ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

DOTTORATO DI RICERCA IN

Matematica

Ciclo XXIX

Settore Concorsuale di afferenza: 01/A3

Settore Scientifico disciplinare: MAT/05

Harnack Inequality and Fundamental Solution for Degenerate Hypoelliptic Operators

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ESAME FINALE ANNO 2017

Contents

Introduction

The aim of this thesis is to give a contribution to the theory of subelliptic operators. We study a class of real second-order PDOs $\mathcal{L}$ in divergence form on $\mathbb{R}^N$ of the following type

$$\mathcal{L} = \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(V(x) A(x) \frac{\partial}{\partial x_j} \right), \tag{1}$$

where $V(x) > 0$ and the matrix $A(x)$ is symmetric and positive semi-definite for every $x \in \mathbb{R}^N$. Further assumptions of the regularity of the coefficients $A(x)$ and $V(x)$ will be clarified later. The above class of PDOs comprises sub-Laplacians on Carnot groups, subelliptic Laplacians on arbitrary Lie groups, elliptic operators in divergence form, as well as the Laplace-Beltrami operator on Riemannian manifolds.

We are interested in establishing *Harnack Inequalities* related to $\mathcal{L}$ in various contexts.

As a first result of the thesis, we describe how we obtained a non-invariant Harnack inequality for (1), passing through a Strong Maximum Principle, following the ideas by Bony in his celebrated paper [16]. In doing so, we require $\mathcal{L}$ to have C^∞ coefficients and to satisfy the following hypotheses:

(NTD) $\mathcal{L}$ is *non-totally degenerate at every point of* $\mathbb{R}^N$, or equivalently (recalling that $A(x)$ is symmetric and positive semi-definite),

$$\text{trace}(A(x)) > 0, \quad \text{for every } x \in \mathbb{R}^N.$$

(HY) $\mathcal{L}$ is *C^∞-hypoelliptic in every open subset of* $\mathbb{R}^N$.

(HY)$_\varepsilon$ *There exists $\varepsilon > 0$ such that $\mathcal{L} - \varepsilon$ is C^∞-hypoelliptic in every open subset of* $\mathbb{R}^N$.

Under these assumptions we prove the following:

Harnack Inequality: *For every connected open set $O \subseteq \mathbb{R}^N$ and every compact subset K of O, there exists a constant $M = M(\mathcal{L}, O, K) \geq 1$ such that*

$$\sup_K u \leq M \inf_K u,$$

for every non-negative $\mathcal{L}$-harmonic function u in O.

Before presenting some further details on our approach (and the roles of our assumptions (NTD), (HY) and $(\text{HY})_\varepsilon$), we recall some references from the literature on *Maximum Principles* and the *Harnack inequality* for operators as in (1).

Starting from the 50's/60's seminal works by De Giorgi [28], Moser [80], Nash [81], Serrin [89], the literature on Harnack inequalities and on regularity issues for divergence-form operators like ours has widely grown in the *uniformly-elliptic case*. The same is true of the vast literature on Hörmander operators, starting from the 60's/70's pioneering papers by Bony [16], Fefferman and Phong [35, 36], Folland [37], Folland and Stein [38], Hörmander [58], Rothschild and Stein [86].

It is during the 80's that many important results on degenerate-elliptic operators under the divergence-form (1) were established, with a special emphasis to the mentioned Harnack Inequality and Maximum Principles; see e.g. the results by: Jerison and Sánchez-Calle [60]; Chanillo and Wheeden [21]; Fabes, Jerison and Kenig [31, 32]; Fabes, Kenig and Serapioni [33]; Franchi and Lanconelli [42, 43]; Gutiérrez [51].

As for the assumptions made in the previous papers on the involved PDOs, in [60] a suitable *subellipticity* hypothesis is assumed, whereas in the other cited papers, operators like ours are considered with very low regularity assumptions on the coefficients, but under the hypothesis that the degeneracy of the principal matrix be controlled on both sides by some appropriate weights: for example, by Muckenhoupt-type weights, [31, 32, 33, 51]; or by doubling weights, [21]; or by a family of diagonal vector fields, [42]. The Muckenhoupt-type condition on the degeneracy is still one of the most frequently assumed hypotheses in obtaining Harnack theorems: see e.g. recent investigations in [27, 74, 93]; see also [64] for a Harnack inequality in the case of the so-called X-elliptic weight condition.

Another type of assumption can be made in facing with potential-theoretic problems for operators $\mathcal{L}$: indeed, very recently a systematic study of the Potential Theory for the harmonic/subharmonic functions related to $\mathcal{L}$ has been carried out in the series of papers [1, 7, 13, 14], under the assumption that $\mathcal{L}$ possesses a smooth, global and positive *fundamental solution*. For the use of the fundamental solution in obtaining the Harnack Inequality for Hörmander sums of squares, see: Citti, Garofalo and Lanconelli [23]; Garofalo and Lanconelli [46, 47]; Pascucci and Polidoro [83, 84]; see also the recent survey by Bramanti, Brandolini, Lanconelli and Uguzzoni, [17], for the same relevant use of the fundamental solution for heat PDOs structured on Hörmander vector fields.

After this long excursus of related references, we now describe our first result. Thanks to the assumptions (NTD) and (HY), we are able to recover Bony's approach in establishing the Strong Maximum Principle for $\mathcal{L}$. Once this has been done, we obtain the Harnack inequality for $\mathcal{L}$ by means of the well-behaved properties of the *Green function* g_ε related to $\mathcal{L} - \varepsilon$: it is at this point that hypothesis $(\text{HY})_\varepsilon$ is required. Some Potential Theoretic results are also used in a crucial way. All this is presented in Chapter 2.

As our assumptions are only (NTD), (HY), $(\text{HY})_\varepsilon$ above, we want to stress that in this thesis we do not require $\mathcal{L}$ to be a Hörmander operator; in particular in Chapter 2 we will show that the Strong Maximum Principle and the Harnack inequality hold true in the *infinitely-degenerate* case as well, nor we make any assumption of subellipticity or Muckenhoupt-weighted degeneracy (see Example 2.1.2); furthermore, we do not assume the existence of any global fundamental solution for $\mathcal{L}$: summing up, *our results are not contained in any of the aforementioned papers*.

As a counterpart of allowing for less assumptions (our hypotheses are, broadly speaking, more qualitative than quantitative), we will have to renounce to lower the regularity of the coefficients (as in Moser-type techniques) or to obtain an *invariant* Harnack inequality (which is roughly put, an inequality with a constant independent of the radius of the balls involved). The main novelty of our work is to obtain the Strong Maximum Principle and the Harnack Inequality for *hypoelliptic* operators with infinitely degenerate coefficients, allowing some eigenvalues of the principal matrix of $\mathcal{L}$ to vanish at infinite order, as in Fediĭ operator, [34],

$$\mathcal{F} := \frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/x_1^2)\,\frac{\partial}{\partial x_2}\Big)^2 \quad \text{in } \mathbb{R}^2$$

(see also Example 2.1.2 for other models of infinitely-degenerate PDOs to which our theory applies). Note that this operator violates the Hörmander maximal rank condition on $\{x_1 = 0\}$, it does not satisfy subelliptic estimates, and its quadratic form does not satisfy Muckenhoupt-type weight conditions. Yet, $\mathcal{F}$ fulfills a maximum propagation principle as one can verify straightforwardly: this is not by chance, indeed, by means of a deep Control Theoretic result by Amano [3] using (HY) and (NTD), we show that a Maximum Propagation holds along the vector fields $X_1, \dots, X_N$ *associated with the rows of the matrix* $A(x)$. The mentioned (long-forgotten) result by Amano ensures that the *sole hypoellipticity of* $\mathcal{L}$ (plus (NTD)) guarantees that the reachable set of $X_1, \dots, X_N$ is the whole space. It is for this reason that the ideas of Bony can be used.

Now we pass to the second main result of the thesis: a Harnack inequality for $\mathcal{L}$ under low regularity assumption.

Currently, it is known that the natural framework for Harnack-type theorems is the setting of *doubling metric spaces*: see e.g., Aimar, Forzani and Toledano [2]; Barlow and Bass [4]; Di Fazio, Gutiérrez and Lanconelli [30]; Grigor'yan and Saloff-Coste [50]; Gutiérrez and Lanconelli [52]; Hebisch and Saloff-Coste [54]; Indratno, Maldonado and Silwal [59]; Kinnunen, Marola, Miranda and Paronetto [62]; Mohammed [77]; Saloff-Coste [88]. In this framework it appears that the Harnack Inequality holds true whenever some axiomatic assumptions are satisfied: a *doubling condition* and a *Poincaré inequality*.

We follow this trend of research and we make the following assumptions:

(E) There exists a family of locally Lipschitz-continuous vector fields $X = \{X_1, \ldots, X_m\}$ on Euclidean space $\mathbb{R}^N$, and two constants $\lambda, \Lambda > 0$ such that

$$\lambda \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \le \langle A(x)\xi, \xi \rangle \le \Lambda \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2, \qquad \forall\ x, \xi \in \mathbb{R}^N,$$

and we consider the *metric of Carnot-Carathéodory* d related to the family X.

If μ is the measure associated with $\mathcal{L}$: $\mathrm{d}\mu = V(x)\mathrm{d}x$ with V as in (1),

(D) $(\mathbb{R}^N, d, \mu)$ is a *doubling metric space*, that is, there exists $Q > 0$ such that

$$\mu(B_d(x, 2r)) \le 2^Q \mu(B_d(x, r)), \qquad \text{for every } x \in \mathbb{R}^N \text{ and every } r > 0.$$

(P) The following *global Poincaré inequality* is satisfied: there exists a constant $C_P > 0$ such that, for every $x \in \mathbb{R}^N$, $r > 0$ and every u which is C^1 in a neighborhood of $B_{2r}(x)$ one has

$$⨍_{B_r(x)} \left| u - u_{B_r(x)} \right| \mathrm{d}\mu \le C_P\, r ⨍_{B_{2r}(x)} |Xu| \, \mathrm{d}\mu.$$

(There is also a further technical topological assumption on $(\mathbb{R}^N, d)$, see Section 4.2).

Under these assumptions, we are able to prove in Chapter 4 the following result:

Non-Homogeneous Invariant Harnack Inequality: *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $g \in L^p(\Omega)$, with $p > Q/2$. Then there exists a structural constant $C > 0$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in* **(E)** *and on p) such that, for every d-ball $B_R(x)$ satisfying $\overline{B_{4R}(x)} \subset \Omega$, one has*

$$\sup_{B_R(x)} u \le C \left(\inf_{B_R(x)} u + R^2 \left(⨍_{B_{4R}(x)} |g|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \right),$$

for any nonnegative W^1_{loc}-weak solution u of $-\mathcal{L}u = g$ in Ω.

We provide a very brief list of related references. In the setting of doubling metric spaces, several authors have dealt with operators related to a family of uniformly subelliptic vector fields: see e.g. Kogoj and Lanconelli [65, 66] where a Harnack inequality was proved for the equation $\mathcal{L}u = 0$; moreover, in [52] Gutiérrez and Lanconelli have showed maximum principle for these operators with lower order terms and, in the case of dilation invariant vector fields, a non-homogeneous Harnack inequality; a yet improved result was obtained by Uguzzoni in [92] where, removing the assumption on the dilation invariance, the author has showed a *local* Harnack inequality under hypothesis of *local doubling condition* and *Poincaré inequality*. The result by Uguzzoni gives a non-homogeneous and invariant Harnack inequality with the only drawback that inequality is *local*, in that it holds for *small radii* and for centres confined to a *compact set*. In this framework our result above Harnack inequality is a further improvement:

it is a *non-homogeneous invariant* Harnack inequality, *with no restriction on the radii and on the centres*.

The most important consequences of this Harnack inequality are (inner and boundary) Hölder estimates and the construction of the Green function on bounded domains. This is accomplished in Section 4.4.1 and Section 5.1.

As a part of our future investigations, the invariant Harnack inequality will be the main tool to show the existence of a *global fundamental solution* for $\mathcal{L}$, as outlined in Chapter 5.

Before giving an outline of the thesis, we would like to underline the crucial role of *Green functions* in this thesis. In Chapter 2 we have been able to construct a Green function thanks to the assumption of *hypoellipticity* of $\mathcal{L}$, and then we have used the Green function (by means of techniques of Potential Theory) in order **to obtain** a non-invariant homogeneous Harnack inequality; with a completely different approach, in Chapter 5 we have proved the existence of a Green function for $\mathcal{L}$ **as a by-product** of the Harnack inequality. Hence, in the framework of Harmonic spaces, the Green function is a **tool** to prove the Harnack inequality; conversely, in the context of doubling metric spaces, the Green function related to $\mathcal{L}$ is an important **consequence** of the Harnack inequality.

Outline of the Thesis.

We conclude the introduction by giving a general outline of the thesis and a short description of our main results.

In Chapter 1 we give some results of Potential Theory: we consider a linear second order PDO $\mathcal{L}$ as in (1) and we assume that $\mathcal{L}$ is endowed with a positive fundamental solution, defined out of the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, with some well-behaved properties. We characterize the solutions to $\mathcal{L}u = 0$ as fixed points of suitable mean-value operators with non-trivial kernels (Koebe-type Theorem) and our aim is to study the topology of family of $\mathcal{L}$-harmonic functions. For this purpose, we obtain a generalization of a classical theorem of Montel, for holomorphic functions, in the subelliptic setting of families of solutions u to $\mathcal{L}u = 0$. Finally, we will show a Heine-Borel theorem for the space of the $\mathcal{L}$-harmonic functions.

In Chapter 2 we prove one of the most important results of the thesis. We consider a PDO $\mathcal{L}$ as in (1) and we prove the Harnack inequality for $\mathcal{L}$ mentioned above. To this aim, a first step is to show the *solvability* of the Dirichlet problem in order to obtain the existence of the Green function for $\mathcal{L}$; then we prove a *Weak Harnack inequality* and we use these results, together with means of Potential Theory, to obtain the Harnack inequality.

In Chapter 3 we will show some further Potential Theoretic results, closely related to the arguments in Chapter 2. In particular, we use the Harnack inequality and the solvability of the Dirichlet problem for $\mathcal{L}$ to prove integral representation theorems and a characterizations of superharmonic functions related to $\mathcal{L}$.

In Chapter 4 we will show another main results of the thesis: we let $\mathcal{L}$ in (1) be associated with a family of vector fields, and we use the Carnot-Carathéodory metric d related to this family. We suppose that $(\mathbb{R}^N, d, \mu)$ is a *doubling metric space*, where $\mathrm{d}\mu$ is $V(x)\mathrm{d}x$, and we further require a global Poincaré inequality. Our study is focused on length spaces, properties of CC metric and Sobolev spaces related to a family of vector fields; several contributions have already been given in the literature for the study of these geometric conditions in the context of PDEs modeled on vector fields, see e.g. Hajłasz and Koskela [53]. In this framework, we prove the mentioned non-homogeneous invariant Harnack inequality, with consequent Hölder-continuous estimates, using the Moser-type technique.

In Chapter 5 we give some results of our future investigations. We use the non-homogeneous Harnack inequality (proved in Chapter 4) to construct a Green function on the bounded domains, following the ideas of Fabes, Jerison and Kenig in [31] (see also Uguzzoni in [92]). Our future aim will be to prove the existence of a *global fundamental solution* for $\mathcal{L}$. Thus we construct a suitable basis for the d-topology on $\mathbb{R}^N$: the idea is to consider the Green functions related to this basis and then, by an exhaustion argument, to show the existence of a global non-negative fundamental solution, continuous out the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, using the invariance of the Harnack inequality.

Chapter 1

Some Potential Theoretic results for subelliptic operators

In this chapter we provide for operators $\mathcal{L}$ in divergence form on $\mathbb{R}^N$ a subelliptic version of a remarkable result, due to P. Koebe, characterizing harmonic functions as fixed points of suitable mean-value integral operators. The presence of non-trivial and possibly unbounded kernels (see (1.1.6)) in this mean-value operators is one of the main novelty with respect to the classical elliptic case. Then we study the topology of the harmonic space related to $\mathcal{L}$, and to this aim we will show a generalization of a classical theorem of Montel in the subelliptic setting of families of solutions u to $\mathcal{L}u = 0$.

1.1 Main assumptions and notation

We need to fix some notations. We shall be concerned with linear second order PDOs in $\mathbb{R}^N$ of the form

$$\mathcal{L} := \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j} \right), \quad x \in \mathbb{R}^N, \tag{1.1.1}$$

where V is a C^1 positive function on $\mathbb{R}^N$, the matrix $A(x) := (a_{i,j}(x))_{i,j \le N}$ is symmetric and positive *semi*-definite at every point $x \in \mathbb{R}^N$, and it has C^1 entries.

Given $\alpha > 0$, if H^α is the α-dimensional Hausdorff measure on $\mathbb{R}^N$, we set

$$\mathrm{d}V^\alpha := V\, \mathrm{d}H^\alpha \tag{1.1.2}$$

to denote the absolutely continuous measure with respect to H^α with density V.

We shall be interested only in the cases $\alpha = N$ and $\alpha = N - 1$.

Our main assumption on $\mathcal{L}$ is the following:

(S) We assume that $\mathcal{L}$ is equipped with a *positive global fundamental solution*

$$\Gamma : D = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \,:\, x \ne y\} \longrightarrow (0, \infty)$$

with the following properties:

(a) Γ is (at least) of class C^3 on D and $\nabla\Gamma(x,\cdot) \neq 0$ on $\mathbb{R}^N \smallsetminus \{x\}$;

(b) for every fixed $x \in \mathbb{R}^N$, we have $\lim\limits_{y\to x} \Gamma(x,y) = \infty$ and $\lim\limits_{y\to\infty} \Gamma(x,y) = 0$;

(c) $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^{2N})$ and, for every $x \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} \Gamma(x,y)\, \mathcal{L}\varphi(y)\, \mathrm{d}V^N(y) = -\varphi(x), \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}). \tag{1.1.3}$$

If $\Omega \subseteq \mathbb{R}^N$ is open, we say that u is *$\mathcal{L}$-harmonic* in Ω if $u \in C^2(\Omega,\mathbb{R})$ and $\mathcal{L}u = 0$ in Ω. The set of the $\mathcal{L}$-harmonic functions in Ω will be denoted by $\mathcal{H}(\Omega)$.

Given any $r > 0$ and any $x \in \mathbb{R}^N$, we introduce the super-level set of Γ

$$\Omega_r(x) := \left\{y \in \mathbb{R}^N \,:\, \Gamma(x,y) > 1/r\right\} \cup \{x\}, \tag{1.1.4}$$

that will be referred to as the *Γ-ball of center x and radius r*.

Observe that, from property **(b)** of the assumption **(S)**, we derive that every Γ-ball $\Omega_r(x)$ is a bounded open neighborhood of x and that

$$\textstyle\bigcap_{r>0} \Omega_r(x) = \{x\}, \qquad \bigcup_{r>0} \Omega_r(x) = \mathbb{R}^N. \tag{1.1.5}$$

Moreover, from property **(a)** we infer that $\partial\Omega_r = \{y : \Gamma(x,y) = 1/r\}$ is a C^3-manifold. (Throughout the chapter, a C^1-assumption on $\partial\Omega_r(x)$ would actually suffice; we use the C^3 hypothesis on Γ only in the proof of Theorem 1.2.3.)

Let $x \in \mathbb{R}^N$ and let us consider the functions, defined for $y \neq x$,

$$\Gamma_x(y) := \Gamma(x,y), \qquad K(x,y) := \frac{\big\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\big\rangle}{|\nabla\Gamma_x(y)|}. \tag{1.1.6}$$

If u is a continuous function on $\partial\Omega_r(x)$, we introduce the following mean-value operator

$$m_r(u)(x) := \int_{\partial\Omega_r(x)} u(y)\, K(x,y)\, \mathrm{d}V^{N-1}(y). \tag{1.1.7}$$

Note that the measure $K(x,y)\,\mathrm{d}V^{N-1}(y)$ is non-negative since A is positive semi-definite (we shall also prove that $\partial\Omega_r(x)$ has measure 1 w.r.t. $K(x,y)\,\mathrm{d}V^{N-1}(y)$).

We end this notational section by recalling some terminology from the theory of topological vector spaces. We only recall, for convenience of reading, a few definitions, referring to [87, Chapter 1] for the missing ones. (This last part of the section only contains basic material, but it is meant to fix notation and definitions.)

Let V be a real vector space and let $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ be a countable family of seminorms on V which is *separating*, that is for every $x \in V \smallsetminus \{0\}$ there exists $n \in \mathbb{N}$ such that $p_n(x) \neq 0$. We denote by $\mathcal{T}(\mathcal{P})$ the smallest topology on V making any $p_n : V \to \mathbb{R}$ continuous and turning V into a topological vector space (t.v.s., for short). Since $\mathcal{P}$ is at most countable, the topological space

$(V, \mathcal{T}(\mathcal{P}))$ is first-countable, hence the convergent sequences characterize $\mathcal{T}(\mathcal{P})$. For example, a sequence $\{x_k\}_k$ in V converges to $x \in V$ w.r.t. $\mathcal{T}(\mathcal{P})$ if and only if, for every fixed $n \in \mathbb{N}$, one has $\lim_{k\to\infty} p_n(x_k - x) = 0$. Moreover, $(V, \mathcal{T}(\mathcal{P}))$ is a locally convex t.v.s., and a base of convex neighborhoods of 0 is given by

$$\{y \in V : p_n(y) < 1/m\}, \quad n, m \in \mathbb{N}.$$

It is well-known that $\mathcal{T}(\mathcal{P})$ coincides with the *metric* topology induced by the distance d on V defined by

$$d(x,y) := \max_{n\in\mathbb{N}} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \qquad x, y \in V. \tag{1.1.8}$$

Clearly, $d(x,y) \le 1$ for every $x, y, \in V$, thus boundedness in the metric space (V,d) is of no relevance. The relevant notion is, instead, the following one.

Definition 1.1.1. A set $E \subseteq V$ is said to be *bounded in the t.v.s.* $(V, \mathcal{T}(\mathcal{P}))$ (or $\mathcal{T}(\mathcal{P})$*-bounded*, for short) if, for every open neighborhood Ω of 0, there exists $s > 0$ such that

$$E \subseteq s\Omega := \{s\,\omega : \omega \in \Omega\}.$$

It is easy to verify that $E \subseteq V$ is $\mathcal{T}(\mathcal{P})$-bounded if and only if every function $p_n|_E : E \to \mathbb{R}$ is bounded (by a constant possibly depending on $n \in \mathbb{N}$).

Here we are only interested in the topologies induced on $V := C(\Omega)$ by the following families of seminorms. We say that a sequence of bounded open sets Ω_n (in the usual Euclidean metric of $\mathbb{R}^N$) is an exhaustion of the open set Ω if

$$\textstyle\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega, \qquad K_n := \overline{\Omega_n} \subset \Omega_{n+1}, \quad \forall\ n \in \mathbb{N}. \tag{1.1.9}$$

With this notation, for every $n \in \mathbb{N}$, we set, for $f \in C(\Omega)$,

$$\begin{aligned} p_n(f) &:= \int_{K_n} |f(x)|\,\mathrm{d}H^N(x), \qquad && \mathcal{P} := \{p_n\}_{n\in\mathbb{N}}, \\ q_n(f) &:= \sup_{x\in K_n} |f(x)|, && \mathcal{Q} := \{q_n\}_{n\in\mathbb{N}}. \end{aligned} \tag{1.1.10}$$

We say that $\mathcal{T}(\mathcal{P})$ and $\mathcal{T}(\mathcal{Q})$ are, respectively, the L^1_{loc}*-topology*, and the L^∞_{loc}*-topology* of $C(\Omega)$. Indeed, from what we recalled above, given functions $f_n, f \in C(\Omega)$ we have $\lim_{n\to\infty} f_n = f$ w.r.t. $\mathcal{T}(\mathcal{P})$ (w.r.t. $\mathcal{T}(\mathcal{Q})$, respectively) if and only if, for every fixed compact set $K \subset \Omega$, one has $\lim_{n\to\infty}(f_n)|_K = f|_K$ in $L^1(K)$ (in $L^\infty(K)$, respectively). This also shows that $\mathcal{T}(\mathcal{P})$ and $\mathcal{T}(\mathcal{Q})$ are independent of the exhausting sequence $\{\Omega_n\}_n$ of Ω.

Clearly, $\mathcal{T}(\mathcal{P}) \subset \mathcal{T}(\mathcal{Q})$, i.e., $\mathcal{T}(\mathcal{Q})$ is a topology (strictly) finer than $\mathcal{T}(\mathcal{P})$. Instead, we shall show that

$$\mathcal{T}(\mathcal{P}) \cap \mathcal{H}(\Omega) = \mathcal{T}(\mathcal{Q}) \cap \mathcal{H}(\Omega).$$

Thanks to the above mentioned characterization of boundedness in terms of the seminorms, we recognize that, given $\mathcal{F} \subseteq C(\Omega)$,

(i) $\mathcal{F}$ is $\mathcal{T}(\mathcal{Q})$-bounded if and only if, for every compact set $K \subset \Omega$, there exists a constant $M(K) > 0$ such that $\sup_K |f| \leq M(K)$, for every $f \in \mathcal{F}$;

(ii) $\mathcal{F}$ is $\mathcal{T}(\mathcal{P})$-bounded if and only if, for every compact set $K \subset \Omega$, there exists a constant $M(K) > 0$ such that $\int_K |f| \,\mathrm{d}H^N \leq M(K)$, for every $f \in \mathcal{F}$.

One of the main results of this chapter will be to relate the notion of a normal family to that of precompactness, using a Montel-type result.

1.2 Integral representations and Koebe-type Theorem

In this section we want to show representation formulas for $\mathcal{L}$, with respect to which we are assuming hypothesis **(S)**.

Thanks to the surface mean-value formula for $\mathcal{L}$, then we can characterize the harmonic functions as fixed points of the mean-value operator in (1.1.7).

Lemma 1.2.1. *Let notation be as in Section 1.1. For every function u of class C^2 on an open set containing the Γ-ball $\overline{\Omega_r(x)}$, we have*

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r}\Big)\, \mathcal{L}u(y)\, \mathrm{d}V^N(y). \tag{1.2.1}$$

We shall refer to (1.2.1) *as the* surface mean-value formula for $\mathcal{L}$. *As a consequence, a function u of class C^2 in the open set $\Omega \subseteq \mathbb{R}^N$ is $\mathcal{L}$-harmonic if and only if*

$$u(x) = m_r(u)(x), \quad \textit{for every } \Gamma\textit{-ball such that } \overline{\Omega_r(x)} \subset \Omega. \tag{1.2.2}$$

Formula (1.2.1) extends the result in [13, Theorem 3.3] to our operators (1.1.1), a class which strictly contains the PDOs considered in [13]. We shall prove Lemma 1.2.1 by exploiting the quasi-divergence form (1.1.1) of $\mathcal{L}$ and integration by parts.

Proof. To begin with, let $\Omega \subset \mathbb{R}^N$ be a bounded open set, with $\partial\Omega$ of class C^1. If $u, d \in C^2(\overline{\Omega}, \mathbb{R})$, we can apply the Divergence Theorem in order to derive, by exploiting the quasi-divergence form (1.1.1) of $\mathcal{L}$ and the symmetry of the matrix A, the following Green-type identity (see also the notation in (1.1.2)):

$$\int_{\Omega} \big(u\,\mathcal{L}d - d\,\mathcal{L}u\big)\, \mathrm{d}V^N = \int_{\partial\Omega} \Big(u\,\langle A\nabla d, N_\Omega\rangle - d\,\langle A\nabla u, N_\Omega\rangle\Big)\, \mathrm{d}V^{N-1}. \tag{1.2.3}$$

Here N_Ω denotes the exterior normal vector on $\partial\Omega$. The choice $d \equiv -1$ yields

$$\int_{\Omega} \mathcal{L}u \, \mathrm{d}V^N = \int_{\partial\Omega} \langle A\nabla u, N_\Omega\rangle \, \mathrm{d}V^{N-1}, \quad \forall\ u \in C^2(\overline{\Omega}, \mathbb{R}). \tag{1.2.4}$$

This proves, in particular, that

$$\int_{\Omega} \mathcal{L}u \, \mathrm{d}V^N = 0, \quad \text{for every } u \in C_0^2(\Omega, \mathbb{R}). \tag{1.2.5}$$

Let $x \in \mathbb{R}^N$ and $r > 0$ be fixed and consider the (regular) open set

$$\Omega_{r,\rho} := \Omega_r(x) \smallsetminus \overline{\Omega_\rho(x)}, \quad \text{for } 0 < \rho < r.$$

Let $u \in C^2(\overline{\Omega_r(x)}, \mathbb{R})$ and choose $d(y) := \Gamma_x(y)$. We are entitled to apply (1.2.3) when $\Omega = \Omega_{r,\rho}$. Recalling that $\mathcal{L}\Gamma_x = 0$ in $\mathbb{R}^N \smallsetminus \{x\}$ (see hypotheses **(S.a)** and **(S.c)** on the fundamental solution Γ), and since $N_\Omega = \mp \nabla\Gamma_x / |\nabla\Gamma_x|$ on $\partial\Omega_r(x)$ and on $\partial\Omega_\rho(x)$, respectively, we obtain

$$\begin{aligned} &- \int_{\Omega_{r,\rho}} \Gamma_x \, \mathcal{L}u \, \mathrm{d}V^N \\ &= - \int_{\partial\Omega_r(x)} u \, \frac{\langle A\nabla\Gamma_x, \nabla\Gamma_x \rangle}{|\nabla\Gamma_x|} \, \mathrm{d}V^{N-1} - \frac{1}{r} \int_{\partial\Omega_r(x)} \langle A\nabla u, N_{\Omega_r(x)} \rangle \, \mathrm{d}V^{N-1} + \\ &\quad + \int_{\partial\Omega_\rho(x)} u \, \frac{\langle A\nabla\Gamma_x, \nabla\Gamma_x \rangle}{|\nabla\Gamma_x|} \, \mathrm{d}V^{N-1} + \frac{1}{\rho} \int_{\partial\Omega_\rho(x)} \langle A\nabla u, N_{\Omega_\rho(x)} \rangle \, \mathrm{d}V^{N-1}. \end{aligned} \tag{1.2.6}$$

Here we used the fact that the exterior normal vector to the domain $\Omega_{r,\rho}$ coincides, on $\partial\Omega_r(x)$, with the exterior normal vector to $\Omega_r(x)$, whereas it coincides, on $\partial\Omega_\rho(x)$, with the opposite of the exterior normal vector to $\Omega_\rho(x)$.

If we introduce the notation

$$J_r(u)(x) := \frac{1}{r} \int_{\partial\Omega_r(x)} \langle A\nabla u, N_{\Omega_r(x)} \rangle \, \mathrm{d}V^{N-1},$$

(1.2.6) can be rewritten as follows (see also (1.1.6) and (1.1.7))

$$- \int_{\Omega_{r,\rho}} \Gamma_x \, \mathcal{L}u \, \mathrm{d}V^N = -m_r(u)(x) - J_r(u)(x) + m_\rho(u)(x) + J_\rho(u)(x). \tag{1.2.7}$$

We now aim to let ρ tend to 0 in (1.2.7).

As for the left-hand side of (1.2.7), we have $\Gamma \in L^1_{\text{loc}}(\mathrm{d}V^N)$ (indeed $L^1_{\text{loc}}(\mathrm{d}H^N) = L^1_{\text{loc}}(\mathrm{d}V^N)$ since V is positive and continuous), whence

$$- \int_{\Omega_r(x) \smallsetminus \overline{\Omega_\rho(x)}} \Gamma_x \, \mathcal{L}u \, \mathrm{d}V^N \xrightarrow{\rho \to 0} - \int_{\Omega_r(x)} \Gamma_x \, \mathcal{L}u \, \mathrm{d}V^N.$$

Moreover, we claim that the last summand in the right-hand side of (1.2.7) vanishes as $\rho \to 0$. First we observe that this is true of $H^N(\Omega_\rho(x))/\rho$; indeed, since $\Gamma \in L^1_{\text{loc}}$,

$$0 \le \frac{H^N(\Omega_\rho(x))}{\rho} = \frac{1}{\rho} \int_{\Omega_\rho(x)} \mathrm{d}H^N(y) \le \int_{\Omega_\rho(x)} \Gamma(x,y) \, \mathrm{d}H^N(y) \xrightarrow{\rho \to 0} 0,$$

in view of $\bigcap_{\rho > 0} \Omega_\rho(x) = \{x\}$. Next, thanks to (1.2.4) we have

$$\lim_{\rho \to 0} |J_\rho(u)(x)| = \lim_{\rho \to 0} \frac{\left| \int_{\Omega_\rho(x)} \mathcal{L}u \, \mathrm{d}V^N \right|}{\rho} \le \lim_{\rho \to 0} \left(\sup_{\Omega_\rho(x)} |V \mathcal{L}u| \, \frac{H^N(\Omega_\rho(x))}{\rho} \right) = 0.$$

Summing up, from (1.2.7) we derive that $\lim_{\rho \to 0} m_\rho(u)(x)$ exists and (1.2.7) gives

$$- \int_{\Omega_r(x)} \Gamma_x \, \mathcal{L}u \, \mathrm{d}V^N = - m_r(u)(x) - \frac{1}{r} \int_{\Omega_r(x)} \mathcal{L}u \, \mathrm{d}V^N + \lim_{\rho \to 0} m_\rho(u)(x). \tag{1.2.8}$$

Before we can calculate the limit of $m_\rho(u)(x)$, we need some preliminary work.

Suppose that $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ and choose $r > 0$ large enough so that the support of u is contained in $\Omega_r(x)$. With these assumptions, note that the left-hand side of (1.2.8) is equal to $-\int_{\mathbb{R}^N} \Gamma(x,y)\,\mathcal{L}u(y)\,\mathrm{d}V^N(y) = u(x)$, since Γ is a fundamental solution for $\mathcal{L}$, see (1.1.3). Moreover, the first summand of the right-hand side of (1.2.8) is null, since $u = 0$ on $\partial\Omega_r(x)$. The same is true of the second summand, thanks to (1.2.5). As a consequence, with the assumption that u is smooth and supported in $\Omega_r(x)$, (1.2.8) is equivalent to

$$u(x) = \lim_{\rho\to 0} m_\rho(u)(x). \tag{1.2.9}$$

A simple argument of cut-off functions implies that (1.2.9) also holds true for any $u \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and any $x \in \mathbb{R}^N$, when u is not necessarily compactly supported. In particular, choosing $u \equiv 1$ we get (recalling (1.1.7))

$$\lim_{\rho\to 0} \int_{\partial\Omega_\rho(x)} K(x,y)\,\mathrm{d}V^{N-1}(y) = 1, \quad \text{for every } x \in \mathbb{R}^N. \tag{1.2.10}$$

This allows us to remove the hypothesis of smoothness of u in (1.2.9) and replace it with the sole continuity of u. Indeed, if $u \in C(\Omega_r(x), \mathbb{R})$, we have

$$m_\rho(u)(x) = \int_{\partial\Omega_\rho(x)} (u(y) - u(x))\,K(x,y)\,\mathrm{d}V^{N-1}(y) + u(x) \int_{\partial\Omega_\rho(x)} K(x,y)\,\mathrm{d}V^{N-1}(y),$$

and, as $\rho \to 0$, the second summand tends to $u(x)$, due to (1.2.10). We claim that the first summand vanishes too. This is a consequence of the following argument: if u is continuous, given $\varepsilon > 0$ (since $\Omega_\rho(x)$ shrinks to $\{x\}$ as $\rho \downarrow 0$), there exists $\overline{\rho} > 0$ such that $\sup_{y\in\partial\Omega_\rho(x)} |u(y) - u(x)| < \varepsilon$, for $\rho \in (0, \overline{\rho})$; hence, if $\rho \in (0, \overline{\rho})$, we have (as $K \geq 0$)

$$\int_{\partial\Omega_\rho(x)} \Big|(u(y) - u(x))\,K(x,y)\Big|\,\mathrm{d}V^{N-1}(y) \leq \varepsilon \int_{\partial\Omega_\rho(x)} K(x,y)\,\mathrm{d}V^{N-1} \xrightarrow{\rho\to 0} \varepsilon.$$

In passing to the limit we invoked again (1.2.10). This proves the claim. We thus have

$$\lim_{\rho\to 0} m_\rho(u)(x) = u(x), \quad \text{for every } u \in C(\Omega_r(x), \mathbb{R}). \tag{1.2.11}$$

Let us now go back to the case $u \in C^2(\overline{\Omega_r(x)}, \mathbb{R})$. Inserting (1.2.11) in (1.2.8) gives

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r}\Big)\,\mathcal{L}u(y)\,\mathrm{d}V^N(y). \tag{1.2.12}$$

This is precisely (1.2.1) in Lemma 1.2.1.

Note that (1.2.12) improves (1.2.10): indeed, taking $u \equiv 1$ in (1.2.12) yields

$$m_r(1)(x) = \int_{\partial\Omega_r(x)} K(x,y)\,\mathrm{d}V^{N-1}(y) = 1, \quad \text{for every } x \in \mathbb{R}^N \text{ and } r > 0. \tag{1.2.13}$$

We pass to the last statement of Lemma 1.2.1. On the one hand, if u is $\mathcal{L}$-harmonic on Ω, formula (1.2.1) directly implies (1.2.2) since $\mathcal{L}u = 0$. On the other hand, if $u \in C^2(\Omega, \mathbb{R})$ is such that $\mathcal{L}u \neq 0$ at some point $x \in \Omega$ (say, to make a choice, $\mathcal{L}u(x) > 0$), there exists $r > 0$ such that $\mathcal{L}u > 0$ in $\Omega_r(x)$ (see (1.1.5)); by (1.2.1) and the positivity of $\Gamma(x,y) - 1/r$ on $\Omega_r(x)$, this gives $u(x) \lneqq m_r(u)(x)$, which contradicts (1.2.2).

This ends the proof. □

We should observe that identity (1.2.2) in Lemma 1.2.1 plays the rôle, for our Γ-balls, played by the Cauchy integral formula for holomorphic functions. Moreover, if A is the $N \times N$ identity matrix and $V \equiv 1$, then $\mathcal{L} = \Delta$ is the classical Laplace operator in $\mathbb{R}^N$; thus $m_r(u)(x)$ is the usual mean-value of u over the sphere $\partial\Omega_r(x)$, and (1.2.2) gives back the Gauss representation theorem for harmonic functions.

We next introduce solid mean-value operators, by a superposition argument. First we need some notation. If $\Omega \subseteq \mathbb{R}^N$ is an open set and if $x \in \Omega$ is fixed, we set

$$R(x) := \sup\{r > 0 : \Omega_r(x) \subset \Omega\}. \tag{1.2.14}$$

Let $x \in \Omega$ be fixed. Let $\varphi : [0, R(x)) \to \mathbb{R}$ be a non-negative L^1 function, with compact support, and such that

$$\int_0^{R(x)} \varphi(\rho)\,\mathrm{d}\rho = 1. \tag{1.2.15}$$

For every continuous function $u : \Omega \to \mathbb{R}$, we set

$$\Phi(u)(x) := \int_0^{R(x)} \varphi(\rho)\, m_\rho(u)(x)\,\mathrm{d}\rho. \tag{1.2.16}$$

The definition is well posed, since, denoted by $[0, r]$ a compact subinterval of $[0, R(x))$ containing the support of φ, one has (see also (1.1.7) and (1.2.13))

$$\int_0^{R(x)} |\varphi(\rho)\, m_\rho(u)(x)|\,\mathrm{d}\rho \le \int_0^r \varphi(\rho) \sup_{\Omega_r(x)} |u|\, m_\rho(1)(x)\,\mathrm{d}\rho = \sup_{\Omega_r(x)} |u| < \infty.$$

Since $\partial\Omega_\rho(x) = \{y : 1/\Gamma(x,y) = \rho\}$, if we insert the very definition (1.1.6) of $K(x,y)$ in $m_\rho(u)(x)$, and if we apply Federer's Coarea Formula, we obtain

$$\begin{aligned}
&\Phi(u)(x) = \\
&= \int_0^{R(x)} \varphi(\rho) \left(\int_{1/\Gamma(x,y)=\rho} u(y) \left\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\right\rangle \frac{V(y)\,\mathrm{d}H^{N-1}(y)}{|\nabla\Gamma_x(y)|} \right) \mathrm{d}\rho \\
&\big(\text{set } f(y) := 1/\Gamma(x,y) \text{ and note that } \nabla\Gamma_x(y) = -\Gamma^2(x,y)\,(\nabla f)(y)\big) \\
&= \int_0^{R(x)} \left(\int_{f(y)=\rho} u(y)\, V(y)\, \varphi\Big(\frac{1}{\Gamma_x(y)}\Big) \frac{\left\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\right\rangle}{\Gamma_x^2(y)} \frac{\mathrm{d}H^{N-1}(y)}{|\nabla f(y)|} \right) \mathrm{d}\rho \\
&= \int_{0<f(y)<R(x)} u(y)\, \varphi\Big(\frac{1}{\Gamma_x(y)}\Big) \frac{\left\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\right\rangle}{\Gamma_x^2(y)}\, V(y)\,\mathrm{d}H^N(y),
\end{aligned}$$

that is, by recalling (1.1.4) and (1.1.2),

$$\Phi(u)(x) = \int_{\Omega_{R(x)}(x)} u(y)\, \varphi\Big(\frac{1}{\Gamma_x(y)}\Big) \frac{\left\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\right\rangle}{\Gamma_x^2(y)}\,\mathrm{d}V^N(y). \tag{1.2.17}$$

Remark 1.2.2. Given $\alpha > -1$, if we take any $r \in (0, R(x))$ and if we set

$$\varphi_r(\rho) := \begin{cases} (\alpha+1)\,\rho^\alpha / r^{\alpha+1}, & \text{if } \rho \in [0, r] \\ 0, & \text{if } \rho \in (0, R(x)), \end{cases}$$

we obtain the family $\{\Phi_r(u)(x)\}_r$ of solid mean-value operators

$$\Phi_r(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y) \frac{\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle}{\Gamma_x^{2+\alpha}(y)} \, \mathrm{d}V^N(y), \quad 0 < r < R(x).$$

When $V \equiv 1$, these are precisely the operators $M_r(u)(x)$ employed in the papers [1, 13]. (and, for the special case of Carnot groups, in [10, 11, 12]) We shall use our more general operators $\Phi(u)(x)$ in the proof of Koebe-type theorem.

Theorem 1.2.3 (Koebe-type Theorem for $\mathcal{H}(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Suppose $u \in C(\Omega)$ satisfies the following condition:*

$$u(x) = m_r(u)(x), \quad \textit{for every } \Gamma\textit{-ball such that } \overline{\Omega_r(x)} \subset \Omega. \tag{1.2.18}$$

Then u is of class C^2 and it is $\mathcal{L}$-harmonic in Ω.

For the case of sub-Laplacians on Carnot groups (a sub-class of our operators (1.1.1)), an analogous result was proved in [15, Theorem 5.6.3], referred to as the Gauss-Koebe-Levi-Tonelli Theorem: identity (1.2.18) for classical harmonic functions in $\mathbb{R}^2$ is traditionally named after Gauss; it was Koebe in [63] who proved that, vice versa, (1.2.18) actually implies harmonicity (see also Kellogg [61] for some extensions of this result); Levi and Tonelli relaxed the continuity hypothesis with an L^1_{loc} assumption, by also replacing (classical) surface mean-values with solid ones. This L^1_{loc} assumption will reappear also in our Theorem 1.3.5 (see Section 1.3).

Proof (of Theorem 1.2.3). With the notation in (1.2.14) for $R(x)$, suppose that $u \in C(\Omega)$ satisfies the following condition:

$$u(x) = m_r(u)(x), \quad \text{for every } r \in (0, R(x)). \tag{1.2.19}$$

It suffices to show that (1.2.19) implies that u is of class C^2 on Ω; indeed, the same argument ending the proof of Lemma 1.2.1 shows that (1.2.19), when $u \in C^2$, implies that u is $\mathcal{L}$-harmonic.

Let Ω_n be a sequence of bounded open sets such as $\overline{\Omega_n} \subset \Omega_{n+1}$ and $\Omega = \bigcup_n \Omega_n$. Fixed any $n \in \mathbb{N}$, it suffices to show that $u \in C^2(\Omega_n)$. To this end, arguing as in the proof of [1, eq. (3.4)], a compactness argument shows that there exists $\varepsilon > 0$ (also depending on n) so small that $\Omega_\varepsilon(x) \subset \Omega$, for every $x \in \Omega_n$. Fixed a, b such that $0 < a < b < \varepsilon$, we take any smooth function $\varphi \geq 0$ supported in $[a, b]$ such that $\int_a^b \varphi = 1$. Since $R(x) \geq \varepsilon$ for every $x \in \Omega_n$, we can define on the whole of Ω_n the function $\Phi(u)(x)$ as in (1.2.16):

$$\Phi(u)(x) = \int_a^b \varphi(r)\, m_r(u)(x)\, \mathrm{d}r, \qquad x \in \Omega_n. \tag{1.2.20}$$

Due to our assumption (1.2.19), if we take $x \in \Omega_n$, if we multiply both sides of (1.2.19) times $\varphi(r)$ and we integrate w.r.t. $r \in [a, b]$, we get

$$u(x) = \Phi(u)(x), \quad \text{for every } x \in \Omega_n. \tag{1.2.21}$$

On the other hand, an application of (1.2.17) gives

$$\Phi(u)(x) = \int_{\Omega_{R(x)}(x)} u(y)\,\varphi\Big(\frac{1}{\Gamma_x(y)}\Big)\,\frac{\big\langle A(y)\nabla\Gamma_x(y),\nabla\Gamma_x(y)\big\rangle}{\Gamma_x^2(y)}\,\mathrm{d}V^N(y). \tag{1.2.22}$$

By our assumption on the support of φ, the integral in the above right-hand side actually performs over the compact Γ-annulus $A_{a,b}(x) := \overline{\Omega_b(x)} \smallsetminus \Omega_a(x)$; with the convention that the integrand function in (1.2.22) is prolonged to be 0 outside $A_{a,b}(x)$, from (1.2.21) and (1.2.22) we derive that

$$u(x) = \int_{\mathbb{R}^N} u(y)\,\varphi\Big(\frac{1}{\Gamma_x(y)}\Big)\,\frac{\big\langle A(y)\nabla\Gamma_x(y),\nabla\Gamma_x(y)\big\rangle}{\Gamma_x^2(y)}\,\mathrm{d}V^N(y), \qquad \forall\ x\in\Omega_n. \tag{1.2.23}$$

By assumption **(S.a)** in Section 1.1, Γ_x is of class C^3 on $A_{a,b}(x)$, and it is bounded on this same set away from zero (indeed, $\Gamma_x(y)\in[1/b,1/a]$ for every $y\in A_{a,b}(x)$). As a consequence, the function

$$(x,y)\mapsto u(y)\,\varphi\Big(\frac{1}{\Gamma_x(y)}\Big)\,\frac{\big\langle A(y)\nabla\Gamma_x(y),\nabla\Gamma_x(y)\big\rangle}{\Gamma_x^2(y)}\,V(y)$$

is of class C^2 w.r.t. $x\in\Omega_n$ and it is continuous and compactly supported w.r.t. $y\in\mathbb{R}^N$. By a simple Dominated Convergence argument applied to (1.2.23), we are therefore entitled to perform two partial derivatives w.r.t. the x variable and to pass them under the integral sign, so that $u\in C^2(\Omega_n)$. This ends the proof. □

1.3 Topology of $\mathcal{H}(\Omega)$

In this section we want to study the topology of $\mathcal{H}(\Omega)$. In particular, we will prove that $\mathcal{H}(\Omega)$ with the L^∞_{loc}-topology inherited from $C(\Omega)$ is a *Heine-Borel space*, that is the compact subsets of $\mathcal{H}(\Omega)$ are precisely the closed and bounded subsets of $\mathcal{H}(\Omega)$ (boundedness is meant in the sense of topological vector spaces). For this purposes, we will extend a celebrated theorem by P. Montel on normal families of holomorphic functions to our subelliptic setting.

Among the normality theorems usually named after Montel, [78], we are interested in the following one, concerning *locally bounded* families (see e.g., [69, Theorem 4, p. 80]): let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega\subseteq\mathbb{C}$, uniformly bounded on the compact subsets of Ω; then $\mathcal{F}$ is a normal family, that is, given any compact set $K\subset\Omega$, every sequence in $\mathcal{F}$ admits a subsequence which is uniformly convergent on K. Therefore, we fix some terminology.

Definition 1.3.1. Let $\Omega\subseteq\mathbb{R}^N$ be an open set, and let $f_n, f:\Omega\to\mathbb{R}$ (with $n\in\mathbb{N}$). We say that $\{f_n\}_n$ is *normally convergent* to f if, for every $\varepsilon>0$ and for every compact set $K\subset\Omega$, there exists $\bar{n}=\bar{n}(\varepsilon,K)\in\mathbb{N}$ such that

$$\sup_{x\in K}|f_n(x)-f(x)|<\varepsilon, \quad \forall\, n\ge\bar{n}.$$

Then, it is clear that normal convergence means uniform convergence on the compact subsets of Ω.

Let $\mathcal{F}$ be a family of real valued functions on Ω; we say that $\mathcal{F}$ is a *normal family* if, for every sequence $\{f_n\}_n$ in $\mathcal{F}$, there exists a subsequence of $\{f_n\}_n$ which is normally convergent to a function $f:\Omega\to\mathbb{R}$. We are interested in characterizing normal families of $\mathcal{L}$-harmonic functions: if $\mathcal{F}$ is such normal family, and if f is the limit function as above, it is not at all obvious whether f is $\mathcal{L}$-harmonic or not. As a consequence of the Koebe-type Theorem 1.2.3, in the following lemma we shall prove that, in fact, $f\in\mathcal{H}(\Omega)$.

