(Preprint) AAS 17-641

RELATIVE MOTION EQUATIONS IN THE LOCAL-VERTICAL LOCAL-HORIZON FRAME FOR RENDEZVOUS IN LUNAR ORBITS

Giovanni Franzini*, and Mario Innocenti†

In this paper, a set of equations for relative motion description in lunar orbits is presented. The local-vertical local-horizon frame is selected to describe the relative dynamics of a chaser approaching a target in lunar orbit, allowing the development of relative guidance and navigation systems for rendezvous and docking. The model considers the Earth and Moon gravitational influence on the two spacecraft, which are assumed to have negligible masses. The proposed equations are intended for the study of rendezvous missions with a future cis-lunar space station, whose development is currently investigated by several space agencies as the next step for space exploration.

INTRODUCTION

The development of a new space outpost in the vicinity of the Moon is one of the objectives of the major national space agencies as a potential gateway for future exploration missions towards the asteroids and Mars, as well as a staging post to access the lunar surface.[1,2] Several studies aimed at the development of the cis-lunar station are currently ongoing, focused on the selection of the most favorable lunar orbits in terms of maintenance cost, and on the station access by the incoming vehicles performing logistic flights, crew transportation missions, or samples return from the Moon surface.[3,4] Station access is particularly challenging, since the relative dynamics in lunar orbits are considerably different from those in lower Earth orbits (LEO), where rendezvous operation technology is well established.[5] As a matter of fact, a vehicle approaching the cis-lunar station will experience the influence of the Moon, as well as the gravitational pull of the Earth. Hence, the classical relative motion models, such as the *Tschauner – Hempel*[6] or the *Clohessy – Wiltshire equations*,[7] that assume the presence of a single primary body, are no longer valid in the cis-lunar scenario.

In this paper, we derive a set of equations for the description of the relative motion of a chaser vehicle with respect to a target spacecraft orbiting around the Moon. The equations set is developed considering the two spacecraft in a three-body scenario, i.e. under the influence of both Moon and Earth gravitational potentials.

Relative motion in three-body setups is usually described by differencing the equations that regulate the motion of the two spacecraft, i.e. the solutions of the *circular* (or *elliptic*) *restricted three-body problem* relative to the target and the chaser. These sets are developed in frame rotating with the primaries, generally referred to as *synodic* or *pulsating reference frame*, and centered

*Ph.D. Candidate, Department of Information Engineering, University of Pisa, Largo Lucio Lazzarino 1, 56122 Pisa, Italy. E-mail: giovanni.franzini@for.unipi.it.

†Full Professor, Department of Information Engineering, University of Pisa, Largo Lucio Lazzarino 1, 56122 Pisa, Italy. E-mail: mario.innocenti@unipi.it.

on one of them, or on their common center of mass, or even in a collinear libration point. The resulting equations describe the relative dynamics in the same frame, and have been adopted for designing formation flying guidance systems, see for example References 8–11. Differently from these equations sets, the model proposed in this paper presents three main differences:

- The use of a *local-vertical local-horizon* frame centered on the target center of mass. This type of frame, widely adopted for relative motion analysis in LEO, is particularly appealing for relative guidance and navigation systems design, since it eases the integration of measurements acquired by target and chaser relative positioning sensors, and allows to better understand and to characterize the chaser trajectories as seen from the target.
- The dynamics are described in terms of spacecraft position and velocity vectors with respect to the Moon. This feature is particularly useful in case one of the spacecraft loses the line of sight with the ground stations on Earth.
- The proposed set is based only on the restricted three body assumption, i.e. the spacecraft have negligible masses with respect to the primaries. Thus, the equations have general validity, and are not restricted to the elliptic or circular three body problem, as in the aforementioned references.

Possible simplifications of the developed equations set are discussed in the paper, and a preliminary analysis of their validity is proposed.

RESTRICTED THREE-BODY DYNAMICS

Consider the three-body system composed by the Earth and the Moon *primary bodies* and a spacecraft i, with masses M_e, M_m, and m_i respectively. Their positions with respect to an *inertial frame* $\mathcal{I}$ is denoted with $\boldsymbol{R}_e$, $\boldsymbol{R}_m$, and $\boldsymbol{R}_i$ respectively..

Each body exerts its gravitational influence on the others, resulting in the following equations of motion for the three bodies,

$$\begin{aligned} m_i \big[\, \ddot{\boldsymbol{R}}_i \,\big]_{\mathcal{I}} &= -G\frac{M_e m_i}{r_{ei}^3}\boldsymbol{r}_{ei} - G\frac{M_m m_i}{r_{mi}^3}\boldsymbol{r}_{mi} \\ M_e \big[\, \ddot{\boldsymbol{R}}_e \,\big]_{\mathcal{I}} &= G\frac{M_m M_e}{r_{em}^3}\boldsymbol{r}_{em} + G\frac{m_i M_e}{r_{ei}^3}\boldsymbol{r}_{ei} \\ M_m \big[\, \ddot{\boldsymbol{R}}_m \,\big]_{\mathcal{I}} &= -G\frac{M_e M_m}{r_{em}^3}\boldsymbol{r}_{em} + G\frac{m_i M_m}{r_{ei}^3}\boldsymbol{r}_{ei} \end{aligned}$$

where $\boldsymbol{r}_{ei} = \boldsymbol{R}_i - \boldsymbol{R}_e$ and $\boldsymbol{r}_{mi} = \boldsymbol{R}_i - \boldsymbol{R}_m$ denote the position of the spacecraft i with respect to the Earth and to the Moon, $\boldsymbol{r}_{em} = \boldsymbol{R}_m - \boldsymbol{R}_e$ is the position of the Moon with respect to the Earth, and G is the *universal gravitational constant*. Relative positions norms are indicated with r_{ei}, r_{mi}, and r_{em}. The notation $\big[\, \ddot{\boldsymbol{R}} \,\big]_{\mathcal{I}}$ denotes the acceleration of the body as seen from the inertial frame.

Spacecraft i equations of motion with respect to the Earth and to the Moon are then given by

$$\big[\, \ddot{\boldsymbol{r}}_{ei} \,\big]_{\mathcal{I}} = \big[\, \ddot{\boldsymbol{R}}_i \,\big]_{\mathcal{I}} - \big[\, \ddot{\boldsymbol{R}}_e \,\big]_{\mathcal{I}} = -G\frac{(M_e + m_i)}{r_{ei}^3}\boldsymbol{r}_{ei} - GM_m\left(\frac{\boldsymbol{r}_{mi}}{r_{mi}^3} + \frac{\boldsymbol{r}_{em}}{r_{em}^3}\right) \tag{1}$$

$$\big[\, \ddot{\boldsymbol{r}}_{mi} \,\big]_{\mathcal{I}} = \big[\, \ddot{\boldsymbol{R}}_i \,\big]_{\mathcal{I}} - \big[\, \ddot{\boldsymbol{R}}_m \,\big]_{\mathcal{I}} = -G\frac{(M_m + m_i)}{r_{mi}^3}\boldsymbol{r}_{mi} - GM_e\left(\frac{\boldsymbol{r}_{ei}}{r_{ei}^3} - \frac{\boldsymbol{r}_{em}}{r_{em}^3}\right) \tag{2}$$

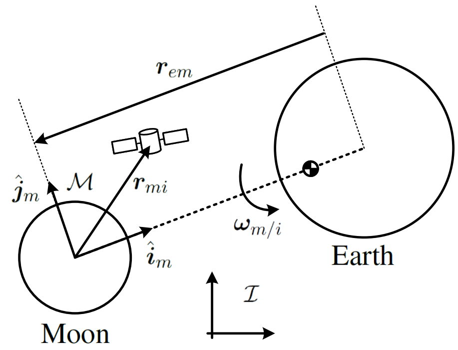


Figure 1: Moon (synodic) reference frame.

