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ON THE GROUP OF EXTENSIONS $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n)})$ OVER A DISCRETE VALUATION RING

By

Takashi KONDO

Abstract. For given group schemes $\mathscr{G}^{(\lambda_i)}$ $(i = 1, 2, \dots)$ deforming the additive group scheme $\mathbf{G}_a$ to the multiplicative group scheme $\mathbf{G}_m$, T. Sekiguchi and N. Suwa constructed extensions:

$$0 \to \mathscr{G}^{(\lambda_2)} \to \mathscr{E}^{(\lambda_1,\lambda_2)} \to \mathscr{G}^{(\lambda_1)} \to 0,$$

$$\dots,$$

$$0 \to \mathscr{G}^{(\lambda_n)} \to \mathscr{E}^{(\lambda_1,\dots,\lambda_n)} \to \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1})} \to 0,$$

$$\dots$$

inductively, by calculating the group of extensions $\mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1})}, \mathscr{G}^{(\lambda_n)})$. Here changing the group schemes, we treat the group $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n)})$ of extensions for any positive integers n. The case of $n = 2, 3$ were studied by D. Horikawa and T. Kondo.

Introduction

T. Sekiguchi and N. Suwa constructed the group schemes deforming the group schemes of Witt vectors to tori in order to unify the Kummer theory and the Artin-Schreier-Witt theory. Let A be a discrete valuation ring with the maximal ideal $\mathfrak{m}$. Then such group schemes of dimension 1 over A are known to be given only by $\mathscr{G}^{(\lambda)} = \mathrm{Spec}\, A[X, 1/(1 + \lambda X)]$ with $\lambda \in \mathfrak{m} \setminus \{0\}$ (cf. [4], [9]). In the higher dimensional case, Sekiguchi and Suwa determined the groups of extensions $\mathscr{E}^{(\lambda_1,\dots,\lambda_n)}$ for $n \geq 2$ successively (cf. [5]).

In this article, we will determine the group of extensions $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n)})$ (Theorem 4.1.2) for any positive integers n. The case of $n = 2, 3$ were determined by D. Horikawa [2] and T. Kondo [3].

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Notations

In this article, ring means a commutative unitary ring.

- p: a fixed prime number
- $\mathbf{G}_{a,A} := \operatorname{Spec} A[T]$: the additive group scheme over a ring A
- $\mathbf{G}_{m,A} := \operatorname{Spec} A[T, 1/T]$: the multiplicative group scheme over a ring A
- $\mathbf{A}_A^n$: the affine space of dimension n over a ring A, endowed with the usual ring scheme structure
- $W(A)$: the ring of Witt vectors over a ring A
- $[a] := (a, 0, 0, \ldots) \in W(A)$: the Teichmüller lifting of $a \in A$
- $\operatorname{Ext}^1(G, H)$: the group of extensions of abelian group schemes G and H
- $H_0^2(G, H)$: the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for group schemes G and H (cf. [1, Chap. II.3 and Chap. III.6]).

1. Witt Vectors

In this section, we recall the fundamental facts on Witt vectors.

1.1. For a non-negative integer n, we denote by $\Phi_n(\mathbf{X}) = \Phi_n(X_0, \ldots, X_n)$ the Witt polynomial:

$$\Phi_n(\mathbf{X}) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^{n-1}X_{n-1}^p + p^n X_n$$

in $\mathbf{Z}[\mathbf{X}] = \mathbf{Z}[X_0, X_1, \ldots]$. We put $\mathbf{W}_{n,\mathbf{Z}} := \operatorname{Spec} \mathbf{Z}[T_0, T_1, \ldots, T_{n-1}]$ and define the map $\Phi^{(n)} : \mathbf{W}_{n,\mathbf{Z}} \to \mathbf{A}_{\mathbf{Z}}^n$ by

$$T_i \mapsto \Phi_i(\mathbf{T}) = \Phi_i(T_0, \ldots, T_i).$$

Proposition 1.1.1. *$\Phi^{(n)}$ induces the ring scheme structure on $\mathbf{W}_{n,\mathbf{Z}}$ uniquely so that it is a ring scheme homomorphism. In particular, $\mathbf{W}_{n,\mathbf{Q}} \simeq \mathbf{A}_{\mathbf{Q}}^n$.*

In fact, the addition σ and the multiplication π of $\mathbf{A}_{\mathbf{Z}}^n$ are given by

$$\sigma^* : T_i \mapsto X_i + Y_i, \quad \pi^* : T_i \mapsto X_i \otimes Y_i$$

with $X_i := T_i \otimes 1$ and $Y_i := 1 \otimes T_i$. Suppose that Σ and Π are the addition and the multiplication which are induced by $\Phi^{(n)}$. Then $\Sigma^*(\Phi_i(\mathbf{T})) = \Phi_i(\mathbf{X}) + \Phi_i(\mathbf{Y})$, $\Pi^*(\Phi_i(\mathbf{T})) = \Phi_i(\mathbf{X})\Phi_i(\mathbf{Y})$ and the following diagrams are commutative:

$$\begin{array}{ccc}
\mathbf{W}_{n,\mathbf{Z}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{W}_{n,\mathbf{Z}} & \xrightarrow{\Phi^{(n)} \times \Phi^{(n)}} & \mathbf{A}^n_{\mathbf{Z}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{A}^n_{\mathbf{Z}} \\
\Sigma \downarrow & & \downarrow \sigma \\
\mathbf{W}_{n,\mathbf{Z}} & \xrightarrow[\Phi^{(n)}]{} & \mathbf{A}^n_{\mathbf{Z}}
\end{array}$$

$$\begin{array}{ccc}
\mathbf{W}_{n,\mathbf{Z}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{W}_{n,\mathbf{Z}} & \xrightarrow{\Phi^{(n)} \times \Phi^{(n)}} & \mathbf{A}^n_{\mathbf{Z}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{A}^n_{\mathbf{Z}} \\
\Pi \downarrow & & \downarrow \pi \\
\mathbf{W}_{n,\mathbf{Z}} & \xrightarrow[\Phi^{(n)}]{} & \mathbf{A}^n_{\mathbf{Z}}
\end{array}.$$

By induction on i, it can be seen that $\Sigma^*(T_i)$, $\Pi^*(T_i) \in \mathbf{Z}[\mathbf{X}, \mathbf{Y}]$, thus $\mathbf{W}_{n,\mathbf{Z}}$ is a ring scheme over $\mathbf{Z}$. We call $\mathbf{W}_{n,\mathbf{Z}}$ the ring scheme of Witt vectors over $\mathbf{Z}$ of length n. We also denote $\Sigma^*(T_i)$ and $\Pi^*(T_i)$ by $S_i(\mathbf{X}, \mathbf{Y})$ and $P_i(\mathbf{X}, \mathbf{Y})$, respectively.

We denote the ring of Witt vectors over a ring A by $W(A)$, and the formal completion of $W(A)$ along the zero section by $\hat{W}(A)$. Then we have

$$\hat{W}(A) = \left\{ (a_0, a_1, \dots) \in W(A) \,\middle|\, \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

1.2. In this subsection, we define some endomorphisms of the additive group $\mathbf{W}_{\mathbf{Z}} := \mathrm{Spec}\,\mathbf{Z}[\mathbf{T}]$. We define the Verschiebung endomorphism $V : \mathbf{W}_{\mathbf{Z}} \to \mathbf{W}_{\mathbf{Z}}$ by

$$T_i \mapsto \begin{cases} T_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}.$$

For $r \geq 0$, we define polynomials $F_r(\mathbf{T}) \in \mathbf{Q}[T_0, \dots, T_{r+1}]$ inductively by

$$\Phi_r(F_0(\mathbf{T}), \dots, F_r(\mathbf{T})) = \Phi_{r+1}(\mathbf{T}).$$

Then $F_r(\mathbf{T}) \in \mathbf{Z}[T_0, \dots, T_{r+1}]$. We define the Frobenius endomorphism $F : \mathbf{W}_{\mathbf{Z}} \to \mathbf{W}_{\mathbf{Z}}$ by

$$T_i \mapsto F_i(\mathbf{T}).$$

F is a ring scheme homomorphism and if A is a $\mathbf{F}_p$-algebra, then $F : W(A) \to W(A)$ is nothing but the usual Frobenius endomorphism. For a ring A and $\lambda \in A$, we define $F^{(\lambda)} : \hat{W}(A) \to \hat{W}(A)$ by

$$F^{(\lambda)}\boldsymbol{a} := (F - [\lambda^{p-1}])\boldsymbol{a} = F\boldsymbol{a} - [\lambda^{p-1}]\boldsymbol{a}$$

and denote the kernel and the cokernel of $F^{(\lambda)}$ by $\hat{W}(A)^{F^{(\lambda)}}$ and $\hat{W}(A)/F^{(\lambda)}$, respectively.

For a vector $\boldsymbol{a} := (a_0, a_1, \dots) \in W(A)$, we define a map $\langle \boldsymbol{a}, \cdot \rangle : W(A) \to W(A)$ by

$$\Phi_n(\langle \boldsymbol{a}, \boldsymbol{x} \rangle) = a_0^{p^n}\Phi_n(\boldsymbol{x}) + pa_1^{p^{n-1}}\Phi_{n-1}(\boldsymbol{x}) + \cdots + p^n a_n \Phi_0(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$. Then we have $\langle \boldsymbol{a}, \cdot \rangle = \sum_{k \geq 0} V^k[a_k]$ and it is an endomorphism (cf. [5, Remark 4.8]).

2. Artin-Hasse Exponential Series

In this section, we review some concepts on Artin-Hasse exponential series from [5].

2.1. We define a formal power series $E_p(T) \in \mathbf{Q}[[T]]$ by

$$E_p(T) := \exp\left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots\right).$$

Then it can be seen $E_p(T) \in \mathbf{Z}_{(p)}[[T]]$ and we call it Artin-Hasse exponential series.

We define a formal power series $E_p(U, \Lambda; T) \in \mathbf{Q}[U, \Lambda][[T]]$ by

$$E_p(U, \Lambda; T) := (1 + \Lambda T)^{U/\Lambda} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{(1/p^k)\{(U/\Lambda)^{p^k} - (U/\Lambda)^{p^{k-1}}\}}.$$

By Artin-Hasse exponential series, we have

$$E_p(U, \Lambda; T) = \begin{cases} \prod_{(k,p)=1} E_p(U\Lambda^{k-1}T^k)^{(-1)^{k-1}/k} & \text{if } p > 2 \\ \prod_{(k,2)=1} E_2(U\Lambda^{k-1}T^k)^{1/k} \left\{ \prod_{(k,2)=1} E_2(U\Lambda^{2k-1}T^{2k})^{1/k} \right\}^{-1} & \text{if } p = 2 \end{cases}$$

and $E_p(U, \Lambda; T) \in \mathbf{Z}_{(p)}[U, \Lambda][[T]]$.

