\ell; is)$, $\text{ch}^{(\text{NS})}(h; is)$ denote the $\mathcal{N} = 4$ massless and massive characters summarized in (A.20), (A.21) and (A.22). To be more precise, since the Gepner points are rational, it turns out that only the discrete values of the angle parameter $\theta = \frac{2\pi r}{N}$ ($r \in \mathbb{Z}_N$) are allowed. In fact, let us recall the schematic decomposition of an $\mathcal{N} = 4$ irrep. by the integral spectral flows as

$$[\text{irrep.}]^{(\text{NS}), \mathcal{N}=4} = \bigoplus_{n \in \mathbb{Z}_N} U_n [\text{irrep.}]^{(\text{NS}), \mathcal{N}=2},$$

where N is defined by (3.12), and we also express the $\mathcal{N} = 2$ Ishibashi state of the A-type as $[[\text{irrep.}]^{(\text{NS})}]_A^{\mathcal{N}=2}$ (defined by the gluing conditions given in the first and second lines in (3.28)). Then, the $\mathcal{N} = 4$ Ishibashi states of A-type with the twist angle $\theta = \frac{2\pi r}{N}$ are written as

$$\left| [\text{irrep.}]^{(\text{NS}); \theta = \frac{2\pi r}{N}} \right\rangle_A = \sum_{n \in \mathbb{Z}_N} (-1)^n e^{2\pi i \frac{r}{N} n} U_n \otimes \tilde{U}_n \left| [\text{irrep.}]^{(\text{NS})} \right\rangle_A^{\mathcal{N}=2}. \quad (3.31)$$

This shows why θ is restricted to discrete values $\theta = \frac{2\pi r}{N}$. The B-type Ishibashi states are similarly constructed.

The Ishibashi states in the RR-sector are obtained by the half-spectral flow from the NSNS ones,

$$\begin{aligned} \left| [\text{irrep.}]^{(\text{R}); \frac{2\pi r}{N}} \right\rangle_A &= U_{1/2} \otimes \tilde{U}_{1/2} \left| [\text{irrep.}]^{(\text{NS}); \frac{2\pi r}{N}} \right\rangle_A, \\ \left| [\text{irrep.}]^{(\text{R}); \frac{2\pi r}{N}} \right\rangle_B &= U_{1/2} \otimes \tilde{U}_{-1/2} \left| [\text{irrep.}]^{(\text{NS}); \frac{2\pi r}{N}} \right\rangle_B. \end{aligned} \quad (3.32)$$

We note the correspondence of the representations,

$$\begin{aligned} U_{\pm 1/2} &: \mathcal{D}_\ell^{(\text{NS})} \longrightarrow \mathcal{D}_{1/2-\ell}^{(\text{R})}, \quad (\ell = 0, 1/2), \\ U_{\pm 1/2} &: \mathcal{C}_h^{(\text{NS})} \longrightarrow \mathcal{C}_{h+\frac{1}{8}}^{(\text{R})}. \end{aligned} \quad (3.33)$$

The R-massive rep. $\mathcal{C}_h^{(\text{R})}$ is generated by doubly degenerated vacua with conformal weight h belonging to an $\text{SU}(2)$ -doublet, as opposed to the NS-one $\mathcal{C}_h^{(\text{NS})}$.

As in the previous analyses on the toroidal models, generic D-branes in our asymmetric orbifold (3.26) are expressed by the boundary states in the form of the orbifold projection with $\sigma^2 = 1$,

$$|B\rangle\rangle = \sqrt{2} \mathcal{P} |B\rangle\rangle_0 \equiv \sqrt{2} \mathcal{P} \left[|B\rangle\rangle_0^{(\text{NS})} + |B\rangle\rangle_0^{(\text{R})} \right], \quad (3.34)$$

where $|B\rangle\rangle_0$ is a (GSO-projected) boundary state describing a D-brane in the unorbifolded theory and $\mathcal{P} \equiv \frac{1+\sigma}{2}$. We assume that $|B\rangle\rangle_0$ describes a half-BPS brane with the Dirichlet conditions for all the transverse coordinates along $\mathbb{R}^{3,1} \times T^2[D_2]$, just for convenience. Namely, $|B\rangle\rangle_0$ is expanded by the Ishibashi states given above for the \mathcal{M} -sector and the

self-overlap is schematically written as

$$\begin{aligned}
{}_0\langle\langle B|e^{-\pi s H^{(c)}}|B\rangle\rangle_0 &= \sum_i \alpha_i \frac{1}{\eta^4} \left[\left(\frac{\theta_3}{\eta}\right)^2 \text{ch}_*^{(\text{NS})}(r_i^{(\text{NS})}; is) - \left(\frac{\theta_4}{\eta}\right)^2 \text{ch}_*^{(\widetilde{\text{NS}})}(r_i^{(\text{NS})}; is) \right. \\
&\quad \left. - \left(\frac{\theta_2}{\eta}\right)^2 \text{ch}_*^{(\text{R})}(r_i^{(\text{R})}; is) \right] \equiv 0, \tag{3.35}
\end{aligned}$$

where $r_i^{(\text{NS})}$ and $r_i^{(\text{R})}$ are unitary irrep.'s of $\mathcal{N} = 4$ SCA related with each other by $U_{\pm 1/2}$ and α_i are some non-trivial coefficients that we are not interested in here. The R.H.S of (3.35) indeed vanishes due to the BPS-property of $|B\rangle\rangle_0$. One can easily confirm that the each term associated to the irrep. $r_i^{(*)}$ actually vanishes.

Therefore, to achieve the vanishing cylinder amplitudes in the asymmetric orbifolds (3.26), it is enough to examine whether or not the amplitude ${}_0\langle\langle B|\sigma e^{-\pi s H^{(c)}}|B\rangle\rangle_0$ vanishes. From now on, we examine this problem in each case of (1) $\sigma_{\mathcal{M}} \equiv \sigma_L^{(3)} \otimes \widehat{\sigma}_R^{(1)}$, (2) $\sigma_{\mathcal{M}} \equiv \sigma_L^{(3)} \otimes \widehat{\sigma}_R^{(3)}$, (3) $\sigma_{\mathcal{M}} \equiv \sigma_L^{(1)} \otimes \widehat{\sigma}_R^{(1)}$, as addressed before. We set $\theta_r \equiv \frac{2\pi r}{N}$, ($r \in \mathbb{Z}_N$) in the following.