We now assume that the mass of the spacecraft i is negligible with respect to the primaries masses, i.e. $m_i \ll M_e$ and $m_i \ll M_m$, i.e. we consider the *restricted three-body problem*. Under this assumption the orbital motion of the two primaries is not affected by the spacecraft, and Eqs. (1) and (2) simplify to,

$$
\begin{aligned}
\left[\, \ddot{\boldsymbol{r}}_{ei} \,\right]_{\mathcal{I}} &= -\mu_e \frac{\boldsymbol{r}_{ei}}{r_{ei}^3} - \mu_m \left(\frac{\boldsymbol{r}_{mi}}{r_{mi}^3} + \frac{\boldsymbol{r}_{em}}{r_{em}^3} \right) \\
\left[\, \ddot{\boldsymbol{r}}_{mi} \,\right]_{\mathcal{I}} &= -\mu_m \frac{\boldsymbol{r}_{mi}}{r_{mi}^3} - \mu_e \left(\frac{\boldsymbol{r}_{ei}}{r_{ei}^3} - \frac{\boldsymbol{r}_{em}}{r_{em}^3} \right)
\end{aligned} \tag{3}
$$

where $\mu_e = GM_e$ and $\mu_m = GM_m$ are the primaries' *gravitational parameters*.

Assume now that the primaries revolve around their common barycenter in elliptic orbits (*elliptic restricted three-body problem*). The motion of the two primaries can then be obtained from the solution of the classical two-body problem. In particular, for the problem at hand, we consider the Moon revolving around the Earth on an elliptic orbit. Moon orbital motion is described by the following parameters:[12]

- Earth-Moon mass ratio, $M_e/M_m = 81.300587$;
- semi-major axis, $a = 384\,400\,\mathrm{km}$;
- eccentricity $e = 0.05490$;
- mean motion $n = 2.661\,699\,5 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$.

The equations of motion for the spacecraft i are generally developed in a frame that rotates with the primaries. A *Moon* or *synodic reference frame* $\mathcal{M}\colon \{\boldsymbol{R}_m; \hat{\boldsymbol{i}}_m, \hat{\boldsymbol{j}}_m, \hat{\boldsymbol{k}}_m\}$ is introduced, with origin in the Moon center of mass, and unit vectors defined as follows,

$$
\hat{\boldsymbol{i}}_m = -\frac{\boldsymbol{r}_{em}}{r_{em}}, \quad \hat{\boldsymbol{j}}_m = \hat{\boldsymbol{k}}_m \times \hat{\boldsymbol{i}}_m, \quad \hat{\boldsymbol{k}}_m = \frac{\boldsymbol{h}_{m/e}}{h_{m/e}}
$$

where $\boldsymbol{h}_{m/e} = \boldsymbol{r}_{em} \times \left[\, \dot{\boldsymbol{r}}_{em} \,\right]_{\mathcal{I}}$ is the *specific angular momentum* of the Moon with respect to the Earth, and $h_{m/e} = \left\| \boldsymbol{h}_{m/e} \right\|$, see Fig. 1. The unit vectors $\hat{\boldsymbol{i}}_m$ - $\hat{\boldsymbol{j}}_m$ lie in the Moon orbital plane. The frame $\mathcal{M}$, i.e. the Earth-Moon system, rotates with respect to an inertial frame of reference with angular velocity equal to $\boldsymbol{\omega}_{m/i} = \omega_{m/i}\hat{\boldsymbol{k}}_m$.

The acceleration of the spacecraft i in the frame $\mathcal{M}$ can be written as follows,

$$\left[\ddot{\boldsymbol{r}}_{mi}\right]_{\mathcal{I}} = \left[\ddot{\boldsymbol{r}}_{mi}\right]_{\mathcal{M}} + 2\boldsymbol{\omega}_{m/i} \times \left[\dot{\boldsymbol{r}}_{mi}\right]_{\mathcal{M}} + \left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{M}} \times \boldsymbol{r}_{mi} + \boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \boldsymbol{r}_{mi}\right) \tag{4}$$

where $2\boldsymbol{\omega}_{m/i} \times \left[\dot{\boldsymbol{r}}_{mi}\right]_{\mathcal{M}}$ is the *Coriolis acceleration* and $\boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \boldsymbol{r}_{mi}\right)$ is the *centripetal acceleration* term.

In the Moon frame we have that $\boldsymbol{r}_{em} = -r_{em}\hat{\boldsymbol{\imath}}_m$, and the spacecraft position vectors with respect to the Moon and to the Earth are defined as follows,

$$\boldsymbol{r}_{mi} = x_i\hat{\boldsymbol{\imath}}_m + y_i\hat{\boldsymbol{\jmath}}_m + z_i\hat{\boldsymbol{k}}_m, \quad \boldsymbol{r}_{ei} = (x_i - r_{em})\hat{\boldsymbol{\imath}}_m + y_i\hat{\boldsymbol{\jmath}}_m + z_i\hat{\boldsymbol{k}}_m$$

Introducing Eq. (3) in Eq. (4), and expressing all the vectors in the Moon reference frame, we obtain the equations of motion for the spacecraft i in the Moon reference frame

$$\begin{cases} \ddot{x}_i - 2\omega_{m/i}\dot{y}_i - \dot{\omega}_{m/i}y_i - \omega_{m/i}^2 x_i = -\mu_m\dfrac{x_i}{r_{mi}^3} - \mu_e\left(\dfrac{x_i - r_{em}}{r_{ei}^3} + \dfrac{1}{r_{em}^2}\right) & \text{(5a)} \\ \ddot{y}_i + 2\omega_{m/i}\dot{x}_i + \dot{\omega}_{m/i}x_i - \omega_{m/i}^2 y_i = -\mu_m\dfrac{y_i}{r_{mi}^3} - \mu_e\dfrac{y_i}{r_{ei}^3} & \text{(5b)} \\ \ddot{z}_i = -\mu_m\dfrac{z_i}{r_{mi}^3} - \mu_e\dfrac{z_i}{r_{ei}^3} & \text{(5c)} \end{cases}$$

where the distances of the spacecraft from the Moon and the Earth are given by

$$r_{ei} = \sqrt{(x_i - r_{em})^2 + y_i^2 + z_i^2}, \quad r_{mi} = \sqrt{x_i^2 + y_i^2 + z_i^2}$$

Eqs. (5) can be normalized expressing the distances in units of the Moon orbit semi-major axis a, time in units of the inverse of the mean angular motion n, i.e. introducing the new time variable $\tau = nt$, and the masses such that $M_e + M_m = 1$. The generic distance x and the associated derivatives are related to the non-dimensional variables $\tilde{x}$ as follows

$$x = a\tilde{x}, \quad \dot{x} = a\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = a\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = an\mathring{\tilde{x}}, \quad \ddot{x} = an\frac{\mathrm{d}\mathring{\tilde{x}}}{\mathrm{d}t} = an^2\mathring{\mathring{\tilde{x}}}$$

where the upper empty circle denotes derivation with respect to the normalized time variable τ. Note that the angular velocity is now expressed in units of n, thus

$$\omega = n\tilde{\omega}, \quad \dot{\omega} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = n^2\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}\tau} = n^2\mathring{\tilde{\omega}}$$

The *normalized gravitational parameter* $\tilde{\mu}$ for the Earth-Moon system is defined as

$$\tilde{\mu} = \frac{\mu_m}{\mu_e + \mu_m} = \left(1 + \frac{M_e}{M_m}\right)^{-1} = 0.012151$$

Since $M_e + M_m = 1$, Moon and Earth gravitational parameters are $\mu_m = \tilde{\mu}$ and $\mu_e = 1 - \tilde{\mu}$ respectively.

Eqs. (5) can now be written in non-dimensional form as follows,

$$
\begin{cases}
\overset{\circ\circ}{\tilde{x}}_i - 2\tilde{\omega}_{m/i}\mathring{\tilde{y}}_i - \mathring{\tilde{\omega}}_{m/i}\tilde{y}_i - \tilde{\omega}^2_{m/i}\tilde{x}_i = -\tilde{\mu}\dfrac{\tilde{x}_i}{\tilde{r}^3_{mi}} - (1-\tilde{\mu})\left(\dfrac{\tilde{x}_i - \tilde{r}_{em}}{\tilde{r}^3_{ei}} + \dfrac{1}{\tilde{r}^2_{em}}\right) & \text{(6a)} \\
\overset{\circ\circ}{\tilde{y}}_i + 2\tilde{\omega}_{m/i}\mathring{\tilde{x}}_i + \mathring{\tilde{\omega}}_{m/i}\tilde{x}_i - \tilde{\omega}^2_{m/i}\tilde{y}_i = -\tilde{\mu}\dfrac{\tilde{y}_i}{\tilde{r}^3_{mi}} - (1-\tilde{\mu})\dfrac{\tilde{y}_i}{\tilde{r}^3_{ei}} & \text{(6b)} \\
\overset{\circ\circ}{\tilde{z}}_i = -\tilde{\mu}\dfrac{\tilde{z}_i}{\tilde{r}^3_{mi}} - (1-\tilde{\mu})\dfrac{\tilde{z}_i}{\tilde{r}^3_{ei}} & \text{(6c)}
\end{cases}
$$

and the normalized distances of the spacecraft from the Earth and the Moon are given by

$$\tilde{r}_{ei} = \sqrt{(\tilde{x}_i - \tilde{r}_{em})^2 + \tilde{y}_i^2 + \tilde{z}_i^2}, \quad \tilde{r}_{mi} = \sqrt{\tilde{x}_i^2 + \tilde{y}_i^2 + \tilde{z}_i^2}$$

respectively.