Moreover, for an infinite sequence of indeterminates $\mathbf{U} = (U_0, U_1, \dots)$, we define formal power series $E_p(\mathbf{U}, \Lambda; T)$ by

$$E_p(\mathbf{U}, \Lambda; T) := \prod_{k=0}^{\infty} E_p(U_k, \Lambda^{p^k}, T^{p^k}) \in \mathbf{Z}_{(p)}[\mathbf{U}, \Lambda][[T]].$$

Then we have

$$E_p(\mathbf{U},\Lambda;T) = (1+\Lambda T)^{U_0/\Lambda} \prod_{k=1}^{\infty} (1+\Lambda^{p^k}T^{p^k})^{(1/p^k\Lambda^{p^k})(\Phi_k(\mathbf{U})-\Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\mathbf{U}))}$$

$$= (1+\Lambda T)^{U_0/\Lambda} \prod_{k=1}^{\infty} (1+\Lambda^{p^k}T^{p^k})^{(1/p^k\Lambda^{p^k})\Phi_{k-1}(F^{(\Lambda)}\mathbf{U})}.$$

Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then for $\boldsymbol{a}, \boldsymbol{b} \in W(A)$, we have

$$E_p(\boldsymbol{a},\lambda;T)E_p(\boldsymbol{b},\lambda;T) = E_p(\boldsymbol{a}+\boldsymbol{b},\lambda;T)$$

with $\boldsymbol{a}+\boldsymbol{b} = (S_0(\boldsymbol{a},\boldsymbol{b}), S_1(\boldsymbol{a},\boldsymbol{b}),\dots)$. Moreover, if $F^{(\lambda)}\boldsymbol{a} = \mathbf{0}$, then

$$E_p(\boldsymbol{a},\lambda;T_0)E_p(\boldsymbol{a},\lambda;T_1) = E_p(\boldsymbol{a},\lambda;T_0+T_1+\lambda T_0T_1).$$

We define a formal power series $F_p(\mathbf{U},\Lambda;T_0,T_1) \in \mathbf{Q}[\mathbf{U},\Lambda][[T_0,T_1]]$ by

$$F_p(\mathbf{U},\Lambda;T_0,T_1) := \prod_{k=1}^{\infty} \left(\frac{(1+\Lambda^{p^k}T_0^{p^k})(1+\Lambda^{p^k}T_1^{p^k})}{1+\Lambda^{p^k}(T_0+T_1+\Lambda T_0T_1)^{p^k}} \right)^{(1/p^k\Lambda^{p^k})\Phi_{k-1}(\mathbf{U})}.$$

Then $F_p(\mathbf{U},\Lambda;T_0,T_1) \in \mathbf{Z}_{(p)}[\mathbf{U},\Lambda][[T_0,T_1]]$ (cf. [5]).

2.2. For $\boldsymbol{a} = (a_0, a_1, \dots) \in W(A)$, we define $\tilde{p}$: $W(A) \to W(A)$ by

$$\tilde{p}(\boldsymbol{a}) := (0, a_0^p, a_1^p, \dots).$$

Moreover, we define $\tilde{p}E_p(\mathbf{U},\Lambda;X)$ and $\tilde{p}F_p(\mathbf{U},\Lambda;X,Y)$ by

$$\tilde{p}E_p(\mathbf{U},\Lambda;X) := E_p(\tilde{p}\mathbf{U},\Lambda;X)$$

and

$$\tilde{p}F_p(\mathbf{U},\Lambda;X,Y) := F_p(\tilde{p}\mathbf{U},\Lambda;X,Y).$$

Proposition 2.2.1 ([5, Lemma 4.10]). *For $k, \ell \in \mathbf{Z}$ with $k \geq 1$ and $\ell \geq 0$, we have*

$$(\tilde{p})^k E_p(\mathbf{U},\Lambda;X) = E_p(\mathbf{U}^{(p^k)}, \Lambda^{p^k}; X^{p^k})$$

$$(\tilde{p})^{k+\ell} F_p(\mathbf{U},\Lambda;X,Y) \equiv (\tilde{p})^{\ell} F_p(\mathbf{U}^{(p^k)}, \Lambda^{p^k}; X^{p^k}, Y^{p^k}) \bmod p^{\ell+1}$$

with $\mathbf{U}^{(p^k)} = (U_0^{p^k}, U_1^{p^k}, \dots)$.

We put

$$\mathbf{V} = (V_0, V_1, \dots) := \left(\frac{U_0}{\Lambda_2}, \frac{U_1}{\Lambda_2}, \dots\right),$$

and define formal power series $\tilde{E}_p(\mathbf{W}, \Lambda_2; E)$ and $\tilde{E}_p(\mathbf{W}, \Lambda_2; F)$ with an infinite sequence of indeterminates $\mathbf{W} = (W_0, W_1, \dots)$ by

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; E) := E^{W_0/\Lambda_2} \prod_{k=1}^{\infty} ((\tilde{p})^k E)^{(1/p^k \Lambda_2^{p^k}) \Phi_{k-1}(F^{(\Lambda_2)}\mathbf{W})}$$

and

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; F) := F^{W_0/\Lambda_2} \prod_{k=1}^{\infty} ((\tilde{p})^k F)^{(1/p^k \Lambda_2^{p^k}) \Phi_{k-1}(F^{(\Lambda_2)}\mathbf{W})}$$

where $E := E_p(\mathbf{U}, \Lambda_1; X)$ and $F := F_p(\mathbf{U}, \Lambda_1; X, Y)$.

Proposition 2.2.2 ([5, Proposition 4.11]). *Under the above notation, we have*

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; E) = E_p(\langle \mathbf{V}, \mathbf{W} \rangle, \Lambda_1, X),$$
$$\tilde{E}_p(\mathbf{W}, \Lambda_2; F) = F_p(\langle \mathbf{V}, \mathbf{W} \rangle, \Lambda_1, X, Y).$$

We define formal power series $G_p(\mathbf{W}, \Lambda_2; E)$ and $G_p(\mathbf{W}, \Lambda_2; F)$ by

$$G_p(\mathbf{W}, \Lambda_2; E) := \prod_{k=1}^{\infty} \left(\frac{1 + (E-1)^{p^k}}{(\tilde{p})^k E} \right)^{(1/p^k \Lambda_2^{p^k}) \Phi_{k-1}(\mathbf{W})}$$

and

$$G_p(\mathbf{W}, \Lambda_2; F) := \prod_{k=1}^{\infty} \left(\frac{1 + (F-1)^{p^k}}{(\tilde{p})^k F} \right)^{(1/p^k \Lambda_2^{p^k}) \Phi_{k-1}(\mathbf{W})}.$$

Then we have

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; E) = \frac{E_p\big(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(E-1)\big)}{\tilde{E}_p(\mathbf{W}, \Lambda_2; E)}$$

and

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; F) = \frac{E_p\big(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(F-1)\big)}{\tilde{E}_p(\mathbf{W}, \Lambda_2; F)}.$$

Moreover, we have $G_p(\mathbf{W}, \Lambda_2; E), G_p(\mathbf{W}, \Lambda_2; F) \in \mathbf{Z}_{(p)}[\mathbf{W}, \mathbf{U}/\Lambda_2, \Lambda_1, \Lambda_2][[X, Y]]$.

Proposition 2.2.3 ([5, Proposition 4.13]). *Under the above notation and an infinite sequence of indeterminates* $\mathbf{A} = (A_0, A_1, \dots)$, *we have*

$$\tilde{E}_p(\mathbf{W}, \Lambda_3; G_p(\mathbf{A}, \Lambda_2; E)) = G_p\left(\left\langle \frac{\mathbf{A}}{\Lambda_3}, \mathbf{W} \right\rangle, \Lambda_2; E\right),$$

$$\tilde{E}_p(\mathbf{W}, \Lambda_3; G_p(\mathbf{A}, \Lambda_2; F)) = G_p\left(\left\langle \frac{\mathbf{A}}{\Lambda_3}, \mathbf{W} \right\rangle, \Lambda_2; F\right).$$

3. Results on $\mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1})}, \mathscr{G}^{(\lambda_n)})$

Sekiguchi and Suwa determined completely the extension groups $\mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1})}, \mathscr{G}^{(\lambda_n)})$ in [5]. In this section, we review some results of [5] which are used in this paper.

3.1. Let $(A, \mathfrak{m})$ be a discrete valuation ring with the maximal ideal $\mathfrak{m}$ such that $\mathrm{ch}(\mathrm{Frac}(A)) = 0$ and $\mathrm{ch}(A/\mathfrak{m}) = p$. Then

$$\mathscr{G}^{(\lambda_1)} := \mathrm{Spec}\, A[X_1, 1/(1 + \lambda_1 X_1)], \quad \lambda_1 \in \mathfrak{m} \backslash \{0\}$$

is a group scheme over A with

$$\begin{aligned} &\text{co-multiplication:} && X_1 \mapsto X_1 \otimes 1 + 1 \otimes X_1 + \lambda_1 X_1 \otimes X_1, \\ &\text{co-unit:} && X_1 \mapsto 0, \\ &\text{co-inverse:} && X_1 \mapsto -X_1/(1 + \lambda_1 X_1). \end{aligned}$$

Moreover, we have the following A-homomorphism $\alpha^{(\lambda_1)}\colon \mathscr{G}^{(\lambda_1)} \to \mathbf{G}_{m,A}$ by

$$\begin{array}{c} A[T, 1/T] \to A[X_1, 1/(1 + \lambda_1 X_1)] \\ T \mapsto 1 + \lambda_1 X_1 \end{array}.$$

In particular, for the generic point η and the special point s of $\mathrm{Spec}\, A$, $\alpha^{(\lambda_1)}$ induces $\alpha_\eta^{(\lambda_1)} : \mathscr{G}_\eta^{(\lambda_1)} \xrightarrow{\sim} \mathbf{G}_{m,K}$ and $\alpha_s^{(\lambda_1)} : \mathscr{G}_s^{(\lambda_1)} \xrightarrow{\sim} \mathbf{G}_{a,k}$ where $K := \mathrm{Frac}(A)$ and $k := A/\mathfrak{m}$.

3.2. Let $A_{\lambda_2} := A/\lambda_2 A$ for $\lambda_2 \in \mathfrak{m}\backslash\{0\}$ and $\iota : \mathrm{Spec}\, A_{\lambda_2} \hookrightarrow \mathrm{Spec}\, A$ the canonical closed immersion. Then since the sequence

$$0 \longrightarrow \mathscr{G}^{(\lambda_2)} \xrightarrow{\alpha^{(\lambda_2)}} \mathbf{G}_{m,A} \xrightarrow{r^{(\lambda_2)}} \iota_* \mathbf{G}_{m,A_{\lambda_2}} \longrightarrow 0 \qquad (*)$$

$$x \mapsto 1 + \lambda_2 x$$

$$t \mapsto t \bmod \lambda_2$$

is exact on small flat site over Spec A, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \mathscr{G}^{(\lambda_2)}) \xrightarrow{\alpha^{(\lambda_2)}} \mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) \xrightarrow{r^{(\lambda_2)}} \mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \iota_* \mathbf{G}_{m,A_{\lambda_2}})$$

$$\xrightarrow{\partial} \mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathscr{G}^{(\lambda_2)}) \xrightarrow{\alpha^{(\lambda_2)}} \mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathbf{G}_{m,A}).$$

We have $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) = 0$ (cf. [5]), thus

$$\mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathscr{G}^{(\lambda_2)}) \simeq \mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \iota_* \mathbf{G}_{m,A_{\lambda_2}})/r^{(\lambda_2)}(\mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \mathbf{G}_{m,A})).$$

Moreover, by $\mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) \simeq \{(1+\lambda_1 X_1)^n \mid n \in \mathbf{Z}\}$, we have

$$\mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathscr{G}^{(\lambda_2)}) \simeq \mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \iota_* \mathbf{G}_{m,A_{\lambda_2}})/\{(1+\lambda_1 X_1)^n \bmod \lambda_2 \mid n \in \mathbf{Z}\}.$$

The following theorem is crucial in the later argument.