(1) $\sigma_{\mathcal{M}} = \sigma_L^{(3)} \otimes \widehat{\sigma}_R^{(1)}$: we first pick up the \mathcal{M} -sector. Because of the gluing conditions (3.28), (3.29), we obtain the equality

$$\sigma_{\mathcal{M}}|*;\theta_r\rangle\rangle_{A(B)} \equiv \sigma_L^{(3)} \otimes \widehat{\sigma}_R^{(1)}|*;\theta_r\rangle\rangle_{A(B)} = \widehat{\sigma}_R^{(1)}\sigma_R^{(3)}|*;\theta_r\rangle\rangle_{A(B)} = \sigma_R^{(2)}|*;\theta_r\rangle\rangle_{A(B)}. \tag{3.36}$$

It is worthwhile to emphasize that this relation does not depend on the angle parameter θ_r at all. Thus, the amplitude from each component of Ishibashi state is eventually evaluated by the $\sigma_R^{(2)}$ -twist irrespective of θ_r , yielding the $\mathcal{N} = 4$ twisted character,

$$\chi_{[0,1]}(h; is) \equiv \frac{2e^{-2\pi s(h-\frac{1}{8})}}{\theta_2(is)}, \tag{3.37}$$

or trivially vanishing one. We summarize necessary formulas for the $\mathcal{N} = 4$ twisted characters in appendix B. In this way, we obtain for the NSNS-sector,

$$\begin{aligned}
{}_{A(B)}\langle\langle \mathcal{D}_0^{(\text{NS})};\theta_r | \sigma_{\mathcal{M}} e^{-\pi s H^{(c)}} | \mathcal{D}_0^{(\text{NS})};\theta_r \rangle\rangle_{A(B)} &= {}_{A(B)}\langle\langle \mathcal{D}_0^{(\text{NS})};\theta_r | (-1)^{f_L} \sigma_{\mathcal{M}} e^{-\pi s H^{(c)}} | \mathcal{D}_0^{(\text{NS})};\theta_r \rangle\rangle_{A(B)} \\
&= \chi_{[0,1]}(h=0; is), \\
{}_{A(B)}\langle\langle \mathcal{D}_{1/2}^{(\text{NS})};\theta_r | \sigma_{\mathcal{M}} e^{-\pi s H^{(c)}} | \mathcal{D}_{1/2}^{(\text{NS})};\theta_r \rangle\rangle_{A(B)} &= {}_{A(B)}\langle\langle \mathcal{D}_{1/2}^{(\text{NS})};\theta_r | (-1)^{f_L} \sigma_{\mathcal{M}} e^{-\pi s H^{(c)}} | \mathcal{D}_{1/2}^{(\text{NS})};\theta_r \rangle\rangle_{A(B)} \\
&= 0, \\
{}_{A(B)}\langle\langle \mathcal{C}_h^{(\text{NS})};\theta_r | \sigma_{\mathcal{M}} e^{-\pi s H^{(c)}} | \mathcal{C}_h^{(\text{NS})};\theta_r \rangle\rangle_{A(B)} &= {}_{A(B)}\langle\langle \mathcal{C}_h^{(\text{NS})};\theta_r | (-1)^{f_L} \sigma_{\mathcal{M}} e^{-\pi s H^{(c)}} | \mathcal{C}_h^{(\text{NS})};\theta_r \rangle\rangle_{A(B)} \\
&= \chi_{[0,1]}(h; is), \tag{3.38}
\end{aligned}$$

where $(-1)^{f_L}$ denotes the twisting for the GSO projection. The fact that $(-1)^{f_L} \sigma_{\mathcal{M}}$ -insertion leads to the equal amplitude is obvious from the boundary conditions for the fermionic currents $G^a(z)$, ($a = 0, 1, 2, 3$).

We also recall that σ includes $(-1_R)^{\otimes 2}$, which just makes the free fermion contribution from the (transverse part of) $\mathbb{R}^{3,1} \times T^2[D_2]$ -sector proportional to $\frac{\theta_3}{\eta} \frac{\theta_4}{\eta}$ for the NSNS-sector,

while $(-1)^{f_L}(-\mathbf{1}_R)^{\otimes 2}$ gives the term $\propto \frac{\theta_4}{\eta} \frac{\theta_3}{\eta}$. On the other hand, the contributions from the RR-sector trivially vanish due to free fermion zero-modes along either of the $\mathbb{R}^{3,1}$ or $T^2[D_2]$ -directions.

Combining all the contributions and taking account of the GSO-projection, we finally obtain

$${}_0\langle\langle B|\sigma e^{-\pi s H^{(c)}}|B\rangle\rangle_0 = \sum_i \alpha'_i \frac{1}{\eta^4} \left[\frac{\theta_3}{\eta} \frac{\theta_4}{\eta} - \frac{\theta_4}{\eta} \frac{\theta_3}{\eta} \right] \chi_{[0,1]}(h_i; i s) \equiv 0. \quad (3.39)$$

In this expression⁸ the summation is taken over all the spin 0 irrep.'s, that is, $\mathcal{C}_h^{(\text{NS})}$ or $\mathcal{D}_0^{(\text{NS})}$, and we assign $h_i = 0$ for the case of $\mathcal{D}_0^{(\text{NS})}$. In this way, we have shown that any boundary states (3.34) associated to $|B\rangle\rangle_0$ satisfying the gluing conditions (3.28) or (3.29) with an arbitrary value of parameter $\theta_r = \frac{2\pi r}{N}$ ($r \in \mathbb{Z}_N$) provide the vanishing self-overlaps,

$$\langle\langle B|e^{-\pi s H^{(c)}}|B\rangle\rangle = 0. \quad (3.40)$$

As in the toroidal case in section 2, the couplings of $|B\rangle\rangle$ and the closed string states are multiplied by the overall factor in $|B\rangle\rangle$. The D-brane tension and the RR charge are hence $\sqrt{2}$ times those in the unorbifolded theory. The open string excitations in the unorbifolded theory remain in the self-overlap of $|B\rangle\rangle_0$, which are tachyon-free.

(2) $\sigma_{\mathcal{M}} = \sigma_L^{(3)} \otimes \hat{\sigma}_R^{(3)}$: in the second case, (3.36) should be replaced with

$$\sigma_{\mathcal{M}}|*;\theta_r\rangle\rangle_{A(B)} \equiv \sigma_L^{(3)} \otimes \hat{\sigma}_R^{(3)}|*;\theta_r\rangle\rangle_{A(B)} = \hat{\sigma}_R^{(3)}\sigma_R^{(3)}|*;\theta_r\rangle\rangle_{A(B)} = e^{\frac{i\pi}{2}FR}|*;\theta_r\rangle\rangle_{A(B)}. \quad (3.41)$$

Thus, the net effect of the twist is just a phase factor for the RR-component of boundary state. Incorporating also the $\mathbb{R}^{3,1} \times T^2$ -sector, the RR-component of the overlap again drops off due to the fermionic zero-modes, and we obtain the following amplitude instead of (3.39),

$${}_0\langle\langle B|\sigma e^{-\pi s H^{(c)}}|B\rangle\rangle_0 = \sum_i \alpha_i \frac{1}{\eta^4} \left[\frac{\theta_3}{\eta} \frac{\theta_4}{\eta} \text{ch}_*^{(\text{NS})}(r_i^{(\text{NS})}; i s) - \frac{\theta_4}{\eta} \frac{\theta_3}{\eta} \text{ch}_*^{(\widetilde{\text{NS}})}(r_i^{(\text{NS})}; i s) \right]. \quad (3.42)$$

At least for generic Gepner models, the R.H.S of (3.42) does not vanish for any value of the moduli parameter θ_r . In fact, R.H.S of (3.42) does not depend on θ_r , and

$$\text{ch}_*^{(\text{NS})}(r_i^{(\text{NS})}; \tau) \neq \text{ch}_*^{(\widetilde{\text{NS}})}(r_i^{(\text{NS})}; \tau),$$

for a generic rep. r_i . Rephrasing more physically, the D-brane tension has been modified by the σ -insertion, while the RR-charge remains the same as in case (1). This causes the mismatch of amplitudes for the graviton and RR-particle exchanges. In this way, we conclude that all of the D-branes in the second case have non-vanishing self-overlaps, as one expects from the general features of non-BPS D-branes.

