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QUANTIFIER ELIMINATION FOR LEXICOGRAPHIC PRODUCTS OF ORDERED ABELIAN GROUPS

By

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Abstract. Let $L_{ag} = \{+, -, 0\}$ be the language of the abelian groups, L an expansion of $L_{ag}(<)$ by relations and constants, and $L_{mod} = L_{ag} \cup \{\equiv_n\}_{n \geq 2}$ where each $\equiv_n$ is defined as follows: $x \equiv_n y$ if and only if $n \mid x - y$. Let H be a structure for L such that $H \mid L_{ag}(<)$ is a totally ordered abelian group and K a totally ordered abelian group. We consider a product interpretation of $H \times K$ with a new predicate I for $\{0\} \times K$ defined by N. Suzuki [9].

Suppose that H admits quantifier elimination in L.

1. If K is a Presburger arithmetic with smallest positive element 1_K then the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, 1) \cup L_{mod}$ with $1^G = (0^H, 1_K)$.
2. If K is dense regular and K/nK is finite for every integer $n \geq 2$ then the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{mod}$ for some set D of constant symbols where $G \models I(d)$ for each $d \in D$.
3. If K admits quantifier elimination in $L_{mod}(<, D)$ for some set D of constant symbols then the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{mod}$ unless K is dense regular with K/nK being infinite for some n.

Conversely, if the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{mod}$ for some

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set D of constant symbols such that $G \models I(d)$ for each $d \in D$ then H admits quantifier elimination in $L \cup L_{\text{mod}}$, and K admits quantifier elimination in $L_{\text{mod}}(<, D)$.

We also discuss the axiomatization of the theory of the product interpretation of $H \times K$.

Introduction

Throughout the paper, "ordered abelian group" will stand for "totally ordered abelian group".

Komori [7] and Weispfenning [12] had shown that the direct product $\mathbf{Z} \times \mathbf{Q}$ equipped with the lexicographic ordering admits quantifier elimination in a language expanding the language of the ordered abelian groups $\{+, -, 0, <\}$. Here, $\mathbf{Z}$ is a Presburger arithmetic (the ordered abelian group of the integers), and $\mathbf{Q}$ a divisible ordered abelian group (the ordered abelian group of rational numbers). They also gave a concrete axiomatization (recursive axiomatization) for the theory of $\mathbf{Z} \times \mathbf{Q}$. Weispfenning [12] extensively studied quantifier elimination in the language

$$\{+, -, 0, <\} \cup \{\equiv_n^i\}_{i \le k, n < \omega} \cup \{I_i\}_{i \le k}$$

where the I_i for $i \le k$ represent convex subgroups such that $I_k \supsetneq I_{k-1} \supsetneq \cdots \supsetneq I_0$ and each $\equiv_n^i$ is a binary relation defined by $x \equiv_n^i y \Leftrightarrow \exists z(I_i(z) \wedge n \mid (x - y - z))$. Suzuki [9] has defined a product interpretation of $H \times K$ in the language $L(I)$ equipped with the lexicographic ordering where H is an L-structure for a language L expanding $\{+, -, 0, <\}$ by adding relation symbols and constant symbols such that the reduct of H to $\{+, -, 0, <\}$ is an ordered abelian group, K is also an ordered abelian group, and I is interpreted as the set $\{0\} \times K$. He has shown that if H admits quantifier elimination in L and K is a divisible ordered abelian group then the product interpretation of $H \times K$ admits quantifier elimination in the language $L(I)$. Moreover, the theory of $H \times K$ is determined by the theory of H and it is recursively axiomatizable if the theory of H is. Tanaka and Yokoyama [11] gave another proof. We will show a similar result when K is a Presburger arithmetic or a dense regular abelian group instead of a divisible ordered abelian group. We also show a similar result when K is an ordered abelian group which admits quantifier elimination in $L_{\text{mod}}(<, D)$ for some set D of constant symbols. In the case that H admits quantifier elimination in $L_{\text{mod}}(<, C)$ for some set C of constant symbols, our results follow from Weispfenning's results [12, 13]. But we believe that our proof is simpler. Choose an ordered abelian group H_0, and let H be an expansion of H_0 by relations and constants which admits quantifier elimination. If the form of the language of H is different from $L_{\mathrm{mod}}(<, C)$ for any set of constant symbols C, then we get a new example of product interpretation of $H \times K$ which admits quantifier elimination.

Tanaka and Yokoyama have shown that if $H \equiv H'$ and $K \equiv K'$ in appropriate languages then $H \times K \equiv H' \times K'$. Let us denote the theory of a structure M by $Th(M)$. We present an axiomatization of $Th(H \times K)$ depending on $Th(H)$ and $Th(K)$. Furthermore, if $Th(H)$ and $Th(K)$ are recursively axiomatizable then so is $Th(H \times K)$.

1. Preliminaries

We follow the notation of Hodges' book [5] in general. Throughout the paper, we use the symbols "$+$", "$-$", "0", "$<$" and "I", where "$+$" is a binary function symbol, "$-$" a unary function symbol, "0" a constant symbol, "$<$" a binary relation symbol, and "I" a unary relation symbol. Let $L_{\mathrm{ag}} = \{+, -, 0\}$. If L is a language, $s_1, s_2, \ldots, s_n$ are new symbols and C is a set of new constant symbols, then $L(s_1, s_2, \ldots, s_n, C)$ denotes the language $L \cup \{s_1, s_2, \ldots, s_n\} \cup C$, and $L(s_1, s_2, \ldots, s_n)$ denotes the language $L \cup \{s_1, s_2, \ldots, s_n\}$. We say that L' is *an expansion of L by relations and constants* if L' can be obtained by adding relation symbols and constant symbols to L.

If L is a language and M is an L-structure, $\mathrm{dom}(M)$ denotes the domain or the universe of M, s^M denotes the interpretation of s in M for each symbol s of L. We often omit "dom" from "$\mathrm{dom}(M)$". Hence, "$x \in M$" will stand for "$x \in \mathrm{dom}(M)$". For a map f and a subset X of the domain of f, $f|X$ denotes the restriction of f to X. If M is an L-structure and $X \subseteq M$, $M|X$ is a structure with domain X such that $R^{M|X} = R^M \cap X^n$ for each n-ary relation symbol R of L, $f^{M|X} = f^M|X^n$ for each n-ary function symbol f of L, and $c^{M|X} = c^M$ for each constant symbol c of L if $c^M \in X$. Note that $f^{M|X}$ might be a partial map on X in general, and $c^{M|X}$ might be non-existing. $M|X$ is an L-substructure of M if $f^{M|X}$ is a total function from X^n to X for every function symbol f of L, and $c^M \in X$ for every constant symbol c of L (i.e., $M|X$ is an L-structure). Let M be an L-structure and M' an expansion of M to a language L'. M' is called a *definitional expansion of* M if every non-logical symbol of L' is definable in M' by an L-formula.

If f is a function and $\bar{a} = (a_1, \ldots, a_n)$ is a tuple of elements $a_1, \ldots, a_n$ from the domain of f, $f(\bar{a})$ denotes the tuple $(f(a_1), \ldots, f(a_n))$. If $\bar{a} = (a_1, \ldots, a_n)$ and b is an element, $\bar{a}\hat{\ }b$ denotes the tuple $(a_1, \ldots, a_n, b)$ and $b\hat{\ }\bar{a}$ denotes the tuple $(b, a_1, \ldots, a_n)$.

If L is a language and M is an L-structure, we also call M a structure for L. If two structures are elementarily equivalent as L-structures, we also say that the two structures are elementarily equivalent for L. If $\bar{y} = (y_1, \dots, y_n)$ is a tuple of variables, $\forall \bar{y} \varphi(\bar{y})$ stands for $\forall y_1 \cdots \forall y_n \varphi(y_1, \dots, y_n)$. To dispense with parentheses in formulas, we follow the following hierarchy of precedences for logical operators and quantifiers. $\neg$ has higher precedence than any other logical operators, $\wedge$ has higher precedence than $\vee$, $\vee$ has higher precedence than $\to$ and $\leftrightarrow$, and the quantifiers $\forall$ and $\exists$ have lower precedence than any logical operators. For example, the formula

$$\forall x, y \quad x^2 = y^2 \wedge x \neq y \to x = -y \wedge x \neq 0$$

stands for

$$(\forall x(\forall y((x^2 = y^2 \wedge x \neq y) \to (x = -y \wedge x \neq 0)))).$$

When we write $s < t$, sometimes we allow s to be $-\infty$ and t to be ∞. We consider $-\infty < t$ and $s < \infty$ to be formulas that are always true. For example, $s < x < t$ with $s = -\infty$ stands for $x < t$, $s < x < t$ with $t = \infty$ stands for $s < x$, and $s < x < t$ with $s = -\infty$ and $t = \infty$ stands for a formula that is always true.

DEFINITION 1.1. An L-structure M *admits quantifier elimination* if for any formula $\varphi(\bar{y})$ of L with a tuple of free variables $\bar{y}$, there is a quantifier-free formula $\psi(\bar{y})$ of L such that

$$M \models \forall \bar{y} \quad \varphi(\bar{y}) \leftrightarrow \psi(\bar{y}).$$

A theory T in L *admits quantifier elimination* if for any formula $\varphi(\bar{y})$ of L with a tuple of free variables $\bar{y}$, there is a quantifier-free formula $\psi(\bar{y})$ of L such that

$$T \vdash \forall \bar{y} \quad \varphi(\bar{y}) \leftrightarrow \psi(\bar{y}).$$

We often consider a definitional expansion M' of M to some extended language L'. When the defining L-formulas of all the new symbols of L' is given, any L-structure can naturally be expanded to an L'-structure. We say that M *admits quantifier elimination in* L' if the definitional expansion M' of M to L' admits quantifier elimination. In the case that L'' is a sublanguage of L', we also say that M *admits quantifier elimination in* L'' if $M'|L''$ admits quantifier elimination.

For the basic definitions and facts on (ordered) abelian groups, we refer the reader to [3] and [4]. Nevertheless, we will review some definitions and facts.

For a set X, id_X denotes the identity map on X. For a term t of L_{ag}, $0 \cdot t$ denotes 0, $1 \cdot t$ denotes t, $2 \cdot t$ denotes $t + t$, $3 \cdot t$ denotes $t + t + t$, and so on. In this way, $m \cdot t$ is defined for any non-negative integer m. For any negative integer m, $m \cdot t$ denotes the term $-(|m| \cdot t)$. We sometimes write mt for $m \cdot t$ when there will be no confusion. Let $L_{\mathrm{mod}} = L_{\mathrm{ag}} \cup \{\equiv_n : n \geq 2\}$ where each $\equiv_n$ is a binary relation defined by $x \equiv_n y \Leftrightarrow \exists z \ (x - y = nz)$. Any abelian group can be considered as an L_{mod}-structure with this definition. For a natural number n, $n|x$ denotes the formula $\exists z \ (x = nz)$.

Definition 1.2 (Abelian Group). An L_{ag}-structure A is called an *abelian group* if

$$A \models \forall x, y, z \quad (x + y) + z = x + (y + z),$$

$$A \models \forall x, y, z \quad x + 0 = 0 + x = x,$$

$$A \models \forall x, y, z \quad x + (-x) = (-x) + x = 0, \quad \text{and}$$

$$A \models \forall x, y \quad x + y = y + x.$$

If an L_{ag}-structure A is an abelian group, L_{ag}-substructure of A is called *a subgroup of A*. If B is a subgroup of an abelian group and $a \in A$, $a + B = \{a + x : x \in B\}$ is called *a coset of B in A*. A coset of B which is different from B is called *a proper coset of B*. For an abelian group A, let $nA = \{nx : x \in A\}$ for an integer n.

Definition 1.3. Suppose an L_{ag}-structure A is an abelian group. A subgroup B of A is called *pure* if for any positive integer n and for any $b \in B$, $A \models \exists x \ (nx = b)$ implies $B \models \exists x \ (nx = b)$. If B is a pure subgroup of A, then B is an L_{mod}-substructure of A.

A subgroup B of an abelian group is called *divisible* if $nB = B$ for every positive integer n. An abelian group A is called *torsion-free* if $A \models \forall x \ (x \neq 0 \rightarrow nx \neq 0)$ for every integer $n > 0$. Suppose A is an abelian group and B and C are subgroups of A. If $A = \{b + c : b \in B, \ c \in C\}$ and $B \cap C = \{0\}$ then we call A the *direct sum* (or the internal direct sum) of B and C and write $A = B \oplus C$. In this case, B is called a *direct summand* of A. C is also a direct summand of A. Every direct summand of an abelian group is a pure subgroup.

Fact 1.4. *Let A be an abelian group and B its subgroup. B is a direct summand of A if and only if there is a group homomorphism* $\pi : A \rightarrow B$ *such that* $\pi \upharpoonright B = \mathrm{id}_B$.