Theorem 3.2.1 ([6, Theorem 2.19.1]). *Let A be a $\mathbf{Z}_{(p)}$-algebra, and $\lambda \in A$ be a nilpotent element. Then the group homomorphism*

$$\hat{W}(A)^{F^{(\lambda)}} \to \mathrm{Hom}(\mathscr{G}^{(\lambda)}, \mathbf{G}_{m,A})$$
$$\boldsymbol{a} \mapsto E_p(\boldsymbol{a}, \lambda; X)$$

and

$$\hat{W}(A)/F^{(\lambda)} \to H_0^2(\mathscr{G}^{(\lambda)}, \mathbf{G}_{m,A})$$
$$\boldsymbol{a} \mapsto F_p(\boldsymbol{a}, \lambda; X, Y)$$

are bijective.

By noting $E_p([\lambda_1], \lambda_1, X_1) = 1 + \lambda_1 X_1$, we have $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathscr{G}^{(\lambda_2)}) \simeq \hat{W}(A_{\lambda_2})^{F^{(\lambda_1)}}/\langle[\lambda_1]\rangle$. This correspondence is given more explicitly as follows.

For $\boldsymbol{u}^1 \bmod \lambda_2 \in \hat{W}(A_{\lambda_2})^{F^{(\lambda_1)}}/\langle[\lambda_1]\rangle$, we put

$$D_1(X_1) := E_p(\boldsymbol{u}^1, \lambda_1; X_1) \bmod \lambda_2.$$

Then D_1 is contained in $\mathrm{Hom}(\mathscr{G}^{(\lambda_1)}, \iota_* \mathbf{G}_{m,A_{\lambda_2}})$, and $\partial D_1 \in \mathrm{Ext}^1(\mathscr{G}^{(\lambda_1)}, \mathscr{G}^{(\lambda_2)})$ is given by the pull-back of the exact sequence $(*)$ by D_1. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}^{(\lambda_2)} & \longrightarrow & \partial D_1 & \longrightarrow & \mathscr{G}^{(\lambda_1)} & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow{\scriptstyle D_1} & & \\
0 & \longrightarrow & \mathscr{G}^{(\lambda_2)} & \xrightarrow[\alpha^{(\lambda_2)}]{} & \mathbf{G}_{m,A} & \xrightarrow[r^{(\lambda_2)}]{} & \iota_* \mathbf{G}_{m,A_{\lambda_2}} & \longrightarrow & 0.
\end{array}$$

Let x_1 and t be local sections of $\mathscr{G}^{(\lambda_1)}$ and $\mathbf{G}_{m,A}$ respectively. Then by $D_1(x_1) = t \bmod \lambda_2$, there exists x_2 such that $t = D_1(x_1) + \lambda_2 x_2$ and we have

$$\partial D_1 = \text{the class of Spec } A\left[X_1, X_2, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}\right].$$

We put $\mathscr{E}^{(\lambda_1,\lambda_2;D_1)} := \mathrm{Spec}\, A\left[X_1, X_2, \dfrac{1}{1+\lambda_1 X_1}, \dfrac{1}{D_1(X_1)+\lambda_2 X_2}\right]$. Then it is an affine group scheme over A.

3.3. Replacing λ_2 by λ_3 with $\lambda_3 \in \mathfrak{m}\backslash\{0\}$ in the exact sequence $(*)$ and apply the same argument to $\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}$, we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \mathscr{G}^{(\lambda_3)}) \xrightarrow{\alpha^{(\lambda_3)}} \mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \mathbf{G}_{m,A})$$

$$\xrightarrow{r^{(\lambda_3)}} \mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}}) \xrightarrow{\partial} \mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \mathscr{G}^{(\lambda_3)}) \longrightarrow 0.$$

By $\mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \mathbf{G}_{m,A}) \simeq \{(1+\lambda_1 X_1)^n (D_1(X_1)+\lambda_2 X_2)^m \mid n, m \in \mathbf{Z}\}$, we have

$$\mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \mathscr{G}^{(\lambda_3)})$$
$$\simeq \mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}})/\{(1+\lambda_1 X_1)^n (D_1(X_1)+\lambda_2 X_2)^m \bmod \lambda_3 \mid n, m \in \mathbf{Z}\}.$$

We put $\boldsymbol{b}_2^3 := \dfrac{1}{\lambda_2} F^{(\lambda_1)} \boldsymbol{u}^1$ and $U^2 := \begin{pmatrix} F^{(\lambda_1)} & -\langle \boldsymbol{b}_2^3, \cdot\rangle \\ 0 & F^{(\lambda_2)} \end{pmatrix}$. Then we have the following theorem.

Theorem 3.3.1 ([5, Theorem 5.1]). *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda_1, \lambda_2, \lambda_3 \in A$. Suppose λ_1 and λ_2 are nilpotent in A_{λ_3}. Then the group homomorphism*

$$\mathrm{Ker}[U^2 : \hat{W}(A_{\lambda_3})^2 \to \hat{W}(A_{\lambda_3})^2] \to \mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}})$$

$$(\boldsymbol{b}_1 \bmod \lambda_3, \boldsymbol{b}_2 \bmod \lambda_3) \mapsto E_p(\boldsymbol{b}_1, \lambda_1; X_1) E_p\left(\boldsymbol{b}_2, \lambda_2; \frac{X_2}{D_1(X_1)}\right) \bmod \lambda_3$$

is bijective.

In particular, under this correspondence, we have

$$([\lambda_1], \mathbf{0}) \mapsto 1 + \lambda_1 X_1$$

and

$$(\boldsymbol{u}^1, [\lambda_2]) \mapsto D_1(X_1) + \lambda_2 X_2.$$

Therefore by Theorem 3.3.1, we have

$$\mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \mathscr{G}^{(\lambda_3)}) \simeq \mathrm{Ker}[U^2 : \hat{W}(A_{\lambda_3})^2 \to \hat{W}(A_{\lambda_3})^2]/\langle([\lambda_1], \mathbf{0}), (\boldsymbol{u}^1, [\lambda_2])\rangle.$$

This correspondence is also given more explicitly as follows.

For $(\boldsymbol{u}_1^2, \boldsymbol{u}_2^2) \in \mathrm{Ker}[U^2 : \hat{W}(A_{\lambda_3})^2 \to \hat{W}(A_{\lambda_3})^2]$, we put

$$D_2(X_1, X_2) := E_p(\boldsymbol{u}_1^2, \lambda_1; X_1)E_p\left(\boldsymbol{u}_2^2, \lambda_2; \frac{X_2}{D_1(X_1)}\right) \bmod \lambda_3.$$

Then D_2 is contained in $\mathrm{Hom}(\mathscr{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}})$ and ∂D_2 is the pull-back of the exact sequence $(*)$ by D_2. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}^{(\lambda_3)} & \longrightarrow & \partial D_2 & \longrightarrow & \mathscr{E}^{(\lambda_1,\lambda_2;D_1)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle D_2} & & \\
0 & \longrightarrow & \mathscr{G}^{(\lambda_3)} & \underset{\alpha^{(\lambda_3)}}{\longrightarrow} & \mathbf{G}_{m,A} & \underset{r^{(\lambda_3)}}{\longrightarrow} & \iota_*\mathbf{G}_{m,A_{\lambda_3}} & \longrightarrow & 0,
\end{array}$$

where

$$\partial D_2 = \text{the class of Spec } A\left[X_1, X_2, X_3, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \frac{1}{D_2(X_1,X_2)+\lambda_3 X_3}\right].$$

We put

$$\mathscr{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} := \text{Spec } A\left[X_1, X_2, X_3, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \frac{1}{D_2(X_1,X_2)+\lambda_3 X_3}\right].$$

Then it is an affine group scheme over A.

3.4. By induction on n and [5, Theorem 5.1], we have

$$\begin{aligned}
&\mathrm{Ext}^1(\mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})}, \mathscr{G}^{(\lambda_n)}) \\
&\simeq \mathrm{Ker}[U^{n-1} : \hat{W}(A_{\lambda_n})^{n-1} \to \hat{W}(A_{\lambda_n})^{n-1}]/\langle([\lambda_1], \mathbf{0}, \dots, \mathbf{0}), (\boldsymbol{u}^1, [\lambda_2], \mathbf{0}, \dots, \mathbf{0}), \dots, \\
&\quad (\boldsymbol{u}_1^{n-3}, \dots, \boldsymbol{u}_{n-3}^{n-3}, [\lambda_{n-2}], \mathbf{0}), (\boldsymbol{u}_1^{n-2}, \dots, \boldsymbol{u}_{n-2}^{n-2}, [\lambda_{n-1}])\rangle
\end{aligned}$$

with

$$\begin{cases} \lambda_1, \dots, \lambda_n \in \mathfrak{m}\backslash\{0\} \\ \boldsymbol{b}_i^n := \dfrac{1}{\lambda_{n-1}}\left(F^{(\lambda_{i-1})}\boldsymbol{u}_{i-1}^{n-2} - \displaystyle\sum_{j=i}^{n-2} \langle \boldsymbol{b}_i^{j+1}, \boldsymbol{u}_j^{n-2}\rangle\right), \quad 2 \le i \le n-2 \\ \boldsymbol{b}_{n-1}^n := \dfrac{1}{\lambda_{n-1}} F^{(\lambda_{n-2})}\boldsymbol{u}_{n-2}^{n-2} \\ U^{n-1} := \begin{pmatrix} F^{(\lambda_1)} & -\langle \boldsymbol{b}_2^3, \cdot\rangle & \cdots & -\langle \boldsymbol{b}_2^n, \cdot\rangle \\ 0 & F^{(\lambda_2)} & \cdots & -\langle \boldsymbol{b}_3^n, \cdot\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F^{(\lambda_{n-1})} \end{pmatrix} \end{cases}$$

and elements of $\operatorname{Ext}^1(\mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};\, D_1,\dots,D_{n-2})}, \mathscr{G}^{(\lambda_n)})$ are the classes of type of group schemes

$$\operatorname{Spec} A\left[X_1, \dots, X_n, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \dots, \frac{1}{D_{n-1}(X_1,\dots,X_{n-1})+\lambda_n X_n}\right]$$

with

$$\begin{cases} (\boldsymbol{u}_1^{n-1}, \dots, \boldsymbol{u}_{n-1}^{n-1}) \in \hat{W}(A_{\lambda_n})^{n-1} \\ F^{(\lambda_i)}\boldsymbol{u}_i^{n-1} - \displaystyle\sum_{j=i+1}^{n-1} \langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^{n-1}\rangle \equiv \boldsymbol{0} \mod \lambda_n, \quad 1 \le i \le n-2 \\ F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^{n-1} \equiv \boldsymbol{0} \mod \lambda_n \\ D_0 := 1 \\ D_{n-1}(X_1, \dots, X_{n-1}) := \displaystyle\prod_{i=1}^{n-1} E_p\left(\boldsymbol{u}_i^{n-1}, \lambda_i; \frac{X_i}{D_{i-1}(X_1,\dots,X_{i-1})}\right) \end{cases}.$$

We put

$$\mathscr{E}^{(\lambda_1,\dots,\lambda_n;\, D_1,\dots,D_{n-1})} := \operatorname{Spec} A\left[X_1, \dots, X_n, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \dots, \frac{1}{D_{n-1}(X_1,\dots,X_{n-1})+\lambda_n X_n}\right]$$

and it is an affine group scheme over A.