⁸We note that the coefficients α'_i are not necessarily equal to those appearing in (3.35), since they would depend on the phases arising from the $\sigma_{\mathcal{M}}$ -actions on the $\mathcal{N} = 4$ primary states.

(3) $\sigma_{\mathcal{M}} = \sigma_L^{(1)} \otimes \widehat{\sigma}_R^{(1)}$: the third case is the most subtle one. When translating the $\sigma_L^{(1)}$ -insertion into that of the right-mover similarly to (3.36), (3.41), we have to take account of the $R(\theta)$ ($\widehat{R}(\theta)$) rotation appearing in the gluing conditions (3.29) ((3.28)). For instance, for the B-type gluing condition, we obtain

$$\sigma_{\mathcal{M}} |*; \theta_r\rangle_B \equiv \sigma_L^{(1)} \otimes \widehat{\sigma}_R^{(1)} |*; \theta_r\rangle_B = \widehat{\sigma}_R^{(1)} \sigma_R^{(1),[\theta_r]} |*; \theta_r\rangle_B, \tag{3.43}$$

instead of (3.36), (3.41), where $\sigma_R^{(1),[\theta_r]}$ denotes the automorphism acting on the $\mathcal{N} = 4$ SCA rotated by $R(\theta_r)$ in the same way as $\sigma_R^{(1)}$.⁹ Obviously the relation (3.43) yields the self-overlap that depends on the parameter θ_r , as opposed to the first and second cases. The resultant amplitude does not vanish generically. However, for the special value $\theta_r = \pm \frac{\pi}{2}$, we find

$$\widehat{\sigma}_R^{(1)} \sigma_R^{(1),[\pm \frac{\pi}{2}]} = \widehat{\sigma}_R^{(1)} \sigma_R^{(2)} = (-1)^{F_R} \sigma_R^{(3)}, \tag{3.44}$$

yielding the cancellation as given in (3.39). The A-type gluing condition is likewise treated.

In this way, we conclude that the D-branes in the third case have the vanishing self-overlaps only for the gluing conditions with $\theta_r \equiv \frac{2\pi r}{N} = \pm \frac{\pi}{2}$, which is possible when $N \in 4\mathbb{Z}_{>0}$.

Absence or presence of tachyonic instabilities. Here we would like to further discuss whether the non-BPS branes considered above could include the tachyonic instabilities. Since it is obvious that no closed string tachyons appear in the relevant boundary states, we should examine the open string excitations in the orbifolded sector. Indeed, it is easy to estimate the lightest excitation in the open string channel. By detailed case studies, it would be possible to write down the formulas of the general spectra, which are however beyond the scope of this paper.

Let us first note common features in the orbifolded sector for the above three cases; (i) the RR contribution to the self-overlap vanishes due to the fermionic zero-modes, implying the lack of GSO-projection for the open string Hilbert space, (ii) the twist by $(-1_R)^{\otimes 2}$ along the $T^2[D_2]$ -direction adds the conformal weight $\frac{1}{4}$ to the open string vacua.

Now, the estimations for the above three cases are summarized as follows;

Case (1): in this case we have the bose-fermi cancellation in the open string spectrum as noted above. Thus, it is enough to consider the NS-sector.

Recall that $\sigma_{\mathcal{M}}$ acts on the $\mathcal{N} = 4$ primary states as the product of the $\mathcal{N} = 2$ involutions for each minimal sector M_{k_i} , which gives rise to the energy shifts bounded from below by $\frac{\hat{c}_i}{8} \equiv \frac{k_i}{8(k_i+2)}$ in the open string spectrum. (See the formula of conformal weight (B.6).) Eventually we find that the minimum value of conformal weight for the open string excitations should satisfy the inequality;

$$h^{(\min)} \geq \frac{1}{4} + \sum_{i=1}^r \frac{\hat{c}_i}{8} = \frac{1}{2}, \tag{3.45}$$

⁹Since the $R(\theta_r)$ -rotation is an outer-automorphism, it seems difficult to write $\sigma_R^{(1),[\theta_r]}$ down explicitly.

and the inequality can be saturated only when all the k_i 's are even. Therefore, the lightest open string excitation could be massless when all k_i 's are even, and always massive if at least some k_i 's are odd. In this way, we conclude that no tachyonic instability emerges in the open string spectrum.

Case (2): $\sigma_{\mathcal{M}}$ again acts on the $\mathcal{N} = 4$ primary states in the same way, whereas it effectively makes the $\mathcal{N} = 4$ SCA invariant, after taking account of the identity (3.41). Thus, the twisted $\mathcal{N} = 4$ character $\chi_{[0,1]}(*; is) \propto \frac{\theta_3 \theta_4}{\eta^3}(is)$ for the case (1) has to be replaced with the untwisted one $\propto \frac{\theta_3^2}{\eta^3}(is)$ for the NS-sector. Making the modular transformation, the net effect just amounts to the shift by $-\frac{1}{8}$ to the R.H.S of (3.45). We thus obtain the inequality

$$h^{(\min)} \geq \frac{1}{4} + \left\{ \sum_{i=1}^r \frac{\hat{c}_i}{8} - \frac{1}{8} \right\} = \frac{3}{8}, \tag{3.46}$$

and open string tachyons would appear. This result is expected since the open string spectrum is non-supersymmetric in this case.

Case (3): again, $\sigma_{\mathcal{M}}$ acts on the $\mathcal{N} = 4$ primary states as the above two cases. On the other hand, by utilizing (3.43), we find that the net effect on the (right-moving) $\mathcal{N} = 4$ SCA by the $\sigma_{\mathcal{M}}$ -insertion amounts to the $SO(2)$ -rotation with the angle parameter $2\theta_r$ on the J_R^1, J_R^2 (and G_R^1, G_R^2) plane, while leaving the other generators intact. Then, the twisted $\mathcal{N} = 4$ character $\propto \frac{\theta_3 \theta_4}{\eta^3}(is)$ for the case (1) is replaced with $\propto \frac{\theta_3(is,0)\theta_3(is,2\theta_r)}{\eta^3}$, which induces the additional energy shift of the amount: $-\frac{1}{8} + \frac{1}{8\pi^2}(2\theta_r)^2$ to the R.H.S of (3.45). The relevant inequality now becomes

$$h^{(\min)} \geq \frac{1}{4} + \left\{ \sum_{i=1}^r \frac{\hat{c}_i}{8} - \frac{1}{8} + \frac{1}{8\pi^2}(2\theta_r)^2 \right\} = \frac{3}{8} + \frac{\theta_r^2}{2\pi^2}. \tag{3.47}$$

This implies that open string tachyons would generically emerge except for the special angle $\theta_r = \frac{\pi}{2}$ for $N \in 4\mathbb{Z}_{>0}$, which realizes the bose-fermi cancellation in the open string spectrum as mentioned above.