Eqs. (6) can be further simplified if we assume the Moon and the Earth rotating around the Earth-Moon barycenter in circular orbits, i.e. we consider the *circular restricted three-body problem* (CR3BP). In this case $\tilde{r}_{em} = 1$, $\tilde{\omega}_{m/i} = 1$, and $\mathring{\tilde{\omega}}_{m/i} = 0$, and Eqs. (6) simplify as follows,

$$
\begin{cases}
\overset{\circ\circ}{\tilde{x}}_i - 2\mathring{\tilde{y}}_i - \tilde{x}_i = -\tilde{\mu}\dfrac{\tilde{x}_i}{\tilde{r}^3_{mi}} - (1-\tilde{\mu})\left(\dfrac{\tilde{x}_i - 1}{\tilde{r}^3_{ei}} + 1\right) & \text{(7a)} \\
\overset{\circ\circ}{\tilde{y}}_i + 2\mathring{\tilde{x}}_i - \tilde{y}_i = -\tilde{\mu}\dfrac{\tilde{y}_i}{\tilde{r}^3_{mi}} - (1-\tilde{\mu})\dfrac{\tilde{y}_i}{\tilde{r}^3_{ei}} & \text{(7b)} \\
\overset{\circ\circ}{\tilde{z}}_i = -\tilde{\mu}\dfrac{\tilde{z}_i}{\tilde{r}^3_{mi}} - (1-\tilde{\mu})\dfrac{\tilde{z}_i}{\tilde{r}^3_{ei}} & \text{(7c)}
\end{cases}
$$

with

$$\tilde{r}_{ei} = \sqrt{(\tilde{x}_i - 1)^2 + \tilde{y}_i^2 + \tilde{z}_i^2}, \quad \tilde{r}_{mi} = \sqrt{\tilde{x}_i^2 + \tilde{y}_i^2 + \tilde{z}_i^2}$$

RELATIVE MOTION IN THE LOCAL-VERTICAL LOCAL-HORIZON FRAME

Equations Development

Consider a *target* and a *chaser* spacecraft, orbiting around the Moon, and subject to both Earth and Moon gravitational influence. The aim of this section is to describe the motion of the chaser relative to the target, in a frame of reference centered on the latter. To this end, the *local-vertical local-horizon* (LVLH) frame $\mathcal{L}$: $\{\boldsymbol{R}_t; \hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\}$ is introduced, with unit vectors defined as follows,

$$\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}}, \quad \hat{\boldsymbol{j}} = -\frac{\boldsymbol{h}_{t/m}}{h_{t/m}}, \quad \hat{\boldsymbol{k}} = -\frac{\boldsymbol{r}_{mt}}{r_{mt}}$$

where $\boldsymbol{r}_{mt}$ denotes the target position with respect to the Moon, $r_{mt} = \|\boldsymbol{r}_{mt}\|$, $\boldsymbol{h}_{t/m} = \boldsymbol{r}_{mt} \times \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}}$ is the target specific angular momentum with respect to the Moon, and $h_{t/m} = \|\boldsymbol{h}_{t/m}\|$. The unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$ in the rendezvous literature are generally referred to as *V-bar*, *H-bar*, and *R-bar*, respectively.[5]

With reference to Fig. 2, chaser position with respect to the Moon is given by

$$\boldsymbol{r}_{mc} = \boldsymbol{r}_{mt} + \boldsymbol{\rho} \tag{8}$$

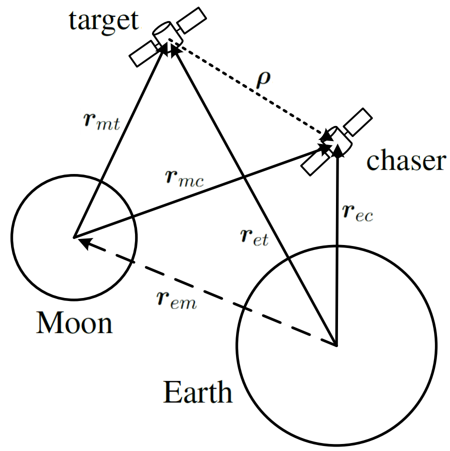


Figure 2: Target and chaser spacecraft in the three-body system.

where $\boldsymbol{\rho}$ is the relative position of the chaser with respect to the target.

The time-derivative of Eq. (8) in the inertial frame is

$$\begin{aligned} \left[\dot{\boldsymbol{r}}_{mc}\right]_{\mathcal{I}} &= \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{I}} \\ &= \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + \boldsymbol{\omega}_{l/i} \times \boldsymbol{\rho} \end{aligned} \tag{9}$$

where $\boldsymbol{\omega}_{l/i}$ is the angular velocity of $\mathcal{L}$ (i.e. of the target) with respect to $\mathcal{I}$. Further derivation of Eq. (9) in $\mathcal{I}$ yields

$$\left[\ddot{\boldsymbol{r}}_{mc}\right]_{\mathcal{I}} = \left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \left[\ddot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + 2\boldsymbol{\omega}_{l/i} \times \left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{I}} \times \boldsymbol{\rho} + \boldsymbol{\omega}_{l/i} \times \left(\boldsymbol{\omega}_{l/i} \times \boldsymbol{\rho}\right) \tag{10}$$

Bearing in mind that $\left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{I}} = \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}}$, and introducing Eq. (3) in Eq. (10), we obtain the nonlinear equations of relative motion in the LVLH frame:

$$\boxed{\begin{aligned} \left[\ddot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + 2\boldsymbol{\omega}_{l/i} \times \left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} \times \boldsymbol{\rho} + \boldsymbol{\omega}_{l/i} \times \left(\boldsymbol{\omega}_{l/i} \times \boldsymbol{\rho}\right) \\ = \mu_m \left(\frac{\boldsymbol{r}_{mt}}{r_{mt}^3} - \frac{\boldsymbol{r}_{mc}}{r_{mc}^3}\right) + \mu_e \left(\frac{\boldsymbol{r}_{et}}{r_{et}^3} - \frac{\boldsymbol{r}_{ec}}{r_{ec}^3}\right) \end{aligned}} \tag{11}$$

where

$$\boldsymbol{r}_{mc} = \boldsymbol{r}_{mt} + \boldsymbol{\rho}, \quad \boldsymbol{r}_{et} = \boldsymbol{r}_{em} + \boldsymbol{r}_{mt}, \quad \boldsymbol{r}_{ec} = \boldsymbol{r}_{em} + \boldsymbol{r}_{mt} + \boldsymbol{\rho}$$

and $r_{ij} = \|\boldsymbol{r}_{ij}\|$.

The angular velocity of the LVLH frame with respect to the inertial frame can be computed as follows,

$$\boldsymbol{\omega}_{l/i} = \boldsymbol{\omega}_{l/m} + \boldsymbol{\omega}_{m/i} \tag{12}$$

where $\boldsymbol{\omega}_{l/m}$ and $\boldsymbol{\omega}_{m/i}$ are the angular velocities of $\mathcal{L}$ with respect to $\mathcal{M}$, and of $\mathcal{M}$ with respect to $\mathcal{I}$, respectively. Consequently,

$$\begin{aligned} \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} &= \left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}} + \left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{L}} \\ &= \left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}} + \left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{M}} - \boldsymbol{\omega}_{l/m} \times \boldsymbol{\omega}_{m/i} \end{aligned} \tag{13}$$

Eq. (11), along with Eqs. (12) and (13) is a nonlinear equations set with time-varying parameters:

- $\boldsymbol{r}_{mt}$, $\boldsymbol{\omega}_{l/m}$, and $\left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}}$, , that depend on the target motion around the Moon;
- $\boldsymbol{r}_{em}$, $\boldsymbol{\omega}_{m/i}$, and $\left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{M}}$, characteristics of the Moon orbital motion.