4. Main Theorem

In this section, we use the notations and the group schemes $\mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})}$ defined in Section 3.

4.1. Let $(A,\mathfrak{m})$ be a discrete valuation ring and $\lambda_0\in\mathfrak{m}\backslash\{0\}$. Then Horikawa determined the extension group $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)},\mathscr{E}^{(\lambda_1,\lambda_2;D_1)})$ and by generalizing it, the author determined the group $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)},\mathscr{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)})$ as in the following theorem.

THEOREM 4.1.1 (Horikawa and Kondo). *Let* $\Theta^{(\lambda_0,\lambda_1,\lambda_2)}$ *be the following map*

$$\Theta^{(\lambda_0,\lambda_1,\lambda_2)}:\hat{W}(A_{\lambda_1})F^{(\lambda_0)}/\langle\ell_1\rangle\times\hat{W}(A_{\lambda_2})/\langle\ell_2\rangle\to\hat{W}(A_{\lambda_2})$$
$$(\boldsymbol{a}_1,\boldsymbol{a}_2)\mapsto F^{(\lambda_0)}\boldsymbol{a}_2-\langle\boldsymbol{b}_1^2,\boldsymbol{u}^1\rangle$$

with

$$\begin{cases}(\ell_1,\ell_2)\in\mathbf{Z}^2\\ \ell_1:=\ell_1[\lambda_0]\equiv\mathbf{0}\mod\lambda_1\\ \ell_2:=\ell_2[\lambda_0]-\left\langle\dfrac{\ell_1}{\lambda_1},\boldsymbol{u}^1\right\rangle\\ \boldsymbol{b}_1^2:=\dfrac{1}{\lambda_1}F^{(\lambda_0)}\boldsymbol{a}_1\end{cases}.$$

Then the group homomorphism

$$\Psi^{(\lambda_0,\lambda_1,\lambda_2)}:\mathrm{Ker}\,\Theta^{(\lambda_0,\lambda_1,\lambda_2)}\to\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)},\mathscr{E}^{(\lambda_1,\lambda_2;D_1)})$$

defined by

$$(\boldsymbol{a}_1,\boldsymbol{a}_2)\mapsto\textit{the class of }\operatorname{Spec}A\left[X_0,X_1,X_2,\frac{1}{1+\lambda_0X_0},\frac{1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)+\lambda_1X_1},\right.$$
$$\left.\frac{1}{E_p(\boldsymbol{a}_2,\lambda_0;X_0)D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\right)+\lambda_2X_2}\right]$$

is bijective.

Moreover, we define the following map $\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)}$:

$$\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)}:\mathrm{Ker}\,\Theta^{(\lambda_0,\lambda_1,\lambda_2)}\times\hat{W}(A_{\lambda_3})/\langle\ell_3\rangle\to\hat{W}(A_{\lambda_3})$$
$$((\boldsymbol{a}_1,\boldsymbol{a}_2),\boldsymbol{a}_3)\mapsto F^{(\lambda_0)}\boldsymbol{a}_3-\langle\boldsymbol{b}_1^2,\boldsymbol{u}_1^2\rangle-\langle\boldsymbol{b}_1^3,\boldsymbol{u}_2^2\rangle$$

with

$$\begin{cases} (\ell_1, \ell_2, \ell_3) \in \mathbf{Z}^3 \\ \ell_3 := \ell_3[\lambda_0] - \left\langle \dfrac{\ell_1}{\lambda_1}, \boldsymbol{u}_1^2 \right\rangle - \left\langle \dfrac{\ell_2}{\lambda_2}, \boldsymbol{u}_2^2 \right\rangle \\ \ell_2 \equiv \mathbf{0} \mod \lambda_2 \\ \boldsymbol{b}_1^3 := \dfrac{1}{\lambda_2}(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1 \rangle) \end{cases}.$$

Then the group homomorphism

$$\Psi^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} : \mathrm{Ker}\,\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} \to \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)})$$

defined by

$(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \mapsto$ *the class of*

$$\mathrm{Spec}\, A\Bigg[X_0, X_1, X_2, X_3, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1}, \\ \frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\right) + \lambda_2 X_2}, \\ \frac{1}{E_p(\boldsymbol{a}_3, \lambda_0; X_0) D_2\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}, \frac{X_2}{E_p(\boldsymbol{a}_2,\lambda_0;X_0)}\right) + \lambda_3 X_3}\Bigg]$$

is bijective.

Our main purpose of this article is to generalize the preceding results. The main results are as follows.

Theorem 4.1.2 (Main Theorem). *We define inductively the map* $\Theta^{(\lambda_0,\dots,\lambda_n)}$:

$$\Theta^{(\lambda_0,\lambda_1,\lambda_2)} : \hat{W}(A_{\lambda_1})F^{(\lambda_0)}/\langle \ell_1 \rangle \times \hat{W}(A_{\lambda_2})/\langle \ell_2 \rangle \to \hat{W}(A_{\lambda_2})$$

$$(\boldsymbol{a}_1, \boldsymbol{a}_2) \mapsto F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1 \rangle$$

with

$$\begin{cases} (\ell_1, \ell_2) \in \mathbf{Z}^2 \\ \ell_1 := \ell_1[\lambda_0] \equiv \mathbf{0} \mod \lambda_1 \\ \ell_2 := \ell_2[\lambda_0] - \left\langle \dfrac{\ell_1}{\lambda_1}, \boldsymbol{u}^1 \right\rangle \\ \boldsymbol{b}_1^2 := \dfrac{1}{\lambda_1} F^{(\lambda_0)} \boldsymbol{a}_1 \end{cases}.$$

and for $n \geq 3$,

$$\Theta^{(\lambda_0,\dots,\lambda_n)} : \mathrm{Ker}\, \Theta^{(\lambda_0,\dots,\lambda_{n-1})} \times \hat{W}(A_{\lambda_n})/\langle \ell_n \rangle \to \hat{W}(A_{\lambda_n})$$

$$((\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-1}),\boldsymbol{a}_n) \mapsto F^{(\lambda_0)}\boldsymbol{a}_n - \sum_{i=1}^{n-1} \langle \boldsymbol{b}_1^{i+1}, \boldsymbol{u}_i^{n-1} \rangle$$

with

$$\begin{cases} (\ell_1,\dots,\ell_n) \in \mathbf{Z}^n \\ \ell_i := \ell_i[\lambda_0] - \sum_{j=1}^{i-1} \left\langle \dfrac{\ell_j}{\lambda_j}, \boldsymbol{u}_j^{i-1} \right\rangle, \quad 3 \leq i \leq n \\ \ell_i \equiv \mathbf{0} \mod \lambda_i, \quad 2 \leq i \leq n-1 \\ \boldsymbol{b}_1^i := \dfrac{1}{\lambda_{i-1}} \left(F^{(\lambda_0)} \boldsymbol{a}_{i-1} - \sum_{j=1}^{i-2} \langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-2} \rangle \right), \quad 3 \leq i \leq n \end{cases}.$$

Then the group homomorphism

$$\Psi^{(\lambda_0,\dots,\lambda_n)} : \mathrm{Ker}\, \Theta^{(\lambda_0,\dots,\lambda_n)} \to \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_0,\dots,\lambda_n; D_1,\dots,D_{n-1})})$$

defined by

$(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) \mapsto$ *the class of*

$$\mathrm{Spec}\, A\left[X_0,\dots,X_n, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)+\lambda_1 X_1}, \right.$$

$$\frac{1}{E_p(\boldsymbol{a}_2,\lambda_0;X_0)D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\right)+\lambda_2 X_2}, \dots,$$

$$\left. \frac{1}{E_p(\boldsymbol{a}_n,\lambda_0;X_0)D_{n-1}\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)},\dots,\frac{X_{n-1}}{E_p(\boldsymbol{a}_{n-1},\lambda_0;X_0)}\right)+\lambda_n X_n}\right]$$

is bijective.

5. Proof of Main Theorem

In this section, we use the notations and the maps $\Theta^{(\lambda_1,\dots,\lambda_n)}$ and $\Psi^{(\lambda_1,\dots,\lambda_n)}$ defined in Section 4.

5.1. We assume the homomorphism $\Psi^{(\lambda_0,\dots,\lambda_k)}$ in Theorem 4.1.2 is isomorphic for $k = 3,\dots,n-1$.