3.4 Points of toroidal orbifolds

Our discussion so far is based mostly only on general properties of the $\mathcal{N} = 4$ SCFT for \mathcal{M} . Thus, we would expect that the spectrum of the non-BPS branes with the vanishing overlaps is unchanged over generic points of the moduli space of $K3$, as long as the asymmetric orbifolding by $\sigma \equiv (-\mathbf{1}_R)^{\otimes 2} \otimes \sigma_{\mathcal{M}}$ is well-defined. The points in our argument were:

- The global symmetry $SU(2)_{\text{diag}}$ preserving G^0 is only identified with an *outer*-automorphisms of the $\mathcal{N} = 4$ SCA.
- We need to pick up a particular $U(1)$ -subalgebra of the $\mathcal{N} = 4$ SCA to define the Ramond sector by the half-spectral flows, which has been generated by J^3 in the above arguments.

Then, only the restricted $SO(2) \subset SU(2)_{\text{diag}}$ twisting is allowed in the gluing conditions (3.28), (3.29), so as to preserve the Ramond sector Hilbert space.

On the other hand, there are special points with the ‘symmetry enhancement’ in the moduli space, at which more general gluing conditions could be solved. For instance, it has been known [29] that the Gepner model $(2)^4$ (Kummer surface) is equivalent with the \mathbb{Z}_2 -orbifold of $T^4[D_4, B_{ij} \equiv 0]$, which is defined as the 4-dim. torus associated to the root lattice of D_4 with the vanishing Kalb-Ramond field.¹⁰ We can reinterpret this system in terms of free bosons and fermions, and thus, the $SU(2)_{\text{diag}}$ is explicitly realized by these free fields. In this special case all the choices of orbifold twisting $\sigma_{\mathcal{M}} = \sigma_L^{(\alpha)} \otimes \hat{\sigma}_R^{(\beta)}$ ($\alpha, \beta = 1, 2, 3$) lead to equivalent superstring vacua, as in the toroidal models studied in section 2. Especially we find the equivalent spectra of the non-BPS D-branes with the vanishing self-overlaps. Indeed, with the help of free field interpretation, one can straightforwardly solve the following equations for the boundary states,

$$\begin{aligned} \left[L_n - \tilde{L}_{-n} \right] |\theta, \varphi\rangle &= 0, & \left[G_r^0 - i\tilde{G}_{-r}^0 \right] |\theta, \varphi\rangle &= 0, \\ \left[G_r^a - iR(\theta, \varphi)^a_b \tilde{G}_{-r}^b \right] |\theta, \varphi\rangle &= 0, & \left[J_n^a + R(\theta, \varphi)^a_b \tilde{J}_{-n}^b \right] |\theta, \varphi\rangle &= 0, \quad (a, b = 1, 2, 3), \end{aligned} \tag{3.48}$$

where $R(\theta, \varphi)$ denotes an arbitrary $SO(3)$ -rotations.

There also exist the $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ -orbifold points within the Gepner models for $K3$ as discussed in [29]. However, such an enhancement of symmetry does not happen for these points, and $SU(2)_{\text{diag}}$ is still identified as outer-automorphisms.

4 Discussion

We have studied the type II string vacua with chiral space-time SUSY constructed as asymmetric orbifolds, focusing on the D-branes on these backgrounds. The simple but crucial idea in this paper is that all the D-branes are non-BPS in any chiral SUSY vacua. As clarified in sections 2 and 3, one can straightforwardly construct the chiral SUSY vacua based on asymmetric orbifolds which accommodate rather generally the non-BPS D-branes with vanishing cylinder amplitudes. This would be hardly realized in the geometrical compactifications of superstring theory.

We have especially investigated the asymmetric orbifolds of $T^2 \times \mathcal{M}$, as well as simpler toroidal models, where $\mathcal{M} = K3$ is described by a general $\mathcal{N} = 4$ SCFT with $c = 6$ defined by the Gepner construction. We have demonstrated in subsection 3.3 that the spectra of such non-BPS D-branes with the bose-fermi cancellation depend notably on the choice of orbifolding, even when the closed string spectra remain unchanged. This feature is in contrast to those of the toroidal asymmetric orbifolds presented in section 2.

In this respect we note that the most of the analyses on the boundary states given in subsection 3.3 are based only on general properties of the $\mathcal{N} = 4$ SCFT for \mathcal{M} , as mentioned in the previous section. Thus, the spectrum of the non-BPS D-branes with vanishing cylinder amplitudes would be unchanged over generic points of the moduli space of $K3$, as

¹⁰To avoid a possible confusion, we here emphasize that $T^4[D_4, B_{ij} \equiv 0]$ differs from the symmetry enhancement point of $\widehat{SO}(8)_1$, which is denoted as ‘ $T^4[D_4]$ ’, say, in (2.1) in the present paper (and also [22]).

long as the asymmetric twist is well-defined. The point in our discussion is summarized in subsection 3.4. The exception would be the orbifold point with symmetry enhancement.

Based on the results in this paper, one may now discuss a possible application to the problem of cosmological constant. As mentioned in the introduction, the cosmological constant induced solely by the non-BPS D-branes would be exponentially suppressed for small string coupling. Furthermore, in a given non-BPS D-brane background, the contributions to the closed-string vacuum amplitude would come only from the diagrams with the external legs sourced by that non-BPS D-brane. The analysis of the loops thus would be much simpler than the case of the bulk SUSY-breaking [1, 3, 9, 20], to control the almost vanishing cosmological constant. It would also be challenging to substantiate the scenario [4], which is based on the analysis on the heterotic dual side and mentioned in the introduction, that the non-BPS D-branes condensate to produce the non-perturbative mismatch of the spectrum. This would also be an interesting problem involving a non-supersymmetric duality. We hope to return to these issues elsewhere.

Acknowledgments

This work is supported in part by JSPS Grant-in-Aid for Scientific Research 24540248 and 17K05406, Japan-Hungary Research Cooperative Program and Japan-Russia Research Cooperative Program from Japan Society for the Promotion of Science (JSPS).

A Summary of conventions

Theta functions.

$$\theta_1(\tau, z) := i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1-q^m)(1-yq^m)(1-y^{-1}q^m), \quad (\text{A.1})$$

$$\theta_2(\tau, z) := \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1-q^m)(1+yq^m)(1+y^{-1}q^m), \quad (\text{A.2})$$

$$\theta_3(\tau, z) := \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}), \quad (\text{A.3})$$

$$\theta_4(\tau, z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}). \quad (\text{A.4})$$

$$\Theta_{m,k}(\tau, z) := \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})}, \quad (\text{A.5})$$

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \quad (\text{A.6})$$

Here, we have set $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$ ($\forall \tau \in \mathbb{H}^+$, $\forall z \in \mathbb{C}$), and used abbreviations, $\theta_i(\tau) \equiv \theta_i(\tau, 0)$ ($\theta_1(\tau) \equiv 0$), $\Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0)$.