Definition 1.5 (Direct Product). Suppose L_{ag}-structures B and C are abelian groups. Let A be an L_{ag}-structure with $\mathrm{dom}(A) = \mathrm{dom}(B) \times \mathrm{dom}(C)$ (a product set) such that $0^A = (0^B, 0^C)$, $(x_1, y_1) +^A (x_2, y_2) = (x_1 +^B x_2, y_1 +^C y_2)$, and $-^A(x, y) = (-^B x, -^C y)$. A is called the *direct product* (or *external direct sum*) of B and C. Let $B' = \{(b, 0^C) : b \in \mathrm{dom}(B)\}$ and $C' = \{(0^B, c) : c \in \mathrm{dom}(C)\}$. $A|B'$ and $A|C'$ are subgroups of A and are isomorphic to B and C respectively as groups (L_{ag}-structures). A is the (internal) direct sum of $A|B'$ and $A|C'$.

Fact 1.6. *Let A be a torsion-free abelian group. Any equation $nx = a$ with $n \in \mathbf{Z}$ and $a \in A$ has at most one solution in A. Intersections of pure subgroups of A are again pure in A. For every subset S of A, there exists a minimal pure subgroup containing S. This subgroup is called* the pure subgroup generated by S.

The following fact is Theorem 38.1 together with Exercise 4 and 5 on p. 162 in [4]. Eklof and Fisher called an abelian group ω_1-*equationally compact* if it satisfies condition (5) of this fact, and pointed out this equivalence [2]. By an *equation over A*, we mean a formula of the form $t = a$ with a term t of L_{ag} (with variables) and $a \in A$. Note that any term of L_{ag} can be considered as a $\mathbf{Z}$-linear combination of variables in abelian groups.

Fact 1.7. *The following conditions on an abelian group A are equivalent*:

(1) *If B is a pure subgroup of C, C/B is countable, and $f : B \to A$ is a group homomorphism, then there is a group homomorphism $g : C \to A$ such that $g \mid B = f$.*
(2) *A is pure-injective: If B is a pure subgroup of C, and $f : B \to A$ a group homomorphism, then there is a group homomorphism $g : C \to A$ such that $g \mid B = f$.*
(3) *A is algebraically compact: If A is a pure subgroup of C then A is a direct summand of C.*
(4) *If every finite subsystem of a system of equations over A has a solution in A, then the whole system is solvable in A.*
(5) *If every finite subsystem of a countable system of equations over A has a solution in A, then the whole system is solvable in A.*

Fact 1.8. *Let A be a torsion-free abelian group. Then for any positive integers m, n,*

(1) $A \models \forall x, y\ x \equiv_n y \leftrightarrow mx \equiv_{mn} my$,
(2) $A \models \forall x, y\ x \equiv_n y \to mx \equiv_n my$, *and*
(3) $A \models \forall x_1, x_2, y_1, y_2\ x_1 \equiv_n y_1 \wedge x_2 \equiv_n y_2 \to x_1 + x_2 \equiv_n y_1 + y_2$.

The following lemma seems to be well-known but we could not find it in the literature. It is essentially due to Presburger [8].

Lemma 1.9. *Suppose G is a torsion-free abelian group. Let $t_1(\bar{y}), \dots, t_n(\bar{y})$ be terms of L_{ag} with tuple $\bar{y}$ of variables, and $l_1, \dots, l_n$ positive integers. Then we can effectively find (by a recursive procedure) a quantifier-free formula $\theta(\bar{y})$ of L_{mod} such that*

$$G \models \forall y \quad \left(\exists x \bigwedge_{i=1,\dots,n} x \equiv_{l_i} t_i(\bar{y}) \right) \leftrightarrow \theta(\bar{y}).$$

Proof. First, we prove a claim.

Claim 1. *Let l and m be any positive integers and let d be the greatest common divisor of l and m. Since l/d and m/d are relatively prime integers, we can choose integers u, v such that $ul/d + vm/d = 1$. Then*

$$G \models \forall x, y, z \quad (x \equiv_l y \wedge x \equiv_m z) \leftrightarrow (x \equiv_{lm/d} (vm/d)y + (ul/d)z \wedge y - z \equiv_d 0).$$

Let $x, y, z \in G$ be arbitrary. Suppose $G \models x \equiv_l y$ and $G \models x \equiv_m z$. Then $G \models (m/d)x \equiv_{ml/d} (m/d)y$ and $G \models (l/d)x \equiv_{ml/d} (l/d)z$. Hence, $G \models (vm/d)x \equiv_{ml/d} (vm/d)y$ and $G \models (ul/d)x \equiv_{ml/d} (ul/d)z$. By adding terms on each side, we have $G \models x \equiv_{ml/d} (vm/d)y + (ul/d)z$.

Also, since $G \models l \mid x - y$, $G \models m \mid x - z$, and $d|l$, m, we have $G \models d \mid x - y$ and $G \models d \mid x - z$, and thus $G \models d \mid y - z$.

Conversely, suppose that $G \models x \equiv_{lm/d} (vm/d)y + (ul/d)z$ and $G \models y - z \equiv_d 0$. Choose $w \in G$ such that $G \models y - z = dw$. Then in G,

$$\begin{aligned} x &\equiv_{lm/d} (vm/d)y + (ul/d)z \\ &= (vm/d + ul/d)y + (ul/d)(z - y) \\ &= 1 \cdot y - ulw \\ &\equiv_l y. \end{aligned}$$

Hence, $G \models x \equiv_l y$. Similarly, $G \models x \equiv_m z$. The claim is proved.

We prove the statement of the lemma by induction on the number n of conjuncts in the scope of "$\exists x$".

If $n = 1$, then we can always choose such x. Therefore, we can choose $0 = 0$ for $\theta(\bar{y})$.

If $n \geq 2$, by Claim 1, we have

$$G \models \forall \bar{y} \quad \left(\exists x \bigwedge_{i=1,\dots,n} x \equiv_{l_i} t_i(\bar{y}) \right) \leftrightarrow t_1(\bar{y}) - t_2(\bar{y}) \equiv_d 0 \wedge \exists x$$

$$x \equiv_{l_1 l_2/d} (vl_2/d)t_1(\bar{y}) + (ul_1/d)t_2(\bar{y}) \wedge \bigwedge_{i=3,\dots,n} x \equiv_{l_i} t_i(\bar{y})$$

where d is the greatest common divisor of l_1 and l_2, and v, u are integers such that $ul_1 + vl_2 = d$. Note that l_1/d and l_2/d are integers.

By induction hypothesis, we can effectively eliminate "$\exists x$" from the subformula

$$\exists x \quad x \equiv_{l_1 l_2/d} (vl_2/d)t_1(\bar{y}) + (ul_1/d)t_2(\bar{y}) \wedge \bigwedge_{i=3,\dots,n} x \equiv_{l_i} t_i(\bar{y}). \qquad \square$$

Quantifier elimination is known for abelian groups by Szmielew [10]. A shorter proof can be found in a Ziegler's paper [14].

FACT 1.10 (Szmielew). *Any abelian group admits quantifier elimination in* L_{mod}.

DEFINITION 1.11 (Ordered Abelian Group). An $L_{\text{ag}}(<)$-structure A is called an *ordered abelian group* if $A|L_{\text{ag}}$ is an abelian group, $<^A$ is a total order on $\text{dom}(A)$, and

$$A \models \forall x, y, z \quad x < y \to x + z < y + z.$$

If an $L_{\text{ag}}(<)$-structure A is an ordered abelian group and B is a subgroup of $A|L_{\text{ag}}$, then the $L_{\text{ag}}(<)$-substructure of A with domain $\text{dom}(B)$ is also an ordered abelian group.

Suppose an $L_{\text{ag}}(<)$-structure A is an ordered abelian group. A subset B of A is called *convex* if for any $a, b \in B$ and for any $x \in A$, $A \models a < x < b$ implies $x \in B$. A *convex subgroup* of A is a subgroup of A whose domain is a convex subset of A. A subset B of A is called *dense* if for any $a, b \in A$, there is an element $x \in B$ such that $A \models a < x < b$. A *dense subgroup* of A is a subgroup of A whose domain is a dense subset of A.

If an $L_{\text{ag}}(<)$-structure A is an ordered abelian group then $A|L_{\text{ag}}$ is a torsion-free abelian group, and any convex subgroup of A is a pure subgroup of A.

The ordered abelian groups which admit quantifier elimination in $L_{\text{mod}}(<)$ together with some set of constant symbols have been classified by Weispfenning [13].

Definition 1.12. An ordered abelian group G is *dense regular* if it satisfies the following equivalent conditions:

(1) For any integer $n \geq 2$,

$$G \models \forall y, z \quad 0 < y \rightarrow \exists x \quad (0 < x < y \wedge x \equiv_n z).$$

(2) For any prime p, pG is dense in G.

(3) G is elementarily equivalent to a dense subgroup of the real numbers $\mathbf{R}$ (a dense Archimedean group).

Remark 1.13. Suppose n is an integer ≥ 2. Then for any ordered abelian group G,

$$G \models \forall y, z \exists x \quad y < x \wedge x \equiv_n z.$$

Proof. Let $y, z \in G$ be arbitrary. If $y < z$ then the statement holds with $x = z$. If $y = z$, choose a positive element d in G. Then the statement holds with $x = z + nd$. If $z < y$, then $0 < y - z$. Then $y - z < n(y - z)$ since $n \geq 2$. Therefore, $y < z + n(y - z) \equiv_n z$. The statement holds with $x = z + n(y - z)$. □

Lemma 1.14. *Let n be an integer ≥ 2. For an ordered abelian group G, the following are equivalent*:

(1) $G \models \forall b, c \;\; 0 < b \rightarrow \exists x \;\; (0 < x < b \wedge x \equiv_n c)$.

(2) $G \models \forall a, b, c \;\; 0 \leq a < b \rightarrow \exists x \;\; (a < x < b \wedge x \equiv_n c)$.

(3) $G \models \forall a, b, c \;\; a < b \rightarrow \exists x \;\; (a < x < b \wedge x \equiv_n c)$.

Proof. We work in G.

(3) $\Rightarrow$ (1) is immediate.

(1) $\Rightarrow$ (2). Let $a, b, c \in G$ be arbitrary with $0 \leq a < b$. By (1), we can choose $x_0 \in G$ such that $0 < x_0 < b - a$ and $x_0 \equiv_n c$. Again by (1), we can choose $x_1 \in G$ such that $0 < x_1 < x_0$ and $x_1 \equiv_n a$. Let $x = a - x_1 + x_0$. Since $a - x_1 \equiv_n 0$, $x \equiv_n x_0 \equiv_n c$. On the other hand, $0 < x_1 < x_0 < b - a$ implies $0 < x_0 - x_1 < b - a$. Hence, $a < a + x_0 - x_1 < b$.

(2) $\Rightarrow$ (3). Let $a, b, c \in G$ be arbitrary with $a < b$. If $0 < b$ then $0 \leq a < b$ or $a < 0 < b$. In either cases, we can choose desired x by (2). If $b \leq 0$, then $0 \leq -b < -a$. By (2), we can choose $x' \in G$ such that $-b < x' < -a$ and $x' \equiv -c \pmod{n}$. Hence, $a < -x' < b$ and $-x' \equiv c \pmod{n}$. □

The additive group of rational numbers $\mathbf{Q}$ is dense regular. There are many dense regular groups. Let p be a prime number, and let $\mathbf{F}_p$ be the prime field of characteristic p. For any abelian group G, G/pG is a $\mathbf{F}_p$-vector space. Let $\beta_p(G) = \dim_{\mathbf{F}_p} G/pG$. $\beta_p(G)$ is called *a Szmielew invariant*. Note that G/nG is finite for every positive integer n if and only if $\beta_p(G)$ is finite for every prime number p.