Lemma 1.3.2. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set and suppose that the sequence $\{f_n\}_n\subset\mathcal{H}(\Omega)$ converges to $f:\Omega\to\mathbb{R}$ in the L^∞_{loc}-topology. Then $f\in\mathcal{H}(\Omega)$.*

Proof. Since $\mathcal{H}(\Omega)\subset C(\Omega)$, one clearly has $f\in C(\Omega)$. We aim to prove that $f\in\mathcal{H}(\Omega)$. By the Koebe-type Theorem 1.2.3 it suffices to show that

$$f(x)=m_r(f)(x),\quad \text{whenever } \overline{\Omega_r(x)}\subset\Omega. \tag{1.3.1}$$

If $\overline{\Omega_r(x)}\subset\Omega$, since $f_n\in\mathcal{H}(\Omega)$ for every $n\in\mathbb{N}$, by Lemma 1.2.1 we derive that

$$f_n(x)=m_r(f_n)(x),\quad \text{for every } n\in\mathbb{N}.$$

We aim to let $n\to\infty$ in the above identity, claiming that this passage to the limit produces (1.3.1). On the one hand, we have $\lim_{n\to\infty}f_n(x)=f(x)$, since f_n converges locally-uniformly, hence everywhere point-wise, to f. We finally show that

$$\lim_{n\to\infty}m_r(f_n)(x)=m_r(f)(x). \tag{1.3.2}$$

This will prove (1.3.1). Now, (1.3.2) is a consequence of the following computation:

$$\begin{aligned}\left|m_r(f_n)(x)-m_r(f)(x)\right| &\le \int_{\partial\Omega_r(x)}\left|f_n(y)-f(y)\right|K(x,y)\,\mathrm{d}V^{N-1}(y)\\ &\le \sup_{\partial\Omega_r(x)}\left|f_n-f\right|\cdot\int_{\partial\Omega_r(x)}K(x,y)\,\mathrm{d}V^{N-1}(y)=\sup_{\partial\Omega_r(x)}\left|f_n-f\right|.\end{aligned}$$

For the last identity we have exploited (1.2.13). The last term in the above estimate vanishes with $n\to\infty$, as $\partial\Omega_r(x)$ is a compact subset of Ω, and since (by construction) f_n converges to f, as $n\to\infty$, uniformly on the compact sets. □

1.3.1 Montel-type Theorem for $\mathcal{H}(\Omega)$

We recall the notion of *locally bounded family*.

Definition 1.3.3. Let $\mathcal{F}$ be a family of real valued functions on $\Omega\subseteq\mathbb{R}^N$ open set; $\mathcal{F}$ is said to be *locally bounded* (for some authors, locally uniformly bounded) if, for every compact set $K\subset\Omega$, there exists $M=M(K)>0$ such that

$$\sup_{x\in K}|f(x)|\le M,\quad \text{for every } f\in\mathcal{F}. \tag{1.3.3}$$

We can introduce the Montel-type Theorem.

Theorem 1.3.4 (Montel-type Theorem for $\mathcal{H}(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Any locally bounded family of $\mathcal{L}$-harmonic functions in Ω is normal.*

This result will straightforwardly derive from the following one (resemblant to the classical Levi-Tonelli result, in that L^∞ norms are replaced by L^1 ones), which is of an independent interest in its own right.

Theorem 1.3.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $\mathcal{F}$ be a family of $\mathcal{L}$-harmonic functions in Ω. Suppose that, for every compact set $K \subset \Omega$, there exists a constant $M = M(K) > 0$ such that*

$$\int_K |f| \, \mathrm{d}H^N \le M, \quad \text{for every } f \in \mathcal{F}. \tag{1.3.4}$$

Then $\mathcal{F}$ is a normal family.

Clearly, Theorem 1.3.5 implies Theorem 1.3.4, as condition (1.3.3) ensures condition (1.3.4) since

$$\int_K |f(x)| \, \mathrm{d}H^N(x) \le \sup_{x \in K} |f(x)| \cdot H^N(K).$$

We observe that in Lemma 1.3.6 we will show that the conditions in Theorems 1.3.4 and 1.3.5 are not only sufficient for normality, but they are also necessary. We shall prove Theorem 1.3.5 by making use of some solid counterparts of the mean-value operators (1.1.7), conveniently modeled on the geometry of the Γ-balls and of the compact subsets of Ω. The proof is unexpectedly delicate, due to the presence of the kernel $K(x,y)$ in (1.1.7), the novelty lying in the (possible) unboundedness of $K(x,y)$ along the diagonal. This fact is not visible in the classical case of harmonic functions (since in this case $K \equiv 1$), nor in the case of sub-Laplacians on Carnot groups, since suitable superpositions can be made in order to obtain mean-value operators with bounded kernels.

Now, we prove Theorem 1.3.5; as already observed, this also provides the proof of the Montel-type Theorem 1.3.4.

Proof (of Theorem 1.3.5). Let notation be as in the statement of Theorem 1.3.5. We consider an exhaustion of Ω by means of bounded open sets Ω_n, and we let K_n be as in (1.1.9). Let $n \in \mathbb{N}$ be fixed and let $\varepsilon_n > 0$ be so small that $\bigcup_{x \in K_n} \overline{\Omega_{\varepsilon_n}(x)}$ lies inside a compact subset of K_{n+1}. (For the existence of ε_n, see the already mentioned arguments in [1, eq. (3.4)].) Fixed a_n, b_n such that $0 < a_n < b_n < \varepsilon_n$, we consider a non-negative cut-off function $\varphi_n \in C_0^\infty(\mathbb{R}, \mathbb{R})$, supported in $[a_n, b_n]$, such that $\int_{\mathbb{R}} \varphi_n = 1$. Since $\mathcal{F} \subseteq \mathcal{H}(\Omega)$, arguing as in the proof of Theorem 1.2.3, we derive, for every $x \in \Omega_n$ and every $f \in \mathcal{F}$,

$$f(x) = \int_{\mathbb{R}^N} f(y) \, \varphi_n\Big(\frac{1}{\Gamma_x(y)}\Big) \, \frac{\big\langle A(y) \nabla \Gamma_x(y), \nabla \Gamma_x(y) \big\rangle}{\Gamma_x^2(y)} \, V(y) \, \mathrm{d}H^N(y). \tag{1.3.5}$$

The above integral extends over the compact set $A_n(x) := \overline{\Omega_{b_n}(x)} \smallsetminus \Omega_{a_n}(x)$, which is a subset of K_{n+1}, for every $x \in K_n$. By our hypothesis (1.3.4), there exists a constant $M(K_{n+1}) > 0$ such that

$$\int_{K_{n+1}} |f| \, \mathrm{d}H^N \le M(K_{n+1}), \qquad \text{for every } f \in \mathcal{F}.$$

Consequently, by means of (1.3.5) we derive the estimate

$$\sup_{x \in K_n} |f(x)| \le \int_{K_{n+1}} |f(y)| \, \mathrm{d}H^N(y) \cdot \sup_{x \in K_n} \Big(\sup_{y \in A_n(x)} \big|\Lambda_n(x,y)\big| \Big) \le M(K_{n+1}) \cdot M_n,$$

where we have set $M_n := \sup\{|\Lambda_n(x,y)| : x \in K_n, y \in A_n(x)\}$ and

$$\Lambda_n(x,y) := \varphi_n\Big(\frac{1}{\Gamma_x(y)}\Big) \frac{\big\langle A(y) \nabla \Gamma_x(y), \nabla \Gamma_x(y) \big\rangle}{\Gamma_x^2(y)} V(y).$$

Since $A_n(x) = \{y : 1/b_n \le \Gamma_x(y) \le 1/a_n\}$, we have

$$M_n \le b_n^2 \, \|\varphi_n\|_\infty \cdot \sup_{y \in K_{n+1}} \big(V(y) \, \|A(y)\|\big) \cdot \sup_{x \in K_n, y \in A_n(x)} |\nabla \Gamma_x(y)|^2 =: M_n'.$$

Here $\|A(y)\|$ stands for the operator norm of the matrix $A(y)$. We crucially remark that the set $\{(x,y) \in \mathbb{R}^{2N} : x \in K_n, y \in A_n(x)\}$ is a compact subset of $\mathbb{R}^{2N}$ far from the diagonal $\{x = y\}$, since it does not intersect the set $\{(x,y) \in \mathbb{R}^{2N} : y \in \Omega_{a_n}(x)\}$ which is a "tubular" neighborhood of the diagonal.

By our regularity assumption on Γ, this proves that M_n' is finite (and independent of $f \in \mathcal{F}$). The arbitrariness of n shows that $\mathcal{F}$ is a locally bounded family of functions. (Indeed, for every compact set $K \subset \Omega$ there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$.) We next prove that $\mathcal{F}$ is also locally equicontinuous. By a simple dominated convergence argument, from (1.3.5) we obtain that, for every $j \in \{1, \dots, N\}$,

$$\frac{\partial f(x)}{\partial x_j} = \int_{\mathbb{R}^N} f(y) \, \frac{\partial \Lambda_n(x,y)}{\partial x_j} \, \mathrm{d}H^N(y), \qquad x \in \Omega_n. \tag{1.3.6}$$

By the same arguments used to prove the local boundedness of $\mathcal{F}$, we can show the existence of a finite constant M_n'', depending on the compact set K_n, such that

$$\sup_{x \in K_n, y \in A_n(x)} \Big| \frac{\partial \Lambda_n(x,y)}{\partial x_j} \Big| \le M_n''.$$

Thus (1.3.6) and the assumption (1.3.4) show that the family of vector-valued functions $\{(\nabla f)|_{K_n} : f \in \mathcal{F}\}$ is uniformly bounded. The arbitrariness of K_n shows that the family $\{\nabla f : f \in \mathcal{F}\}$ is locally bounded on the compact subsets of Ω. A standard argument based of Lagrange's Mean Value Theorem yields the equicontinuity of $\mathcal{F}$ on the compact subsets of Ω.

We are now in a position to prove that $\mathcal{F}$ is a normal family. Indeed, given a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}$, the family $\{f_n|_{K_1}\}_n$ is uniformly bounded and equicontinuous; thus, by the Arzelà-Ascoli Theorem, we can select a subsequence $\{f_{n(1,k)}\}_{k \in \mathbb{N}}$ which is uniformly convergent on K_1 to a function, say $g_1 : K_1 \to \mathbb{R}$. The family $\{f_{n(1,k)}|_{K_2}\}_{k \in \mathbb{N}}$ is uniformly bounded and equicontinuous, so that we can select a subsequence $\{f_{n(2,k)}\}_{k\in\mathbb{N}}$ of $\{f_{n(1,k)}\}_{k\in\mathbb{N}}$ which is uniformly convergent on K_2 to a function, say $g_2 : K_2 \to \mathbb{R}$. Since $K_1 \subset K_2$ we have $g_1 \equiv g_2$ on K_1. Inductively, for every $j \in \mathbb{N}$ we can construct sequences

$$\{f_n\}_{n\in\mathbb{N}}, \quad \{f_{n(1,k)}\}_{k\in\mathbb{N}}, \quad \{f_{n(2,k)}\}_{k\in\mathbb{N}}, \quad \dots \quad \{f_{n(j,k)}\}_{k\in\mathbb{N}}, \quad \dots$$

where each is a subsequence of the preceding one, and such that $\{f_{n(j,k)}\}_{k\in\mathbb{N}}$ is uniformly convergent on K_j to a function, say $g_j : K_j \to \mathbb{R}$. We define

$$f : \Omega \to \mathbb{R}, \quad f|_{K_j} := g_j \quad \text{for every } j \in \mathbb{N}.$$

Due to our discussion above, this definition is well-posed and f is continuous on Ω; moreover, the Cantor-diagonal sequence $\{f_{n(k,k)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{f_n\}_n$ which converges uniformly to f on every K_j, for any $j \in \mathbb{N}$. This proves that $\mathcal{F}$ is normal; by Lemma 1.3.2 we know that $f \in \mathcal{H}(\Omega)$ since it is the L^∞_{loc}-limit of a subsequence of $\mathcal{L}$-harmonic functions. □

1.3.2 Heine-Borel Theorem for $\mathcal{H}(\Omega)$

In order to introduce our last main result in this section, we want to recall some notations of Section 1.1 to restate the Theorems 1.3.4 and 1.3.5 with the usual terminology of the theory of topological vector spaces.

With $\mathcal{T}(\mathcal{P})$ and $\mathcal{T}(\mathcal{Q})$ we denote, respectively, the L^1_{loc}-topology and the L^∞_{loc}-topology on $X := C(\Omega)$. Then, a subset $\mathcal{F}$ of X is *bounded* in the topological vector space $(X, \mathcal{T}(\mathcal{Q}))$ if and only if $\mathcal{F}$ is locally bounded, i.e., if and only if, for every compact set $K \subset \Omega$, there exists $M = M(K) > 0$ such that

$$\sup_{x\in K} |f(x)| \le M, \qquad \text{for every } f \in \mathcal{F},$$

that is (1.3.3) is fulfilled.

Furthermore, $\mathcal{F}$ is *bounded* in the topological vector space $(X, \mathcal{T}(\mathcal{P}))$ if and only if, for every compact set $K \subset \Omega$, there exists $M = M(K) > 0$ such that

$$\int_K |f(x)| \mathrm{d}H^N(x) \le M, \qquad \text{for every } f \in \mathcal{F},$$

that is (1.3.4) is fulfilled.

Hence, Theorems 1.3.4 and 1.3.5 can be restated by saying that *bounded subsets of the topological vector spaces $(\mathcal{H}(\Omega), \mathcal{T}(\mathcal{Q}))$ and $(\mathcal{H}(\Omega), \mathcal{T}(\mathcal{P}))$ are normal families*.

Moreover, since normal convergence is evidently equivalent to the convergence with respect to the L^∞_{loc}-topology, a family $\mathcal{F} \subseteq C(\Omega)$ is normal if and only if $\mathcal{F}$ is a precompact set (i.e., it has compact closure) in the topological space $(C(\Omega), \mathcal{T}(\mathcal{Q}))$. Even if the L^∞_{loc}-topology is, in general, strictly finer than the L^1_{loc}-topology, they coincide on $\mathcal{H}(\Omega)$, as the following useful result states.

Lemma 1.3.6 (Topology of $\mathcal{H}(\Omega)$ and normality)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The topology of $\mathcal{H}(\Omega)$ as a subspace of $(C(\Omega), \mathcal{T}(\mathcal{Q}))$ coincides with the topology of $\mathcal{H}(\Omega)$ as a subspace of $(C(\Omega), \mathcal{T}(\mathcal{P}))$. With these equivalent topologies, $\mathcal{H}(\Omega)$ is a complete subspace of $C(\Omega)$, hence it is a Fréchet space.*

Furthermore, given a set $\mathcal{F} \subseteq \mathcal{H}(\Omega)$, the following conditions are equivalent:

1. *$\mathcal{F}$ is a normal family;*
2. *$\mathcal{F}$ is a precompact subset of $\mathcal{H}(\Omega)$ (in the topologies $\mathcal{T}(\mathcal{P})$ or $\mathcal{T}(\mathcal{Q})$);*
3. *$\mathcal{F}$ is $\mathcal{T}(\mathcal{Q})$-bounded, i.e., for every compact set $K \subset \Omega$, there exists a constant $M(K) > 0$ such that $\sup_K |f| \leq M(K)$, for every $f \in \mathcal{F}$;*
4. *$\mathcal{F}$ is $\mathcal{T}(\mathcal{P})$-bounded, i.e., for every compact set $K \subset \Omega$, there exists a constant $M(K) > 0$ such that $\int_K |f| \, \mathrm{d}H^N \leq M(K)$, for every $f \in \mathcal{F}$.*

The proof is split in three steps.

Proof. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $X := C(\Omega)$. We remind that $\mathcal{T}(\mathcal{Q})$ is the L^∞_{loc}-topology of X, while $\mathcal{T}(\mathcal{P})$ is the L^1_{loc}-topology of X.

STEP I. We begin with showing that the topology of $\mathcal{H}(\Omega)$ as a subspace of $(X, \mathcal{T}(\mathcal{Q}))$ coincides with the topology of $\mathcal{H}(\Omega)$ as a subspace of $(X, \mathcal{T}(\mathcal{P}))$. This amounts to proving that the following map is a homeomorphism

$$\iota : (\mathcal{H}(\Omega), \mathcal{T}(\mathcal{Q})) \to (\mathcal{H}(\Omega), \mathcal{T}(\mathcal{P})), \quad \iota(f) := f.$$

The continuity of ι is trivial since $\mathcal{T}(\mathcal{P}) \subset \mathcal{T}(\mathcal{Q})$. Since $\mathcal{P}$ and $\mathcal{Q}$ are countable families of seminorms, $\mathcal{T}(\mathcal{Q})$ and $\mathcal{T}(\mathcal{P})$ are first-countable topologies. Therefore the continuity of ι^{-1} can be proved sequentially. Given $f_n, f \in \mathcal{H}(\Omega)$ such that

$$\lim_{n\to\infty} f_n = f \text{ w.r.t. } \mathcal{T}(\mathcal{P}), \tag{1.3.7}$$

we need to show that $\lim_{n\to\infty} f_n = f$ w.r.t. $\mathcal{T}(\mathcal{Q})$ too. If K is any compact subset of Ω, by definition of $\mathcal{T}(\mathcal{P})$ we have that $f_n|_K$ converges to $f|_K$ in $L^1(K)$. In particular, $\{f_n|_K\}_n$ is a bounded set in the norm of $L^1(K)$. Due to the arbitrariness of K, we are in a position to apply Theorem 1.3.5 to $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$, deriving that $\mathcal{F}$ is a normal family. This is equivalent to saying that every subsequence $\{f_{n(k)}\}_k$ of $\{f_n\}_n$ admits a subsequence $\{f_{n(k(j))}\}_j$ which converges w.r.t. $\mathcal{T}(\mathcal{Q})$ to some function g. Since $\mathcal{T}(\mathcal{P}) \subset \mathcal{T}(\mathcal{Q})$, we have $\lim_{j\to\infty} f_{n(k(j))} = g$ in $\mathcal{T}(\mathcal{P})$ too. Now, by assumption (1.3.7) we derive that $g = f$. Summing up, we demonstrated that every subsequence of $\{f_n\}_n$ admits a further subsequence which $\mathcal{T}(\mathcal{Q})$-converges to f. This is possible if and only if $\{f_n\}_n$ itself is $\mathcal{T}(\mathcal{Q})$-convergent to f.

STEP II. Next we show that $\mathcal{H}(\Omega)$ is a closed subspace of $C(\Omega)$ w.r.t. the $\mathcal{T}(\mathcal{Q})$-topology; since, as it is well-known, $(C(\Omega), \mathcal{T}(\mathcal{Q}))$ is a Fréchet space, this will prove that $\mathcal{H}(\Omega)$ is a complete subspace of $(C(\Omega), \mathcal{T}(\mathcal{Q}))$, hence a Fréchet subspace if equipped with the $\mathcal{T}(\mathcal{Q})$-topology (or, equivalently, with the $\mathcal{T}(\mathcal{P})$-topology; see Step I). Now, the fact that $\mathcal{H}(\Omega)$ is a $\mathcal{T}(\mathcal{Q})$-closed subspace of $C(\Omega)$ is exactly the statement of Lemma 1.3.2.

STEP III. Finally, given a set $\mathcal{F} \subseteq \mathcal{H}(\Omega)$, we are left to proving that conditions (1)-to-(4) in the last part of Lemma 1.3.6 are equivalent.

(1)⇒(2): Conditions (1) and (2) are equivalent; indeed, in every metrizable space, precompactness of a set $\mathcal{F}$ is equivalent to the condition that every sequence in $\mathcal{F}$ admits a convergent subsequence. Now, $(\mathcal{H}(\Omega), \mathcal{T}(\mathcal{Q}))$ is indeed a metrizable space since $\mathcal{T}(\mathcal{Q})$ is induced by a metric (see Section 1.1).

(2)⇒(3): This is generally true in topological vector spaces;[1] for completeness, we provide a direct proof. Let $\mathcal{F}$ be a precompact subset of $(\mathcal{H}(\Omega), \mathcal{T}(\mathcal{Q}))$ and suppose, by contradiction, that there exists a compact set $K \subset \Omega$ and a sequence $\{f_n\}_n \subset \mathcal{F}$ such that $\sup_K |f_n| > n$ for every $n \in \mathbb{N}$; since $\overline{\mathcal{F}}$ is $\mathcal{T}(\mathcal{Q})$-compact, we can select a subsequence $\{f_{n(k)}\}_k$ which is $\mathcal{T}(\mathcal{Q})$-convergent in $\mathcal{H}(\Omega)$. In particular, $\{\sup_K |f_{n(k)}|\}_k$ must be bounded, and this conflicts with the condition $\sup_K |f_{n(k)}| > n(k)$ for every $k \in \mathbb{N}$, and the fact that $n(k) \to \infty$ as $k \to \infty$.

(3)⇒(4): This is a direct consequence of $\int_K |f| \,\mathrm{d}H^N \le \sup_K |f| \cdot H^N(K)$.

(4)⇒(1): This is the precisely the statement of Theorem 1.3.5. □

From Lemma 1.3.6 we straightforwardly derive the following result.

Theorem 1.3.7 (Heine-Borel for $\mathcal{H}(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The set $\mathcal{H}(\Omega)$ of the $\mathcal{L}$-harmonic functions in Ω endowed with the L^1_{loc}-topology inherited from $C(\Omega)$ (ore, equivalently, endowed with the L^∞_{loc}-topology) is a Heine-Borel topological vector space.*

Proof. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We equip $\mathcal{H}(\Omega)$ with the L^∞_{loc}-topology $\mathcal{T}(\mathcal{Q})$ inherited from $C(\Omega)$ (this is equivalent to equip it with the L^1_{loc}-topology, see Lemma 1.3.6). Since (in every topological vector space) any compact set is closed and bounded, in order to prove that $\mathcal{H}(\Omega)$ is a Heine-Borel space we are left to show that a closed and $\mathcal{T}(\mathcal{Q})$-bounded set $\mathcal{F} \subset \mathcal{H}(\Omega)$ is compact. From condition (3) in Lemma 1.3.6, the $\mathcal{T}(\mathcal{Q})$-boundedness of $\mathcal{F}$ implies that $\overline{\mathcal{F}}$ is compact; since $\mathcal{F} = \overline{\mathcal{F}}$ (for $\mathcal{F}$ is closed), the proof is complete. □

[1] If $\overline{\mathcal{F}}$ is compact, then $\overline{\mathcal{F}}$ is $\mathcal{T}(\mathcal{Q})$-bounded whence $\mathcal{F}$ is $\mathcal{T}(\mathcal{Q})$-bounded.

Chapter 2

Harnack Inequality for degenerate hypoelliptic operators

In this chapter we consider a class of hypoelliptic second-order partial differential operators $\mathcal{L}$ in divergence form on $\mathbb{R}^N$, for which we have showed the Strong and Weak Maximum Principles in [5]; here our aim is to prove the Harnack Inequality for $\mathcal{L}$.

In order to prove the Harnack theorem, we need to show the solvability of the Dirichlet problem for $\mathcal{L}$ on a basis of the Euclidean topology; then we prove a *Weak Harnack inequality* and we use these results, together with means of Potential Theory, to obtain the Harnack inequality.

2.1 Main assumptions and notation

We shall be concerned with linear second order partial differential operators (PDOs, in the sequel), possibly degenerate-elliptic, in divergence form

$$\mathcal{L} := \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j}\Big), \qquad x \in \mathbb{R}^N, \tag{2.1.1}$$

where V is a C^∞ strictly positive function on $\mathbb{R}^N$, the matrix $A(x) := (a_{i,j}(x))_{i,j}$ is symmetric and *positive semi-definite* at every point $x \in \mathbb{R}^N$, and it has real-valued C^∞ entries. In particular, $\mathcal{L}$ is formally self-adjoint on $L^2(\mathbb{R}^N, \mathrm{d}\nu)$ with respect to the measure $\mathrm{d}\nu(x) = V(x)\,\mathrm{d}x$, which clarifies the rôle of V.

In order to describe our results, we need to fix some notation and definition: we say that a linear second order PDO on $\mathbb{R}^N$

$$L := \sum_{i,j=1}^{N} \alpha_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \beta_i(x) \frac{\partial}{\partial x_i} + \gamma(x) \tag{2.1.2}$$

is *non-totally degenerate* at a point $x \in \mathbb{R}^N$ if the matrix $(\alpha_{i,j}(x))_{i,j}$ (which will be referred to as the principal matrix of L) is non-vanishing. We observe that the principal matrix of an operator $\mathcal{L}$ of the form (2.1.1) is precisely $A(x) = (a_{i,j}(x))_{i,j}$.

We also remind that L is said to be *(C^∞-)hypoelliptic* in an open set $\Omega \subseteq \mathbb{R}^N$ if, for every $u \in \mathcal{D}'(\Omega)$, every open set $U \subseteq \Omega$ and every $f \in C^\infty(U, \mathbb{R})$, the equation $Lu = f$ in U implies that u is (a function-type distribution associated with) a C^∞ function on U.

In the sequel, if $\Omega \subseteq \mathbb{R}^N$ is open, we say that u is *L-harmonic* (resp., *L-subharmonic*) in Ω if $u \in C^2(\Omega, \mathbb{R})$ and $Lu = 0$ (resp., $Lu \geq 0$) in Ω. The set of the L-harmonic functions in Ω will be denoted by $\mathcal{H}_L(\Omega)$. We observe that, if L is hypoelliptic on every open subset of $\mathbb{R}^N$, then $\mathcal{H}_L(\Omega) \subset C^\infty(\Omega, \mathbb{R})$; under this hypoellipticity assumption, $\mathcal{H}_L(\Omega)$ has important topological properties, which will be crucially used in the sequel (Remark 2.3.9).

In order to introduce our first main result we assume the following hypotheses on $\mathcal{L}$:

(NTD) $\mathcal{L}$ is *non-totally degenerate at every point of* $\mathbb{R}^N$, or equivalently (recalling that $A(x)$ is symmetric and positive semi-definite),

$$\text{trace}(A(x)) > 0, \quad \text{for every } x \in \mathbb{R}^N. \tag{2.1.3}$$

(HY) $\mathcal{L}$ is *C^∞-hypoelliptic in every open subset of* $\mathbb{R}^N$.

Under these two assumptions we have showed in [5] the *Strong Maximum Principle for $\mathcal{L}$* (see also Section 2.2).

Condition (NTD), if compared to the Muckenhoupt-type weights on the degeneracies of $A(x)$, does not allow a *simultaneous* vanishing of the eigenvalues of $A(x)$, but it has the advantage of permitting a very fast vanishing of small eigenvalues (see Example 2.1.2) together with a very fast growing of large eigenvalues (see Example 2.1.1); both phenomena can happen at an exponential rate (e.g., like e^{-1/x^2} as $x \to 0$ in the first case, and like e^x as $x \to \infty$ in the second case), which is not allowed when Muckenhoupt weights are involved.

Meaningful examples of operators satisfying hypotheses (NTD) and (HY), providing prototype PDOs to which our theory applies and a motivation for our investigation, are now described in the following two examples.

Example 2.1.1. The following PDOs satisfy the assumptions (NTD) and (HY).

(a.) If $\mathbb{R}^N$ is equipped with a Lie group structure $\mathbb{G} = (\mathbb{R}^N, *)$, and if we fix a set $X := \{X_1, \dots, X_m\}$ of Lie-generators for the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ (this means that the smallest Lie algebra containing X is equal to $\mathfrak{g}$), then a direct computation shows that

$$\mathcal{L}_X := -\sum_{j=1}^{m} X_j^* X_j \tag{2.1.4}$$

is of the form (2.1.1), where $V(x)$ is the density of the Haar measure ν on $\mathbb{G}$, and $(a_{i,j})_{i,j}$ is equal to $S\,S^T$, where S is the $N \times m$ matrix whose columns are given by the coefficients of the vector fields $X_1, \dots, X_m$; here X_j^* denotes the (formal) adjoint of X_j in the Hilbert space $L^2(\mathbb{R}^N, \mathrm{d}\nu)$. Most importantly, $\mathcal{L}_X$ in (2.1.4) satisfies the assumptions (NTD) and (HY) above. Indeed:

- The non-total-degeneracy is a consequence of X being a set of Lie-generators of $\mathfrak{g}$.
- $\mathcal{L}_X$ is a Hörmander operator, of the form $\sum_{j=1}^m X_j^2 + X_0$, where X_0 is a linear combination (with smooth coefficients) of $X_1, \dots, X_m$. Therefore $\mathcal{L}_X$ is hypoelliptic due to Hörmander's Hypoellipticity Theorem, [58], jointly with the cited fact that X is a set of Lie-generators of $\mathfrak{g}$.

The density V need not be identically 1, as for example for the Lie group $(\mathbb{R}^2, *)$, where

$$(x_1, x_2) * (y_1, y_2) = (x_1 + y_1 e^{x_2}, x_2 + y_2),$$

since in this case $V(x) = e^{-x_2}$. The left-invariant PDO associated with the set of generators $X = \{e^{x_2} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ has fast-growing coefficients:

$$\mathcal{L}_X = e^{2x_2} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_2}.$$

Note that the eigenvalues of the principal matrix of $\mathcal{L}_X$ are e^{2x_2} and 1, so that the largest eigenvalue cannot be controlled (for $x_2 > 0$) by any integrable weight.

(b.) More generally (arguing as above), if $X = \{X_1, \dots, X_m\}$ is a family of smooth vector fields in $\mathbb{R}^N$ satisfying Hörmander's Rank Condition, if $\mathrm{d}\nu(x) = V(x)\,\mathrm{d}x$ is the Radon measure associated with any positive smooth density V on $\mathbb{R}^N$, then the operator $-\sum_{j=1}^m X_j^* X_j$ is of the form (2.1.1) and it satisfies (NTD) and (HY). Here X_j^* denotes the formal adjoint of X_j in $L^2(\mathbb{R}^N, \mathrm{d}\nu)$. The PDOs of this form naturally arise in CR Geometry and in the function theory of several complex variables (see [60]).

The above examples show that geometrically meaningful PDOs belonging to the class of our concern actually fall in the hypoellipticity class of the Hörmander operators. Nonetheless, hypotheses (NTD) and (HY) are general enough to comprise *non-Hörmander* and *non-subelliptic* operators, as it is shown in the next example. Applications to this kind of *infinitely-degenerate* PDOs also furnish one of the main motivation for our study.

Example 2.1.2. Let us consider the class of operators in $\mathbb{R}^2$ defined by

$$\mathcal{L}_a = \frac{\partial^2}{\partial x_1^2} + \Big(a(x_1) \frac{\partial}{\partial x_2}\Big)^2, \tag{2.1.5a}$$

with $a \in C^\infty(\mathbb{R}, \mathbb{R})$, a even, nonnegative, nondecreasing on $[0, \infty)$ and vanishing only at 0. Then $\mathcal{L}_a$ satisfies (NTD) (obviously) and (HY), thanks to a result by Fediĭ, [34]. Note that $\mathcal{L}_a$ does not satisfy Hörmander's Rank Condition at $x_1 = 0$ if all the derivatives of a vanish at 0, as for $a(x_1) = \exp(-1/x_1^2)$. Other examples of operators satisfying our assumptions (NTD) and (HY) but failing to be Hörmander operators can be found, e.g., in the following papers: Bell and Mohammed [9]; Christ [22, Section 1]; Kohn [67]; Kusuoka and Stroock [70, Theorem 8.41]; Morimoto [79]. Explicit examples are, for instance,

$$\frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/|x_1|)\,\frac{\partial}{\partial x_2}\Big)^2 + \Big(\exp(-1/|x_1|)\,\frac{\partial}{\partial x_3}\Big)^2 \qquad \text{in } \mathbb{R}^3, \tag{2.1.5b}$$

$$\frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/\sqrt{|x_1|})\,\frac{\partial}{\partial x_2}\Big)^2 + \frac{\partial^2}{\partial x_3^2} \qquad \text{in } \mathbb{R}^3, \tag{2.1.5c}$$

$$\frac{\partial^2}{\partial x_2^2} + \Big(x_2\,\frac{\partial}{\partial x_1}\Big)^2 + \frac{\partial^2}{\partial x_4^2} + \Big(\exp(-1/\sqrt[3]{|x_1|})\,\frac{\partial}{\partial x_3}\Big)^2 \qquad \text{in } \mathbb{R}^4. \tag{2.1.5d}$$

For the hypoellipticity of (2.1.5b) see [22]; for (2.1.5c) see [70]; for (2.1.5d) see [79]. Later on, in proving the Harnack Inequality, we shall add another hypothesis to (NTD) and (HY) and, as we shall show, the operators from (2.1.5a) to (2.1.5d) (and those in Example 2.1.1) will fulfil this assumption as well. Hence our main results fully apply to these PDOs.

Moreover, since the PDOs (2.1.5a)-to-(2.1.5d) *are not subelliptic* (see Remark 2.1.3), they do not fall in the class considered by Jerison and Sánchez-Calle in [60]. Finally, note that the smallest eigenvalue in all the above examples vanishes very quickly (like $\exp(-1/|x|^\alpha)$ for $x \to 0$, with positive α) and it cannot be bounded from below by any weight $w(x)$ with locally integrable reciprocal function.

In order to prove the main result of the chapter (namely, the Harnack Inequality for $\mathcal{L}$), we shall need a further assumption, very similar to (HY) (and, indeed, equivalent to it in many important cases), together with some technical results on the solvability of the Dirichlet problem related to $\mathcal{L}$. Our next assumption is the following one:

(HY)$_\varepsilon$ *There exists $\varepsilon > 0$ such that $\mathcal{L} - \varepsilon$ is C^∞-hypoelliptic in every open subset of $\mathbb{R}^N$.*

For operators $\mathcal{L}$ satisfying hypotheses (NTD), (HY) and (HY)$_\varepsilon$ we are able to prove the Harnack Inequality (see Theorem 2.4.3).

We postpone the description of the relationship between assumptions (HY) and (HY)$_\varepsilon$ (and their actual equivalence for large classes of operators: for subelliptic PDOs, for instance) in Remark 2.1.3 below. Instead, we anticipate the rôle of the perturbation $\mathcal{L} - \varepsilon$ of the operator $\mathcal{L}$: this is motivated by a crucial comparison argument (which we generalize to our setting), due to Bony [16, Proposition 7.1, p.298], giving the lower bound

$$u(x_0) \geq \varepsilon \int_\Omega u(y)\, k_\varepsilon(x_0, y)\, V(y)\, \mathrm{d}y \qquad \forall\, x_0 \in \Omega, \tag{2.1.6}$$

for every nonnegative $\mathcal{L}$-harmonic function u on the open set Ω which possesses a Green kernel $k_\varepsilon(x, y)$ relative to the perturbed operator $\mathcal{L} - \varepsilon$ (see Theorem 2.3.7 for the notion of a Green kernel, and see Lemma 2.4.1 for the proof of (2.1.6)). This lower bound, plus some topological facts on hypoellipticity, is the key ingredient for a *weak* Harnack Inequality related to $\mathcal{L}$, as we shall explain.

Some remarks on assumption $(\text{HY})_\varepsilon$ are now in order.

Remark 2.1.3. Hypothesis $(\text{HY})_\varepsilon$ is implicit in hypothesis (HY) for notable classes of operators, whence our assumptions for the validity of the Harnack Inequality for $\mathcal{L}$ reduce to (NTD) and (HY) solely: namely, *(HY) implies $(HY)_\varepsilon$ in the following cases:*

- for Hörmander operators, and, more generally, for second order *subelliptic* operators (in the usual sense of fulfilling a subelliptic estimate, see e.g., [60, 67]); indeed, any operator L in these classes of PDOs is hypoelliptic (see Hörmander [58], Kohn and Nirenberg [68]), and L still belongs to these classes after the addition of a smooth zero-order term;

- for operators with *real-analytic coefficients.* Indeed, in the C^ω case, one can apply known results by Oleĭnik and Radkevič ensuring that, for a general C^ω operator L as in (2.1.2), hypoellipticity is equivalent to the verification of Hörmander's Rank Condition for the vector fields $X_0, X_1, \dots, X_N$ obtained by rewriting L as $\sum_{i=1}^N \partial_i(X_i)+X_0+\gamma$; this condition is clearly invariant under any change of the zero-order term γ of L so that (HY) and $(\text{HY})_\varepsilon$ are indeed equivalent.

The problem of establishing, in general, whether (HY) implies $(\text{HY})_\varepsilon$ seems non-trivial and it is postponed to future investigations.[1] In this regard we remind that, for example, in the complex coefficient case the presence of a zero-order term (even a small ε) may drastically alter hypoellipticity (see for instance the example given by Stein in [90] and the very recent paper [82] by Parmeggiani).

We explicitly remark that the operators (2.1.5a)-to-(2.1.5d) are *not* subelliptic (nor C^ω), yet they satisfy hypotheses (NTD), (HY) and $(\text{HY})_\varepsilon$. The lack of subellipticity is a consequence of the characterization of the subelliptic PDOs due to Fefferman and Phong [35, 36] (see also [67, Prop.1.3] or [60, Th.2.1 and Prop.2.1], jointly with the presence of a coefficient with a zero of infinite order in (2.1.5a)-to-(2.1.5d)). The second assertion concerning the verification of $(\text{HY})_\varepsilon$ (the other hypotheses being already discussed) derives from the following result by Kohn, [67]: any operator of the form

$$L_1 + \lambda(x)\, L_2 \quad \text{in } \mathbb{R}^n_x \times \mathbb{R}^m_y$$

is hypoelliptic, where $\lambda \in C^\infty(\mathbb{R}_x)$, $\lambda \geq 0$ has a zero of infinite order at 0 (and no other zeroes of infinite order), and L_1 (operating in $x \in \mathbb{R}^n$) and L_2 (operating in $y \in \mathbb{R}^m$) are general second order PDOs (as in (2.1.2)) with smooth coefficients and they are assumed to be subelliptic. It is straightforward to recognize that by subtracting ε to any PDO in (2.1.5a)-to-(2.1.5d) we get an operator of the form $(L_1 - \varepsilon) + \lambda(x)\, L_2$, where λ has the required features, L_2 is uniformly

[1] It appears that having some quantitative information on the loss of derivatives may help in facing this question (personal communication by A. Parmeggiani).

elliptic (indeed, a classical Laplacian in all the examples), and $L_1 - \varepsilon$ is a uniformly elliptic operator (cases (2.1.5a)-to-(2.1.5c)) or it is a Hörmander operator (case (2.1.5d)).

2.2 The Strong and Weak Maximum Principles

The aim of this section is to give some recall on the Strong and Weak Maximum Principle for $\mathcal{L}$ (for proofs of the main results see [5]). Clearly, a fundamental tool is played by a suitable Hopf-type lemma, furnished in Lemma 2.2.1. (For a recent interesting survey on maximum principles and Hopf-type results for uniformly elliptic operators, see López-Gómez [73].)

First the relevant definition and notation: given an open set $\Omega \subseteq \mathbb{R}^N$ and a relatively closed set F in Ω, we say that ν is *externally orthogonal to F at y*, and we write

$$\nu \perp F \text{ at } y, \tag{2.2.1}$$

if: $y \in \Omega \cap \partial F$; $\nu \in \mathbb{R}^N \smallsetminus \{0\}$; $\overline{B(y+\nu, |\nu|)}$ is contained in Ω and it intersects F only at y. Here and throughout this chapter $B(x_0, r)$ is the Euclidean ball in $\mathbb{R}^N$ of centre x_0 and radius $r > 0$; moreover $|\cdot|$ will denote the Euclidean norm on $\mathbb{R}^N$. The notation (2.2.1) does not explicitly refer to externality, but this will not create any confusion in the sequel. It is not difficult to recognize that if Ω is connected and if F is a proper (relatively closed) subset of Ω, then there always exist couples (y, ν) such that $\nu \perp F$ at y.

Lemma 2.2.1 (of Hopf-type for $\mathcal{L}$). *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1) *with C^1 coefficients $V > 0$ and $a_{i,j}$, and let us set $A(x) := (a_{i,j}(x))_{i,j}$. (We remind that $A(x) \geq 0$ for every $x \in \mathbb{R}^N$.) Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set. Then, the following facts hold.*

(1) *Let $u \in C^2(\Omega, \mathbb{R})$ be such that $\mathcal{L}u \geq 0$ on Ω; let us suppose that*

$$F(u) := \left\{ x \in \Omega : u(x) = \max_{\Omega} u \right\} \tag{2.2.2}$$

is a proper subset of Ω. Then

$$\langle A(y)\nu, \nu \rangle = 0 \quad \textit{whenever } \nu \perp F(u) \textit{ at } y. \tag{2.2.3}$$

(2) *Suppose $c \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative on $\mathbb{R}^N$, and let us set $\mathcal{L}_c := \mathcal{L} - c$. Let $u \in C^2(\Omega, \mathbb{R})$ be such that $\mathcal{L}_c u \geq 0$ on Ω; let us suppose that $F(u)$ in* (2.2.2) *is a proper subset of Ω and that $\max_\Omega u \geq 0$. Then* (2.2.3) *holds true.*

Our main result under conditions (NTD) and (HY) is the following one.

Theorem 2.2.2 (Strong Maximum Principle for $\mathcal{L}$). *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1), *with C^∞ coefficients $V > 0$ and $(a_{i,j})_{i,j} \geq 0$, and that it satisfies* (NTD) *and* (HY). *Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set. Then, the following facts hold.*

(1) *Any function* $u \in C^2(\Omega, \mathbb{R})$ *satisfying* $\mathcal{L}u \geq 0$ *on* Ω *and attaining a maximum in* Ω *is constant throughout* Ω*.*

(2) *If* $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ *is nonnegative on* $\mathbb{R}^N$*, and if we set*

$$\mathcal{L}_c := \mathcal{L} - c, \tag{2.2.4}$$

then any function $u \in C^2(\Omega, \mathbb{R})$ *satisfying* $\mathcal{L}_c u \geq 0$ *on* Ω *and attaining a nonnegative maximum in* Ω *is constant throughout* Ω*.*

The rôle of the nonnegativity of the zero-order term c in the above statement (2) in obtaining Strong Maximum Principles is well-known (see e.g., Pucci and Serrin [85]).

Remark 2.2.3. We have seen that, in order to obtain the SMP and WMP for $\mathcal{L} - c$, it is also sufficient to replace the hypothesis on the hypoellipticity of $\mathcal{L}$ with the (more natural hypothesis of the) hypoellipticity of $\mathcal{L} - c$, still under assumption (NTD) and the divergence-form structure of $\mathcal{L}$; see Remark 2.2.7 for the precise result.

The proof of the SMP in Theorem 2.2.2 follows a rather classical scheme, in that it rests on a Hopf Lemma for $\mathcal{L}$ (see Lemma 2.2.1). However, the passage from the Hopf Lemma to the SMP is, in general, non-trivial and the same is true in our framework. For example, in the paper [16] by Bony, where Hörmander operators are considered, this passage is accomplished by means of a maximum propagation principle, crucially based on Hörmander's Rank Condition, the latter ensuring a connectivity property (the so-called *Chow's Connectivity Theorem* for Hörmander vector fields). The novelty in our setting is that, since hypotheses (NTD) and (HY) do *not* necessarily imply that $\mathcal{L}$ is a Hörmander operator (see for instance Example 2.1.2), we have to supply for a lack of geometric information.

We are able to supply the lack of Hörmander's Rank Condition by using a notable control-theoretic property encoded in the hypoellipticity assumption (HY), proved by Amano in [3]: indeed, thanks to the hypothesis (NTD), we are entitled to use [3, Theorem 2] which states that (HY) ensures the *controllability* of the ODE system

$$\dot{\gamma} = \xi_0 X_0(\gamma) + \sum_{i=1}^N \xi_i X_i(\gamma), \qquad (\xi_0, \xi_1, \dots, \xi_N) \in \mathbb{R}^{1+N},$$

on every open and connected subset of $\mathbb{R}^N$. Here $X_1, \dots, X_N$ denote the vector fields associated with the rows of the principal matrix of $\mathcal{L}$, whereas X_0 is the drift vector field obtained by writing $\mathcal{L}$ (this being always possible) in the form

$$\mathcal{L}u = \sum_{i=1}^N \frac{\partial}{\partial x_i}(X_i u) + X_0 u.$$

By definition of a controllable system, Amano's controllability result provides another geometric *connectivity property* (a substitute for Chow's Theorem): any couple of points can be joined by a continuous path which is piece-wise an integral curve of some vector field Y belonging to $\text{span}_{\mathbb{R}}\{X_0, X_1, \dots, X_N\}$. The SMP will then follow if we show that there is a propagation of the maximum of any $\mathcal{L}$-subharmonic function u along all integral curves γ_Y of every $Y \in \text{span}_{\mathbb{R}}\{X_0, X_1, \dots, X_N\}$. In other words, we need to show that if the set $F(u)$ of the maximum points of u intersects any such γ_Y, then γ_Y is wholly contained in $F(u)$: briefly, if this happens we say that $F(u)$ is Y-invariant. In its turn, this Y-invariance property can be characterized (see Bony, [16, §2]) in terms of a tangentiality property of Y with respect to $F(u)$.

Now, the self-adjoint structure of our PDO $\mathcal{L}$ in (2.1.1) ensures that X_0 is a linear combination with smooth coefficients of $X_1, \dots, X_N$. Hence, by the very definition of tangentiality, the tangentiality of X_0 w.r.t. $F(u)$ will be inherited from the tangentiality of $X_1, \dots, X_N$ w.r.t. $F(u)$. By means of the above argument of controllability/propagation, this allows us to reduce the proof of the SMP to showing that any of the vector fields $X_1, \dots, X_N$ is tangent to $F(u)$. Luckily, this tangentiality is a consequence of the choice of $X_1, \dots, X_N$ as deriving from the rows of the principal matrix of $\mathcal{L}$, together with the Hopf-type Lemma 2.2.1 for $\mathcal{L}$.

Remark 2.2.4. We explicitly remark that, as it is proved by Amano in [3, Theorem 1], the above controllability property ensures the validity of the Hörmander Rank Condition only on an open *dense* subset of $\mathbb{R}^N$ which may fail to coincide with the whole of $\mathbb{R}^N$. This actual possible lack of the Hörmander Rank Condition is clearly exhibited in Example 2.1.2 (of non-Hörmander operators which nonetheless satisfy our assumptions (NTD) and (HY), and hence the SMP).

To the best of our knowledge, Amano's controllability result for hypoelliptic non-totally-degenerate operators has been long forgotten in the literature; only recently, it has been used by B. Abbondanza and A. Bonfiglioli [1] in studying the Dirichlet problem for $\mathcal{L}$, and in obtaining Potential Theoretic results for the harmonic sheaf related to $\mathcal{L}$.

As a Corollary of Theorem 2.2.2 we immediately get the following result.

Corollary 2.2.5 (Weak Maximum Principle for $\mathcal{L}$). *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1), *with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and that it satisfies* (NTD) *and* (HY). *Suppose also that $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ is nonnegative on $\mathbb{R}^N$ (the case $c \equiv 0$ is allowed), and let us set $\mathcal{L}_c := \mathcal{L} - c$. Then, $\mathcal{L}_c$ satisfies the* Weak Maximum Principle *on every bounded open set $\Omega \subseteq \mathbb{R}^N$, that is:*

$$\begin{cases} u \in C^2(\Omega, \mathbb{R}) \\ \mathcal{L}_c u \geq 0 \text{ on } \Omega \\ \limsup_{x \to x_0} u(x) \leq 0 \text{ for every } x_0 \in \partial\Omega \end{cases} \implies \quad u \leq 0 \text{ on } \Omega. \tag{2.2.5}$$

As a consequence, if $\Omega \subseteq \mathbb{R}^N$ is bounded, and if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is nonnegative and such that $\mathcal{L}_c u \geq 0$ on Ω, then one has $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$.

Since Amano's results on hypoellipticity/controllability are independent of the presence of a zero-order term, we have the following remarks.

Remark 2.2.6. *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1)*, with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and that it satisfies* (NTD)*. Let $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be nonnegative and suppose that the operator $\mathcal{L}_c := \mathcal{L} - c$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

If $\Omega \subseteq \mathbb{R}^N$ is a connected open set, then any function $u \in C^2(\Omega, \mathbb{R})$ satisfying $\mathcal{L}_c u \geq 0$ on Ω and attaining a nonnegative maximum in Ω is constant throughout Ω.

Remark 2.2.7. *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1)*, with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and that it satisfies* (NTD)*. Let $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be nonnegative and suppose that the operator $\mathcal{L}_c := \mathcal{L} - c$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

Then $\mathcal{L}_c$ satisfies the Weak Maximum Principle on every bounded open set $\Omega \subseteq \mathbb{R}^N$.

As a consequence, if $\Omega \subseteq \mathbb{R}^N$ is bounded, and if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is nonnegative and such that $\mathcal{L}_c u \geq 0$ on Ω, then one has $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$.

2.3 The Dirichlet problem for $\mathcal{L}$

Before describing the approach to the Harnack Inequality in Section 2.4, inspired by the techniques in [16], we state the main needed technical tools on the solvability of the Dirichlet problem for $\mathcal{L}$ and for the perturbed operator $\mathcal{L} - \varepsilon$.

Lemma 2.3.1. *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1)*, with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and that $\mathcal{L}$ satisfies* (NTD)*. Let $\varepsilon \geq 0$ be fixed (the case $\varepsilon = 0$ being admissible). We set $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$ and we assume that $\mathcal{L}_\varepsilon$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

Then, there exists a basis for the Euclidean topology of $\mathbb{R}^N$, independent of ε, made of open and connected sets Ω (with Lipschitz boundary) with the following properties: for every continuous function f on $\overline{\Omega}$ and for every continuous function φ on $\partial\Omega$, there exists one and only one solution $u \in C(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon u = -f & \text{on } \Omega \quad \textit{(in the weak sense of distributions)}, \\ u = \varphi & \text{on } \partial\Omega \quad \textit{(point-wise)}. \end{cases} \tag{2.3.1}$$

Furthermore, if $f, \varphi \geq 0$ then $u \geq 0$ as well. Finally, if f belongs to $C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$, then the same is true of u, and u is a classical solution of (2.3.1)*.*

We prove this theorem for a considerably larger class of operators than the $\mathcal{L}_\varepsilon$ above; see Theorem 2.3.2: our slightly more general framework (we indeed deal with general hypoelliptic operators which are non-totally degenerate at every point) compared to the one considered by Bony in [16] (where Hörmander operators are concerned) does not present much more difficulties than the one in [16, Section 5], and the proof is given for the sake of completeness only.

Theorem 2.3.2. *Suppose that L is an operator on $\mathbb{R}^N$ of the form*

$$L = \sum_{i,j=1}^{N} \alpha_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \beta_i \frac{\partial}{\partial x_i} + \gamma, \tag{2.3.2}$$

with $\alpha_{i,j}, \beta_i, \gamma \in C^\infty(\mathbb{R}^N, \mathbb{R})$, with $(\alpha_{i,j})$ symmetric and positive semi-definite. We assume that L is non-totally degenerate at every $x \in \mathbb{R}^N$ and that L is C^∞-hypoelliptic in every open set.

Then there exists a basis for the Euclidean topology of $\mathbb{R}^N$ made of open sets Ω with the following properties: for every continuous function f on $\overline{\Omega}$ and for every continuous function φ on $\partial\Omega$, there exists one and only one solution $u \in C(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} Lu = -f & \text{on } \Omega \text{ (in the weak sense of distributions)}, \\ u = \varphi & \text{on } \partial\Omega \text{ (point-wise)}. \end{cases} \tag{2.3.3}$$

Furthermore, if $f, \varphi \geq 0$ then $u \geq 0$ as well. Finally, if f belongs to $C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$, then the same is true of u, and u is a classical solution of (2.3.3).

