

# Modeling the Joint Distribution of Income and Consumption in Italy

## A Copula-Based Approach with $\kappa$ -Generalized Margins

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### Abstract

This chapter elaborates a new parametric model for the joint distribution of income and consumption. The model combines estimates for the marginal distributions of income and consumption and a parametric copula function to capture the dependence structure between the two variates. Specifically, we apply the “symmetrized Joe-Clayton” copula to model the dependence between income and consumption margins whose non-identical distributions belong to the “ $\kappa$ -generalized” family. Using data from the Bank of Italy’s Survey on Household Income and Wealth for the period 1987–2014, we find that the proposed copula-based approach accounts well for the complex dependence between income and consumption observed in our samples. The chapter also points to further developments that are specific to the field of welfare economics.

**Keywords:** Italy, consumption, personal income,  $\kappa$ -generalized distribution, dependence, symmetrized Joe-Clayton copula

### 1. Introduction

The focus of this chapter is to develop and fit a flexible parametric model for the bivariate distribution of income and consumption in Italy. Since the independence between income and consumption is not the most appropriate assumption to work with, we study the joint distribution of the two variables by separately estimating the univariate marginal distribution models for income and consumption, and by estimating a parametric copula function to capture information about the dependence between the two dimensions. This approach is appealing as copulas are easily estimated using maximum likelihood techniques, and there are many alternatives available in the literature that capture a wide range of dependence structures beyond simply correlation. In addition, copulas are flexible in that they can be applied to any specification of the marginal distributions, including allowing for the latter to have different specifications. This provides an attractive method for capturing the dependence structure contained in the joint distribution of income and consumption of actual samples.

Using copulas to model multivariate distributions is extremely popular in the finance and actuarial context, particularly for capturing dependence among stocks. However, copula-based approaches have rarely been applied in welfare economics—but see [3] on potential applications. There are some notable exceptions: the approach used by [12] and [33] to analyze the correlation between the incomes of spouses is (implicitly) copula-based; [25] use a copula-based framework to measure the extent of re-ranking through taxation; [9] estimate a parametric copula to describe individual earnings trajectories and income mobility in France; [70], [71] and [84] utilize copulas to measure association between income and health, whereas [50] and [36] take advantage of the copula paradigm to analyze the dependence between income and wealth; a copula-based approach was also considered, *inter alia*, by [81], [26], [69], and [4] for assessing inequality and poverty under dependent dimensions of well-being.

As far as income and consumption are concerned, the only attempt that we are aware of in the current literature is by [28] and [29], who apply a copula-based approach to the measurement of household financial fragility in Italy. However, our work is different than their approach because of the distinctive parametric assumptions we make for both the uni-variate margins and the copula function that summarizes the existing dependence structure. Furthermore, we are better positioned to take a long-term perspective since we use the same data but for a longer time span than the single appraisal period as in [28] and [29].

The organization of the chapter is as follows. Section 2.1 describes the data set used and provides a preliminary inspection of the degree of dependence between income and consumption in Italy. Section 2.2 motivates the choice of  $\kappa$ -generalized models for the income and consumption distributions, whereas Section 2.3 briefly reviews the theory of copulas and discusses the reasons of our interest in the “symmetrized Joe-Clayton” specification for modeling the association between the two variables. Estimation results and analysis of the parametric model for the bi-variate distribution of income and consumption are presented in Section 3. Finally, Section 4 concludes and points to possible extensions of this work for the future.

## 2. Data and methodology

### 2.1. The Italian personal-income and consumption data

Income and consumption data are drawn from the Survey on Household Income and Wealth (SHIW), a representative survey of the Italian resident population conducted by the Bank of Italy since the mid-1960s to gather data on income, saving, consumption expenditure, wealth, demographics and labor force participation of Italian house-

holds.<sup>1</sup>

The SHIW was carried out yearly until 1987 (except for 1985) and every two years thereafter (the survey for 1997 was shifted to 1998). The sample used in the most recent waves comprises about 8,000 households (20,000 individuals), distributed over around 300 Italian municipalities.

The data set employed in this chapter includes fourteen independent cross sections of Italian households covering the period 1987–2014, for a total of 111,118 observations. While income and consumption data are available also for years prior to 1987, we choose to focus on data collected from 1987 onwards because of a major overhaul of the survey that took place in 1986–87, when the design of the questionnaire was entirely revised, the sample size was raised to double that of previous waves, and the income definition underwent significant changes that hinder temporal comparisons (income from financial assets started to be recorded only in 1987).

The basic definition of income provided by the SHIW is net of taxation and social security contributions. It is the sum of four main components: compensation of employees; pensions and net transfers; net income from self-employment; property income (including income from buildings and income from financial assets). The SHIW variable recording household consumption expenditure, in turn, is obtained by aggregating household expenditures for durable and non-durable goods. According to the definition of the Bank of Italy, expenditures for non-durable goods correspond to all spending on both food and non-food items, plus non-monetary income integrations (fringe benefits) and imputed rents.<sup>2</sup> Household expenditures for durable goods correspond to items belonging to the following categories: means of transport, furniture, and precious objects.

The variables analyzed here focus on total income and consumption of the households surveyed. Since in some waves there were cases of zero and/or negative figures, we dropped such observations and kept only strictly-positive amounts of income and consumption.<sup>3</sup> Furthermore, income and consumption figures have been adjusted

<sup>1</sup>The data (with documentation in English) are freely available at: <https://www.bancaditalia.it/statistiche/tematiche/indagini-famiglie-imprese/bilanci-famiglie/index.html>. We refer the reader to the works of [11] and [10] for details on design of the survey, data quality, and main changes in the sample and variable definitions. See also the *Supplement to the Statistical Bulletin* available from <https://www.bancaditalia.it/pubblicazioni/indagine-famiglie/index.html>, which sets out the main results of the survey waves that the Bank of Italy has carried out.