FACT 1.15 (Zakon). *For any function f from the set of prime numbers to $\omega \cup \{\omega\}$, there is a dense regular group G such that $\beta_p(G) = f(p)$ for any prime number p. Here ω is the first infinite ordinal number.*

PROOF. We present a construction by Weispfenning [12]. Let $\{r_{p,n} : p$ is a prime, $n < \omega\}$ be a set of linearly independent real numbers over $\mathbf{Q}$. Let $\mathbf{Z}_p = \{a/b \in \mathbf{Q} : b \not\equiv 0 \pmod{p}\}$, and

$$G = \bigoplus_{p:\text{prime}} \bigoplus_{n<f(p)} \mathbf{Z}_p \cdot r_{p,n}.$$

Then G is a dense subgroup of the additive group of the real number field and $\beta_p(G) = f(p)$ for every prime p. □

FACT 1.16 (Weispfenning). *Let G be an ordered abelian group, and D a pure subgroup of G. Consider each element of D as a constant symbol. Then G admits quantifier elimination in $L_{\mathrm{mod}}(<, D)$ if and only if*

(1) *G is dense regular or*
(2) *there exists a finite sequence $\{G_i\}_{0 \le i \le m}$ of convex subgroups of G and a sequence $\{(k_i, d_i)\}_{1 \le i \le m}$ such that*
    (i) $G_m = G$;
    (ii) *for $1 \le i \le m$, k_i is a positive integer, $d_i \in D$, $d_i \in G_i - G_{i-1}$, G_i/G_{i-1} is a $\mathbf{Z}$-group with smallest positive element $1_i + G_{i-1}$, $k_i \cdot 1_i - d_i \in G_{i-1}$;*
    (iii) *G_0 is dense regular, and for every prime p, $\beta_p(G_0)$ is finite and every coset of pG_0 in G_0 has a representative in D.*

The following is a corollary to this fact.

FACT 1.17 (Weispfenning). *Let G be an ordered abelian group.*

(1) *G admits quantifier elimination in $L_{\mathrm{mod}}(<)$ if and only if G is dense regular.*
(2) *Let d be an element of G. G admits quantifier elimination in $L_{\mathrm{mod}}(<, d)$ if and only if G is dense regular, or there exists a divisible convex subgroup G_0 of G and an integer $k \ne 0$ such that G/G_0 is a $\mathbf{Z}$-group with smallest positive element $1 + G_0$ and $k \cdot 1 - d \in G_0$.*

2. Product Interpretations

Definition 2.1 (Lexicographic Product). Let $L_{ag}(<)$-structures B and C be ordered abelian groups. An $L_{ag}(<)$-structure A is called *the lexicographic product of B and C* if $A|L_{ag}$ is the direct product of abelian groups $B|L_{ag}$ and $C|L_{ag}$, and for any $x, y \in A$ with $x = (x_B, x_C)$, $y = (y_B, y_C)$, $A \models x < y$ if and only if

$$B \models x_B < y_B \quad \text{or}$$

$$B \models x_B = y_B \quad \text{and} \quad C \models x_C < y_C.$$

Now, we will introduce the notion of product interpretation for the direct product of two ordered abelian groups. The definition was given in [9] and [11]. The following is a slightly generalized one.

Definition 2.2 (Extended Product Interpretation). Let L be an expansion of $L_{ag}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose that H is an L-structure such that $H \,|\, L_{ag}(<)$ is an ordered abelian group, K an $L_{ag}(<, D)$-structure such that $K \,|\, L_{ag}(<)$ is an ordered abelian group. Let I be a new unary relation symbol which does not appear in L. A structure G for $L(I, D)$ is called *an extended product interpretation of $H \times K$ with new predicate I*, if

1. $G \,|\, L_{ag}(<)$ is a lexicographic product of $H \,|\, L_{ag}(<)$ and $K \,|\, L_{ag}(<)$,
2. for each constant symbol $c \in L$, there is an element $c_K \in K$ such that $c^G = (c^H, c_K)$, and $c_1^H = c_2^H$ implies $c_1^G = c_2^G$ for any constant symbols $c_1, c_2 \in L$,
3. $((x_1, y_1), \ldots, (x_n, y_n)) \in R^G$ if and only if $(x_1, \ldots, x_n) \in R^H$ for each relation symbol R of $L - \{<\}$,
4. $I^G = \{(0^H, x) : x \in K\}$, and
5. $d^G = (0^H, d^K)$ for each constant symbol $d \in D$.

Note that $K \cong G \,|\, I^G$ as $L_{mod}(<, D)$-structures. An extended product interpretation of $H \times K$ is not unique because of condition 2. If $c^G = (c^H, 0^K)$ for each constant symbol $c \in L$, then G is called *the product interpretation of $H \times K$ with new predicate I* [9, 11].

Lemmas 2.3 and 2.8 below are essentially proved by Tanaka and Yokoyama [11].

Lemma 2.3. *Let L be an expansion of $L_{ag}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose that H is an*

L-structure such that $H \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group, K an $L_{\mathrm{mod}}(<, D)$-structure such that $K \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I. If $\varphi(\bar{x})$ is a quantifier-free formula of L with an n-tuple $\bar{x}$ of variables, there is a quantifier-free formula $\varphi^*(\bar{x})$ of $L(I)$ such that for any tuple $\bar{g} = (g_1, \ldots, g_n) \in G^n$ with $g_i = (g_{i,H}, g_{i,K})$ for $i = 1, \ldots, n$, $H \models \varphi(\bar{g}_H)$ if and only if $G \models \varphi^*(\bar{g})$, where $\bar{g}_H = (g_{1,H}, \ldots, g_{n,H})$.

Proof. Let $\varphi(\bar{x})$ be a quantifier-free formula of L with a tuple $\bar{x}$ of n variables. Then $\varphi(\bar{x})$ is a Boolean combination of formulas of forms $t(\bar{x}) = 0$, $0 < t(\bar{x})$, and $R(s_1(\bar{x}), \ldots, s_l(\bar{x}))$, where $t(\bar{x})$, $s_1(\bar{x}), \ldots, s_l(\bar{x})$ are terms of L and R is an l-ary relation symbol of L.

Let $\bar{g} = (g_1, \ldots, g_n)$ be an arbitrary tuple from G with $g_i = (g_{i,H}, g_{i,K})$ for $i = 1, \ldots, n$, and let $\bar{g}_H = (g_{1,H}, \ldots, g_{n,H})$ and $\bar{g}_K = (g_{1,K}, \ldots, g_{n,K})$.

We can write $t(\bar{x}) = t_1(\bar{x}) + t_2(\bar{c})$ where $t_1(\bar{x})$ is a term of L_{ag}, $t_2(\bar{z})$ a term of L_{ag} with a p-tuple $\bar{z}$ of variables, and $\bar{c} = (c_1, \ldots, c_p)$ a tuple of constant symbols of L. Choose $c_{i,K} \in K$ such that $c_i^G = (c_i^H, c_{i,K})$ for $i = 1, \ldots, p$ and let $\bar{c}_K = (c_{1,K}, \ldots, c_{p,K})$. Then $t^G(\bar{g}) = (t^H(\bar{g}_H), t_1^K(\bar{g}_K) + t_2^K(\bar{c}_K))$. Hence,

$$H \models t(\bar{g}_H) = 0 \Leftrightarrow G \models I(t(\bar{g})), \quad \text{and}$$

$$H \models 0 < t(\bar{g}_H) \Leftrightarrow G \models 0 < t(\bar{g}) \wedge \neg I(t(\bar{g})).$$

Similarly, we have

$$H \models R(s_1(\bar{g}_H), \ldots, s_l(\bar{g}_H)) \Leftrightarrow G \models R(s_1(\bar{g}), \ldots, s_l(\bar{g})).$$

Let $\varphi^*(\bar{x})$ be the formula obtained from $\varphi(\bar{x})$ by replacing $t(\bar{x}) = 0$ and $0 < t(\bar{x})$ with $I(t(\bar{x}))$ and $0 < t(\bar{x}) \wedge \neg I(t(\bar{x}))$, respectively. Then $H \models \varphi(\bar{g}_H)$ if and only if $G \models \varphi^*(\bar{g})$. □

Definition 2.4 (Unnested atomic formula). Let L be a language. By *an unnested atomic formula* $\varphi(\bar{x})$ where $\bar{x}$ is a tuple of variables, we mean an atomic formula of one of the following forms:

$u = v$;	
$c = v$	for some constant symbol c of L;
$f(\bar{z}) = y$	for some function symbol f of L;
$R(\bar{z})$	for some relation symbol R of L.

Here, u, v, y are variables from $\bar{x}$, and $\bar{z}$ a tuple of variables from $\bar{x}$.

Definition 2.5 (Partial isomorphism). Let A and B be structures for a language L. A partial map f from A to B is called *a partial L-isomorphism* if for any tuple $\bar{a}$ from the domain of f and for any unnested formula $\varphi(\bar{x})$ of L with a tuple $\bar{x}$ of free variables such that the length of $\bar{x}$ is equal to the length of $\bar{a}$,

$$A \models \varphi(\bar{a}) \Leftrightarrow B \models \varphi(f(\bar{a})).$$

Note that since $u = v$ is an unnested formula, a partial L-isomorphism is a one-to-one map.

We are going to define $A \approx_k B$, which is defined in [5], p. 102. We define it in a different way, but they are equivalent essentially by [5], Lemma 3.3.1.

Definition 2.6. Let A and B be structures for a language L, $\bar{a}$ a tuple from A, and $\bar{b}$ a tuple from B. Suppose that $\bar{a}$ and $\bar{b}$ have the same length. For any integer $k \geq 0$, we define $(A, \bar{a}) \approx_k (B, \bar{b})$ *for* L by induction on k as the following:

$(A, \bar{a}) \approx_0 (B, \bar{b})$ for L if there is a partial L-isomorphism f from A to B such that $f(\bar{a}) = \bar{b}$.

Suppose $k > 0$. $(A, \bar{a}) \approx_k (B, \bar{b})$ for L if for every element c of A there is an element d of B such that $(A, \bar{a}\hat{\ }c) \approx_{k-1} (B, \bar{b}\hat{\ }d)$ for L, and for every element d of B there is an element c of A such that $(A, \bar{a}\hat{\ }c) \approx_{k-1} (B, \bar{b}\hat{\ }d)$ for L.

For $k \geq 1$, $A \approx_k B$ for L if $(A, (\ )) \approx_k (B, (\ ))$ for L where $(\ )$ is the empty tuple.

The following is Corollary 3.3.3 in [5].

Fact 2.7 (Fraïssé-Hintikka). *Let A and B be structures for a finite language L. Then the following are equivalent*:

(1) *$A \equiv B$ for L.*

(2) *$A \approx_k B$ for L for every integer $k \geq 1$.*

Lemma 2.8. *Let L be an expansion of $L_{\mathrm{ag}}(<)$ by predicates and constants and I a new unary predicate. Suppose that $H \equiv H'$ for L, and $K \equiv K'$ for $L_{\mathrm{ag}}(<, D)$ for some set D of new constant symbols. Then the following hold.*

(1) *The product interpretations $H \times K$ and $H' \times K'$ with new predicate I are elementarily equivalent.*

(2) *If G is an extended product interpretation of $H \times K$ with new predicate I, G' is an extended product interpretation of $H' \times K'$ with new predicate I,*

and for each constant symbol c in L there is a constant symbol $d_c \in D \cup \{0\}$ such that $c^G = (c^H, d_c^K)$ and $c^{G'} = (c^{H'}, d_c^{K'})$, then $G \equiv G'$ for the language $L(I, D)$.

Proof. It is enough to prove (2). Let G and G' be as above. We only have to show that $G \equiv G'$ for any finite sublanguage L' of $L(I, D)$ such that $L_{ag}(<, I) \subseteq L'$. We can assume that for any constant symbol $c \in L \cap L'$, there is a constant symbol $d \in (D \cap L') \cup \{0\}$ such that $c^G = (c^H, d^K)$ and $c^{G'} = (c^{H'}, d^{K'})$.

Claim 1. *Let $a_i \in H$, $a_i' \in H'$, $b_i \in K$ and $b_i' \in K'$ for $i = 1, \dots, m$ with $m \geq 0$. For any integer $k \geq 0$, if $(H, (a_1, a_2, \dots, a_m)) \approx_k (H', (a_1', a_2', \dots, a_m'))$ for $L \cap L'$ and $(K, (b_1, b_2, \dots, b_m)) \approx_k (H', (b_1', b_2', \dots, b_m'))$ for $L_{ag}(<, D) \cap L'$ then $(G, (g_1, g_2, \dots, g_m)) \approx_k (G', (g_1', g_2', \dots, g_m'))$ for L' where $g_i = (a_i, b_i)$ and $g_i' = (a_i', b_i')$ for $i = 1, \dots, m$.*

We prove the claim by induction on k.

Suppose $k = 0$. Assume $m > 0$. By the assumption, there is a partial $(L \cap L')$-isomorphism f_1 from H to H' such that $f_1(a_i) = a_i'$ for $i = 1, \dots, m$, and there is a partial $(L_{ag}(<, D) \cap L')$-isomorphism f_2 from K to K' such that $f_2(b_i) = b_i'$ for $i = 1, \dots, m$. Let f be a partial map from G to G' defined by $f(g_i) = f((a_i, b_i)) = (a_i', b_i') = (f_1(a_i), f_2(b_i)) = g_i'$ for $i = 1, \dots, m$. It is straightforward to prove that f is well-defined and it is a partial L'-isomorphism. We show that f is a partial $C \cup \{I\}$-isomorphism where C is the set of constant symbols of $L \cap L'$. The remaining cases can be treated similarly.