Finally, if the zero-order term γ of L is non-positive on $\mathbb{R}$, the above basis $\{\Omega\}$ does not depend on γ. If $\gamma < 0$, the basis $\{\Omega\}$ only depends on the principal matrix $(\alpha_{i,j})$ of L.

The key step is to construct a basis for the Euclidean topology of $\mathbb{R}^N$ as follows:

Lemma 2.3.3. *Let $A(x) = (a_{i,j}(x))$ be a matrix with real-valued continuous entries on $\mathbb{R}^N$, which is symmetric, positive semi-definite and non-vanishing at a point $x_0 \in \mathbb{R}^N$.*

Then, there exists a basis of connected open neighborhoods $\mathcal{B}_{x_0}$ of x_0 such that any $\Omega \in \mathcal{B}_{x_0}$ satisfies the following property: for every $y \in \partial\Omega$ there exists $\nu \in \mathbb{R}^N \smallsetminus \{0\}$ such that $\overline{B(y+\nu, |\nu|)}$ intersects $\overline{\Omega}$ at y only, and such that

$$\langle A(y)\,\nu, \nu \rangle > 0. \tag{2.3.4}$$

Proof. By the assumptions on $A(x_0)$ there exists a unit vector h_0 such that

$$\langle A(x_0) h_0, h_0 \rangle > 0. \tag{2.3.5}$$

Following the idea of Bony [16], we choose the neighborhood basis $\mathcal{B}_{x_0} = \{\Omega(\varepsilon)\}$ as follows:

$$\Omega(\varepsilon) := B(x_0 + \varepsilon^{-1} h_0, \varepsilon^{-1} + \varepsilon^2) \cap B(x_0 - \varepsilon^{-1} h_0, \varepsilon^{-1} + \varepsilon^2).$$

It suffices to show that there exists $\overline{\varepsilon} > 0$ such that every $\Omega(\varepsilon)$ with $0 < \varepsilon \leq \overline{\varepsilon}$ satisfies the requirement of the lemma. Now, the set $\Omega(\varepsilon)$ (which is trivially an open neighborhood of x_0) shrinks to $\{x_0\}$ as ε shrinks to 0. Moreover, every $y \in \partial\Omega(\varepsilon)$ belongs to one at least of the spheres $\partial B(x_0 \pm \varepsilon^{-1} h_0, \varepsilon^{-1} + \varepsilon^2)$; accordingly, we choose

$$\nu = \nu_\varepsilon(y) := \frac{y - (x_0 \pm \varepsilon^{-1} h_0)}{\varepsilon^{-1} + \varepsilon^2}$$

to get the geometric condition $\overline{B(y+\nu,|\nu|)} \cap \overline{\Omega(\varepsilon)} = \{y\}$. It obviously holds that $\nu_\varepsilon(y)$ tends to $h(x_0)$ as $\varepsilon \to 0$ (uniformly for bounded x_0, y, h_0), so that (2.3.4) follows from (2.3.5) by continuity arguments, for any $0 \le \varepsilon \le \overline{\varepsilon}$, with $\overline{\varepsilon}$ conveniently small. □

We proceed with the proof of Theorem 2.3.2 by constructing, for any given $x_0 \in \mathbb{R}^N$, a basis of neighborhoods of x_0 as required. The crucial step is to reduce L to some equivalent operator $\widetilde{L}$ with zero-order term $\widetilde{L}(1)$ which is strictly negative around x_0. We observe that this procedure is not necessary if $\gamma = L(1)$ is already known to be negative on $\mathbb{R}^N$. In general, we let

$$\widetilde{L}u := w\,L(w\,u), \quad \text{where } w(x) = 1 - M\,|x-x_0|^2,$$

with $M \gg 1$ to be chosen. Let us denote by $B(x_0)$ the Euclidean ball of centre x_0 and radius $1/\sqrt{M}$. It is readily seen that the second order parts of L and $\widetilde{L}$ are equal, modulo the factor w^2. This shows that $\widetilde{L}$ is non-totally degenerate at any point of $B(x_0)$ and that the principal matrix of $\widetilde{L}$ is symmetric and positive semi-definite at any point of $B(x_0)$. Since

$$\widetilde{L}(1)(x) = w^2(x)\,\gamma(x) - 2M\text{trace}(A(x)) - 2M\sum_{i=1}^{N}\beta_i(x)\,(x-x_0)_i,$$

if we choose M so large that $M > \gamma(x_0)/(2\,\text{trace}(A(x_0)))$ (we remind that $\text{trace}(A(x)) > 0$ at any x since L is non-totally degenerate at any point), then $\widetilde{L}(1)(x_0) < 0$. By continuity, there exists $r > 0$ small enough such that $B'(x_0) := B(x_0, r) \subseteq B(x_0)$ and such that $\widetilde{L}(1) < 0$ on the closure of $B'(x_0)$. We explicitly remark (and this will prove the final statement of the theorem) that the condition $\gamma \le 0$ allows us to take $M = 1$ for all x_0 and to use the bound

$$\widetilde{L}(1)(x) \le -2\text{trace}(A(x)) - 2\sum_{i=1}^{N}\beta_i(x)\,(x-x_0)_i,$$

in order to chose r independently of γ.

Remark 2.3.4. Classical arguments, [71], show that, due to the strict negativity of $\widetilde{L}(1)$ on $B'(x_0)$, the operator $\widetilde{L}$ satisfies the Weak Maximum Principle on every open subset of $B'(x_0)$, that is:

$$\begin{cases} \Omega \subset B'(x_0),\ u \in C^2(\Omega,\mathbb{R}) \\ \widetilde{L}u \ge 0 \text{ on } \Omega \\ \limsup\limits_{x\to y} u(x) \le 0 \text{ for every } y \in \partial\Omega \end{cases} \implies \quad u \le 0 \text{ on } \Omega. \tag{2.3.6}$$

The rest of the proof consists in demonstrating the following statement:

(S) *there exists a basis $\mathcal{B}_{x_0}$ of neighborhoods Ω of x_0 all contained in $B'(x_0)$ with the properties required in Theorem 2.3.2 relative to $\widetilde{L}$ (in place of L).*

Once this is proved, given any $\Omega \in \mathcal{B}_{x_0}$, any $f \in C(\overline{\Omega},\mathbb{R})$ and any $\varphi \in C(\partial\Omega,\mathbb{R})$, we obtain the solution $\widetilde{u}$ of the problem

$$\begin{cases} \widetilde{L}\widetilde{u} = -w\,f & \text{on } \Omega \text{ (in the weak sense of distributions)}, \\ \widetilde{u} = \varphi/w & \text{on } \partial\Omega \text{ (point-wise)}; \end{cases} \tag{2.3.7}$$

then we set $u := w\widetilde{u}$, and a simple verification shows that u solves (2.3.3), so that existence is proved. As for uniqueness, it suffices to observe that for any fixed $\Omega \in \mathcal{B}_{x_0}$, to any solution u of (2.3.3) on Ω, there corresponds a solution $\widetilde{u} = u/w$ of (2.3.7) (which is unique, as it is claimed in (S)). Finally all the other requirements on u in the statement of Theorem 2.3.2 are satisfied, since w is positive and smooth on $\Omega \subseteq B(x_0)$.

Remark 2.3.5. *We remark that the operator $\widetilde{L}$ is C^∞-hypoelliptic on every open subset of $B(x_0)$.*

Indeed, for any open sets V, V' such that $V \subseteq V' \subseteq B(x_0)$, a distribution $u \in \mathcal{D}'(V')$ such that $\widetilde{L}u = f \in C^\infty(V, \mathbb{R})$ satisfies $L(w\,u) = f/w \in C^\infty(V, \mathbb{R})$; thus, by the hypoellipticity of L, we infer that $w\,u \in C^\infty(V, \mathbb{R})$ so that $u \in C^\infty(V, \mathbb{R})$ (recalling that $w \neq 0$ on $B(x_0)$).

We are then left to prove statement (S). From now on we choose a neighborhood basis $\mathcal{B}_{x_0}$ of x_0 consisting of open sets (contained in $B'(x_0)$) as in Lemma 2.3.3 relative to the principal matrix $\widetilde{A}$ of the operator $\widetilde{L}$ (the matrix $\widetilde{A}(x_0)$ is symmetric, positive semi-definite and non vanishing, as already discussed). We will show that any $\Omega \in \mathcal{B}_{x_0}$ has the requirements in statement (S). For the uniqueness part, it suffices to use in a standard way the WMP in Remark 2.3.4 jointly with the hypoellipticity condition in Remark 2.3.5. As for existence, we split the proof in several steps and, to simplify the notation, we write P instead of $\widetilde{L}$.

(I): *f smooth and $\varphi \equiv 0$.* We fix Ω as above, $f \in C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ and $\varphi \equiv 0$. We use a standard elliptic approximation argument. For every $n \in \mathbb{N}$ we set

$$P_n := P + \frac{1}{n}\sum_{j=1}^{N}\Big(\frac{\partial}{\partial x_j}\Big)^2.$$

We observe that:

- P_n is uniformly elliptic on $\mathbb{R}^N$;
- the zero-order term $P_n(1) = P(1)$ $(= \widetilde{L}(1))$ is (strictly) negative on Ω;
- Ω satisfies an exterior ball condition, due to Lemma 2.3.3;
- $f \in C^\infty(\Omega, \mathbb{R})$.

These conditions imply the existence (see e.g., Gilbarg and Trudinger [49]) of a classical solution $u_n \in C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} P_n u_n = -f & \text{on } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $c_0 > 0$ be such that $P(1) < -c_0$ on the closure of $B'(x_0)$. With this choice, we observe that (setting $\|f\|_\infty = \sup_{\overline{\Omega}}|f|$)

$$\begin{cases} P_n\Big(\pm u_n - \dfrac{\|f\|_\infty}{c_0}\Big) = \mp f - \dfrac{\|f\|_\infty}{c_0}P(1) \geq \mp f + \dfrac{\|f\|_\infty}{c_0}c_0 \geq 0 & \text{on } \Omega \\ \pm u_n - \dfrac{\|f\|_\infty}{c_0} = -\dfrac{\|f\|_\infty}{c_0} \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as in Remark 2.3.4, the Weak Maximum Principle for P_n proves that

$$\|u_n\|_\infty = \sup_{x\in\overline{\Omega}} |u_n(x)| \le \frac{\|f\|_\infty}{c_0} \quad \text{uniformly for every } n \in \mathbb{N}. \tag{2.3.8}$$

This provides us with a subsequence of u_n (still denoted by u_n) and a function $u \in L^\infty(\Omega)$ such that u_n tends to u in the weak* topology, that is

$$\lim_{n\to\infty} \int_\Omega u_n\, h = \int_\Omega u\, h, \quad \text{for all } h \in L^1(\Omega). \tag{2.3.9}$$

Moreover one knows that

$$\|u\|_{L^\infty(U)} \le \limsup_{n\to\infty} \|u_n\|_{L^\infty(U)}, \quad \text{for all } U \subseteq \Omega. \tag{2.3.10}$$

From (2.3.9) it easily follows that

$$\int_\Omega u\, P^* \psi = -\int_\Omega f\, \psi, \quad \text{for all } \psi \in C_0^\infty(\Omega, \mathbb{R}).$$

This means that $Pu = -f$ in the weak sense of distributions. As P is hypoelliptic on every open set (Remark 2.3.5), we infer that u can be modified on a null set in such a way that $u \in C^\infty(\Omega, \mathbb{R})$. Thus $Pu = -f$ in the classical sense on Ω. We aim to prove that u can be continuously prolonged to 0 on $\partial\Omega$. To this end, given any $y \in \partial\Omega$, in view of Lemma 2.3.3 (and the choice of Ω), there exists $\nu \in \mathbb{R}^N \smallsetminus \{0\}$ such that $\overline{B(y+\nu, |\nu|)}$ intersects $\overline{\Omega}$ at y only, and such that (see (2.3.4))

$$\langle \widetilde{A}(y)\, \nu, \nu \rangle > 0. \tag{2.3.11}$$

As in the Hopf-type Lemma 2.2.1, we consider the function

$$w(x) := e^{-\lambda|x-(y+\nu)|^2} - e^{-\lambda|\nu|^2},$$

where λ is a positive real number chosen in a moment. For every n and for every x one has

$$\begin{aligned} P_n w(x) &= Pw(x) + \frac{1}{n}\, e^{-\lambda|x-(y+\nu)|^2}\Big(4\lambda^2 |x-(y+\nu)|^2 - 2\lambda N\Big) \\ &\ge Pw(x) - 2\lambda N e^{-\lambda|x-(y+\nu)|^2}. \end{aligned} \tag{2.3.12}$$

If we set $P = \sum_{i,j} \widetilde{a}_{i,j}\partial_{i,j} + \sum_j \widetilde{b}_j \partial_j + \widetilde{c}$, a simple computation shows that

$$\begin{aligned} &\Big(Pw(x) - 2\lambda N e^{-\lambda|x-(y+\nu)|^2}\Big)\Big|_{x=y} \\ &= e^{-\lambda|\nu|^2}\left(4\lambda^2 \langle \widetilde{A}(y)\nu, \nu\rangle - 2\lambda \sum_{j=1}^N \big(\widetilde{a}_{j,j}(y) - \widetilde{b}_j(y)\nu_j\big) - 2\,\lambda\, N\right). \end{aligned}$$

Thanks to (2.3.11), there exists $\lambda \gg 1$ such that the above right-hand side is strictly positive. Therefore, due to (2.3.12) there exist $\varepsilon > 0$ and an open ball $V = B(y, \delta)$ (with ε and δ *independent of* n) such that

$$P_n w(x) \ge \varepsilon \quad \text{for every } x \in V \text{ and every } n \in \mathbb{N}. \tag{2.3.13}$$

We are willing to apply the Weak Maximum Principle for the operator P_n on the open set $\Omega \cap V$, and for the functions $M\,w \pm u_n$, where $M \gg 1$ is chosen as follows. First we have

$$P_n(M\,w \pm u_n) = M\,P_n w \pm P_n u_n = M\,P_n w \mp f \geq M\,\varepsilon \mp f \geq M\,\varepsilon - \|f\|_\infty, \quad \text{in } \Omega \cap V.$$

Consequently we first chose $M > \|f\|_\infty/\varepsilon$. Then we study the behavior of $M\,w \pm u_n$ on

$$\partial(\Omega \cap V) = [V \cap \partial\Omega] \cup [\overline{\Omega} \cap \partial V] =: \Gamma_1 \cup \Gamma_2.$$

Firstly, on Γ_1 we have $M\,w \pm u_n = M\,w \leq 0$ since $\Gamma_1 \subseteq \mathbb{R}^N \smallsetminus B(y+\nu, |\nu|)$. Secondly, on Γ_2,

$$M\,w \pm u_n \leq M\,\max_{\Gamma_2} w + \|u_n\|_\infty \overset{(2.3.8)}{\leq} M\,\max_{\Gamma_2} w + \frac{\|f\|_\infty}{c_0}.$$

Since Γ_2 is a compact set on which w is strictly negative, we have $\max_{\Gamma_2} w < 0$ and the further choice $M \geq -\|f\|_\infty/(c_0 \max_{\Gamma_2} w)$ yields $M\,w \pm u_n \leq 0$ on Γ_2. Summing up,

$$\begin{cases} P_n(M\,w \pm u_n) \geq 0 & \text{on } \Omega \cap V \\ M\,w \pm u_n \leq 0 & \text{on } \partial(\Omega \cap V). \end{cases}$$

The Weak Maximum Principle yields $M\,w \pm u_n \leq 0$ on $\Omega \cap V$, that is (since $w < 0$ on Ω)

$$|u_n(x)| \leq M\,|w(x)| \quad \text{for every } x \in \Omega \cap V \text{ and for every } n \in \mathbb{N}.$$

Since $w(y) = 0$, for every $\sigma > 0$ there exists an open neighborhood $W \subset V$ of y such that $\|w\|_{L^\infty(W)} < \sigma$; the above inequality then gives $\|u_n\|_{L^\infty(W\cap\Omega)} \leq M\,\sigma$. Jointly with (2.3.10) we deduce that $\|u\|_{L^\infty(W\cap\Omega)} \leq M\,\sigma$, so that $\lim_{\Omega \ni x \to y} u(x) = 0$. From the arbitrariness of y, we obtain that u prolongs to be 0 on $\partial\Omega$ with continuity.

In order to complete the proof of (S), we are left to show that if $f \in C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ is nonnegative, then the unique solution $u \in C(\overline{\Omega}, \mathbb{R})$ of

$$\begin{cases} Pu = -f & \text{on } \Omega \text{ (in the weak sense of distributions)} \\ u = 0 & \text{on } \partial\Omega \text{ (point-wise)} \end{cases}$$

is nonnegative as well. From the hypoellipticity of P (see Remark 2.3.5), we already know that $u \in C^\infty(\Omega, \mathbb{R})$, and we can apply the WMP to $-u$ (see Remark 2.3.4) to get $-u \leq 0$.

(II): *f and φ smooth.* We fix Ω as above, and f is in $C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ and φ is the restriction to $\partial\Omega$ of some function Φ which is smooth and defined on an open neighborhood of $\overline{\Omega}$. As in Step (I), we consider the unique solution $v \in C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ of

$$\begin{cases} Pv = -f - P\Phi & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and we observe that $u = v + \Phi$ is the (unique) classical solution of

$$\begin{cases} Pu = -f & \text{on } \Omega \\ u = \Phi|_{\partial\Omega} = \varphi & \text{on } \partial\Omega. \end{cases}$$

If furthermore $f, \varphi \geq 0$, the nonnegativity of u is a consequence of the WMP as in Step (I).

(III): *f and φ continuous*. Finally we consider $f \in C(\overline{\Omega}, \mathbb{R})$ and $\varphi \in C(\partial\Omega, \mathbb{R})$. By the Stone-Weierstrass Theorem, there exist polynomial functions f_n, φ_n uniformly converging to f, φ respectively on $\overline{\Omega}, \partial\Omega$ as $n \to \infty$. As in Step (II), for every $n \in \mathbb{N}$ we consider the unique classical solution u_n of

$$\begin{cases} Pu_n = -f_n & \text{on } \Omega \\ u_n = \varphi_n & \text{on } \partial\Omega. \end{cases}$$

From the fact that $-c_0 := \max_{\overline{\Omega}} P(1) < 0$, we can argue as in Step (I), obtaining the estimate

$$\|u_n - u_m\|_{C(\overline{\Omega})} \leq \max\left\{\frac{1}{c_0}\, \|f_n - f_m\|_{C(\overline{\Omega})}, \|\varphi_n - \varphi_m\|_{C(\partial\Omega)}\right\}.$$

This proves that there exists the uniform limit $u := \lim_{n\to\infty} u_n$ in $C(\overline{\Omega}, \mathbb{R})$. Clearly one has: $u = \varphi$ point-wise on $\partial\Omega$ and $Pu = -f$ in the weak sense of distributions on Ω. From the hypoellipticity of P (Remark 2.3.5) we infer that f smooth implies u smooth. Finally, suppose that $f, \varphi \geq 0$. By the Tietze Extension Theorem, we prolong f out of $\overline{\Omega}$ to a *continuous* function F on $\mathbb{R}^N$; we consider a mollifying sequence $F_n \in C^\infty(\mathbb{R}^N, \mathbb{R})$ uniformly converging to F on the compact sets of $\mathbb{R}^N$. Since mollification preserves the sign, the fact that $F|_{\overline{\Omega}} \equiv f \geq 0$ on $\overline{\Omega}$ gives that $F_n \geq 0$ on $\overline{\Omega}$. As above in this Step, we solve the problem

$$\begin{cases} PU_n = -F_n & \text{on } \Omega \\ U_n = \varphi & \text{on } \partial\Omega, \end{cases} \qquad \text{with} \quad U_n \in C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R}),$$

and we get that U_n uniformly converges on $\overline{\Omega}$ to the unique continuous solution u of

$$\begin{cases} Pu = -f & \text{in } \mathcal{D}'(\Omega) \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

From the WMP for $-U_n$ (recalling that $F_n \geq 0$ and $\varphi \geq 0$), we derive $U_n \geq 0$ on $\overline{\Omega}$; this gives $u(x) = \lim_{n\to\infty} U_n(x) \geq 0$ for all $x \in \overline{\Omega}$. This completes the proof. □

2.3.1 The Green function and the Green kernel for $\mathcal{L} - \varepsilon$

Thanks to the existence of the weak solution of the Dirichlet problem for $\mathcal{L}_\varepsilon$ on a bounded open set Ω, we can define the associated Green operator as usual:

Definition 2.3.6 (Green operator and Green measure). Let $\varepsilon \geq 0$ be fixed, and let $\mathcal{L}_\varepsilon$ and Ω satisfy, respectively, the hypothesis and the thesis of Lemma 2.3.1. We consider the operator (depending on $\mathcal{L}_\varepsilon$ and Ω; we avoid keeping track of the dependency on Ω in the notation)

$$G_\varepsilon : C(\overline{\Omega}, \mathbb{R}) \longrightarrow C(\overline{\Omega}, \mathbb{R}) \tag{2.3.14}$$

mapping $f \in C(\overline{\Omega}, \mathbb{R})$ into the function $G_\varepsilon(f)$ which is the unique distributional solution u in $C(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon u = -f & \text{on } \Omega \quad \text{(in the weak sense of distributions)}, \\ u = 0 & \text{on } \partial\Omega \quad \text{(point-wise)}. \end{cases} \tag{2.3.15}$$

We call G_ε *the Green operator related to* $\mathcal{L}_\varepsilon$ *and to the open set* Ω.

By the Riesz Representation Theorem (which is applicable thanks to the monotonicity properties in Lemma 2.3.1 with respect to the function f), for every $x \in \overline{\Omega}$ there exists a (nonnegative) Radon measure $\lambda_{x,\varepsilon}$ on $\overline{\Omega}$ such that

$$G_\varepsilon(f)(x) = \int_{\overline{\Omega}} f(y)\,\mathrm{d}\lambda_{x,\varepsilon}(y), \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}). \tag{2.3.16}$$

We call $\lambda_{x,\varepsilon}$ *the Green measure related to* $\mathcal{L}_\varepsilon$ *(to the open set* Ω *and to the point* x*)*.

Let $\mathcal{L}$ be as in (2.1.1); in this chapter, we set once and for all

$$\mathrm{d}\nu(x) := V(x)\,\mathrm{d}x, \tag{2.3.17}$$

that is, ν is the (Radon) measure on $\mathbb{R}^N$ associated with the (positive) density V in (2.1.1), absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$. It is clear that the measure ν plays the following key rôle:

$$\int \varphi\,\mathcal{L}\psi\,\mathrm{d}\nu = \int \psi\,\mathcal{L}\varphi\,\mathrm{d}\nu, \quad \text{for every } \varphi, \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}), \tag{2.3.18}$$

thus making $\mathcal{L}$ (formally) self-adjoint in the space $L^2(\mathbb{R}^N, \mathrm{d}\nu)$. We observe that (in general) our operators $\mathcal{L}$ in (2.1.1) are not *classically* self-adjoint; indeed the classical adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is related to $\mathcal{L}$ by the following identity (a consequence of (2.3.18))

$$\mathcal{L}^* u = V\,\mathcal{L}(u/V), \quad \text{for every } u \text{ of class } C^2. \tag{2.3.19}$$

The possibility of dealing with non-identically 1 densities V (as in the case of Lie groups, see Example 2.1.1-(a)) makes it more convenient to decompose the Green measure $\lambda_{x,\varepsilon}$ with respect to ν in (2.3.17), rather than w.r.t. Lebesgue measure. Hence we prove the following:

Theorem 2.3.7 (Green kernel). *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1), *with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and that $\mathcal{L}$ satisfies* (NTD). *Let $\varepsilon \geq 0$ be fixed. We set $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$ and we assume that $\mathcal{L}_\varepsilon$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

Let Ω be an open set as in Lemma 2.3.1. If G_ε and $\lambda_{x,\varepsilon}$ are the Green operator and the Green measure related to $\mathcal{L}_\varepsilon$ (Definition 2.3.6), there exists a function $k_\varepsilon : \Omega \times \Omega \to \mathbb{R}$, smooth and positive out of the diagonal of $\Omega \times \Omega$, such that the following representation holds true:

$$G_\varepsilon(f)(x) = \int_\Omega f(y)\,k_\varepsilon(x,y)\,\mathrm{d}\nu(y), \quad \textit{for every } x \in \Omega, \tag{2.3.20}$$

and for every $f \in C(\overline{\Omega}, \mathbb{R})$. We call k_ε the Green kernel related to *$\mathcal{L}_\varepsilon$ (and to the open set Ω).*

Furthermore, we have the following properties:

(i) *Symmetry of the Green kernel:*

$$k_\varepsilon(x,y) = k_\varepsilon(y,x) \quad \textit{for every } x,y \in \Omega. \tag{2.3.21}$$

(ii) *For every fixed $x \in \Omega$, the function $k_\varepsilon(x,\cdot)$ is $\mathcal{L}_\varepsilon$-harmonic in $\Omega \smallsetminus \{x\}$; moreover $G_\varepsilon(\mathcal{L}_\varepsilon \varphi) = -\varphi = \mathcal{L}_\varepsilon(G_\varepsilon(\varphi))$ for any $\varphi \in C_0^\infty(\Omega,\mathbb{R})$, that is*

$$\begin{aligned} -\varphi(x) &= \int_\Omega \mathcal{L}_\varepsilon \varphi(y)\, k_\varepsilon(x,y)\,\mathrm{d}\nu(y) \\ &= \mathcal{L}_\varepsilon\Big(\int_\Omega \varphi(y)\, k_\varepsilon(x,y)\,\mathrm{d}\nu(y)\Big), \qquad \textit{for every } \varphi \in C_0^\infty(\Omega,\mathbb{R}). \end{aligned} \tag{2.3.22}$$

(iii) *For every fixed $x \in \Omega$, one has*

$$\lim_{y\to y_0} k_\varepsilon(x,y) = 0 \quad \textit{for any } y_0 \in \partial\Omega. \tag{2.3.23}$$

(iv) *For every fixed $x \in \Omega$, the functions $k_\varepsilon(x,\cdot) = k_\varepsilon(\cdot,x)$ are in $L^1(\Omega)$, and $k_\varepsilon \in L^1(\Omega\times\Omega)$.*

The key ingredients in the proof of the above results are the following facts:

- the hypoellipticity of $\mathcal{L}_\varepsilon$ (as assumed in the hypothesis) which will imply the hypoellipticity of the *classical* adjoint of $\mathcal{L}_\varepsilon$ (see Remark 2.3.8);
- the C^∞-topology on the space of the $\mathcal{L}_\varepsilon$-harmonic functions is the same as the L^1_{loc}-topology, another consequence of the hypoellipticity of $\mathcal{L}_\varepsilon$ (Remark 2.3.9);
- the fact that $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{R}^N, \mathrm{d}\nu)$ (see (2.3.18)) so that the same is true of $\mathcal{L}_\varepsilon$ (this will be crucial in proving the symmetry of the Green kernel);
- the Strong Maximum Principle for the perturbed operator $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$, which we obtain as a consequence of our previous Strong Maximum Principle for $\mathcal{L}$ in Theorem 2.2.2 (see precisely Remark 2.2.6, where *nonnegative* maxima are considered): this is a key step for the proof of the *positivity* of k_ε;
- the Schwartz Kernel Theorem (used for the regularity of the Green kernel).

In the first part of the proof (Steps I–III) we follow the classical scheme by Bony (see [16, Theorem 6.1]), hence we skip many details; it is instead in Step IV that a slight difference is presented, in that we exploit the measure $\mathrm{d}\nu(x) = V(x)\,\mathrm{d}x$ in order to obtain the symmetry property of the Green kernel even when our operator $\mathcal{L}$ is not (classically) self-adjoint. The problem of the behavior of the Green kernel along the diagonal is more subtle, as it is shown by Fabes, Jerison and Kenig in [31] who proved that, for divergence-form operators as in (2.1.1) (when $V \equiv 1$ and, roughly put, when the degeneracy of $A(x)$ is controlled by a suitable weight) the limit of the Green kernel along the diagonal need not be infinite.

We are ready for the proof.

Proof (of Theorem 2.3.7). We fix an operator $\mathcal{L}$ of the form (2.1.1), with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and we assume that $\mathcal{L}$ satisfies (NTD). Moreover, we also fix $\varepsilon \geq 0$ (note that the case $\varepsilon = 0$ is allowed) and we set $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$; we assume that $\mathcal{L}_\varepsilon$ is hypoelliptic on every open subset of $\mathbb{R}^N$. Finally, Ω is a fixed open set as in Lemma 2.3.1, such that the Dirichlet problem (2.3.1) is (uniquely) solvable.

From Lemma 2.3.1, we know that there exists a monotone operator G_ε (which we called the Green operator related to $\mathcal{L}_\varepsilon$ and Ω); since $\varepsilon \geq 0$ is fixed, in all this section we drop the subscript ε in $G_\varepsilon, k_\varepsilon, \lambda_{x,\varepsilon}$ and we simply write G, k, λ_x. Hence we are given the monotone operator

$$G : C(\overline{\Omega}, \mathbb{R}) \longrightarrow C(\overline{\Omega}, \mathbb{R})$$

mapping $f \in C(\overline{\Omega}, \mathbb{R})$ into the unique function $G(f) \in C(\overline{\Omega}, \mathbb{R})$ satisfying

$$\begin{cases} \mathcal{L}_\varepsilon(G(f)) = -f & \text{on } \Omega \quad \text{(in the weak sense of distributions)}, \\ G(f) = 0 & \text{on } \partial\Omega \quad \text{(point-wise)}. \end{cases} \tag{2.3.24}$$

We also know that the (Riesz) representation

$$G(f)(x) = \int_{\overline{\Omega}} f(y)\,\mathrm{d}\lambda_x(y) \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}) \text{ and every } x \in \overline{\Omega} \tag{2.3.25}$$

holds true, with a unique Radon measure λ_x defined on $\overline{\Omega}$ (which we called the Green measure related to $\mathcal{L}_\varepsilon$, Ω and x).

Finally, we set $\mathrm{d}\nu(x) := V(x)\,\mathrm{d}x$ and we observe that (as in (2.3.18))

$$\int \varphi\,\mathcal{L}_\varepsilon\psi\,\mathrm{d}\nu = \int \psi\,\mathcal{L}_\varepsilon\varphi\,\mathrm{d}\nu, \qquad \text{for every } \varphi, \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}). \tag{2.3.26}$$

STEP I. We fix $x \in \Omega$. We begin by proving that λ_x is absolutely continuous with respect to the Lebesgue measure on $\overline{\Omega}$. To this end, let $\varphi \in C_0^\infty(\Omega, \mathbb{R})$; by (2.3.24) it is clear that $G(\mathcal{L}_\varepsilon\varphi) = -\varphi$, so that (see (2.3.25))

$$-\varphi(x) = \int_{\overline{\Omega}} \mathcal{L}_\varepsilon\varphi(y)\,\mathrm{d}\lambda_x(y), \qquad \text{for every } \varphi \in C_0^\infty(\overline{\Omega}, \mathbb{R}).$$

If we consider λ_x as a distribution on Ω in the standard way, this identity boils down to

$$(\mathcal{L}_\varepsilon)^*\lambda_x = -\mathrm{Dir}_x \quad \text{in } \mathcal{D}'(\Omega), \tag{2.3.27}$$

where Dir_x denotes the Dirac mass at x, and $(\mathcal{L}_\varepsilon)^*$ is the classical adjoint operator of $\mathcal{L}_\varepsilon$. It is noteworthy to observe that, in general, $(\mathcal{L}_\varepsilon)^*$ is neither equal to $\mathcal{L}_\varepsilon$ nor of the form $\widetilde{\mathcal{L}} - \varepsilon$ for any $\widetilde{\mathcal{L}}$ a divergence operator as in (2.1.1).

However, the following crucial property of $(\mathcal{L}_\varepsilon)^*$ is fulfilled:

Remark 2.3.8. *The operator $(\mathcal{L}_\varepsilon)^*$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

Indeed, let $U \subseteq W$ be open sets and let $u \in \mathcal{D}'(W)$ be such that $(\mathcal{L}_\varepsilon)^* u = h$ in $\mathcal{D}'(U)$, where $h \in C^\infty(U, \mathbb{R})$. This gives the following chain of identities (here $\psi \in C_0^\infty(U, \mathbb{R})$ is arbitrary)

$$\int h\,\psi = \langle u, \mathcal{L}_\varepsilon \psi \rangle = \langle u, \mathcal{L}\psi - \varepsilon\psi \rangle \overset{(2.3.19)}{=} \Big\langle u, \frac{\mathcal{L}^*(V\psi)}{V} - \varepsilon\psi \Big\rangle$$
$$= \Big\langle \frac{u}{V}, \mathcal{L}^*(V\psi) - \varepsilon\psi\, V \Big\rangle = \Big\langle \frac{u}{V}, (\mathcal{L}_\varepsilon)^*(V\psi) \Big\rangle.$$

If we write $\int h\,\psi = \int \frac{h}{V}\,(\psi\, V)$, and if we observe that $C_0^\infty(U, \mathbb{R}) = \{\psi\, V \,:\, \psi \in C_0^\infty(U, \mathbb{R})\}$, the above computation shows that $\mathcal{L}_\varepsilon(u/V) = h/V$ in $\mathcal{D}'(U)$. The hypoellipticity of $\mathcal{L}_\varepsilon$ now gives $u/V \in C^\infty(U, R)$ whence $u \in C^\infty(U, R)$, as V is smooth and positive.

Identity (2.3.27) gives in particular $(\mathcal{L}_\varepsilon)^* \lambda_x = 0$ in $\mathcal{D}'(\Omega \smallsetminus \{x\})$; thanks to Remark 2.3.8, this ensures the existence of $g_x \in C^\infty(\Omega \smallsetminus \{x\}, \mathbb{R})$ such that the distribution λ_x restricted to $\Omega \smallsetminus \{x\}$ is the function-type distribution associated with the function g_x; equivalently

$$\int \varphi(y)\,\mathrm{d}\lambda_x(y) = \int \varphi(y)\,g_x(y)\,\mathrm{d}y, \quad \text{for every } \varphi \in C_0^\infty(\Omega \smallsetminus \{x\}, \mathbb{R}). \tag{2.3.28}$$

Clearly $g_x \geq 0$ on $\Omega \smallsetminus \{x\}$ and $(\mathcal{L}_\varepsilon)^* g_x = 0$ in $\Omega \smallsetminus \{x\}$. This temporarily proves that λ_x coincides with $g_x(y)\,\mathrm{d}y$ on $\Omega \smallsetminus \{x\}$. We claim that this is also true throughout Ω. This will follow if we show that $C := \lambda_x(\{x\}) = 0$. Clearly, by the definition of C, on Ω we have

$$\lambda_x = C\,\mathrm{Dir}_x + (\lambda_x)|_{\Omega \smallsetminus \{x\}} = C\,\mathrm{Dir}_x + g_x(y)\,\mathrm{d}y.$$

Treating this as an identity between distributions on Ω, we apply the operator $(\mathcal{L}_\varepsilon)^*$ to get

$$C\,(\mathcal{L}_\varepsilon)^*\mathrm{Dir}_x = -\mathrm{Dir}_x - (\mathcal{L}_\varepsilon)^*(g_x(y)\,\mathrm{d}y).$$

Here we used (2.3.27). We now proceed as follows:

- we multiply both sides by a C^∞ function χ compactly supported in Ω and $\chi \equiv 1$ near x;
- we compute the Fourier transform of the tempered distributions obtained as above;
- on the left-hand side we obtain a function-type distribution associated with function
$$y \mapsto C\,e^{-i\langle x,y\rangle}\Big(-\sum_{i,j} a_{i,j}(x)\,y_i y_j + \{\text{polynomial in } y \text{ of degree} \leq 1\}\Big),$$
where $(a_{i,j})$ is the principal matrix of $\mathcal{L}$;
- on the right-hand side we obtain a function-type distribution associated with a function which is the sum of $y \mapsto -e^{-i\langle x,y\rangle}$ with a function of the form
$$y \mapsto -\sum_{i,j} \alpha_{i,j}(x,y)\,y_i y_j + \{\text{polynomial in } y \text{ of degree} \leq 1\},$$
where
$$\alpha_{i,j}(x,y) = -\int g_x(\xi)\,\chi(\xi)\,a_{i,j}(\xi)\,e^{-i\langle \xi,y\rangle}\,\mathrm{d}\xi.$$
By the Riemann-Lebesgue Theorem one has $\alpha_{i,j}(x,y) \longrightarrow 0$ as $|y| \to \infty$. This implies that $C = 0$, since at least one of the entries of $(a_{i,j}(x))$ is non-vanishing, due to the (NTD) hypothesis on $\mathcal{L}$.

We have therefore proved that, for any $x \in \Omega$,

$$\mathrm{d}\lambda_x(y) = g_x(y)\,\mathrm{d}y \text{ on } \Omega. \tag{2.3.29}$$

Since λ_x is a finite measure (recalling that $\overline{\Omega}$ is compact), from (2.3.29) we get $g_x \in L^1(\Omega)$ for every $x \in \Omega$.

STEP II. We next show that $\lambda_x(\partial\Omega) = 0$ for any $x \in \overline{\Omega}$. For small $\delta > 0$, we let D_δ denote the closed δ-neighborhood of $\partial\Omega$ of the points in $\mathbb{R}^N$ having distance from $\partial\Omega$ less than or equal to δ; we then choose a function $F \in C(\mathbb{R}^N, [0,1])$ which is identically 1 on $\partial\Omega$ and is supported in the interior of D_δ. We denote by f the restriction of F to $\overline{\Omega}$. From (2.3.25) we have

$$0 \le G(f)(x) = \int_{\overline{\Omega}} f(y)\,\mathrm{d}\lambda_x(y) \le \int_{\overline{\Omega}} \mathrm{d}\lambda_x(y) = G(1)(x), \quad \text{for every } x \in \overline{\Omega}. \tag{2.3.30}$$

For any $x \in \overline{\Omega}$ we have

$$\begin{aligned}\lambda_x(\partial\Omega) &= \int_{\partial\Omega} \mathrm{d}\lambda_x(y) = \int_{\partial\Omega} f(y)\,\mathrm{d}\lambda_x(y) \le \int_{\overline{\Omega}} f(y)\,\mathrm{d}\lambda_x(y) = G(f)(x)\\ &\le \sup_{\overline{\Omega}} G(f) = \max\left\{ \sup_{\overline{\Omega}\cap D_\delta} G(f), \sup_{\overline{\Omega}\setminus D_\delta} G(f) \right\} =: \max\{\mathrm{I}, \mathrm{II}\}.\end{aligned}$$

We claim that I and II in the above right-hand side are bounded from above by $\sup_{\overline{\Omega}\cap D_\delta} G(1)$. This is true of I, due to (2.3.30); as for II we invoke the last assertion in Remark 2.2.7 applied to:

- the hypoelliptic operator $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$,
- the bounded open set $\Omega_1 := \overline{\Omega} \setminus D_\delta$,
- the nonnegative function $G(f)$, which satisfies $\mathcal{L}_\varepsilon G(f) = -f = 0$ on Ω_1 both weakly and strongly due to the hypoellipticity of $\mathcal{L}_\varepsilon$.

The mentioned Remark 2.2.7 then ensures that the values of $G(f)$ on $\overline{\Omega} \setminus D_\delta$ are bounded from above by the values of $G(f)$ on the boundary of this set, so that II $\le$ I. Summing up,

$$\lambda_x(\partial\Omega) \le \max\{\mathrm{I}, \mathrm{II}\} \le \sup_{\overline{\Omega}\cap D_\delta} G(1).$$

As δ goes to 0, the right-hand side tends to $\sup_{\partial\Omega} G(1) = 0$ by (2.3.24). This gives the desired $\lambda_x(\partial\Omega) = 0$, for any $x \in \overline{\Omega}$. By collecting together (2.3.29) and $\lambda_x(\partial\Omega) = 0$, we infer that (for every $f \in C(\overline{\Omega}, \mathbb{R})$ and $x \in \Omega$)

$$G(f)(x) \overset{(2.3.25)}{=} \int_{\overline{\Omega}} f(y)\,\mathrm{d}\lambda_x(y) = \int_\Omega f(y)\,\mathrm{d}\lambda_x(y) \overset{(2.3.29)}{=} \int_\Omega f(y)\,g_x(y)\,\mathrm{d}y.$$

This proves the identity

$$G(f)(x) = \int_\Omega f(y)\,g_x(y)\,\mathrm{d}y, \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}) \text{ and every } x \in \Omega. \tag{2.3.31}$$

If $\varphi \in C_0^\infty(\Omega, \mathbb{R})$, since we know that $G(\mathcal{L}_\varepsilon \varphi) = -\varphi$, we get

$$-\varphi(x) = \int_\Omega \mathcal{L}_\varepsilon \varphi(y)\, g_x(y)\, \mathrm{d}y, \quad \text{for every } x \in \Omega. \tag{2.3.32}$$

This is equivalent to

$$(\mathcal{L}_\varepsilon)^* g_x = -\mathrm{Dir}_x \quad \text{for every } x \in \Omega. \tag{2.3.33}$$

STEP III. If g_x is as in Step I, we are ready to set

$$g : \Omega \times \Omega \longrightarrow [0, \infty], \qquad g(x,y) := \begin{cases} g_x(y) & \text{if } x \neq y \\ \infty & \text{if } x = y. \end{cases}$$

Hence the representation (2.3.31) becomes

$$G(f)(x) = \int_\Omega f(y)\, g(x,y)\, \mathrm{d}y, \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}) \text{ and every } x \in \Omega. \tag{2.3.34}$$

We aim to prove that g is smooth outside the diagonal of $\Omega \times \Omega$.

Remark 2.3.9. Let O be any open subset of $\mathbb{R}^N$. *The hypoellipticity of a general PDO L as in* (2.1.2) *ensures the equality of the topologies on $\mathcal{H}_L(O)$ inherited by the Fréchet spaces $C^\infty(O)$ and $L^1_{\mathrm{loc}}(O)$.*

Indeed, let $\mathcal{X}$ and $\mathcal{Y}$ denote respectively the topological space $\mathcal{H}_L(O)$ with the topologies inherited by $C^\infty(O)$ and $L^1_{\mathrm{loc}}(O)$. Then $\mathcal{X}$ and $\mathcal{Y}$ are Fréchet spaces, since, if a sequence $u_n \in \mathcal{H}_L(O)$ converges to u uniformly on the compact sets of Ω or, more generally in L^1_{loc},

$$0 = \int u_n\, L^* \varphi \xrightarrow{n\to\infty} \int u\, L^* \varphi, \qquad \forall\ \varphi \in C_0^\infty(O, \mathbb{R}).$$

Now, the identity map $\iota : \mathcal{X} \to \mathcal{Y}$ is trivially linear, bijective and continuous, whence, by the Open Mapping Theorem, ι is a homeomorphism, whence the mentioned topologies coincide.

We next resume our main proof. The set $\{g_x\}_{x\in\Omega}$ is bounded in $L^1(\Omega)$, since

$$0 \le \int_\Omega g_x(y)\, \mathrm{d}y = G(1)(x) \le \max_{\overline{\Omega}} G(1).$$

A fortiori, the set $\{g_x\}_{x\in\Omega}$ is also bounded in the topological vector space $L^1_{\mathrm{loc}}(\Omega)$. We next fix two disjoint open sets U, W with closures contained in Ω. The family of the restrictions

$$\left\{ (g_x)\big|_U \right\}_{x \in W}$$

is contained in the space of the $(\mathcal{L}_\varepsilon)^*$-harmonic functions on U. By Remark 2.3.9, the set $\mathcal{G}$ is also bounded in the topological vector space

$$\mathcal{H}_{(\mathcal{L}_\varepsilon)^*}(U), \quad \text{endowed with the } C^\infty\text{-topology}.$$

This means that, for every compact set $K \subset U$ and for every $m \in \mathbb{N}$, there exists a constant $C(K,m) > 0$ such that

$$\sup_{|\alpha| \le m}\ \sup_{y \in K} \left| \left(\frac{\partial}{\partial y}\right)^\alpha g(x,y) \right| \le C(K,m), \quad \text{uniformly for } x \in W. \tag{2.3.35}$$

Following Bony [16, Section 6], we introduce the operator F transforming any distribution T compactly supported in U into the function on W defined by

$$F(T) : W \longrightarrow \mathbb{R}, \qquad F(T)(x) := \langle T, g_x \rangle \quad (x \in W).$$

The definition is well-posed since $g_x \in C^\infty(U, \mathbb{R})$ (and T is compactly supported in U). We claim that $F(T) \in C^\infty(W, \mathbb{R})$. Once this is proved, by the Schwartz Kernel Theorem (see e.g., [29, Section 11] or [91, Chapter 50]), we can conclude that $g(x, y)$ is smooth on $W \times U$. By the arbitrariness of the disjoint open sets U, W this proves that $g(x, y)$ is smooth out of the diagonal of $\Omega \times \Omega$, as desired.

As for the proof of the claimed $F(T) \in C^\infty(W, \mathbb{R})$, we can take (say, by some appropriate convolution) a sequence of continuous functions f_n, supported in U, converging to T in the weak sense of distributions; due to the compactness of the supports (of the f_n and of T),

$$\lim_{n\to\infty} \int_U f_n\, \varphi = \langle T, \varphi \rangle, \quad \text{for every } \varphi \in C^\infty(U, \mathbb{R}).$$

We are hence entitled to take $\varphi = g_x$ (for any fixed $x \in W$). From (2.3.34) we get

$$\lim_{n\to\infty} G(f_n)(x) = \langle T, g_x \rangle = F(T)(x), \quad \text{for any } x \in W. \tag{2.3.36}$$

We now prove that $F(T) \in L^\infty(W)$; this follows from the next calculation (here $C > 0$ and $m \in \mathbb{N}$ are constants depending on T and on the compact set $\overline{U}$)

$$\|F(T)\|_{L^\infty} = \sup_{x\in W} |\langle T, g_x \rangle| \le \sup_{x\in W} C \sum_{|\alpha|\le m} \sup_{y\in\overline{U}} \left| \Big(\frac{\partial}{\partial y}\Big)^\alpha g(x, y) \right| \overset{(2.3.35)}{\le} \widetilde{C}(\overline{U}, m) < \infty.$$

We finally prove that $\mathcal{L}_\varepsilon(F(T)) = 0$ in the weak sense of distributions on W; by the hypoellipticity of $\mathcal{L}_\varepsilon$ this will yield the smoothness of $F(T)$ on W. We aim to show that,

$$\int_W F(T)(x)\, (\mathcal{L}_\varepsilon)^* \varphi(x)\, \mathrm{d}x = 0 \qquad \text{for any } \varphi \in C_0^\infty(W).$$

Now, the left-hand side is (by (2.3.36))

$$\int \lim_{n\to\infty} G(f_n)(x)\, (\mathcal{L}_\varepsilon)^* \varphi(x)\, \mathrm{d}x.$$

If a dominated convergence can be applied, this is equal to

$$\lim_{n\to\infty} \int_W G(f_n)(x)\, (\mathcal{L}_\varepsilon)^* \varphi(x)\, \mathrm{d}x \text{(2.3.24)}= -\lim_{n\to\infty} \int_W f_n(x)\, \varphi(x)\, \mathrm{d}x = 0,$$

the last equality descending from the fact that the f_n are supported in U for every n. We are then left with showing that the dominated convergence is fulfilled: this is a consequence of (2.3.35), of the boundedness of $F(T)$ on W, and of the fact that the convergence in (2.3.36) is indeed uniform w.r.t. $x \in W$ (a general result of distribution theory: the uniform convergence for sequences of distributions on bounded sets).

STEP IV. We are finally ready to introduce our kernel

$$k:\Omega\times\Omega\longrightarrow[0,\infty),\qquad k(x,y):=\frac{g(x,y)}{V(y)}. \tag{2.3.37}$$

Clearly, from (2.3.34) and (2.3.18) we immediately have

$$G(f)(x)=\int_\Omega f(y)\,k(x,y)\,\mathrm{d}\nu(y),\quad \text{for every } f\in C(\overline{\Omega},\mathbb{R}) \text{ and every } x\in\Omega. \tag{2.3.38}$$

This gives the representation (2.3.20) whilst (2.3.22) follows from (2.3.32).

The integrability of $k(x,\cdot)$ in Ω is a consequence of $g_x\in L^1(\Omega)$ (and the positivity of the continuous function V on $\mathbb{R}^N$). Moreover, k is smooth on $\Omega\times\Omega$ deprived of the diagonal by Step III. Also, the nonnegative function k is integrable on $\Omega\times\Omega$ as this computation shows:

$$0\le\int_{\Omega\times\Omega}k(x,y)\,\mathrm{d}x\mathrm{d}y=\int_\Omega\Big(\int_\Omega\frac{1}{V(y)}\,k(x,y)\,\mathrm{d}\nu(y)\Big)\mathrm{d}x\overset{(2.3.38)}{=}\int_\Omega G(1/V)(x)\,\mathrm{d}x<\infty,$$

the last inequality following from the continuity of $G(1/V)$ on the compact set $\overline{\Omega}$.

For fixed $x\in\Omega$, the $\mathcal{L}_\varepsilon$-harmonicity of the function $k(x,\cdot)$ in $\Omega\smallsetminus\{x\}$ is a consequence of the following computation

$$0\overset{(2.3.33)}{=}(\mathcal{L}_\varepsilon)^*g_x\overset{(2.3.19)}{=}V\,\mathcal{L}_\varepsilon\Big(\frac{g_x}{V}\Big)\overset{(2.3.37)}{=}V\,\mathcal{L}_\varepsilon(k(x,\cdot)).$$

The fact that V is positive then gives $\mathcal{L}_\varepsilon(k(x,\cdot))=0$ in $\Omega\smallsetminus\{x\}$. From the SMP for $\mathcal{L}_\varepsilon=\mathcal{L}-\varepsilon$ in Remark 2.2.6, we deduce that the nonnegative function $k(x,\cdot)$ (which is $\mathcal{L}_\varepsilon$-harmonic in $\Omega\smallsetminus\{x\}$) cannot attain the (minimal) value 0; therefore $k(x,\cdot)>0$ on the connected open set $\Omega\smallsetminus\{x\}$.