<sup>2</sup>If a household dwelling is neither owned nor rented, but occupied in usufruct or free of charge, the total consumption expenditure for that household will include an imputed rent, i.e. an amount corresponding to the rent that could be charged for such a dwelling.

<sup>3</sup>This exclusion affected only a tiny fraction of the data—on average, 0.28% and 0.01% of the observations on income and consumption, respectively—and left us with a total of 110,870 observations. Accordingly, the sampling weights of households have been re-calibrated in such a way that estimates from the samples after deletion of non-positive records are forced to fit the initial population-level distribution of certain

for differences in household size using the “modified OECD” equivalence scale and weighted by the provided sampling weights.<sup>45</sup> Finally, we deflated all monetary aggregates (expressed in Euros) so as to obtain *real* distributions of income and consumption. To do so, we employed the consumption deflator for resident households provided by the Italian statistical office (ISTAT).<sup>6</sup> The base year for the deflator is 2010.

Information on the association between income and consumption in our samples is shown in Figure 1, where we plot summary indicators of correlation such as Pearson’s  $\rho$ , Spearman’s  $\rho_s$ , and Kendall’s  $\tau$ . The Pearson’s correlation coefficient gives us an indication of the linear relationship between income and consumption. The others—Spearman’s  $\rho_s$  and Kendall’s  $\tau$ —are rank correlation indicators that are often preferred to Pearson’s  $\rho$  for non-normal data, since they are less sensitive to extreme data (e.g. [56] and [23]; see also discussion in Section 2.3). Overall, we observe a strong positive dependence between income and consumption in Italy that is generally greater than 0.5 in all samples, but that dependence varied considerably over time. Indeed, regardless of the indicator used, two regime changes of temporal evolution are clearly identified: correlation was high in the early part of the period, then lowered in central years, and raised at last during the recent economic crisis.

To test for the presence of time-varying dependence, we perform a structural change analysis of correlation coefficients over the whole period using the procedure proposed by [5] and [6], henceforth BP. A key feature of the methodology developed by these authors is that it allows to test for multiple breaks at unknown dates. The model considered here is the linear regression model with  $m$  breaks (or, equivalently,  $m + 1$

characteristics known from external sources. The external information used in calibration is: gender, age group (under 26, 26–45, 46–65, and over 65), geographical area (North, Centre, South and Islands) and size of the municipality of residence (under 20,000 inhabitants, 20,000–40,000, 40,000–500,000, and over 500,000). Weighting adjustments have been implemented using the R function `calib` from the library `sampling` [79].

<sup>45</sup>The “modified OECD” equivalence scale allocates points to each person in a household by taking the first adult as having a weight of 1 point, whereas each additional person who is 14 years or older is allocated 0.5 points, and each child under the age of 14 is allocated 0.3 points [35]. Equivalized household figures are derived by dividing total household income and consumption by a factor equal to the sum of the equivalence points allocated to household members. Unlike the old OECD scale, the modified one gives less weight to any additional household member, allowing for higher economies of scale.

<sup>5</sup>We use person-level adjusted weights (the product of household weights and the number of household members) when generating income and consumption indicators for the total population. Results from estimation of distributional parameters, presented in Section 3, have also been weighted to obtain population-level estimates and account for the SHIW survey design.

<sup>6</sup>Available at: <https://www.bancaditalia.it/statistiche/tematiche/indagini-famiglie-imprese/bilanci-famiglie/tavole-principali-risultati/index.html>.