Bosonic building blocks. Here we summarize the notation of the building blocks used in the main text according to [22]. Associated to the basic representation of $(\widehat{D_r})_1$

($r \in 2\mathbb{Z}_{>0}$), we set

$$\chi_{(a,b)}^{D_r}(\tau) := \begin{cases} \frac{1}{2\eta(\tau)^r} \left\{ \theta_3(\tau)^r + e^{\frac{i\pi r}{4}a} \theta_4(\tau)^r \right\}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \frac{1}{2\eta(\tau)^r} \left\{ \theta_3(\tau)^r + e^{\frac{i\pi r}{4}b} \theta_2(\tau)^r \right\}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \frac{1}{2\eta(\tau)^r} \left\{ \theta_4(\tau)^r + e^{\frac{i\pi r}{4}(a+b-1)} \theta_2(\tau)^r \right\}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1). \end{cases} \quad (\text{A.7})$$

We also define the following functions,

$$\chi_{(a,b)}^{D_r,[-]}(\tau) := \begin{cases} \frac{1}{2\eta(\tau)^r} \left\{ \theta_3(\tau)^r - e^{\frac{i\pi r}{4}a} \theta_4(\tau)^r \right\}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \frac{1}{2\eta(\tau)^r} \left\{ \theta_3(\tau)^r - e^{\frac{i\pi r}{4}b} \theta_2(\tau)^r \right\}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \frac{1}{2\eta(\tau)^r} \left\{ \theta_4(\tau)^r - e^{\frac{i\pi r}{4}(a+b-1)} \theta_2(\tau)^r \right\}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1), \end{cases} \quad (\text{A.8})$$

which are associated to the vector representation of $(\widehat{D_r})_1$.

For $(\widehat{A_1})_1$, we introduce

$$\chi_{(a,b)}^{A_1}(\tau) := \begin{cases} \frac{1}{2} \left\{ \chi_+^{A_1}(\tau) + e^{\frac{i\pi}{2}a} \chi_-^{A_1}(\tau) \right\}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \frac{1}{\sqrt{2}} \left\{ \chi_0^{A_1}(\tau) + e^{\frac{i\pi}{2}b} \chi_1^{A_1}(\tau) \right\}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \frac{1}{\sqrt{2}} \left\{ \chi_0^{A_1}(\tau) + e^{\frac{i\pi}{2}(a+b-1)} \chi_1^{A_1}(\tau) \right\}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1), \end{cases} \quad (\text{A.9})$$

where we set

$$\chi_{\pm}^{A_1}(\tau) := \chi_0^{A_1}(\tau) \pm \chi_1^{A_1}(\tau), \quad (\text{A.10})$$

and the $(\widehat{A_1})_1$ -characters are given as

$$\begin{aligned} \chi_0^{A_1}(\tau) &:= \frac{\theta_3(2\tau)}{\eta(\tau)} \equiv \frac{\Theta_{0,1}(\tau)}{\eta(\tau)}, & (\text{basic rep.}), \\ \chi_1^{A_1}(\tau) &:= \frac{\theta_2(2\tau)}{\eta(\tau)} \equiv \frac{\Theta_{1,1}(\tau)}{\eta(\tau)}, & (\text{spin } 1/2 \text{ rep.}). \end{aligned} \quad (\text{A.11})$$

On the other hand, we define

$$\tilde{\chi}_{(a,b)}^{A_1}(\tau) := \begin{cases} \sqrt{\frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^2}}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \sqrt{\frac{\theta_3(\tau)\theta_2(\tau)}{\eta(\tau)^2}}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \sqrt{\frac{\theta_4(\tau)\theta_2(\tau)}{\eta(\tau)^2}}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1), \end{cases} \quad (\text{A.12})$$

which are interpretable as the $(\widehat{A_1})_1$ -characters twisted by the involution $\rho_{A_1}^{(\alpha)} \equiv e^{-i\pi\frac{\ell}{2}} e^{i\pi J_0^\alpha}$, ($\alpha = 1, 2, 3$) for the spin $\ell/2$ -integrable representation of $(\widehat{A_1})_1$.

Fermionic building blocks. To describe the supersymmetric chiral blocks for the free fermions, we introduce the notation

$$\mathcal{J}(\tau) := \frac{1}{2\eta(\tau)^4} \left\{ \theta_3(\tau)^4 - \theta_4(\tau)^4 - \theta_2(\tau)^4 \right\} (\equiv 0), \quad (\text{A.13})$$

and associated to the reflection of four components $(-\mathbf{1}_L)^{\otimes 4}$,

$$\begin{aligned}
 f_{(a,b)}(\tau) &:= q^{\frac{1}{4}a^2} e^{\frac{i\pi}{2}ab} \left(\frac{\theta_1\left(\tau, \frac{a\tau+b}{2}\right)}{\eta(\tau)} \right)^2 \left(\frac{\theta_1(\tau, 0)}{\eta(\tau)} \right)^2 \\
 &\equiv \begin{cases} e^{\frac{i\pi}{2}ab} \frac{1}{2\eta(\tau)^4} \{\theta_3(\tau)^2 \theta_4(\tau)^2 - \theta_4(\tau)^2 \theta_3(\tau)^2 + 0\}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1), \\ e^{\frac{i\pi}{2}ab} \frac{1}{2\eta(\tau)^4} \{\theta_3(\tau)^2 \theta_2(\tau)^2 + 0 - \theta_2(\tau)^2 \theta_3(\tau)^2\}, & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}), \\ -e^{\frac{i\pi}{2}ab} \frac{1}{2\eta(\tau)^4} \{0 + \theta_2(\tau)^2 \theta_4(\tau)^2 - \theta_4(\tau)^2 \theta_2(\tau)^2\}, & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}+1), \\ \mathcal{J}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}). \end{cases}
 \end{aligned} \tag{A.14}$$

In the second line, each term corresponds to the NS, $\widetilde{\text{NS}}$, R sectors with keeping this order. These trivially vanish, as is consistent with the space-time SUSY. They satisfy the modular covariance of the form,

$$\begin{aligned}
 f_{(a,b)}(\tau)|_S &\equiv f_{(a,b)}\left(-\frac{1}{\tau}\right) = f_{(b,-a)}(\tau), \\
 f_{(a,b)}(\tau)|_T &\equiv f_{(a,b)}(\tau+1) = -e^{-2\pi i \frac{1}{6}} f_{(a,a+b)}(\tau).
 \end{aligned} \tag{A.15}$$