If $G \models I(g_i)$ then $a_i = 0^H$ since $g_i = (a_i, b_i)$. We have $f(g_i) = f((0^H, b_i)) = (f_1(0^H), f_2(b_i)) = (0^{H'}, b_i')$. Hence, $G' \models I(f(g_i))$. By symmetry, $G \models I(g_i)$ if and only if $G' \models I(f(g_i))$. Therefore, f is a partial $\{I\}$-isomorphism.

Suppose $G \models g_i = c$ for a constant symbol $c \in L \cap L'$. Then $g_i = (c^H, d_c^K)$ for some $d_c \in D \cap L'$. We have $f(g_i) = f((c^H, d_c^K)) = (f_1(c^H), f_2(d_c^K)) = (c^{H'}, d_c^{K'})$. Hence, $G' \models f(g_i) = c$. By symmetry, $G \models g_i = c$ if and only if $G' \models f(g_i) = c$. Therefore, f is a partial C-isomorphism.

Now, we turn to the induction step. Suppose $k > 0$. We are going to show that $(G, (g_1, g_2, \dots, g_m)) \approx_k (G', (g_1', g_2', \dots, g_m'))$ for L'. By symmetry, it is enough to show that for any $g_{m+1} \in G$, there is $g_{m+1}' \in G'$ such that $(G, (g_1, g_2, \dots, g_m, g_{m+1})) \approx_{k-1} (G', (g_1', g_2', \dots, g_m', g_{m+1}'))$ for L'.

Let $g_{m+1} = (a_{m+1}, b_{m+1}) \in G$ be arbitrary. Since

$$(H, (a_1, a_2, \dots, a_m)) \approx_k (H', (a_1', a_2', \dots, a_m'))$$

for $L \cap L'$ and $a_{m+1} \in H$, we can choose $a'_{m+1} \in H'$ such that

$$(H, (a_1, a_2, \dots, a_m, a_{m+1})) \approx_{k-1} (H', (a'_1, a'_2, \dots, a'_m, a'_{m+1}))$$

for $L \cap L'$. Also, since

$$(K, (b_1, b_2, \dots, b_m)) \approx_k (K', (b'_1, b'_2, \dots, b'_m))$$

for $L_{\mathrm{ag}}(<, D) \cap L'$, we can choose $b'_{m+1} \in K'$ such that

$$(K, (b_1, b_2, \dots, b_m, b_{m+1})) \approx_{k-1} (H', (b'_1, b'_2, \dots, b'_m, b'_{m+1}))$$

for $L_{\mathrm{ag}}(<, D) \cap L'$. Let $g'_{m+1} = (a'_{m+1}, b'_{m+1})$. Then by the induction hypothesis,

$$(G, (g_1, g_2, \dots, g_m, g_{m+1})) \approx_{k-1} (G', (g'_1, g'_2, \dots, g'_m, g'_{m+1}))$$

for L'. We have proved the claim.

Now we turn to the proof of the lemma. Let $k \geq 1$ be any integer. Since $H \equiv H'$ for $L \cap L'$ and $K \equiv K'$ for $L_{\mathrm{ag}}(<, D) \cap L'$, we have $H \approx_k H'$ for $L \cap L'$ and $K \approx_k K'$ for $L_{\mathrm{ag}}(<, D) \cap L'$ by Fact 2.7. Hence, $G \approx_k G'$ for L' by Claim 1. Since $G \approx_k G'$ for L' for any integer $k \geq 1$, $G \equiv G'$ for L' by Fact 2.7. □

Lemma 2.9. *If G is an ordered abelian group, A a convex subgroup of G, B a subgroup of G, and $G = B \oplus A$ as an abelian group, then G is isomorphic to the lexicographic product of B and A.*

Proof. Assume $b + a \leq b' + a'$ with $b, b' \in B$ and $a, a' \in A$.

Suppose $b < b'$ is not the case. Then $b \geq b'$ and we have $0 \leq b - b' \leq a' - a \in A$. Hence, $b - b' \in A$ by convexity of A and thus $b - b' \in A \cap B = \{0\}$. Hence, $b = b'$ and $a \leq a'$. □

Proposition 2.10 (Theory of an Extended Product Interpretation). *Let L be an expansion of $L_{\mathrm{ag}}(<)$ by predicates and constants, and H a structure for L such that $H \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group, and K an $L_{\mathrm{ag}}(<, D)$-structure for some set D of constant symbols such that $K \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group. Let G be an extended product interpretation of $H \times K$ with a new predicate I. Suppose that for each constant symbol $c \in L$, there is a constant symbol $d_c \in D$ such that $c^G = (c^H, d_c^K)$. Then $M \equiv G$ for $L(I, D)$ if and only if M satisfies the following axioms:*

1. *$M \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group;*
2. *I^M is a convex subgroup;*
3. *$I^M \equiv K$ for $L_{\mathrm{ag}}(<, D)$;*

4. *for each relation symbol R of $L - \{<\}$, truth value of R is fixed modulo I, i.e., if R has the arity m,*

$$M \models \forall x_1, \dots, x_m \forall y_1, \dots, y_m$$
$$I(y_1) \wedge \cdots \wedge I(y_m) \to (R(x_1, \dots, x_m) \leftrightarrow R(x_1 - y_1, \dots, x_m - y_m));$$

5. $M/I^M \equiv H$ *for* L;
6. *for each term $t(y_1, \dots, y_n)$ of L_{ag} and a tuple $(c_1, \dots, c_n)$ of constant symbols of L,*

$$M \models t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}) \neq 0 \to \neg I(t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}))$$

and for each positive integer n,

$$M \models \forall x \quad I(x) \wedge n \mid x + t(c_1 - d_{c_1}, \dots, c_n - d_{c_n})$$
$$\to n \mid x \wedge n \mid t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}).$$

Note that assuming condition 4, M/I^M *can naturally be considered as an* L*-structure.*

In particular, if the theory of H in L and the theory of K in $L_{\mathrm{ag}}(<, D)$ are recursively axiomatizable and the function mapping each constant symbol c of L to a constant symbol d_c of D is a recursive function, then the theory of G in $L(I, D)$ is recursively axiomatizable.

PROOF. It is straitforward to check that G satisfies the axioms 1–6.

Let M be any model of the axioms 1–6. To show that $M \equiv G$ for $L(I, D)$, we can replace M by an elementary extension of M. So, we can assume that M is ω_1-saturated. Let us denote the $L_{\mathrm{ag}}(<, D)$-substructure of M with domain I^M by I^M also. Let C be the set of constant symbols of L and P the pure subgroup of M generated by $\{(c - d_c)^M : c \in C\}$. Then $P \cap I^M = \{0\}$ and $P \oplus I^M$ is a pure subgroup of M by Axiom 6. Therefore, there is a group homomorphism g from $P \oplus I^M$ to I^M such that $g \mid I^M = \mathrm{id}$ and $g(x) = 0^M$ for every $x \in P$. Since M is ω_1-saturated, I^M satisfies condition (5) of Fact 1.7 (ω_1-equationally compact). Hence, I^M is pure-injective by Fact 1.7. Therefore, we can extend g to a homomorphism $g' : M \to I^M$. Since $g'(x) = g(x) = x$ for every $x \in I^M$, $M = \mathrm{Ker}\, g' \oplus I^M$. Since $P \subseteq \mathrm{Ker}(g) \subseteq \mathrm{Ker}(g')$, M is isomorphic to an extended product interpretation of $\mathrm{Ker}(g')$ and I^M by Lemma 2.9, and $\mathrm{Ker}(g') \equiv H$ as L-structures by Axiom 4. Therefore, $M \cong \mathrm{Ker}(g') \times I^M \equiv G$ in the language $L(I, D)$ by Lemma 2.8. □

3. Lemmas for Quantifier Elimination

In this section, we present some lemmas used in common later.

Remark 3.1. Suppose that $L = L'(C)$ for some set C of constant symbols. Then to show that a theory T admits quantifier elimination in L, it is enough to show that every existential formula of L' is equivalent to a quantifier-free formula of $L = L'(C)$ modulo T.

Lemma 3.2. *Let L be an expansion of $L_{ag}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L-structure such that $H \,|\, L_{ag}(<)$ is an ordered abelian group, K an $L_{ag}(<, D)$-structure such that $K \,|\, L_{ag}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I. Let L_{R} be the set of relation symbols of L other than $<$. Then the following are equivalent*:

(1) *G admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$.*

(2) *Let x be a variable and $\bar{y}$ an n-tuple of variables. Suppose that p, q are natural numbers such that $p \le q$, m is a non-zero integer, $\varphi(x, \bar{y})$ a conjunction of literals of $L_{\mathrm{R}}(+, -, 0, I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1, \dots, q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞, $\Psi_1(x, \bar{y})$ the formula*

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

Then for any n-tuple $\bar{a}$ from G, each of the statements $G \models \exists x \varphi(x, \bar{a})$, $G \models \exists x \Psi_1(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ for some quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\mathrm{mod}}$.

Proof. Let C be the set of constant symbols of L. Let L' be the language $L_{\mathrm{R}}(I) \cup L_{\mathrm{mod}}$. Then $L(I, D) \cup L_{\mathrm{mod}} = L'(C \cup D)$. By Remark 3.1, it is enough to show that any existential formula of $L_{\mathrm{R}}(I) \cup L_{\mathrm{mod}}$ is equivalent to a quantifier-free formula of $L(I, D) \cup L_{\mathrm{mod}}$ modulo the theory of G.

Since G is totally ordered by $<^G$, any quantifier-free formula of $L_{\mathrm{R}}(I) \cup L_{\mathrm{mod}}$ with free variables $x\hat{\ }\bar{y}$ is equivalent to a disjunction of formulas of forms $\Psi_1(x, \bar{y})$ and $\Psi_2(x, \bar{y})$ allowing m to be 0. In the case with $m = 0$, it is enough to eliminate the quantifier from $\exists x \varphi(x, \bar{y})$. Now, the lemma is clear. □

The statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ of Lemma 3.2 (2) are reduced by the following lemma.

LEMMA 3.3. *Assume the assumption of Lemma* 3.2, *and the assumption of Lemma* 3.2 (2). *Let* $\varphi^1(x, \bar{y})$ *be the formula obtained from* $\varphi(x, \bar{y})$ *by replacing each subformula "*$I(t)$*" with "*$t = 0$*". Then the following hold*:

(1) *Let* $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ *be an arbitrary n-tuple from G, and* $\bar{a}_H$ *the n-tuple* $(b_1, \dots, b_n)$. *Then* $G \models \exists x \varphi(x, \bar{a})$ *if and only if* $H \models \exists x_1 \varphi^1(x_1, \bar{a}_H)$.

(2) *Let* $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ *be an arbitrary n-tuple from G, and* $\bar{a}_H$ *the n-tuple* $(b_1, \dots, b_n)$. *Then* $G \models \exists x \Psi_2(x, \bar{a})$ *if and only if the conjunction of the following statements holds*:

$$H \models \exists x_1 \quad m x_1 = s(\bar{a}_H) \wedge \varphi^1(x_1, \bar{a}_H),$$

$$G \models s(\bar{a}) \equiv_m 0 \wedge \bigwedge_{1 \le i \le p} s(\bar{a}) \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} s(\bar{a}) \equiv_{l_j} t_j(\bar{a}).$$

(3) *If H admits quantifier elimination in L then for any n-tuple* $\bar{a}$ *from G, each of the statements* $G \models \exists x \varphi(x, \bar{a})$ *and* $G \models \exists x \Psi_2(x, \bar{a})$ *is equivalent to a Boolean combination of statements of the form* $G \models \theta(\bar{a})$ *with a quantifier-free formula* $\theta(\bar{y})$ *of* $L(I) \cup L_{\mathrm{mod}}$.

PROOF. (1) and (2) are immediate. We have (3) by (1), (2) and Lemma 2.3. □

Statement $G \models \exists x \Psi_1(x, \bar{a})$ of Lemma 3.2 (2) will be reduced with several lemmas.

LEMMA 3.4. *Assume the assumption of Lemma* 3.2, *and the assumption of Lemma* 3.2 (2). *Let* $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ *be an arbitrary n-tuple from G, and* $\bar{a}_H$ *the n-tuple* $(b_1, \dots, b_n)$. *Then* $G \models \exists x \Psi_1(x, \bar{a})$ *is equivalent to the disjunction of the following statements* (a) *and* (b):

(a) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ *and* $G \models \exists x \Psi_1(x, \bar{a})$.

(b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ *and* $G \models \exists x \Psi_1(x, \bar{a})$.

Statement (a) of Lemma 3.4 is reduced by the following lemma.