In the following section, the expression of $\boldsymbol{\omega}_{l/m}$, and $\left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}}$ are derived, in order to complete the description of the relative dynamics in the LVLH frame.

Target Angular Velocity and Acceleration with Respect to the Moon

In this section, we look for an analytical expression of the LVLH frame angular velocity and acceleration vectors with respect to the Moon frame, that exploits only kinematics relationships. To this end, the same consideration adopted by Casotto in Reference 13 are here used to express $\boldsymbol{\omega}_{l/m}$ and $\left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}}$ in terms of the position, the velocity, the acceleration, and the jerk of the target with respect to the Moon, hence using measurements taken during its motion in the vicinity of Moon.

Consider the time-derivatives of the LVLH frame unit vectors as seen from the Moon frame:

$$\left[\dot{\hat{\boldsymbol{i}}}\right]_{\mathcal{M}} = \boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{i}}, \quad \left[\dot{\hat{\boldsymbol{j}}}\right]_{\mathcal{M}} = \boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{j}}, \quad \left[\dot{\hat{\boldsymbol{k}}}\right]_{\mathcal{M}} = \boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{k}}$$

Left vectorial multiplication of the previous expressions by the relative unit vector gives the following expressions,

$$\begin{aligned}
\hat{\boldsymbol{i}} \times \left[\dot{\hat{\boldsymbol{i}}}\right]_{\mathcal{M}} &= \hat{\boldsymbol{i}} \times \left(\boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{i}}\right) = \boldsymbol{\omega}_{l/m} - \left(\boldsymbol{\omega}_{l/m} \cdot \hat{\boldsymbol{i}}\right) \hat{\boldsymbol{i}} \\
\hat{\boldsymbol{j}} \times \left[\dot{\hat{\boldsymbol{j}}}\right]_{\mathcal{M}} &= \hat{\boldsymbol{j}} \times \left(\boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{j}}\right) = \boldsymbol{\omega}_{l/m} - \left(\boldsymbol{\omega}_{l/m} \cdot \hat{\boldsymbol{j}}\right) \hat{\boldsymbol{j}} \\
\hat{\boldsymbol{k}} \times \left[\dot{\hat{\boldsymbol{k}}}\right]_{\mathcal{M}} &= \hat{\boldsymbol{k}} \times \left(\boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{k}}\right) = \boldsymbol{\omega}_{l/m} - \left(\boldsymbol{\omega}_{l/m} \cdot \hat{\boldsymbol{k}}\right) \hat{\boldsymbol{k}}
\end{aligned}$$

that can be summed up obtaining

$$\hat{\boldsymbol{i}} \times \left[\dot{\hat{\boldsymbol{i}}}\right]_{\mathcal{M}} + \hat{\boldsymbol{j}} \times \left[\dot{\hat{\boldsymbol{j}}}\right]_{\mathcal{M}} + \hat{\boldsymbol{k}} \times \left[\dot{\hat{\boldsymbol{k}}}\right]_{\mathcal{M}} = 2\boldsymbol{\omega}_{l/m} \tag{14}$$

The time-derivative of the unit vector $\hat{\boldsymbol{k}}$ is

$$\left[\dot{\hat{\boldsymbol{k}}}\right]_{\mathcal{M}} = -\frac{1}{r_{mt}} \left(\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} + \dot{r}_{mt} \hat{\boldsymbol{k}}\right) \tag{15}$$

Noting that $\boldsymbol{r}_{mt} = -r_{mt}\hat{\boldsymbol{k}}$ and $\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{L}} = -\dot{r}_{mt}\hat{\boldsymbol{k}}$, we can write

$$\begin{aligned}
\dot{r}_{mt} &= -\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{L}} \cdot \hat{\boldsymbol{k}} \\
&= -\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{k}} + \left(\boldsymbol{\omega}_{l/m} \times \boldsymbol{r}_{mt}\right) \cdot \hat{\boldsymbol{k}} \\
&= -\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{k}} + \boldsymbol{\omega}_{l/m} \cdot \left(\boldsymbol{r}_{mt} \times \hat{\boldsymbol{k}}\right) \\
&= -\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{k}}
\end{aligned} \tag{16}$$

Substitution of Eq. (16) into Eq. (15) gives,

$$\begin{aligned}
\left[\dot{\hat{\boldsymbol{k}}}\right]_{\mathcal{M}} &= -\frac{1}{r_{mt}} \left(\left(\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}}\right) \hat{\boldsymbol{i}} + \left(\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{j}}\right) \hat{\boldsymbol{j}}\right) \\
&= -\frac{1}{r_{mt}} \left(\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}}\right) \hat{\boldsymbol{i}}
\end{aligned} \tag{17}$$

Note that $\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{j}} = 0$ since the target velocity as seen from the frame $\mathcal{M}$ is perpendicular to the specific angular momentum $\boldsymbol{h}_{t/m}$.

For the unit vector $\hat{\boldsymbol{j}}$ we have that

$$\begin{aligned}
\left[\,\dot{\hat{\boldsymbol{j}}}\,\right]_{\mathcal{M}} &= -\frac{1}{h_{t/m}}\left(\left[\,\dot{\boldsymbol{h}}_{t/m}\,\right]_{\mathcal{M}} + \dot{h}_{t/m}\hat{\boldsymbol{k}}\right) \\
&= -\frac{1}{h_{t/m}}\left(\left(\left[\,\dot{\boldsymbol{h}}_{t/m}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}}\right)\hat{\boldsymbol{i}} + \left(\left[\,\dot{\boldsymbol{h}}_{t/m}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{k}}\right)\hat{\boldsymbol{k}}\right) \\
&= -\frac{1}{h_{t/m}}\left(\left(\boldsymbol{r}_{mt} \times \left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\right) \cdot \hat{\boldsymbol{i}}\right)\hat{\boldsymbol{i}} \\
&= -\frac{1}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \left(\hat{\boldsymbol{i}} \times \boldsymbol{r}_{mt}\right)\right)\hat{\boldsymbol{i}} \\
&= -\frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{j}}\right)\hat{\boldsymbol{i}} \qquad (18)
\end{aligned}$$

where we exploited the following results,

$$\begin{aligned}
\left[\,\dot{\boldsymbol{h}}_{t/m}\,\right]_{\mathcal{L}} &= -\dot{h}_{t/m}\hat{\boldsymbol{j}} = -\left(\left[\,\dot{\boldsymbol{h}}_{t/m}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{j}}\right)\hat{\boldsymbol{j}} \\
\left[\,\dot{\boldsymbol{h}}_{t/m}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{k}} &= \left(\boldsymbol{r}_{mt} \times \left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\right) \cdot \hat{\boldsymbol{k}} = 0
\end{aligned}$$

the latter justified by the fact that $\boldsymbol{r}_{mt} \times \left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}$ is perpendicular to $\boldsymbol{r}_{mt}$, i.e. to $\hat{\boldsymbol{k}}$.

Eventually, the time-derivative of $\hat{\boldsymbol{i}}$ is given by

$$\begin{aligned}
\left[\,\dot{\hat{\boldsymbol{i}}}\,\right]_{\mathcal{M}} =& \left[\,\dot{\hat{\boldsymbol{j}}}\,\right]_{\mathcal{M}} \times \hat{\boldsymbol{k}} + \hat{\boldsymbol{j}} \times \left[\,\dot{\hat{\boldsymbol{k}}}\,\right]_{\mathcal{M}} \\
=& \frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{j}}\right)\hat{\boldsymbol{j}} + \frac{1}{r_{mt}}\left(\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}}\right)\hat{\boldsymbol{k}} \qquad (19)
\end{aligned}$$

Substitution of Eqs. (17), (18), and (19) into Eq. (14), yields

$$\boxed{\boldsymbol{\omega}_{l/m} = \omega^{y}_{l/m}\hat{\boldsymbol{j}} + \omega^{z}_{l/m}\hat{\boldsymbol{k}} = \left(-\frac{1}{r_{mt}}\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}}\right)\hat{\boldsymbol{j}} + \left(\frac{r_{mt}}{h_{t/m}}\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{j}}\right)\hat{\boldsymbol{k}}} \qquad (20)$$

Note that in Eq. (20) the component of the angular velocity along the V-bar direction is zero due to the definition of the LVLH frame.

The components of the angular acceleration in the LVLH frame can be obtained by direct derivation of Eq. (20).