Let $(x_1, \dots, x_n)$ be a local section of $\mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})}$ and t be a local section of $\mathbf{G}_{m,A}$, respectively. We define group scheme homomorphisms ρ and β by

$$\rho : \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})} \to \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A}$$

$$(x_1, \dots, x_n) \mapsto ((x_1, \dots, x_{n-1}), D_{n-1}(x_1, \dots, x_{n-1}) + \lambda_n x_n)$$

and

$$\beta : \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A} \to \iota_* \mathbf{G}_{m,A_{\lambda_n}}$$

$$((x_1, \dots, x_{n-1}), t) \mapsto D_{n-1}(x_1, \dots, x_{n-1})^{-1} t \mod \lambda_n.$$

Then the sequence

$$0 \to \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})} \xrightarrow{\rho} \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A} \xrightarrow{\beta} \iota_* \mathbf{G}_{m,A_{\lambda_n}} \to 0 \qquad (**)$$

is exact on small flat site over $\operatorname{Spec} A$ and we have an exact sequence:

$$0 \to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})}) \xrightarrow{\rho} \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A})$$

$$\xrightarrow{\beta} \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m,A_{\lambda_n}}) \xrightarrow{\partial} \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})})$$

$$\xrightarrow{\rho^*} \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A}) \xrightarrow{\beta^*} \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m,A_{\lambda_n}}) \to \cdots$$

5.2. By using the previous long exact sequence we make the following diagram consisting of group homomorphisms, which will be used to prove Theorem 4.1.2.

$$\begin{array}{ccc}
\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A}) & \xrightarrow{\varphi_1} & \hat{W}(A_{\lambda_n})^{F^{(\lambda_0)}} \\
\| & & \wr \downarrow \psi_1 \\
\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A}) & \xrightarrow[\beta]{} & \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m,A_{\lambda_n}})
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\varphi_2} & \mathrm{Ker}\, \Theta^{(\lambda_0,\dots,\lambda_n)} & \xrightarrow{\varphi_3} & \mathrm{Ker}\, \Theta^{(\lambda_0,\dots,\lambda_{n-1})} \\
 & \downarrow \Psi^{(\lambda_0,\dots,\lambda_n)} & & \wr \downarrow \Psi^{(\lambda_0,\dots,\lambda_{n-1})} \\
\xrightarrow[\partial]{} & \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})}) & \xrightarrow[\rho^*]{} & \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A})
\end{array}$$

$$\begin{array}{cc}
\xrightarrow{\varphi_4} & \hat{W}(A_{\lambda_n})/F^{(\lambda_0)} \\
 & \wr \downarrow \psi_2 \\
\xrightarrow[\beta^*]{} & \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m,A_{\lambda_n}})
\end{array}$$

where

$$\begin{cases} \varphi_1 := \psi_1^{-1} \circ \beta \\ \varphi_2 : \boldsymbol{a}_n \mapsto (\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{a}_n) \\ \varphi_3 : (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \mapsto (\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}) \\ \varphi_4 : (\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}) \mapsto -\sum_{i=1}^{n-1} \langle \boldsymbol{b}_1^{i+1}, \boldsymbol{u}_i^{n-1} \rangle \end{cases}$$

and ψ_1, ψ_2 are isomorphisms by Subsection 3.2.

Note that $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m, A_{\lambda_n}}) \simeq H_0^2(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m, A_{\lambda_n}})$ (cf. [3]), λ_i is nilpotent in A_{λ_n} for $i = 0, \dots, n-1$, and the map $\Psi^{(\lambda_0, \dots, \lambda_{n-1})}$ is isomorphic by the induction hypothesis.

To prove the exactness of the first horizontal line and the commutativity of this diagram, we prepare some notations and some results.

Definition 5.2.1. We put $E := E_p(\mathbf{W}_E, \Lambda_E; X_E)$ and $G := G_p(\mathbf{W}_G, \Lambda_G; X_G)$ and define $\tilde{p}(EG)$ by

$$\tilde{p}(EG) := E_p(\tilde{p}\mathbf{W}_E, \Lambda_E, X_E) G_p(\tilde{p}\mathbf{W}_G, \Lambda_G, X_G).$$

If $\mathbf{W}_G = \mathbf{0}$, then the definition of $\tilde{p}$ is equivalent to one in Subsection 2.2.

Then we have

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; EG) = \frac{E_p\big(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(EG - 1)\big)}{\tilde{E}_p(\mathbf{W}, \Lambda_2; EG)}$$

and

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; EG) = \tilde{E}_p(\mathbf{W}, \Lambda_2; E) \tilde{E}_p(\mathbf{W}, \Lambda_2; G).$$

Moreover, we put $F := F_p(\mathbf{W}_F, \Lambda_F; X_F, Y_F)$. Then by the following definition:

$$\tilde{p}(FG) := F_p(\tilde{p}\mathbf{W}_F, \Lambda_F, X_F, Y_F) G_p(\tilde{p}\mathbf{W}_G, \Lambda_G, X_G),$$

we have

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; FG) = \frac{E_p\big(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(FG - 1)\big)}{\tilde{E}_p(\mathbf{W}, \Lambda_2; FG)}$$

and

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; FG) = \tilde{E}_p(\mathbf{W}, \Lambda_2; F) \tilde{E}_p(\mathbf{W}, \Lambda_2; G).$$

Next we decide the group $\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1, \dots, \lambda_{n-1}; D_1, \dots, D_{n-2})})$ for $n \geq 2$.

Let Ω_1 be an element of $\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{G}^{(\lambda_1)})$. Then by Subsection 3.2, there exists $\ell_1 \in \mathbf{Z}$ such that $1 + \lambda_1\Omega_1 = (1 + \lambda_0 X_0)^{\ell_1} = E_p(\ell_1[\lambda_0], \lambda_0; X_0) \equiv 1 \mod \lambda_1$, we have

$$\Omega_1 = \frac{1}{\lambda_1}\{E_p(\ell_1[\lambda_0], \lambda_0; X_0) - 1\}$$

with $\ell_1[\lambda_0] \equiv \mathbf{0} \mod \lambda_1$. Therefore we have

$$\begin{aligned}\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{G}^{(\lambda_1)}) &= \{\Omega_1 \mid \ell_1 \in \mathbf{Z}, \ell_1 \equiv \mathbf{0} \mod \lambda_1\} \\ &\simeq \{\ell_1 \in \mathbf{Z} \mid \ell_1 \equiv \mathbf{0} \mod \lambda_1\}.\end{aligned}$$

By the exact sequence

$$0 \to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\lambda_2;D_1)}) \to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})$$

induced by $(**)$ and $\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \simeq \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{G}^{(\lambda_1)}) \times \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathbf{G}_{m,A})$, if an element $(\Omega_1, E_p(\ell_2[\lambda_0], \lambda_0; X_0)) \in \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{G}^{(\lambda_1)} \times \mathbf{G}_{m,A})$ is in $\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\lambda_2;D_1)})$, then there exists Ω_2 such that $D_1(\Omega_1) + \lambda_2\Omega_2 = E_p(\ell_2[\lambda_0], \lambda_0; X_0)$. Moreover, by the equalities

$$\begin{aligned}
&D_1(\Omega_1)^{-1}E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= E_p\left(-\boldsymbol{u}^1, \lambda_1; \frac{1}{\lambda_1}\{E_p(\ell_1[\lambda_0], \lambda_0; X_0) - 1\}\right)E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= \tilde{E}_p(-\boldsymbol{u}^1, \lambda_1; E_p(\ell_1[\lambda_0], \lambda_0; X_0)) \\
&\quad \cdot G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\ell_1[\lambda_0], \lambda_0; X_0))E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= E_p\left(-\left\langle \frac{\ell_1[\lambda_0]}{\lambda_1}, \boldsymbol{u}^1 \right\rangle, \lambda_0; X_0\right) \\
&\quad \cdot G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\ell_1[\lambda_0], \lambda_0; X_0))E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= E_p(\ell_2, \lambda_0; X_0) \mod \lambda_2,
\end{aligned}$$

we have $\Omega_2 = \frac{1}{\lambda_2}\{E_p(\ell_2[\lambda_0], \lambda_0; X_0) - D_1(\Omega_1)\}$ with $\ell_2 = \ell_2[\lambda_0] - \left\langle \frac{\ell_1}{\lambda_1}, \boldsymbol{u}^1 \right\rangle \equiv \mathbf{0}$ mod λ_2. Therefore we have

$$\begin{aligned}\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\lambda_2;D_1)}) &= \left\{(\Omega_1, \Omega_2) \,\middle|\, \begin{array}{l}(\ell_1, \ell_2) \in \mathbf{Z}^2 \\ \ell_i \equiv \mathbf{0} \mod \lambda_i \text{ for } i = 1, 2\end{array}\right\} \\ &\simeq \{(\ell_1, \ell_2) \in \mathbf{Z}^2 \mid \ell_i \equiv \mathbf{0} \mod \lambda_i \text{ for } i = 1, 2\}.\end{aligned}$$

Moreover, by induction on n, we have the following proposition.

Proposition 5.2.2. *For $n \geq 3$, we put*

$$\Omega_i := \begin{cases} \dfrac{1}{\lambda_1}\{E_p(\ell_1, \lambda_0; X_0) - 1\} & \text{if } i = 1 \\ \dfrac{1}{\lambda_i}\{E_p(\ell_i[\lambda_0], \lambda_0; X_0) - D_{i-1}(\Omega_1, \ldots, \Omega_{i-1})\} & \text{if } 2 \leq i \leq n-1 \end{cases}$$

with $\ell_i \equiv \mathbf{0}$ for $i = 1, \ldots, n-1$. Then we have

$$\begin{aligned} &\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1, \ldots, \lambda_{n-1}; D_1, \ldots, D_{n-2})}) \\ &\quad = \left\{ (\Omega_1, \ldots, \Omega_{n-1}) \,\middle|\, \begin{array}{l} (\ell_1, \ldots, \ell_{n-1}) \in \mathbf{Z}^{n-1} \\ \ell_i \equiv \mathbf{0} \bmod \lambda_i \text{ for } i = 1, \ldots, n-1 \end{array} \right\} \\ &\quad \simeq \{(\ell_1, \ldots, \ell_{n-1}) \in \mathbf{Z}^{n-1} \mid \ell_i \equiv \mathbf{0} \bmod \lambda_i \text{ for } i = 1, \ldots, n-1\}. \end{aligned}$$

To prove Proposition 5.2.2, we use the following lemma.

Lemma 5.2.3. *Under the above notation, for $n \in \mathbf{Z}$ with $n \geq 2$, we have the following equalities*:

(1) $n = 2$

$$D_1(\Omega_1) = E_p\left(\left\langle \frac{\ell_1}{\lambda_1}, \boldsymbol{u}^1 \right\rangle, \lambda_0; X_0\right) G_p(F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\ell_1, \lambda_0; X_0))$$

(2) $n \geq 3$

$$\begin{aligned} &D_{n-1}(\Omega_1, \ldots, \Omega_{n-1}) \\ &\quad = E_p\left(\sum_{i=1}^{n-1} \left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^{n-1} \right\rangle, \lambda_0; X_0\right) \\ &\qquad \cdot \prod_{i=1}^{n-2} G_p\Bigg(F^{(\lambda_i)}\boldsymbol{u}_i^{n-1} - \sum_{j=i+1}^{n-1} \langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^{n-1} \rangle, \lambda_i; \\ &\qquad\qquad E_p(\ell_i[\lambda_0], \lambda_0; X_0) D_{i-1}(\Omega_1, \ldots, \Omega_{i-1})^{-1}\Bigg) \\ &\qquad \cdot G_p(F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^{n-1}, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0) D_{n-2}(\Omega_1, \ldots, \Omega_{n-2})^{-1}). \end{aligned}$$

Proof. If $n = 2$ or 3, then Lemma 5.2.3 is true by [2] and [3]. Hence we assume $n \geq 4$.