A crucial step consists in proving the symmetry property (2.3.21). We take any nonnegative $\varphi\in C_0^\infty(\Omega,\mathbb{R})$ and we set (note the reverse order of x and y, if compared to $G(\varphi)$)

$$\Phi(x)=\int_\Omega\varphi(y)\,k(y,x)\,\mathrm{d}\nu(y),\qquad x\in\Omega.$$

We claim that $\Phi\ge G(\varphi)$ on Ω; once the claim is proved, from (2.3.38) we infer that

$$\int_\Omega\varphi(y)\,k(x,y)\,\mathrm{d}\nu(y)\le\int_\Omega\varphi(y)\,k(y,x)\,\mathrm{d}\nu(y),\qquad x\in\Omega.$$

The arbitrariness of φ will then give $k(x,y)\le k(y,x)$ (recalling that $\mathrm{d}\nu=V(y)\,\mathrm{d}y$ with positive V) for every $y\in\Omega$; since $x,y\in\Omega$ are arbitrary, we get $k(x,y)=k(y,x)$ on $\Omega\times\Omega$. We prove the claim. We observe that Φ is continuous on Ω and that $\mathcal{L}_\varepsilon\Phi=-\varphi$ in $\mathcal{D}'(\Omega)$, as the following computation shows ($\psi\in C_0^\infty(\Omega,\mathbb{R})$ is arbitrary):

$$\begin{aligned}
\int_\Omega\Phi(x)\,(\mathcal{L}_\varepsilon)^*\psi(x)\,\mathrm{d}x&=\int_\Omega\varphi(y)\Big(\int_\Omega k(y,x)\,\,(\mathcal{L}_\varepsilon)^*\psi(x)\,\mathrm{d}x\Big)\mathrm{d}\nu(y)\\
&=\int_\Omega\varphi(y)\Big(\int_\Omega k(y,x)\,\,\frac{(\mathcal{L}_\varepsilon)^*\psi(x)}{V(x)}\,\mathrm{d}\nu(x)\Big)\mathrm{d}\nu(y)\\
&\overset{(2.3.19)}{=}\int_\Omega\varphi(y)\Big(\int_\Omega k(y,x)\,\,\mathcal{L}_\varepsilon\Big(\frac{\psi(x)}{V(x)}\Big)\,\mathrm{d}\nu(x)\Big)\mathrm{d}\nu(y)\\
&\overset{(2.3.22)}{=}-\int_\Omega\varphi(y)\,\frac{\psi(y)}{V(y)}\,\mathrm{d}\nu(y)=-\int_\Omega\varphi(y)\,\psi(y)\,\mathrm{d}y.
\end{aligned}$$

From the hypoellipticity of $\mathcal{L}_\varepsilon$ we get $\Phi \in C^\infty(\Omega, \mathbb{R})$ and $\mathcal{L}_\varepsilon \Phi = -\varphi$ point-wise. We now apply the WMP in Remark 2.2.7 to the operator $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$ and to the function $G(\varphi) - \Phi$: this function is smooth and $\mathcal{L}_\varepsilon$-harmonic on Ω, and $G(\varphi) - \Phi \le G(\varphi)$ on Ω (since Φ is nonnegative), so that

$$\limsup_{x\to x_0}(G(\varphi) - \Phi)(x) \le \limsup_{x\to x_0} G(\varphi)(x) = 0 \quad \text{for every } x_0 \in \partial\Omega.$$

Therefore $G(\varphi) - \Phi \le 0$ on Ω as claimed.

We finally prove (2.3.23). Due to the symmetry property of k, (2.3.23) will follow if we show that, given $x_0 \in \Omega$ and $y_0 \in \partial\Omega$, one has

$$\lim_{n\to\infty} k(y_n, x_0) = 0, \tag{2.3.39}$$

for every sequence y_n in Ω converging to y_0. To this end, we fix an open set Ω' containing x_0 and with closure contained in Ω, and it is non-restrictive to suppose that $y_n \notin \Omega'$ for every n. The functions

$$k_n : \Omega' \longrightarrow \mathbb{R}, \qquad k_n(x) := k(y_n, x), \quad x \in \Omega'$$

are smooth and $\mathcal{L}_\varepsilon$-harmonic in Ω'. We also have $k_n \longrightarrow 0$ in $L^1(\Omega')$, as it follows from

$$\begin{aligned} 0 &\le \int_{\Omega'} k_n(x)\,\mathrm{d}x \le \int_\Omega k(y_n, x)\,\mathrm{d}x = \int_\Omega \frac{g(y_n, x)}{V(x)}\,\mathrm{d}x \\ &\le \sup_\Omega \frac{1}{V} \int_\Omega g(y_n, x)\,\mathrm{d}x = \sup_\Omega \frac{1}{V}\, G(1)(y_n) \xrightarrow{n\to\infty} 0. \end{aligned}$$

From Remark 2.3.9 we get that $k_n \longrightarrow 0$ in the Fréchet space $\mathcal{H}_{\mathcal{L}_\varepsilon}(\Omega')$ with the C^∞-topology, so that $k_n \longrightarrow 0$ uniformly on the compact sets of Ω' and in particular point-wise on Ω'. □

2.4 The Harnack Inequality

In this section we will prove the main result of this chapter.

We begin by proving the next crucial lemma. This is the first time that, broadly speaking, the PDOs $\mathcal{L}$ and the perturbed $\mathcal{L} - \varepsilon$ clearly interact.

Lemma 2.4.1. *Let $\mathcal{L}$ be as in* (2.1.1) *and let it satisfy* (NTD) *and* (HY)$_\varepsilon$. *Let Ω be an open set in $\mathbb{R}^N$ as in the thesis of Lemma 2.3.1, and let Ω' be an open set containing $\overline{\Omega}$. Finally, we denote by k_ε the Green kernel related to $\mathcal{L}_\varepsilon$ and to the set Ω (as in Theorem 2.3.7).*

Then we have the estimate

$$u(x) \ge \varepsilon \int_\Omega u(y)\, k_\varepsilon(x, y)\,\mathrm{d}\nu(y), \quad \forall\, x \in \Omega, \tag{2.4.1}$$

holding true for every smooth nonnegative $\mathcal{L}$-harmonic function u in Ω'.

Proof. We consider the function $v(x) = \int_\Omega u(y)\, k_\varepsilon(x, y)\,\mathrm{d}\nu(y)$ on Ω. From (2.3.20) (and the definition of Green operator) we know that $v = G_\varepsilon(u)$, where G_ε is the Green operator related to $\mathcal{L}_\varepsilon$ (and to the open set Ω); moreover, since u is smooth (by assumption) on $\overline{\Omega}$, we know from Lemma 2.3.1 (and the hypoellipticity of $\mathcal{L}_\varepsilon$) that $v \in C^\infty(\Omega) \cap C(\overline{\Omega})$ is the solution of

$$\begin{cases} \mathcal{L}_\varepsilon v = -u & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4.2}$$

This gives $\mathcal{L}_\varepsilon(\varepsilon\, v - u) = -\varepsilon\, u - (\mathcal{L} - \varepsilon)u = -\varepsilon\, u + \varepsilon\, u = 0$ on Ω; moreover, on $\partial\Omega$, $\varepsilon\, v - u = -u \le 0$, by the nonnegativity of u. By the WMP in Remark 2.2.7, we get $\varepsilon\, v - u \le 0$ on Ω which is equivalent to (2.4.1). □

We are ready for the proof of the Weak Harnack Inequality (for higher order derivatives)[2].

Theorem 2.4.2 (**Weak Harnack inequality for derivatives**). *Let $\mathcal{L}$ satisfy* (NTD), (HY) *and* (HY)$_\varepsilon$. *Then, for every connected open set $O \subseteq \mathbb{R}^N$, every compact subset K of O, every $m \in \mathbb{N} \cup \{0\}$ and every $y_0 \in O$, there exists a positive $C(y_0) = C(\mathcal{L}, \varepsilon, O, K, m, y_0)$ such that*

$$\sum_{|\alpha| \le m} \sup_{x \in K} \Big| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \Big| \le C(y_0)\, u(y_0), \tag{2.4.3}$$

for every nonnegative $\mathcal{L}$-harmonic function u in O.

Proof. We distinguish two cases: $y_0 \notin K$ and $y_0 \in K$. The second case can be reduced to the former. Indeed, let us assume we have already proved the theorem in the former case, and let $y_0 \in K$. If we take any $y_0' \in O \smallsetminus K$, and we consider the inequality

$$u(y_0') \le C'\, u(y_0),$$

resulting from (2.4.3) by considering $m = 0$ and the compact set $\{y_0'\}$, we get

$$\sum_{|\alpha| \le m} \sup_{x \in K} \Big| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \Big| \overset{(2.4.3)}{\le} C\, u(y_0') \le C\, C'\, u(y_0).$$

We are therefore entitled to assume that $y_0 \notin K$. By the aid of a classical argument (with a chain of suitable small open sets $\{\Omega_n\}_{n=1}^p$ covering a connected compact set containing $K \cup \{y_0\}$), it is not restrictive to assume that $K \cup \{y_0\} \subset \Omega \subset \overline{\Omega} \subset O$, where Ω is one of the basis open sets constructed in Lemma 2.3.1.

Let $x_0 \in K$ be arbitrarily fixed. The function $k_\varepsilon(x_0, \cdot)$ (the Green kernel related to $\mathcal{L}_\varepsilon$ and Ω) is *strictly positive* in $\Omega \smallsetminus \{x_0\}$ (this is a consequence of the SMP applied to the $\mathcal{L}_\varepsilon$-harmonic function $k_\varepsilon(x_0, \cdot)$; see Theorem 2.3.7). In particular, since $y_0 \notin K$, we infer that $k_\varepsilon(x_0, y_0) > 0$. Hence, there exist a neighborhood W of x_0 (contained in Ω) and a constant $\mathbf{c} = \mathbf{c}(\varepsilon, y_0, x_0) > 0$ such that

$$\inf_{z \in W} k_\varepsilon(z, y_0) \ge \mathbf{c} > 0. \tag{2.4.4}$$

[2]The naming 'Weak' or 'Strong' Harnack Inequality is non-standard: for example some authors refer to weak Harnack inequalities when at least one side of (2.4.7) is replaced by some L^p-norm of u; we follow the naming from Potential Theory used by Loeb and Walsh in [72], with the hope that this does not lead to any ambiguity.

Our assumptions allow us to apply Lemma 2.4.1: hence, for every nonnegative $u \in \mathcal{H}_{\mathcal{L}}(O)$, we have the following chain of inequalities

$$\begin{aligned} u(y_0) &\overset{(2.4.1)}{\geq} \varepsilon \int_{\Omega} u(z)\, k_\varepsilon(y_0,z)\,\mathrm{d}\nu(z) \geq \varepsilon \int_W u(z)\, k_\varepsilon(y_0,z)\,\mathrm{d}\nu(z) \\ &\overset{(2.3.21)}{=} \varepsilon \int_W u(z)\, k_\varepsilon(z,y_0)\,\mathrm{d}\nu(z) \overset{(2.4.4)}{\geq} \varepsilon\,\mathbf{c} \int_W u(z)\,\mathrm{d}\nu(z) \geq \varepsilon\,\mathbf{c}\,\inf_W V \int_W u(z)\,\mathrm{d}z. \end{aligned}$$

Summing up, for every $x_0 \in K$ there exist a neighborhood W of x_0 and a constant $\mathbf{c}_1 > 0$ (also depending on x_0 but independent of u) such that

$$u(y_0) \geq \mathbf{c}_1 \int_W u(z)\,\mathrm{d}z, \tag{2.4.5}$$

for every nonnegative $u \in \mathcal{H}_{\mathcal{L}}(O)$.

Next, from Remark 2.3.9, we know that the hypothesis (HY) for $\mathcal{L}$ ensures the equality of the topologies on $\mathcal{H}_{\mathcal{L}}(W)$ inherited by the Fréchet spaces $C^\infty(W)$ and $L^1_{\mathrm{loc}}(W)$. In particular, to any chosen open neighborhood U of x_0 (with $\overline{U} \subset W$) we are given a positive constant $\mathbf{c}_2 = \mathbf{c}_2(U,W,m)$ such that

$$\sum_{|\alpha|\leq m} \sup_{x\in U} \Big|\frac{\partial^\alpha u(x)}{\partial x^\alpha}\Big| \leq \mathbf{c}_2 \int_W u(z)\,\mathrm{d}z, \tag{2.4.6}$$

for every nonnegative $u \in \mathcal{H}_{\mathcal{L}}(O)$. Gathering together (2.4.5) and (2.4.6), we infer that, for every $x_0 \in K$ there exist a neighborhood U of x_0 and a constant $\mathbf{c}_3 > 0$ (again depending on x_0 but independent of u) such that

$$u(y_0) \geq \mathbf{c}_3 \sum_{|\alpha|\leq m} \sup_{x\in U} \Big|\frac{\partial^\alpha u(x)}{\partial x^\alpha}\Big|,$$

for every nonnegative $u \in \mathcal{H}_{\mathcal{L}}(O)$. The compactness of K allows us to derive (2.4.3) from the latter inequality, and a covering argument. □

Our aim is to prove the following result:

Theorem 2.4.3 (**Strong Harnack Inequality**). *Suppose that $\mathcal{L}$ is an operator of the form* (2.1.1), *with C^∞ coefficients $V > 0$ and $(a_{i,j}) \geq 0$, and suppose it satisfies hypotheses* (NTD), (HY) *and* (HY)$_\varepsilon$.

Then, for every connected open set $O \subseteq \mathbb{R}^N$ and every compact subset K of O, there exists a constant $M = M(\mathcal{L}, O, K) \geq 1$ such that

$$\sup_K u \leq M \inf_K u, \tag{2.4.7}$$

for every nonnegative $\mathcal{L}$-harmonic function u in O.

If $\mathcal{L}$ is subelliptic or if it has C^ω coefficients, then assumption (HY)$_\varepsilon$ *can be dropped.*

The last assertion follows from Remark 2.1.3.

The main step towards the Strong Harnack Inequality is the following Theorem 2.4.4 from Potential Theory. A proof of a more general abstract version of this useful result, in the framework of axiomatic harmonic spaces, can be found in the survey notes [18, pp.20–24] by Brelot, where this theorem is attributed to G. Mokobodzki. (See also a further improvement to harmonic spaces which are not necessarily second-countable, by Loeb and Walsh, [72]). Instead of appealing to an abstract Potential-Theoretic statement, we prefer to formulate the result under the following more specific form (where a harmonic sheaf related to a smooth PDO is considered).

Theorem 2.4.4. *Let L be a second order linear PDO in $\mathbb{R}^N$ with smooth coefficients. Suppose the following conditions are satisfied.*

(Regularity) *There exists a basis $\mathcal{B}$ for the Euclidean topology of $\mathbb{R}^N$ (consisting of bounded open sets) such that, for every $\Omega \in \mathcal{B} \setminus \{\varnothing\}$ and for every $\varphi \in C(\partial\Omega, \mathbb{R})$, there exists a unique L-harmonic function $H_\varphi^\Omega \in C^2(\Omega) \cap C(\overline{\Omega})$ solving the Dirichlet problem*

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

and satisfying $H_\varphi^\Omega \geq 0$ whenever $\varphi \geq 0$.

(Weak Harnack Inequality) *For every connected open set $O \subseteq \mathbb{R}^N$, every compact subset K of O and every $y_0 \in O$, there exists a constant $C(y_0) = C(L, O, K, y_0) > 0$ such that*

$$\sup_K u \leq C(y_0)\, u(y_0),$$

for every nonnegative L-harmonic function u in O.

Then, the following Strong Harnack Inequality *for L holds: for every connected open set O and every compact subset K of O there exists a constant $M = M(L, O, K) \geq 1$ such that*

$$\sup_K u \leq M \inf_K u, \tag{2.4.8}$$

for every nonnegative L-harmonic function u in O.

Proof. As anticipated, the proof is based in an essential way on the ideas by Mokobodzki-Brelot in [18, Chapter I], ensuring the equivalence of the Strong Harnack Inequality with a series of properties comprising the Weak Harnack Inequality, provided some assumptions are fulfilled. We furnish some details in order to be oriented through these equivalent properties.

We denote by $\mathcal{H}_L$ the harmonic sheaf on $\mathbb{R}^N$ defined by $O \mapsto \mathcal{H}_L(O)$ (here $O \subseteq \mathbb{R}^N$ is any open set). Under the assumptions of (Regularity) and (Weak Harnack Inequality), Brelot proves that (see [18, pp.22–24]), for any connected open set $O \subseteq \mathbb{R}^N$, and any $x_0 \in O$, the set

$$\Phi_{x_0} := \Big\{ h \in \mathcal{H}_L(O) \,:\, h \geq 0, \quad h(x_0) = 1 \Big\} \tag{2.4.9}$$

is equicontinuous at x_0. The proof of this fact rests on some results of Functional Analysis related to the family of the so-called harmonic measures $\{\mu_x^\Omega\}_{x \in \partial\Omega}$ associated with L (and on basic properties of the harmonic sheaf $\mathcal{H}_L$). Next, we show how to prove (2.4.8) starting from the equicontinuity of Φ_{x_0} at x_0. Indeed, let $K \subset O$, where K is compact and O is an open and connected subset of $\mathbb{R}^N$. By possibly enlarging K, we can suppose that K is connected as well. Let $u \in \mathcal{H}_L(O)$ be nonnegative. If $u \equiv 0$ then (2.4.8) is trivial; if u is not identically zero then (from the Weak Harnack Inequality) one has $u > 0$ on O. For every $x \in K$, the equicontinuity of Φ_x ensures the existence of $\delta(x) > 0$ such that (with the choice $h = u/u(x)$ in (2.4.9))

$$\frac{1}{2}\,u(x) \le u(\xi) \le \frac{3}{2}\,u(x), \quad \text{for all } \xi \in B_x := B(x,\delta(x)). \tag{2.4.10}$$

From the open cover $\{B_x\}_{x\in K}$ we can extract a finite subcover $B_{x_1},\dots,B_{x_p}$ of K. It is also non-restrictive (since K is connected) to assume that the elements of this subcover are chosen in such a way that

$$B_{x_1}\cap B_{x_2} \ne \varnothing, \quad (B_{x_1}\cup B_{x_2})\cap B_{x_3} \ne \varnothing, \quad \dots \quad (B_{x_1}\cup\cdots\cup B_{x_{p-1}})\cap B_{x_p} \ne \varnothing.$$

From (2.4.10) it follows (2.4.8) with K replaced by B_{x_1} (with $M = 3$); since B_{x_1} intersects B_{x_2}, one can use again (2.4.10) in order to prove (2.4.8) with K replaced by $B_{x_1}\cup B_{x_2}$ (with $M = 3^2$); by proceeding in an inductive way, one can prove (2.4.8) with K replaced by $B_{x_1}\cup\cdots\cup B_{x_p}$ (and $M = 3^p$), and this finally proves (2.4.8), since $B_{x_1}\cup\cdots\cup B_{x_p}$ covers K. □

Remark 2.4.5. Following Brelot [18, pp.14–17], it being understood that axiom (Regularity) in Theorem 2.4.4 holds true, the axiom (Weak Harnack Inequality) can be replaced by any of the following equivalent assumptions (see also Constantinescu and Cornea [25]):

(Brelot Axiom) For every connected open set $O \subseteq \mathbb{R}^N$, if $\mathcal{F}$ is an up-directed[3] family of L-harmonic functions in O, then $\sup_{u\in\mathcal{F}} u$ is either $+\infty$ or it is L-harmonic in O.

(Harnack Principle) For every connected open set $O \subseteq \mathbb{R}^N$, if $\{u_n\}_n$ is a non-decreasing sequence of L-harmonic functions in O, then $\lim_{n\to\infty} u_n$ is either $+\infty$ or it is an L-harmonic function in O.

We are ready to derive our main result for this section: due to all our preliminary results, the proof is now a few lines argument.

Proof (of Harnack Inequality, Theorem 2.4.3). Due to Theorem 2.4.4, it suffices to prove that our operator $\mathcal{L}$ as in the statement of Theorem 2.4.3 satisfies the properties named (Regularity) and (Weak Harnack Inequality) in Theorem 2.4.4: the former is a consequence of Lemma 2.3.1 (with $f = 0$), whilst the latter follows from Theorem 2.4.2. □

We remark that topological properties similar to those mentioned above for the space of the $\mathcal{L}$-harmonic functions are also valid when $\mathcal{L}$ in (2.1.1) is *not necessarily hypoelliptic*, provided

[3] $\mathcal{F}$ is said to be up-directed if for any $u, v \in \mathcal{F}$ there exists $w \in \mathcal{F}$ such that $\max\{u,v\} \le w$.

that it possesses a global positive fundamental solution (not necessarily smooth): see e.g. [7], where Montel-type results are proved (in the sense of [78]), jointly with the equivalence of the topologies induced on $\mathcal{H}_{\mathcal{L}}(\Omega)$ by L^1_{loc} and by L^∞_{loc}, under no hypoellipticity assumptions.

Chapter 3

Integral Representation of Superharmonic functions

In this chapter we want to study the integral representation and characterization of superharmonic functions related to a real second-order PDO in divergence form on $\mathbb{R}^N$. In particular, we consider the *hypoelliptic* operator $\mathcal{L}$ in (2.1.1) and we use the Harnack inequality proved in Chapter 2 in order to prove *global* and *local* representation theorems for superharmonic functions, and to characterize a superharmonic function u as a L^1_{loc}-function such that $\mathcal{L}u \le 0$ in the weak sense of distributions.

More precisely, throughout the chapter we assume the following hypotheses on $\mathcal{L}$:

(NTD) $\mathcal{L}$ is *non-totally degenerate at every point of* $\mathbb{R}^N$, or equivalenty (recalling that $A(x)$ is symmetric and positive semi-definite),

$$\text{trace}(A(x)) > 0, \quad \text{for every } x \in \mathbb{R}^N.$$

(HY) $\mathcal{L}$ is C^∞*-hypoelliptic in every open subset of* $\mathbb{R}^N$.

(HY)$_\varepsilon$ *There exists* $\varepsilon > 0$ *such that* $\mathcal{L} - \varepsilon$ *is* C^∞*-hypoelliptic in every open subset of* $\mathbb{R}^N$.

We remind that under these hypotheses we have showed the solvability of the Dirichlet problem on a basis of Euclidean topology and the Harnack inequality for $\mathcal{L}$ (see Sections 2.3 and 2.4).

We recall the following definitions.

Definition 3.0.1 (Regular set)**.** We say that an open set $\omega \subseteq \mathbb{R}^N$ is *regular* if for any $f \in C(\overline{\omega})$ and $\varphi \in C(\partial\omega)$ there exists a unique solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = -f & \text{on } \omega \quad \text{(in the weak sense of distributions)}, \\ u = \varphi & \text{on } \partial\omega \quad \text{(point-wise)}. \end{cases} \tag{3.0.1}$$

Definition 3.0.2 (Strongly regular set)**.** We say that an open set $\omega \subseteq \mathbb{R}^N$ is *strongly regular* (below SR) if for any $y \in \partial\omega$ there exists an outer normal vector for ω in y non characteristic for $\mathcal{L}$, i.e. a vector $\rho \neq 0$ such that the open ball $B(y+\rho, |\rho|)$ contains no points of ω and

$$\sum_{i,j=1}^{N} a_{ij}(y)\rho_i\rho_j > 0.$$

In the same way as in [16], it can be proved that any SR set is a regular set. Furthermore it is clear that if ω_1, ω_2 are SR sets, then $\omega_1 \cap \omega_2$ is a SR set.

Remark 3.0.3. Let ω be a regular open set. In Lemma 2.3.1, for any $f \in C(\overline{\omega})$, we have showed the existence and uniqueness of the distributional solution for the Dirichlet problem

$$\begin{cases} \mathcal{L}u = -f & \text{on } \omega \quad \text{(in the weak sense of distributions),} \\ u = 0 & \text{on } \partial\omega \quad \text{(point-wise).} \end{cases} \tag{3.0.2}$$

In particular, we have showed that there exists a basis of SR connected open sets of $\mathbb{R}^N$ such that, for any ω SR set, the solution of the Dirichlet problem (3.0.2) can be represented in the following way

$$u(x) = Gf(x) = \int_\omega k(x,y)f(y)\mathrm{d}\nu(y), \qquad \text{for every } x \in \omega, \tag{3.0.3}$$

where G is the *Green operator* and k is the *Green kernel* related to $\mathcal{L}$ and to the open set ω.

We know that k is a positive smooth function out of the diagonal $\omega \times \omega$; on this diagonal we put:

$$k(y,y) = \liminf_{y \neq x \to y} k(x,y). \tag{3.0.4}$$

In this chapter we want to give a characterization of superharmonic functions w.r.t. $\mathcal{L}$, showing that u is superharmonic if and only if $u \in L^1_{\text{loc}}$ and $\mathcal{L}u \leq 0$ in the sense of distributions. Furthermore, we will prove the representation theorems for superharmonic functions. To this aim, we need to introduce some notation of Potential Theory (for further details see [15]).

Let Ω be an open set of $\mathbb{R}^N$, and we consider the map

$$\Omega \longmapsto \mathcal{H}_{\mathcal{L}}(\Omega).$$

It is easy to see that this map is a *harmonic sheaf* on $\mathbb{R}^N$. Moreover, thanks to the hypothesis on the operator $\mathcal{L}$ and its construction in (2.1.1), it can be proved that this harmonic sheaf gives to $\mathbb{R}^N$ a structure of *harmonic space*, in which the axiom of Brelot holds. Below we will write $\mathcal{H}(\Omega)$ in place of $\mathcal{H}_{\mathcal{L}}(\Omega)$.

We introduce the following definitions.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We remind that a function $u : \Omega \to \,]-\infty, +\infty\,]$ is called *lower semicontinuous* (l.s.c.) at $x \in \Omega$ if

$$u(x) = \liminf_{y \to x} u(y) := \sup_{V \in \mathcal{U}_x} \left(\inf_{V \cap \Omega} u \right),$$

where $\mathcal{U}_x$ denotes the family of the neighborhoods of x.

A function $u : \Omega \to [-\infty, +\infty\,[$ is called *upper semicontinuous* (u.s.c.) at $x \in \Omega$ if

$$u(x) = \limsup_{y \to x} u(y) := \inf_{V \in \mathcal{U}_x} \left(\sup_{V \cap \Omega} u \right).$$

Definition 3.0.4 (Hyperharmonic Function)**.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set. A l.s.c. function $u : \Omega \to \,]-\infty, +\infty\,]$ is called *hyperharmonic function* in Ω if for every regular[1] open set $U \subseteq \overline{U} \subseteq \Omega$ we have

$$H_u^U(x) := \int_{\partial U} u(y) \mathrm{d}\mu_x^U(y) \le u(x) \qquad \text{for any } x \in U, \tag{3.0.5}$$

where μ_x^U denotes the $\mathcal{L}$-harmonic measure related to U and x.

We shall denote by $\mathcal{H}^*(\Omega)$ the set of the hyperharmonic functions in Ω.

A function $v : \Omega \to [-\infty, +\infty[$ will be called *hypoharmonic* if $-v \in \mathcal{H}^*(\Omega)$. We denote by $\mathcal{H}_*(\Omega) := -\mathcal{H}^*(\Omega)$ the family of hypoharmonic functions in Ω.

Remark 3.0.5. We want to remind that a function $u : \Omega \to \,]-\infty, +\infty\,]$ is l.s.c. in Ω if and only if the set

$$A(t) := \{x \in \Omega \,:\, u(x) > t\}$$

is an open set in Ω, for any $t \in \mathbb{R}$.

Definition 3.0.6 (Superharmonic Function)**.** Let u be a hyperharmonic function in Ω. We say that u is a *superharmonic function* in Ω if, for every regular open set $U \subseteq \overline{U} \subseteq \Omega$, the function H_u^U in (3.0.5) is harmonic in U. The set of the superhamonic functions in Ω will be denoted by $\overline{\mathcal{S}}(\Omega)$.

A function $v : \Omega \to [-\infty, +\infty[$ will be said *subharmonic* in Ω if $-v \in \overline{\mathcal{S}}(\Omega)$. We denote by $\underline{\mathcal{S}}(\Omega) := -\overline{\mathcal{S}}(\Omega)$ the set of the subharmonic functions in Ω.

Remark 3.0.7. Since the harmonic sheaf $\mathcal{H}$ satisfies the axiom of Brelot, it can be proved the following characterization of $\overline{\mathcal{S}}(\Omega)$:

$$u \in \overline{\mathcal{S}}(\Omega) \text{ if and only if } u \in \mathcal{H}^*(\Omega) \text{ and the set } \{x \in \Omega \,:\, u(x) < \infty\} \text{ is dense in } \Omega.$$

Moreover, as a consequence of the Weak Maximum Principle for $\mathcal{L}$ (see Corollary 2.2.5), we know that if $u \in C^2(\Omega; \mathbb{R})$ we have:

$$u \in \overline{\mathcal{S}}(\Omega) \iff \mathcal{L}u \le 0 \text{ in } \Omega.$$

[1]Since $(\mathbb{R}^N, \mathcal{H})$ is a harmonic space and $\mathcal{L}$ satisfies (HY), it is easy to show that the *regular open sets* seen in the classical sense of Potential Theory are equivalent to our regular open sets that we have introduced.

In the end we want to introduce the following definition.

Definition 3.0.8 (Potential Function)**.** Let $u \in \overline{\mathcal{S}}(\Omega)$, $u \geq 0$. We say that u is a *potential* on Ω if the greatest harmonic minorant of u in Ω is the zero function. We shall denote by $\mathcal{P}(\Omega)$ the set of the potential functions in Ω.

The following result gives us necessary and sufficient conditions so that a function u is a potential (see [26, Proposition 2.2.1]).

Proposition 3.0.9. *Let u be a superharmonic function on an open SR set ω such that $u \geq 0$. The following assertions are equivalent:*

(i) $u \in \mathcal{P}(\omega)$;

(ii) *if v is a hyperharmonic function on ω for which $u + v \geq 0$, then $v \geq 0$;*

(iii) *if v is a hypoharmonic minorant of u, then $v \leq 0$.*

The most important results of this chapter are the following theorems.

Theorem A (Characterization Superharmonic Functions)**.** *Let Ω be an open subset of $\mathbb{R}^N$ and $u : \Omega \to \,]-\infty, +\infty\,]$. Then the following statements are equivalent:*

(i) $u \in \overline{\mathcal{S}}(\Omega)$*, more precisely: there exists $v \in \overline{\mathcal{S}}(\Omega)$ such that $u = v$ a.e. in Ω.*

(ii) $u \in L^1_{\mathrm{loc}}(\Omega)$ *and $\mathcal{L}u \leq 0$ in $\mathcal{D}'(\Omega)$.*

Observe that (ii) means

$$\int_\Omega u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x \leq 0, \qquad \text{for any } \varphi \in C_0^\infty(\Omega),\, \varphi \geq 0.$$

Now we denote with $\mathcal{M}^+(\Omega)$ the set of non negative Radon measure on Ω.

Theorem B (Local Representation Theorem)**.** *Let Ω be an open set, ω be an open SR set such that $\omega \subseteq \overline{\omega} \subseteq \Omega \subseteq \mathbb{R}^N$ and $u \in \overline{\mathcal{S}}(\Omega)$. Then there exists a unique $\mu \in \mathcal{M}^+(\omega)$ and a unique $h \in \mathcal{H}(\omega)$ such that $\mu(\omega) < +\infty$ and*

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x) \qquad \textit{for almost every } x \in \omega, \tag{3.0.6}$$

where k is the Green kernel for ω, and V is the smooth positive function in (2.1.1).

Theorem C (Global Representation Theorem)**.** *Let ω be an open SR set such that $\omega \subseteq \mathbb{R}^N$, and let K be a compact set with $K \subseteq \omega$. If $u \in \overline{\mathcal{S}}(\omega) \cap \mathcal{H}(\omega \setminus K)$, then there exists a unique $\mu \in \mathcal{M}^+(\omega)$ and a unique $h \in \mathcal{H}(\omega)$ such that $\mu(\omega) = \mu(K) < \infty$ and*

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x) \qquad \textit{for almost every } x \in \omega, \tag{3.0.7}$$

where k denotes the Green kernel for ω, and V is the smooth positive function in (2.1.1).

If furthermore $u \in \mathcal{P}(\omega)$ then (3.0.7) *holds with $h \equiv 0$.*

3.1 Notions of Potential Theory for the Green operator and its kernel

Here we want to prove some result for the Green operator and its kernel related to $\mathcal{L}$ and a SR open set ω.

Lemma 3.1.1. *Let ω be a SR open set of $\mathbb{R}^N$. For every $y \in \omega$, there exists a sequence $\{p_n\}$ of potentials on ω such that:*

(i) *there exists a compact set $C \subseteq \omega$ such that* $\mathrm{supp}_{\mathcal{H}}(p_n) \subseteq C$, *for any $n \in \mathbb{N}$, that is*

$$p_n \in \mathcal{H}(\omega \smallsetminus C), \quad \textit{for any } n \in \mathbb{N};$$

(ii) $\lim_{n\to\infty} p_n(x) = k(x,y)$ *uniformly on compact sets of $\omega \smallsetminus \{y\}$.*

Proof. Fix $y \in \omega$ and let r be a positive number such that $\overline{B(y,2r)} \subseteq \omega$.

We consider now a sequence $\{f_n\} \subseteq C_0^\infty(\mathbb{R}^N;\mathbb{R})$ such that:

1. $f_n \geq 0$ in $\mathbb{R}^N$, for any $n \in \mathbb{N}$;
2. $\mathrm{supp}(f_n) \subseteq \overline{B(y,\frac{r}{n})} \subseteq \omega$, for any $n \in \mathbb{N}$;
3. $\int f_n(t)\mathrm{d}\nu(t) = 1$, for any $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, we put

$$p_n(x) := G(f_n)(x) = \int_\omega f_n(t)k(x,t)\mathrm{d}\nu(t), \qquad \forall\, x \in \overline{\omega}. \tag{3.1.1}$$

We want to prove that $\{p_n\}$ is a sequence of potentials on ω such that the properties (i) and (ii) are satisfied.

Thanks to hypothesis on $\{f_n\}$, we know that $\{p_n\} \subseteq C(\overline{\omega};\mathbb{R}) \cap C^\infty(\omega;\mathbb{R})$ and $p_n \geq 0$ on ω, for any $n \in \mathbb{N}$. Moreover, for any $n \in \mathbb{N}$ we have

$$\mathcal{L}p_n(x) = \mathcal{L}(G(f_n))(x) = -f_n(x) \leq 0, \quad \forall\, x \in \omega,$$

hence $p_n \in \overline{\mathcal{S}}(\omega)$, thanks to Remark 3.0.7.

Now fix $n \in \mathbb{N}$; if $h \in \mathcal{H}(\omega)$ such that $h \leq p_n$ in ω, for every $\xi \in \partial\omega$ we have

$$\limsup_{x\to\xi} h(x) \leq \limsup_{x\to\xi} p_n(x) = p_n(\xi) = 0,$$

since $p_n \in C(\overline{\omega},\mathbb{R})$ and $p_n = G(f_n) = 0$ on $\partial\omega$. Therefore, we can apply the Weak Maximum Principle (for $\mathcal{L}$) and we get $h \leq 0$ in ω.

Then we have showed that

$$\sup\{h \in \mathcal{H}(\omega) \,:\, h \leq p_n \text{ in } \omega\} = 0,$$

so $p_n \in \mathcal{P}(\omega)$, for any $n \in \mathbb{N}$.

We prove now point (i). Observe that, for any $n \in \mathbb{N}$,

$$\mathcal{L}p_n(x) = \mathcal{L}(G(f_n))(x) = -f_n(x) = 0, \qquad \text{for any } x \in \omega \setminus \overline{B(y, 2r)},$$

then we have obtained point (i), with $C := \overline{B(y, 2r)}$.

In the end, we want to show point (ii).

Let $K \subseteq \omega \setminus \{y\}$ be a compact set. Since $y \notin K$, there exists $j \in \mathbb{N}$ such that $\overline{B(y, \frac{r}{j})} \cap K = \emptyset$; hence $K \times \overline{B(y, \frac{r}{j})} \subseteq (\omega \times \omega) \setminus \Delta$, where $\Delta := \{(x, y) \in \omega \times \omega \ : \ x = y\}$. We know that k is a continuous function on $(\omega \times \omega) \setminus \Delta$, then for any $\varepsilon > 0$ there exists $m = m(\varepsilon) \in \mathbb{N}$ such that for any $t \in \overline{B(y, \frac{r}{j})}$, with $|t - y| < \frac{r}{m}$, we have

$$|k(x, t) - k(x, y)| < \varepsilon, \quad \forall\, x \in K.$$

Therefore, for any $n \geq \max\{j, m\}$ and $x \in K$, we get

$$\begin{aligned} |p_n(x) - k(x, y)| &= \left| \int_\omega (k(x, t) - k(x, y))\, f_n(t) \mathrm{d}\nu(t) \right| \leq \int_\omega |k(x, t) - k(x, y)|\, f_n(t) \mathrm{d}\nu(t) = \\ &= \int_{B(y, \frac{r}{n})} |k(x, t) - k(x, y)|\, f_n(t) \mathrm{d}\nu(t) < \varepsilon \left(\int_{B(y, \frac{r}{n})} f_n(t) \mathrm{d}\nu(t) \right) = \varepsilon. \end{aligned}$$

Then we have showed that $p_n(x) \to k(x, y)$ uniformly on K, as $n \to \infty$, and this proves point (ii). □

Proposition 3.1.2. *Let ω be a SR open set of $\mathbb{R}^N$ and $y \in \omega$; we put $k_y(x) := k(x, y)$ for any $x \in \omega$. Then k_y is a nonnegative superharmonic function on ω such that $k_y \in \mathcal{H}(\omega \setminus \{y\})$.*

Proof. Since k is a nonnegative smooth function on $(\omega \times \omega) \setminus \Delta$, where $\Delta = \{(x, y) \in \omega \times \omega \ : \ x = y\}$, k_y is a nonnegative l.s.c. function on ω (see Remark 3.0.3 and (3.0.4)). In particular, we know that $0 < k_y(x) < +\infty$ for any $x \in \omega \setminus \{y\}$.

Let U be a regular open set such that $\overline{U} \subseteq \omega$ and $\partial U \subseteq \omega \setminus \{y\}$; now we choose a sequence $\{p_n\}$ of potentials on ω as in Lemma 3.1.1.

Since $\{p_n\} \subseteq \overline{\mathcal{S}}(\omega)$, for any $n \in \mathbb{N}$, we have

$$p_n(x) \geq \int_{\partial U} p_n(t) \mathrm{d}\mu_x^U(t), \qquad \forall\, x \in U.$$

Now, thanks to point (ii) of Lemma 3.1.1, as $n \to \infty$ we get

$$k_y(x) \geq \int_{\partial U} k_y(t) \mathrm{d}\mu_x^U(t), \tag{3.1.2}$$

for any $x \in U \setminus \{y\}$.

Therefore, we have showed that $k_y \in L^1(\partial U, \mu_x^U)$, for any $x \in U \setminus \{y\}$, and so the function

$$U \ni x \longmapsto H_{k_y}^U(x) := \int_{\partial U} k_y(t) \mathrm{d}\mu_x^U(t)$$

is harmonic in U. Moreover, if $y \notin U$ then (3.1.2) holds for any $x \in U$.

On the other hand, if $y \in U$, thanks to continuity of $H^U_{k_y}$ on U, we have:

$$k_y(y) = \liminf_{\omega\smallsetminus\{y\}\ni x\to y} k_y(x) = \liminf_{U\smallsetminus\{y\}\ni x\to y} k_y(x) \geq$$
$$\geq \liminf_{U\smallsetminus\{y\}\ni x\to y} H^U_{k_y}(x) = H^U_{k_y}(y) = \int_{\partial U} k_y(t)\mathrm{d}\mu^U_y(t).$$

Then, in any case, we get that (3.1.2) holds for any $x \in U$.

Now we know that, for any $x_0 \in \omega$, the family

$$\mathcal{B}(x_0) = \{U \text{ regular open set} : x_0 \in U \subseteq \overline{U} \subseteq \omega,\ \partial U \subseteq \omega \smallsetminus \{y\}\}$$

is a base of neighborhoods of x_0, and moreover, thanks to (3.1.2), we get

$$k_y(x_0) \geq \int_{\partial U} k_y(t)\mathrm{d}\mu^U_{x_0}(t), \qquad \text{for any } U \in \mathcal{B}(x_0).$$

Hence we can say that $k_y \in \overline{\mathcal{S}}(\omega)$, thanks to Remark 3.0.7.

In the end, since we know that k_y is harmonic on $\omega \smallsetminus \{y\}$ but not all ω, we obtain that $k_y \in \mathcal{H}(\omega \smallsetminus \{y\})$. □

Proposition 3.1.3. *Let ω be a SR open set of $\mathbb{R}^N$. Then, for any $y \in \omega$, the function defined on ω $k_y(\cdot) := k(\cdot, y)$ is a potential on ω.*

Proof. From Proposition 3.1.2 we know that k_y is a nonnegative superharmonic function on ω. To prove that $k_y \in \mathcal{P}(\omega)$, it is sufficient to show that for any $\varphi \in \mathcal{H}(\omega)$, such that $\varphi \leq k_y$ on ω, we have $\varphi \leq 0$ on ω.

Let U be a regular open set such that $y \in U \subseteq \overline{U} \subseteq \omega$. We put

$$P(x) := G(1)(x) = \int_\omega k(x,t)\mathrm{d}\nu(t),$$

for any $x \in \omega$; then we know that $P \in C(\overline{\omega};\mathbb{R}) \cap C^\infty(\omega,\mathbb{R})$.

As in Lemma 3.1.1, we can prove that $P \in \mathcal{P}(\omega)$. Moreover, thanks to Strong Maximum Principle (see Theorem 2.2.2) related to $\mathcal{L}$ and connected components of ω, we get $P > 0$ on ω.

Now we want to prove that there exists $M > 0$ such that

$$k_y(x) \leq MP(x), \qquad \forall\, x \in \omega \smallsetminus U. \tag{3.1.3}$$

Since $y \in U$, we can say that k_y is continuous on $\partial U \subseteq \omega \smallsetminus \{y\}$. Hence, if we put $\lambda := \max_{\partial U} k_y$ and $m := \min_{\partial U} P$, we have $\lambda, m > 0$ and

$$k_y(x) \leq \lambda = \frac{\lambda}{m} m \leq \frac{\lambda}{m} P(x), \quad \forall\, x \in \partial U.$$

Now we consider the function $u := k_y - MP$ in ω, with $M := \frac{\lambda}{m} > 0$.

Note that $u \in C^\infty(\omega \smallsetminus \{y\})$, and in particular u is a smooth function on $\omega \smallsetminus U \subseteq \omega \smallsetminus \{y\}$. Moreover, u is subharmonic on $\omega \smallsetminus \overline{U}$, because $k_y \in \mathcal{H}(\omega \smallsetminus \{y\})$ and $MP \in \overline{\mathcal{S}}(\omega)$.

A consequence is that

$$\limsup_{\omega \smallsetminus \overline{U} \ni x \to \xi} u(x) = u(\xi) = k_y(\xi) - MP(\xi) \leq 0, \qquad \text{for any } \xi \in \partial U,$$

then for the Weak Maximum Principle we get that $u \leq 0$ on $\omega \smallsetminus \overline{U}$, and so we have showed (3.1.3).

Now, fix $\varphi \in \mathcal{H}(\omega)$ such that $\varphi \leq k_y$ in ω.

From (3.1.3), we have $\varphi \leq MP$ on $\omega \smallsetminus U$. Then, thanks to Weak Maximum Pronciple related to U and applied to the subharmonic function $v := \varphi - MP \in C^\infty(\omega)$, we get

$$\varphi(x) \leq MP(x), \qquad \forall\, x \in \omega.$$

Since $MP \in \mathcal{P}(\omega)$, we have $\varphi \leq 0$ on ω and then $k_y \in \mathcal{P}(\omega)$. □

Now we are ready to prove a main result for the Green kernel $k(x, y)$.

Proposition 3.1.4. *Let ω be a SR open set and k be the Green kernel related to $\mathcal{L}$ and ω. Then k is l.s.c. on $\omega \times \omega$.*

Proof. Observe that the function k is smooth out of the diagonal $\omega \times \omega$, then to show that k is l.s.c. on $\omega \times \omega$ it is sufficient to prove that for any $x_0 \in \omega$ and for any $\lambda < k(x_0, x_0)$, there exists a neighborhood V of x_0 such that $k(x, y) > \lambda$ for any $(x, y) \in V \times V$. In fact, if we prove this, thanks to Remark 3.0.5 we show that the function k is l.s.c. on the diagonal $\omega \times \omega$, and then k is l.s.c. on $\omega \times \omega$.

Fix $x_0 \in \omega$ and $\lambda \in \mathbb{R}$ such that $\lambda < k(x_0, x_0)$. From Proposition 3.1.2 we know that $k_{x_0} \in \overline{\mathcal{S}}(\omega)$, then there exist a real number $\beta > \lambda$ and a regular open set $V_0 \subseteq \overline{V_0} \subseteq \omega$, such that $x_0 \in V_0$ and

$$k_{x_0}(t) \geq \beta > \lambda, \quad \text{for any } t \in \overline{V_0}.$$

We choose $\alpha > 0$ such that $\beta(1 - \alpha) > \lambda$; it can be proved that there exists a connected regular open set $\omega_0 \subseteq \overline{\omega_0} \subseteq V_0$, such that $x_0 \in \omega_0$ and $\mu_{x_0}^{\omega_0}(\partial\omega_0) > 1 - \alpha$. Then we get

$$\int_{\partial\omega_0} k_{x_0}(t) \mathrm{d}\mu_{x_0}^{\omega_0}(t) \geq \beta \int_{\partial\omega_0} \mathrm{d}\mu_{x_0}^{\omega_0}(t) = \beta\mu_{x_0}^{\omega_0}(\partial\omega_0) > \beta(1 - \alpha) > \lambda.$$

Hence we put

$$2\varepsilon = \int_{\partial\omega_0} k_{x_0}(\xi) \mathrm{d}\mu_{x_0}^{\omega_0}(\xi) - \lambda;$$

it is clear that $\varepsilon > 0$.

Remind that k is continuous out of the diagonal $\omega \times \omega$. Then, if we fix $\xi \in \partial\omega_0$, there exists an open neighborhood U of x_0 such that $\overline{U} \subseteq \omega_0$ and

$$|k(\xi, y) - k(\xi, x_0)| < \varepsilon \Big(\int_{\partial\omega_0} \mathrm{d}\mu_{x_0}^{\omega_0} \Big)^{-1}, \quad \text{for any } y \in U.$$

So, for any $y \in U$, we have

$$\begin{aligned}\int_{\partial\omega_0} k(\xi,y)\mathrm{d}\mu_{x_0}^{\omega_0}(\xi) &\geq \int_{\partial\omega_0} k(\xi,x_0)\mathrm{d}\mu_{x_0}^{\omega_0}(\xi) - \int_{\partial\omega_0} |k(\xi,x_0)-k(\xi,y)|\mathrm{d}\mu_{x_0}^{\omega_0}(\xi) = \\ &= 2\varepsilon + \lambda - \int_{\partial\omega_0} |k(\xi,x_0)-k(\xi,y)|\mathrm{d}\mu_{x_0}^{\omega_0}(\xi) > \\ &> 2\varepsilon + \lambda - \varepsilon = \lambda + \varepsilon.\end{aligned} \tag{3.1.4}$$

If $y \in U$ and $z \in \omega_0$, we put:

$$u_y(z) = H_{k_y}^{\omega_0}(z) = \int_{\partial\omega_0} k(\xi,y)\mathrm{d}\mu_z^{\omega_0}(\xi). \tag{3.1.5}$$

It is obvious that u_y is harmonic in ω_0, since ω_0 is a regular set. Moreover by (3.1.4), for any $y \in U$, we have

$$u_y(x_0) > \lambda + \varepsilon.$$

Now, we want to show that for any $z \in \omega_0$, the set $\{u_y(z) \,:\, y \in U\}$ is bounded. In fact, fixed $z \in \omega_0$, it's clear that

$$\begin{aligned}|u_y(z)| &\leq \int_{\partial\omega_0} |k(\xi,y)|\mathrm{d}\mu_z^{\omega_0}(\xi) \leq \\ &\leq \left(\sup_{\eta\in U,\xi\in\partial\omega_0} k(\xi,\eta)\right)\int_{\partial\omega_0} \mathrm{d}\mu_z^{\omega_0}(\xi) = c(z) < +\infty,\end{aligned}$$

where the constant $c(z)$ depends only on z.

Making use of Theorem 2.4.2 we can prove that the set $\mathscr{F} := \{u_y \,:\, y \in U\}$ is equibounded and equicontinuous on any convex compact subset of ω_0. Let $K \subseteq \omega_0$ be a convex compact set, then we have:

(i) let $x \in \omega_0$ be a fixed point; from Weak Harnack Inequality we know that there exists a positive constant $C = C(\mathcal{L},\omega_0,K,x)$ such that

$$\sup_K |u_y| \leq Cu_y(x) \leq C\cdot c(x), \text{ for any } y \in U.$$

Therefore, if we put $M := C\cdot c(x) > 0$, we have showed that

$$|u_y(z)| \leq M, \quad \forall\, y \in U \text{ and } \forall\, z \in K,$$

that is $\mathscr{F}$ is a equibounded family on K;

(ii) let $x \in \omega_0$ be a fixed point, and fix $z_0 \in K$; we want to prove that $\mathscr{F}$ is equicontinuous in z_0.

From Weak Harnack Inequality we know that there exists a positive constant C such that

$$\sum_{j=1}^N \sup_K |\partial_j u_y| \leq \sup_K |u_y| + \sum_{j=1}^N \sup_K |\partial_j u_y| \leq Cu_y(x) \leq C\cdot c(x),$$

for any $y \in U$. If we put $M := C\cdot c(x) > 0$, we get

$$\|\nabla u_y(z)\| \leq M, \quad \forall\, y \in U,\, \forall\, z \in K.$$

On the other hand, if $z \in K$, since K is a convex set we have $[z, z_0] \subseteq K$; then from Mean Value Theorem we know that there exists $\xi \in \mathrm{int}[z, z_0]$ such that

$$|u_y(z) - u_y(z_0)| \le \|\nabla u_y(\xi)\| \cdot \|z - z_0\| \le M \|z - z_0\|,$$

for any $y \in U$.

Hence $\mathscr{F}$ is a equicontinuous family on K.

In particular, there exists a neighborhood W of the point x_0 such that

$$|u_y(x) - u_y(x_0)| < \varepsilon, \qquad \text{for any } x \in W \text{ and } y \in U,$$

from which it follows $u_y(x) > u_y(x_0) - \varepsilon > \lambda + \varepsilon - \varepsilon = \lambda$ for any $x \in W$ and $y \in U$, that is

$$\int_{\partial\omega_0} k(\xi, y) \mathrm{d}\mu_x^{\omega_0}(\xi) > \lambda, \qquad \text{for any } (x, y) \in W \times U.$$

On the other hand, since k_y is superharmonic in ω, we have

$$k_y(x) \ge \int_{\partial\omega_0} k_y(\xi) \mathrm{d}\mu_x^{\omega_0}(\xi),$$

for any $x \in W$ and $y \in U$, then we have obtained that $k(x, y) > \lambda$, for any $(x, y) \in W \times U$, which is what we wanted to show. □

Let ω be an open SR set, k the Green kernel for ω. For any $\mu \in \mathcal{M}^+(\omega)$, we put:

$$G\mu(x) := \int_\omega k(x, y) \mathrm{d}\mu(y), \qquad \text{for any } x \in \omega. \tag{3.1.6}$$

We can to prove that $G\mu$ is integrable in ω and moreover, it is a potential.