Lemma 3.5. *Assume the assumption of Lemma* 3.2, *and the assumption of Lemma* 3.2 (2). *Let* $\varphi^1(x, \bar{y})$ *be the formula obtained from* $\varphi(x, \bar{y})$ *by replacing each subformula* "$I(t)$" *with* "$t = 0$".

Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ *be an arbitrary n-tuple from* G, $\bar{a}_H$ *the n-tuple* $(b_1, \dots, b_n)$, *and* $\bar{a}_K$ *the n-tuple* $(c_1, \dots, c_n)$. *Then the following statements* (1) *and* (2) *are equivalent*:

(1) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ *and* $G \models \Psi_1(x, \bar{a})$.

(2) *For some* $W \subseteq \{1, \dots, p\}$,

$$H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) \wedge \exists x_1$$

$$s_1(\bar{a}_H) \leq mx_1 \leq s_2(\bar{a}_H) \wedge \varphi^1(x_1, \bar{a}_H) \wedge \bigwedge_{k \in W^c} mx_1 \not\equiv_{l_k} t_k(\bar{a}_H)$$

$$\wedge \bigwedge_{i \in W} mx_1 \equiv_{l_i} t_i(\bar{a}_H) \wedge \bigwedge_{p+1 \leq j \leq q} mx_1 \equiv_{l_j} t_j(\bar{a}_H)$$

and

$$K \models \exists x_2 \bigwedge_{i \in W} mx_2 \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_2 \equiv_{l_j} t_j(\bar{a}_K).$$

Proof. (1) $\Rightarrow$ (2). Assume (1). Then there is $x = (x_H, x_K) \in G$ such that

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}) \wedge \varphi(x, \bar{a}).$$

First, we have

$$H \models s_1(\bar{a}_H) \leq mx_H \leq s_2(\bar{a}_H) \wedge \varphi^1(x_H, \bar{a}).$$

Let $W = \{1 \leq i \leq p : H \models mx_H \equiv_{l_i} t_i(\bar{a}_H)\}$. For $i \in W$, if $K \models mx_K \equiv_{l_i} t_i(\bar{a}_K)$ then $G \models m(x_H, x_K) \equiv_{l_i} (t_i(\bar{a}_H), t_i(\bar{a}_K))$. Therefore, $K \models mx_K \not\equiv_{l_i} t_i(\bar{a}_K)$ for $i \in W$. (2) holds with $x_1 = x_H \in H$ and $x_2 = x_K \in K$.

(2) $\Rightarrow$ (1). Assume (2). Choose $W \subseteq \{1, \dots, p\}$, $x_1 \in H$ and $x_2 \in K$ such that

$$H \models s_1(\bar{a}_H) \leq mx_1 \leq s_2(\bar{a}_H) \wedge \varphi^1(x_1, \bar{a}_H)$$

$$\wedge \bigwedge_{k \in W^c} mx_1 \not\equiv_{l_k} t_k(\bar{a}_H) \wedge \bigwedge_{i \in W} mx_1 \equiv_{l_i} t_i(\bar{a}_H) \wedge \bigwedge_{p+1 \leq j \leq q} mx_1 \equiv_{l_j} t_j(\bar{a}_H)$$

and

$$K \models \bigwedge_{i \in W} mx_2 \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_2 \equiv_{l_j} t_j(\bar{a}_K).$$

Since $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ and $H \models s_1(\bar{a}_H) \le mx_1 \le s_2(\bar{a}_H)$, we have $H \models mx_1 < s_2(\bar{a}_H)$ or $H \models mx_1 = s_2(\bar{a}_H)$.

Case $H \models mx_1 < s_2(\bar{a}_H)$. Let l be a common multiple of $l_1, \dots, l_q$ and m. By Remark 1.13, we can choose an element $d \in K$ satisfying $K \models d \equiv_l 0$ and $K \models s_1(\bar{a}_K) - mx_2 < d$. Since $K \models d \equiv_l 0$, $K \models d \equiv_{|m|} 0$. Pick $d' \in K$ such that $K \models d = md'$. Put $x_K = x_2 + d' \in K$ and $x = (x_1, x_K)$. Then $K \models s_1(\bar{a}_K) < mx_2 + d = m(x_2 + d') = mx_K$. Since $H \models s_1(\bar{a}_H) \le mx_1$, $K \models s_1(\bar{a}_K) < mx_K$, and $s_1^G(\bar{a}) = (s_1^H(\bar{a}_H), s_1^K(\bar{a}_K))$, we have $G \models s_1(\bar{a}) < mx$. Since $H \models mx_1 < s_2(\bar{a}_H)$, we have $G \models mx < s_2(\bar{a})$.

Since $K \models d \equiv_{l_i} 0$ for each l_i, we have $K \models mx_K = mx_2 + d \equiv_{l_i} mx_2$ for each i. Hence,

$$K \models \bigwedge_{i \in W} mx_K \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \le j \le q} mx_K \equiv_{l_j} t_j(\bar{a}_K).$$

Therefore, we have (1):

$$H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) \quad \text{and}$$

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}) \wedge \varphi(x, \bar{a}).$$

Case $H \models mx_1 = s_2(\bar{a}_H)$. Let l be a common multiple of $l_1, \dots, l_q$ and m. By Remark 1.13, we can choose an element $d \in K$ satisfying $K \models d \equiv_l 0$ and $K \models -s_2(\bar{a}_K) + mx_2 < d$. Since $K \models d \equiv_l 0$, $K \models d \equiv_{|m|} 0$. Pick $d' \in K$ such that $K \models d = md'$. Put $x_K = x_2 - d' \in K$ and $x = (x_1, x_K)$. Then $K \models mx_K = m(x_2 - d') = mx_2 - d < s_2(\bar{a}_K)$. Since $H \models mx_1 = s_2(\bar{a}_H)$, $K \models mx_K < s_2(\bar{a}_K)$, and $s_2^G(\bar{a}) = (s_2^H(\bar{a}_H), s_2^K(\bar{a}_K))$, we have $G \models mx < s_2(\bar{a})$. Since $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) = mx_1$, we have $G \models s_1(\bar{a}) < mx$.

Now, with an argument similar to the case $H \models mx_1 < s_2(\bar{a}_H)$, we can deduce (1). □

Lemma 3.6. *Assume the assumption of Lemma* 3.2, *and the assumption of Lemma* 3.2 (2). *Suppose H admits quantifier elimination in L and for any positive integer l, K/lK is finite and there is a set D_l of variable-free terms of $L_{\mathrm{ag}}(D)$ such that $D_l^K = \{d^K : d \in D_l\}$ forms a set of representatives of the proper cosets of lK in K. Let $\psi(x, \bar{y})$ be a formula of L, and W a subset of $\{1, \dots, p\}$. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \dots, b_n)$ and $\bar{a}_K = (c_1, \dots, c_n)$. Then the conjunction of the statements*

(e) $$H \models \exists x_1 \quad \psi(x_1, \bar{a}_H) \wedge \bigwedge_{i \in W} mx_1 \equiv_{l_i} t_i(\bar{a}_H) \wedge \bigwedge_{p+1 \le j \le q} mx_1 \equiv_{l_j} t_j(\bar{a}_H)$$

and

$$\text{(f)} \qquad K \models \exists x_2 \bigwedge_{i \in W} mx_2 \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \le j \le q} mx_2 \equiv_{l_j} t_j(\bar{a}_K)$$

is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ *with a quantifier-free formula* $\theta(\bar{y})$ *of* $L(I, D) \cup L_{\text{mod}}$.

Proof. Let l be an arbitrary integer such that $l \ge 2$, and let D_l be a set of variable-free terms of $L_{\text{ag}}(D)$ such that $D_l^K = \{d^K : d \in D_l\}$ forms a set of representatives of the proper cosets of lK in K. Then (f) is equivalent to (f1):

$$\text{(f1)} \qquad K \models \exists x_2 \bigwedge_{i \in W} \left(\bigvee_{d \in D_{l_i}} mx_2 \equiv_{l_i} t_i(\bar{a}_K) + d \right) \wedge \bigwedge_{p+1 \le j \le q} mx_2 \equiv_{l_j} t_j(\bar{a}_K).$$

Assuming (e), (f1) is equivalent to

$$\text{(f2)} \qquad G \models \exists x \bigwedge_{i \in W} \left(\bigvee_{d \in D_{l_i}} mx \equiv_{l_i} t_i(\bar{a}) + d \right) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}).$$

Hence, the conjuction of (e) and (f) is equivalent to the conjunction of (e) and (f2).

By the assumption that H admits quantifier elimination in L and Lemma 2.3, (e) is equivalent to a statement of the form $G \models \theta(\bar{a})$ with $\theta(\bar{y})$ a quantifier-free formula of $L(I)$.

It is enough to show that (f2) is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with $\theta(\bar{y})$ a quantifier-free formula of $L(I, 1) \cup L_{\text{mod}}$. (f2) is equivalent to a finite disjunction of statements of the form

$$\text{(f3)} \qquad G \models \exists x \bigwedge_{1 \le i \le n'} mx \equiv_{l_i'} t_i'(\bar{a})$$

with terms $t_i'(\bar{y})$ of $L_{\text{ag}}(D)$.

By Lemma 1.9,

$$G \models \forall z_1, \dots, z_{n'} \quad \left(\exists x \bigwedge_{i=1,\dots,n'} x \equiv_{l_i'} z_i \right) \leftrightarrow \theta_2(z_1, \dots, z_{n'})$$

for some quantifier-free formula $\theta_2(z_1, \dots, z_{n'})$ in L_{mod}. Therefore, (f3) is equivalent to

$$G \models \theta_2(t_1(\bar{a}), \dots, t_{n'}(\bar{a}))$$

with a quantifier-free formula $\theta_2(t_1'(\bar{y}), \dots, t_{n'}'(\bar{y}))$ of $L_{\text{mod}}(D)$. The lemma is proved. □

4. Products with a Presburger Arithmetic

Definition 4.1. An ordered abelian group G is called a *Presburger arithmetic* or a **Z**-*group* if it is elementarily equivalent to the structure $\mathbf{Z}$ of integers for $L_{\mathrm{ag}}(<)$.

Theorem 4.2. *Let L be an expansion of $L_{\mathrm{ag}}(<)$ by predicates and constants, and H an L-structure such that $H \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group and H admits quantifier elimination in L, and K a Presburger arithmetic* (**Z**-*group*) *with smallest positive element 1_K. Then any extended product interpretation G of $H \times K$ with new predicate I admits quantifier elimination in $L(I,d) \cup L_{\mathrm{mod}}$ with a new constant symbol d when d^G is any non-zero multiple of $(0^H, 1_K)$.*

Moreover, if there is a recursive procedure for quantifier elimination of H in L and there is a recursive map f from the set C of constant symbols of L to K such that $c^G = (c^H, f(c))$ for each $c \in C$, then there is a recursive procedure for quantifier elimination of G in $L(I,d) \cup L_{\mathrm{mod}}$.

Proof. First, we introduce a constant symbol 1 such that $1^G = (0^H, 1_K)$. In G, d can be represented as $m_0 \cdot 1$ for some non-zero integer m_0. At some stage, we use 1 for quantifier elimination an then eliminate the constant 1 using d.

We show the statement of Lemma 3.2 (2). Let x be a variable and $\bar{y}$ an n-tuple of variables. Suppose that p, q, and m are natural numbers with $p \le q$, $\varphi(x,\bar{y})$ is a conjunction of literals of $L_{\mathrm{R}}(+,-,0,I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1,\dots,q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞, $\Psi_1(x,\bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x,\bar{y}),$$

and $\Psi_2(x,\bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x,\bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x,\bar{y})$.

By Lemma 3.3, we have the following:

Claim 1. *For any n-tuple $\bar{a}$ from G, each of the statements $G \models \exists x \varphi(x,\bar{a})$ and $G \models \exists x \Psi_2(x,\bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I)$.*

Now, we turn to the reduction of $G \models \exists x \Psi_1(x, \bar{a})$ for any n-tuple $\bar{a}$ from G.

Claim 2. *Let l be a common multiple of all the l_i's and m. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \dots, b_n)$. Then the following statements* (b) *and* (b1) *are equivalent*:

(b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ *and* $G \models \Psi_1(x, \bar{a})$.

(b1) $H \models \exists x_1 \; s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$, *and for some natural number k such that* $1 \leq k \leq l$,

$$G \models s_1(\bar{a}) + k \cdot 1 < s_2(\bar{a}) \wedge s_1(\bar{a}) + k \cdot 1 \equiv_m 0 \wedge \bigwedge_{1 \leq i \leq p} s_1(\bar{a}) + k \cdot 1 \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} s_1(\bar{a}) + k \cdot 1 \equiv_{l_j} t_j(\bar{a}).$$

Proof of Claim 2. Suppose (b) holds. Choose $x = (x_H, x_K) \in G$ such that

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \varphi(x, \bar{a}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

Since $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$, we have $H \models s_1(\bar{a}_H) = mx_H = s_2(\bar{a}_H)$. Hence, $H \models \exists x_1 \; s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$.