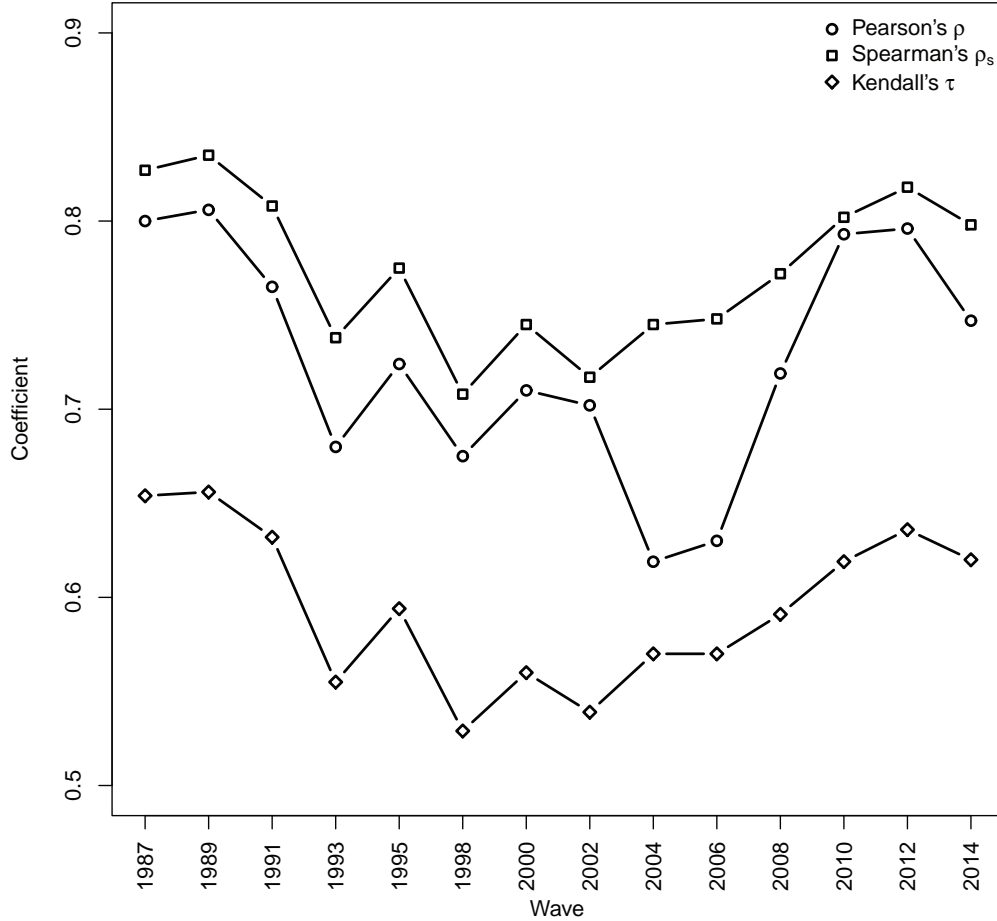


Figure 1 The association between income and consumption in Italy, 1987–2014

regimes)

$$y_t = \beta_j + u_t, \quad t = T_{j-1} + 1, \dots, T_j, \quad j = 1, \dots, m + 1, \quad (1)$$

where  $T_0 = 0$  and  $T_{m+1} = T$  by convention ( $T$  is the number of yearly observations).<sup>7</sup> In other words, within the regime  $j$  the Pearson's  $\rho$ , the Spearman's  $\rho_s$ , or the Kendall's  $\tau$  equals the regime-specific mean  $\beta_j$  plus a stationary error term  $u_t$ , which may have a different distribution across regimes. The goal of the analysis is to determine the

<sup>7</sup>To increase significance of findings, we use linear interpolation to estimate missing data for years in which the SHIW waves are not available. This leads us to enlarge sample size from 14 to 28 time observations.

optimal number and location of the structural break points  $T_j$ ,  $j = 1, \dots, m$ , by minimizing the within-regime sums of squares. By default, our implementation of BP’s technique derives the appropriate number of breaks as the one achieving the lowest Bayesian information criterion score [75].<sup>8</sup>

The results can be visualized in Figure 2, which shows time series plots for any of the three measures of dependence stated previously (gray solid lines) along with the estimated break points (black dashed lines) and the regime-specific means in each resulting data segment (black solid lines). As can be seen, the period under consideration has two clear breaks, which correspond to 1991 and 2008. For all correlation indicators, we find statistically significant evidence of a break in the earlier two dates, with a  $p$ -value lower than 0.05.<sup>9</sup>

Thus, we can conclude that there is evidence against constant dependence structure over time for the SHIW income-consumption data. This provides a solid motivation for considering copula-based multivariate models that are able to reproduce the analyzed pattern of time-varying dependence. However, as it will be shown in Section 2.3, the association between income and consumption in any single year is more complex than can be captured by single summary measures like linear correlation or rank correlation, because the strength of dependence between the two variables in the bottom tail of their joint distribution is different from what comes out of the upper tail. Hence, our “ideal” copula-based model should also be able to accommodate asymmetric dependence in the tails of the bi-variate distribution of income and consumption.

## 2.2. The $\kappa$ -generalized distribution for margins

The interest in finding parametric models for the size distribution of income has a long history. A natural starting point in this area of inquiry was the observation that the number of persons in a population whose incomes exceed  $x$  is often well approximated by  $Cx^{-\alpha}$ , for some real  $C$  and positive  $\alpha$ , as Pareto argued over 100 years ago [63, 64, 66, 65]. Since the early studies of Pareto, numerous empirical works have shown that the power-law tail is a ubiquitous feature of income distribution. However, even 100 years after Pareto’s observation, the understanding of the shape of income distribution is still far to be complete and definitive. This reflects the fact that there are

<sup>8</sup>When implementing the BP’s procedure for structural change, the maximal number of breaks to be calculated is a parameter to be fixed by the researcher. For our data, we allow simultaneous calculation for up to  $m = 2$  breaks. The technique suggested by BP has been implemented in a unified way in the package `strucchange` [87, 86, 85] for the statistical software R [72], which is the one we rely upon in the present study.

<sup>9</sup>The values of the  $F$  statistic for testing against a single-shift alternative of known timing—the so-called “Chow test” [14]—amounted in fact to 12.58, 19.66, and 23.02 in 1991 and to 8.39, 6.55, and 5.51 in 2008 for, respectively, the Pearson’s  $\rho$ , the Spearman’s  $\rho_s$ , and the Kendall’s  $\tau$ , which exceed in all cases their respective 5% critical values.