By induction on n,

$$\prod_{i=1}^{n-1} E_p\left(\boldsymbol{u}_i^n, \lambda_i; \frac{\Omega_i}{D_{i-1}(\Omega_1,\dots,\Omega_{i-1})}\right)$$

$$= E_p\left(\sum_{i=1}^{n-1}\left\langle\frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^n\right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(F^{(\lambda_i)}\boldsymbol{u}_i^n - \sum_{j=i+1}^{n-1}\langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0)D_{i-1}(\Omega_1,\dots,\Omega_{i-1})^{-1}\right)$$

$$\cdot G_p(F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^n, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0)D_{n-2}(\Omega_1,\dots,\Omega_{n-2})^{-1}).$$

Moreover, by Proposition 2.2.2, Proposition 2.2.3 and Definition 5.2.1,

$$D_n(\Omega_1,\dots,\Omega_n) = \prod_{i=1}^{n} E_p\left(\boldsymbol{u}_i^n, \lambda_i; \frac{\Omega_i}{D_{i-1}(\Omega_1,\dots,\Omega_{i-1})}\right)$$

$$= \prod_{i=1}^{n-1} E_p\left(\boldsymbol{u}_i^n, \lambda_i; \frac{\Omega_i}{D_{i-1}(\Omega_1,\dots,\Omega_{i-1})}\right)$$

$$\cdot E_p\left(\boldsymbol{u}_n^n, \lambda_n, \frac{\Omega_n}{D_{n-1}(\Omega_1,\dots,\Omega_{n-1})}\right)$$

(by the induction hypothesis, Subsection 3.4 and the definition of Ω_i for $i = 1, 2, \dots, n$,)

$$= E_p\left(\sum_{i=1}^{n-1}\left\langle\frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^n\right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(F^{(\lambda_i)}\boldsymbol{u}_i^n - \sum_{j=i+1}^{n-1}\langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0)D_{i-1}(\Omega_1,\dots,\Omega_{i-1})^{-1}\right)$$

$$\cdot G_p(F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^n, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0)D_{n-2}(\Omega_1,\dots,\Omega_{n-2})^{-1})$$

$$\cdot E_p\left(\boldsymbol{u}_n^n, \lambda_n, \frac{1}{\lambda_n}\{E_p(\ell_n[\lambda_0], \lambda_0; X_0)D_{n-1}(\Omega_1,\dots,\Omega_{n-1})^{-1} - 1\}\right)$$

(by Definition 5.2.1,)

$$= E_p\left(\sum_{i=1}^{n-1}\left\langle\frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^n\right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(F^{(\lambda_i)}\boldsymbol{u}_i^n - \sum_{j=i+1}^{n-1}\langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0)D_{i-1}(\Omega_1,\dots,\Omega_{i-1})^{-1}\right)$$

$$\cdot\, G_p(F^{(\lambda_{n-1})}\boldsymbol{u}^n_{n-1}, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0) D_{n-2}(\Omega_1, \dots, \Omega_{n-2})^{-1})$$

$$\cdot\, G_p(F^{(\lambda_n)}\boldsymbol{u}^n_n, \lambda_n; E_p(\ell_n[\lambda_0], \lambda_0; X_0) D_{n-1}(\Omega_1, \dots, \Omega_{n-1})^{-1})$$

$$\cdot\, \tilde{E}_p(\boldsymbol{u}^n_n, \lambda_n; E_p(\ell_n[\lambda_0], \lambda_0; X_0) D_{n-1}(\Omega_1, \dots, \Omega_{n-1})^{-1})$$

(by the above hypothesis,)

$$= E_p\left(\sum_{i=1}^{n-1}\left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}^n_i \right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(F^{(\lambda_i)}\boldsymbol{u}^n_i - \sum_{j=i+1}^{n-1} \langle \boldsymbol{b}^{j+1}_{i+1}, \boldsymbol{u}^n_j \rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0) D_{i-1}(\Omega_1, \dots, \Omega_{i-1})^{-1}\right)$$

$$\cdot\, G_p(F^{(\lambda_{n-1})}\boldsymbol{u}^n_{n-1}, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0) D_{n-2}(\Omega_1, \dots, \Omega_{n-2})^{-1})$$

$$\cdot\, \tilde{E}_p\Bigg(\boldsymbol{u}^n_n, \lambda_n; E_p(\ell_n, \lambda_0; X_0)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(-F^{(\lambda_i)}\boldsymbol{u}^{n-1}_i + \sum_{j=i+1}^{n-1} \langle \boldsymbol{b}^{j}_{i}, \boldsymbol{u}^{n-1}_j \rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0) D_{i-1}(\Omega_1, \dots, \Omega_{i-1})^{-1}\right)\Bigg)$$

$$\cdot\, G_p(-F^{(\lambda_{n-1})}\boldsymbol{u}^{n-1}_{n-1}, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0) D_{n-2}(\Omega_1, \dots, \Omega_{n-2})^{-1})$$

$$\cdot\, G_p(F^{(\lambda_n)}\boldsymbol{u}^n_n, \lambda_n; E_p(\ell_n[\lambda_0], \lambda_0; X_0) D_{n-1}(\Omega_1, \dots, \Omega_{n-1})^{-1})$$

(by Proposition 2.2.2, Proposition 2.2.3 and Definition 5.2.1,)

$$= E_p\left(\sum_{i=1}^{n-1}\left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}^n_i \right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(F^{(\lambda_i)}\boldsymbol{u}^n_i - \sum_{j=i+1}^{n-1} \langle \boldsymbol{b}^{j+1}_{i+1}, \boldsymbol{u}^n_j \rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0) D_{i-1}(\Omega_1, \dots, \Omega_{i-1})^{-1}\right)$$

$$\cdot\, G_p(F^{(\lambda_{n-1})}\boldsymbol{u}^n_{n-1}, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0) D_{n-2}(\Omega_1, \dots, \Omega_{n-2})^{-1})$$

$$\cdot\, G_p(F^{(\lambda_n)}\boldsymbol{u}^n_n, \lambda_n; E_p(\ell_n[\lambda_0], \lambda_0; X_0) D_{n-1}(\Omega_1, \dots, \Omega_{n-1})^{-1})$$

$$\cdot\, E_p\left(\left\langle \frac{\ell_n}{\lambda_n}, \boldsymbol{u}^n_n \right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-2} G_p\left(-\left\langle \frac{1}{\lambda_n}\left(F^{(\lambda_i)}\boldsymbol{u}_i^{n-1} - \sum_{j=i+1}^{n-1} \langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^{n-1}\rangle\right), \boldsymbol{u}_i^n\right\rangle, \lambda_i;\right.$$

$$\left. E_p(\ell_i[\lambda_0], \lambda_0; X_0) D_{i-1}(\Omega_1, \dots, \Omega_{i-1})^{-1}\right)$$

$$\cdot G_p\left(-\left\langle \frac{1}{\lambda_n} F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^{n-1}, \boldsymbol{u}_n^n\right\rangle, \lambda_{n-1}; E_p(\ell_{n-1}[\lambda_0], \lambda_0; X_0) D_{n-2}(\Omega_1, \dots, \Omega_{n-2})^{-1}\right)$$

$$= E_p\left(\sum_{i=1}^{n}\left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^n\right\rangle, \lambda_0; X_0\right)$$

$$\cdot \prod_{i=1}^{n-1} G_p\left(F^{(\lambda_i)}\boldsymbol{u}_i^n - \sum_{j=i+1}^{n} \langle \boldsymbol{b}_{i+1}^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_i; E_p(\ell_i[\lambda_0], \lambda_0; X_0) D_{i-1}(\Omega_1, \dots, \Omega_{i-1})^{-1}\right)$$

$$\cdot G_p(F^{(\lambda_n)}\boldsymbol{u}_n^n, \lambda_n; E_p(\ell_n[\lambda_0], \lambda_0; X_0) D_{n-1}(\Omega_1, \dots, \Omega_{n-1})^{-1}).$$

Therefore Lemma 5.2.3 is true.

By Lemma 5.2.3 and Subsection 3.4, we have the following equality:

$$D_{n-1}(\Omega_1, \dots, \Omega_{n-1}) \equiv E_p\left(\sum_{i=1}^{n-1}\left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^{n-1}\right\rangle, \lambda_0; X_0\right) \mod \lambda_n.$$

We show Proposition 5.2.2 as follows.

By the exact sequence:

$$0 \to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n; D_1,\dots,D_{n-1})}) \to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1}; D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A})$$
$$\to \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m,A_{\lambda_{n-1}}}),$$

the canonical isomorphism:

$$\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1}; D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A})$$
$$\simeq \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1}; D_1,\dots,D_{n-2})}) \times \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathbf{G}_{m,A}),$$

and the induction hypothesis for n, if an element

$$((\Omega_1, \dots, \Omega_{n-1}), E_p(\ell_n[\lambda_0], \lambda_0; X_0)) \in \mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1}; D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A})$$

is in $\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n; D_1,\dots,D_{n-1})})$, then there exists Ω_n such that $D_{n-1}(\Omega_1,\dots,\Omega_{n-1}) + \lambda_n\Omega_n = E_p(\ell_n[\lambda_0], \lambda_0; X_0)$. Moreover, by Lemma 5.2.3, we have

$$\Omega_n = \frac{1}{\lambda_n}\{E_p(\ell_n[\lambda_0], \lambda_0; X_0) - D_{n-1}(\Omega_1,\dots,\Omega_{n-1})\}$$

with $\ell_n = \ell_n[\lambda_0] - \sum_{i=1}^{n-1}\left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^{n-1}\right\rangle \equiv \mathbf{0} \bmod \lambda_n$.

Therefore Proposition 5.2.2 is true and

$$\begin{aligned}
&\mathrm{Hom}(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1}; D_1,\dots,D_{n-2})}) \times \mathbf{G}_{m,A} \\
&= \left\{ ((\Omega_1,\dots,\Omega_{n-1}), E_p(\ell_n[\lambda_0], \lambda_0, X_0)) \,\middle|\, \begin{matrix} (\ell_1,\dots,\ell_n) \in \mathbf{Z}^n \\ \ell_i \equiv \mathbf{0} \bmod \lambda_i \text{ for } i = 1,\dots,n-1 \end{matrix} \right\} \\
&\simeq \{(\ell_1,\dots,\ell_n) \in \mathbf{Z}^n \mid \ell_i \equiv \mathbf{0} \bmod \lambda_i \text{ for } i = 1,\dots,n-1\}.
\end{aligned}$$

By Proposition 5.2.2 and Lemma 5.2.3, we have the following equalities:

$$\begin{aligned}
&\beta((\Omega_1,\dots,\Omega_{n-1}))E_p(\ell_n[\lambda_0], \lambda_0; X_0) \\
&\quad = D_{n-1}(\Omega_1,\dots,\Omega_{n-1})^{-1} E_p(\ell_n[\lambda_0], \lambda_0; X_0) \bmod \lambda_n \\
&\quad \equiv E_p\left(-\sum_{i=1}^{n-1}\left\langle \frac{\ell_i}{\lambda_i}, \boldsymbol{u}_i^{n-1}\right\rangle, \lambda_0; X_0\right) E_p(\ell_n[\lambda_0], \lambda_0; X_0) \bmod \lambda_n \\
&\quad = E_p(\ell_n, \lambda_0; X_0).
\end{aligned}$$

Therefore we have $\mathrm{Im}\,\varphi_1 = \mathrm{Ker}\,\varphi_2$. The equality $\mathrm{Im}\,\varphi_2 = \mathrm{Ker}\,\varphi_3$ is trivial.