Lemma 3.1.5. *Let $\omega \subseteq \Omega$ be an open SR set; let k the Green kernel for ω. Let $\mu \in \mathcal{M}^+(\omega)$ be such that $\mu(\omega) < +\infty$. Then $G\mu \in L^1(\omega)$ and $\mathcal{L}G\mu = -\frac{1}{V}\mu$ in $\mathcal{D}'(\omega)$, where V is the smooth positive function in* (2.1.1).

Proof. We prove that $G\mu$ is integrable on ω.

By (3.1.6) and Tonelli's theorem, we have:

$$\begin{aligned}\int_\omega G\mu(x)\mathrm{d}x &= \int_\omega \left(\int_\omega k(x, y)\mathrm{d}\mu(y) \right) \mathrm{d}x = \int_\omega \left(\int_\omega k(x, y)\mathrm{d}x \right) \mathrm{d}\mu(y) = \\ &= \int_\omega \left(\int_\omega k(y, x)\mathrm{d}x \right) \mathrm{d}\mu(y),\end{aligned}$$

where the last equality is been obtained by the symmetry of k.

Now we want to remind that for our operators $\mathrm{d}\nu(x) := V(x)\mathrm{d}x$, so we have

$$\begin{aligned}\int_\omega G\mu(x)\mathrm{d}x &= \int_\omega \left(\int_\omega k(y, x)\mathrm{d}x \right) \mathrm{d}\mu(y) = \int_\omega \left(\int_\omega k(y, x) \frac{1}{V(x)} \mathrm{d}\nu(x) \right) \mathrm{d}\mu(y) = \\ &= \int_\omega G(1/V)(y)\mathrm{d}\mu(y),\end{aligned}$$

where in the last equality we have used the identity (3.0.3) for the Green operator G.

Observe that $G(1/V) \in C(\overline{\omega})$ and $\overline{\omega}$ is a compact set; then

$$\int_\omega G\mu(x)\mathrm{d}x = \int_\omega G(1/V)(y)\mathrm{d}\mu(y) \le C\mu(\omega) < +\infty,$$

hence $G\mu \in L^1(\omega)$.

Now we can consider $G\mu \in \mathcal{D}'(\omega)$, so for any $\varphi \in C_0^\infty(\omega)$ we have:

$$\begin{aligned}\langle \mathcal{L}G\mu, \varphi\rangle &= \langle G\mu, \mathcal{L}^*\varphi\rangle = \int_\omega G\mu(x)\mathcal{L}^*\varphi(x)\mathrm{d}x = \int_\omega\left(\int_\omega k(x,y)\mathrm{d}\mu(y)\right)\mathcal{L}^*\varphi(x)\mathrm{d}x = \\ &= \int_\omega\left(\int_\omega k(x,y)\mathcal{L}^*\varphi(x)\mathrm{d}x\right)\mathrm{d}\mu(y) = \int_\omega\left(\int_\omega k(x,y)V(x)\mathcal{L}(\varphi/V)(x)\mathrm{d}x\right)\mathrm{d}\mu(y) = \\ &= \int_\omega\left(\int_\omega k(x,y)\mathcal{L}(\varphi/V)(x)\mathrm{d}\nu(x)\right)\mathrm{d}\mu(y) = \\ &= \int_\omega\left(\int_\omega \mathcal{L}(\varphi/V)(x)k(y,x)\mathrm{d}\nu(x)\right)\mathrm{d}\mu(y) = -\int_\omega \frac{\varphi(y)}{V(y)}\mathrm{d}\mu(y) = \langle -(1/V)\mu, \varphi\rangle,\end{aligned}$$

where we have used (2.3.19), for the expression of the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$, and (2.3.22).

Therefore we have showed that

$$\langle \mathcal{L}G\mu, \varphi\rangle = \langle -(1/V)\mu, \varphi\rangle, \qquad \text{for any } \varphi \in C_0^\infty(\omega),$$

so we get $\mathcal{L}G\mu = -\frac{1}{V}\mu$ in $\mathcal{D}'(\omega)$. □

We want to introduce the following important definition.

Definition 3.1.6. Let Ω be an open set. Given $u \in \mathcal{H}^*(\Omega)$ and a regular open set $W \subseteq \overline{W} \subseteq \Omega$, define $u_W : \Omega \to \,]-\infty, +\infty\,]$ in the following way:

$$u_W(x) := \begin{cases} u(x), & \text{for } x \notin W, \\ \int_{\partial W} u\mathrm{d}\mu_x^W, & \text{for } x \in W. \end{cases} \tag{3.1.7}$$

The function u_W is called the *Perron-regularization* of u related to W.

The Perron-regularization of a hyperharmonic function has many important properties.

Proposition 3.1.7. *Suppose that $u \in \mathcal{H}^*(\Omega)$ and let W be a regular open set such that $W \subseteq \overline{W} \subseteq \Omega$, then:*

(i) $u_W \le u$ *in* Ω,

(ii) $u_W \in \mathcal{H}^*(\Omega)$,

(iii) $u_W \le v_W$ *if* $u, v \in \mathcal{H}^*(\Omega)$ *and* $u \le v$.

Moreover, if $u \in \overline{\mathcal{S}}(\Omega)$, *then*

(iv) $u_W \in \overline{\mathcal{S}}(\Omega)$ *and* $u_W \in \mathcal{H}(W)$.

The proof of this result can be seen in [15, Theorem 6.5.6].

Now, we want to give the following definition.

Definition 3.1.8 (Perron Set Generated by a Function)**.** Let u be a superharmonic function on Ω such that u possesses a subharmonic minorant, and let $\mathscr{B} = \{B_j\}_{j\in\mathbb{N}}$ be a covering of Ω; the following set of functions

$$\mathscr{F} := \left\{u_{B_{i_1},B_{i_2},\dots,B_{i_n}} \;:\; \{B_{i_k}\}_{k=1,\dots,n} \text{ is a finite sequence in } \mathscr{B}\right\}$$

is called the *Perron set generated by u and $\mathscr{B}$.*

Remark 3.1.9. Let $u \geq 0$ be a superharmonic function on Ω, and let $\mathscr{B} = \{B_j\}_{j\in\mathbb{N}}$ be a basis of open SR sets for Ω such that, for any $n \in \mathbb{N}$, the set $A_n := \{j \in \mathbb{N} : B_j = B_n\}$ is infinity.

We define by recurrence the following sequence:

$$u_1 = u_{B_1}, \quad u_{j+1} = (u_j)_{B_{j+1}};$$

thanks to Proposition 3.1.7, we can observe that $0 \leq u_{j+1} \leq u_j$ and $u_j \leq u$, for any $j \in \mathbb{N}$. Then, if we put $u_\infty := \lim_{j\to\infty} u_j$, it is clear that $u_\infty = \inf_{j\in\mathbb{N}} u_j$.

Now we want to consider the Perron set $\mathscr{F}$ generated by u and $\mathscr{B}$ as in the Definition 3.1.8. It is obvious that $\{u_j\} \subset \mathscr{F}$, then $\inf \mathscr{F} \leq \inf_{j\in\mathbb{N}} u_j$. We want to show that $\inf \mathscr{F} = \inf_{j\in\mathbb{N}} u_j$; to this end, we will prove that $u_\infty \in \mathcal{H}(\Omega)$.

Fix $n \in \mathbb{N}$ and let $\{j_k\} \subseteq A_n$ be such that $j_k \leq j_{k+1}$; then $\{u_{j_k}\}_k$ is a decreasing subsequence of $\{u_j\}$, so $\lim_{k\to\infty} u_{j_k} = u_\infty$. Moreover $\{u_{j_k}\} \subset \mathcal{H}(B_n)$, since for any $k \in \mathbb{N}$

$$u_{j_k} = (u_{j_{k-1}})_{B_{j_k}}$$

and by point (iv) of Proposition 3.1.7 we have $u_{j_k} \in \mathcal{H}(B_{j_k})$; but $B_{j_k} = B_n$, for any $k \in \mathbb{N}$, so we obtain that $u_{j_k} \in \mathcal{H}(B_n)$, for any $k \in \mathbb{N}$. In the end, it is clear that the sequence $\{u_{j_k}\}$ is a down directed family[2] and $u_\infty > -\infty$ in a dense subset of Ω; then, thanks to a note result of Potential Theory (see [15]), we have that $u_\infty \in \mathcal{H}(B_n)$, and it is true for any $n \in \mathbb{N}$.

Therefore, we have showed that $u_\infty \in \mathcal{H}(\Omega)$. On the other hand, we know that $u_\infty \leq u$ on Ω, and by [26, Theorem 2.2.2] we have that $\inf \mathscr{F}$ is the greatest harmonic minorant of u in Ω, so we get that $u_\infty \leq \inf \mathscr{F}$ and this gives us the thesis.

Hence, u_∞ is the greatest harmonic minorant of u in Ω.

Proposition 3.1.10. *Let ω be an open SR set and k the Green kernel for ω. Let $\mu \in \mathcal{M}^+(\omega)$ be such that $\mu(\omega) < +\infty$. Then $G\mu \in \mathcal{P}(\omega)$.*

Proof. First observe that $G\mu$ is l.s.c. on ω (see [45, Lemma 2.2.1]), then from Lemma 3.1.5 we know that $G\mu$ is finite on a dense subset D of ω.

[2] A family $\mathcal{F}$ is called down directed if for any $v, w \in \mathcal{F}$ there exists a function $f \in \mathcal{F}$ such that

$$v \geq f \text{ and } w \geq f.$$

Hence, thanks to Remark 3.0.7, we need to show that $G\mu$ is a hyperharmonic function to prove that $G\mu \in \overline{\mathcal{S}}^+(\omega)$.

If U is a regular open set such that $\overline{U} \subseteq \omega$, for any $x \in U$ we have:

$$\begin{aligned} H^U_{G\mu}(x) &= \int_{\partial U} G\mu(\xi)\mathrm{d}\mu^U_x(\xi) = \int_{\partial U}\left(\int_\omega k(\xi,y)\mathrm{d}\mu(y)\right)\mathrm{d}\mu^U_x(\xi) = \\ &= \int_\omega\left(\int_{\partial U} k(\xi,y)\mathrm{d}\mu^U_x(\xi)\right)\mathrm{d}\mu(y) = \int_\omega H^U_{k_y}(x)\mathrm{d}\mu(y) \le \\ &\le \int_\omega k_y(x)\mathrm{d}\mu(y) = \int_\omega k(x,y)\mathrm{d}\mu(y), \end{aligned}$$

where we obtain the last inequality thanks to Proposition 3.1.4. Therefore we know that

$$H^U_{G\mu}(x) \le \int_\omega k(x,y)\mathrm{d}\mu(y) = G\mu(x), \quad \text{for any } x \in U,$$

then $G\mu$ is hyperharmonic in ω, and so $G\mu \in \overline{\mathcal{S}}^+(\omega)$.

Now, to prove that $G\mu \in \mathcal{P}(\omega)$, we show that the greatest harmonic minorant of $G\mu$ is identically 0.

Let $\mathcal{B} = \{B_j\}$ be a basis of open SR sets for ω as in the Remark 3.1.9, and now we take $u := G\mu$. We want to consider the sequence $\{u_j\}$ as in the Remark 3.1.9; we have showed that the function $u_\infty := \lim_{j\to\infty} u_j$ is the greatest harmonic minorant of u. Our aim is to prove that $u_\infty \equiv 0$.

For any fixed $y \in \omega$ and $j \in \mathbb{N}$, we define $k_j(\cdot, y)$ as in the Remark 3.1.9; hence we observe that:

$$k_{j+1}(x,y) = (k_j(x,y))_{B_{j+1}} = \begin{cases} k_j(x,y) & \text{if } x \notin B_{j+1}, \\ \int_{\partial B_{j+1}} k_j(\xi,y)\mathrm{d}\mu^{B_{j+1}}_x(\xi) & \text{if } x \in B_{j+1}. \end{cases}$$

Therefore, if we define

$$\lambda_x = \begin{cases} \delta_x & \text{if } x \notin B_{j+1}, \\ \mu^{B_{j+1}}_x & \text{if } x \in B_{j+1} \end{cases}$$

we have

$$k_{j+1}(x,y) = \int_{\partial B_{j+1}} k_j(\xi,y)\mathrm{d}\lambda_x(\xi), \quad \text{for any } x \in \omega.$$

Since $\lim_{x\to x_0}\lambda_x = \lambda_{x_0}$ in $\mathcal{M}^+(\omega)$, thanks to [45, Lemma 2.2.1] we can say that k_j is l.s.c. as a function of (x,y).

By Proposition 3.1.3, we know that $k_y := k(\cdot, y) \in \mathcal{P}(\omega)$, so for any $y \in \omega$ we have:

$$\lim_{j\to\infty} k_j(x,y) = 0, \quad \forall\, x \in \omega.$$

Now we want to prove by induction that the following equality

$$u_j(x) = \int_\omega k_j(x,y)\mathrm{d}\mu(y) \qquad \text{for any } x \in \omega \tag{3.1.8}$$

holds for any $j \in \mathbb{N}$.

By definition we know that

$$u_1(x) = \begin{cases} u(x) = \int_\omega k(x,y)\mathrm{d}\mu(y) & \text{if } x \notin B_1, \\ \int_{\partial B_1} \left(\int_\omega k(\xi,y)\mathrm{d}\mu(y)\right) \mathrm{d}\mu_x^{B_1}(\xi) & \text{if } x \in B_1. \end{cases}$$

Then, if $x \in B_1$, by Fubini's Theorem we obtain:

$$u_1(x) = \int_\omega \left(\int_{\partial B_1} k(\xi,y)\mathrm{d}\mu_x^{B_1}(\xi)\right)\mathrm{d}\mu(y) = \int_\omega k_1(x,y)\mathrm{d}\mu(y).$$

On the other hand, if $x \notin B_1$, we have $k_1(x,y) = k(x,y)$, hence (3.1.8) is true for $j = 1$. Now we suppose that (3.1.8) is true for $j \in \mathbb{N}$, and we show that it holds for $j+1$.

If $x \notin B_{j+1}$ we know that $k_{j+1}(x,y) = k_j(x,y)$, then we have:

$$u_{j+1}(x) = u_j(x) = \int_\omega k_j(x,y)\mathrm{d}\mu(y) = \int_\omega k_{j+1}(x,y)\mathrm{d}\mu(y).$$

On the other hand, if $x \in B_{j+1}$, we observe that:

$$u_{j+1}(x) = \int_{\partial B_{j+1}} u_j(\xi)\mathrm{d}\mu_x^{B_{j+1}}(\xi) = \int_{\partial B_{j+1}} \left(\int_\omega k_j(\xi,y)\mathrm{d}\mu(y)\right)\mathrm{d}\mu_x^{B_{j+1}}(\xi) =$$
$$= \int_\omega \left(\int_{\partial B_{j+1}} k_j(\xi,y)\mathrm{d}\mu_x^{B_{j+1}}(\xi)\right)\mathrm{d}\mu(y) = \int_\omega k_{j+1}(x,y)\mathrm{d}\mu(y),$$

so we have showed that (3.1.8) holds for any $j \in \mathbb{N}$.

By Proposition 3.1.7 we can say that $0 \le k_j(x,y) \le k(x,y)$, for any $x, y \in \omega$ and $j \in \mathbb{N}$, and $k \in L^1(\omega \times \omega)$; moreover, we have seen that $\lim_{j\to\infty} k_j(x,y) = 0$, then by Lebesgue's theorem on the dominated convergence and (3.1.8) it follows that

$$u_\infty(x) = \lim_{j\to\infty} u_j(x) = \lim_{j\to\infty} \int_\omega k_j(x,y)\mathrm{d}\mu(y) = 0 \qquad \text{for almost every } x \in \omega,$$

hence $u_\infty = 0$ for almost every $x \in \omega$.

On the other hand, $u_\infty \in \mathcal{H}(\omega)$ and $\mathcal{L}$ satisfies (HY), then $u_\infty \equiv 0$ on ω. Therefore, $G\mu \in \mathcal{P}(\omega)$.

□

3.2 Integral Representation Theorems

In order to prove Theorems B and C, we give some important result.

Theorem 3.2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $u \in L^1_{\text{loc}}(\Omega)$ be such that $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$. Then there exists $\mu \in \mathcal{M}^+(\Omega)$ such that for any open SR set ω, with $\omega \subseteq \overline{\omega} \subseteq \Omega$, we have:*

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x) \qquad \textit{for almost every } x \in \omega, \tag{3.2.1}$$

where k is the Green kernel related to $\mathcal{L}$ and ω, and $h \in \mathcal{H}(\omega)$.

Proof. Since $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$, there exists a unique $\mu \in \mathcal{M}^+(\Omega)$ such that $-\mathcal{L}u = \mu$. Let ω be an open SR set with $\overline{\omega} \subseteq \Omega$; we put $\mu_\omega := \mu_{|_\omega}$. It is clear that $\mu_\omega \in \mathcal{M}^+(\omega)$ and $\mu_\omega(\omega) < +\infty$. Moreover, we get $\mathcal{L}u = -\mu_\omega$ in $\mathcal{D}'(\omega)$.

On the other hand, by Lemma 3.1.5 we know that $\mathcal{L}(G(V\mu_\omega)) = -\mu_\omega$ in $\mathcal{D}'(\omega)$. Hence, we have

$$\mathcal{L}\left(u - G(V\mu_\omega)\right) = 0 \text{ in } \mathcal{D}'(\omega),$$

then there exists $h \in \mathcal{H}(\omega)$ such that

$$u(x) - G(V\mu_\omega)(x) = h(x), \qquad \text{for almost every } x \in \omega.$$

Therefore, we get (3.2.1) if we remind that

$$G(V\mu_\omega)(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu_\omega(y) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y).$$

□

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We have need to show some result in order to prove that if $u \in \overline{\mathcal{S}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ then $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$.

We define in $L^1_{\mathrm{loc}}(\Omega)$ the set

$$\mathcal{S}_2(\Omega) := \overline{\left\{v \in \overline{\mathcal{S}}(\Omega) \,:\, v \in C^2(\Omega,\mathbb{R})\right\}}$$

equipped with the seminorm

$$v \longmapsto \int_K |v(x)|\mathrm{d}x, \quad K \subset \Omega \text{ a compact set.}$$

Lemma 3.2.2. *If $u \in \mathcal{S}_2(\Omega)$, then $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$, that is*

$$\int_\Omega u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x \le 0, \quad \forall\, \varphi \in C_0^\infty(\Omega,\mathbb{R}) \text{ with } \varphi \ge 0.$$

Proof. If $u \in \mathcal{S}_2(\Omega)$, then there exists a sequence $\{u_n\} \subseteq \overline{\mathcal{S}}(\Omega) \cap C^2(\Omega,\mathbb{R})$ such that $u_n \longrightarrow u$, as $n \to \infty$, in $L^1_{\mathrm{loc}}(\Omega)$.

Therefore, we know that $\mathcal{L}u_n \le 0$ in Ω, for any $n \in \mathbb{N}$ (see Remark 3.0.7). In particular, it is obvious that $\mathcal{L}u_n \le 0$ in $\mathcal{D}'(\Omega)$, for any $n \in \mathbb{N}$; hence we have

$$\lim_{n\to\infty} \int_\Omega u_n(x)\mathcal{L}^*\varphi(x)\mathrm{d}x = \int_\Omega u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x,$$

for any $\varphi \in C_0^\infty(\Omega,\mathbb{R})$, with $\varphi \ge 0$, and we get $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$. □

Lemma 3.2.3. *Let u, v be superharmonic functions on Ω such that $u, v \in C^2(\Omega,\mathbb{R})$. If $\varphi \in C^2(\mathbb{R})$ is a concave function such that $|\varphi'(x)| \le 1$, for any $x \in \mathbb{R}$, then the function $w := u + v + \varphi \circ (u - v)$ is a superharmonic function in Ω and $w \in C^2(\Omega,\mathbb{R})$.*

Proof. It is obvious that $w \in C^2(\Omega,\mathbb{R})$, so we need to prove that $w \in \overline{\mathcal{S}}(\Omega)$. To this end, we can prove that $\mathcal{L}w \le 0$ in Ω.

By construction of $\mathcal{L}$ in (2.1.1), for any $x \in \Omega$, we get

$$\begin{aligned}
\mathcal{L}w(x) &= \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(V(x)a_{i,j}(x)\frac{\partial w(x)}{\partial x_j}\right) = \\
&= \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(V(x)a_{i,j}(x)\left(\frac{\partial u(x)}{\partial x_j} + \frac{\partial v(x)}{\partial x_j} + \frac{\partial \varphi((u-v)(x))}{\partial x_j}\right)\right) = \\
&= \mathcal{L}u(x) + \mathcal{L}v(x) + \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(V(x)a_{i,j}(x)\varphi'((u-v)(x))\frac{\partial (u-v)(x)}{\partial x_j}\right) = \\
&= \mathcal{L}u(x) + \mathcal{L}v(x) + \varphi'((u-v)(x))\mathcal{L}(u-v)(x) + \\
&+ \varphi''((u-v)(x)) \sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial (u-v)(x)}{\partial x_i}\frac{\partial (u-v)(x)}{\partial x_j} = \\
&= \mathcal{L}u(x)\left[1 + \varphi'((u-v)(x))\right] + \mathcal{L}v(x)\left[1 - \varphi'((u-v)(x))\right] + \\
&+ \varphi''((u-v)(x)) \sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial (u-v)(x)}{\partial x_i}\frac{\partial (u-v)(x)}{\partial x_j},
\end{aligned}$$

since φ is a concave function on $\mathbb{R}$ and the matrix $A(x) = (a_{i,j}(x))$ is positive semi-definite at every point $x \in \mathbb{R}^N$, the last term of the equation is non positive on Ω; moreover, $\mathcal{L}u, \mathcal{L}v \le 0$ in Ω and $|\varphi'| \le 1$ in $\mathbb{R}$, so also the first two terms of the equation are non positive. Therefore, we obtain $\mathcal{L}w \le 0$ in Ω, which gives $w \in \overline{\mathcal{S}}(\Omega)$. □

Lemma 3.2.4. *Let $u, v \in \mathcal{S}_2(\Omega)$ and let $\varphi \in C^2(\mathbb{R})$ be a concave function such that $|\varphi'(x)| \le 1$, for any $x \in \mathbb{R}$. If we put $w := u + v + \varphi \circ (u - v)$, then $w \in \mathcal{S}_2(\Omega)$.*

Proof. We know that $L^1_{\text{loc}}(\Omega)$ is a metrizable space, so we can think in the following way.

Since $u, v \in \mathcal{S}_2(\Omega)$, there exist the sequences $\{u_n\}, \{v_n\} \subseteq \overline{\mathcal{S}}(\Omega) \cap C^2(\Omega, \mathbb{R})$ such that $u_n \longrightarrow u$ and $v_n \longrightarrow v$ in $L^1_{\text{loc}}(\Omega)$, as $n \to \infty$.

On the other hand, if we fix $x \in \Omega$ and $n \in \mathbb{N}$, we can apply the Mean Value Theorem to the function φ in the interval of extremes $(u_n - v_n)(x)$ and $(u - v)(x)$; then we get

$$|\varphi((u_n - v_n)(x)) - \varphi((u-v)(x))| = |(u_n - v_n)(x) - (u-v)(x)| \cdot |\varphi'(c)| \le |(u_n - v_n)(x) - (u-v)(x)|,$$

which gives

$$|\varphi((u_n - v_n)(x)) - \varphi((u-v)(x))| \le |u_n(x) - u(x)| + |v_n(x) - v(x)|, \tag{3.2.2}$$

for any $x \in \Omega$ and $n \in \mathbb{N}$.

Now, if we put $w_n := u_n + v_n + \varphi \circ (u_n - v_n)$ for any $n \in \mathbb{N}$, it is clear that $w_n \longrightarrow w$ in $L^1_{\text{loc}}(\Omega)$, as $n \to \infty$. Moreover, by Lemma 3.2.3 we know that $w_n \in \overline{\mathcal{S}}(\Omega) \cap C^2(\Omega, \mathbb{R})$, for any $n \in \mathbb{N}$; therefore, we obtain $w \in \mathcal{S}_2(\Omega)$. □

Lemma 3.2.5. *If $u, v \in \mathcal{S}_2(\Omega)$, then $\inf\{u, v\} \in \mathcal{S}_2(\Omega)$.*

Proof. It is known that $2\inf\{u, v\} = u + v - |u - v|$.

Now we consider a sequence $\{\varphi_n\} \subseteq C^2(\mathbb{R})$ of concave functions such that $|\varphi_n'| \le 1$ in $\mathbb{R}$, for any $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \varphi_n'(t) = -|t|, \quad \text{uniformly in } \mathbb{R}.$$

For example, we can choose the function $\varphi_n(t) = \frac{1}{n} - \sqrt{t^2 + \frac{1}{n^2}}$ for any $n \in \mathbb{N}$.

It is clear that $\varphi_n \circ (u-v) \longrightarrow -|u-v|$ in $L^1_{\text{loc}}(\Omega)$, as $n \to \infty$. Now we put $w_n := u+v+\varphi_n \circ (u-v)$ for any $n \in \mathbb{N}$; by Lemma 3.2.4 we know that $w_n \in \mathcal{S}_2(\Omega)$, for any $n \in \mathbb{N}$, and moreover $w_n \longrightarrow 2\inf\{u,v\}$ in $L^1_{\text{loc}}(\Omega)$, as $n \to \infty$, then we get that $\inf\{u,v\} \in \mathcal{S}_2(\Omega)$. □

Corollary 3.2.6. *If u is locally the lower envelope in Ω of a finite number of superharmonic functions of class C^2, then $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$.*

Proof. Thanks to Lemma 3.2.5, we know that for any $x \in \Omega$ there exists an open set $W_x \subseteq \Omega$ neighborhood of x such that $u \in \mathcal{S}_2(W_x)$; hence, from Lemma 3.2.2 we get that $\mathcal{L}u \le 0$ in $\mathcal{D}'(W_x)$.

It is clear that the family $\{W_x\}$ is a covering of Ω; then there there exists a sequence $\{\rho_j\} \subset C_0^\infty(\Omega)$, with $\rho_j \ge 0$, such that

1. supp$\rho_j \subseteq W_j$, for any $j \in \mathbb{N}$;
2. $\sum_{j=1}^\infty \rho_j(x) = 1$ for every $x \in \Omega$;
3. to every compact $A \subset \Omega$ correspond an integer m and an open set $U \supset A$ such that

$$\rho_1(x) + \ldots + \rho_m(x) = 1, \qquad \forall\, x \in U.$$

Fix $\varphi \in C_0^\infty(\Omega;\mathbb{R})$ with $\varphi \ge 0$ on Ω; we want to prove that $\langle \mathcal{L}u, \varphi\rangle \le 0$.

We put $K :=$supp$\varphi \subseteq \Omega$, and for any $j \in \mathbb{N}$ we consider the positive smooth functions

$$\varphi_j(x) = \rho_j(x)\varphi(x), \quad \forall\, x \in \Omega.$$

It is easy to see that for any $j \in \mathbb{N}$, $\varphi_j \in C_0^\infty(W_j)$, then we know that

$$\int_{W_j} u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x \le 0, \quad \text{for any } j \in \mathbb{N}. \tag{3.2.3}$$

On the other hand we have:

$$\begin{aligned}\int_\Omega u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x &= \int_U u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x = \int_U u(x)\mathcal{L}^*\left(\sum_{j=1}^m \rho_j(x)\varphi(x)\right)\mathrm{d}x = \\ &= \sum_{j=1}^m \int_U u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x = \sum_{j=1}^m \int_{U\cap W_j} u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x \le \\ &\le \sum_{j=1}^m \int_{W_j} u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x \le 0,\end{aligned}$$

then we have showed that $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$. □

Lemma 3.2.7. *Let $U_1, U_2, \ldots, U_p$ be p regular open sets such that $\overline{U_i} \subseteq \Omega$, for $i = 1, 2, \ldots, p$, and define $U = \bigcup_{i=1}^p U_i$. If $u \in \overline{\mathcal{S}}(\Omega)$ strictly, that is u is not harmonic in any regular open set of Ω, and $v = \inf\{u_{U_1}, u_{U_2}, \ldots, u_{U_p}\}$, then $v \in \overline{\mathcal{S}}(\Omega) \cap L^1_{\text{loc}}(U)$ (it is also a continuous function on U) and $\mathcal{L}v \leq 0$ in $\mathcal{D}'(U)$.*

Proof. Since $u \in \overline{\mathcal{S}}(\Omega)$, by Proposition 3.1.7, we know that $u_{U_i} \in \overline{\mathcal{S}}(\Omega)$, for any $i = 1, \ldots, p$; hence $v \in \overline{\mathcal{S}}(\Omega)$.

We fix $x_0 \in U = \bigcup_{i=1}^p U_i$, so there exists $q \in \mathbb{N}$, with $1 \leq q \leq p$, such that $x_0 \in U_1 \cap \ldots \cap U_q \cap (\Omega \smallsetminus U_{q+1}) \cap \ldots \cap (\Omega \smallsetminus U_p)$. Since u is strictly superharmonic in Ω, by construction of the Perron regularization of u and Proposition 3.1.7, we have

$$u_{U_1}(x_0), \ldots, u_{U_q}(x_0) < u(x_0) = u_{U_{q+1}}(x_0) = \ldots = u_{U_p}(x_0). \tag{3.2.4}$$

Moreover, $u_{U_i} \in \mathcal{H}(U_i)$, for any $i = 1, \ldots, p$; in particular, u_{U_i} is a continuous function in x_0, for $i = 1, \ldots, q$, and u_{U_j} is a l.s.c. function in x_0, for $j = q+1, \ldots, p$. Therefore, thanks to (3.2.4), there exists a neighborhood $W \subseteq \left(\bigcap_{i=1}^q U_i\right) \cap \left(\bigcap_{j=q+1}^p \Omega \smallsetminus U_j\right)$ of the point x_0 such that $u_{U_i} < u_{U_j}$ on W for any $i = 1, \ldots, q$ and $j = q+1, \ldots, p$. Now we can observe that $\inf\{u_{U_1}, \ldots, u_{U_q}\} \leq u_{U_k}$ on W, for any $k = 1, \ldots, p$, and so we get that $\inf\{u_{U_1}, \ldots, u_{U_q}\} \leq v$ on W. On the other hand, it is obvious that $v \leq \inf\{u_{U_1}, \ldots, u_{U_q}\}$, then

$$v = \inf\{u_{U_1}, \ldots, u_{U_q}\} \quad \text{on } W.$$

Therefore, we have obtained that v is locally in U the lower envelope of a finite number of harmonic functions; then we can apply Corollary 3.2.6 and we obtain that $\mathcal{L}v \leq 0$ in $\mathcal{D}'(U)$. Moreover, it is clear that $v \in L^1_{\text{loc}}(U)$, since v is a continuous function on U (v is locally the lower envelope of a finite number of smooth functions). □

Now we are ready to prove the following result.

Proposition 3.2.8. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $u \in \overline{\mathcal{S}}(\Omega) \cap L^1_{\text{loc}}(\Omega)$, then $\mathcal{L}u \leq 0$ in $\mathcal{D}'(\Omega)$.*

Proof. We will prove the proposition in two steps.

STEP I We want to show that for any $x_0 \in \Omega$, there exists $\omega \subseteq \overline{\omega} \subseteq \Omega$ bounded open set such that $x_0 \in \omega$ and $\mathcal{L}u \leq 0$ in $\mathcal{D}'(\omega)$.

Fix $x_0 \in \Omega$, then there exists a regular open set $\omega \subseteq \overline{\omega} \subseteq \Omega$ such that $x_0 \in \omega$. At first we suppose that u is a strictly superharmonic function in Ω.

Let $\varphi \in C_0^\infty(\omega; \mathbb{R})$ be a nonnegative function on ω. If we put $K := \text{supp}\varphi \subseteq \omega$ and fix $n \in \mathbb{N}$, then we can cover K with a finite number of regular open sets with diameter $\leq \frac{1}{n}$. By Lemma 3.2.7, we can match a superharmonic function v_n on Ω, such that:

(i) $v_n \leq u$ on Ω, then in particular $v_n \leq u$ on K;

(ii) $v_n \in L^1(K)$ and

$$\int_K v_n(x)\mathcal{L}^*\varphi(x)\mathrm{d}x \le 0.$$

Therefore, for $n \in \mathbb{N}$ we get the sequence $\{v_n\}$; thanks to construction of v_n, it is clear that when $n \to \infty$ (that is the diameters tend to zero) we have $v_n \longrightarrow u$ point-wise in Ω.

On the other hand, let $m \le 0$ be a constant such that $m \le \inf_{\overline{\omega}} u$. Since $\mathcal{L}$ is homogeneous, m is a harmonic function; then, for any $W \subseteq \overline{W} \subseteq \omega$ regular open set, we have

$$m = m_W(x) \le u_W(x) \le u(x), \qquad \forall\, x \in \omega.$$

Hence, thanks to construction of v_n, we get $m \le v_n$ on K, for any $n \in \mathbb{N}$.

Now we observe that we can apply the Dominated Convergence Theorem (remind that $u \in L^1_{\mathrm{loc}}(\Omega)$):

$$\lim_{n\to\infty} \int_K v_n(x)\mathcal{L}^*\varphi(x)\mathrm{d}x = \int_K u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x,$$

and by point (ii) we get

$$\int_\omega u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x \le 0;$$

thanks to arbitrariness of φ, we have showed that $\mathcal{L}u \le 0$ in $\mathcal{D}'(\omega)$.

Now we need to show the general case, when $u \in \overline{\mathcal{S}}(\Omega)$.

If we fix $x_0 \in \Omega$, then there exists a SR open set ω_0 such that $x_0 \in \omega_0 \subseteq \overline{\omega_0} \subseteq \Omega$. Now, we know that there exists a unique solution $v \in C(\overline{\omega_0}) \cap C^\infty(\omega_0)$ of the following problem:

$$\begin{cases} \mathcal{L}v = -1 & \text{on } \omega_0 \\ v = 0 & \text{on } \partial\omega_0 \quad \text{(point-wise)}. \end{cases}$$

Fix now $\varepsilon > 0$ and put $u_\varepsilon(x) = u(x) + \varepsilon v(x)$ for any $x \in \omega_0$.

Observe that $u, v \in \overline{\mathcal{S}}(\omega_0) \cap L^1_{\mathrm{loc}}(\omega_0)$, then $u_\varepsilon \in \overline{\mathcal{S}}(\omega_0) \cap L^1_{\mathrm{loc}}(\omega_0)$. Moreover, since $\mathcal{L}v = -1 < 0$ on ω_0 and $v \in C^\infty(\omega_0)$, we can say that the function v is strictly superharmonic on ω_0. Therefore, it is clear that the function u_ε is strictly superharmonic on ω_0; thanks to the first part of the proof, we know that for any $x \in \omega_0$, there exists a bounded open set $W \subseteq \overline{W} \subseteq \omega_0$ such that $x \in W$ and $\mathcal{L}u_\varepsilon \le 0$ in $\mathcal{D}'(W)$. In particular, we have:

$$\int_W (u(x) + \varepsilon v(x))\mathcal{L}^*\varphi(x)\mathrm{d}x \le 0, \qquad \forall\, \varphi \in C_0^\infty(W;\mathbb{R}),\ \varphi \ge 0 \text{ on } W \text{ and } \forall\, \varepsilon > 0.$$

Hence, it is easy to see that as $\varepsilon \to 0$ we get

$$\int_W u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x \le 0,$$

for any $\varphi \in C_0^\infty(W;\mathbb{R})$, with $\varphi \ge 0$ on W; so $\mathcal{L}u \le 0$ in $\mathcal{D}'(W)$.

Therefore, in correspondence with $x_0 \in \omega_0 \subseteq \Omega$ there exists a bounded open set $\omega \subseteq \overline{\omega} \subseteq \omega_0 \subseteq \Omega$ such that $x_0 \in \omega$ and $\mathcal{L}u \le 0$ in $\mathcal{D}'(\omega)$, that is the claim.

STEP II We want to prove that $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$.

Fix $\varphi \in C_0^\infty(\Omega;\mathbb{R})$ such that $\varphi \ge 0$ on Ω, and we put $K := \operatorname{supp}\varphi \subseteq \Omega$. In the STEP I we have showed that for any $x_0 \in \Omega$, there exists $\omega \subseteq \overline{\omega} \subseteq \Omega$ bounded open set such that $x_0 \in \omega$ and $\mathcal{L}u \le 0$ in $\mathcal{D}'(\omega)$. It is clear that as $x_0 \in \Omega$, we get a collection of bounded open sets whose union is Ω; then we can consider a locally finite partition of unity $\{\rho_j\}$ in Ω as in the proof of Corollary 3.2.6. Hence, in correspondence to K, we have:

1. $\operatorname{supp}\rho_j \subseteq \omega_j$, for any $j \in \mathbb{N}$;
2. $\sum_{j=1}^{\infty} \rho_j(x) = 1$ for every $x \in \Omega$;
3. in correspondence to K, there exist an integer m and an open set $U \supset K$ such that

$$\rho_1(x) + \ldots + \rho_m(x) = 1, \qquad \forall\, x \in U.$$

Now, for any$j \in \mathbb{N}$ we put

$$\varphi_j(x) := \rho_j(x)\varphi(x), \quad \text{for any } x \in \Omega.$$

It is obvious that for any $j \in \mathbb{N}$, $\varphi_j \in C_0^\infty(\omega_j;\mathbb{R})$ and $\varphi_j \ge 0$ on ω_j. Then from STEP I we know that

$$\int_{\omega_j} u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x \le 0.$$

Therefore we get:

$$\begin{aligned}\int_\Omega u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x &= \int_U u(x)\mathcal{L}^*\varphi(x)\mathrm{d}x = \int_U u(x)\mathcal{L}^*(\sum_{j=1}^m \rho_j(x)\varphi(x))\mathrm{d}x = \\ &= \sum_{j=1}^m \int_U u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x = \sum_{j=1}^m \int_{U\cap\omega_j} u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x \le \\ &\le \sum_{j=1}^m \int_{\omega_j} u(x)\mathcal{L}^*\varphi_j(x)\mathrm{d}x \le 0,\end{aligned}$$

hence we have $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$.

□

Now we can prove an important consequence of these results.

Corollary 3.2.9. *Let ω be a SR open set and $y \in \omega$. If $u \in \mathcal{P}(\omega) \cap L^1_{\text{loc}}(\omega)$ such that $u \in \mathcal{H}(\omega \setminus \{y\})$, then there exists $C := C(y) > 0$ such that*

$$u(x) = Ck_y(x), \quad \textit{for any } x \in \omega.$$

Proof. Since $u \in \mathcal{P}(\omega)$, in particular we have that $u \in \overline{\mathcal{S}}(\omega) \cap L^1_{\text{loc}}(\omega)$. Thanks to Proposition 3.2.8, we know that $\mathcal{L}u \le 0$ in $\mathcal{D}'(\omega)$. Therefore, since $u \in \mathcal{H}(\omega \setminus \{y\})$, there exists $c := c(y) > 0$ such that

$$\mathcal{L}u = -c\delta_y \quad \text{in } \mathcal{D}'(\omega).$$

Since $\mathcal{L}k_y = -\frac{1}{V(y)}\delta_y$ in $\mathcal{D}'(\omega)$ (see (2.3.22)), as in the proof of Theorem 3.2.1, we can prove that there exists $h \in \mathcal{H}(\omega)$ such that

$$u(x) = Ck(x,y) + h(x), \tag{3.2.5}$$

for almost every $x \in \omega$, with $C = cV(y) > 0$ (note that C is a positive constant that depends only on y).

Since $u \in \mathcal{H}(\omega \setminus \{y\})$, thanks to continuity we have that (3.2.5) holds for any $x \in \omega \setminus \{y\}$.

On the other hand, if W is a regular open set such that $y \in W \subseteq \overline{W} \subseteq \omega$, then we consider the Perron-regularization function u_W of u in ω. Observe that:

(i) for any $x \in \omega \setminus W$, we have $x \neq y$ and then

$$u_W(x) = u(x) = Ck(x,y) + h(x);$$

(ii) for any $x \in W$, we get

$$\begin{aligned} u_W(x) &= \int_{\partial W} u(t) \mathrm{d}\mu_x^W(t) = \int_{\partial W} (Ck_y(t) + h(t))\, \mathrm{d}\mu_x^W(t) = \\ &= C \cdot \int_{\partial W} k(t,y) \mathrm{d}\mu_x^W(t) + h(x) = C(k_y)_W(x) + h(x). \end{aligned}$$

Therefore, we have showed that

$$u_W(x) = C(k_y)_W(x) + h(x), \quad \text{for any } x \in \omega.$$

Moreover, we know that

$$\lim_{\mathrm{diam}(W)\to 0} u_W(x) = u(x) \text{ and } \lim_{\mathrm{diam}(W)\to 0} (k_y)_W(x) = k_y(x), \quad \forall\, x \in \omega;$$

hence, we get that $u(y) = Ck_y(y) + h(y)$ and so (3.2.5) holds on any point of ω.

Now, since $u, k_y \in \mathcal{P}(\omega)$, we have (see Proposition 3.0.9):

- the function $h \geq 0$ on ω, because $Ck_y(x) + h(x) \geq 0$, for any $x \in \omega$, and $h \in \mathcal{H}(\omega)$;
- for any $x \in \omega$, we have

  $$h(x) = u(x) - Ck_y(x) \leq u(x),$$

  then $h \leq 0$ on ω.

Therefore, thanks last points we obtain that $h \equiv 0$ on ω, and so $u(x) = Ck(x,y)$, for any $x \in \omega$. □

Making use of the Proposition 3.2.8, we can prove as in the Theorem 3.2.1 the following theorems of representation.

Theorem 3.2.10. *Let Ω be an open set in $\mathbb{R}^N$, and let ω be an open SR set with $\overline{\omega} \subseteq \Omega$. If $u \in \overline{\mathcal{S}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$, then there exist a unique $\mu \in \mathcal{M}^+(\omega)$ and a unique $h \in \mathcal{H}(\omega)$ such that:*

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x) \qquad \textit{for almost every } x \in \omega, \tag{3.2.6}$$

where k is the Green kernel related to $\mathcal{L}$ and ω.

Proof. Since $u \in \overline{\mathcal{S}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$, from Proposition 3.2.8 we know that $\mathcal{L}u \leq 0$ in $\mathcal{D}'(\Omega)$. In particular, we have $\mathcal{L}u \leq 0$ in $\mathcal{D}'(\omega)$; then, there exists a unique $\mu \in \mathcal{M}^+(\omega)$ such that $\mathcal{L}u = -\mu$ in $\mathcal{D}'(\omega)$.

On the other hand, it is known that $\mathcal{L}(G(V\mu)) = -\mu$ in $\mathcal{D}'(\omega)$; hence, there exists $h \in \mathcal{H}(\omega)$ such that

$$u(x) = G(V\mu)(x) + h(x), \qquad \text{for almost every } x \in \omega,$$

that is (3.2.6) holds.

In the end, we want to prove that h is unique. Suppose that ψ is a harmonic function on ω such that

$$u(x) = G(V\mu)(x) + \psi(x), \qquad \text{for almost every } x \in \omega.$$

Then, it is clear that $\psi(x) = h(x)$ for almost every $x \in \omega$; since $\psi, h \in \mathcal{H}(\omega)$, thanks to hypoellipticity of $\mathcal{L}$, we get $\psi \equiv h$ on ω. □

Theorem 3.2.11. *Let ω be an open SR set, K a compact subset of ω. If $u \in \overline{\mathcal{S}}(\omega) \cap \mathcal{H}(\omega \smallsetminus K) \cap L^1_{\mathrm{loc}}(\omega)$, then there exist a unique $\mu \in \mathcal{M}^+(\omega)$ and a unique $h \in \mathcal{H}(\omega)$ such that: $\mu(\omega) = \mu(K) < +\infty$ and*

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x) \qquad \textit{for almost every } x \in \omega, \tag{3.2.7}$$

where k is the Green kernel related to $\mathcal{L}$ and ω.

Proof. Since $u \in \overline{\mathcal{S}}(\omega) \cap L^1_{\mathrm{loc}}(\omega)$, thanks to the proof of Theorem 3.2.10, we know that there exist a unique $\mu \in \mathcal{M}^+(\omega)$ and a unique $h \in \mathcal{H}(\omega)$ such that (3.2.7) holds.

Now we want to prove that $\mu(\omega) = \mu(K)$, that is u and μ have the same support[3] on ω.

Since $u \in \mathcal{H}(\omega \smallsetminus K)$, by (3.2.7) we get $\mathcal{L}(G(V\mu)) = 0$ in $\mathcal{D}'(\omega \smallsetminus K)$, and so $\mu = 0$ on $\omega \smallsetminus K$, that is $\mu(\omega \smallsetminus K) = 0$. Therefore, we have $\mu(\omega) = \mu(K) < +\infty$, because $K \subset \omega$ is a compact set. □

Corollary 3.2.12. *Let ω be a SR open set and $K \subset \omega$ be a compact set. If $u \in \mathcal{P}(\omega) \cap \mathcal{H}(\omega \smallsetminus K) \cap L^1_{\mathrm{loc}}(\omega)$, then there exist a unique $\mu \in \mathcal{M}^+(\omega)$ such that*

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y), \qquad \textit{for almost every } x \in \omega, \tag{3.2.8}$$

where k is the Green kernel related to $\mathcal{L}$ and ω.

Proof. Since $u \in \overline{\mathcal{S}}(\omega) \cap L^1_{\mathrm{loc}}(\omega)$, thanks to Theorem 3.2.11, we know that there exist a unique $\mu \in \mathcal{M}^+(\omega)$ and a unique $h \in \mathcal{H}(\omega)$ such that $u(x) = G(V\mu)(x) + h(x)$, for almost every $x \in \omega$.

On the other hand, we know that $u, G(V\mu) \in \mathcal{P}(\omega)$ (see Proposition 3.1.10), so we put

$$\varphi(x) := u(x) - G(V\mu)(x), \qquad \text{for any } x \in \omega.$$

Observe that $h(x) = \varphi(x)$ for almost every $x \in \omega$; moreover, it is clear that $\varphi \in \mathcal{H}(\omega)$, hence $\varphi \in C^\infty(\omega)$.

We want to prove that $\varphi \equiv 0$ on ω. We know that (see Proposition 3.0.9):

[3] We are considering the harmonic support $\mathrm{supp}_{\mathcal{H}} u$ of u in ω.

(i) $G(V\mu) \geq 0$ on ω, then $\varphi \leq u$ on ω; since $u \in \mathcal{P}(\omega)$ and $\varphi \in \mathcal{H}(\omega)$, we can say that $\varphi \leq 0$ on ω.

(ii) $u \geq 0$ on ω, so we have

$$\varphi(x) + G(V\mu)(x) \geq 0, \quad \forall\, x \in \omega.$$

Since $G(V\mu) \in \mathcal{P}(\omega)$ and $\varphi \in \mathcal{H}(\omega)$, we get $\varphi \geq 0$ on ω.

Therefore, we have showed that $\varphi \equiv 0$ on ω; then $h(x) = 0$ for almost every $x \in \omega$, and by continuity we have $h \equiv 0$ on ω. Hence, we have showed (3.2.8). □

Note that if we prove the following inclusion

$$\overline{\mathcal{S}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega),$$

then by Theorem 3.2.10 and 3.2.11 we get Theorem B and C.

This last result will be the object of the next part.

3.3 Characterization of Superharmonic Functions

In order to prove Theorem A, we have need to show the following results.

Lemma 3.3.1. *Let Ω be an open set in $\mathbb{R}^N$ and ω be an open SR set such that $\overline{\omega} \subseteq \Omega$. If $u \in C^\infty(\omega) \cap C(\overline{\omega})$ such that $\mathcal{L}u \leq 0$ and $u \geq 0$ on Ω, and k_ε is the Green kernel relative to $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$ (where $\varepsilon > 0$ in the hypothesis (HY)$_\varepsilon$) and ω, then*

$$u(x) \geq \varepsilon \int_\omega k_\varepsilon(x,y)u(y)\mathrm{d}\nu(y) \qquad \text{for any } x \in \omega. \tag{3.3.1}$$

Proof. We consider the function $v(x) = \int_\omega u(y)k_\varepsilon(x,y)\mathrm{d}\nu(y)$ for any $x \in \omega$. Thanks to definition of Green operator, it is clear that $v = G_\varepsilon(u)$, where G_ε is the Green operator related to $\mathcal{L}_\varepsilon$ and ω. Since $u \in C^\infty(\omega) \cap C(\overline{\omega})$, we know that $v \in C^\infty(\omega) \cap C(\overline{\omega})$ is the classical solution of

$$\begin{cases} \mathcal{L}_\varepsilon v = -u & \text{on } \omega \\ v = 0 & \text{on } \partial\omega, \end{cases}$$

(see Lemma 2.3.1).

Hence, we get $\mathcal{L}_\varepsilon(\varepsilon v - u) = -\varepsilon u - (\mathcal{L} - \varepsilon)u = -\mathcal{L}u \geq 0$ on ω. On the other hand, $\varepsilon v - u = -u \leq 0$ on $\partial\omega$. Now we can apply the Weak Maximum Principle for $\mathcal{L}_\varepsilon$ (see Remark 2.2.7), and we get that $u \geq \varepsilon v$ on ω, that is (3.3.1). □

We introduce the following notion.

Definition 3.3.2 (Balayage)**.** Let Ω be an open set of $\mathbb{R}^N$ and $A \subseteq \Omega$. If $u \in \overline{\mathcal{S}}(\Omega)$ and $u \geq 0$ in Ω, then we can define the *reduced function* of u in A the following way:

$$\mathbf{R}^A_u := \inf\left\{\varphi \in \overline{\mathcal{S}}(\Omega)\,:\, \varphi \geq 0 \text{ in } \Omega \text{ and } \varphi \geq u \text{ in } A\right\}.$$

We called *balayage of* u on A the function

$$\widehat{\mathbf{R}}_u^A := \widehat{(\mathbf{R}_u^A)},$$

that is the lower regularization of $\mathbf{R}_u^A$ in Ω.