Since $G \models I(s_2(\bar{a}) - s_1(\bar{a}))$, we have $G \models mx = s_1(\bar{a}) + z$ for some $z \in I^G$ with $G \models 0 < z$. Let $z = (0^H, z_K)$. Since K is a **Z**-group, there is an integer k such that $1 \leq k \leq l$ and $K \models k \cdot 1 \equiv_l z_K$. Also, $K \models k \cdot 1 \leq z_K$ because 1^K is the least positive element of K. Therefore, $G \models s_1(\bar{a}) + k \cdot 1 \leq s_1(\bar{a}) + z = mx < s_2(\bar{a})$. Also, $G \models s_1(\bar{a}) + k \cdot 1 \equiv_l mx$. By the choice of l, we have $G \models s_1(\bar{a}) + k \cdot 1 \equiv_m mx \equiv_m 0$ and $G \models s_1(\bar{a}) + k \cdot 1 \equiv_{l_i} mx$ for each i. Therefore, we have (b1).

Conversely, suppose (b1) holds. Choose $x_1 \in H$ and a positive integer k as in (b1). Since $G \models s_1(\bar{a}) + k \cdot 1 \equiv_m 0$, there is $x \in G$ such that $G \models mx = s_1(\bar{a}) + k \cdot 1$. Let $x = (x_1', x_2)$. Then clearly, $H \models mx_1 = s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1'$, and thus $x_1' = x_1$. Hence $G \models \varphi(x, \bar{a})$. Note also that $G \models s_1(\bar{a}) < s_1(\bar{a}) + k \cdot 1$ by $k \geq 1$. Replacing $s_1(\bar{a}) + k \cdot 1$ with mx, we get (b). The claim is proved.

Claim 3. *For any n-tuple $\bar{a}$ from G, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, 1) \cup L_{\mathrm{mod}}$.*

Proof of Claim 3. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \dots, b_n)$. By Lemma 3.4, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the statements (a), (b) of Lemma 3.4. Statement (a) of

Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I,1) \cup L_{\text{mod}}$ by Lemma 3.6 with $D = \{1\}$ and Lemma 2.3. Statement (b) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I,1) \cup L_{\text{mod}}$ by Claim 2 and Lemma 2.3. The claim is proved.

CLAIM 4. *G admits quantifier elimination in $L(I,d) \cup L_{\text{mod}}$ if $d^G = m_0 \cdot 1^G$ with an integer $m_0 \neq 0$.*

PROOF OF CLAIM 4. 1 occurs only in subformulas of one of the forms $s(\bar{y}) = t(\bar{y})$, $s(\bar{y}) \equiv_l t(\bar{y})$ and $s(\bar{y}) < t(\bar{y})$ with terms $s(\bar{y})$, $t(\bar{y})$ of $L_{\text{ag}}(1)$. For any n-tuple $\bar{a}$ from G, $G \models s(\bar{a}) = t(\bar{a}) \leftrightarrow |m_0|s(\bar{a}) = |m_0|t(\bar{a})$, $G \models s(\bar{a}) \equiv_l t(\bar{a}) \leftrightarrow |m_0|s(\bar{a}) \equiv_{l|m_0|} |m_0|t(\bar{a})$, and $G \models s(\bar{a}) < t(\bar{a}) \leftrightarrow |m_0|s(\bar{a}) < |m_0|t(\bar{a})$. Since $|m_0|s(\bar{y})$ and $|m_0|t(\bar{y})$ can be considered as terms of $L_{\text{ag}}(d)$, G admits quantifier elimination in $L(I,d) \cup L_{\text{mod}}$. □

5. Products with a Dense Regular Group

THEOREM 5.1. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \varnothing$. Suppose H is an L-structure such that $H \mid L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<,D)$-structure such that $K \mid L_{\text{ag}}(<)$ is a dense regular ordered abelian group, and K/nK is finite and every proper coset of nK intersects with $D^K = \{d^K : d \in D\}$ for any integer $n \geq 2$. If H admits quantifier elimination in L then any extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I,D) \cup L_{\text{mod}}$.*

Moreover, if there is a recursive procedure for quantifier elimination of H in L and for quantifier elimination of K in $L_{\text{mod}}(<,D)$, and there is a recursive map f from the set C of constant symbols of L to K such that $c^G = (c^H, f(c))$ for each $c \in C$, then there is a recursive procedure for quantifier elimination of G in $L(I,D) \cup L_{\text{mod}}$.

PROOF. We show the statement of Lemma 3.2 (2). Let x be a variable and $\bar{y}$ an n-tuple of variables. Suppose that p, q are natural numbers such that $p \leq q$, m is a non-zero integer, $\varphi(x,\bar{y})$ is a conjunction of literals of $L_{\text{R}}(+,-,0,I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1,\ldots,q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞, $\Psi_1(x,\bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x,\bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

By Lemma 3.3, we have the following:

Claim 1. *For any n-tuple $\bar{a}$ from G, each of the statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I)$.*

Now, we turn to the reduction of $G \models \exists x \Psi_1(x, \bar{a})$ for any n-tuple $\bar{a}$ from G.

Claim 2. *Let l be a common multiple of all the l_i's and m, and D_l a subset of D such that $D_l^K = \{d^K : d \in D\}$ forms a set of representatives of all the proper cosets of lK in K. Let $\bar{a} = ((b_1, c_1), \ldots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \ldots, b_n)$. Then the following statements* (b) *and* (b1) *are equivalent*:

(b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ *and* $G \models \Psi_1(x, \bar{a})$.

(b1) $H \models \exists x_1 \; s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$, *and for some* $d \in D_l \cup \{0\}$,

$$G \models s_1(\bar{a}) < s_2(\bar{a}) \wedge s_1(\bar{a}) + d \equiv_m 0$$
$$\wedge \bigwedge_{1 \le i \le p} s_1(\bar{a}) + d \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} s_1(\bar{a}) + d \equiv_{l_j} t_j(\bar{a}).$$

Proof of Claim 2. Suppose (b) holds. Choose $x = (x_H, x_K) \in G$ such that

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \varphi(x, \bar{a}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}).$$

Since $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$, we have $H \models s_1(\bar{a}_H) = mx_H = s_2(\bar{a}_H)$. Hence, $H \models \exists x_1 \; s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$.

Since $G \models I(s_2(\bar{a}) - s_1(\bar{a}))$, we have $G \models mx = s_1(\bar{a}) + z$ for some $z \in I^G$. Since $D_l \cup \{0\}$ is a set of representatives of all the cosets of lK in K and $K \cong I^G$, there is $d \in D_l$ such that $I^G \models z \equiv_l d$, and thus $G \models z \equiv_l d$. Therefore, $G \models s_1(\bar{a}) + d \equiv_l mx$. By the choice of l, we have $G \models s_1(\bar{a}) + d \equiv_m mx \equiv_m 0$ and $G \models s_1(\bar{a}) + d \equiv_{l_i} mx$ for each i. Therefore, we have (b1).

Conversely, suppose (b1) holds. Choose $x_1 \in H$ and $d \in D_l \cup \{0\}$ as in (b1). We have $G \models 0 < s_2(\bar{a}) - s_1(\bar{a})$ and $G \models I(s_2(\bar{a}) - s_1(\bar{a}))$. Since K is dense regular and $K \cong G | I^G$ as $L_{\mathrm{mod}}(D)$-structures, we can pick $x_2 \in I^G$ such that $G \models 0 < x_2 < s_2(\bar{a}) - s_1(\bar{a}) \wedge x_2 \equiv_l d$.

Then we have

$$G \models s_1(\bar{a}) < s_1(\bar{a}) + x_2 < s_2(\bar{a}) \wedge s_1(\bar{a}) + x_2 \equiv_m 0$$
$$\wedge \bigwedge_{1 \le i \le p} s_1(\bar{a}) + x_2 \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} s_1(\bar{a}) + x_2 \equiv_{l_j} t_j(\bar{a}).$$

Let $x = (x_H, x_K) \in G$ be such that $G \models mx = s_1(\bar{a}) + x_2$. Since $x_2 \in I^G$, $x_2 = (0, z)$ for some $z \in K$. Hence, $s_1^G(\bar{a}) + x_2 = (s_1^H(\bar{a}_H), s_1^K(\bar{a}_K) + d)$. Therefore, $H \models mx_H = s_1(\bar{a}_H)$. Since $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$, we have $H \models mx_H = s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1$. Hence, $H \models x_H = x_1$. Therefore, $G \models \varphi(x, \bar{a})$ since $H \models \varphi^1(x_H, \bar{a}_H)$. Now, we have (b). The claim is proved.

CLAIM 3. *For any n-tuple $\bar{a}$ from G, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$.*

PROOF OF CLAIM 3. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \dots, b_n)$. By Lemma 3.4, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the statements (a), (b) of Lemma 3.4. Statement (a) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$ by Lemma 3.6 and Lemma 2.3. Statement (b) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$ by Claim 2 and Lemma 2.3. The claim is proved. □

For the case that K is a dense regular ordered abelian group such that K/nK is infinite for some n, we have the following.

THEOREM 5.2. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants. Suppose H is an L-structure such that $H \mid L_{\text{ag}}(<)$ is a divisible ordered abelian group, and K an $L_{\text{ag}}(<)$-structure which is a dense regular ordered abelian group. If H admits quantifier elimination in L then any extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I) \cup L_{\text{mod}}$.*

PROOF. We show the statement of Lemma 3.2 (2). Let x be a variable and $\bar{y}$ an n-tuple of variables. Suppose that p, q are natural numbers such that $p \le q$, m is a non-zero integer, $\varphi(x, \bar{y})$ a conjunction of literals of $L_{\mathrm{R}}(+, -, 0, I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1, \ldots, q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞, $\Psi_1(x, \bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

By Lemma 3.3, we have the following:

Claim 1. *For any n-tuple $\bar{a}$ from G, each of the statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I)$.*

Now, we turn to the reduction of $G \models \exists x \Psi_1(x, \bar{a})$ for any n-tuple $\bar{a}$ from G.

Claim 2. *Let $\bar{a} = ((b_1, c_1), \ldots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \ldots, b_n)$. Then the following statements* (a) *and* (a1) *are equivalent*:

(a) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ *and* $G \models \exists x \Psi_1(x, \bar{a})$.

(a1) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) \wedge \exists x_1 \; s_1(\bar{a}_H) \le mx_1 \le s_2(\bar{a}_H) \wedge \varphi^1(x, \bar{a})$ *and*

$$G \models \exists x \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}).$$

Proof of Claim 2. (a) $\Rightarrow$ (a1) is immediate.

(a1) $\Rightarrow$ (a). Suppose (a1) holds. Let $\bar{a}_K = (c_1, \ldots, c_n)$. Choose $x = (x_H, x_K) \in G$ such that

$$G \models \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}).$$

Since H is divisible, $mx_H \equiv_{l_i} t_i(\bar{a}_H)$ for $i = 1, \ldots, p, \ldots, q$. Therefore,

$$K \models \bigwedge_{1 \le i \le p} mx_K \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}_K).$$

Now, we can show (a) by an argument similar to the proof of (2) $\Rightarrow$ (1) for Lemma 3.5. Claim 2 is proved.

CLAIM 3. *Let* $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ *be an arbitrary n-tuple from* G, *and put* $\bar{a}_H = (b_1, \dots, b_n)$. *Then the following statements* (b) *and* (b1) *are equivalent*:

(b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ *and* $G \models \exists x \Psi_1(x, \bar{a})$.

(b1) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H) \wedge \exists x_1 \ mx_1 \le s_1(\bar{a}_H) \wedge \varphi^1(x, \bar{a})$ *and*

$$G \models s_1(\bar{a}) < s_2(\bar{a}) \wedge \exists x \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}).$$

PROOF OF CLAIM 3. (b) $\Rightarrow$ (b1) is immediate.

(b1) $\Rightarrow$ (b). Suppose (b1) holds. Let $\bar{a}_K = (c_1, \dots, c_n)$. As in Claim 2, we can choose $x_K \in K$ such that

$$K \models \bigwedge_{1 \le i \le p} mx_K \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}_K).$$

Let l be a common multiple of all the l_i's. Choose $d \in K$ such that $K \models 0 < d < s_2(\bar{a}_K) - s_1(\bar{a}_K) \wedge d \equiv_l x_K$. Let $x = (s_1^H(\bar{a}_H), s_1^K(\bar{a}_K) + d)$. Then we have (b). The claim is proved.