For $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) \in \mathrm{Ker}\,\Theta^{(\lambda_0,\dots,\lambda_n)}$, we have

$$\begin{aligned}
(\varphi_4 \circ \varphi_3)(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) &= -\sum_{i=1}^{n-1}\langle \boldsymbol{b}_1^{i+1}, \boldsymbol{u}_i^{n-1}\rangle \\
&\equiv -F^{(\lambda_0)}\boldsymbol{a}_n \bmod \lambda_n.
\end{aligned}$$

Therefore we have $\mathrm{Im}\,\varphi_3 \subset \mathrm{Ker}\,\varphi_4$. On the other hand, if there exists $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-1}) \in \mathrm{Ker}\,\varphi_4$, then there exists $\boldsymbol{a}_n \in \hat{W}(A_{\lambda_n})$ such that $F^{(\lambda_0)}\boldsymbol{a}_n \equiv \sum_{i=1}^{n-1}\langle \boldsymbol{b}_1^{i+1}, \boldsymbol{u}_i^{n-1}\rangle \bmod \lambda_n$ and we have $\varphi_3(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-1})$. Therefore we have $\mathrm{Im}\,\varphi_3 = \mathrm{Ker}\,\varphi_4$.

Next we show the commutativity of the diagram. $\beta = \psi_1 \circ \varphi_1$ is trivial. For $\boldsymbol{a}_n \in \hat{W}(A)^{F^{(\lambda_0)}}$, we have

$$(\Psi^{(\lambda_0,\dots,\lambda_n)} \circ \varphi_2)(\boldsymbol{a}_n) = \text{the class of Spec } A\left[X_0, \dots, X_n, \frac{1}{1+\lambda_0 X_0}, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \dots, \frac{1}{D_{n-2}(X_1,\dots,X_{n-2})+\lambda_{n-1}X_{n-1}}, \frac{1}{E_p(\boldsymbol{a}_n,\lambda_0;X_0)D_{n-1}(X_1,\dots,X_{n-1})+\lambda_n X_n}\right]$$

and

$$\psi_1(\boldsymbol{a}_n) = E_p(\boldsymbol{a}_n,\lambda_0;X_0) \mod \lambda_n.$$

We put $E := E_p(\boldsymbol{a}_n,\lambda_0;X_0) \mod \lambda_n$. Then ∂E is the pull-back of (∗∗) by E and we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})} & \longrightarrow & \partial E & \longrightarrow & \mathscr{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow E & & \\
0 & \longrightarrow & \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})} & \xrightarrow{\rho} & \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A} & \xrightarrow{\beta} & \iota_*\mathbf{G}_{m,A_{\lambda_n}} & \longrightarrow & 0.
\end{array}$$

Let $(x_1,\dots,x_{n-1})$, t and x_0 be local sections of $\mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})}$, $\mathbf{G}_{m,A}$ and $\mathscr{G}^{(\lambda_0)}$, respectively. Then we have

$$E(x_0) \equiv D_{n-1}(x_1,\dots,x_{n-1})^{-1} t \mod \lambda_n.$$

Because it is equivalent to $E(x_0)D_{n-1}(x_1,\dots,x_{n-1}) \equiv t \mod \lambda_n$, there exists x_n such that

$$t = E(x_0)D_{n-1}(x_1,\dots,x_{n-1}) + \lambda_n x_n.$$

Therefore we have

$$\partial E = \text{the class of Spec } A\left[X_0, \dots, X_n, \frac{1}{1+\lambda_0 X_0}, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \dots, \frac{1}{D_{n-2}(X_1,\dots,X_{n-2})+\lambda_{n-1}X_{n-1}}, \frac{1}{E_p(\boldsymbol{a}_n,\lambda_0;X_0)D_{n-1}(X_1,\dots,X_{n-1})+\lambda_n X_n}\right]$$

and we have $\partial \circ \psi_1 = \Psi^{(\lambda_0,\dots,\lambda_n)} \circ \varphi_2$.

By $\mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A}) \simeq \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})})$, ρ^* is the push-down map

$$\pi_* : \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})}) \to \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})})$$

by the canonical projection $\pi : \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})} \to \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})}$ and we have a commutative diagram with the exact horizontal lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{E}^{(\lambda_1,\dots,\lambda_n;D_1,\dots,D_{n-1})} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & {\scriptstyle\pi}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} & \longrightarrow & \pi_*\mathscr{E} & \longrightarrow & \mathscr{G}^{(\lambda_0)} & \longrightarrow & 0.
\end{array}$$

Let $\bar{\mathscr{E}}$ be the class of $\mathscr{E}$. Then we have

$$\rho^*(\bar{\mathscr{E}}) = \overline{\pi_*\mathscr{E}} = \text{the class of Spec } A\Bigg[X_0, \dots, X_{n-1}, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1},$$
$$\frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right) + \lambda_2 X_2}, \dots,$$
$$\frac{1}{E_p(\boldsymbol{a}_{n-1}, \lambda_0; X_0) D_{n-2}\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}, \dots, \frac{X_{n-2}}{E_p(\boldsymbol{a}_{n-2}, \lambda_0; X_0)}\right) + \lambda_{n-1} X_{n-1}}\Bigg].$$

Therefore we have $\rho^* \circ \Psi^{(\lambda_0,\dots,\lambda_n)} = \Psi^{(\lambda_0,\dots,\lambda_{n-1})} \circ \varphi_3$ by the induction hypothesis.

Let $\mathscr{E} \in \mathrm{Ext}^1(\mathscr{G}^{(\lambda_0)}, \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})})$. Then the exact sequence

$$0 \to \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \to \mathscr{E} \to \mathscr{G}^{(\lambda_0)} \to 0$$

induces the exact sequence:

$$0 \to \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A} \to \mathscr{E} \times \mathbf{G}_{m,A} \to \mathscr{G}^{(\lambda_0)} \to 0,$$

and we have a commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{E}^{(\lambda_1,\dots,\lambda_{n-1};D_1,\dots,D_{n-2})} \times \mathbf{G}_{m,A} & \longrightarrow & \mathscr{E} \times \mathbf{G}_{m,A} & \longrightarrow & \mathscr{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & {\scriptstyle\beta}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \iota_*\mathbf{G}_{m,A_{\lambda_n}} & \longrightarrow & \beta_*(\mathscr{E} \times \mathbf{G}_{m,A}) & \longrightarrow & \mathscr{G}^{(\lambda_0)} & \longrightarrow & 0.
\end{array}$$

Because the group structure of $\mathscr{E}$ is induced by the group scheme homomorphism:

$$\mathscr{E} \to \mathbf{G}_{m,A}^n$$
$$(x_0,\dots,x_{n-1}) \mapsto \Bigg(1+\lambda_0 x_0, E_p(\boldsymbol{a}_1,\lambda_0;x_0)+\lambda_1 x_1,$$
$$E_p(\boldsymbol{a}_2,\lambda_0;x_0)D_1\left(\frac{x_1}{E_p(\boldsymbol{a}_1,\lambda_0;x_0)}\right)+\lambda_2 x_2,\dots,$$
$$D_{n-2}\left(\frac{x_1}{E_p(\boldsymbol{a}_1,\lambda_0;x_0)},\dots,\frac{x_{n-2}}{E_p(\boldsymbol{a}_{n-2},\lambda_{n-2};x_0)}\right)+\lambda_{n-1}x_{n-1}\Bigg),$$

we have

$$(x_0,\dots,x_{n-1}) = (x_0,0,\dots,0)+\left(0,\frac{x_1}{E_p(\boldsymbol{a}_1,\lambda_0;x_0)},\dots,\frac{x_{n-1}}{E_p(\boldsymbol{a}_{n-1},\lambda_0;x_0)}\right)$$

with a local section $(x_0,\dots,x_{n-1}) \in \mathscr{E}$ and

$$\mathscr{E}\times\mathbf{G}_{m,A} \to \beta_*(\mathscr{E}\times\mathbf{G}_{m,A})$$
$$((x_0,\dots,x_{n-1}),t) \mapsto \left(x_0, D_{n-1}\left(\frac{x_1}{E_p(\boldsymbol{a}_1,\lambda_0;x_0)},\dots,\frac{x_{n-1}}{E_p(\boldsymbol{a}_{n-1},\lambda_0;x_0)}\right)^{-1}t\right)$$

for a local section $t\in\mathbf{G}_{m,A}$.

By the group scheme homomorphism $\mathscr{E}\to\mathbf{G}_{m,A}^n$, we have

$$((x_0,0,\dots,0),1) \mapsto ((1+\lambda_0 x_0, E_p(\boldsymbol{a}_1,\lambda_0;x_0),\dots,E_p(\boldsymbol{a}_{n-1},\lambda_0;x_0)),1).$$

We put $((X_0,\dots,X_{n-1}),1) := ((x,0,\dots,0),1)((x',0,\dots,0),1)$. Then we have

$$\Bigg(\Bigg(1+\lambda_0 X_0, E_p(\boldsymbol{a}_1,\lambda_0;X_0)+\lambda_1 X_1, E_p(\boldsymbol{a}_2,\lambda_0;X_0)D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\right)+\lambda_2 X_2,\dots,$$
$$E_p(\boldsymbol{a}_{n-1},\lambda_0;X_0)D_{n-2}\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)},\dots,\frac{X_{n-2}}{E_p(\boldsymbol{a}_{n-2},\lambda_0;X_0)}\right)\Bigg),1\Bigg)$$
$$=((1+\lambda_0(x+x'+\lambda_0 xx'), E_p(\boldsymbol{a}_1,\lambda_0;x)E_p(\boldsymbol{a}_1,\lambda_0;x'),$$
$$E_p(\boldsymbol{a}_2,\lambda_0;x)E_p(\boldsymbol{a}_2,\lambda_0;x'),\dots,E_p(\boldsymbol{a}_{n-1},\lambda_0;x)E_p(\boldsymbol{a}_{n-1},\lambda_0;x')),1)$$

and

$$\begin{cases} X_0 = x + x' + \lambda_0 x x' \\ X_1 = \dfrac{1}{\lambda_1}\{E_p(\boldsymbol{a}_1, \lambda_0; x)E_p(\boldsymbol{a}_1, \lambda_0; x') - E_p(\boldsymbol{a}_1, \lambda_0; X_0)\} \\ X_2 = \dfrac{1}{\lambda_2}\left\{E_p(\boldsymbol{a}_2, \lambda_0; x)E_p(\boldsymbol{a}_2, \lambda_0; x') - E_p(\boldsymbol{a}_2, \lambda_0; X_0)D_1\left(\dfrac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right)\right\} \\ \qquad \cdots \\ X_{n-1} = \dfrac{1}{\lambda_{n-1}}\left\{E_p(\boldsymbol{a}_{n-1}, \lambda_0; x)E_p(\boldsymbol{a}_{n-1}, \lambda_0; x') \right. \\ \qquad \left. - E_p(\boldsymbol{a}_{n-1}, \lambda_0; X_0)D_{n-2}\left(\dfrac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}, \dots, \dfrac{X_{n-2}}{E_p(\boldsymbol{a}_{n-2}, \lambda_0; X_0)}\right)\right\}. \end{cases}$$

Therefore the cocycle F on $\mathscr{G}^{(\lambda_0)} \times \iota_* \mathbf{G}_{m, A_{\lambda_n}}$ giving $\beta_*(\mathscr{E} \times \mathbf{G}_{m,A})$ is given by

$$F(x, x') = D_{n-1}\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}, \dots, \frac{X_{n-1}}{E_p(\boldsymbol{a}_{n-1}, \lambda_0; X_0)}\right)^{-1}.$$

Next we analyze the right hand side of this equality.