The balayage has the following properties (see [15] and [57]):

Proposition 3.3.3. *Let $u \in \overline{\mathcal{S}}(\Omega)$ be a non-negative function on Ω, and let $A \subseteq \Omega$. Then:*

(i) $0 \le \widehat{\mathbf{R}}_u^A \le u$ *in* Ω*;*

(ii) $\widehat{\mathbf{R}}_u^A = u$ *in* $\mathrm{int}A$*;*

(iii) $\widehat{\mathbf{R}}_u^A \in \overline{\mathcal{S}}(\Omega)$ *and* $\widehat{\mathbf{R}}_u^A \in \mathcal{H}(\Omega \setminus A)$*;*

(iv) *if* $\overline{A} \subseteq \Omega$ *then* $\widehat{\mathbf{R}}_u^A$ *is a potential in* Ω*.*

Lemma 3.3.4. *Let ω and ω_0 be SR open sets such that $\overline{\omega_0} \subseteq \omega$ and ω_0 is a connected set. If $x_0 \in \omega_0$ and $K \subseteq \omega_0$ is a compact set, then there exists a constant $c = c(\omega_0, K, x_0, \varepsilon) > 0$ such that*

$$u(x_0) \ge c \int_K u(y)\mathrm{d}\nu(y), \tag{3.3.2}$$

for any $u \in \overline{\mathcal{S}}(\omega) \cap L^1_{\mathrm{loc}}(\omega)$ and $u \ge 0$ in ω.

Proof. Let $u \in \overline{\mathcal{S}}(\omega) \cap L^1_{\mathrm{loc}}(\omega)$ with $u \ge 0$ in ω. We can study the following cases.

(I) Suppose that $u = G(f)$, where G is the Green operator related to $\mathcal{L}$ and ω and $f \in C^\infty(\omega) \cap C(\overline{\omega})$, with $f \ge 0$ in ω.

In this case we can apply Lemma 2.3.1, so we get $u \in C^\infty(\omega) \cap C(\overline{\omega})$, $u \ge 0$ and $\mathcal{L}u = -f \le 0$ in ω. From Lemma 3.3.1 we know that

$$u(x_0) \ge \varepsilon \int_{\omega_0} k_\varepsilon(x_0, y)u(y)\mathrm{d}\nu(y) \ge \varepsilon \int_K k_\varepsilon(x_0, y)u(y)\mathrm{d}\nu(y).$$

Observe that $k_\varepsilon(x_0, \cdot)$ is a positive continuous function on $\omega \setminus \{x_0\}$ and a l.s.c. on ω, then

$$u(x_0) \ge \varepsilon \inf_{z \in K} k_\varepsilon(x_0, z) \cdot \int_K u(y)\mathrm{d}\nu(y),$$

so if we choose $c := \varepsilon \inf_K k_\varepsilon(x_0, \cdot) > 0$, we get (3.3.2).

(II) Suppose that $u \in \mathcal{P}(\omega)$ such that $\mathrm{supp}_{\mathcal{H}} u = A$ is a compact set contained in $\omega \setminus \{x_0\}$ (that is $u \in \mathcal{H}(\omega \setminus A)$). Therefore, applying Theorem 3.2.11, we know that there exist $\mu \in \mathcal{M}^+(\omega)$ and $h \in \mathcal{H}(\omega)$ such that $\mu(\omega) = \mu(A) < +\infty$ and

$$u(x) = \int_\omega k(x, y)V(y)\mathrm{d}\mu(y) + h(x),$$

for almost every $x \in \omega$ and for any $x \in \omega \setminus A$.

Since $u \in \mathcal{P}(\omega)$ we can say that $h \equiv 0$ on ω, thanks to Corollary 3.2.12, and so we have that $u(x) = G(V\mu)(x)$ for almost every $x \in \omega$ and for any $x \in \omega \setminus A$; in particular, since $A \subset \omega \setminus \{x_0\}$, we can say that $u(x_0) = G(V\mu)(x_0)$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ be a positive function such that $\int \varphi(x)\mathrm{d}x = 1$. For any $j \in \mathbb{N}$ we put

$$\varphi_j(x) = j^N \varphi(jx), \qquad \text{for every } x \in \mathbb{R}^N.$$

We choose φ such that $\mathrm{supp}(\mu * \varphi_j) \subseteq \omega \setminus \{x_0\}$, for any $j \in \mathbb{N}$. The sequence $\{\varphi_j\}$ is called *approximation of identity on* $\mathbb{R}^N$, and from a known result of Functional Analysis we have

$$\lim_{j\to\infty} \mu * \varphi_j = \mu \quad \text{(in the sense of distribution).}$$

Since $k(x_0, \cdot)$ is a smooth positive function in $\omega \setminus \{x_0\}$, we get

$$\lim_{j\to\infty} \int_\omega k(x_0, y)V(y)\mathrm{d}(\mu * \varphi_j)(y) = \int_\omega k(x_0, y)V(y)\mathrm{d}\mu(y), \tag{3.3.3}$$

that is $G(V(\mu * \varphi_j))(x_0) \to G(V\mu)(x_0) = u(x_0)$, as $j \to \infty$.

On the other hand, the convolution $\mu * \varphi_j \in C^\infty(\omega) \cap C(\overline{\omega})$, for any $j \in \mathbb{N}$, so from case **(I)** we have

$$G(V(\mu * \varphi_j))(x_0) \geq c \int_K G(V(\mu * \varphi_j))(x)\mathrm{d}\nu(x), \tag{3.3.4}$$

for any $j \in \mathbb{N}$.

Now we put

$$\Phi(y) = \int_K k(x, y)\mathrm{d}\nu(x), \qquad \forall\, x \in \omega.$$

It is known that $\mathrm{d}\nu(x) = V(x)\mathrm{d}x$, so if we call λ the Lebesgue's measure restricted to K, thanks to the symmetry of k we get that $\Phi = G(V\lambda)$ on ω. From Proposition 3.1.10 we can say that $\Phi \in \mathcal{P}(\omega)$, in particular Φ is l.s.c. in ω.

On the other hand, let $\{f_n\} \subseteq C_0(\omega)$ be a decreasing sequence of positive functions such that $f_n \to \chi_K$ in ω, as $n \to \infty$. If we consider the sequence $\{G(f_n)\} \subseteq C(\overline{\omega})$, we know that $\{G(f_n)\}$ is decreasing and $G(f_n) \geq 0$. From the Theorem of Beppo-Levi, for any $x \in \omega$, we get

$$\lim_{n\to\infty} \int_\omega k(x, y)f_n(y)\mathrm{d}\nu(y) = \int_\omega k(x, y)\chi_K(y)\mathrm{d}\nu(y),$$

that is $G(f_n) \to \Phi$ in ω, as $n \to \infty$, hence $\Phi = \inf_n G(f_n)$ on ω.

We want to prove that Φ is u.s.c. in ω, that is

$$\Phi(x) = \inf_{U \in \mathcal{U}_x} \left(\sup_{U\cap\omega} \Phi \right), \quad \forall\, x \in \omega.$$

Fix $x \in \omega$ and $n \in \mathbb{N}$; let $U \in \mathcal{U}_x$ and $t > 0$, then there exists $y_t \in U \cap \omega$ such that $\Phi(y_t) > \sup_{U\cap\omega} \Phi - t$. Therefore we get

$$\sup_{U\cap\omega} \Phi < \Phi(y_t) + t \leq G(f_n)(y_t) + t \leq \sup_{U\cap\omega} G(f_n) + t,$$

then $\sup_{U\cap\omega}\Phi \le \sup_{U\cap\omega} G(f_n) + t$, so when $t \to 0$ we have

$$\sup_{U\cap\omega}\Phi \le \sup_{U\cap\omega} G(f_n),$$

for any $U \in \mathcal{U}_x$.

Hence we get

$$\inf_{U\in\mathcal{U}_x}\left(\sup_{U\cap\omega}\Phi\right) \le \inf_{U\in\mathcal{U}_x}\left(\sup_{U\cap\omega} G(f_n)\right) = G(f_n)(x),$$

where the last equality is obtained by continuity of $G(f_n)$ on ω. Thanks to arbitrariness of $n \in \mathbb{N}$, we can see that

$$\inf_{U\in\mathcal{U}_x}\left(\sup_{U\cap\omega}\Phi\right) \le \Phi(x),$$

and by arbitrariness of $x \in \omega$ we get that Φ is u.s.c. in ω. Therefore, Φ is a continuous function on ω.

Now, from the continuity of Φ on ω we get:

$$\begin{aligned}\int_K G(V(\mu*\varphi_j))(x)\mathrm{d}\nu(x) &= \int_K\left(\int_\omega k(x,y)V(y)\mathrm{d}(\mu*\varphi_j)(y)\right)\mathrm{d}\nu(x) = \\ &= \int_\omega\left(\int_K k(x,y)\mathrm{d}\nu(x)\right)V(y)\mathrm{d}(\mu*\varphi_j)(y) = \\ &= \int_\omega \Phi(y)V(y)\mathrm{d}(\mu*\varphi_j)(y) \longrightarrow \int_\omega \Phi(y)V(y)\mathrm{d}\mu(y), \text{ as } j\to\infty.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_K G(V\mu)(x)\mathrm{d}\nu(x) &= \int_K\left(\int_\omega k(x,y)V(y)\mathrm{d}\mu(y)\right)\mathrm{d}\nu(x) = \\ &= \int_\omega\left(\int_K k(x,y)\mathrm{d}\nu(x)\right)V(y)\mathrm{d}\mu(y) = \int_\omega \Phi(y)V(y)\mathrm{d}\mu(y).\end{aligned}$$

Therefore, we have showed that

$$\lim_{j\to\infty}\int_K G(V(\mu*\varphi_j))(x)\mathrm{d}\nu(x) = \int_K G(V\mu)(x)\mathrm{d}\nu(x) = \int_K u(x)\mathrm{d}\nu(x),$$

since $u(x) = G(V\mu)(x)$ for almost every $x \in \omega$.

Then, by (3.3.3) and (3.3.4) we get (3.3.2).

(III) Suppose that $u \in \mathcal{P}(\omega)$ and its support is a compact set in ω.

Let W be a regular open set in ω such that $x_0 \in W \subseteq \overline{W} \subseteq \omega$; in this case we can consider the Perron regularization function u_W. Thanks to properties of u_W, we know that $u_W \in \overline{\mathcal{S}}(\omega)\cap\mathcal{H}(V)$ and $0 \le u_W \le u$ on ω. Then, we get that $u_W \in L^1_{\text{loc}}(\omega)$ (since $u \in L^1_{\text{loc}}(\omega)$) and $u_W \in \mathcal{P}(\omega)$. In fact, if $\varphi \in \mathcal{H}(\omega)$ such that $\varphi \le u_W$ on ω, we have $\varphi \le u$ on ω and so $\varphi \le 0$, since $u \in \mathcal{P}(\omega)$.

Moreover, it is clear that the harmonic support of u_W is a compact set in ω such that $\mathrm{supp}_{\mathcal{H}} u_W \cap W = \varnothing$, because $u_W \in \mathcal{H}(W)$. Hence $\mathrm{supp}_{\mathcal{H}} u_W \subseteq \omega \setminus \{x_0\}$, and now we can apply the case **(II)** at the function u_W:

$$u_W(x_0) \ge c\int_K u_W(x)\mathrm{d}\nu(x),$$

since $u_W \leq u$ on ω, we get

$$u(x_0) \geq c \int_K u_W(x) \mathrm{d}\nu(x). \tag{3.3.5}$$

We know that when the diameter of W tends to 0, we have $u_W \to u$ in ω, so we can apply the Theorem of Dominated Convergence and from (3.3.5) we get (3.3.2).

(IV) Suppose the general case: $u \in \overline{\mathcal{S}}(\omega) \cap L^1_{\text{loc}}(\omega)$, with $u \geq 0$ on ω.

In this case we consider the balayage $\widehat{\mathbf{R}}^{\omega_0}_u$ of u in ω_0. From Proposition 3.3.3 we know that $0 \leq \widehat{\mathbf{R}}^{\omega_0}_u \leq u$ in ω, then $\widehat{\mathbf{R}}^{\omega_0}_u \in L^1_{\text{loc}}(\omega)$; moreover, $\widehat{\mathbf{R}}^{\omega_0}_u \in \overline{\mathcal{S}}(\omega)$, in particular $\widehat{\mathbf{R}}^{\omega_0}_u$ is a potential in ω, since $\overline{\omega_0} \subseteq \omega$. At last, $\widehat{\mathbf{R}}^{\omega_0}_u \in \mathcal{H}(\omega \smallsetminus \overline{\omega_0})$, that is $\widehat{\mathbf{R}}^{\omega_0}_u$ has a compact harmonic support in ω.

Then we can apply the case **(III)** at the function $\widehat{\mathbf{R}}^{\omega_0}_u$, and we get

$$\widehat{\mathbf{R}}^{\omega_0}_u(x_0) \geq c \int_K \widehat{\mathbf{R}}^{\omega_0}_u(x) \mathrm{d}\nu(x). \tag{3.3.6}$$

On the other hand, we know that $\widehat{\mathbf{R}}^{\omega_0}_u = u(x)$ for any $x \in \omega_0$, then by (3.3.6) we get (3.3.2).

□

Now we are ready to prove an important result.

Theorem 3.3.5. *Let Ω be an open set of $\mathbb{R}^N$. Then*

$$\overline{\mathcal{S}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega).$$

Proof. Let $u \in \overline{\mathcal{S}}(\Omega)$ and $K \subseteq \Omega$ be a compact set. We can suppose without loss of generality that there exists an open SR set ω such that $K \subseteq \omega \subseteq \overline{\omega} \subseteq \Omega$.

Since u is a l.s.c. function on Ω, u is l.s.c. on $\overline{\omega}$; then u attains its minimum $m \in \mathbb{R}$ on $\overline{\omega}$. Note that $\mathcal{L}$ is homogeneous, hence $u - m \in \overline{\mathcal{S}}(\omega)$ and $u - m \geq 0$ on ω. Therefore, we can suppose $u \in \overline{\mathcal{S}}(\omega)$ such that $u \geq 0$ on ω.

Now we consider a connected open SR set ω_0 such that $\omega_0 \subseteq \overline{\omega_0} \subseteq \omega$ and $K \subseteq \omega_0$. Let x_0 be a point in $\omega_0 \smallsetminus K$ such that $u(x_0) < +\infty$.

For every $n \in \mathbb{N}$, we put

$$u_n := \inf\{u, n\} \quad \text{on } \omega,$$

so we have a increasing sequence $\{u_n\}$ in ω such that $u_n \in \overline{\mathcal{S}}(\omega) \cap L^1_{\text{loc}}(\omega)$ and $u_n \geq 0$, for any $n \in \mathbb{N}$, because $\mathcal{L}$ is homogeneous and u is a superharmonic function in ω. Moreover, $u_n \to u$ in ω, as $n \to \infty$.

Then we can apply Lemma 3.3.4:

$$u_n(x_0) \geq c \int_K u_n(x) \mathrm{d}\nu(x), \quad \forall\, n \in \mathbb{N},$$

and thanks to Theorem of Beppo-Levi and the construction of $\{u_n\}$, we get

$$u(x_0) \geq c \int_K u(x) \mathrm{d}\nu(x),$$

therefore we have

$$\int_K u(x)\mathrm{d}x = \int_K u(x)\frac{1}{V(x)}\mathrm{d}\nu(x) \le \sup_{y\in K}\left(\frac{1}{V(y)}\right)\cdot\int_K u(x)\mathrm{d}\nu(x) \le$$
$$\le \sup_{y\in K}\left(\frac{1}{V(y)}\right)\cdot\frac{1}{c}u(x_0) < +\infty,$$

that is $u \in L^1_{\mathrm{loc}}(\Omega)$. □

As a consequence, we have:

Corollary 3.3.6. *Let Ω be an open set of $\mathbb{R}^N$. If $u \in \overline{\mathcal{S}}(\Omega)$, then $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$.*

Proof. In fact, thanks to Theorem 3.3.5, we know that if $u \in \overline{\mathcal{S}}(\Omega)$ then $u \in L^1_{\mathrm{loc}}(\Omega)$. Now we can apply Proposition 3.2.8, and so we have $\mathcal{L}u \le 0$ in $\mathcal{D}'(\Omega)$. □

Therefore, in Theorem A we have showed that (i) ⇒ (ii). Now, we want to show that (ii) ⇒ (i) in Theorem A. To this aim, we have need to prove some result.

Theorem 3.3.7. *Let Ω be an open set of $\mathbb{R}^N$ and $y \in \Omega$. Suppose that there exists a potential $P \in \mathcal{P}(\Omega)$, such that $P > 0$ on Ω.*

Then, all potential with harmonic support in $\{y\}$ are proportional.

Proof. We study the following cases:

(i) Let ω be a SR open set such that $y \in \omega \subseteq \overline{\omega} \subseteq \Omega$. If $p \in \mathcal{P}(\omega) \cap \mathcal{H}(\omega \smallsetminus \{y\})$, in particular we have that $p \in \overline{\mathcal{S}}(\omega) \subseteq L^1_{\mathrm{loc}}(\omega)$; then, by Corollary 3.2.9, we get that there exists a constant $c > 0$ such that

$$p(x) = ck_y(x), \quad \text{for any } x \in \omega.$$

Since $k_y \in \mathcal{P}(\omega)$, we can say that p is proportional to a potential on ω.

(ii) Let P_1 and P_2 be potentials on Ω such that $P_1, P_2 \in \mathcal{H}(\Omega \smallsetminus \{y\})$. We want to prove that P_1 and P_2 are proportional on Ω.

Let ω be a SR open set such that $y \in \omega \subseteq \overline{\omega} \subseteq \Omega$. Thanks to [57, Theorem 16.4], we know that there exist a unique $p_1 \in \mathcal{P}(\omega) \cap \mathcal{H}(\omega \smallsetminus \{y\})$, in correspondence of P_1, and a unique $p_2 \in \mathcal{P}(\omega) \cap \mathcal{H}(\omega \smallsetminus \{y\})$, in correspondence of P_2, such that:

$$P_1(x) = p_1(x) + h_1(x) \quad \text{and} \quad P_2(x) = p_2(x) + h_2(x),$$

for any $x \in \omega$, where $h_1, h_2 \in \mathcal{H}(\omega)$.

Since $p_1, p_2 \in \mathcal{P}(\omega) \cap \mathcal{H}(\omega \smallsetminus \{y\})$, from case **(i)** we know that there exist positive constants c_1, c_2 such that

$$p_1(x) = c_1 k(x,y) \text{ and } p_2(x) = c_2 k(x,y) \quad \text{for any } x \in \omega.$$

Therefore, we get

$$P_2(x) = \bar{c}P_1(x) + h_2(x) - \bar{c}h_1(x), \quad \text{for any } x \in \omega, \tag{3.3.7}$$

where $\bar{c} = c_2/c_1 > 0$. We want to prove that $P_2 = \bar{c}P_1$ on Ω.

We put

$$h(x) := \begin{cases} P_2(x) - \bar{c}P_1(x) & \text{if } x \in \Omega \setminus \omega, \\ h_2(x) - \bar{c}h_1(x) & \text{if } x \in \omega. \end{cases}$$

Hence, $h : \Omega \to \mathbb{R}$ and we can show that $h \in \mathcal{H}(\Omega)$. In fact, $h \in \mathcal{H}(\omega)$, since $h_1, h_2 \in \mathcal{H}(\omega)$. On the other hand, $P_1, P_2 \in \mathcal{H}(\Omega \setminus \{y\})$ and $y \in \omega$, then $h \in \mathcal{H}(\Omega \setminus \omega)$. Therefore, we have obtained that h is harmonic on Ω.

It is clear that

$$P_2(x) = \bar{c}P_1(x) + h(x), \quad \text{for any } x \in \Omega.$$

Now, if we prove that $h \equiv 0$, we get the proportionality between P_1 and P_2 on Ω.

Observe that:

- since $P_1 \in \mathcal{P}(\Omega)$ and $P_2 \geq 0$ on Ω, we have $h \geq 0$ on Ω;
- $h = P_2 - \bar{c}P_1 \leq P_2$ on Ω, and $P_2 \in \mathcal{P}(\Omega)$, then $h \leq 0$ on Ω.

Therefore, we get $h \equiv 0$ on Ω and so $P_2(x) = \bar{c}P_1(x)$ for any $x \in \Omega$.

□

Remark 3.3.8. Thanks to Theorem 3.3.7, if we suppose that there exists a positive potential on Ω, then we can apply [57, Theorem 18.1], and for any $y \in \Omega$, we can choose a potential $p_y \in \mathcal{P}(\Omega)$ such that $p_y \in \mathcal{H}(\Omega \setminus \{y\})$ and the function $y \mapsto p_y(x)$ is continuous on $\Omega \setminus \{x\}$, for any $x \in \Omega$.

Let ω be a SR open set such that $y \in \omega \subseteq \overline{\omega} \subseteq \Omega$. From [57, Theorem 16.4], we can say that there exists a unique $p \in \mathcal{P}(\omega) \cap \mathcal{H}(\omega \setminus \{y\})$ such that

$$p_y(x) = p(x) + h(x), \quad \text{for any } x \in \omega,$$

where $h \in \mathcal{H}(\omega)$.

We know that there exists $c := c(y) > 0$ such that $p = ck(x,y)$ on ω, hence we get

$$p_y(x) = c(y)k(x,y) + h(x), \quad \text{for any } x \in \omega.$$

As a function of $y \in \omega$, we can say that $c(y)$ is a continuous positive function on ω.

Then we have $\mathcal{L}p_y = -c(y)/V(y)\delta_y$ in $\mathcal{D}'(\omega)$; in the proof of Corollary 3.2.9 we have showed that $c(y) = \tilde{c}(y)V(y)$, so we get

$$\mathcal{L}p_y = -\tilde{c}(y)\delta_y, \quad \text{in } \mathcal{D}'(\omega).$$

Now we can extend $\tilde{c}$ on Ω to obtain

$$\mathcal{L}p_y = -\tilde{c}(y)\delta_y, \quad \text{in } \mathcal{D}'(\Omega).$$

Moreover, we can choose p_y such that $\mathcal{L}p_y = -\delta_y$ in $\mathcal{D}'(\Omega)$.

Now, if $P \in \mathcal{P}(\Omega)$ we know that it has a unique integral representation on Ω (see [57, Theorem 18.2]):

$$P(x) = \int_\Omega p_y(x)\mathrm{d}\mu(y), \quad \text{for any } x \in \Omega, \tag{3.3.8}$$

where $\mu \in \mathcal{M}^+(\Omega)$.

It is easy to prove that

$$\mathcal{L}P = -\mu \quad \text{in } \mathcal{D}'(\Omega).$$

In fact, for any $\varphi \in C_0^\infty(\Omega)$ we have:

$$\begin{aligned}\langle \mathcal{L}P, \varphi\rangle = \langle P, \mathcal{L}^*\varphi\rangle = \int_\Omega P(x)\mathcal{L}^*\varphi(x)\mathrm{d}x = \int_\Omega \left(\int_\Omega p_y(x)\mathrm{d}\mu(y)\right)\mathcal{L}^*\varphi(x)\mathrm{d}x = \\ = \int_\Omega \left(\int_\Omega p_y(x)\mathcal{L}^*\varphi(x)\mathrm{d}x\right)\mathrm{d}\mu(y) = -\int_\Omega \varphi(y)\mathrm{d}\mu(y) = \langle -\mu, \varphi\rangle,\end{aligned}$$

hence we get $\mathcal{L}P = -\mu$ in $\mathcal{D}'(\Omega)$.

Proposition 3.3.9. *Let Ω be an open set of $\mathbb{R}^N$. If $u_1, u_2 \in \overline{\mathcal{S}}(\Omega)$ and $u_1 = u_2$ a.e. on Ω, then $u_1(x) = u_2(x)$ for any $x \in \Omega$.*

Proof. Let U be a regular connected open set such that $U \subseteq \overline{U} \subseteq \Omega$. Since $u_1, u_2 \in \overline{\mathcal{S}}(\Omega)$, they are l.s.c. on $\overline{U}$; then it is clear that the following set is not empty:

$$\mathscr{F}_i := \{\varphi \in \mathcal{H}(U) \,:\, \varphi \le u_i \text{ on } U\} \ne \varnothing,$$

for $i = 1, 2$. Fix $i \in \{1, 2\}$, we want to consider the greatest harmonic minorant of u_i in U

$$h_i := \sup_{\varphi \in \mathscr{F}_i} \varphi \in \mathcal{H}(U).$$

If $h_i \equiv u_i$, for $i = 1, 2$, then we have $h_1 = h_2$ a.e. on U. Hence, by continuity, we have $h_1 \equiv h_2$ on U and so $u_1 \equiv u_2$ on U.

If there exists $i \in \{1, 2\}$, for example $i = 1$, such that $h_1 \ne u_1$ on U, we get $u_1 - h_1 > 0$ on U; moreover, $u_1 - h_1 \in \mathcal{P}(U)$. In fact, if $\varphi \in \mathcal{H}(U)$ s.t. $\varphi \le u_1 - h_1$ on U, then we get $\varphi + h_1 \le h_1$ and so $\varphi \le 0$ on U.

Therefore, we can apply Theorem 3.3.7 and thanks to Remark 3.3.8, we can say that $u_1 - h_1$ has a unique integral representation as in (3.3.8) on U:

$$u_1(x) - h_1(x) = \int_U p_y(x)\mathrm{d}\mu_1(y), \quad \text{for any } x \in U,$$

where $\mu_1 \in \mathcal{M}^+(U)$.

On the other hand, also the function $u_2 - h_2$ is a potential on U, hence it admits a unique integral representation on U:

$$u_2(x) - h_2(x) = \int_U p_y(x)\mathrm{d}\mu_2(y), \qquad \text{for any } x \in U,$$

where $\mu_2 \in \mathcal{M}^+(U)$.

Now, we have seen in Remark 3.3.8 that $\mathcal{L}(u_i - h_i) = -\mu_i$ in $\mathcal{D}'(U)$, for $i = 1, 2$. Since $u_1 = u_2$ a.e. on U, we get

$$-\mu_1 = \mathcal{L}(u_1 - h_1) = \mathcal{L}(u_1) = \mathcal{L}(u_2) = \mathcal{L}(u_2 - h_2) = -\mu_2,$$

then $\mu_1 = \mu_2$ on U.

Therefore we have that $u_1 - u_2 = h_1 - h_2$ on U, but $u_1 = u_2$ a.e. on U, then $h_1(x) = h_2(x)$ for almost every $x \in U$; thanks to continuity we get $h_1 \equiv h_2$ on U, so $u_1 \equiv u_2$ on U.

In the end, since the regular connected open sets U in Ω are a covering of Ω, we get $u_1(x) = u_2(x)$ for any $x \in \Omega$. □

Now we are ready to prove the last result of this section.

Theorem 3.3.10. *Let Ω be an open set of $\mathbb{R}^N$. If $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\mathcal{L}u \leq 0$ in $\mathcal{D}'(\Omega)$, then there exists a function $v \in \overline{\mathcal{S}}(\Omega)$ such that*

$$u(x) = v(x), \qquad \textit{for almost every } x \in \Omega.$$

Proof. Let ω be a SR open set such that $\omega \subseteq \overline{\omega} \subseteq \Omega$. From Theorem 3.2.1 we know that there exists $\mu \in \mathcal{M}^+(\Omega)$ such that:

$$u(x) = \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x), \qquad \text{for almost every } x \in \omega,$$

where $h \in \mathcal{H}(\omega)$ and k is the Green kernel related to $\mathcal{L}$ and ω.

For any $\omega \subseteq \overline{\omega} \subseteq \Omega$ SR open set, we put

$$v_\omega(x) := \int_\omega k(x,y)V(y)\mathrm{d}\mu(y) + h(x), \qquad \text{for any } x \in \omega.$$

Now we construct the function $v : \Omega \to]-\infty, +\infty]$ such that

$$v(x) := v_\omega(x), \qquad \text{for any } x \in \omega \text{ and for any SR open set } \omega. \tag{3.3.9}$$

Since the SR open sets $\omega \subseteq \overline{\omega} \subseteq \Omega$ are a covering of Ω, we want to show that (3.3.9) is well posed.

If ω_1, ω_2 are SR open sets in Ω such that $\omega_1 \cap \omega_2 \neq \varnothing$, we have $v = v_{\omega_i}$ on ω_i, for $i = 1, 2$; since $u = v_{\omega_i}$ a.e. on ω_i, for $i = 1, 2$, we get that

$$v_{\omega_1}(x) = v_{\omega_2}(x), \qquad \text{for almost every } x \in \omega_1 \cap \omega_2.$$

It is clear that $v_\omega \in \overline{\mathcal{S}}(\omega)$, for any SR open set $\omega \subseteq \overline{\omega} \subseteq \Omega$. Then, thanks to Proposition 3.3.9, we have that

$$v_{\omega_1}(x) = v_{\omega_2}(x), \quad \forall\, x \in \omega_1 \cap \omega_2,$$

so we can say that (3.3.9) is well posed on Ω.

In the end, since $v \in \overline{\mathcal{S}}(\omega)$, for any SR open set ω in Ω, we get that $v \in \overline{\mathcal{S}}(\Omega)$. Then, we have showed that there exists a function $v \in \overline{\mathcal{S}}(\Omega)$, such that $u(x) = v(x)$ for almost every $x \in \Omega$. This completes the proof. □

Chapter 4

Harnack Inequality in Doubling-Poincaré spaces

In this chapter we prove a *non-homogeneous invariant* Harnack inequality in the setting of doubling metric spaces. We consider a real second-order PDO in divergence form on $\mathbb{R}^N$ associated with a family of vector fields.

In the first section we will give some review on control distances, length spaces and doubling measures; then we will study the notions of Sobolev spaces (related to a family of vector fields) and *weak solutions* in W^1-sense. Finally, in the last section we will prove the non-homogeneous invariant Harnack inequality, using the Moser iterative technique (see e.g. [49]), with consequent Hölder-continuous estimates.

4.1 Recalls on control distances, length spaces and doubling measures

In order to prove the main result of this chapter, we need to give some recalls about the notions of control distances, length spaces and doubling measures.

4.1.1 The control distance

Let $X = \{X_1, \dots, X_m\}$ be a family of locally Lipschitz-continuous vector fields in Euclidean space $\mathbb{R}^N$. We recall the definition of control distance (or Carnot-Carathéodory distance) d_X associated with X. In the sequel we shall also briefly use the term *X-distance* for d_X.

First we fix a definition: we say that an $\mathbb{R}^N$-valued continuous curve γ connects x and y if γ is defined on some compact interval $[a, b]$ (with $a \leq b$), and $\gamma(a) = x$ and $\gamma(b) = y$.

We say that a piece-wise C^1 curve $\gamma : [0,1] \to \mathbb{R}^N$ is an X-*trajectory* if

$$\dot{\gamma}(t) = \sum_{j=1}^{m} a_j(t)\, X_j(\gamma(t)) \quad \text{for almost every } t \in [0,1], \tag{4.1.1}$$

for suitable real-valued functions $a_1, \dots, a_m$ on $[0,1]$, and

$$\ell_X(\gamma) := \sup_{t\in[0,1]} \left(\sum_{j=1}^{m} |a_j(t)|^2 \right)^{1/2} < \infty.$$

In this case, for any $x, y \in \mathbb{R}^N$, we set

$$d_X(x,y) := \inf \left\{ \ell_X(\gamma) \,\middle|\, \gamma \text{ is an } X\text{-trajectory connecting } x \text{ and } y \right\}. \tag{4.1.2}$$

It is understood that, whenever the above set in curly braces is empty, one sets $d_X(x,y) := \infty$. To the contrary, if (for every $x, y \in \mathbb{R}^N$) this set is never empty, we say that $\mathbb{R}^N$ is X-*connected*. In the latter case, d_X is a genuine distance on $\mathbb{R}^N$.

Remark 4.1.1. The above definition of d_X is equivalent to the following one: we say that a piece-wise C^1 curve $\gamma : [0,T] \to \mathbb{R}^N$ (with $T \geq 0$) is X-*subunit* if (4.1.1) holds true, jointly with

$$\sup_{t\in[0,1]} \left(\sum_{j=1}^{m} |a_j(t)|^2 \right)^{1/2} \leq 1.$$

Then it can be easily proved that[1]

$$d_X(x,y) = \inf \left\{ T \,\middle|\, \gamma : [0,T] \to \mathbb{R}^N \text{ is an } X\text{-subunit curve connecting } x \text{ and } y \right\}. \tag{4.1.3}$$

With this useful characterization of d_X one obtains that, if $\gamma : [0,T] \to \mathbb{R}^N$ is X-subunit, then

$$d_X(\gamma(t_1), \gamma(t_2)) \leq t_2 - t_1, \quad \text{whenever } 0 \leq t_1 \leq t_2 \leq T. \tag{4.1.4}$$

It is less obvious that a piece-wise C^1 curve $\gamma : [0,T] \to \mathbb{R}^N$ is X-subunit if and only if (for almost every $t \in [0,T]$)

$$\langle \dot{\gamma}(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \qquad \forall\ \xi \in \mathbb{R}^N.$$

The following important fact holds true:

Remark 4.1.2. Let (M,d) be a metric space; we say that a curve $\gamma : [a,b] \to M$ is *d-rectifiable* if

$$\ell_d(\gamma) := \sup \left\{ \sum_{j=1}^{n} d(\gamma(t_{j-1}), \gamma(t_j)) \,\middle|\, \{a = t_0 < t_1 < \dots < t_n = b\} \text{ is a partition of } [a,b] \right\}$$

[1]The cited equivalence is trivial: if $\gamma : [0,T] \to \mathbb{R}^N$ is X-subunit, then

$$\mu : [0,1] \longrightarrow \mathbb{R}^N, \quad \mu(t) := \gamma(T\,t)$$

is an X-trajectory with $\ell_X(\mu) \leq T$; viceversa, if $\gamma : [0,1] \to \mathbb{R}^N$ is an X-trajectory (with $\ell_X(\gamma) \neq 0$), then

$$\mu : [0,T] \longrightarrow \mathbb{R}^N, \quad \mu(t) := \gamma(t/\ell_X(\gamma))$$

is X-subunit, if one takes $T = \ell_X(\gamma)$.

is finite. Then (M,d) is said to be a *length space* if, for every $x,y \in M$, one has

$$d(x,y) = \inf\left\{ \ell_d(\gamma) \;\middle|\; \gamma : [a,b] \to M \text{ is a continuous } d\text{-rectifiable curve connecting } x \text{ and } y \right\}.$$

It is part of the definition of a length space to require that the set in the above rhs is always non-void.

Going back to X-distances, it is not difficult to show that[2] if $\mathbb{R}^N$ is X-connected, then $(\mathbb{R}^N, d_X)$ *is a length space*, i.e.,

$$d_X(x,y) = \inf\left\{ \ell_{d_X}(\gamma) \middle| \gamma : [a,b] \to \mathbb{R}^N \text{ is continuous, } d_X\text{-rectifiable and connects } x,y \right\}. \tag{4.1.5}$$

4.1.2 Known facts on length spaces

For the recalls in this section, see e.g., [20, Chapter 1]. Throughout this section (M,d) is a length space; in the sequel it is understood that M is equipped with the metric topology. From the very definition of $\ell_d(\gamma)$, it is not difficult to show the additivity property of ℓ_d: if γ is d-rectifiable, then

$$\ell_d(\gamma) = \sum_{i=1}^{n} \ell_d\big(\gamma|_{[t_{i-1},t_i]}\big), \tag{4.1.6}$$

for any partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$. We also have the following lower semi-continuity property: if $\gamma, \gamma_n : [a,b] \to M$ are curves such that γ_n point-wise converges to γ, then

$$\liminf_{n\to\infty} \ell_d(\gamma_n) \geq \ell_d(\gamma). \tag{4.1.7}$$

As for the Riemann integral, we have the following *mesh property* of ℓ_d (a consequence of the definition of a length space and of the Heine-Borel theorem): if $\gamma : [a,b] \to M$ is continuous and d-rectifiable, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for any partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ with $\sup_{1\leq j\leq n} |t_j - t_{j-1}| \leq \delta(\varepsilon)$, then

$$\ell_d(\gamma) - \sum_{j=1}^{n} d(\gamma(t_{j-1}), \gamma(t_j)) < \varepsilon. \tag{4.1.8}$$

In the sequel, we employ the usual notation for the open ball of centre $x \in M$ and radius $r > 0$:

$$B_d(x,r) := \{y \in M : d(x,y) < r\}.$$

Whereas in an arbitrary metric space this is not always the case, in a length space we have

$$\overline{B_d(x,r)} = \{y \in M : d(x,y) \leq r\} \quad \text{and} \quad \partial B_d(x,r) = \{y \in M : d(x,y) = r\}.$$

[2]The inequality $d_X(x,y) \leq \ell_{d_X}(\gamma)$ (if γ is as in the rhs of (4.1.5)) is a trivial consequence of the triangle inequality; vice versa, if $\gamma : [0,T] \to \mathbb{R}^N$ is X-subunit and connects x and y, one uses the inequality

$$\sum_{j=1}^{n} d_X(\gamma(t_{j-1}), \gamma(t_j)) \overset{(4.1.4)}{\leq} \sum_{j=1}^{n} (t_j - t_{j-1}) = T.$$

The latter (besides showing that an X-subunit curve is d_X-rectifiable) easily implies that the infimum in the rhs of (4.1.5) is less than or equal to the rhs of (4.1.3), which is $d_X(x,y)$.

Indeed, it suffices to show that if $d(x,y) = r$ then there exist $y_n \in B_d(x,r)$ such that $y_n \to y$. To this end, one takes d-rectifiable curves $\gamma_n : [a_n, b_n] \to M$ connecting x and y, and such that $\lim_n \ell_d(\gamma_n) = d(x,y) = r$. From the Intermediate Value Theorem, there exists $\tau_n \in]a_n, b_n[$ such that

$$d(x, \gamma_n(\tau_n)) = \frac{n-1}{n}\, r.$$

The choice $y_n := \gamma_n(\tau_n)$ does the required job, as

$$r \le d(x, y_n) + d(y_n, y) \le \ell_d\big(\gamma_n|_{[a_n,\tau_n]}\big) + \ell_d\big(\gamma_n|_{[\tau_n,b_n]}\big) \overset{(4.1.6)}{=} \ell_d(\gamma_n) \longrightarrow r.$$

We now provide some recalls on arc-length parameterizations. We let $\gamma : [\alpha, \beta] \to M$ be a continuous d-rectifiable curve with $\ell_d(\gamma) > 0$. Let us consider the map

$$[\alpha, \beta] \ni t \mapsto f(t) := \ell_d\big(\gamma|_{[\alpha,t]}\big).$$

By the additivity property (4.1.6) we infer

$$f(t_2) - f(t_1) = \ell_d\big(\gamma|_{[t_1,t_2]}\big), \qquad \text{for } \alpha \le t_1 \le t_2 \le \beta,$$

which has the following consequences:

- f is non-decreasing;
- if $f(t_1) = f(t_2)$ then $\gamma(t_1) = \gamma(t_2)$;
- f is continuous (for the proof of this fact, one may benefit of the mesh-property (4.1.8) of ℓ_d).

All these properties entitle us to set the following definition:

$$\Gamma : [0, \ell_d(\gamma)] \longrightarrow M, \qquad s \mapsto \Gamma(s) := \gamma(t(s)),$$

where, for any $s \in [0, \ell_d(\gamma)]$, $t(s) \in [\alpha, \beta]$ has been chosen in some way so that $f(t(s)) = s$, i.e.,

$$\ell_d\big(\gamma|_{[\alpha,t(s)]}\big) = s.$$

The way $t(s)$ is chosen does not affect the definition of $\Gamma(s)$. We can also assume that $s \mapsto t(s)$ is non-decreasing. It is not difficult to prove that Γ is continuous; this follows from

$$d(\Gamma(s_2), \Gamma(s_1)) \le s_2 - s_1, \quad \text{for } 0 \le s_1 \le s_2 \le \ell_d(\gamma). \tag{4.1.9}$$

The additivity property (4.1.6) also ensures that

$$\ell_d\big(\Gamma|_{[t_1,t_2]}\big) = \ell_d\Big(\gamma|_{[t(s_1),t(s_2)]}\Big) = s_2 - s_1, \tag{4.1.10}$$

whenever $0 \le s_1 \le s_2 \le \ell_d(\gamma)$ (for the first equality see [20, eq.(5.13) p.22]). We say that Γ is the *arc-length parameterization* of γ. Clearly Γ is d-rectifiable (due to (4.1.9)).

We have the following compactness result:

Lemma 4.1.3. *Let $(\mathbb{R}^N, d)$ be a length space. Suppose $\gamma_n : [\alpha, \beta] \to \mathbb{R}^N$ is a sequence of continuous curves satisfying the following properties:*

1. *there exists $M > 0$ such that $\ell_d(\gamma_n) \le M$, for every $n \in \mathbb{N}$;*
2. *there exists a compact subset of $\mathbb{R}^N$ containing $\gamma_n([\alpha, \beta])$, for every $n \in \mathbb{N}$.*

Then there exists a subsequence $(n_k)_k$ and re-parameterizations $\widetilde{\gamma}_{n_k}$ of γ_{n_k}, all defined on $[0,1]$, such that, as $k \to \infty$, the sequence $\widetilde{\gamma}_{n_k}$ uniformly converges on $[0,1]$ to a continuous d-rectifiable curve $\widetilde{\gamma}$.

Indeed, we first extract a subsequence, which we still denote by γ_k, such that $\gamma_k(\alpha)$ converges as $k \to \infty$; then we consider the arc-length parameterization Γ_k of γ_k, and we re-scale it by setting

$$\widetilde{\gamma}_k(s) := \Gamma_k(s\, \ell_d(\gamma_k)), \quad s \in [0,1].$$

It is then easy to show that the family $\{\widetilde{\gamma}_k\}_k$ is equi-bounded and equi-continuous (the latter follows from (4.1.9)); an application of the Arzelà-Ascoli Theorem proves the lemma.

4.1.3 Doubling spaces

We assume that (M, d) is a metric space equipped with a measure satisfying the following global doubling assumption:

(D) there exists a measure μ on M such that (M, d, μ) *is a doubling metric space*, that is, there exists $A > 1$ such that

$$\mu(B_d(x, 2r)) \le A\, \mu(B_d(x, r)), \qquad \text{for every } x \in M \text{ and every } r > 0. \tag{4.1.11}$$

Since d will always be understood, we shall also frequently use the notations $B(x, r)$ and $B_r(x)$ to denote the d-ball $B_d(x, r)$. Moreover, as it is customary, we set $A = 2^Q$, i.e.,

$$Q := \log_2 A,$$

so that (4.1.11) becomes $\mu(B_{2r}(x)) \le 2^Q \mu(B_r(x))$. For the sake of future references, we now state some generalizations of (D). First, an iteration argument gives[3]

$$\mu(B(x, R)) \le 2^Q \Big(\frac{R}{r}\Big)^Q \mu(B(x, r)), \qquad \text{for every } x \in M \text{ and } 0 < r \le R; \tag{4.1.12}$$

we can also allow for different centres, as long as a ball is contained in the other:[4]

$$\mu(B(y, R)) \le 4^Q \Big(\frac{R}{r}\Big)^Q \mu(B(x, r)), \qquad \text{whenever } B(x, r) \subseteq B(y, R); \tag{4.1.13}$$

[3] This follows by iterating (4.1.11) n times, with $n \in \mathbb{N}$ such that $n - 1 \le \log_2(R/r) < n$, so that $r/2 \le R/2^n < r$, whence $\mu(B(x, R/2^n)) \le \mu(B(x, r))$ and $2^{nQ} \le 2^Q (R/r)^Q$.

[4] The triangle inequality gives $B(y, R) \subseteq B(x, 2R)$ so that (4.1.13) follows from (4.1.12).

and we can also improve the latter for a more general geometry of the balls involved:[5]

$$\mu(B(y,R)) \le 8^Q \Big(\frac{R}{r}\Big)^Q \mu(B(x,r)), \qquad \text{whenever } y \in M,\ x \in B(y,R) \text{ and } 0 < r \le R. \quad (4.1.14)$$

As a consequence of (D), we infer that (M,d) is a *homogeneous space* in the sense of [24, Ch.III], which amounts to the following property:

Corollary 4.1.4. *In the doubling metric space (M,d,μ), any d-ball $B_d(x,r)$ can contain at most 18^Q pair-wise disjoint d-balls of radius $r/2$. Furthermore, there exists an integer $n \le 18^Q$ such that, for every $x \in M$ and every $r > 0$, $B_d(x,r)$ contains at most n points $x_1,\dots,x_n$ such that $d(x_i,x_j) \ge r/2$ for every $i \ne j$.*

More generally, if n is as above, for any $h \in \mathbb{N}$ and any $x \in M$ and $r > 0$, $B_d(x,r)$ contains at most n^h points $x_1,\dots,x_{n^h}$ such that $d(x_i,x_j) \ge r/2^h$ for every $i \ne j$.

Indeed, let us choose $i \in \{1,\dots,n\}$ minimizing the measures of $B(x_1,r/4),\dots,B(x_n,r/4)$; let us also observe that these balls are pair-wise disjoint and all contained in $B_d(x,r+r/4)$, so that

$$\mu(B(x_1,r/4)) + \cdots + \mu(B(x_n,r/4)) \le \mu(B_d(x,r+r/4)) \le \mu(B(x_i,9r/4)).$$

By the minimality property of i, the above lhs is greater than $n\mu(B(x_i,r/4))$, whereas (due to (4.1.12)) the far rhs is smaller than $18^Q\mu(B(x_i,r/4))$. This prescribes the bound $n \le 18^Q$. The last statement of the corollary can be proved by induction on h (see [24, p.68]).

Remark 4.1.5. As a consequence of the last statement of Corollary 4.1.4, it easily follows that *any bounded set in the doubling metric space (M,d) is also totally-bounded*: indeed, if $\varepsilon > 0$, given a ball $B_d(x,r)$ we chose $h \gg 1$ such that $r/2^h < \varepsilon$ so that (with the notation in the cited corollary relative to the ball $B_d(x,r)$)

$$B_d(x,r) \subseteq \textstyle\bigcup_{j=1}^{n^h} B_d(x_j,r/2^h) \subseteq \bigcup_{j=1}^{n^h} B_d(x_j,\varepsilon).$$

4.1.4 The segment property

In the sequel we assume that Euclidean space $\mathbb{R}^N$ is equipped with the structure of a length space, which we occasionally denote by (M,d) to preserve the taste of general metric-space theory, which is also endowed with the structure of a doubling metric space (M,d,μ) by means of a measure μ, and (M,d) further satisfies the following topological assumption:

(T) the topology of the metric space $(\mathbb{R}^N,d)$ coincides with the usual Euclidean topology of $\mathbb{R}^N$, and $(\mathbb{R}^N,d)$ is a complete metric space.

Remark 4.1.6. Under all the above assumptions we claim that *a set $A \subset \mathbb{R}^N$ is compact (in the Euclidean topology) if and only if it is closed and bounded in* $(\mathbb{R}^N,d)$. Indeed, since the Euclidean

[5] The triangle inequality gives $B(x,R) \subseteq B(y,2R)$ so that (4.1.14) follows from (4.1.13).

topology of $\mathbb{R}^N$ coincides with the metric topology due to (T), A is Euclidean-compact iff it is compact in the metric space $(\mathbb{R}^N, d)$; since the latter is complete again by assumption (T), A is compact in $(\mathbb{R}^N, d)$ iff it is closed and totally-bounded in $(\mathbb{R}^N, d)$; the claim now follows from Remark 4.1.5.

Arguing analogously, one can prove that *a set $A \subset \mathbb{R}^N$ is bounded in $(\mathbb{R}^N, d)$ if and only if it is bounded in the Euclidean metric.*

From the last assertion we infer that $\mathbb{R}^N$ is unbounded wrt d; this easily shows that

$$B(x,r) \setminus B(x,\lambda r) \neq \varnothing, \qquad \forall\ x \in \mathbb{R}^N,\ r > 0,\ \lambda \in (0,1). \tag{4.1.15}$$

We have the following remarkable property:

Theorem 4.1.7 (Segment property)**.** *Let $\mathbb{R}^N$ be equipped with the structure of a doubling metric space (M, d, μ), which is also a length space, and it satisfies the topological assumption* (T).

Then, for every $x, y \in \mathbb{R}^N$, there exists a continuous d-rectifiable curve $\gamma : [0,1] \to \mathbb{R}^N$ connecting x and y, with $\ell_d(\gamma) = d(x,y)$ and such that

$$d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y), \quad \forall\ t \in [0,1]. \tag{4.1.16}$$

Proof. Given $x \neq y$ we set $r := d(x,y)$. By the definition of a length space, there exists a sequence $\gamma_n : [0,1] \to \mathbb{R}^N$ of continuous d-rectifiable curves connecting x and y such that $\lim_n \ell_d(\gamma_n) = r$. For large n and for any $t \in [0,1]$ we have

$$2r \geq \ell_d(\gamma_n) \geq d(x,\gamma_n(t)) + d(\gamma_n(t),y) \geq d(x,\gamma_n(t)).$$

This shows that $\gamma_n([0,1]) \subseteq \overline{B_d(x,2r)}$, and the latter is a compact set in the Euclidean $\mathbb{R}^N$ (due to Remark 4.1.6). We can apply Lemma 4.1.3 and infer the existence of a sequence $\{\psi_k\}_k$ (obtained as re-parameterizations of some subsequence $\{\gamma_{n_k}\}_k$) uniformly converging to a continuous d-rectifiable curve ψ on $[0,1]$. From (4.1.7) we get

$$d(x,y) \leq \ell_d(\psi) \leq \liminf_{k\to\infty} \ell_d(\psi_k) = \liminf_{k\to\infty} \ell_d(\gamma_{n_k}) = r = d(x,y).$$

Finally, from the additivity property (4.1.6) we get

$$d(x,y) \leq d(x,\psi(t)) + d(\psi(t),y) \leq \ell_d\big(\psi|_{[0,t]}\big) + \ell_d\big(\psi|_{[t,1]}\big) = \ell_d\big(\psi|_{[0,1]}\big) = d(x,y).$$

This proves (4.1.16), ending the proof. □

Remark 4.1.8. Using (4.1.10), it is not difficult[6] to show that the arc-length parameterization $\Gamma(s)$ of the curve $\gamma(t)$ in Theorem 4.1.7 has the following properties:

$\Gamma : [0, d(x,y)] \to \mathbb{R}^N$ is a continuous d-rectifiable curve connecting x and y satisfying:

$$d(x,y) = d(x,\Gamma(s)) + d(\Gamma(s),y), \quad \text{for every } s \in [0,d(x,y)]; \tag{4.1.17}$$

$$d(\Gamma(s_1),\Gamma(s_2)) = \ell_d\big(\Gamma|_{[s_1,s_2]}\big) = s_2 - s_1, \quad \text{for every } 0 \le s_1 \le s_2 \le d(x,y). \tag{4.1.18}$$

Remark 4.1.9. Any d-ball is a *John domain* (for the general definition see e.g., [53, Section 9.1]). More precisely, given an arbitrary d-ball $B(y,r)$, for any $x \in B(y,r)$ we consider the curve $\Gamma(s)$ as in Remark (4.1.8). For any $\xi \notin B(y,r)$ we have

$$R \le d(y,\xi) \le d(y,\Gamma(s)) + d(\Gamma(s),\xi),$$

so that

$$d(\Gamma(s),\xi) \ge R - d(y,\Gamma(s)) > d(x,y) - d(y,\Gamma(s)) \overset{(4.1.17)}{=} d(x,\Gamma(s)) = d(\Gamma(0),\Gamma(s)) \overset{(4.1.18)}{=} s.$$

This gives

$$\operatorname{dist}_d\big(\Gamma(s), \mathbb{R}^N \smallsetminus B(y,r)\big) := \inf_{\xi \notin B(y,r)} d(\Gamma(s),\xi) \ge s, \quad \forall\ s \in [0,d(x,y)],$$

which ensures that $B(y,r)$ is a John domain.