CLAIM 4. *For any n-tuple* $\bar{a}$ *from* G, $G \models \exists x \Psi_1(x, \bar{a})$ *is equivalent to a Boolean combination of statements of the form* $G \models \theta(\bar{a})$ *with a quantifier-free formula* $\theta(\bar{y})$ *of* $L(I) \cup L_{\mathrm{mod}}$.

PROOF OF CLAIM 4. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n-tuple from G, and put $\bar{a}_H = (b_1, \dots, b_n)$.

By Lemma 3.4, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the statements (a) and (b) of Lemma 3.4. By Fact 1.10, the statement

$$G \models \exists x \bigwedge_{1 \le i \le p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \le j \le q} mx \equiv_{l_j} t_j(\bar{a}).$$

is equivalent to a statement of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of L_{mod}. Hence, the statement (a) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I) \cup L_{\mathrm{mod}}$ by Claim 2 and Lemma 2.3, and the statement (b) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I) \cup L_{\mathrm{mod}}$ by Claim 3 and Lemma 2.3. The claim is proved. □

QUESTION 5.3. Is there any ordered abelian group H other than divisible ordered abelian group such that an extended product interpretation of $H \times K$ admits quantifier elimination?

Example 5.4. Let R be a dense regular ordered abelian group such that R/nR is infinite for some $n > 0$. Let H_0 be the lexicographic product $\mathbf{Z} \times R$. H_0 does not admit quantifier elimination in $L_{\text{mod}}(<)$. Let H be a definitional expansion of H_0 such that H admits quantifier elimination in the expanded language L. Note that L is different from $L_{\text{mod}}(<, D)$ for any set D of constant symbols. Let K be a $\mathbf{Z}$-group or a dense regular group such that K/nK is finite for any integer $n > 0$. Then any extended product interpretation of $H \times K$ admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ for some set $D \subseteq K$ of constants.

6. Products with a Quantifier Eliminable Group

The following two lemmas appear in [13] in some different forms.

Lemma 6.1. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L-structure such that $H \mid L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<, D)$-structure such that $K \mid L_{\text{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I. Suppose H has the smallest positive element 1_H and there is a constant symbol c of L such that $c^H = k \cdot 1_H$ for some integer $k \neq 0$. Then I is equivalent to a quantifier-free formula of $L_{\text{ag}}(<, c)$ in G.*

Proof. Suppose c is a constant symbol of L such that $c^H = k \cdot 1_H$ with an integer $k \neq 0$. Without loss of generality, we can assume that $k > 0$. Since $c^G = (k \cdot 1_H, c_K)$ for some $c_K \in K$, we have $G \models \forall x \ (I(x) \leftrightarrow -c < kx < c)$. □

Lemma 6.2. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L-structure such that $H \mid L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<, D)$-structure such that $K \mid L_{\text{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I. Suppose further that n is an integer and there is a binary relation $\equiv_n'$ of L such that $H \models \forall x, y \ (x \equiv_n' y \leftrightarrow n \mid (x - y))$. Then the following hold*:

(1) $G \models \forall x, y \ x \equiv_n' y \leftrightarrow \exists z \ (I(z) \wedge x - y - z \equiv_n 0)$.

(2) *If K/nK is finite and every coset of nK in K has a representative of the form t^K for some term t of $L_{\text{ag}}(D)$, then the relation $\equiv_n'$ is definable by a quantifier-free formula of $L_{\text{mod}}(D)$ in G.*

(3) *Suppose K is a $\mathbf{Z}$-group and let 1_K be the smallest positive element of K. If $K \models d = k \cdot 1_K$ for some $d \in D$ with an integer $k \neq 0$, then the relation $\equiv_n'$ is definable by a quantifier-free formula of $L_{\text{mod}}(d)$ in G.*

Proof. (1) Let x, y be arbitrary elements of G. Then we can write $x = (x_H, x_K)$ and $y = (y_H, y_K)$ for some $x_H, y_H \in H$ and $x_K, y_K \in K$. Suppose $G \models x \equiv'_n y$. Then by the definition of an extended product interpretation, $H \models x_H \equiv'_n y_H$, and thus $H \models n \mid (x_H - y_H)$. Let $z = (0^H, x_K -^K y_K)$. Then $z \in G$ and $G \models I(z) \wedge x - y - z \equiv_n 0$.

Conversely, suppose $G \models I(z) \wedge x - y - z \equiv_n 0$ for some $z \in G$. Since $G \models I(z)$, $z = (0^H, z_K)$ for some $z_K \in K$. Hence, $(x - y - z)^G = (x_H -^H y_H, u)$ for some $u \in K$. Since $G \models n \mid (x - y - z)$, $H \models n \mid (x_H - y_H)$. Therefore, $G \models x \equiv'_n y$.

(2) Let S be a finite set of terms of $L_{\mathrm{ag}}(D)$ such that the set $S^K = \{t^K : t \in S\}$ forms a set of representatives of all the cosets of nK in K. Then by (1),

$$G \models \forall x, y \quad x \equiv'_n y \leftrightarrow \bigvee_{t \in S} x - y \equiv_n t.$$

(3) Introduce a constant symbol 1 such that 1^K is the smallest positive element of K. Let $S = \{0, 1, 2 \cdot 1, \ldots, (n-1) \cdot 1\}$. Then S^K forms a set of representatives of all the cosets of nK in K.

Let $d \in D$ be such that $K \models d = k \cdot 1$ with an integer $k \neq 0$. Then for each $i < n$ and for any $x, y \in G$, $G \models x - y \equiv_n i \cdot 1$ if and only if $G \models k(x - y) \equiv_{kn} i \cdot d$. By this and (2), the relation $\equiv'_n$ is definable by a quantifier-free formula of $L_{\mathrm{mod}}(d)$ in G. □

Theorem 6.3. *Let L be an expansion of $L_{\mathrm{ag}}(<)$ by predicates and constants, and H an L-structure such that $H \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group. Suppose K is an ordered abelian group and $D \subseteq K$ a pure subgroup of K such that K admits quantifier elimination in $L_{\mathrm{mod}}(<, D)$ but K is not dense regular.*

If H admits quantifier elimination in L then any extended product interpretation of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$.

Proof. Since K admits quantifier elimination in the language $L_{\mathrm{mod}}(<, D)$, by Fact 1.16, there is a finite sequence $\{G_i\}_{0 \le i \le m}$ of convex subgroups of K and a sequence $\{(k_i, d_i)\}_{1 \le i \le m}$ such that (i) $G_m = K$; (ii) for $1 \le i \le m$, k_i is a positive integer, $d_i \in D$, $d_i \in G_i - G_{i-1}$, G_i / G_{i-1} is a $\mathbf{Z}$-group with smallest positive element $1_i + G_{i-1}$, $k_i \cdot 1_i - d_i \in G_{i-1}$; and (iii) G_0 is dense regular, and for every prime p, $\beta_p(G_0)$ is finite and every coset of pG_0 in G_0 has a representative in D.

Introduce a new predicate I_i representing G_i for each $i \le m$. Let K' be an ω_1-saturated elementary extension of K in the expanded language $L_{\mathrm{ag}}(<, D) \cup \{I_i\}_{i \le m}$. Let $G'_i = I_i(K')$ for each $i = 1, \ldots, m$. By Fact 1.7, for each $i = 1, \ldots, m$, there is a subgroup A_i of G_i' such that $G_i' = A_i \oplus G_{i-1}'$. $A_i \cong G_i'/G_{i-1}'$ is a **Z**-group. Let 1_{A_i} be the smallest positive element of A_i for each $i = 1, \dots, m$. Then $k_i \cdot 1_{A_i} - d_i^{K'} \in G_{i-1}'$ for each $i = 1, \dots, m$.

Now, let G be an arbitrary extended product interpretation of $H \times K$ with a new predicate I. For each constant symbols c of L, we have $c^G = (c^H, c_K)$ for some $c_K \in K$ by the definition of an extended product interpretation. Let G' be an extended product interpretation of $H \times K' \mid L_{\mathrm{mod}}(<, D)$ with new predicate I such that $c^{G'} = (c^H, c_K)$ for each constant symbols c of L. Then $G' \equiv G$ for $L(I, D)$ by Lemma 2.8. To show that G admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$, it is enough to show that G' admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$. Let $D_{G_0} = \{d \in D : d^K \in G_0\}$. Then by Remark 3.1, it is enough to show that the reduct G'' of G' to $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0})$ admits quantifier elimination in $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0}) \cup L_{\mathrm{mod}}$.

Consider A_m as a structure for $L_{\mathrm{ag}}(<, d_m)$ by $d_m^{A_m} = k_m \cdot 1_{A_m}$. Let B_m be a structure for $L(I, d_m)$ which is an extended product interpretation of $H \times A_m$ with new predicate I such that $c^{B_m} = (c^H, c_{A_m})$ for each constant symbol of L where $c^G = (c^H, c_K)$ with $c_K = c_{A_m} + c_{G_{m-1}'}$, $c_{A_m} \in A_m$ and $c_{G_{m-1}'} \in G_{m-1}'$. By Theorem 4.2, B_m admits quantifier elimination in the language $L(I, d_m) \cup L_{\mathrm{mod}}$. Let $L_{\mathrm{mod}}^m = \{\equiv_n^m : n \geq 2\}$ and consider B_m as a structure for $L(I, d_m) \cup L_{\mathrm{mod}}^m$ with $B_m \models \forall x, y$ $(x \equiv_n^m y \leftrightarrow x \equiv_n y)$ for each integer $n \geq 2$. B_m admits quantifier elimination in the language $L(I, d_m) \cup L_{\mathrm{mod}}^m$. Since K' is isomorphic to the lexicographic product of A_m and G_{m-1}' by Lemma 2.9, G'' is isomorphic to a reduct of an extended product interpretation of $B_m \times G_{m-1}'$ with new predicate I_{m-1}. Here, G_{m-1}' is considered as a structure for $L_{\mathrm{ag}}(<, d_{m-1}, \dots, d_1, D_{G_0})$.

Now, consider A_{m-1} as a structure for $L_{\mathrm{ag}}(<, d_{m-1})$ by $d_{m-1}^{A_{m-1}} = k_{m-1} \cdot 1_{A_{m-1}}$. Let B_{m-1} be a structure for $L(I, I_{m-1}, d_m, d_{m-1})$ which is an extended product interpretation of $B_m \times A_m$ with new predicate I_{m-1} such that $c^{B_{m-1}} = (c^H, c_{A_m}, c_{A_{m-1}})$ for each constant symbol of L where $c_K = c_{A_m} + c_{A_m-1} + c_{G_{m-2}'}$, $c_{A_m} \in A_m$, $c_{A_{m-1}} \in A_{m-1}$, and $c_{G_{m-1}'} \in G_{m-1}'$. By Theorem 4.2, B_{m-1} admits quantifier elimination in the language $L(I, I_{m-1}, d_m, d_{m-1}) \cup L_{\mathrm{mod}}^m \cup L_{\mathrm{mod}}$. I_{m-1} is definable by a quantifier-free formula of $L_{\mathrm{ag}}(d_m)$ in B_{m-1} by Lemma 6.1, and each relation of L_{mod}^m is definable by a quantifier-free formula of $L_{\mathrm{ag}}(d_{m_1})$ in B_{m-1} by Lemma 6.2. Therefore, B_{m-1} admits quantifier elimination in the language $L(I, d_m, d_{m-1}) \cup L_{\mathrm{mod}}$. Let $L_{\mathrm{mod}}^{m-1} = \{\equiv_n^{m-1} : n \geq 2\}$ and consider B_{m-1} as a structure for $L(I, d_m, d_{m-1}) \cup L_{\mathrm{mod}}^{m-1}$ with $B_{m-1} \models \forall x, y$ $(x \equiv_n^{m-1} y \leftrightarrow x \equiv_n y)$ for each integer $n \geq 2$. B_{m-1} admits quantifier elimination in the language $L(I, d_m, d_{m-1}) \cup L_{\mathrm{mod}}^{m-1}$. Since G_{m-1}' is isomorphic to the lexicographic product of A_{m-1} and G_{m-2}' by Lemma 2.9, G'' is isomorphic to a reduct of an extended product interpretation of $B_{m-1} \times G_{m-2}'$

with new predicate I_{m-2}. Here, G'_{m-2} is considered as a structure for $L_{ag}(<, d_{m-2}, \dots, d_1, D_{G_0})$.