Lemma 5.2.4. *For positive integers i and n with $n \geq 2$ and $i \leq n-1$, we put G_i^{n-1} by*

(1) $n = 2, 3$

$$\begin{cases} G_1^1 := G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\ G_1^2 := G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2 + \langle \boldsymbol{b}_2^3, \boldsymbol{u}_2^2 \rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\ G_2^2 := G_p(-F^{(\lambda_2)}\boldsymbol{u}_2^2, \lambda_2; F_p(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1 \rangle, \lambda_0; x, x')G_1^1) \end{cases}$$

(2) $n \geq 4$

(i) $i = 1$

$$G_1^{n-1} := G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_1^{n-1} + \sum_{j=1}^{n-2} \langle \boldsymbol{b}_2^{j+2}, \boldsymbol{u}_{j+1}^{n-1} \rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')\right)$$

(ii) $2 \leq i \leq n-2$

$$G_i^{n-1} := G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_i^{n-1} + \sum_{j=i}^{n-2} \langle \boldsymbol{b}_{i+1}^{j+2}, \boldsymbol{u}_{j+1}^{n-1} \rangle, \lambda_i; \right.$$
$$\left. F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1} \rangle, \lambda_0; x, x'\right) \prod_{j=1}^{i-1} G_j^{i-1}\right)$$

(iii) $i = n-1$

$$G_{n-1}^{n-1} := G_p\left(-F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^{n-1}, \lambda_{n-1};\right.$$
$$\left. F_p\left(F^{(\lambda_0)}\boldsymbol{a}_{n-1} - \sum_{j=1}^{n-2}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-2}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-2} G_j^{n-2}\right).$$

Then we have

$$D_{n-1}\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}, \dots, \frac{X_{n-1}}{E_p(\boldsymbol{a}_{n-1}, \lambda_0; X_0)}\right)^{-1}$$
$$= F_p\left(-\sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-1} G_j^{n-1}.$$

Proof. If $n = 2$ or 3, then Lemma 5.2.3 is true by [2] and [3]. Hence we assume $n \geq 4$.

$$D_n\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}, \dots, \frac{X_n}{E_p(\boldsymbol{a}_n, \lambda_0; X_0)}\right)^{-1}$$

(by the induction hypothesis, Subsection 3.4 and the definition of X_i for $i = 1, 2, \dots, n-1$,)

$$= F_p\left(-\sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n}\rangle, \lambda_0; x, x'\right)$$
$$\cdot G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_1^n + \sum_{j=1}^{n-2}\langle \boldsymbol{b}_2^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')\right)$$
$$\cdot G_p\left(-F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^n, \lambda_{n-1}; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_{n-1} - \sum_{j=1}^{n-2}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-2}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-2} G_j^{n-2}\right)$$
$$\cdot \prod_{i=2}^{n-2} G_p\left(-F^{(\lambda_i)}\boldsymbol{u}_i^n + \sum_{j=i}^{n-2}\langle \boldsymbol{b}_{i+1}^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_i;\right.$$
$$\left. F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{i-1} G_j^{i-1}\right)$$
$$\cdot E_p\left(-\boldsymbol{u}_n^n, \lambda_n; \frac{X_n}{D_{n-1}\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}, \dots, \frac{X_{n-1}}{E_p(\boldsymbol{a}_{n-1}, \lambda_0; X_0)}\right)E_p(\boldsymbol{a}_n, \lambda_0; X_0)}\right)$$

(by the induction hypothesis, Subsection 3.4 and the definition of X_i for $i = 1, 2, \dots, n$,)

$$= F_p\left(-\sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_0; x, x'\right)$$

$$\cdot G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_1^n + \sum_{j=1}^{n-2}\langle \boldsymbol{b}_2^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')\right)$$

$$\cdot G_p\left(-F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^n, \lambda_{n-1}; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_{n-1} - \sum_{j=1}^{n-2}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-2}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-2} G_j^{n-2}\right)$$

$$\cdot \prod_{i=2}^{n-2} G_p\left(-F^{(\lambda_i)}\boldsymbol{u}_i^n + \sum_{j=i}^{n-2}\langle \boldsymbol{b}_{i+1}^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_i;\right.$$

$$\left. F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{i-1} G_j^{i-1}\right)$$

$$\cdot E_p\left(-\boldsymbol{u}_n^n, \lambda_n; \frac{1}{\lambda_n}\left[F_p\left(F^{(\lambda_0)}\boldsymbol{a}_n - \sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-1}\rangle, \lambda_0; x, x'\right)\right.\right.$$

$$\cdot G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_1^{n-1} + \sum_{j=1}^{n-2}\langle \boldsymbol{b}_2^{j+2}, \boldsymbol{u}_{j+1}^{n-1}\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')\right)$$

$$\cdot G_p\left(-F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^{n-1}, \lambda_{n-1}; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_{n-1} - \sum_{j=1}^{n-2}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-2}\rangle, \lambda_0; x, x'\right)\cdot\prod_{j=1}^{n-2} G_j^{n-2}\right)$$

$$\cdot \prod_{i=2}^{n-2} G_p\left(-F^{(\lambda_i)}\boldsymbol{u}_i^{n-1} + \sum_{j=i}^{n-2}\langle \boldsymbol{b}_{i+1}^{j+2}, \boldsymbol{u}_{j+1}^{n-1}\rangle, \lambda_i;\right.$$

$$\left.\left.\left. F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{i-1} G_j^{i-1}\right) - 1\right]\right)$$

(by Proposition 2.2.2, Proposition 2.2.3 and Definition 5.2.1,)

$$= F_p\left(-\sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_0; x, x'\right)$$

$$\cdot G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_1^n + \sum_{j=1}^{n-2}\langle \boldsymbol{b}_2^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')\right)$$

$$\cdot\, G_p\left(-F^{(\lambda_{n-1})}\boldsymbol{u}_{n-1}^n, \lambda_{n-1}; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_{n-1} - \sum_{j=1}^{n-2}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-2}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-2} G_j^{n-2}\right)$$

$$\cdot\prod_{i=2}^{n-2} G_p\Bigg(-F^{(\lambda_i)}\boldsymbol{u}_i^n + \sum_{j=i}^{n-2}\langle \boldsymbol{b}_{i+1}^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_i;$$

$$F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{i-1} G_j^{i-1}\Bigg)$$

$$\cdot\, F_p(-\langle \boldsymbol{b}_1^{n+1}, \boldsymbol{u}_n^n\rangle, \lambda_0; x, x')G_p(\langle \boldsymbol{b}_1^2, \boldsymbol{u}_n^n\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))$$

$$\cdot\, G_p\left(\langle \boldsymbol{b}_n^{n+1}, \boldsymbol{u}_n^n\rangle, \lambda_{n-1}; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_{n-1} - \sum_{j=1}^{n-2}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-2}\rangle, \lambda_0, x, x'\right)\prod_{j=1}^{n-2} G_j^{n-2}\right)$$

$$\cdot\prod_{i=2}^{n-2} G_p\left(\langle \boldsymbol{b}_{i+1}^{n+1}, \boldsymbol{u}_n^n\rangle, \lambda_i; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{i-1} G_j^{i-1}\right)$$

$$\cdot\, G_p\left(-F^{(\lambda_n)}\boldsymbol{u}_n^n, \lambda_n; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_n - \sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-1} G_j^{n-1}\right)$$

$$= F_p\left(-\sum_{j=1}^{n}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_0; x, x'\right)$$

$$\cdot\, G_p\left(-F^{(\lambda_1)}\boldsymbol{u}_1^n + \sum_{j=1}^{n-2}\langle \boldsymbol{b}_2^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')\right)$$

$$\cdot\prod_{i=2}^{n-1} G_p\Bigg(-F^{(\lambda_i)}\boldsymbol{u}_i^n + \sum_{j=i}^{n-1}\langle \boldsymbol{b}_{i+1}^{j+2}, \boldsymbol{u}_{j+1}^n\rangle, \lambda_i;$$

$$F_p\left(F^{(\lambda_0)}\boldsymbol{a}_i - \sum_{j=1}^{i-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{i-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{i-1} G_j^{i-1}\Bigg)$$

$$\cdot\, G_p\left(-F^{(\lambda_n)}\boldsymbol{u}_n^n, \lambda_n; F_p\left(F^{(\lambda_0)}\boldsymbol{a}_n - \sum_{j=1}^{n-1}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^{n-1}\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n-1} G_j^{n-1}\right)$$

$$= F_p\left(-\sum_{j=1}^{n}\langle \boldsymbol{b}_1^{j+1}, \boldsymbol{u}_j^n\rangle, \lambda_0; x, x'\right)\prod_{j=1}^{n} G_j^n.$$

By Lemma 5.2.4, Subsection 3.4 and Theorem 4.1.1, we have

$$F(x, x') \equiv F_p\left(-\sum_{i=1}^{n-1} \langle \boldsymbol{b}_1^{i+1}, \boldsymbol{u}_i^{n-1} \rangle, \lambda_0; x, x'\right) \bmod \lambda_n.$$

Therefore we have $\beta^* \circ \Psi^{(\lambda_0, \dots, \lambda_n)} = \psi_2 \circ \varphi_4$. □

Theorem 4.1.2 has been proved.

References

[1] M. Demazure and P. Gabriel, Groups algébriques, Tome 1, Masson-North-Holland, Paris-Amsterdam, 1970.
[2] D. Horikawa, On the extensions of some group schemes, Master thesis (in Japanese), Chuo University, 2002.
[3] Takashi Kondo, On the extensions of group schemes deforming $\mathbf{G}_a$ to $\mathbf{G}_m$, Tokyo Journal of Mathematics, 2010.
[4] T. Sekiguchi, F. Oort and N. Suwa, On the deformation of Artin-Schreier to Kummer, Ann. Scient. Éc. Norm. Sup., 4^e série **22** (1989), 345–375.
[5] T. Sekiguchi and N. Suwa, On the unified Kummer-Artin-Schreier-Witt theory, Prépublication No. 11, Université de Bordeaux (1999).
[6] T. Sekiguchi and N. Suwa, A note on extensions of algebraic and formal groups, IV, Tohoku Math. J. **53** (2001), 203–240.
[7] T. Sekiguchi and N. Suwa, A note on extensions of algebraic and formal groups, V, Japan. J. Math. Vol. 29, No. 2, pp. 221–284 December 2003.
[8] Jean-Pierre Serre, Algebraic Groups and Class Fields, GTM., 117, Springer-Verlag, 1988.
[9] William C. Waterhouse and B. Weisfeiler, One-dimensional Affine group schemes, Journal of Algebra. **66** (1980).

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