We next define the non-supersymmetric chiral block twisted by the two component reflection $(-\mathbf{1}_L)^{\otimes 2}$,

$$\begin{aligned}
 g_{(a,b)}(\tau) &:= (-1)^{ab} \epsilon_{(a,b)}^{[-2]} \left[\widetilde{\chi}_{(a,b)}^{A_1}(\tau) \right]^2 \chi_{(a,b)}^{D_2,[-]}(\tau) \\
 &\equiv \begin{cases} e^{-\frac{i\pi}{4}ab} \frac{1}{2\eta(\tau)^4} \left\{ \theta_3(\tau)^3 \theta_4(\tau) - (-1)^{\frac{a}{2}} \theta_4(\tau)^3 \theta_3(\tau) + 0 \right\}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1) \\ e^{\frac{i\pi}{4}ab} \frac{1}{2\eta(\tau)^4} \left\{ \theta_3(\tau)^3 \theta_2(\tau) + 0 - (-1)^{\frac{b}{2}} \theta_2(\tau)^3 \theta_3(\tau) \right\}, & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}) \\ -e^{\frac{i\pi}{4}ab} \frac{1}{2\eta(\tau)^4} \left\{ 0 + \theta_4(\tau)^3 \theta_2(\tau) + i(-1)^{\frac{a+b}{2}} \theta_2(\tau)^3 \theta_4(\tau) \right\}, & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}+1) \\ \mathcal{J}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}), \end{cases}
 \end{aligned} \tag{A.16}$$

and also for the twisting by $(-1)^{FL}$,

$$\begin{aligned}
 h_{(a,b)}(\tau) &:= q^{\frac{a^2}{2}} e^{i\pi ab} \left(\frac{\theta_1\left(\tau, \frac{a\tau+b}{2}\right)}{\eta(\tau)} \right)^4 \\
 &\equiv \begin{cases} \frac{1}{2\eta(\tau)^4} \{\theta_3(\tau)^4 - \theta_4(\tau)^4 + \theta_2(\tau)^4\} \equiv \left(\frac{\theta_2(\tau)}{\eta(\tau)} \right)^4, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1), \\ \frac{1}{2\eta(\tau)^4} \{\theta_3(\tau)^4 + \theta_4(\tau)^4 - \theta_2(\tau)^4\} \equiv \left(\frac{\theta_4(\tau)}{\eta(\tau)} \right)^4, & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}), \\ -\frac{1}{2\eta(\tau)^4} \{\theta_3(\tau)^4 + \theta_4(\tau)^4 + \theta_2(\tau)^4\} \equiv -\left(\frac{\theta_3(\tau)}{\eta(\tau)} \right)^4, & (a, b \in 2\mathbb{Z}+1), \\ \mathcal{J}(\tau), & (a, b \in 2\mathbb{Z}). \end{cases}
 \end{aligned} \tag{A.17}$$

Again they satisfy the modular covariance in the same sense as (A.15).

We also introduce slightly modified chiral blocks,

$$\mathbf{f}_{(a,b)}(\tau) := \begin{cases} f_{(a,b)}(\tau), & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1), \\ h_{(\frac{a}{2}, \frac{b}{2})}, & (a, b \in 2\mathbb{Z}), \end{cases} \tag{A.18}$$

$$\mathbf{g}_{(a,b)}(\tau) := \begin{cases} g_{(a,b)}(\tau), & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1), \\ h_{(\frac{a}{2}, \frac{b}{2})}, & (a, b \in 2\mathbb{Z}). \end{cases} \tag{A.19}$$

They correspond to the cases of $[(-\mathbf{1}_L)^{\otimes 4}]^2 = (-1)^{F_L}$, and $[(-\mathbf{1}_L)^{\otimes 2}]^2 = (-1)^{F_L}$, respectively, and behave modular covariantly as above.

Characters for the $\mathcal{N} = 4$ SCA with $c = 6$. The character formulas of the unitary irrep.'s of the $\mathcal{N} = 4$ SCA with $c = 6$ (level 1) are given in [30, 31], and we exhibit them here. We focus on the NS-sector:

Massive representation $\mathcal{C}_h^{(\text{NS})}$.

$$\text{ch}^{\mathcal{N}=4,(\text{NS})}(h; \tau, z) = q^{h-\frac{1}{8}} \frac{\theta_3(\tau, z)^2}{\eta(\tau)^3} \quad (\text{for } \mathcal{C}_h^{(\text{NS})}). \quad (\text{A.20})$$

Massless representations $\mathcal{D}_\ell^{(\text{NS})}$.

$$\text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\ell = \frac{1}{2}; \tau, z\right) = q^{-1/8} \sum_{n \in \mathbb{Z}} \frac{1}{1+yq^{n-1/2}} q^{\frac{n^2}{2}} y^n \frac{\theta_3(\tau, z)}{\eta(\tau)^3} \quad (\text{for } \mathcal{D}_{1/2}^{(\text{NS})}), \quad (\text{A.21})$$

$$\text{ch}_0^{\mathcal{N}=4,(\text{NS})}(\ell=0; \tau, z) = q^{-1/8} \sum_{n \in \mathbb{Z}} \frac{(1-q)q^{\frac{n^2}{2}+n-\frac{1}{2}} y^{n+1}}{(1+yq^{n+1/2})(1+yq^{n-1/2})} \frac{\theta_3(\tau, z)}{\eta(\tau)^3} \quad (\text{for } \mathcal{D}_0^{(\text{NS})}). \quad (\text{A.22})$$

The R-sector characters are obtained by the 1/2-spectral flow. Namely,

$$\begin{aligned} \text{ch}^{\mathcal{N}=4,(\text{R})}(h; \tau, z) &= q^{\frac{1}{4}} y \text{ch}^{\mathcal{N}=4,(\text{NS})}\left(h - \frac{1}{4}; \tau, z + \frac{\tau}{2}\right), \quad (\text{for } \mathcal{C}_h^{(\text{R})}), \\ \text{ch}_0^{\mathcal{N}=4,(\text{R})}(\ell; \tau, z) &= q^{\frac{1}{4}} y \text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\frac{1}{2} - \ell; \tau, z + \frac{\tau}{2}\right), \quad (\text{for } \mathcal{D}_\ell^{(\text{R})}). \end{aligned} \quad (\text{A.23})$$

B Twisted characters of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFTs

In this appendix we summarize the definitions of the twisted characters of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal algebras, according to [24, 25].

$\mathcal{N} = 2$ twisted characters for the minimal model M_k . We consider the characters of the $\mathcal{N} = 2$ SCA, twisted by the \mathbb{Z}_2 -automorphism

$$\sigma_L^{\mathcal{N}=2} : T \longrightarrow T, \quad J \longrightarrow -J, \quad G^\pm \longrightarrow G^\mp, \quad (\text{B.1})$$

and express them as $\text{ch}_{[S,T]}^{(\alpha)}$, where α are the spin structures, and $S, T \in \mathbb{Z}_2$ signify the spatial and temporal boundary conditions associated with the $\sigma^{\mathcal{N}=2}$ -twist ($S, T = 1$ means twisted, and $S, T = 0$ means untwisted). We then have the following identities,

$$\text{ch}_{[0,1]}^{(\text{NS})}(\tau) = \text{ch}_{[0,1]}^{(\widetilde{\text{NS}})}(\tau), \quad \text{ch}_{[1,0]}^{(\text{NS})}(\tau) = \text{ch}_{[1,0]}^{(\text{R})}(\tau), \quad \text{ch}_{[1,1]}^{(\widetilde{\text{NS}})}(\tau) = \text{ch}_{[1,1]}^{(\text{R})}(\tau), \quad (\text{B.2})$$