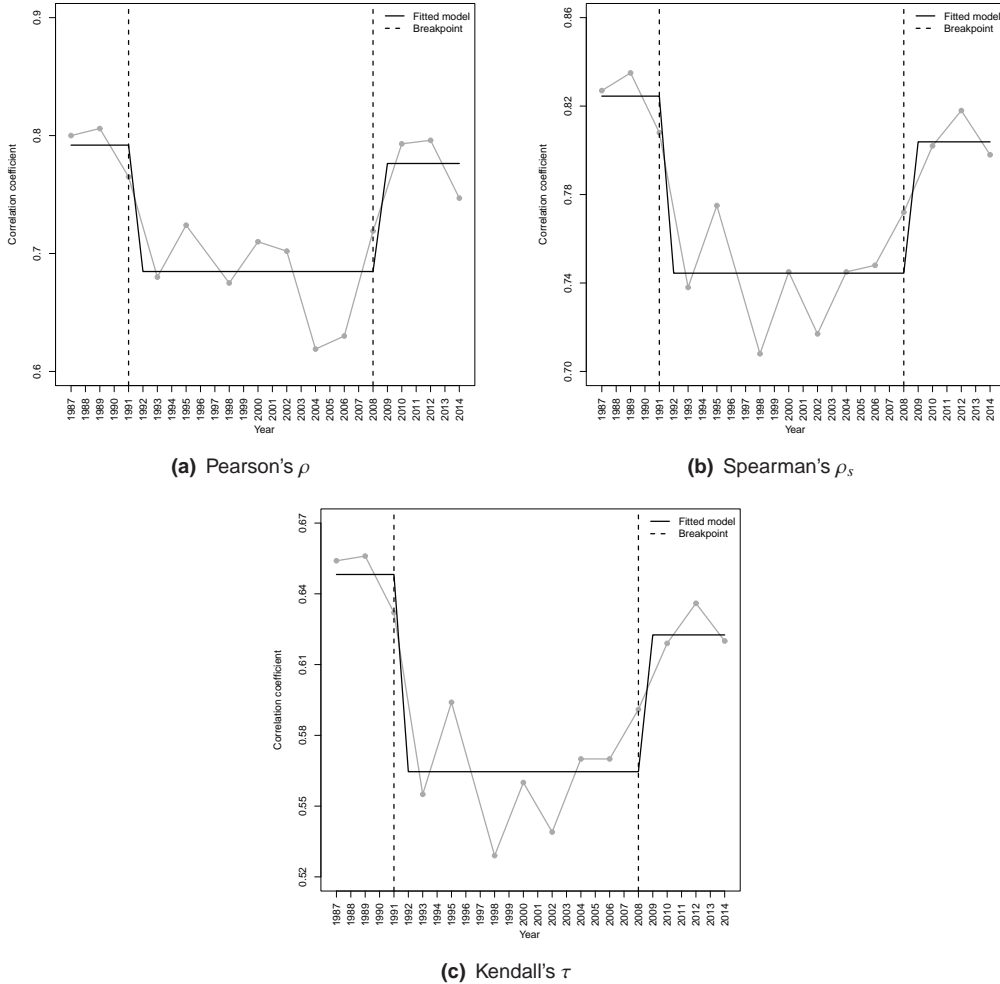


Figure 2 Structural breaks in correlation coefficients, 1987–2014

two distributions, one for the rich, following the Pareto’s power law, and one for the vast majority of people, which appears to be governed by a completely different law.

Over the years, research in the field has considered a wide variety of functional forms as possible models for the size distribution of income, some of which aim at providing a unified framework for the description of real-world data—including the heavy tails present in empirical income distributions [52]. Among these, the “ $\kappa$ -generalized distribution” was found to work remarkably well [17, 15, 18, 19, 20, 16, 22]. First proposed in 2007, and further developed over successive years, this model finds its roots in the context of generalized statistical mechanics [43, 44, 45, 46, 47, 48]. Within this

theoretical framework, the ordinary exponential function  $\exp(x)$  generalizes into the function  $\exp_\kappa(x)$  defined through

$$\exp_\kappa(x) = \left( \sqrt{1 + \kappa^2 x^2} + \kappa x \right)^{\frac{1}{\kappa}}, \quad x \in \mathbb{R}, \quad \kappa \in [0, 1). \quad (2)$$

We recall briefly that in the  $\kappa \rightarrow 0$  limit the function (2) reduces to the ordinary exponential, i.e.  $\exp_0(x) = \exp(x)$ , and for  $x \rightarrow 0$ —independently on the value of  $\kappa$ —behaves very similarly with the ordinary exponential. On the other hand, the most interesting property of  $\exp_\kappa(x)$  for modeling the size distribution of income and wealth is the power-law asymptotic behavior

$$\exp_\kappa(x) \underset{x \rightarrow \pm\infty}{\sim} |2\kappa x|^{\pm \frac{1}{|\kappa|}}. \quad (3)$$

Given (2), the  $\kappa$ -generalized distribution is defined in terms of the following cumulative distribution function (CDF)

$$F(x; \alpha, \beta, \kappa) = 1 - \exp_\kappa \left[ - \left( \frac{x}{\beta} \right)^\alpha \right], \quad x > 0, \quad \alpha, \beta > 0, \quad \kappa \in [0, 1), \quad (4)$$

where  $\{\alpha, \beta, \kappa\}$  are parameters. The corresponding probability density function (PDF) reads as

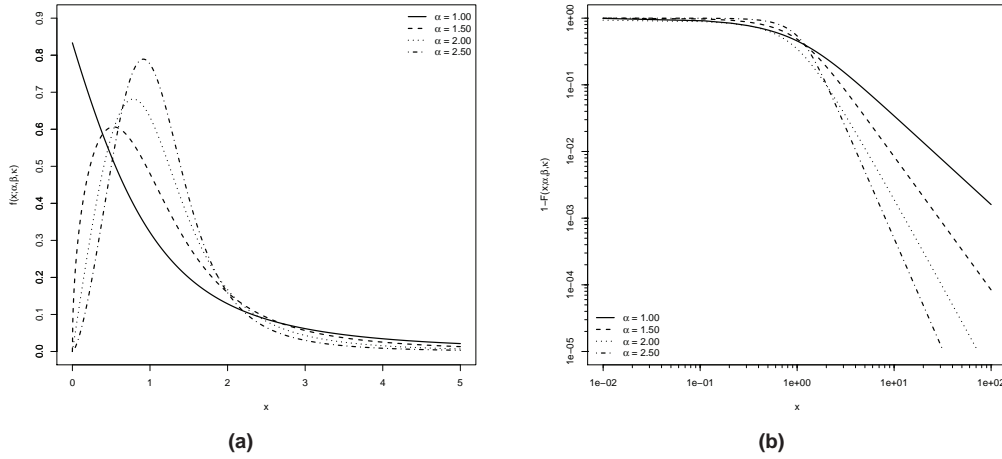
$$f(x; \alpha, \beta, \kappa) = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1} \frac{\exp_\kappa \left[ - \left( \frac{x}{\beta} \right)^\alpha \right]}{\sqrt{1 + \kappa^2 \left( \frac{x}{\beta} \right)^{2\alpha}}}. \quad (5)$$

The distribution defined through (4) and (5) can be viewed as a generalization of the Weibull distribution, which recovers in the  $\kappa \rightarrow 0$  limit. Consequently, the exponential law is also a special limiting case of the  $\kappa$ -generalized distribution, since it is a special case of the Weibull with  $\alpha = 1$ . For  $x \rightarrow 0^+$ , the  $\kappa$ -generalized behaves similarly to the Weibull distribution, whereas for large  $x$  it presents a Pareto’s power-law tail, hence satisfying the weak Pareto’s law [55].