Finally we have the following useful result (see Di Fazio, Gutiérrez, Lanconelli, [30]):

Theorem 4.1.10 (Global reverse doubling)**.** *Let the assumptions of Theorem 4.1.7 apply. There exists $\delta \in (0,1)$ (only depending on the doubling constant Q) such that*

$$\mu(B_d(x,r)) \le \delta\, \mu(B_d(x,2r)), \quad \textit{for every } x \in \mathbb{R}^N \textit{ and } r > 0. \tag{4.1.19}$$

Proof. Let $1 < \eta < 2\theta < 2$ and let $y \in B(x,2\theta r) \smallsetminus B(x,\eta r)$ (see (4.1.15)). If $\sigma > 0$ is smaller than $\min\{2-2\theta, \eta - 1\} < 1$ we have $B(y,\sigma r) \subset B(x,2r) \smallsetminus B(x,r)$. From (4.1.13) we get

$$\mu(B(x,2r)) \ge \mu(B(x,r)) + \mu(B(y,\sigma r)) \ge \mu(B(x,r)) + 2^{-Q}(\sigma/4)^Q\, \mu(B(x,2r)),$$

proving (4.1.16) with the choice $\delta := 1 - (\sigma/8)^Q$. □

[6] The analogue of the segment property holds for Γ due to the chain of inequalities:

$$d(x,y) \le d(x,\Gamma(s)) + d(\Gamma(s),y) \le \ell_d(\Gamma_{[0,s]}) + \ell_d(\Gamma_{[s,d(x,y)]}) = \ell_d(\Gamma_{[0,d(x,y)]}) = \ell_d(\Gamma) = \ell_d(\gamma) = d(x,y).$$

Moreover one has

$$\begin{aligned} d(x,y) &\le d(x,\Gamma(s_1)) + d(\Gamma(s_1),\Gamma(s_2)) + d(\Gamma(s_2),y) \\ &\le \ell_d(\Gamma_{[0,s_1]}) + \ell_d(\Gamma_{[s_1,s_2]}) + \ell_d(\Gamma_{[s_2,d(x,y)]}) = \ell_d(\Gamma) = d(x,y), \end{aligned}$$

so that

$$d(x,y) = d(x,\Gamma(s_1)) + d(\Gamma(s_1),\Gamma(s_2)) + d(\Gamma(s_2),y) =: a + b + c$$
$$d(x,y) = \ell_d(\Gamma_{[0,s_1]}) + \ell_d(\Gamma_{[s_1,s_2]}) + \ell_d(\Gamma_{[s_2,d(x,y)]}) =: A + B + C.$$

Since $a \le A$, $b \le B$ and $c \le C$, the latter are all equalities and in particular $d(\Gamma(s_1),\Gamma(s_2)) = \ell_d(\Gamma_{[s_1,s_2]}) = s_2 - s_1$.

4.2 The assumptions on the operator

In this short section we fix the assumptions on the operators that we shall consider throughout the sequel. We assume that $\mathcal{L}$ is a divergence-form operator on $\mathbb{R}^N$ (with nonnegative characteristic form, possibly degenerate) under the following form

$$\mathcal{L} = \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j}\Big), \tag{4.2.1}$$

where $a_{i,j} = a_{j,i}$ are measurable functions (for every $i, j \le N$) with $A(x) := (a_{i,j}(x))_{i,j}$ positive semidefinite for every $x \in \mathbb{R}^N$, and $V > 0$ is a C^1 function on $\mathbb{R}^N$. Due to the low regularity of the coefficients of $\mathcal{L}$, we shall obviously consider solutions and sub-/super-solutions of $\mathcal{L}u = f$ in an appropriate weak sense that will be specified in the sequel.

Attached with $\mathcal{L}$, we have a natural (Borel) measure, namely

$$\mathrm{d}\mu(x) := V(x)\,\mathrm{d}x, \tag{4.2.2}$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^N$. In many of the following results, the C^1 assumption on V may be relaxed, requiring V to be a locally bounded and measurable function. Next we assume that the possible degeneracy of the matrix $A(x)$ be controlled by well-behaved vector fields, in the following sense: we assume that there exists a family of locally Lipschitz-continuous vector fields $X = \{X_1, \dots, X_m\}$ on Euclidean space $\mathbb{R}^N$, and two constants $\lambda, \Lambda > 0$ such that

$$\lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2, \qquad \forall\ x, \xi \in \mathbb{R}^N. \tag{4.2.3}$$

Finally, we make our assumptions on the X-control distance: we assume that $\mathbb{R}^N$ is X-connected (so that $(\mathbb{R}^N, d_X)$ is a length space, see (4.1.5)), and the associated X-distance d_X satisfies the following assumptions.

(T) The topology of the metric space $(\mathbb{R}^N, d_X)$ coincides with the usual Euclidean topology of $\mathbb{R}^N$, and $(\mathbb{R}^N, d_X)$ is a complete metric space.

For brevity, we shall write d instead of d_X.

(D) If μ is the measure (4.2.2) associated with $\mathcal{L}$, then $(\mathbb{R}^N, d, \mu)$ is a doubling metric space, that is, there exists $Q > 0$ such that

$$\mu(B_d(x, 2\,r)) \le 2^Q\, \mu(B_d(x, r)), \qquad \text{for every } x \in \mathbb{R}^N \text{ and every } r > 0. \tag{4.2.4}$$

The ball $B_d(x, r)$ will be denoted indifferently by $B(x, r)$ or $B_r(x)$. With no restrictions on the validity of (4.2.4), we shall assume that $Q > 2$.

(P) The following *global Poincaré inequality* is satisfied: there exists a constant $C_P > 0$ such that, for every $x \in \mathbb{R}^N$, $r > 0$ and every u which is C^1 in a neighborhood of $B_{2r}(x)$ one has

$$⨍_{B_r(x)} \left|u - u_{B_r(x)}\right| \mathrm{d}\mu \le C_P\, r\, ⨍_{B_{2r}(x)} |Xu|\,\mathrm{d}\mu. \tag{4.2.5}$$

Here μ is as in (4.2.2), and we throughout use the following notations:

$$\fint_{B_r(x)} \{\cdots\} \, \mathrm{d}\mu = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \{\cdots\} \, \mathrm{d}\mu, \qquad u_{B_r(x)} := \fint_{B_r(x)} u \, \mathrm{d}\mu,$$

and $|Xu| := \sqrt{\sum_{j=1}^m |X_j u|^2}$.

Remark 4.2.1. Condition (4.2.3) has been introduced by Kogoj and Lanconelli in [65], and the operators $\mathcal{L}$ satisfying it have been called *X-elliptic*. Meaningful examples of operators satisfying the assumptions above are contained in [52, Section 6.1] by Gutiérrez and Lanconelli, also comprising operators previously considered by Franchi and Lanconelli [40, 41]. For other examples see also [92, Section 1].

The role of the density V comes from the need to allow for second order operators coming from applications to Lie groups; indeed, one can find in [5, Example 1.1] relevant examples of operators under the form (4.2.1), where $V \neq 1$ is the density of the Haar measure of a Lie group G, and $X_1, \ldots, X_m$ is a family of generators of the Lie algebra of G. The same kind of operators (coming from Lie group theory) have also been investigated in [1, 7].

A set of hypotheses similar to ours is considered by Kogoj and Lanconelli in [65, 66], where scale-invariant Harnack inequalities for the homogeneous equation $\mathcal{L}u = 0$ are obtained.

Remark 4.2.2. Due to assumptions (T) and (D), we know that the segment property in Theorem 4.1.7 and the reverse doubling property in Theorem 4.1.10 hold true for our space $(\mathbb{R}^N, d_X, \mathrm{d}\mu)$, and the latter is a homogeneous space in the sense of Corollary 4.1.4.

4.2.1 A Poincaré-Sobolev inequality

Arguing as in [53], starting from assumption (P) one can prove the following result, a global Poincaré-Sobolev-type inequality:

Lemma 4.2.3. *Let the assumptions in Section 4.2 be satisfied. Let us fix throughout the notation*

$$q := \frac{2Q}{Q-2}. \tag{4.2.6}$$

Then, there exists a constant C (only depending on the doubling constant Q in (4.2.4) *and on the Poincaré constant C_P in* (4.2.5)*) such that*

$$\left(\fint_{B_r(x)} \left| u - u_{B_r(x)} \right|^q \mathrm{d}\mu \right)^{1/q} \le C\, r \left(\fint_{B_{10r}(x)} |Xu|^2 \, \mathrm{d}\mu \right)^{1/2}, \tag{4.2.7}$$

for every $x \in \mathbb{R}^N$, $r > 0$ and every u which is C^1 in a neighborhood of $B_{10r}(x)$.

Proof. This follows by arguing as in [53, Theorem 5.1, p.22]. We remark that, in order to use the arguments in [53], some results on the maximal function in metric spaces are required (see Theorems 1.8 and 2.2 in [55]). □

Remark 4.2.4. Inequality (4.2.7) can be improved to a genuine Poincaré-Sobolev inequality, that is with $B_r(x)$ in place of $B_{10r}(x)$ in the right-hand side, by arguing as in [53, Corollary 9.8]. To this end, however, it is also crucially required to invoke (together with the segment property (4.1.16)), the fact that any d-ball is a John domain (see Remark 4.1.9). Since we do not need all of this machinery, and only Lemma 4.2.3 is needed, we shall not further improve the latter lemma.

As it is expected, Lemma 4.2.3 allows us to obtain a (global) Sobolev inequality, given in the next result. First we fix a notation: if μ is as in (4.2.2), given any $p > 0$, any measurable set $A \subseteq \mathbb{R}^N$ and any measurable function u on A, we set

$$\|u\|_{L^p(A)} := \left(\int_A |u|^p \,\mathrm{d}\mu\right)^{1/p} \quad \text{and} \quad \|u\|^*_{L^p(A)} := \left(\fint_A |u|^p \,\mathrm{d}\mu\right)^{1/p}.$$

When A is understood, we shall also use the notations (resp.) $\|u\|_p$ and $\|u\|_p^*$.

Theorem 4.2.5 (Global Sobolev inequality)**.** *Let the assumptions in Section 4.2 be satisfied. Let q be as in* (4.2.6)*. Then, there exists a constant C (only depending on the doubling constant Q in* (4.2.4) *and on the Poincaré constant C_P in* (4.2.5)*) such that*

$$\|u\|_{L^q(B(x,r))} \leq \frac{C\,r}{\mu(B(x,r))^{1/Q}}\,\|Xu\|_{L^2(B(x,r))}, \tag{4.2.8}$$

$$\|u\|^*_{L^q(B(x,r))} \leq C\,r\,\|Xu\|^*_{L^2(B(x,r))}, \tag{4.2.9}$$

for every $x \in \mathbb{R}^N$, $r > 0$ and every $u \in C_0^1(B(x,r))$.

Finally, if $\Omega \subset \mathbb{R}^N$ is a bounded open set, there exists a constant $C(\Omega) > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C(\Omega)\,\|Xu\|_{L^2(\Omega)}, \quad \textit{for every } u \in C_0^1(\Omega). \tag{4.2.10}$$

Proof. Let x, r, u be as in the assertion. By trivially prolonging u outside $B_r := B(x,r)$, from Hölder inequality one has

$$\|u\|_{L^q(B_r)} \leq \left\|u - u_{B_{2r}}\right\|_{L^q(B_{2r})} + \left(\frac{\mu(B_r)}{\mu(B_{2r})}\right)^{1-1/q} \|u\|_{L^q(B_r(x))}.$$

From the reverse doubling inequality (4.1.19) one gets

$$\begin{aligned}
\|u\|_{L^q(B_r)} &\leq \frac{\mu(B_{2r})^{1/q}}{1-\delta^{1-1/q}}\left\|u - u_{B_{2r}}\right\|^*_{L^q(B_{2r})} \qquad \text{(by (4.2.7))}\\
&\leq \frac{2C\,r}{1-\delta^{1-1/q}}\,\frac{\mu(B_{2r})^{1/q}}{\mu(B_{20r})^{1/2}}\,\|Xu\|_{L^2(B_{20r})}\\
&\quad \text{(we use } u \in C_0^1(B_r)\text{, the doubling condition and } \mu(B_{20r}) \geq \mu(B_r))\\
&\leq \frac{2^{Q/q+1}C\,r}{1-\delta^{1-1/q}}\,\frac{\mu(B_r)^{1/q}}{\mu(B_r)^{1/2}}\,\|Xu\|_{L^2(B_r)}.
\end{aligned}$$

This is (4.2.8) since $1/2 - 1/q = 1/Q$. The latter identity also shows that (4.2.9) follows from (4.2.8). Finally, from Remark 4.1.6 we know that Ω (which is bounded in $\mathbb{R}^N$) is also bounded in $(\mathbb{R}^N, d)$, so there exists $B(0,r)$ containing Ω (with $r = r(\Omega)$); if $u \in C_0^1(\Omega)$, then it can be trivially prolonged to a function in $C_0^1(B(0,r))$. Thus (4.2.10) follows from (4.2.8). □

4.3 X-Sobolev spaces and W^1-weak solutions for $\mathcal{L}$

As is usually done when dealing with X-control distances, we need to consider the appropriate X-Sobolev spaces. We tacitly understand that the assumptions in Section 4.2 on $X = \{X_1, \dots, X_m\}$, μ and $d = d_X$ be satisfied. L^p spaces are meant wrt the measure μ in (4.2.2). We also assume throughout this section that Ω is a fixed open subset of $\mathbb{R}^N$.

Let $j \in \{1, \dots, m\}$ and let us define the formal L^2-adjoint of X_j (as a linear first order operator) as the unique operator X_j^* (possibly containing first and zero order terms) such that

$$\int_{\mathbb{R}^N} \psi\, X_j \varphi \,\mathrm{d}\mu = \int_{\mathbb{R}^N} \varphi\, X_j^* \psi \,\mathrm{d}\mu, \qquad \forall\ \psi, \varphi \in C_0^\infty(\mathbb{R}^N). \tag{4.3.1}$$

Since any X_j is locally Lipschitz-continuous and since the density V of $\mathrm{d}\mu(x) = V(x)\,\mathrm{d}x$ is C^1, X_j^* is (uniquely) well-posed. Then we recall that, given $u \in L^2(\Omega)$ and $j \in \{1, \dots, m\}$, we define $X_j u$ (in the weak sense) whenever there exists a function $\phi_j \in L^2(\Omega)$ (denoted by $X_j u$) such that

$$\int_{\Omega} \psi\, \phi_j \,\mathrm{d}\mu = \int_{\Omega} u\, X_j^* \psi \,\mathrm{d}\mu, \qquad \forall\ \psi \in C_0^\infty(\mathbb{R}^N). \tag{4.3.2}$$

Throughout the sequel, we always understand that the components of $Xu = (X_1 u, \dots, X_m u)$ are meant in the above weak sense. As usual, $|Xu| = \sqrt{\sum_{j=1}^m |X_j u|^2}$. To avoid cumbersome notations, we write $\|Xu\|_2$ in place of the L^2-norm of $|Xu|$.

Definition 4.3.1. We define $W^1(\Omega, X)$ as the vector space of the functions $u \in L^2(\Omega)$ such that $X_j u$ exists and belongs to $L^2(\Omega)$, for any $j = 1, \dots, m$. On $W^1(\Omega, X)$ we consider the norm

$$\|u\|_{W^1} := \sqrt{\|u\|_2^2 + \|Xu\|_2^2}.$$

We denote by $W^1_{\mathrm{loc}}(\Omega, X)$ the set of the functions u belonging to $W^1(\Omega', X)$, for any open set Ω' whose closure is a compact subset of Ω.

Finally, we denote by $W^1_0(\Omega)$ the closure of $C^1_0(\Omega)$ wrt $\|\cdot\|_{W^1}$. We write $W^1(\Omega)$ shortly for $W^1(\Omega, X)$, and W^1 whenever Ω is understood. The same for W^1_0 or W^1_{loc}.

Clearly, $\|\cdot\|_{W^1}$ is a norm induced by the scalar product

$$\langle u, v\rangle_{W^1} := \int_\Omega u\, v \,\mathrm{d}\mu + \int_\Omega \sum_{j=1}^m X_j u\, X_j v \,\mathrm{d}\mu, \qquad u, v \in W^1(\Omega).$$

On W^1 we shall also consider the equivalent norm $\|u\|_2 + \|Xu\|_2$. By an abuse of notation, the latter will also be denoted by $\|\cdot\|_{W^1}$.

It is a simple exercise to check that $(W^1(\Omega, X), \|\cdot\|_{W^1})$ is a Hilbert space, hence the same is true of $W^1_0(\Omega)$ with the induced norm.

Remark 4.3.2. A profound result (of Meyer-Serrin type) is that

$$C^\infty(\Omega) \cap W^1(\Omega, X) \textit{ is dense in } (W^1(\Omega, X), \|\cdot\|_{W^1}).$$

This is proved by Garofalo and Nhieu in [48] when μ is Lebesgue measure. In our case $\mathrm{d}\mu = V\,\mathrm{d}x$, the same fact holds true, due to a result by Franchi, Hajłasz and Koskela [39, Section 3] where measures μ of our form are considered.

Proposition 4.3.3. *Let Ω be a bounded open set. Then the $\|\cdot\|_{W^1}$ norm on $W^1_0(\Omega)$ is equivalent to*

$$\|u\|_{W^1_0} := \|Xu\|_2, \quad u \in W^1_0(\Omega), \tag{4.3.3}$$

and there exists $C(\Omega) > 0$ such that

$$\|u\|_2 \le C(\Omega)\,\|Xu\|_2, \quad \forall\, u \in W^1_0(\Omega). \tag{4.3.4}$$

Proof. If $u_n \in C^1_0(\Omega)$ is a sequence converging to u in W^1, we have

$$\|u_n\|_2 \le (\mu(\Omega))^{\frac{q-1}{2q}}\,\|u_n\|_q \overset{(4.2.10)}{\le} (\mu(\Omega))^{\frac{q-1}{2q}}\,C(\Omega)\,\|Xu_n\|_2 =: C'(\Omega)\,\|Xu_n\|_2.$$

By letting $n \to \infty$ we infer $\|u\|_2 \le C'(\Omega)\,\|Xu\|_2$, and the proof is complete. □

Theorem 4.3.4 (W^1_0-Sobolev and W^1-Poincaré inequalities)**.** *With the same constants C, C_P as in* (4.2.9) *and in* (4.2.5), *we have*

$$\|u\|^*_{L^q(B_r(x))} \le C\,r\,\|Xu\|^*_{L^2(B_r(x))}, \tag{4.3.5}$$

for any $x \in \mathbb{R}^N$, $r > 0$ and any $u \in W^1_0(B_r(x))$. If $\Omega \subseteq \mathbb{R}^N$ is an open set, we have

$$⨍_{B_r(x)} \left|u - u_{B_r(x)}\right| \mathrm{d}\mu \le C_P\, r\, ⨍_{B_{2r}(x)} |Xu|\,\mathrm{d}\mu, \tag{4.3.6}$$

whenever $\overline{B_{2r}(x)} \subset \Omega$, and for every $u \in W^1(\Omega, X)$.

Proof. If $u \in W^1_0(B_r(x))$ and $u_n \in C^1_0(B_r(x))$ is a sequence converging to u in W^1, from (4.2.9) applied to $u_n - u_m$, and the fact that $Xu_n \to Xu$ in L^2, we infer that $(u_n)_n$ is a Cauchy sequence in L^q. Since $u_n \to u$ in L^2, we get $u_n \to u$ in L^q as well, so that (4.3.5) follows from a density argument from (4.2.9). As for (4.3.6), one can argue analogously, by using a sequence $u_n \in C^1 \cap W^1(\Omega)$ converging to u in W^1 (see Remark 4.3.2), and using the fact that $\|Xu_n - Xu_m\|^*_{L^1(B_{2r}(x))} \le \|Xu_n - Xu_m\|^*_{L^2(B_{2r}(x))}$. □

The following fact will be extremely relevant for the proof of the Harnack inequality:

Remark 4.3.5. The following cut-off argument has been proved (crucially, for our purposes) by Kogoj and Lanconelli in [66, Theorem 10], under the same assumptions that we have done for the metric $d = d_X$ (see Section 4.2):

Given any $x_0 \in \mathbb{R}^N$ and any $0 < R_1 < R_2 < \infty$, there exists $\eta \in W^1_0(B_{R_2}(x_0), X)$ such that

1. $0 \le \eta \le 1$, $\eta \equiv 1$ *on* $B_{R_1}(x_0)$, η *is compactly supported in* $B_{R_2}(x_0)$*;*
2. $|X\eta| \le \frac{2}{R_2 - R_1}$ *almost everywhere on* $B_{R_2}(x_0)$.

For the latter inequality, it is required the crucial estimate $|Xd(x_0,\cdot)| \le 1$ (a.e.) first proved by Franchi, Serapioni and Serra Cassano [44, Proposition 2.9]; for the existence of cut-off functions in are particular cases see [23, 41, 75].

In the sequel Ω will always denote an open subset of $\mathbb{R}^N$. Moreover, the assumptions of Section 4.2 hold true, and $\mathcal{L}$ is the operator in (4.2.1).

We consider the bilinear operator $L : C^1(\Omega) \times C^1_0(\Omega) \longrightarrow \mathbb{R}$ defined by

$$L(u,v) := \int_\Omega \big\langle A(x)\nabla u(x), \nabla v(x)\big\rangle \,\mathrm{d}\mu(x), \qquad u \in C^1(\Omega),\ v \in C^1_0(\Omega). \tag{4.3.7}$$

Here $A(x) = (a_{i,j}(x))_{i,j}$ is the symmetric matrix associated with $\mathcal{L}$, and μ is as in (4.2.2). From our assumption (4.2.3) and due to $A(x) \ge 0$ for any x, we get

$$|\langle A\nabla u, \nabla v\rangle| \le \sqrt{\langle A\nabla u, \nabla u\rangle}\cdot\sqrt{\langle A\nabla v, \nabla v\rangle} \overset{(4.2.3)}{\le} \Lambda\,|Xu|\cdot|Xv|,$$

so that $|L(u,v)| \le \Lambda\|Xu\|_2\,\|Xv\|_2 \le \Lambda\|u\|_{W^1}\|v\|_{W^1_0}$. Hence, by density, L can be (uniquely) prolonged to an operator

$$L : W^1(\Omega) \times W^1_0(\Omega) \longrightarrow \mathbb{R}.$$

We fix once and for all a function

$$g \in L^p(\Omega), \quad \text{with } p > Q/2. \tag{4.3.8}$$

We consider the linear operator $F_g : C^1_0(\Omega) \longrightarrow \mathbb{R}$ defined by

$$F_g(v) := \int_\Omega v\,g\,\mathrm{d}\mu(x), \qquad v \in C^1_0(\Omega). \tag{4.3.9}$$

If Ω is bounded, from the Sobolev inequality (4.2.10), we get[7]

$$|F_g(v)| \le \|v\|_q\,\|g\|_{q'} \le C(\Omega,Q,p)\,\|v\|_q\,\|g\|_p \overset{(4.2.10)}{\le} C(\Omega,Q,p,g)\,\|Xv\|_{W^1_0},$$

so that, again by density, F_g can be (uniquely) prolonged to an operator

$$F_g : W^1_0(\Omega) \longrightarrow \mathbb{R}.$$

Definition 4.3.6 (W^1-solution for $\mathcal{L}$)**.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let g satisfy (4.3.8).

(a) If Ω is bounded, we say that u is a W^1-*weak solution* of $-\mathcal{L}u = g$ in Ω iff $u \in W^1(\Omega)$ and $L(u,v) = F_g(v)$ for every $v \in W^1_0(\Omega)$.

Clearly we say that a function u is a W^1-*weak subsolution* to $-\mathcal{L}u = g$ in Ω iff $u \in W^1(\Omega)$ and $L(u,v) \le F_g(v)$ for every $v \in W^1_0(\Omega)$, with $v \ge 0$.

(b) For an arbitrary Ω, we say that u is a W^1_{loc}-*weak solution* of $-\mathcal{L}u = g$ in Ω iff $u \in W^1_{\mathrm{loc}}(\Omega)$ and u is a W^1-weak solution of $-\mathcal{L}u = g$ in O, for any bounded open set O such that $\overline{O} \subset \Omega$.

[7] We use Hölder inequality jointly with $q' = 2Q/(Q+2) < Q/2 < p$.

Remark 4.3.7. Let Ω be bounded. Clearly, $-\mathcal{L}u = g$ in the W^1-weak sense if and only if there exists a sequence $u_n \in C^1(\Omega)$ with $u_n \to u$ in W^1 such that, for any sequence $v_n \in C_0^1(\Omega)$ possessing a limit in W_0^1, then it holds that

$$\lim_{n\to\infty} \int_\Omega \Big(\langle A\nabla u_n, \nabla v_n \rangle - g\, v_n \Big)\, \mathrm{d}\mu = 0.$$

4.4 The non-homogeneous, invariant Harnack inequality

The aim of this section is to prove the following result:

Theorem 4.4.1 (Non-homogeneous, invariant Harnack inequality). *Let the assumptions in Section 4.2 be satisfied for $\mathcal{L}$ and for the doubling metric space $(\mathbb{R}^N, d_X, \mu)$. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $g \in L^p(\Omega)$, with $p > Q/2$.*

Then there exists a structural constant $C > 0$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in (4.2.3) *and on p) such that, for every d-ball $B_R(x)$ satisfying $\overline{B_{4R}(x)} \subset \Omega$, one has*

$$\sup_{B_R(x)} u \le C \left(\inf_{B_R(x)} u + R^2 \|g\|^*_{L^p(B_{4R}(x))} \right), \tag{4.4.1}$$

for any nonnegative W^1_{loc}-weak solution u of $-\mathcal{L}u = g$ in Ω.

Remark 4.4.2. In the particular case when $g \equiv 0$, one obtains the homogeneous, invariant Harnack inequalities obtained by Kogoj and Lanconelli in [65, 66] (in [66] the operators involved are more general than ours, in that they may contain first order terms). Again in the homogeneous case $g \equiv 0$, an invariant Harnack inequality under *local* doubling/Poincaré has been proved by Gutiérrez and Lanconelli in [52], for balls of *small* radii. In the same paper [52], the authors obtain a non-homogeneous invariant Harnack inequality, under the presence of some dilation-invariance property on the vector fields X involved.

The summand $R^2 \|g\|^*_{L^p(B_{4R}(x))}$ is bounded by above by

$$\frac{R^2}{\mu(B_R(x))^{1/p}}\, \|g\|_{L^p(\Omega)};$$

when R is small and x lies in a compact set $K \subset \Omega$, there exists a constant $C(Q, K) > 0$ such that (due to the doubling inequality (4.1.14)) the latter does not exceed

$$C(Q, K)\, R^{2-Q/p} \|g\|_{L^p(\Omega)}.$$

Thus, our inequality (4.4.1) contains the analogous non-homogeneous, invariant Harnack inequality by Uguzzoni in [92], where it is considered the particular case when x is confined in some compact set $K \subset \Omega$ and R is very small. Roughly put, these more restrictive assumptions are the drawback of the local doubling/Poincaré assumptions made in [92].

As is expected, the proof of Theorem 4.4.1 is long and laborious, and it is based on the Moser iterative technique. This machinery is by no means new in the PDE literature, so we skip the largest part of the details. Much is based on the following lemma (and on its proof):

Lemma 4.4.3. *Let the assumptions of Theorem 4.4.1 hold. Any (not necessarily nonnegative) W^1_{loc}-weak solution u of $-\mathcal{L}u = g$ in Ω is locally bounded.*

Proof. Let $B := B_{4R}(x)$ and suppose $\overline{B} \subset \Omega$. We set $\overline{u} := u^+ + \sigma$ (with $u^+ = \max\{u, 0\}$) with $\sigma > 0$ to be chosen. If $n \in \mathbb{N}$ and $\alpha \geq 1$ are arbitrary, we consider the function $H = H_n$

$$H : [\sigma, \infty) \to \mathbb{R}, \quad H(s) := s^\alpha \, \chi_{[\sigma,n]}(s) + (\alpha \, n^{\alpha-1}(s-n) + n^\alpha)\, \chi_{(n,\infty)}(s).$$

It is not difficult to see that $(H_n)_n$ is non-decreasing and C^1, and it point-wise converges to s^α. Finally, given a nonnegative cut-off function $\eta \in C^1_0(B)$, we set

$$v := \eta^2\, G(\overline{u}), \quad \text{where } G(t) := \int_\sigma^t (H'(s))^2 \, \mathrm{d}s.$$

One has $v \in W^1_0(B)$. Since u solves $-\mathcal{L}u = g$ in the W^1_{loc}-weak sense, we have $L(u,v) = F_g(v)$. For regular u, v (say $u = u_n$, $v = v_n$ as in Remark 4.3.7), one has

$$\begin{aligned} F_g(v) - L(u,v) &= \int_B g\,\eta^2 G(\overline{u})\,\mathrm{d}\mu - \int_B \langle A\nabla u, \nabla(\eta^2 G(\overline{u}))\rangle \,\mathrm{d}\mu \\ &\overset{(4.2.3)}{\leq} \int_B |g|\,\eta^2 G(\overline{u})\,\mathrm{d}\mu - \lambda \int_{B\cap\{u>0\}} \eta^2 G'(\overline{u})|Xu|^2\,\mathrm{d}\mu + 2\Lambda \int_B \eta\, G(\overline{u})|Xu|\,|X\eta|\,\mathrm{d}\mu. \end{aligned}$$

By a limit argument (recall that $u = u_n, v = v_n$), and by using $G(t) \leq t\, G'(t)$, one gets

$$\int_{B\cap\{u>0\}} \eta^2 G'(\overline{u})|Xu|^2\,\mathrm{d}\mu \leq \frac{1}{\lambda}\int_{B\cap\{u>0\}} \Big(|g|\,\eta^2\,\overline{u}\,G'(\overline{u}) + 2\Lambda\,\eta\,\overline{u}\,G'(\overline{u})\,|Xu|\,|X\eta|\Big)\,\mathrm{d}\mu.$$

We set $a := \sqrt{|g|/(\lambda\,\sigma)}$. By an interpolation argument,[8] and as $|g|/\lambda \leq a^2\,\overline{u}$, we get

$$\begin{aligned} \frac{1}{2}\int_{B\cap\{u>0\}} \eta^2 G'(\overline{u})|Xu|^2\,\mathrm{d}\mu &\leq \int_{B\cap\{u>0\}} \Big(5\,\eta^2\,a^2\overline{u}^2\,G'(\overline{u}) + 16\frac{\Lambda^2}{\lambda^2}\,\overline{u}^2\,G'(\overline{u})\,|X\eta|^2\Big)\,\mathrm{d}\mu \\ &\leq C(\Lambda,\lambda)\int_B (\overline{u}\,H'(\overline{u}))^2(|X\eta|^2 + \eta^2 a^2)\,\mathrm{d}\mu. \end{aligned}$$

As $G'(\overline{u})|Xu|^2\chi_{\{u>0\}} = |X(H(\overline{u}))|^2$ and $sH'(s) \leq \alpha\, H(s)$, this gives

$$\Big\|\eta\,|X(H(\overline{u}))|\Big\|^*_{L^2(B)} \leq C\,\alpha\Big(\Big\|H(\overline{u})\,|X\eta|\Big\|^*_{L^2(B)} + \Big\|H(\overline{u})\,\eta\,a\Big\|^*_{L^2(B)}\Big). \tag{4.4.2}$$

We apply the Sobolev inequality (4.3.5) to $\eta\, H(\overline{u}) \in W^1_0(B)$; thanks to (4.4.2) we easily get

$$\Big\|\eta\, H(\overline{u})\Big\|^*_{L^q(B)} \leq C_1\, R\,(\alpha+1)\Big(\Big\|H(\overline{u})\,|X\eta|\Big\|^*_{L^2(B)} + \Big\|H(\overline{u})\,\eta\,a\Big\|^*_{L^2(B)}\Big). \tag{4.4.3}$$

Via the interpolation $\|w\|^*_s \leq \epsilon\,\|w\|^*_r + \epsilon^{-\nu}\,\|w\|^*_h$ (holding true for $h \leq s \leq r$ and $\nu = \frac{1/h - 1/s}{1/s - 1/r}$), with the choices $s = 2p/(p-1)$, $h = 2$, $r = q$, $w = \eta\, H(\overline{u})$ one gets

$$\|H(\overline{u})\,\eta\,a\|^*_2 \leq \|a\|^*_{2p}\Big(\epsilon\|\eta\, H(\overline{u})\|^*_q + \epsilon^{Q/(Q-2p)}\|\eta\, H(\overline{u})\|^*_2\Big). \tag{4.4.4}$$

[8] We use $AB \leq \frac{1}{8}\,A^{1/2} + 8\,B^{1/2}$.

We choose

$$\epsilon := (2C_1\, a^* \,(1+\alpha))^{-1} \quad \text{where} \quad a^* := R\,\|a\|^*_{L^{2p}(B_{4R}(x))} \quad \text{and} \quad \sigma := R^2\,\|g\|^*_{L^p(B_{4R}(x))}.$$

With the above choice of σ one actually has

$$a^* = R\Big(\fint_{B_{4R}(x)} \frac{|g|^p}{\lambda^p \sigma^p}\Big)^{\frac{1}{2p}} = \frac{R}{\sqrt{\lambda}\sqrt{\sigma}}\Big(\fint_{B_{4R}(x)} |g|^p\Big)^{\frac{1}{2p}} = \frac{1}{\sqrt{\lambda}},$$

so a^* is a structural constant. By inserting (4.4.4) into (4.4.3) one gets

$$\Big\| \eta\, H(\overline{u}) \Big\|^*_{L^q(B)} \le C(1+\alpha)^{1+\nu}\Big(\Big\|\eta\, H(\overline{u})\Big\|^*_{L^2(B)} + R\,\Big\| |X\eta|\, H(\overline{u})\Big\|^*_{L^2(B)}\Big),$$

where C depends on $Q, C_P, \Lambda, \lambda, p$ and where $\nu = Q/(2p-Q)$. Recalling that $H = H_n$, by letting $n\to\infty$ (and by monotone convergence) we infer

$$\Big\| \eta\, \overline{u}^\alpha \Big\|^*_{L^q(B)} \le C(1+\alpha)^{1+\nu}\Big(\Big\|\eta\, \overline{u}^\alpha\Big\|^*_{L^2(B)} + R\,\Big\| |X\eta|\, \overline{u}^\alpha\Big\|^*_{L^2(B)}\Big). \tag{4.4.5}$$

It is legitimate to take as η a cut-off function as in Remark 4.3.5, relative to x, R_1, R_2 with $R \le R_1 < R_2 \le 2R$. From (4.4.5), the doubling condition and the distinguished properties of η, we easily get

$$\|\overline{u}^\alpha\|^*_{L^q(B_{R_1})} \le C(1+\alpha)^{1+\nu}\Big(1 + \frac{R}{R_2 - R_1}\Big)\,\|\overline{u}^\alpha\|^*_{L^2(B_{R_2})}, \qquad \forall\ R \le R_1 < R_2 \le 2R, \tag{4.4.6}$$

where the centre x of the d-balls is understood. Inequality (4.4.6) is the starting point for Moser's iterative technique.

We introduce the function (with $R>0$ and $s \in \mathbb{R}\smallsetminus\{0\}$)

$$\phi(s,R) := \Big(\fint_{B_R(x)} |\overline{u}|^s \,\mathrm{d}\mu\Big)^{1/s}. \tag{4.4.7}$$

Clearly one has

$$\lim_{s\to\infty} \phi(s,R) = \sup_{B_R(x)} \overline{u}, \qquad \lim_{s\to-\infty} \phi(s,R) = \inf_{B_R(x)} \overline{u}.$$

Inequality (4.4.6) becomes

$$\phi(\alpha\, q, R_1) \le \Big(C(1+\alpha)^{1+\nu}\Big(1 + \frac{R}{R_2 - R_1}\Big)\Big)^{1/\alpha} \phi(2\alpha, R_2), \tag{4.4.8}$$

holding true for any $\alpha > 1$ and $R \le R_1 < R_2 \le 2R$. Given $t \in (2,q)$, for any $n\in\mathbb{N}$ we set

$$\alpha_n = t\,(q/2)^n, \quad \rho_n = R(1 + 2^{-n}).$$

We apply (4.4.8) with the triple (α, R_1, R_2) first equal to $(t/2, \rho_1, 2R)$ and then, iteratively, equal to $(\alpha_{n-1}/2, \rho_n, \rho_{n-1})$. One gets (for some $\overline{C} > 1$)

$$\phi(\alpha_n, R) \le (\overline{C}\, q)^{2(1+\nu)\sum_{k=1}^n k(2/q)^{k-1}} \phi(t, 2R), \qquad \forall\ n \ge 2.$$

Since $q > 2$, this gives (letting $n\to\infty$)

$$\sup_{B_R(x)} \overline{u} \le C' \Big(\fint_{B_{2R}(x)} |\overline{u}|^t \,\mathrm{d}\mu\Big)^{1/t}.$$

Letting $t \to 2^+$ (and being $\overline{u} = u^+ + \sigma > u^+$), we infer

$$\sup_{B_R(x)} u^+ \leq C' \, \|\overline{u}\|^*_{L^2(B_{2R}(x))} < \infty,$$

whence $u^+ \in L^\infty(B_R(x))$. Since $-u$ is a $W^1_{\rm loc}$-weak solution of $-\mathcal{L}u = -g$, the same argument gives $u^- \in L^\infty(B_R(x))$, and the proof is complete. □

The next step for the proof of the Harnack inequality is the next lemma, where a gain in summability is established for the $W^1_{\rm loc}$-solution u.

Lemma 4.4.4. *Let the assumptions of Theorem 4.4.1 be satisfied and let u be any nonnegative $W^1_{\rm loc}$-weak solution of $-\mathcal{L}u = g$ on Ω, with $g \in L^p(\Omega)$ (with $p > Q/2$). Suppose also that $\overline{B_{4R}(x)} \subseteq \Omega$. Let us also set (as in the proof of Lemma 4.4.3) that $\overline{u} = u + \sigma$, with $\sigma = R^2 \, \|g\|^*_{L^p(B_{4R}(x))}$.*

The following facts hold true:

(a) For every $s \in (1, q/2)$, there exists a constant $C(s) > 0$ such that

$$\sup_{B_R(x)} \overline{u} \leq C(s) \, \|\overline{u}\|^*_{L^s(B_{2R}(x))}. \tag{4.4.9}$$

(b) For every $p_0 \in (0,1)$, there exists a constant $C(p_0) > 0$ such that

$$\Big(\fint_{B_{3R}(x)} \overline{u}^{-p_0} \, \mathrm{d}\mu \Big)^{-1/p_0} \leq C(p_0) \inf_{B_r(x)} \overline{u}. \tag{4.4.10}$$

(c) For every p_0, s such that $0 < p_0 < 1 < s < q/2$, there exists a constant $C(p_0, s) > 0$ such that

$$\|\overline{u}\|^*_{L^s(B_{2R}(x))} \leq C(p_0, s) \Big(\fint_{B_{3R}(x)} \overline{u}^{p_0} \, \mathrm{d}\mu \Big)^{1/p_0}. \tag{4.4.11}$$

Here the constants $C(s), C(p_0), C(p_0, s)$ depend also on the structural doubling/Poincaré constants Q, C_P, on the ellipticity constants λ, Λ, on p (the summability exponent of g), but are otherwise independent of x, R and u.

Proof. We only give a sketch of the proof, since basically the main technique is the same as in the proof of Lemma 4.4.3. We set $B := B_{4R}(x)$ as in the assertion. Let us consider a nonnegative cut-off function $\eta \in W^1_0(B)$ and any $\alpha \in \mathbb{R} \setminus \{0\}$. Let us set

$$w := \begin{cases} \overline{u}^{\frac{\alpha+1}{2}} & \text{if } \alpha \neq -1, \\ \log \overline{u} & \text{if } \alpha = -1. \end{cases}$$

One can argue as in the proof of Lemma 4.4.3, this time by using in a crucial way the nonnegativity of u in order to define suitable test-functions v of the form $\eta^2 \, \overline{u}^\alpha$ to be implemented in the equality $L(u, v) = F_g(v)$. As a consequence, it is possible to prove that (where $a = \sqrt{|g|/(\lambda\sigma)}$)

$$\text{if } \alpha = -1: \qquad \int_B \eta^2 \, |Xw|^2 \, \mathrm{d}\mu \leq 64 \int_B \Big(\frac{\Lambda^2}{\lambda^2} \, |X\eta|^2 + a^2 \, \eta^2 \Big) \mathrm{d}\mu; \tag{4.4.12}$$

$$\text{if } \alpha \neq -1: \qquad \big\|\eta \, w\big\|^*_{L^q(B)} \leq C \, (1 + |1+\alpha|)^{1+\nu} \Big(\big\|w \, \eta\big\|^*_{L^2(B)} + R \, \big\|w \, |X\eta|\big\|^*_{L^2(B)} \Big). \tag{4.4.13}$$

Here $\nu = Q/(2p-Q)$ and C depends on $Q, C_P, \lambda, \Lambda, p$ and α.

By the aid of a cut-off function η as [66] (see Remark 4.3.5), starting from (4.4.13), we can prove the following fact:

$$\|w\|^*_{L^q(B_{R_1})} \le C\,(1+|1+\alpha|)^{1+\nu}\Big(1+\frac{R}{R_2-R_1}\Big)\|w\|^*_{L^2(B_{R_2})}, \tag{4.4.14}$$

holding true for $\alpha \in \mathbb{R} \smallsetminus \{0,-1\}$, and for $R \le R_1 < R_2 \le 2R$ (the centre x of the d-balls is understood).

The proofs of our three inequalities (4.4.9), (4.4.10), (4.4.11) now follow three different lines, all based on Moser-type iterative techniques. The notation $\phi(s,R)$ as in (4.4.7) is understood.

Proof of (4.4.9)*:* Let $s \in (1, q/2)$ be fixed. If $\alpha > -1$, raising (4.4.14) to the power $\frac{2}{\alpha+1}$, we get

$$\phi\Big((\alpha+1)\frac{q}{2}, R_1\Big) \le \Big(C\,(2+\alpha)^{1+\nu}\Big(1+\frac{R}{R_2-R_1}\Big)\Big)^{\frac{2}{\alpha+1}}\phi(\alpha+1,R_2).$$

A suitable iteration of this inequality yields

$$\phi(\gamma_n,R) \le (3C\,q)^{4(1+\nu)\sum_{k=1}^n k(2/q)^{k-1}}\,\phi(s,2R),$$

where $\gamma_n = s\,(q/2)^n$ (the iteration is also based on the choice of the radii $R_n = R(1+2^{-n})$ and $\alpha+1=\gamma_{n-1}$). Letting $n\to\infty$ one gets (4.4.9).

Proof of (4.4.10)*:* Let $p_0 \in (0,1)$ be fixed. If $\alpha < -1$, raising (4.4.14) to the negative power $2/(\alpha+1)$, we get

$$\phi(\alpha+1,R_2) \le \Big(C\,(1+|1+\alpha|)^{1+\nu}\Big(1+\frac{R}{R_2-R_1}\Big)\Big)^{\frac{2}{|\alpha+1|}}\phi\Big((\alpha+1)\frac{q}{2},R_1\Big).$$

A suitable iteration of this inequality yields (taking into account first the doubling property)

$$\phi(-p_0,3R) \le C(Q,p_0)\,\phi(-p_0,2R) \le (2C\,q)^{\frac{4}{p_0}\,(1+\nu)\sum_{k=1}^n k(2/q)^{k-1}}\,\phi(\gamma_n,R),$$

where $\gamma_n = -p_0\,(q/2)^n$ (the iteration is also based on the choice of the radii $R_n = R(1+2^{-n})$ and $\alpha+1=\gamma_{n-1}$). Letting $n\to\infty$ one gets (4.4.10).

Proof of (4.4.11)*:* Let $0 < p_0 < 1 < s < q/2$ be fixed. A slight modification in the radii appearing in (4.4.14) gives

$$\phi\Big((\alpha+1)\frac{q}{2}, R_1\Big) \le \Big(C\,(2+\alpha)^{1+\nu}\Big(1+\frac{R}{R_2-R_1}\Big)\Big)^{\frac{2}{\alpha+1}}\phi(\alpha+1,R_2),$$

this time with $2R \le R_1 < R_2 \le 3R$ (and $\alpha > -1$). A suitable iteration of this inequality yields

$$\phi(s,2R) \le (3C\,q)^{\frac{4s}{p_0}\,(1+\nu)\sum_{k=1}^n k(2/q)^{k-1}}\,\phi(p_0,3R),$$

which proves (4.4.10), it sufficing to choose the least n such that

$$s \le \Big(\frac{q}{2}\Big)^n\frac{p_0}{s}.$$

The iteration is based on the choices $\gamma_n = (q/2)^n\,p_0/s$, $R_n = R(1+2^{-n})$ and $\alpha+1=\gamma_{n-1}$.

The proof of the lemma is complete. □

The last step in the proof of the Harnack inequality is given by the next result, resting on some John-Nirenberg type estimates.

Lemma 4.4.5. *Let the assumptions and notations in Lemma 4.4.4 hold. Then there exists $p_0 \in (0,1)$ and a constant $C'(p_0) > 0$ such that*

$$\left(\fint_{B_{3R}(x)} \overline{u}^{p_0} \,\mathrm{d}\mu \right)^{1/p_0} \leq C'(p_0) \left(\fint_{B_{3R}(x)} \overline{u}^{-p_0} \,\mathrm{d}\mu \right)^{-1/p_0}. \tag{4.4.15}$$

Here C has the same parameter-dependence as in Lemma 4.4.4.

Proof. Let $B(z,2\rho) \subseteq B(x,4R)$. We now consider (4.4.12) in the proof of Lemma 4.4.4 (where $w = \log \overline{u}$), and we choose a suitable cut-off function as in [66] (see Remark 4.3.5): indeed, we can take a nonnegative $\eta \in W_0^1(B_{2\rho}(z))$ such that $\eta \equiv 1$ on $B_\rho(z)$, $\eta \equiv 0$ outside $B_{2\rho}(z)$ and $|X\eta| \leq 2/\rho$ in $B_{2\rho}(z)$. Simple estimates based on (4.4.12) and on the properties of η give

$$\fint_{B_\rho(z)} |Xw|^2 \,\mathrm{d}\mu \leq C' \Big(\frac{1}{\rho^2} + \fint_{B_{2\rho}(z)} a^2 \,\mathrm{d}\mu \Big), \tag{4.4.16}$$

where as usual $a := \sqrt{|g|/(\lambda\,\sigma)}$ and C' is a constant as in the assertion of Theorem 4.4.1. Since $p > 1$, the choice $\sigma = R^2 \, \|g\|^*_{L^p(B_{4R}(x))}$ yields

$$\fint_{B_{2\rho}(z)} a^2 \,\mathrm{d}\mu \leq \frac{1}{\lambda\, R^2} \frac{\|g\|^*_{L^p(B_{2\rho}(x))}}{\|g\|^*_{L^p(B_{4R}(x))}}.$$

By inserting this in (4.4.16) and by doubling we get

$$\fint_{B_\rho(z)} |Xw|^2 \,\mathrm{d}\mu \leq C' \left(\frac{1}{\rho^2} + \frac{C(Q,p)}{\lambda\, R^2} \Big(\frac{R}{\rho}\Big)^{Q/p} \right) \leq C'' \left(\frac{1}{\rho^2} + \frac{1}{\lambda\, R^2} \Big(\frac{R}{\rho}\Big)^{Q/p} \right).$$

From $p > Q/2$ and $\rho \leq 4R$ we get $R^{Q/p-2}/\rho^{Q/p} \leq 4^{2-Q/p}/\rho^2$; we have therefore obtained

$$\fint_{B_\rho(z)} |Xw|^2 \,\mathrm{d}\mu \leq \frac{2C''}{\rho^2}, \qquad \text{whenever } B(z,2\rho) \subseteq B(x,4R). \tag{4.4.17}$$

From the Poincaré inequality (4.3.6) for $w \in W_0^1(B_{4R}(x))$, we infer from (4.4.17) that

$$\fint_{B_\rho(z)} \left| w - w_{B_\rho(z)} \right| \mathrm{d}\mu \leq C_P\, \rho \, \|Xw\|^*_{L^1(B_{2\rho}(z))} \leq C_P\, \rho \, \|Xw\|^*_{L^2(B_{2\rho}(z))} \overset{(4.4.17)}{\leq} \widetilde{C}.$$

Summing up

$$\fint_{B_\rho(z)} \left| w - w_{B_\rho(z)} \right| \mathrm{d}\mu \leq \widetilde{C}, \qquad \text{whenever } B(z,4\rho) \subseteq B(x,4R). \tag{4.4.18}$$

Now, due to our global doubling and Poincaré assumptions, we are entitled to apply Theorems 0.3 and 0.4 in the paper by Bukley [19]; the latter results allow us to infer from (4.4.18) the following John-Nirenberg type estimate: there exists $p_0 \in (0,1)$ such that

$$\fint_{B_{3R}(x)} \exp\left(p_0 \left| w - w_{B_{3R}(x)} \right| \right) \mathrm{d}\mu \leq C, \tag{4.4.19}$$

with the usual dependence of C on the structural parameters. Let us drop the notation of the centre x in the d-balls. Recalling that $w = \log \overline{u}$ we have

$$\begin{aligned}\fint_{B_{3R}} \overline{u}^{-p_0}\,\mathrm{d}\mu \cdot \fint_{B_{3R}} \overline{u}^{p_0}\,\mathrm{d}\mu &= \fint_{B_{3R}} \exp\Big(p_0(-w + w_{B_{3R}})\Big)\,\mathrm{d}\mu \cdot \fint_{B_{3R}} \exp\Big(p_0(w - w_{B_{3R}})\Big)\,\mathrm{d}\mu \\ &\le \bigg(\fint_{B_{3R}} \exp\Big(p_0|w - w_{B_{3R}}|\Big)\,\mathrm{d}\mu \bigg)^2 \le C^2.\end{aligned}$$

By raising to the power $1/p_0$ we get (4.4.15). □

Once Lemmas 4.4.4 and 4.4.5 are established, the proof of the Harnack inequality is straightforward.