$$\text{ch}_{[0,1]}^{(\text{R})}(\tau) = \text{ch}_{[0,1]}^{(\widetilde{\text{R}})}(\tau), \quad \text{ch}_{[1,0]}^{(\widetilde{\text{NS}})}(\tau) = \text{ch}_{[1,0]}^{(\widetilde{\text{R}})}(\tau), \quad \text{ch}_{[1,1]}^{(\text{NS})}(\tau) = \text{ch}_{[1,1]}^{(\widetilde{\text{R}})}(\tau), \quad (\text{B.3})$$

and denote the twisted characters in the first line (B.2) as ‘ $\chi_{[0,1]}(\tau)$ ’, ‘ $\chi_{[1,0]}(\tau)$ ’ and ‘ $\chi_{[1,1]}(\tau)$ ’ for brevity. Especially, for the minimal models M_k , they are presented in [24, 25] (based on [32–34, 41–43]) as

$$\begin{aligned}
 \chi_{\ell[0,1]}^k(\tau) &= \begin{cases} \frac{2}{\theta_2(\tau)} \left(\Theta_{2(\ell+1),4(k+2)}(\tau) + (-1)^k \Theta_{2(\ell+1)+4(k+2),4(k+2)}(\tau) \right), & (\ell : \text{even}), \\ 0, & (\ell : \text{odd}). \end{cases} \\
 \chi_{\ell[1,0]}^k(\tau) &= \frac{1}{\theta_4(\tau)} \left(\Theta_{\ell+1-\frac{k+2}{2},k+2}(\tau) - \Theta_{-(\ell+1)-\frac{k+2}{2},k+2}(\tau) \right) \\
 &= \frac{1}{\theta_4(\tau)} \left(\Theta_{2(\ell+1)-(k+2),4(k+2)}(\tau) + \Theta_{2(\ell+1)+3(k+2),4(k+2)}(\tau) \right. \\
 &\quad \left. - \Theta_{-2(\ell+1)-(k+2),4(k+2)}(\tau) - \Theta_{-2(\ell+1)+3(k+2),4(k+2)}(\tau) \right), \\
 \chi_{\ell[1,1]}^k(\tau) &= \frac{1}{\theta_3(\tau)} \left(\Theta_{2(\ell+1)-(k+2),4(k+2)}(\tau) + (-1)^k \Theta_{2(\ell+1)+3(k+2),4(k+2)}(\tau) \right. \\
 &\quad \left. + (-1)^\ell \Theta_{-2(\ell+1)-(k+2),4(k+2)}(\tau) + (-1)^{k+\ell} \Theta_{-2(\ell+1)+3(k+2),4(k+2)}(\tau) \right). \quad (\text{B.4})
 \end{aligned}$$

The conformal weights of the ground states corresponding to the first characters are

$$h = h_\ell \equiv \frac{\ell(\ell+2)}{4(k+2)}, \quad (\text{B.5})$$

while those for the second and third ones are given by

$$h = h_\ell^t \equiv \frac{k-2+(k-2\ell)^2}{16(k+2)} + \frac{1}{16}. \quad (\text{B.6})$$

Note that only the states with the vanishing U(1)-charges can contribute to the relevant characters. Note also that $\chi_{k-\ell[1,0]}^k = \chi_{\ell[1,0]}^k$, $\chi_{k-\ell[1,1]}^k = \chi_{\ell[1,1]}^k$. Due to these relations the corresponding fields are identified, leaving only $\ell = 0, 1, \dots, [\frac{k}{2}]$ as independent primary fields.

The modular transformations of the twisted $\mathcal{N} = 2$ characters are

$$\begin{aligned}
 \chi_{\ell[0,1]}^k(\tau+1) &= e^{2\pi i \left(h_\ell - \frac{k}{8(k+2)} \right)} \chi_{\ell[0,1]}^k(\tau), \quad \chi_{\ell[0,1]}^k \left(-\frac{1}{\tau} \right) = \sum_{\ell'=0}^k (-1)^{\ell'/2} S_{\ell,\ell'} \chi_{\ell'[0,1]}^k(\tau), \\
 \chi_{\ell[1,0]}^k(\tau+1) &= e^{2\pi i \left(h_\ell^t - \frac{k}{8(k+2)} \right)} \chi_{\ell[1,0]}^k(\tau), \quad \chi_{\ell[1,0]}^k \left(-\frac{1}{\tau} \right) = \sum_{\ell'=0}^k S_{\ell,\ell'} (-1)^{\ell'/2} \chi_{\ell'[0,1]}^k(\tau), \\
 \chi_{\ell[1,1]}^k(\tau+1) &= e^{2\pi i \left(h_\ell^t - \frac{k}{8(k+2)} \right)} \chi_{\ell[1,0]}^k(\tau), \quad \chi_{\ell[1,1]}^k \left(-\frac{1}{\tau} \right) = \sum_{\ell'=0}^k \widehat{S}_{\ell,\ell'} \chi_{\ell'[1,1]}^k(\tau). \quad (\text{B.7})
 \end{aligned}$$

Here $S_{\ell,\ell'} \equiv \sqrt{\frac{2}{k+2}} \sin \left(\frac{\pi(\ell+1)(\ell'+1)}{k+2} \right)$ is the modular S-matrix of the SU(2) WZW model at level k , and $\widehat{S}_{\ell,\ell'} \equiv e^{\frac{\pi i}{2}(\ell+\ell'-\frac{k}{2})} S_{\ell,\ell'}$.

Let us briefly comment on the remaining minimal model characters appearing in the second line (B.3). For example, for the [0,1]-type boundary condition in the R-sector, almost all the characters vanish, except for the special representation generated by the non-degenerate Ramond ground state with $h = \frac{\hat{c}}{8}$, $Q = 0$, that is, $\ell = \frac{k}{2}$, $m = \pm(\frac{k}{2} + 1)$ with $k \in 2\mathbb{Z}_{>0}$. The corresponding character equal ± 1 , where the sign ambiguity is just due to the action of $\sigma_L^{\mathcal{N}=2}$ on primary states.

$\mathcal{N} = 4$ twisted characters. We next summarize the twisted $\mathcal{N} = 4$ characters defined by the $\sigma_L^{(1)}$ and $\sigma_L^{(3)}$ -twists in the unitary irrep.'s of the $\mathcal{N} = 4$ SCA with $c = 6$. We first focus on the $\sigma_L^{(3)}$ -twist for the boundary conditions given in (B.2). The key formula is the spectral flow decomposition of the $\mathcal{N} = 4$ characters by the $\mathcal{N} = 2$ ones [30, 31], written schematically as

$$\text{ch}^{\mathcal{N}=4,(\text{NS})}(*; \tau, z) = \sum_{n \in \mathbb{Z}} q^{n^2} y^{2n} \text{ch}^{\mathcal{N}=2,(\text{NS})}(*; \tau, z + n\tau), \quad (\text{B.8})$$

for the NS-sector, where $n \in \mathbb{Z}$ is identified with the n -th spectral flow sector. It is again the simplest to evaluate the case of $[S, T] = [0, 1]$ (i.e. with the insertion of $\sigma_L^{(3)}$ into the trace). This just yields an extra phase factor $(-1)^n$ in each n -th spectral flow sector in the decomposition (B.8), and we obtain the desired character formulas (by setting $z = 0$):