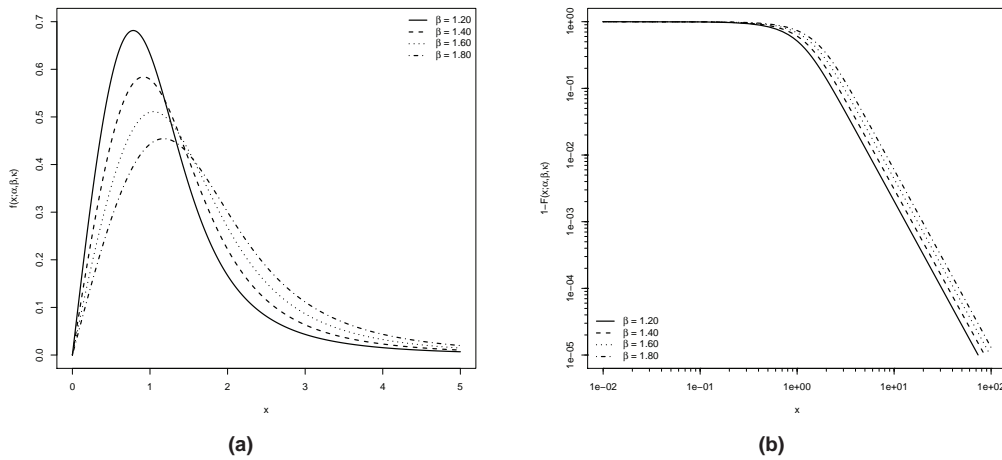
Figures 3 to 5 illustrate the behavior of the  $\kappa$ -generalized PDF and complementary CDF,  $1 - F(x; \alpha, \beta, \kappa)$ , for various parameter values. The exponent  $\alpha$  quantifies the curvature (shape) of the distribution, which is less (more) pronounced for lower (higher) values of the parameter, as seen in Figure 3.<sup>10</sup> The constant  $\beta$  is a characteristic scale, since its value determines the scale of the probability distribution: if  $\beta$  is small, then the distribution will be more concentrated around the mode; if  $\beta$  is large, then it will be more spread out (Figure 4). Finally, as Figure 5 shows, the parameter  $\kappa$  measures the fatness of the upper tail: the larger (smaller) its magnitude, the fatter

<sup>10</sup>It should be noted that for  $\alpha = 1$  the density exhibits a pole at the origin, whereas for  $\alpha > 1$  there exists an interior mode.





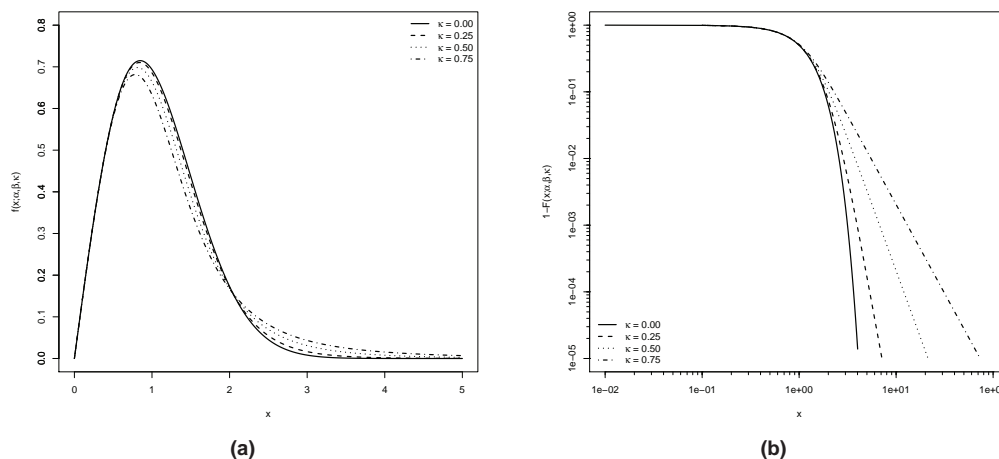
**Figure 3** Plot of the  $\kappa$ -generalized PDF (a) and complementary CDF (b) for some different values of  $\alpha$  ( $= 1.00, 1.50, 2.00, 2.50$ ) and fixed  $\beta$  ( $= 1.20$ ) and  $\kappa$  ( $= 0.75$ ). The complementary CDF is plotted on doubly-logarithmic axes, which is the standard way of emphasizing the right-tail behavior of a distribution. Notice that the curvature (shape) of the distribution becomes less (more) pronounced when the value of  $\alpha$  decreases (increases). The case  $\alpha = 1.00$  corresponds to the standard exponential distribution



**Figure 4** Plot of the  $\kappa$ -generalized PDF (a) and complementary CDF (b) for some different values of  $\beta$  ( $= 1.20, 1.40, 1.60, 1.80$ ) and fixed  $\alpha$  ( $= 2.00$ ) and  $\kappa$  ( $= 0.75$ ). The complementary CDF is plotted on doubly-logarithmic axes, which is the standard way of emphasizing the right-tail behavior of a distribution. Notice that the distribution spreads out (concentrates) as the value of  $\beta$  increases (decreases)

(thinner) the tail.