The angular acceleration along the H-bar is given by,

$$\begin{aligned}
\dot{\omega}^{y}_{l/m} &= -\frac{1}{r_{mt}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}} + \left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \left[\,\dot{\hat{\boldsymbol{i}}}\,\right]_{\mathcal{M}} - \frac{\dot{r}_{mt}}{r_{mt}}\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}}\right) \\
&= -\frac{1}{r_{mt}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{i}} + \left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}} \cdot \left[\,\dot{\hat{\boldsymbol{i}}}\,\right]_{\mathcal{M}} + \dot{r}_{mt}\omega^{y}_{l/m}\right) \qquad (21)
\end{aligned}$$

The term $\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\left[\,\dot{\hat{\boldsymbol{i}}}\,\right]_{\mathcal{M}}$ can be simplified as follows

$$
\begin{aligned}
\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\left[\,\dot{\hat{\boldsymbol{i}}}\,\right]_{\mathcal{M}} &= \left(\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{L}}+\boldsymbol{\omega}_{l/m}\times\boldsymbol{r}_{mt}\right)\cdot\left(\frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right)\hat{\boldsymbol{j}}+\frac{1}{r_{mt}}\left(\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}}\right)\hat{\boldsymbol{k}}\right)\\
&= \left(-\dot{r}_{mt}\hat{\boldsymbol{k}}-r_{mt}\omega^{y}_{l/m}\hat{\boldsymbol{i}}\right)\cdot\left(\omega^{z}_{l/m}\hat{\boldsymbol{j}}-\omega^{y}_{l/m}\hat{\boldsymbol{k}}\right)\\
&= \dot{r}_{mt}\omega^{y}_{l/m}
\end{aligned}
$$

and substituted into Eq. (21), obtaining

$$
\boxed{\dot{\omega}^{y}_{l/m}=-\frac{1}{r_{mt}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}}+2\dot{r}_{mt}\omega^{y}_{l/m}\right)} \tag{22}
$$

The angular acceleration along $\hat{\boldsymbol{k}}$ is

$$
\dot{\omega}^{z}_{l/m}=\frac{r_{mt}}{h_{t/m}}\left(\frac{\dot{r}_{mt}}{r_{mt}}\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}+\left[\,\dddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}+\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\left[\,\dot{\hat{\boldsymbol{j}}}\,\right]_{\mathcal{M}}-\frac{\dot{h}_{t/m}}{h_{t/m}}\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right) \tag{23}
$$

Noting that,

$$
\begin{aligned}
\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\left[\,\dot{\hat{\boldsymbol{j}}}\,\right]_{\mathcal{M}} &= \left(\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}}\right)\hat{\boldsymbol{i}}+\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right)\hat{\boldsymbol{j}}+\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{k}}\right)\hat{\boldsymbol{k}}\right)\\
&\quad\cdot\left(-\frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right)\hat{\boldsymbol{i}}\right)\\
&= -\frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}}\right)\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right)
\end{aligned}
$$

Eq. (23) can be written as

$$
\dot{\omega}^{z}_{l/m}=\frac{r_{mt}}{h_{t/m}}\left(\frac{\dot{r}_{mt}}{r_{mt}}\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}+\left[\,\dddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}-\frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}}+\frac{\dot{h}_{t/m}}{r_{mt}}\right)\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right) \tag{24}
$$

Considering that $\boldsymbol{h}_{t/m}=-h_{t/m}\hat{\boldsymbol{j}}$, and that

$$
\begin{aligned}
\boldsymbol{h}_{t/m} &= \boldsymbol{r}_{mt}\times\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\\
&= -r_{mt}\hat{\boldsymbol{k}}\times\left(\left[\,\dot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{L}}+\boldsymbol{\omega}_{l/m}\times\boldsymbol{r}_{mt}\right)\\
&= -r_{mt}\hat{\boldsymbol{k}}\times\left(-\dot{r}_{mt}\hat{\boldsymbol{k}}-\omega^{y}_{l/m}r_{mt}\hat{\boldsymbol{i}}\right)\\
&= r^{2}_{mt}\omega^{y}_{l/m}\hat{\boldsymbol{j}}
\end{aligned} \tag{25}
$$

by differentiation of Eq. (25), and recalling Eq. (22), the derivative of the specific angular momentum norm can be obtained:

$$
\dot{h}_{t/m}=-2r_{mt}\dot{r}_{mt}\omega^{y}_{l/m}-r^{2}_{mt}\dot{\omega}^{y}_{l/m}=r_{mt}\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}} \tag{26}
$$

Introduction of Eq. (26) in Eq. (24) yields the expression of the angular acceleration along $\hat{\boldsymbol{k}}$:

$$
\boxed{\dot{\omega}^{z}_{l/m}=\frac{r_{mt}}{h_{t/m}}\left(\frac{\dot{r}_{mt}}{r_{mt}}\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}+\left[\,\dddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}-2\frac{r_{mt}}{h_{t/m}}\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{i}}\right)\left(\left[\,\ddot{\boldsymbol{r}}_{mt}\,\right]_{\mathcal{M}}\cdot\hat{\boldsymbol{j}}\right)\right)} \tag{27}
$$

To compute the target jerk, consider the following relationship between $\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}}$ and $\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}$:

$$\begin{aligned}\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} =& \left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} - 3\left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{I}} \times \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} - 3\boldsymbol{\omega}_{m/i} \times \left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} - \left[\ddot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{I}} \times \boldsymbol{r}_{mt} \\ &- 3\boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}}\right) - 2\left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{I}} \times \left(\boldsymbol{\omega}_{m/i} \times \boldsymbol{r}_{mt}\right) \\ &- \boldsymbol{\omega}_{m/i} \times \left(\left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{I}} \times \boldsymbol{r}_{mt}\right) - \boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \boldsymbol{r}_{mt}\right)\right)\end{aligned}$$

The jerk in $\mathcal{I}$ can be obtained using the chain rule, i.e.

$$\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} = \frac{\partial\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}{\partial \boldsymbol{r}_{mt}}\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \frac{\partial\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}{\partial\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \frac{\partial\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}{\partial \boldsymbol{r}_{em}}\left[\dot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}} + \frac{\partial\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}{\partial\left[\dot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}}}\left[\ddot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}} \quad (28)$$

Since the target moves in a conservative field, the gradient of $\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}$ with respect to $\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}$ and $\left[\dot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}}$ is zero, and Eq. (28) reduces to

$$\begin{aligned}\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} =& \frac{\partial\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}{\partial \boldsymbol{r}_{mt}}\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \frac{\partial\left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}}}{\partial \boldsymbol{r}_{em}}\left[\dot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}} \\ =& -\mu_m \frac{\partial}{\partial \boldsymbol{r}_{mt}}\left[\frac{\boldsymbol{r}_{mt}}{r_{mt}^3}\right]\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} \\ &- \mu_e\left(\frac{\partial}{\partial \boldsymbol{r}_{mt}}\left[\frac{\boldsymbol{r}_{mt} + \boldsymbol{r}_{em}}{\|\boldsymbol{r}_{mt} + \boldsymbol{r}_{em}\|^3}\right]\left(\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} + \left[\dot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}}\right) - \frac{\partial}{\partial \boldsymbol{r}_{em}}\left[\frac{\boldsymbol{r}_{em}}{r_{em}^3}\right]\left[\dot{\boldsymbol{r}}_{em}\right]_{\mathcal{I}}\right)\end{aligned}$$

where $\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} = \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} + \boldsymbol{\omega}_{m/i} \times \boldsymbol{r}_{mt}$, and

$$\frac{\partial}{\partial \boldsymbol{r}}\left[\frac{\boldsymbol{r}}{r^3}\right] = \frac{1}{r^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}\boldsymbol{r}^T}{r^2}\right)$$

with $\boldsymbol{I}$ denoting the identity matrix.

SIMPLIFICATION OF THE EQUATIONS OF RELATIVE MOTION

Due to the nonlinearity of the gravitational acceleration and the presence of several time-varying parameters, the equations of relative motions derived in the previous section, namely Eqs. (11) along with angular velocity and acceleration vectors given by Eqs. (20), (22), and (27), may be difficult to use for the development of guidance and navigation systems. Two possible simplifications are here discussed, aimed at linearizing the equations set and at reducing the number of time-varying parameters.