Proof (of Theorem 4.4.1). Let the assumptions and notations in Theorem 4.4.1 hold. Let $p_0 \in (0,1)$ be as in Lemma 4.4.5. Since $q > 2$, we can fix any $s \in (1, q/2)$. We have the following chain of inequalities:

$$\begin{aligned}\sup_{B_r(x)} \overline{u} &\overset{(4.4.9)}{\le} C(s)\, \|\overline{u}\|^*_{L^s(B_{2R}(x))} \\ &\overset{(4.4.11)}{\le} C(s)\, C(p_0, s) \Big(\fint_{B_{3R}(x)} \overline{u}^{p_0}\,\mathrm{d}\mu \Big)^{1/p_0} \\ &\overset{(4.4.15)}{\le} C(s)\, C(p_0, s)\, C'(p_0) \Big(\fint_{B_{3R}(x)} \overline{u}^{-p_0}\,\mathrm{d}\mu \Big)^{-1/p_0} \\ &\overset{(4.4.10)}{\le} C(s)\, C(p_0, s)\, C'(p_0)\, C(p_0) \inf_{B_r(x)} \overline{u}.\end{aligned}$$

Since $\overline{u} = u + \sigma = u + R^2 \|g\|^*_{L^p(B_{4R}(x))}$, the far right-hand side of the above chain of inequalities is the right-hand side of (4.4.1); moreover, as $u \le \overline{u}$, the far left-hand side in the above inequalities is no less than the left-hand side of (4.4.1): the proof of the Harnack inequality (4.4.1) is compelete. □

4.4.1 Applications: Inner and boundary Hölder estimates

Our aim is to prove inner and boundary Hölder estimates, using the non-homogeneous invariant Harnack inequality proved in Section 4.4 (Theorem 4.4.1). We will follow the arguments in [49, Chapter 8].

In the sequel we require that $\mathcal{L}$ satisfies the assumptions in Section 4.2.
A first result is the following estimate.

Theorem 4.4.6. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $g \in L^p(\Omega)$, with $p \ge Q/2$.*

Then there exist structural constants $C > 0$ and $0 < \alpha < 1$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in (4.2.3) *and on p) such that, for every d-ball $B_R(x_0)$ satisfying $\overline{B_R(x_0)} \subset \Omega$, one has*

$$\mathrm{osc}_{B_r(x_0)} u \le C r^\alpha \left(R^{-\alpha} \sup_{B_R(x_0)} |u| + R^{2-\alpha} \|g\|^*_{L^p(B_R(x_0))} \right) \qquad \forall\, r \in [0, R], \tag{4.4.20}$$

for any W^1_{loc}-weak solution u of $-\mathcal{L}u = g$ in Ω.

In order to prove the previous theorem, we need to give the following result.

Lemma 4.4.7. *Let* $\omega :]0,R] \to \mathbb{R}$ *be a non-decreasing function, and let* $\sigma :]0,R] \to \mathbb{R}$ *be a function such that there exists* $\bar{c} > 0$ *for which* $\sigma(r_1) \le \bar{c}\sigma(r_2)$ *for any* $r_1, r_2 \in]0,R]$*, with* $r_1 \le r_2$*. Suppose that there exist* $\gamma, \tau \in]0,1[$ *satisfying the following condition:*

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r) \qquad \forall\, r \le R, \tag{4.4.21}$$

then there exists $C_0 := C_0(\gamma,\tau) > 0$ *such that, for every* $\nu \in]0,1[$*, one has:*

$$\omega(r) \le C_0\left(\left(\frac{r}{R}\right)^{\alpha}\omega(R) + \sigma\left(r^{\nu}R^{1-\nu}\right)\right) \qquad \forall\, r \le R, \tag{4.4.22}$$

where $\alpha := (1-\nu)\frac{\log\gamma}{\log(1/4)} \in]0,1[$.

The proof of this last result is an adaptation of the arguments in [49, Lemma 8.23].

Proof (of Theorem 4.4.6). Fix $B(x_0,R) \subset \overline{B(x_0,R)} \subset \Omega$ and let u be a W^1_{loc}-weak solution of $-\mathcal{L}u = g$ in Ω. We consider $\rho \le R/4$ and we put

$$M_0 := \sup_{B_R(x_0)} |u|,\; M_1 := \sup_{B_\rho(x_0)} u,\; m_1 := \inf_{B_\rho(x_0)} u,\; M_4 := \sup_{B_{4\rho}(x_0)} u \text{ and } m_4 := \inf_{B_{4\rho}(x_0)} u.$$

By Lemma 4.4.3 we know that $u \in L^\infty_{\mathrm{loc}}(\Omega)$, then $M_0, M_i, m_i \in \mathbb{R}$, for $i = 1,4$.

We have:

$$\mathcal{L}(M_4 - u) = -\mathcal{L}u = g \;\text{ (in } W^1_{\mathrm{loc}}\text{-weak sense)}$$
$$\mathcal{L}(u - m_4) = \mathcal{L}u = -g \;\text{ (in } W^1_{\mathrm{loc}}\text{-weak sense)},$$

then we can apply the non-homogeneous invariant Harnack inequality in Theorem 4.4.1 to the functions $M_4 - u,\ u - m_4 \in W^1_{\mathrm{loc}}(\Omega, X)$, which are non-negative functions in $B(x_0, 4\rho)$. Hence, there exists a constant $C > 0$ such that

$$\sup_{B_\rho(x_0)} (M_4 - u) \le C\left(\inf_{B_\rho(x_0)} (M_4 - u) + \rho^2 \|g\|^*_{L^p(B_{4\rho}(x_0))}\right)$$
$$\sup_{B_\rho(x_0)} (u - m_4) \le C\left(\inf_{B_\rho(x_0)} (u - m_4) + \rho^2 \|g\|^*_{L^p(B_{4\rho}(x_0))}\right),$$

that together give

$$M_4 - m_4 \le C\left(M_4 - m_4 - (M_1 - m_1) + 2\rho^2 \|g\|^*_{L^p(B_{4\rho}(x_0))}\right);$$
$$M_1 - m_1 \le \left(1 - \frac{1}{C}\right)(M_4 - m_4) + 2\rho^2 \|g\|^*_{L^p(B_{4\rho}(x_0))}. \tag{4.4.23}$$

If we put

$$\omega(\rho) := \mathrm{osc}_{B_\rho(x_0)} u \;\text{ and }\; k(\rho) := 2\rho^2 \|g\|^*_{L^p(B_{4\rho}(x_0))},$$

we can rewrite (4.4.23) in the following way

$$\omega(\rho) \le \left(1 - \frac{1}{C}\right)\omega(4\rho) + k(\rho), \qquad \forall\, \rho \le \frac{R}{4},$$

or equivalently

$$\omega\left(\frac{1}{4}r\right) \le \left(1-\frac{1}{C}\right)\omega(r) + k\left(\frac{1}{4}r\right), \qquad \forall\, r \le R.$$

Therefore we can apply Lemma 4.4.7, with $\gamma = (1-1/C)$, $\tau = 1/4$ and $\sigma(r) = k(r/4)$,[9] and we obtain that there exists $C_0 > 0$ such that, for every $\nu \in]0,1[$, one has

$$\omega(r) \le C_0\left(\left(\frac{r}{R}\right)^{\alpha}\omega(R) + \sigma\left(r^{\nu}R^{1-\nu}\right)\right) \qquad \forall\, r \le R, \tag{4.4.24}$$

with $\alpha = (1-\nu)\frac{\log\gamma}{\log(1/4)} \in]0,1[$.

We put $\delta := 1 - \frac{Q}{2p} > 0$, and we choose $\nu \in]0,1[$ such that $\alpha < \nu\delta$. Hence, for every $r \le R$, by (4.4.24) we get

$$\begin{aligned}
\omega(r) &\overset{(4.4.24)}{\le} C_0\left(r^{\alpha}R^{-\alpha}\omega(R) + \frac{1}{8}\left(r^{\nu}R^{1-\nu}\right)^2 \|g\|^*_{L^p(B_{r^{\nu}R^{1-\nu}}(x_0))}\right) = \\
&= C_0 r^{\alpha}\left(R^{-\alpha}\omega(R) + \frac{1}{8}r^{-\alpha+2\nu}R^{2-2\nu}\|g\|^*_{L^p(B_{r^{\nu}R^{1-\nu}}(x_0))}\right) \le \\
&\overset{(D)}{\le} C_0 r^{\alpha}\left(R^{-\alpha}\omega(R) + \frac{1}{8}r^{-\alpha+2\nu}R^{2-2\nu}\bar{C}R^{(Q\nu)/p}r^{-(Q\nu)/p}\|g\|^*_{L^p(B_R(x_0))}\right) \le \\
&\le C r^{\alpha}\left(R^{-\alpha}\omega(R) + r^{2\delta\nu-\alpha}R^{2-2\delta\nu}\|g\|^*_{L^p(B_R(x_0))}\right) \le \\
&\le C r^{\alpha}\left(R^{-\alpha}\omega(R) + R^{2-\alpha}\|g\|^*_{L^p(B_R(x_0))}\right),
\end{aligned}$$

which gives (4.4.20). This completes the proof.

□

An immediate consequence of Theorem 4.4.6 is the following result.

Corollary 4.4.8. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $g \in L^p(\Omega)$, with $p \ge Q/2$.*

Then there exist structural constants $C > 0$ and $0 < \alpha < 1$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in (4.2.3) *and on p) such that, for every d-ball $B_R(x_0)$ satisfying $\overline{B_{3R}(x_0)} \subset \Omega$, one has*

$$\sup_{x,y\in B_R(x_0),\, x\ne y}\frac{|u(x)-u(y)|}{d(x,y)^{\alpha}} \le C\left(R^{-\alpha}\sup_{B_{3R}(x_0)}|u| + R^{2-\alpha}\|g\|^*_{L^p(B_{3R}(x_0))}\right), \tag{4.4.25}$$

for any W^1_{loc}-weak solution u of $-\mathcal{L}u = g$ in Ω.

In the sequel we want to prove local estimates at the boundary of a bounded open set of $\mathbb{R}^N$. To this aim we want to recall the following notions.

[9] It is easy to prove that there exists $\bar{c} := \bar{c}(Q) > 0$ such that $k(r_1) \le \bar{c}k(r_2)$, for any $r_1 \le r_2 \le R$. Indeed, we use condition (D) to obtain the following inequalities:

$$\begin{aligned}
k(r_1) = 2r_1^2\|g\|^*_{L^p(B_{4r_1}(x_0))} &\le c(Q,p)r_1^2\left(\frac{r_2}{r_1}\right)^{Q/p}\|g\|^*_{L^p(B_{4r_2}(x_0))} \le \\
&\le c(Q)r_2^2\|g\|^*_{L^p(B_{4r_2}(x_0))} = \bar{c}k(r_2),
\end{aligned}$$

where in the last inequality we have used $p \ge Q/2$ and $r_1 \le r_2$.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, and let $l \in \mathbb{R}$. If $u \in W^1(\Omega, X)$, we say that $u \leq l$ on $\partial\Omega$ iff $(u-l)^+ \in W_0^1(\Omega, X)$; thus we define

$$\sup_{\partial\Omega} u := \inf\{l \in \mathbb{R} \,:\, u \leq l \text{ on } \partial\Omega\} \tag{4.4.26}$$

$$\inf_{\partial\Omega} u := \sup\{l \in \mathbb{R} \,:\, l \leq u \text{ on } \partial\Omega\}. \tag{4.4.27}$$

Finally, we say that $u = 0$ on $\partial\Omega$ iff $u \leq 0$ and $u \geq 0$ on $\partial\Omega$.

Proposition 4.4.9. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $g \in L^p(\Omega)$, with $p \geq Q/2$.*

Then there exist structural constants $C > 0$ and $p_0 \in]0,1[$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in (4.2.3) *and on p) such that, for every d-ball $B(x_0, R)$ satisfying $\Omega \cap B(x_0, 4R) \neq \varnothing$, one has*

$$\|u_m^-\|^*_{L^{p_0}(B_{3R}(x_0))} \leq C \left(\inf_{B_R(x_0)} u_m^- + R^2 \, \|\tilde{g}\|^*_{L^p(B_{4R}(x_0))} \right), \tag{4.4.28}$$

for any non-negative W^1-weak solution u of $-\mathcal{L}u = g$ in $\Omega \cap B(x_0, 4R)$, where $\tilde{g}$ is the trivial extension of g on $\mathbb{R}^N$, and

$$u_m^-(x) = \begin{cases} \inf\{u(x), m\} & \text{if } x \in \Omega \\ m & \text{if } x \notin \Omega, \end{cases}$$

with $m := \inf_{\partial\Omega \cap B_{4R}(x_0)} u$.

An analogous result is the following.

Proposition 4.4.10. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $g \in L^p(\Omega)$, with $p \geq Q/2$.*

Then there exists a structural constant $C > 0$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in (4.2.3) *and on p) such that, for every d-ball $B_R(x_0)$ one has*

$$\sup_{B_R(x_0)} u_M^+ \leq C \left(\|u_M^+\|^*_{L^s(B_{2R}(x_0))} + R^2 \, \|\tilde{g}\|^*_{L^p(B_{4R}(x_0))} \right) \quad \forall\, s \in \left]1, \frac{q}{2}\right[\,, \tag{4.4.29}$$

for any W^1-weak subsolution u of $-\mathcal{L}u = g$ in Ω, where $\tilde{g}$ is the trivial extension of g on $\mathbb{R}^N$ and

$$u_M^+(x) := \begin{cases} \sup\{u(x), M\} & \text{if } x \in \Omega \\ M & \text{if } x \notin \Omega, \end{cases}$$

with $M := \sup_{\partial\Omega \cap B_{2R}(x_0)} u^+$.

Similar arguments seen in Theorem 4.4.1 have been used to prove Proposition 4.4.9 and Proposition 4.4.10 (see also [49, Chapter 8]), so we don't provide the proofs of the previous results.

An immediate consequence of Proposition 4.4.10 is the following.

Corollary 4.4.11. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, and let $g \in L^p(\Omega)$, with $p \geq Q/2$.*

Fix a d-ball $B_R(x_0)$ with $x_0 \in \partial\Omega$ and let $u \in W_0^1(\Omega \cap B_{4R}(x_0), X)$ be a W^1-weak solution of $-\mathcal{L}u = g$ in $\Omega \cap B_{2R}(x_0)$, then $u \in L^\infty(\Omega \cap B_R(x_0))$.

Finally, as a consequence of Proposition 4.4.9, we can prove a local estimate at the boundary for W^1-weak solution; in this case we need to suppose a suitable condition on the boundary. The proof of the following result is an adaptation of the ideas in [49, Theorem 8.27].

Theorem 4.4.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let $g \in L^p(\Omega)$, with $p \geq Q/2$.*

Let $x_0 \in \partial\Omega$ and suppose that there exist $R_0 > 0$ and $\vartheta \in]0,1[$ such that:

$$\mu\left(B(x_0, r) \setminus \Omega\right) \geq \vartheta \mu\left(B(x_0, r)\right) \qquad \forall\, r \in]0, R_0[. \tag{4.4.30}$$

Then there exist structural constants $C > 0$ and $0 < \alpha < 1$ (only depending on the doubling/Poincaré constants Q, C_P, on the X-ellipticity constants λ, Λ in (4.2.3), on ϑ and on p) such that, for every d-ball $B_R(x_0)$ one has

$$\mathrm{osc}_{\Omega\cap B_\rho(x_0)} u \leq C\rho^\alpha \left(\tilde{R}^{-\alpha} \sup_{\Omega \cap B_{\tilde{R}}(x_0)} |u| + \tilde{R}^{2-\alpha} \|\tilde{g}\|^*_{L^p(\Omega\cap B_{\tilde{R}}(x_0))} \right) \quad \forall\, \rho \in]0, \tilde{R}[, \tag{4.4.31}$$

for any W^1-weak solution u of $-\mathcal{L}u = g$ in $\Omega \cap B_{2R}(x_0)$, with $u \in W_0^1(\Omega \cap B_{4R}(x_0), X)$, where $\tilde{R} := \min\{R_0, R\}$.

A direct application of the assumption (P) is the following result.

Lemma 4.4.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $x_0 \in \partial\Omega$ and $R > 0$. We suppose that there exist $R_0 > 0$ and $\vartheta \in]0,1[$ such that, for every $y_0 \in \partial\Omega \cap B_{2R}(x_0)$ one has*

$$\mu\left(B(y_0, r) \setminus \Omega\right) \geq \vartheta \mu\left(B(y_0, r)\right) \qquad \forall\, r \in]0, R_0]. \tag{4.4.32}$$

If $u \in W_0^1(\Omega \cap B_{4R}(x_0), X) \cap C\left(\overline{\Omega \cap B_R(x_0)}\right)$ then $u(x) = 0$ for every $x \in \partial\Omega \cap B_R(x_0)$.

Proof. Assume by contradiction that there exists $y_0 \in \partial\Omega \cap B_R(x_0)$ such that $u(y_0) \neq 0$, suppose $u(y_0) > 0$ to fix ideas. Since $u \in C(\overline{\Omega \cap B_R(x_0)})$, there exist $r, \delta > 0$ such that $u(y) \geq \delta$ for every $y \in \overline{\Omega \cap B_r(y_0)}$. We put

$$w(x) := \begin{cases} \min\{u(x), \delta\} & \text{if } x \in \Omega \cap B_{4R}(x_0) \\ 0 & \text{if } x \notin \Omega \cap B_{4R}(x_0). \end{cases}$$

We observe that $w \in W^1(\mathbb{R}^N, X)$, moreover we have

$$w(x) = \begin{cases} \delta & \text{if } x \in \Omega \cap B_r(y_0) \\ 0 & \text{if } x \in B_r(y_0) \setminus \Omega, \end{cases}$$

then $|Xw| = 0$ a. e. in $B_r(y_0)$. Hence, if we consider $\rho < r/2, R_0$, by Poincaré inequality in Theorem 4.3.4 we get

$$0 \leq \fint_{B_\rho(y_0)} |w - w_{B_rho}|\, \mathrm{d}\mu \leq C_P \rho \fint_{B_{2\rho}(y_0)} |Xw|\, \mathrm{d}\mu = 0,$$

which gives $\left|w - w_{B_\rho}\right| = 0$ a. e. in $B_\rho(y_0) \subset B_r(y_0)$. Observe that

$$w_{B_\rho} = \frac{1}{\mu(B_\rho(y_0))} \int_{\Omega \cap B_\rho(y_0)} w \, \mathrm{d}\mu = \delta \, \frac{\mu(\Omega \cap B_\rho(y_0))}{\mu(B_\rho(y_0))} > 0,$$

then $w = w_{B_\rho} > 0$ a. e. in $B_\rho(y_0)$, but this is a contradiction thanks to (4.4.32). This completes the proof. □

Finally, we prove the last result of this section.

Corollary 4.4.14. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let $g \in L^p(\mathbb{R}^N)$, with $p \geq Q/2$.*

Fix $R > 0$ and $x_0 \in \partial\Omega$, and we suppose that there exist $R_0 > 0$ and $\vartheta \in]0,1[$ such that, for every $y_0 \in \partial\Omega \cap B_{2R}(x_0)$, one has

$$\mu\left(B(y_0, r) \setminus \Omega\right) \geq \vartheta \mu\left(B(y_0, r)\right) \qquad \forall\, r \in]0, R_0]. \tag{4.4.33}$$

If $u \in W_0^1(\Omega \cap B_{4R}(x_0), X)$ is a W^1-weak solution of $-\mathcal{L}u = g$ in $\Omega \cap B_{2R}(x_0)$, then $u \in C(\overline{\Omega \cap B_R(x_0)})$ and $u(x) = 0$ for every $x \in \partial\Omega \cap B_R(x_0)$.

Proof. Fix $\bar{x} \in \Omega \cap \overline{B_R(x_0)}$; we want to prove that u is continuous in $\bar{x}$. We put $\Omega' := \Omega \cap B_{2R}(x_0)$, and we observe that u is a W^1_{loc}-weak solution of $-\mathcal{L}u = g$ in Ω'; hence, we can apply Corollary 4.4.8 and we get that u is continuous in $\bar{x} \in \Omega'$. Therefore we have showed that u is continuous in $\Omega \cap \overline{B_R(x_0)}$.

Finally, we want to prove that u is continuous in $\partial\Omega \cap \overline{B_R(x_0)}$.

By Theorem 4.4.12 we get that there exists $\lim_{x \to x_0} u(x)$ and it is finite; thus we put

$$u(x_0) := \lim_{x \to x_0} u(x).$$

Let us fix $y_0 \in \partial\Omega \cap \overline{B_R(x_0)}$; by Theorem 4.4.12 we still get that there exists $\lim_{x \to y_0} u(x)$ and it is finite, thus in the same way we put

$$u(y_0) := \lim_{x \to y_0} u(x).$$

Then, it is easy to prove that the function u is continuous in every $y_0 \in \partial\Omega \cap \overline{B_R(x_0)}$.
Therefore, we obtain that $u \in C(\overline{\Omega \cap B_R(x_0)})$, and thanks Lemma 4.4.13 we have $u(x) = 0$ for every $x \in \partial\Omega \cap B_R(x_0)$. This completes the proof. □

Chapter 5

The Green function for some subelliptic operators

In this chapter our aim is to give our most recent results related to subelliptic operators. In particular, using the non-homogeneous invariant Harnack inequality proved in Chapter 4, we can construct the Green function related to our operator on any bounded domain satisfying a suitable condition on the boundary.

Finally, the main goal of our future investigation is to prove the existence of a continuous non-negative global fundamental solution for $\mathcal{L}$. To this aim, we need to construct a suitable basis for the d-topology on $\mathbb{R}^N$; here, we want to give a sketch of the proof and an idea of the arguments that we will use to show the existence of a global fundamental solution.

We consider the real second-order PDO $\mathcal{L}$ seen in Chapter 4,

$$\mathcal{L} = \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(V(x) a_{i,j}(x) \frac{\partial}{\partial x_j} \right), \quad x \in \mathbb{R}^N, \tag{5.0.1}$$

and we suppose the same assumptions that we have used to prove the non-homogeneous invariant Harnack inequality and consequent Hölder-continuous estimates.

5.1 The Green function on bounded domains

In order to prove the existence of a global fundamental solution for $\mathcal{L}$ in (5.0.1), we have to deal with the study of the Green function related to $\mathcal{L}$.

We shall consider a fixed bounded open set $\Omega \subset \mathbb{R}^N$, satisfying the following condition uniformly at any point of the boundary $x_0 \in \partial\Omega$: there exist $\rho, \vartheta > 0$ (not depending on x_0) such that

$$\mu(B(x_0,r)\setminus\Omega) \ge \vartheta\mu(B(x_0,r)) \qquad \text{for every } r \in]0,\rho]. \tag{5.1.1}$$

Let us fix $p > \frac{Q}{2}$ and $2 \le p < \infty$.

Now we can construct the Green operator related to $\mathcal{L}$.

Theorem 5.1.1 (Green Operator)**.** *For every $h \in L^p(\Omega)$ there exists a unique W_0^1-weak solution $u := G(h)$ to $-\mathcal{L}u = h$ in Ω. Moreover $u \in C(\overline{\Omega})$ and $u \equiv 0$ on $\partial\Omega$.*

Therefore $G : L^p(\Omega) \to C(\overline{\Omega})$ defines a bounded linear operator, so that its adjoint $G^* : \mathcal{M}(\overline{\Omega}) \to L^{p'}(\Omega)$ is a bounded linear operator, where $\mathcal{M}(\overline{\Omega})$ is the set of the finite real Borel measures supported in $\overline{\Omega}$ and p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Furthermore we have

$$G(h) \ge 0 \qquad \text{for any } h \in L^p(\Omega) \text{ with } h \ge 0; \tag{5.1.2}$$

$$G^*(\nu) \ge 0 \qquad \text{for any } \nu \in \mathcal{M}(\overline{\Omega}) \text{ with } \nu \ge 0; \tag{5.1.3}$$

$$G(h) = G^*(h) \qquad \text{for any } h \in L^p(\Omega). \tag{5.1.4}$$

We call G the Green operator *related to $\mathcal{L}$ and Ω.*

Proof. We know that $W_0^1(\Omega, X)$ is a Hilbert space; moreover, it is easy to see that $L(\cdot,\cdot)$ is a coercive symmetric continuous bilinear form on $W_0^1(\Omega, X)$ and F_h is a linear continuous functional on $W_0^1(\Omega, X)$, then we can apply the Lax-Milgram Theorem. Hence, there exists a unique W_0^1-weak solution $G(h)$ to $-\mathcal{L}u = h$ in Ω.

We want to prove that the function $u := G(h)$ is continuous up to the boundary of Ω and vanishes on $\partial\Omega$.

Let us fix $x_0 \in \Omega$, then there exists $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$; by Corollary 4.4.8 we know that u is continuous on $B(x_0, r/3)$, in particular u is a continuous function in x_0. Hence, $u \in C(\Omega)$ thanks the arbitrariness of x_0.

Fix $x_0 \in \partial\Omega$ and $r > 0$. Let η be a cut-off function such that $\eta \in C_0^1(B(x_0, 4r))$ and

(i) $\eta \equiv 1$ on $B(x_0, 2r)$;

(ii) $0 \le \eta \le 1$ on $B(x_0, 4r)$.

We put $\psi = \eta u$; it is clear that $\psi \in W_0^1(\Omega \cap B(x_0, 4r), X)$, since $u \in W_0^1(\Omega, X)$ and $\eta \in C_0^1(B(x_0, 4r))$. Moreover, thanks the construction of η, we have that $\psi \equiv u$ on $\Omega \cap B(x_0, 2r)$; then $-\mathcal{L}\psi = h$ in Ω (in the weak sense of W^1) and Ω satisfies condition (5.1.1) uniformly at any point of the boundary $\partial\Omega$, so we can apply Corollary 4.4.14 and we have $\psi \in C(\overline{\Omega \cap B(x_0,r)})$ and $\psi(x) = 0$ for any $x \in \partial\Omega \cap B(x_0, r)$. Therefore, thanks the arbitrariness of x_0, we get that $u \in C(\overline{\Omega})$ and $u(x) = 0$ for any $x \in \partial\Omega$.

We have constructed a linear operator $G : L^p(\Omega) \to C(\overline{\Omega})$ which is also bounded. Indeed, for any $h \in L^p(\Omega)$, we know that $u := G(h)$ is a W_0^1-weak solution to $-\mathcal{L}u = h$ in Ω, then we can apply the Maximum Principle in [52, Theorem 3.1], that is there exists a constant $C > 0$ (not depending on h) such that

$$\sup_{\Omega} u^+ \le \sup_{\partial\Omega} u^+ + C \|h\|_{L^p(\Omega)} .$$

Moreover, $u^+ = 0$ on $\partial\Omega$, since $u^+ \in W_0^1(\Omega, X)$; hence

$$\sup_{\Omega} u^+ \le C \|h\|_{L^p(\Omega)} ,$$

and we get that there exists $M > 0$ such that

$$\|G(h)\|_{L^\infty(\Omega)} \le M \|h\|_{L^p(\Omega)} , \quad \text{for any } h \in L^p(\Omega).$$

Therefore, the operator $G : L^p(\Omega) \to C(\overline{\Omega})$ is a bounded linear operator between Banach spaces and we can consider its adjoint $G^* : \mathcal{M}(\overline{\Omega}) \to L^{p'}(\Omega)$, a bounded linear operator satisfying the following relation:

$$\langle G^*(\nu), h\rangle = \langle \nu, G(h)\rangle , \quad \text{for any } h \in L^p(\Omega) \text{ and } \nu \in \mathcal{M}(\overline{\Omega}). \tag{5.1.5}$$

Let's start by proving (5.1.2).

We fix $h \in L^p(\Omega)$, such that $h \ge 0$, and we put $w := -G(h)$ then we have:

$$\begin{aligned} L(w, v) &= \int_\Omega \langle A(x)\nabla w(x), \nabla v(x)\rangle \,\mathrm{d}\mu(x) = -\int_\Omega \langle A(x)\nabla G(h)(x), \nabla v(x)\rangle \,\mathrm{d}\mu(x) = \\ &= \int_\Omega -h(x)v(x)\mathrm{d}\mu(x) \le 0, \end{aligned}$$

for any $v \in W_0^1(\Omega, X)$, with $v \ge 0$. Hence $w \in W_0^1(\Omega, X)$ is a W^1-weak subsolution of $\mathcal{L}w = 0$ in Ω, and thanks the Maximum Principle [52, Theorem 3.1] we get

$$\sup_{\Omega} w^+ \le \sup_{\partial\Omega} w^+.$$

On the other hand $w^+ = 0$ on $\partial\Omega$, then $w^+ = 0$ on Ω or equivalently, $G(h) \ge 0$ on Ω.

Now, we want to prove (5.1.3). Fix $\nu \in \mathcal{M}^+(\overline{\Omega})$ and let h be a non-negative function such that $h \in L^p(\Omega)$, by condition (5.1.5) and (5.1.2) we get:

$$\int_\Omega h(x)G^*(\nu)\mathrm{d}\mu(x) = \langle G^*(\nu), h\rangle = \langle \nu, G(h)\rangle = \int_\Omega G(h)(x)\mathrm{d}\nu(x) \ge 0,$$

then we have showed that $G^*(\nu) \ge 0$.

Finally, we prove (5.1.4). We fix $h \in L^p(\Omega)$, by condition (5.1.5), construction of G and symmetry of $L(\cdot,\cdot)$, for any $\varphi \in L^p(\Omega)$ we have:

$$\begin{aligned} \int_\Omega \varphi(x)G^*(h)(x)\mathrm{d}\mu(x) &= \langle G^*(h), \varphi\rangle = \langle h, G(\varphi)\rangle = \int_\Omega h(x)G(\varphi)(x)\mathrm{d}\mu(x) = \\ &= L\left(G(h), G(\varphi)\right) = L\left(G(\varphi), G(h)\right) = \int_\Omega \varphi(x)G(h)\mathrm{d}\mu(x), \end{aligned}$$

hence for any $h \in L^p(\Omega)$, we have $G(h) = G^*(h)$. □

Definition 5.1.2 (Green function)**.** For every $y \in \Omega$ we define

$$g_y := G^*(\delta_y), \tag{5.1.6}$$

where δ_y denotes the Dirac measure supported at y.

We call $g(x,y) := g_y(x)$ the *Green function* of Ω.

Theorem 5.1.3. *For every $y \in \Omega$ and every small $r > 0$, $g_y \in W^1(\Omega \smallsetminus \overline{B(y,r)}, X)$ and it is a non-negative W^1-weak solution to $\mathcal{L} g_y = 0$ in $\Omega \smallsetminus \overline{B(y,r)}$. Moreover, $g_y \in C(\overline{\Omega} \smallsetminus \{y\})$ and $g_y \equiv 0$ on $\partial\Omega$.*

Let us define $\Delta := \{(x,y) \in \Omega \times \Omega \,:\, x = y\}$, then $g \in C((\Omega \times \Omega) \smallsetminus \Delta)$ and we have:

$$G(h)(y) = \int_\Omega g(x,y) h(x) \mathrm{d}\mu(x) \qquad \text{for every } h \in L^p(\Omega) \text{ and } y \in \Omega, \tag{5.1.7}$$

$$G^*(\nu)(x) = \int_\Omega g(x,y) \mathrm{d}\nu(y) \qquad \text{a.e. } x \in \Omega, \text{ for any fixed } \nu \in \mathcal{M}(\Omega), \tag{5.1.8}$$

$$g(x,y) = g(y,x) \geq 0 \qquad \text{for any } x, y \in \Omega, \text{ with } x \neq y. \tag{5.1.9}$$

Proof. Let us fix $y \in \Omega$.

By (5.1.3) of Theorem 5.1.1 and Definition 5.1.2, we have $g_y \geq 0$ on Ω. Moreover, thanks Definition 5.1.2 and (5.1.5), for any $h \in L^p(\Omega)$, we get (5.1.7):

$$G(h)(y) = \langle \delta_y, G(h) \rangle = \langle G^*(\delta_y), h \rangle = \int_\Omega g_y(x) h(x) \mathrm{d}\mu(x).$$

Now, we want to approximate the function g_y by the sequence $u_n := u_n^y = G(f_n)$, where

$$f_n(x) := f_n^y(x) = \frac{1}{\mu\left(B(y,1/n)\right)} \chi_{B(y,1/n)}(x), \qquad \forall\, x \in \Omega \text{ and } \forall\, n \in \mathbb{N}.$$

Recalling that d induces the Euclidean topology, for every $\psi \in L^p(\Omega)$, by (5.1.4) and (5.1.5) we obtain:

$$\begin{aligned}\int_\Omega u_n(x)\psi(x)\mathrm{d}\mu(x) &= \int_\Omega G(f_n)(x)\psi(x)\mathrm{d}\mu(x) = \int_\Omega G^*(f_n)\psi(x)\mathrm{d}\mu(x) = \\ &= \int_\Omega f_n(x) G(\psi)(x)\mathrm{d}\mu(x) = \fint_{B(y,\frac{1}{n})} G(\psi)(x)\mathrm{d}\mu(x),\end{aligned}$$

the last term tends to $G(\psi)(y)$ when $n \to \infty$, hence by (5.1.7) we get that

$$\lim_{n\to\infty} \int_\Omega u_n(x)\psi(x)\mathrm{d}\mu(x) = \int_\Omega g_y(x)\psi(x)\mathrm{d}\mu(x) \quad \forall\, \psi \in L^p(\Omega),$$

that is $u_n \to g_y$ weakly in $L^{p'}(\Omega)$. In particular, u_n is a bounded sequence in $L^{p'}(\Omega)$.

On the other hand, u_n is a non-negative W^1-weak solution to $\mathcal{L} u_n = 0$ in $\Omega \smallsetminus \overline{B(y,1/n)}$, thanks to construction of u_n and (5.1.2).

Fix now $0 < r < 1$ such that $\overline{B(y,2r)} \subseteq \Omega$; we put $\Omega_r := \Omega \smallsetminus \overline{B(y,r)}$. Let us fix $x \in \partial B(y,r)$ and $n > \frac{4}{r}$, then we have $B(x, r/2) \subseteq \Omega \smallsetminus \overline{B(y,1/n)}$. Thus we can apply the Harnack inequality in Theorem 4.4.1, and we obtain

$$\sup_{B(x,\frac{r}{8})} u_n \le C \inf_{B(x,\frac{r}{8})} u_n \le C \fint_{B(x,\frac{r}{8})} u_n(z)\mathrm{d}\mu(z) \le C \frac{1}{\mu\left(B(x,\frac{r}{8})\right)^{\frac{1}{p'}}} \|u_n\|_{p'} \le$$
$$\overset{\text{(D)}}{\le} \frac{\overline{C}}{\mu\left(B(y,\frac{9r}{8})\right)^{\frac{1}{p'}}} \|u_n\|_{p'} \le \frac{\overline{C}}{\mu\left(B(y,r)\right)^{\frac{1}{p'}}} \|u_n\|_{p'} \le$$
$$\le \frac{\overline{C}}{\mu\left(B(y,r)\right)^{\frac{1}{p'}}} \|u_n\|_{L^{p'}(\Omega)} \le \frac{\overline{C}}{\mu\left(B(y,r)\right)^{\frac{1}{p'}}} \sup_{n\in\mathbb{N}} \|u_n\|_{L^{p'}(\Omega)} =: M_y,$$

where M_y is a positive constant not depending on $n \in \mathbb{N}$. Hence, we have obtained that

$$u_n(x) \le \sup_{B(x,\frac{r}{8})} u_n \le M_y, \qquad \forall\, x \in \partial B(y,r) \text{ and } \forall\, n > \frac{4}{r}. \tag{5.1.10}$$

Therefore we have $u_n - M_y \le 0$ on $\partial B(y,r)$ and $u_n = 0$ on $\partial\Omega$, then $(u_n - M_y)^+ = 0$ on $\partial\Omega_r$ and $(u_n - M_y)^+ \in C(\overline{\Omega})$, since $u_n \in C(\overline{\Omega})$. Hence, we get $(u_n - M_y)^+ \in W_0^1(\Omega_r, X)$, for any $n > 4/r$.

On the other hand, we observe that $\mathcal{L}(u_n - M_y) = \mathcal{L}u_n = 0$ in $\Omega \setminus \overline{B(y,1/n)}$, then $u_n - M_y$ is a W^1-weak solution to $\mathcal{L}(u_n - M_y) = 0$ on Ω_r, since $\Omega_r \subseteq \Omega \setminus \overline{B(y,1/n)}$ for any $n > 4/r$, and we can apply the Maximum principle of [52, Theorem 3.1] obtaining

$$0 \le \sup_{\Omega_r}(u_n - M_y)^+ \le \sup_{\partial\Omega_r}(u_n - M_y)^+ = 0,$$

thus

$$u_n \le M_y \quad \text{ on } \Omega_r, \text{ for every } n > \frac{4}{r}. \tag{5.1.11}$$

Now we want to prove that u_n is a bounded sequence in $W^1(\Omega_{2r}, X)$.

Let us consider $\eta \in C^1(\Omega)$ a cut-off function such that $0 \le \eta \le 1$ and

(i) $\eta \equiv 1$ on Ω_{2r};

(ii) $\eta \equiv 0$ on $B(y, \frac{3}{2}r)$.

We put $v := \eta^2 u_n$ for every $n > 4/r$, then $v \in W_0^1(\Omega_r, X)$ is a test function. Since $\mathcal{L}u_n = 0$ on Ω_r (in the weak sense of W^1), we have $L(u_n, v) = 0$. Then, using the X-ellipticity of $\mathcal{L}$ and supposing u_n, v smooth functions, we get:

$$-L(u_n,v) = -\int_{\Omega_r} \left\langle A\nabla u_n, \nabla(\eta^2 u_n)\right\rangle \mathrm{d}\mu = -2\int_{\Omega_r} \eta u_n \left\langle A\nabla u_n, \nabla\eta\right\rangle \mathrm{d}\mu - \int_{\Omega_r} \eta^2 \left\langle A\nabla u, \nabla\eta\right\rangle \mathrm{d}\mu \le$$
$$\le 2\Lambda \int_{\Omega_r} \eta u_n |Xu_n||X\eta| \mathrm{d}\mu - \lambda \int_{\Omega_r} \eta^2 |Xu_n|^2 \mathrm{d}\mu,$$

and by approximation we obtain that

$$\int_{\Omega_r} \eta^2 |Xu_n|^2 \mathrm{d}\mu \le 2\frac{\Lambda}{\lambda} \int_{\Omega_r} \eta u_n |Xu_n||X\eta| \mathrm{d}\mu. \tag{5.1.12}$$

By interpolation we have:

$$\left(u_n\,|X\eta|\,\frac{2\Lambda}{\lambda}\right)(\eta|Xu_n|) \le \frac{4}{\varepsilon}\left(\frac{\Lambda}{\lambda}\right)^2 u_n^2|Xu_n|^2 + \varepsilon\eta^2|Xu_n|^2, \quad \forall\, \varepsilon > 0.$$

If we choose $\varepsilon = \frac{1}{2}$, by (5.1.12) we get

$$\int_{\Omega_r} \eta^2 |Xu_n|^2 \mathrm{d}\mu \leq 8 \left(\frac{\Lambda}{\lambda}\right)^2 \int_{\Omega_r} u_n^2 |X\eta|^2 \mathrm{d}\mu + \frac{1}{2} \int_{\Omega_r} \eta^2 |Xu_n|^2 \mathrm{d}\mu,$$

which gives

$$\frac{1}{2} \int_{\Omega_r} \eta^2 |Xu_n|^2 \mathrm{d}\mu \leq 8 \left(\frac{\Lambda}{\lambda}\right)^2 M_y^2 \int_{\Omega_r} |X\eta|^2 \mathrm{d}\mu,$$

where in the last inequality we have used (5.1.11). Moreover $\Omega_{2r} \subseteq \Omega_r$, then we obtain

$$\int_{\Omega_{2r}} \eta^2 |Xu_n|^2 \mathrm{d}\mu \leq 16 \left(\frac{\Lambda}{\lambda}\right)^2 M_y^2 \int_{\Omega_r} |X\eta|^2 \mathrm{d}\mu,$$

which gives (recalling (i)):

$$\|Xu_n\|^2_{L^2(\Omega_{2r})} \leq C(\Lambda, \lambda, y), \quad \forall\, n > \frac{4}{r}, \tag{5.1.13}$$

where $C > 0$ does not depend on $n \in \mathbb{N}$.

Therefore, using (5.1.11) and (5.1.13), we obtain that u_n is a bounded sequence in $W^1(\Omega_{2r}, X)$. Then there exists $w \in W^1(\Omega_{2r}, X)$ such that $u_n \to w$ weakly in $W^1(\Omega_{2r}, X)$, in particular u_n converges weakly to w in $L^2(\Omega_{2r})$. Since $p \geq 2$, $L^p(\Omega_{2r}) \subseteq L^2(\Omega_{2r})$ and we obtain $u_n \to w$ weakly in $L^{p'}(\Omega_{2r})$. On the other hand, we have already showed that $u_n \to g_y$ weakly in $L^{p'}(\Omega)$, in particular in $L^{p'}(\Omega_{2r})$; then necessarily $w = g_y$. Thus $u_n \to g_y$ weakly in $W^1(\Omega_{2r}, X)$. Now we observe that, for any $\varphi \in W_0^1(\Omega_{2r}, X)$, $L(\cdot, \varphi) : W^1(\Omega_{2r}, X) \to \mathbb{R}$ is a bounded linear functional; hence, it is sufficient to let $n \to \infty$ in the equality $L(u_n, \varphi) = 0$, for every $\varphi \in W_0^1(\Omega_{2r}, X)$, to prove that $\mathcal{L} g_y = 0$ in Ω_{2r} (in the weak sense of W^1). Then, by Corollary 4.4.8 we get that g_y is a continuous function in Ω_{2r}.

We want to prove the continuity of g_y up to $\partial\Omega$.

Let us fix $x_0 \in \partial\Omega$ and let $\psi \in C_0^\infty(B(x_0, 4\varepsilon))$ be a cut-off function such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $B(x_0, 2\varepsilon)$, where we have chosen $\varepsilon > 0$ such that $B(x_0, 4\varepsilon) \cap \overline{B(y, 2r)} = \varnothing$.

Since $u_n \in W^1(\Omega \setminus \overline{B(y, 1/n)})$ we have $\psi u_n \in W_0^1(\Omega \cap B(x_0, 4\varepsilon), X)$, for any $n \in \mathbb{N}$. Moreover, thanks to boundedness of the sequence $\{u_n\}$ in $W^1(\Omega_{2r}, X)$, it is easy to show that $\{\psi u_n\}$ is a bounded sequence in $W_0^1(\Omega \cap B(x_0, 4\varepsilon), X)$. Then there exists $v \in W_0^1(\Omega \cap B(x_0, 4\varepsilon), X)$ such that $\psi u_n \to v$ weakly in $W_0^1(\Omega \cap B(x_0, 4\varepsilon), X)$; in particular, ψu_n converges weakly to v in $W^1(\Omega \cap B(x_0, 2\varepsilon), X)$ and so necessarily $v = g_y$ in $\Omega \cap B(x_0, 2\varepsilon)$.

Therefore, $v \in W_0^1(\Omega \cap B(x_0, 4\varepsilon), X)$ is a W^1-weak solution to $\mathcal{L} v = 0$ in $\Omega \cap B(x_0, 2\varepsilon)$. Hence, by Corollary 4.4.14 we have $v \in C(\overline{\Omega \cap B(x_0, \varepsilon)})$ and $v \equiv 0$ on $\partial\Omega \cap B(x_0, \varepsilon)$. Thus, g_y is a continuous function up to $\partial\Omega$ and g_y vanishes on $\partial\Omega$.

Hence, we have showed that $g_y \in C(\overline{\Omega} \setminus \{y\})$ and $g_y(x) = 0$ for every $x \in \partial\Omega$.

The continuity of g in the couple $(x, y) \in (\Omega \times \Omega) \setminus \Delta$ can be obtained by adapting the arguments in [31, Proposition 2.6] and in [92, Theorem 3.4].

In order to prove (5.1.8) we fix $\nu \in \mathcal{M}(\overline{\Omega})$ and $h \in L^p(\Omega)$. Since g is a non-negative and continuous function in $(\Omega \times \Omega) \setminus \Delta$, where Δ is a set of $\mathrm{d}\mu(x) \times \mathrm{d}\nu(y)$-measure zero, we can apply Fubini's theorem and we get:

$$\langle G^*(\nu), h\rangle \overset{(5.1.5)}{=} \int_\Omega G(h)(y)\mathrm{d}\nu(y) \overset{(5.1.7)}{=} \int_\Omega \left(\int_\Omega g(x,y)h(x)\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y) = \\ = \int_\Omega h(x)\left(\int_\Omega g(x,y)\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x).$$

Therefore, we have

$$\int_\Omega h(x)G^*(\nu)(x)\mathrm{d}\mu(x) = \int_\Omega h(x)\left(\int_\Omega g(x,y)\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x), \qquad \forall\, h \in L^p(\Omega),$$

which gives (5.1.8).

Finally, we want to prove the symmetry of g in the couple (x,y).
Fix $x_0, y_0 \in \Omega$ such that $x_0 \neq y_0$. Let us consider a function $h \in L^p(\Omega)$ supported in a neighborhood $B(y_0,\rho_2)$ of y_0, such that there exists $\rho_1 > 0$ for which $B(x_0,\rho_1) \cap B(y_0,\rho_2) = \varnothing$. We consider the function

$$F(x) := \int_{B(y_0,\rho_2)} g(x,y)h(y)\mathrm{d}\mu(y), \quad \forall\, x \in B(x_0,\rho_1).$$

Since g is continuous in $(\Omega\times\Omega)\setminus\Delta$, the function F is well defined and continuous in $B(x_0,\rho_1)$.

On the other hand, we know that

$$F(x) = \int_\Omega g(x,y)h(y)\mathrm{d}\mu(y) \overset{(5.1.8)}{=} G^*(h)(x) \overset{(5.1.4)}{=} G(h)(x) \overset{(5.1.7)}{=} \int_\Omega g(y,x)h(y)\mathrm{d}\mu(y),$$

for almost every $x \in B(x_0,\rho_1)$. By continuity of F and $G(h)$ in $B(x_0,\rho_1)$, we get $F(x) = G(h)(x)$ for any $x \in B(x_0,\rho_1)$, which gives

$$\int_{B(y_0,\rho_2)} \left(g(x,y) - g(y,x)\right)h(y)\mathrm{d}\mu(y) = 0 \quad \text{for any } x \in B(x_0,\rho_1).$$

By arbitrariness of $h \in L^p$, we have $g(\cdot,x) = g(x,\cdot)$ a.e. in $B(y_0,\rho_2)$ and thanks to continuity of g out of the diagonal, we obtain $g(x,y) = g(y,x)$ for any $(x,y) \in B(x_0,\rho_1)\times B(y_0,\rho_2)$. Finally, by arbitrariness of (x_0,y_0) we get (5.1.9). □

5.2 Towards a global fundamental solution

In this section we want to give a sketch of the existence proof of a global fundamental solution for the operator $\mathcal{L}$ in (5.0.1). This argument will be the object of our future investigation.

The first step is the construction of a basis of bounded open sets for the d-topology, satisfying the condition (5.1.1). In particular, for every $x_0 \in \mathbb{R}^N$, we want to prove the existence of a basis $\mathscr{B} := \{\Omega_n\}_{n\in\mathbb{N}}$ of bounded open sets for the d-topology on $\mathbb{R}^N$, such that:

(1) $B(x_0, n-\frac{1}{2}) \subseteq \Omega_n \subseteq B(x_0,n)$, for every $n \in \mathbb{N}$;

(2) $\Omega_n \subseteq \Omega_{n+1}$, for every $n \in \mathbb{N}$;

(3) there exists $\vartheta(Q) > 0$ such that, for any $y_0 \in \partial\Omega_n$, we have

$$\mu\left(B(y_0, r) \smallsetminus \Omega_n\right) \geq \vartheta\mu\left(B(y_0, r)\right), \quad \text{for every } r \leq \frac{1}{2} \text{ and } n \in \mathbb{N}. \tag{5.2.1}$$

Furthermore, for any $n \in \mathbb{N}$, the set Ω_n can be obtained in the following way:

$$\Omega_n := B(x_0, n) \smallsetminus \bigcup_{j=1}^{p} \overline{B(x_j, 1/2)}, \tag{5.2.2}$$

where $x_j \in \partial B(x_0, n)$ and $\{B(x_j, 1/2) : j = 1, \ldots, p(Q, n)\}$ is a finite covering of $\partial B(x_0, n)$. Clearly it is easy to prove conditions (1) and (2); in the proof of condition (3) the idea will be to use the *segment property* and the doubling condition (D).

In order to prove the existence of global fundamental solution for $\mathcal{L}$, after the construction of a suitable basis for d-topology on $\mathbb{R}^N$ we need to consider the Green functions $g_n(\cdot, \cdot)$ related to any bounded open set Ω_n of the basis. If we consider the trivial extension of any g_n out the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, the idea is to use the Maximum principle in [52] to prove that $\{g_n\}$ is a non-decreasing sequence. Hence, we put

$$\Gamma(x, y) := \lim_{n \to \infty} g_n(x, y), \qquad \text{for every } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ with } x \neq y,$$

and we will show that Γ is a continuous function out the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, using the invariance of the Harnack inequality in Chapter 4.

Finally, to prove that Γ is a global fundamental solution for $\mathcal{L}$, we will use the *representation formulas* for the Green operator and its adjoint (see Theorem 5.1.3), with the suitable construction of the measure μ related to $\mathcal{L}$ (see (4.2.2)).

Acknowledgments

During my Ph.D. studies I have met many people with whom I studied, worked, shared beautiful and less beautiful experiences. I would like to thank everyone.

First of all, I would like to thank the advisor of this thesis, Prof. Andrea Bonfiglioli. He was a wise guidance during these years and I learned a lot working with him; I would like to thank him for his time, his kindness, his helpfulness and his support.

Then, I wish to thank especially Prof. Ermanno Lanconelli, for his wise advices and his support in my work. I would also like to thank Prof. Giovanna Citti and Prof. Alberto Parmeggiani; their help was invaluable in these years for all of us young students who aim at working in the research field.

I would like to thank the referees of this thesis, Prof. Giovanna Citti, Prof. Giuseppe Di Fazio and Prof. Sergio Polidoro; their valuable advices have been very important for the improvement of this work.

I wish to thank all friends which have shared this journey with me, specially Serena Federico, Benedetta Franceschiello and Lorenzo Ruffoni. Their moral support was very important for me; these years have been really special thanks to their friendship.

I also have to thank my colleague Stefano Biagi, together we have shared exciting work experiences; thanks for his company and his support during these years. Thanks to Alessia Kogoj for her useful advices, her support and kindness.

I wish to thank the people which I have met during my short stay at Henry Poincaré Institut in Paris, especially Davide Barilari; thanks to their hospitality, they made me feel at home.

Finally, I would like to thank my family and my fiance; they are the pillars of my life. Thanks for their support, patience and unconditional love. I got this thanks to you.

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