Expressions that facilitate the analysis of the associated moments and various tools for the measurement of inequality have been reported for the  $\kappa$ -generalized distribution [15, 18, 19, 20, 16, 22]. These expressions are functions of the parameters in the model



**Figure 5** Plot of the  $\kappa$ -generalized PDF (a) and complementary CDF (b) for some different values of  $\kappa$  ( $= 0.00, 0.25, 0.50, 0.75$ ) and fixed  $\alpha$  ( $= 2.00$ ) and  $\beta$  ( $= 1.20$ ). The complementary CDF is plotted on doubly-logarithmic axes, which is the standard way of emphasizing the right-tail behavior of a distribution. Notice that the upper tail of the distribution fattens (thins) as the value of  $\kappa$  increases (decreases). The case  $\kappa = 0.00$  corresponds to the Weibull (stretched exponential) distribution

and prove useful in the analysis of population characteristics.

The  $\kappa$ -generalized distribution was also successfully used in a three-component mixture model for analyzing the singularities of survey data on *net* wealth, i.e. the value of gross wealth minus total debt, which present highly significant frequencies of households or individuals with null and/or negative wealth [21, 16, 22]. The support of the  $\kappa$ -generalized mixture model for net wealth distribution is the real line  $\mathbb{R} = (-\infty, \infty)$ , thus allowing to describe the subset of economic units with nil and negative net worth. Furthermore, four-parameter variants exist that contain as a particular case the  $\kappa$ -generalized model for income distribution [61].

During the last decade, there have been several applications of  $\kappa$ -generalized models to real-world data on income and wealth distribution. Of special interest are papers fitting several distributions to the same data, with an eye on relative performance. From comparative studies such as [19], who considered the distribution of household income in Italy for the years 1989 to 2006, it emerges that model (5) typically outperforms its three-parameter competitors such as the Singh-Maddala [77] and Dagum type I [24] distributions, apart from the generalized beta II (GB2) which has an extra parameter.<sup>11</sup> The model was also fitted by [20] to data from other household bud-

<sup>11</sup>The GB2 is a quite general family of parametric models for the size distribution of income that nests most of the functional forms previously considered in the size distributions literature as special or limiting cases [58]. In particular, both the Singh-Maddala and Dagum type I distributions are special cases of the

get surveys, namely Germany 1984–2007, Great Britain 1991–2004, and the United States 1980–2005. In a remarkable number of cases, the distribution of household income follows the  $\kappa$ -generalized more closely than the Singh-Maddala and Dagum type I. In particular, the fit is statistically superior in the right tail of data with respect to the other competitors in many instances. Another example of comparative study is [60], who considered US and Italian income data for the 2000s. He found the three-parameter  $\kappa$ -generalized model to yield better estimates of income inequality even when the goodness-of-fit is inferior to that of distributions in the GB2 family. The excellent fit of the  $\kappa$ -generalized distribution and its ability in providing relatively more accurate estimation of income inequality have recently been confirmed in a book by [16], who utilize household income data for 45 countries selected from the most recent waves of the LIS Database (<http://www.lisdatacenter.org/>).

The previously mentioned works were mainly concerned with the distribution of household incomes. In an interesting contribution by [21], the  $\kappa$ -generalized distribution was used in a three-component mixture to model the US net wealth data for 1984–2011. Both graphical procedures and statistical methods indicate an overall good approximation of the data. The authors also highlight the relative merits of their specification with respect to finite mixture models based upon the Singh-Maddala and Dagum type I distributions for the positive values of net wealth. Similar results were recently obtained by [16] when analyzing net wealth data for 9 countries selected from the most recent waves of the LWS Database (<http://www.lisdatacenter.org/>).

Finally, four-parameter extensions of the  $\kappa$ -generalized distribution were used by [61] to analyze household income/consumption data for approximately 20 countries selected from Waves IV to VI of the LIS Database. To provide a comparison with alternative four-parameter models of income distribution, the GB2 and the double Pareto-lognormal (dPIN) distribution introduced by [74] were also fitted to the same data sets. In almost all cases, the new variants of the  $\kappa$ -generalized distribution outperform the other four-parameter models for both the income and consumption variables. In particular, they show an empirical tendency to estimate inequality indices more accurately than they counterparts do.

Given the excellent performance of the  $\kappa$ -generalized family of distributions, documented through several years of research, we shall assume in the following that consumption and income data can be modeled by non-identical three-parameter  $\kappa$ -generalized distributions, henceforth denoted by  $F_c(x_c; \alpha_c, \beta_c, \kappa_c)$  and  $F_i(x_i; \alpha_i, \beta_i, \kappa_i)$ , where the subscripts  $c$  and  $i$  clearly refer to consumption and income, respectively.

Parameter estimation will be performed using the maximum goodness-of-fit (MGF) estimation method [54], also known as the “minimum distance estimation method”

GB2.

[42, 82, 83]. MGF estimation consists in maximizing goodness of fit—or minimizing a goodness-of-fit distance—between the empirical distribution function (EDF) of the sample and the CDF of the specified distribution. This method is suitable for estimating distribution parameters of data characterized by both skewness and heavy tails, since in such cases other commonly used estimation techniques (e.g. maximum likelihood approach) can lose their optimality properties [54].