Circular Restricted Three-Body Problem Assumption

Under the assumption of primaries revolving in circular orbits, the number of time-varying parameters in Eqs. (11) reduces. As a matter of fact, in the CR3BP setup $\boldsymbol{r}_{em}$ is constant, $\boldsymbol{\omega}_{m/i} = n\boldsymbol{k}_m$, and $\left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{M}} = \boldsymbol{0}$. Therefore, the angular velocity and acceleration of the LVLH frame with respect to the inertial frame simplify as follows,

$$\boldsymbol{\omega}_{l/i} = \boldsymbol{\omega}_{l/m} + n\boldsymbol{k}_m, \quad \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} = \left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}} - \boldsymbol{\omega}_{l/m} \times n\boldsymbol{k}_m$$

Furthermore, since $\left[\ddot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{M}} = \boldsymbol{0}$, the computation of the jerk $\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}}$ in Eq. (27) simplifies as follows,

$$\begin{aligned}\left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} =& \left[\dddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{I}} - 3\boldsymbol{\omega}_{m/i} \times \left[\ddot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} - 3\boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}}\right) \\ &- \boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \left(\boldsymbol{\omega}_{m/i} \times \boldsymbol{r}_{mt}\right)\right)\end{aligned}$$

Linearization of the Earth and Moon Gravitational Acceleration

Consider the gravitational acceleration on the chaser, due to the i-th primary,

$$\boldsymbol{g}_i(\boldsymbol{r}_{ic}) = -\mu_i \frac{\boldsymbol{r}_{ic}}{r_{ic}^3}, \quad \boldsymbol{r}_{ic} = \boldsymbol{r}_{it} + \boldsymbol{\rho}$$

This can be linearized by means of a Taylor expansion to first order around the target position with respect to the primary i:

$$\boldsymbol{g}_i(\boldsymbol{r}_{ic}) \approx \boldsymbol{g}_i(\boldsymbol{r}_{it}) + \left.\frac{\partial \boldsymbol{g}_i(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_{it}} (\boldsymbol{r}_{ic} - \boldsymbol{r}_{it}), \quad \frac{\partial \boldsymbol{g}_i(\boldsymbol{r})}{\partial \boldsymbol{r}} = -\frac{\mu_i}{r^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}\boldsymbol{r}^T}{r^2}\right)$$

Hence, the right-hand side of Eq. (11) can be approximated as follows,

$$\mu_m\left(\frac{\boldsymbol{r}_{mt}}{r_{mt}^3} - \frac{\boldsymbol{r}_{mc}}{r_{mc}^3}\right) + \mu_e\left(\frac{\boldsymbol{r}_{et}}{r_{et}^3} - \frac{\boldsymbol{r}_{ec}}{r_{ec}^3}\right) \approx \frac{\mu_m}{r_{mt}^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}_{mt}\boldsymbol{r}_{mt}^T}{r_{mt}^2}\right)\boldsymbol{\rho} + \frac{\mu_m}{r_{et}^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}_{et}\boldsymbol{r}_{et}^T}{r_{et}^2}\right)\boldsymbol{\rho}$$

The *linear time-varying equations of relative motion* are then

$$\boxed{\left[\ddot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + 2\boldsymbol{\Omega}_{l/i}\left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{L}} + \left(\left[\dot{\boldsymbol{\Omega}}_{l/i}\right]_{\mathcal{L}} + \boldsymbol{\Omega}_{l/i}^2 - \frac{\mu_m}{r_{mt}^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}_{mt}\boldsymbol{r}_{mt}^T}{r_{mt}^2}\right) - \frac{\mu_m}{r_{et}^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}_{et}\boldsymbol{r}_{et}^T}{r_{et}^2}\right)\right)\boldsymbol{\rho} = \boldsymbol{0}} \tag{29}$$

where $\boldsymbol{\Omega}_{l/i}$ denotes the skew-symmetric matrix associated to $\boldsymbol{\omega}_{l/i}$.

Simplified Equations Set Comparison

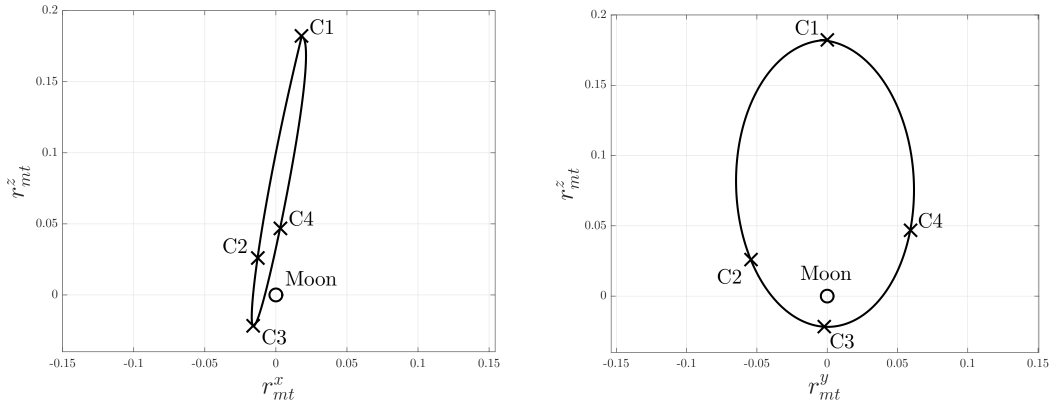
A preliminary analysis of the error introduced by the simplifications previously discussed is here proposed. In particular, by means of numerical simulations we compared the linear time-varying equations, Eq. (29), and the same equations with the assumption of the CR3BP, for simplifying the angular velocity and acceleration vectors $\boldsymbol{\omega}_{l/i}$ and $\left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}}$. The equations were implemented in Simulink, and integrated using ode4 Runge-Kutta algorithm, with step size equal to $30\,\mathrm{s}$, over a time span of 6 hours.

Four initial target position and velocity conditions were chosen, named C1 through C4, see Table 1. The initial conditions belong to the Earth-Moon L_2 *near rectilinear halo orbit* shown in Fig. 3, obtained by means of the state transition matrix for the CR3BP provided in Reference 14, Chapter 6.7. Such orbits are of particular interest, since they are characterized by close passages over a lunar pole, and maintain constant line of sight with the Earth. In particular, the orbit chosen for this comparison provides access to the lunar south pole, that is of particular scientific interest.[15]

For each initial target condition, we compared the solution of the simplified equations sets, denoted with $(\tilde{\boldsymbol{\rho}}(t), \left[\dot{\tilde{\boldsymbol{\rho}}}\right]_{\mathcal{M}}(t))$, against the one of the exact set, i.e. Eqs. (11) with angular velocity and acceleration terms given by Eqs. (20), (22), and (27), denoted with $(\boldsymbol{\rho}(t), \left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{M}}(t))$. As error indexes, the norm of the position and velocity error is considered,

$$e_{\boldsymbol{\rho}}(t) = \|\tilde{\boldsymbol{\rho}}(t) - \boldsymbol{\rho}(t)\|, \quad e_{\dot{\boldsymbol{\rho}}}(t) = \left\|\left[\dot{\tilde{\boldsymbol{\rho}}}\right]_{\mathcal{M}}(t) - \left[\dot{\boldsymbol{\rho}}\right]_{\mathcal{M}}(t)\right\|$$

Relative motion is simulated starting from two different initial relative position for every target initial condition. More specifically, a +V-bar and a +R-bar point were chosen, $1\,\mathrm{km}$ away from the target, with zero relative velocity in both cases.



(a) Projection on $\hat{\boldsymbol{\imath}}_m$-$\hat{\boldsymbol{k}}_m$ plane. (b) Projection on $\hat{\boldsymbol{\jmath}}_m$-$\hat{\boldsymbol{k}}_m$ plane.

Figure 3: Target halo orbit.

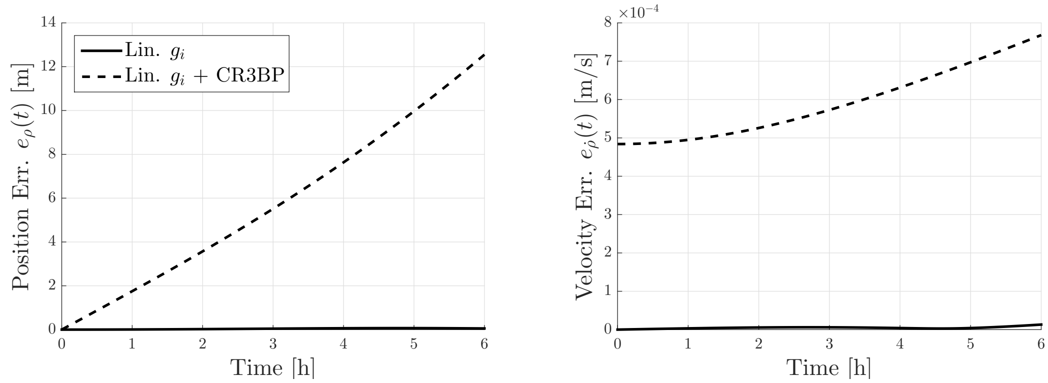
Table 1: Initial target position and velocity expressed in $\mathcal{M}$ (normalized units).