Massive representation $\mathcal{C}_h^{(\text{NS})}$.

$$\begin{aligned} \text{Tr}_{\mathcal{C}_h^{(\text{NS})}}[\sigma_L^{(3)} q^{L_0 - \frac{1}{4}}] &= q^{h - \frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \frac{\theta_3(\tau)}{\eta(\tau)^3} = q^{h - \frac{1}{8}} \frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^3} \\ &\equiv \frac{2q^{h - \frac{1}{8}}}{\theta_2(\tau)} =: \chi_{[0,1]}(h; \tau). \end{aligned} \quad (\text{B.9})$$

Massless representations $\mathcal{D}_\ell^{(\text{NS})}$.

$$\text{Tr}_{\mathcal{D}_{1/2}^{(\text{NS})}}[\sigma_L^{(3)} q^{L_0 - \frac{1}{4}}] = q^{-1/8} \sum_{n \in \mathbb{Z}} (-1)^{n+1} \frac{1}{1 + q^{n-1/2}} \frac{\theta_3(\tau)}{\eta(\tau)^3} \equiv 0, \quad (\text{B.10})$$

$$\begin{aligned} \text{Tr}_{\mathcal{D}_0^{(\text{NS})}}[\sigma_L^{(3)} q^{L_0 - \frac{1}{4}}] &= q^{-1/8} \sum_{n \in \mathbb{Z}} (-1)^n \frac{(1-q)q^{\frac{n^2}{2} + n - \frac{1}{2}}}{(1 + q^{n+1/2})(1 + q^{n-1/2})} \frac{\theta_3(\tau)}{\eta(\tau)^3} \\ &= q^{-1/8} \frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^3} \equiv \chi_{[0,1]}(h = 0; \tau, z). \end{aligned} \quad (\text{B.11})$$

The second line of (B.11) follows from the identity

$$\frac{(1-q)q^{n-\frac{1}{2}}}{(1 + q^{n+1/2})(1 + q^{n-1/2})} = 1 - \frac{1}{1 + q^{n-\frac{1}{2}}} - \frac{q^{n+\frac{1}{2}}}{1 + q^{n+\frac{1}{2}}}. \quad (\text{B.12})$$

We next consider the $\sigma_L^{(1)}$ -twist. Since the $\sigma_L^{(1)}$ -twist acts as $J(\equiv 2J^3) \rightarrow -J$ on the $U(1)_R$ -current of the underlying $\mathcal{N} = 2$ SCA, the spectral flow sectors of $n \neq 0$ cannot contribute when $\sigma_L^{(1)}$ is inserted into the trace. Thus, the wanted characters should be equal to the ones for the $\mathcal{N} = 2$ non-degenerate representations, that is,

$$\text{Tr}_{\mathcal{C}_h^{(\text{NS})}}[\sigma_L^{(1)} q^{L_0 - \frac{1}{4}}] = \frac{q^{h-1/8}}{\eta(\tau)} \cdot \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}} \cdot \sqrt{\frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^2}} \equiv \chi_{[0,1]}(h; \tau), \quad (\text{B.13})$$

$$\text{Tr}_{\mathcal{D}_{1/2}^{(\text{NS})}}[\sigma_L^{(1)} q^{L_0 - \frac{1}{4}}] = 0, \quad (\text{B.14})$$

$$\text{Tr}_{\mathcal{D}_0^{(\text{NS})}}[\sigma_L^{(1)} q^{L_0 - \frac{1}{4}}] = \frac{q^{-1/8}}{\eta(\tau)} \cdot \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}} \cdot \sqrt{\frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^2}} \equiv \chi_{[0,1]}(h = 0; \tau). \quad (\text{B.15})$$

They indeed coincide with those of $\sigma_L^{(3)}$ -twisting (B.9), (B.10) and (B.11).¹¹ The $\sigma_L^{(2)}$ -twisting leads to the same formulas, too.

The character formulas for other boundary conditions are just determined by the modular transformations. We denote the spin structures as well as the boundary conditions of $\sigma_L^{(\alpha)}$ such as $\{\text{NS}, [S, T]\}$. Starting from the character formula of $\{\text{NS}, [0, 1]\}$ given above, we find that there are three types of non-trivial characters $\chi_{[0,1]}(h; \tau)$, $\chi_{[1,0]}(h; \tau)$, $\chi_{[1,1]}(h; \tau)$;

$$\begin{aligned}
 \{\text{NS}, [0, 1]\}, \{\widetilde{\text{NS}}, [0, 1]\} : & \quad \chi_{[0,1]}(h; \tau) \equiv \frac{2q^{\frac{p^2}{2}}}{\theta_2(\tau)}, & \left(h = \frac{p^2}{2} + \frac{1}{8}\right), \\
 \{\text{NS}, [1, 0]\}, \{\text{R}, [1, 0]\} : & \quad \chi_{[1,0]}(h; \tau) \equiv \frac{2q^{\frac{p^2}{2}}}{\theta_4(\tau)}, & \left(h = \frac{p^2}{2} + \frac{1}{4}\right), \\
 \{\widetilde{\text{NS}}, [1, 1]\}, \{\text{R}, [1, 1]\} : & \quad \chi_{[1,1]}(h; \tau) \equiv \frac{2q^{\frac{p^2}{2}}}{\theta_3(\tau)}, & \left(h = \frac{p^2}{2} + \frac{1}{4}\right). \quad (\text{B.16})
 \end{aligned}$$

There still remain the boundary conditions presented in (B.3). We briefly describe them although only the ones listed in (B.16) are necessary in the main text,

$$\text{Tr}_{c_h^{(\text{R})}} \left[\sigma_L^{(\alpha)} q^{L_0 - \frac{1}{4}} \right] = \text{Tr}_{\mathcal{D}_{1/2}^{(\text{R})}} \left[\sigma_L^{(\alpha)} q^{L_0 - \frac{1}{4}} \right] = 0, \quad \text{Tr}_{\mathcal{D}_0^{(\text{R})}} \left[\sigma_L^{(\alpha)} q^{L_0 - \frac{1}{4}} \right] = \pm 1, \quad (\text{B.17})$$

($\forall \alpha = 1, 2, 3$). The sign ambiguity in the formula for $\mathcal{D}_0^{(\text{R})}$ is due to the same reason as above. We also obtain the same results for the $\{\widetilde{\text{R}}, (0, 1)\}$ -characters. It is trivial to modular transform these results to obtain the remaining ones $\{\widetilde{\text{NS}}, [1, 0]\}$, $\{\text{NS}, [1, 1]\}$ ($\{\widetilde{\text{R}}, [1, 0]\}$, $\{\widetilde{\text{R}}, [1, 1]\}$).

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¹¹This coincidence would be anticipated. However, it is not necessarily self-evident because the automorphisms $\sigma_L^{(1)}$ and $\sigma_L^{(3)}$ are interpolated only by an *outer*-automorphism of the $\mathcal{N} = 4$ SCA, as opposed to the case of e.g. $\widehat{\text{SU}}(2)_k$.

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