Repeating this argument, we get a structure B_1 for $L(I, d_m, d_{m-1}, \dots, d_1) \cup L^1_{\mathrm{mod}}$ with $L^1_{\mathrm{mod}} = \{\equiv^1_n : n \geq 2\}$ such that $B_1 \models \forall x, y\ (x \equiv^1_n y \leftrightarrow x \equiv_n y)$ for each integer $n \geq 2$, B_1 admits quantifier elimination in its language, and G'' is isomorphic to a reduct of an extended product interpretation B_0 of $B_1 \times G'_0$ with new predicate I_0. Here, G'_0 is considered as a structure for $L_{ag}(<, D_{G_0})$. B_0 admits quantifier elimination in the language $L(I, I_0, d_m, d_{m-1}, \dots, d_1) \cup L^1_{\mathrm{mod}} \cup L_{\mathrm{mod}}$ by Theorem 5.1. I_0 is definable by a quantifier-free formula of $L_{ag}(d_1)$ in B_0 by Lemma 6.1, and each relation of L^1_{mod} is definable by a quantifier-free formula of $L_{ag}(D_{G_0})$ in B_0 by Lemma 6.2. Therefore, B_0 admits quantifier elimination in the language $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0}) \cup L_{\mathrm{mod}}$ by Theorem 5.1. Since G'' is isomorphic to the reduct of B_0 to the language $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0})$, G'' admits quantifier elimination in $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0}) \cup L_{\mathrm{mod}}$. □

Finally, we show partial converses.

THEOREM 6.4. *Let L be an expansion of $L_{ag}(<)$ by predicates and constants, and H an L-structure such that $H \,|\, L_{ag}(<)$ is an ordered abelian group. Suppose K is an ordered abelian group and $D \subseteq K$ a pure subgroup of K.*

If an extended product interpretation of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$ then H admits quantifier elimination in $L \cup L_{\mathrm{mod}}$.

PROOF. Suppose an extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$. We show that H admits quantifier elimination in $L \cup L_{\mathrm{mod}}$. Let x be a variable and $\bar{y}$ a tuple of variables. Let $\exists x \varphi(x, \bar{y})$ be a formula of $L \cup L_{\mathrm{mod}}$, where $\varphi(x, \bar{y})$ is quantifier-free. Since $\varphi(x, \bar{y})$ is a quantifier-free formula of $L \cup L_{\mathrm{mod}}$, the formula $\varphi(x, \bar{y})$ is a Boolean combination of formulas of the forms $mx = t(\bar{y})$, $mx < t(\bar{y})$, $mx + t(\bar{y}) \equiv_n 0$ and $R(s_1(x, \bar{y}), \dots, s_l(x, \bar{y}))$, where R is a relation symbol of $L - \{<\}$, l, m, n are integers such that l is the arity of R and $n \geq 2$, and $t(\bar{y})$, $s_1(x, \bar{y}), \dots, s_l(x, \bar{y})$ are terms of L.

Let $\varphi^*(x, \bar{y})$ be a formula obtained from $\varphi(x, \bar{y})$ by replacing $mx = t(\bar{y})$, $mx < t(\bar{y})$ and $mx + t(\bar{y}) \equiv_n 0$ with $I(t(\bar{y}) - mx)$, $mx < t(\bar{y}) \wedge \neg I(t(\bar{y}) - mx)$, and $\exists z (I(mx + t(\bar{y}) - nz))$, respectively. Let $\bar{h} = (h_1, \dots, h_n)$ be a tuple of elements from the ordered abelian group H. Then, we have

$$H \models \exists x \varphi(x, \bar{h}) \Leftrightarrow G \models \exists x \varphi^*(x, \bar{h}_G),$$

where $\bar{h}_G = ((h_1, 0^K), \dots, (h_n, 0^K))$. Since the ordered abelian group G admits quantifier elimination in the language $L(I, D) \cup L_{\mathrm{mod}}$, there exists some quantifier-free formula $\psi(\bar{y})$ in $L(I, D) \cup L_{\mathrm{mod}}$ such that

$$G \models \exists x \varphi^*(x, \bar{h}_G) \Leftrightarrow G \models \psi(\bar{h}_G).$$

Because $\psi(\bar{y})$ is a quantifier-free formula of $L(I, D) \cup L_{\mathrm{mod}}$, the formula $\psi(\bar{y})$ is a Boolean combination of formulas of the forms $t(\bar{y}) = 0$, $t(\bar{y}) < 0$, $t(\bar{y}) \equiv_n 0$, $R(s_1(\bar{y}), \dots, s_l(\bar{y}))$ and $I(t(\bar{y}))$, where l, n are positive integers, t, $s_1, \dots, s_l$ are terms of $L(D)$ and R is an l-ary relation symbol of L other than "$<$". Let $t(\bar{y}) = t_1(\bar{y}) + t_2(\bar{c}) + d$, where $t_1(\bar{y})$ is a term of L_{ag}, $t_2(\bar{z})$ a term of L_{ag} with a p-tuple $\bar{z}$ of variables, $\bar{c} = (c_1, \dots, c_p)$ is a tuple of constant symbols from L, and $d \in D$. Choose $c_{i,K} \in K$ such that $c_i^G = (c_i^H, c_{i,K})$ for each $i = 1, \dots, p$ and let $\bar{c}_K = (c_{1,K}, \dots, c_{p,K})$. Note that $t_2(\bar{c})^G = (t_2(\bar{c})^H, t_2^K(\bar{c}_K))$. Then,

$$G \models t_1(\bar{h}_G) + t_2(\bar{c}) + d = 0 \Leftrightarrow \begin{cases} H \models t_1(\bar{h}) + t_2(\bar{c}) = 0 & \text{if } K \models t_2(\bar{c}_K) + d = 0 \\ H \models \neg(0 = 0) & \text{if } K \models t_2(\bar{c}_K) + d \neq 0, \end{cases}$$

$$G \models t_1(\bar{h}_G) + t_2(\bar{c}) + d < 0 \Leftrightarrow \begin{cases} H \models t_1(\bar{h}) + t_2(\bar{c}) < 0 & \text{if } K \models t_2(\bar{c}_K) + d \geq 0 \\ H \models t_1(\bar{h}) + t_2(\bar{c}) \leq 0 & \text{if } K \models t_2(\bar{c}_K) + d < 0, \end{cases}$$

$$G \models t_1(\bar{h}_G) + t_2(\bar{c}) + d \equiv_n 0 \Leftrightarrow \begin{cases} H \models t_1(\bar{h}) + t_2(\bar{c}) \equiv_n 0 & \text{if } K \models t_2(\bar{c}_K) + d \equiv_n 0 \\ H \models \neg(0 = 0) & \text{if } K \models t_2(\bar{c}_K) + d \not\equiv_n 0, \end{cases}$$

$$G \models R(s_1(\bar{h}_G), \dots, s_l(\bar{h}_G)) \Leftrightarrow H \models R(s_1^*(\bar{h}), \dots, s_l^*(\bar{h})),$$

$$G \models I(t_1(\bar{h}_G) + t_2(\bar{c}) + d) \Leftrightarrow H \models t_1(\bar{h}) + t_2(\bar{c}) = 0,$$

where $s_i^*(\bar{y})$ is the term obtained from $s_i(\bar{y})$ by replacing each element of D with 0.

Therefore, there exists some quantifier-free formula $\psi'(\bar{y})$ in $L \cup L_{\mathrm{mod}}$ such that $G \models \psi(\bar{h}_G) \Leftrightarrow H \models \psi'(\bar{h})$. It follows that H admits quantifier elimination in $L \cup L_{\mathrm{mod}}$. □

THEOREM 6.5. *Let L be an expansion of $L_{\mathrm{ag}}(<)$ by predicates and constants, and H an L-structure such that $H \mid L_{\mathrm{ag}}(<)$ is an ordered abelian group. Suppose K is an ordered abelian group and $D \subseteq K$ a pure subgroup of K.*

If an extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\mathrm{mod}}$ and there is a constant symbol $d_c \in D$ such that $c^G = (c^H, d_c^K)$ for each constant symbol c of L, then K admits quantifier elimination in $L_{\mathrm{mod}}(<, D)$.

PROOF. Suppose an extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I,D) \cup L_{\mathrm{mod}}$. We show that K admits quantifier elimination in $L_{\mathrm{mod}}(<,D)$. Let $\exists x\theta(x,\bar{y})$ be a formula of $L_{\mathrm{mod}}(<,D)$, where $\theta(x,\bar{y})$ is a quantifier-free formula of $L_{\mathrm{mod}}(<,D)$. Then the formula $\theta(x,\bar{y})$ is a Boolean combination of formulas of the forms $mx = t(\bar{y})$, $mx < t(\bar{y})$, and $mx + t(\bar{y}) \equiv_n 0$, where m and n are integers such that $n \geq 2$, and t is a term of $L_{\mathrm{ag}}(D)$. Let $\bar{k} = (k_1, \dots, k_n)$ be a tuple of elements from the ordered abelian group K. Let $\bar{k}_G = ((0,k_1), \dots, (0,k_n))$. Then, we have

$$K \models \exists x\varphi(x,\bar{k}) \Leftrightarrow G \models \exists x \ I(x) \wedge \varphi(x,\bar{k}_G).$$

Since the ordered abelian group G admits quantifier elimination in the language $L(I,D) \cup L_{\mathrm{mod}}$, there exists some quantifier-free formula $\tau(\bar{y})$ of $L(I,D) \cup L_{\mathrm{mod}}$ such that

$$G \models \exists x \ I(x) \wedge \varphi(x,\bar{k}_G) \Leftrightarrow G \models \tau(\bar{k}_G).$$

Because $\tau(\bar{y})$ is a quantifier-free formula of $L(I,D) \cup L_{\mathrm{mod}}$, the formula $\tau(\bar{y})$ is a Boolean combination of the forms $t(\bar{y}) = 0$, $t(\bar{y}) < 0$, $t(\bar{y}) \equiv_n 0$, $R(s_1(\bar{y}), \dots, s_l(\bar{y}))$ and $I(t(\bar{y}))$, where l, n are positive integers, t, $s_1, \dots, s_l$ are terms of $L(D)$ and R is an l-ary relation symbol of L. Let $t(\bar{y}) = t_1(\bar{y}) + t_2(\bar{c}) + d$, where $t_1(\bar{y})$ is a term of L_{ag}, $t_2(\bar{z})$ a term of L_{ag} with a p-tuple $\bar{z}$ of variables, $\bar{c} = (c_1, \dots, c_p)$ is a tuple of constant symbols from L, and $d \in D$. Put $\bar{0} = (0, \dots, 0)$. Choose $d_{c_i} \in D$ such that $c_i^G = (c_i^H, d_{c_i}^K)$ for each $i = 1, \dots, p$ and let $\bar{d}_{\bar{c}} = (d_{c_1}, \dots, d_{c_p})$. Note that $t_2(\bar{c})^G = (t_2(\bar{c})^H, t_2(\bar{d}_{\bar{c}})^K)$. Then,

$$G \models t_1(\bar{k}_G) + t_2(\bar{c}) + d = 0 \Leftrightarrow \begin{cases} K \models t_1(\bar{k}) + t_2(\bar{d}\bar{c}) + d = 0 & \text{if } H \models t_2(\bar{c}) = 0 \\ K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) \neq 0, \end{cases}$$

$$G \models t_1(\bar{k}_G) + t_2(\bar{c}) + d < 0 \Leftrightarrow \begin{cases} K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) > 0 \\ K \models t_1(\bar{k}) + t_2(\bar{d}\bar{c}) + d < 0 & \text{if } H \models t_2(\bar{c}) = 0 \\ K \models 0 = 0 & \text{if } H \models t_2(\bar{c}) < 0, \end{cases}$$

$$G \models t_1(\bar{k}_G) + t_2(\bar{c}) + d \equiv_n 0 \Leftrightarrow \begin{cases} K \models t_1(\bar{k}) + t_2(\bar{d}\bar{c}) + d \equiv_n 0 & \text{if } H \models t_2(\bar{c}) \equiv_n 0 \\ K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) \not\equiv_n 0, \end{cases}$$

$$G \models R(s_1(\bar{k}_G), \dots, s_l(\bar{k}_G)) \Leftrightarrow \begin{cases} K \models 0 = 0 & \text{if } H \models R(s_1^*(\bar{0}), \dots, s_l^*(\bar{0})) \\ K \models \neg(0 = 0) & \text{if } H \models \neg R(s_1^*(\bar{0}), \dots, s_l^*(\bar{0})), \end{cases}$$

$$G \models I(t_1(\bar{k}_G) + t_2(\bar{c}) + d) \Leftrightarrow \begin{cases} K \models 0 = 0 & \text{if } H \models t_2(\bar{c}) = 0 \\ K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) \neq 0, \end{cases}$$

where $s_i^*(\bar{y})$ is the term obtained from $s_i(\bar{y})$ by replacing d with 0.

Therefore, there exists some quantifier-free formula $\tau'(\bar{y})$ in $L_{\text{mod}}(<, D)$ such that $G \models \tau(\bar{k}_G) \Leftrightarrow K \models \tau'(\bar{k})$. It follows that K admits quantifier elimination in $L_{\text{mod}}(<, D)$. □

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