	$\boldsymbol{r}_{mt} \cdot \hat{\boldsymbol{\imath}}_m$	$\boldsymbol{r}_{mt} \cdot \hat{\boldsymbol{\jmath}}_m$	$\boldsymbol{r}_{mt} \cdot \hat{\boldsymbol{k}}_m$	$\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{\imath}}_m$	$\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{\jmath}}_m$	$\left[\dot{\boldsymbol{r}}_{mt}\right]_{\mathcal{M}} \cdot \hat{\boldsymbol{k}}_m$
C1	0.0182	0	0.1821	0	-1.1015	0
C2	-0.0129	-0.0543	0.0258	-0.6272	1.8163	-4.4384
C3	-0.0160	-0.0021	-0.0218	0.8374	8.8544	-0.5448
C4	0.0030	0.0591	0.0470	0.8013	0.7355	3.8722

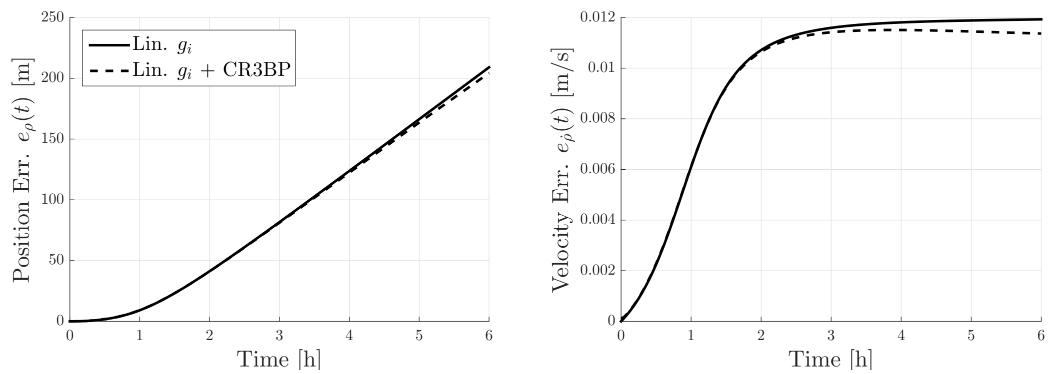
Simulation results are shown in Fig. 4 and 5. The linearized equations set show the best performance in almost all the simulations. However, for the initial condition C3 the linearized set with the CR3BP simplification produced less error. Nevertheless, the difference between the two simplified sets of equations is minimal, especially during the first 3 hours, approximately the interval of interest for the terminal phase of a rendezvous mission. With reference to this type of scenario, we can conclude that the both the simplified sets can be used for the preliminary design of relative guidance and navigation systems, provided that the mission lasts less than 1-2 hours, since within this period the error remains reasonably bounded. In addition, it must be noted that in a real scenario the navigation filters will prevent the error growth introduced by the linearized equations, by proper integration of the measurements coming from the chaser sensor suite, and from the target if a cooperative scenario is considered.

CONCLUSIONS

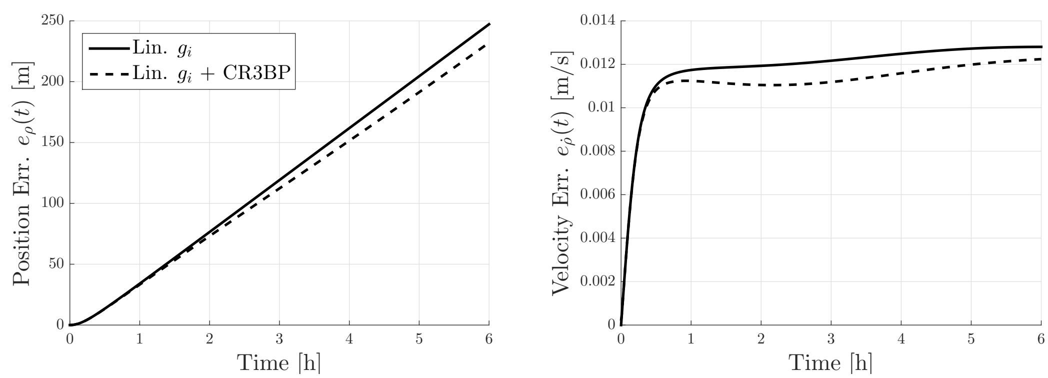
A set of nonlinear time-varying equations for the description of relative motion in the LVLH frame in the restricted three-body problem scenario was derived. The set presents several time-varying parameters, depending on the Earth-Moon motion and on the target motion around the Moon. The latter require the computation of the target angular velocity and acceleration vectors during its motion around the Moon. An analytical expression for these quantities was provided, based on the position, velocity, acceleration, and jerk of the target with respect to the Moon. Possible simplifications for the derived equations of relative motion were discussed. These involve the relaxation of hypothesis on the Earth-Moon motion, or the linearization of the nonlinear terms appearing



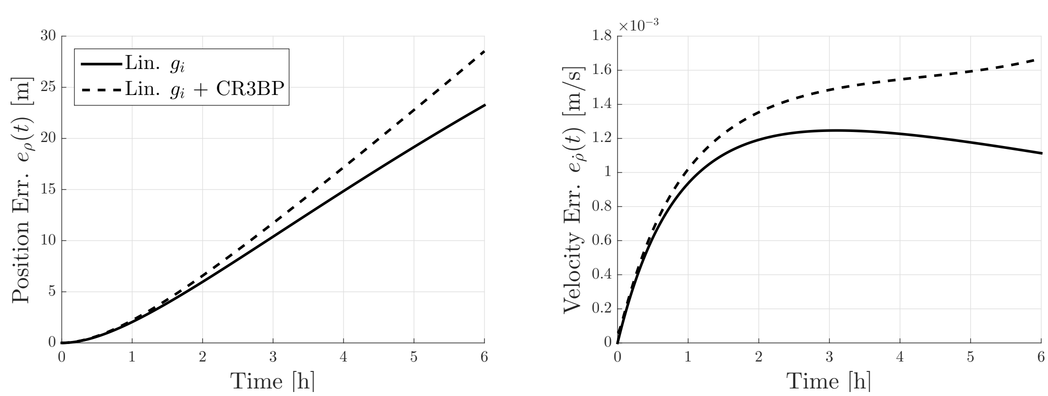
(a) Target initial position and velocity C1.



(b) Target initial position and velocity C2.

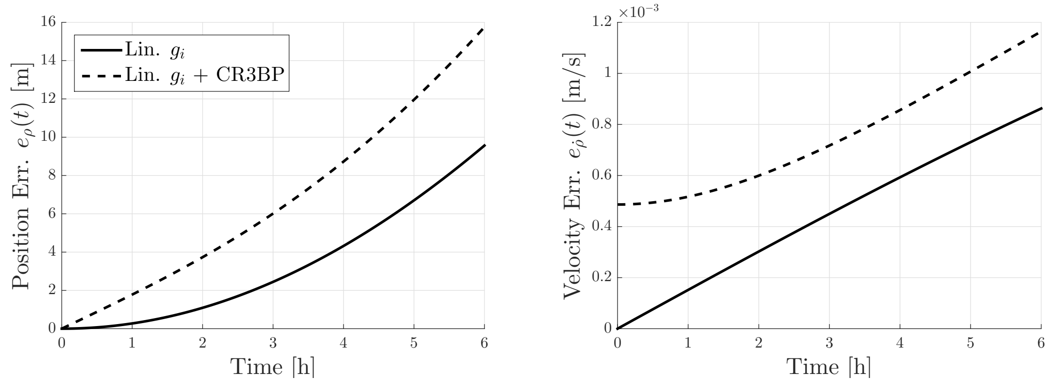


(c) Target initial position and velocity C3.