In what follows, the distance measure that will be minimized in order to fit the  $\kappa$ -generalized distribution to the SHIW micro-data on income and consumption is the so-called “right-tail Anderson-Darling statistic of second degree”, defined as

$$\text{AD2R} = 2 \sum_{j=1}^n \ln(1 - z_j) + \frac{1}{n} \sum_{j=1}^n \frac{2j - 1}{1 - z_{n+1-j}}, \quad (6)$$

where  $z_j = F(x_j; \boldsymbol{\gamma})$  is the point-wise  $\kappa$ -generalized CDF of income or consumption,  $\boldsymbol{\gamma} = \{\alpha, \beta, \kappa\}$  is the unknown parameters vector and  $n$  is the sample size. The AD2R, one of the variants of the Anderson and Darling’s distance [2] proposed by [54], assigns more weight to the right tail of the distribution, and thus is particularly indicated to accommodate both heavy-tailedness and positive skewness in data.

Minimization of the AD2R statistics with respect to the unknown parameters of the  $\kappa$ -generalized CDF (4) will be performed by numerical methods using optimization routines from the `fitdistrplus` package [27] implemented in the R programming language [72].

### 2.3. The symmetrized Joe-Clayton copula

Often the issue of dependence between random variates is addressed through the concept of correlation. However, for non-normal variables more complex, non-linear dependence structures can arise when considering their joint distributions (e.g. [80]).<sup>12</sup> Copula-based multivariate models are becoming an increasingly popular approach to modeling joint distributions as they make it possible for a wide range of dependence structures to be captured beyond simply correlation.

Popularized by Sklar [78], copula-based models allow the researcher to specify the

<sup>12</sup>Non-normality is usually the case when analyzing income and consumption data, because of skewness and fat tails (kurtosis) in their distributions. An obvious consequence is that correlation can be misleading when analyzing their degree of association. For example, Pearson’s correlation coefficient—by far the most widely applied correlation concept in statistics—is known to be sensitive towards extreme events, which are more likely to occur with fat tails than is predicted by normal distribution (e.g. [56] and [23]). Furthermore, Pearson’s correlation coefficient measures the degree of linear association between two random variates, but usually this does not sufficiently describe association between non-normally distributed random variables [32]. In particular, the concept of correlation is not defined for some heavy-tailed distributions whose second moments do not exist (see e.g. [73]).

models for the marginal distributions separately from the dependence structure (copula) that links these distributions. In particular, a substantial advantage of copula-based methods is that the models for the marginal distributions may come from different families. This frees researchers from considering only existing multivariate distributions, and allows for a much greater degree of flexibility in forming the joint distribution.

More in detail, a copula is a multivariate distribution which is defined on the  $[0, 1]^d$  hypercube, where each of the  $d$  marginal variates is uniformly distributed.<sup>13</sup> That is, given a set of  $d$  random variates  $X_1, \dots, X_d$  with cumulative distribution function  $F_1(x_1), \dots, F_d(x_d)$ , then each can be transformed into marginal variates defined on the unit interval  $[0, 1]$  using  $u_i \sim F_i(x_i)$ , for  $i = 1, \dots, d$ . Each variate also has an inverse cumulative distribution function such that, for  $i = 1, \dots, d$ ,  $x_i \sim F_i^{-1}(u_i)$ .

Under Sklar’s theorem [78], if the joint cumulative distribution of  $X_1, \dots, X_d$  is given by some function  $H(x_1, \dots, x_d)$ , then there exists a copula function  $C(u_1, \dots, u_d)$  with margins  $F_1(x_1), \dots, F_d(x_d)$  such that

$$\begin{aligned} H(x_1, \dots, x_d) &= H(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)) \\ &= C(F_1(x_1), \dots, F_d(x_d)) \\ &= C(u_1, \dots, u_d). \end{aligned} \tag{7}$$

Thus, the joint distribution is expressed in terms of its respective marginal distributions and a function  $C$  that binds them together. This makes modeling the dependence between the uniformly distributed margins equivalent to modeling the dependence between the variates themselves. In case that the multivariate distribution has a density  $h$ , and this is available, it holds further that

$$h(x_1, \dots, x_d) = c(u_1, \dots, u_d) \times f_1(x_1) \times \dots \times f_d(x_d), \tag{8}$$

where

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \tag{9}$$

is the density of the copula.

If the marginal distributions  $F_1(x_1), \dots, F_d(x_d)$  are continuous, then the corresponding copula in Equation (7) is unique. If  $F_1(x_1), \dots, F_d(x_d)$  are not all continuous, the joint distribution function can always be expressed as (7), although in such a case the copula is not unique [76, ch. 6].

There exist many copula functions that could be used in dependence modeling, especially for the bi-variate case—a fairly exhaustive list is contained, e.g., in [40,

<sup>13</sup>[38] and [59] are the two comprehensive treatments on this topic. A detailed review and discussion of copula theory is also given, among others, in [31], [80], [1], [7], [68], and [40].

ch. 4].<sup>14</sup> All of these functions depend on one or more parameters, say  $\theta$ , which are called association parameters and are related to the degree of dependence between margins. Common measures of the amount of association between two variables, such as Kendall’s  $\tau$  and Spearman’s  $\rho_s$  among others, are usually expressed as function of the association parameters. For instance, Kendall’s  $\tau$  can be written as  $\tau(\theta) = 4 \int_0^1 \int_0^1 C(u_c, u_i; \theta) c(u_c, u_i; \theta) du_c du_i - 1$ , while Spearman’s  $\rho_s$  is written in terms of copula as  $\rho_s(\theta) = 12 \int_0^1 \int_0^1 C(u_c, u_i; \theta) du_c du_i - 3$  [59, ch. 5]. In particular, these measures do not depend on parameters of the marginal distributions but on the association parameters only.<sup>15</sup>

It is worth noticing, however, that the  $\tau$  and  $\rho_s$  values corresponding to the  $\theta$  domain do not necessarily cover the whole dependence interval  $[-1, 1]$ , and the range of dependence that can be really achieved varies for different copulas. Therefore, a key point to consider should be choosing an appropriate copula from the competitive functions whose association parameter lies within a range that allows  $\tau$  and  $\rho_s$  to cover at least their empirical values, or more generally, the positive dependence domain  $[0, 1]$ .<sup>16</sup>