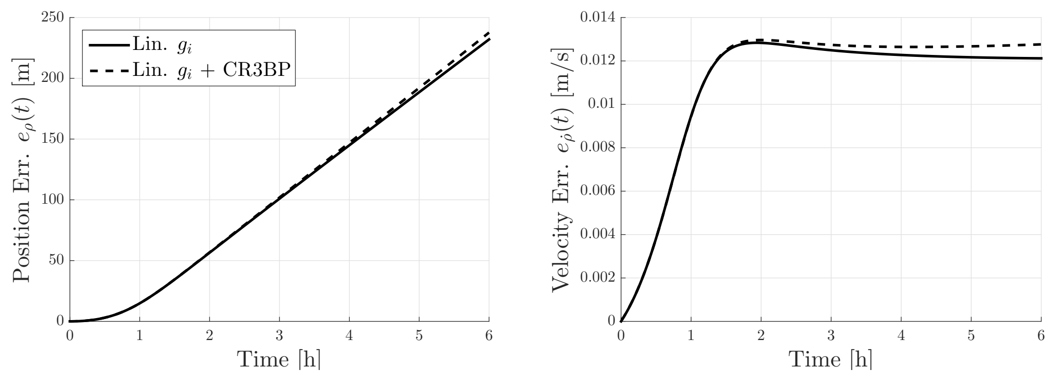


(d) Target initial position and velocity C4.

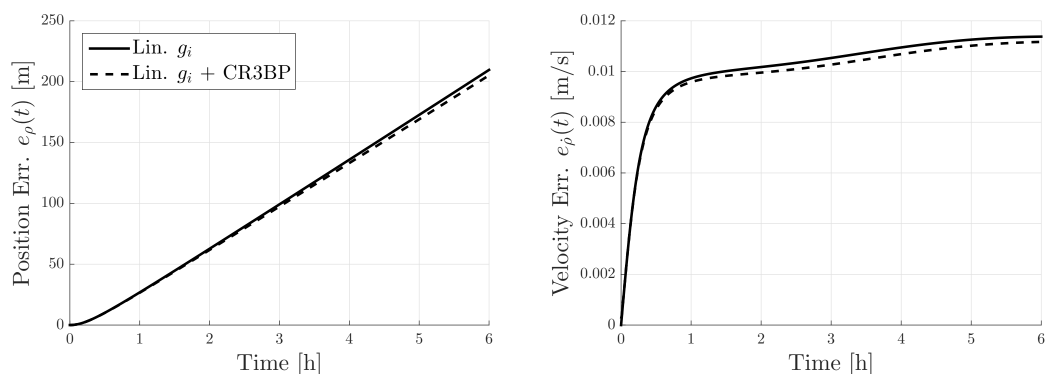
Figure 4: Simplified equations set comparison, +V-bar point: $\boldsymbol{\rho} = (1\,\mathrm{km})\hat{\boldsymbol{\imath}}$, $\left[\,\dot{\boldsymbol{\rho}}\,\right]_{\mathcal{L}} = \mathbf{0}$.



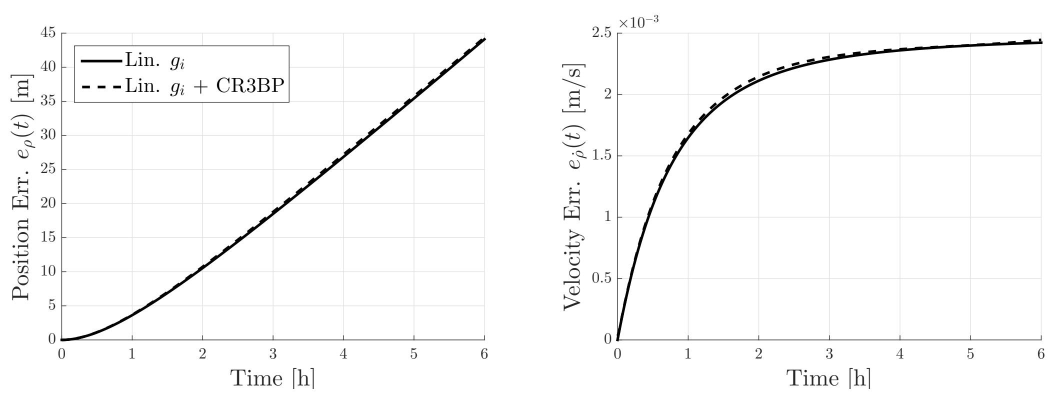
(a) Target initial position and velocity C1.



(b) Target initial position and velocity C2.



(c) Target initial position and velocity C3.



(d) Target initial position and velocity C4.

Figure 5: Simplified equations set comparison, +R-bar point: $\boldsymbol{\rho} = (1\,\mathrm{km})\hat{\boldsymbol{k}}$, $\left[\,\dot{\boldsymbol{\rho}}\,\right]_{\mathcal{L}} = \mathbf{0}$.

in the equations, more specifically of the gravitational accelerations. A preliminary comparison showed that these simplifications may be used in order to ease the development of relative guidance systems, provided that proper navigation filter are designed in order to prevent the error growth due to the linearization error.

REFERENCES

[1] The International Space Exploration Coordination Group, "The global exploration roadmap," www.nasa.gov/sites/default/files/files/GER-2013_Small.pdf, 2013. [Online; accessed 02-Aug-2017].

[2] M. Landgraf, J. Carpenter, and H. Sawada, "HERACLES concept - An international lunar exploration study," *Annual Meeting of the Lunar Exploration Analysis Group*, Vol. 1863 of *LPI Contributions*, Columbia (MD), USA, 2015, p. 2039.

[3] N. Murakami, S. Ueda, T. Ikenaga, M. Maeda, T. Yamamoto, and H. Ikeda, "Practical rendezvous scenario for transportation missions to cis-lunar station in Earth-Moon L2 halo orbit," *Proc. 25th International Symposium on Space Flight Dynamics*, Munich, Germany, 2015.

[4] S. F. R. Carnà, L. Bucci, and M. Lavagna, "Earth-Moon multiPurpose orbiting infrastructure," *Advances in the Astronautical Sciences*, Vol. 158, 2016, pp. 1637–1658. Paper AAS 16-488.

[5] W. Fehse, *Automated Rendezvous and Docking of Spacecraft*. Cambridge University Press, 2003.

[6] J. Tschauner and P. Hempel, "Rendezvous zu einem in elliptischer Bahn umlaufenden Ziel [Rendezvous with a target in an elliptical orbit]," *Astronautica Acta*, Vol. 11, No. 2, 1965, pp. 104–109.

[7] W. H. Clohessy and R. S. Wiltshire, "Terminal guidance system for satellite rendezvous," *Journal of the Aerospace Sciences*, Vol. 27, No. 9, 1960, pp. 653–658.

[8] R. J. Luquette and R. M. Sanner, "A nonlinear, six-degree of freedom, precision formation control algorithm, based on restricted three body dynamics," *Advances in the Astronautical Sciences*, Vol. 113, 2003, pp. 105–114. Paper AAS 03-007.

[9] H. Peng, J. Zhao, Z. Wu, and W. Zhong, "Optimal periodic controller for formation flying on libration point orbits," *Acta Astronautica*, Vol. 69, No. 7, 2011, pp. 537–550.

[10] M. Li, H. Peng, and W. Zhong, "Optimal control of loose spacecraft formations near libration points with collision avoidance," *Nonlinear Dynamics*, Vol. 83, No. 4, 2016, pp. 2241–2261.

[11] G. Franzini, M. Tannous, and M. Innocenti, "Spacecraft relative motion control using the state-dependent Riccati equation technique," *Proc. 10th International ESA Conference on Guidance, Navigation, and Control Systems*, Salzburg, Austria, 2017.

[12] C. F. Yoder, "Astrometric and geodetic properties of Earth and the Solar system," *Global Earth Physics: A Handbook of Physical Constants* (T. Ahrens, ed.), American Geophysical Union, 1995.

[13] S. Casotto, "The equations of relative motion in the orbital reference frame," *Celestial Mechanics and Dynamical Astronomy*, Vol. 124, No. 3, 2016, pp. 215–234.

[14] W. Koon, M. Lo, J. Marsden, and S. Ross, *Dynamical Systems, the Three-Body Problem and Space Mission Design*. Marsden Books, 2011.

[15] D. J. Grebow, M. T. Ozimek, K. C. Howell, and D. C. Folta, "Multibody orbit architectures for lunar south pole coverage," *Journal of Spacecraft and Rockets*, Vol. 45, No. 2, 2008, pp. 344–358.