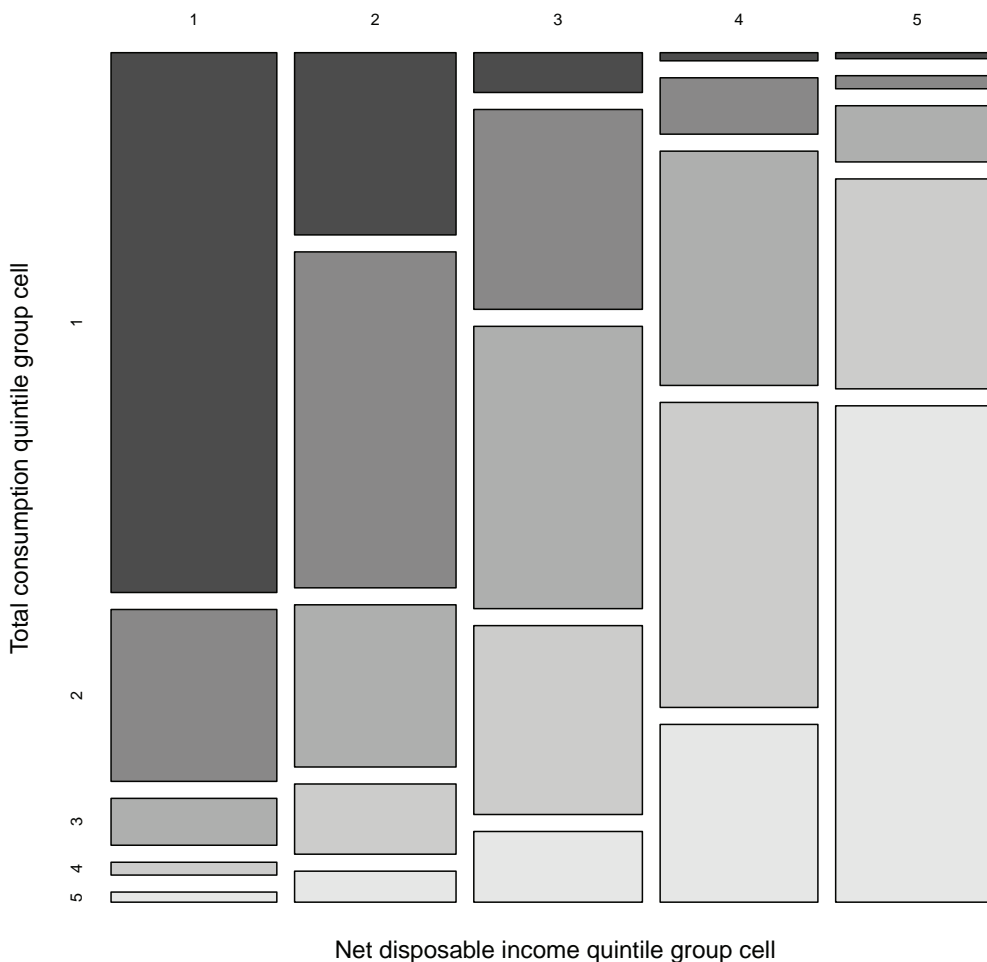
Information on “tail dependence” is also useful for making initial decisions on the types of copulas that may be suitable for a given data set, since many copula models—such as the normal and Frank ones—impose zero tail dependence in both tails, whereas other copulas impose zero tail dependence in one of their tails (e.g. right for the Clayton copula and left for the Gumbel copula).<sup>17</sup> Tail dependence is a measure of the strength of dependence in the joint upper (lower) tail of a bi-variate distribution. Informally, in our application it measures the probability that large (or

<sup>14</sup>Henceforth, we will concentrate on the bi-variate case, i. e. when  $d = 2$ , since it will be later considered in the empirical analysis. Accordingly, by “copula” we will always mean bi-variate copulas for modeling the dependence between income and consumption distributions.

<sup>15</sup>The Pearson’s correlation coefficient is a poor measure of the association between two variables. In particular, it is not invariant under general non-linear, strictly increasing transformations of the variables—e.g.  $\rho(X_c, X_i) \neq \rho[\exp(X_c), \exp(X_i)]$ —and is affected by the marginal distributions of the data (see, *inter alia*, [49], [32], [80], and [68]). This is equivalent to imposing that a better measure of dependence should be obtained as a function of the *ranks* of the data only, which is in turn equivalent to it being a function solely of the copula, and not the marginal distributions. Both Kendall’s  $\tau$  and Spearman’s  $\rho_s$  are invariant under non-linear, strictly increasing transformations and, as seen in the main text, they can be expressed in terms of the associated copula.

<sup>16</sup>For instance, the Farlie-Gumbel-Morgenstern copula allows only for a limited degree of dependence (Kendall’s  $\tau$  is restricted to  $[-\frac{2}{9}, \frac{2}{9}]$  and Spearman’s  $\rho_s$  to  $[-\frac{1}{3}, \frac{1}{3}]$ ), which reduces its appeal for use in applications (e.g. [40, p. 213]). Similar considerations hold also for the Ali-Mikhail-Haq copula, one of the members of the so-called Archimedean family of copulas, whose range for Kendall’s  $\tau$  is restricted to  $[-0.18, 0.33]$  and for Spearman’s  $\rho_s$  to  $[-0.27, 0.48]$ , so that it can only model weak dependence (e.g. [59]).

<sup>17</sup>For more on this issue, see [59, ch. 4], [40, ch. 2], and [68].

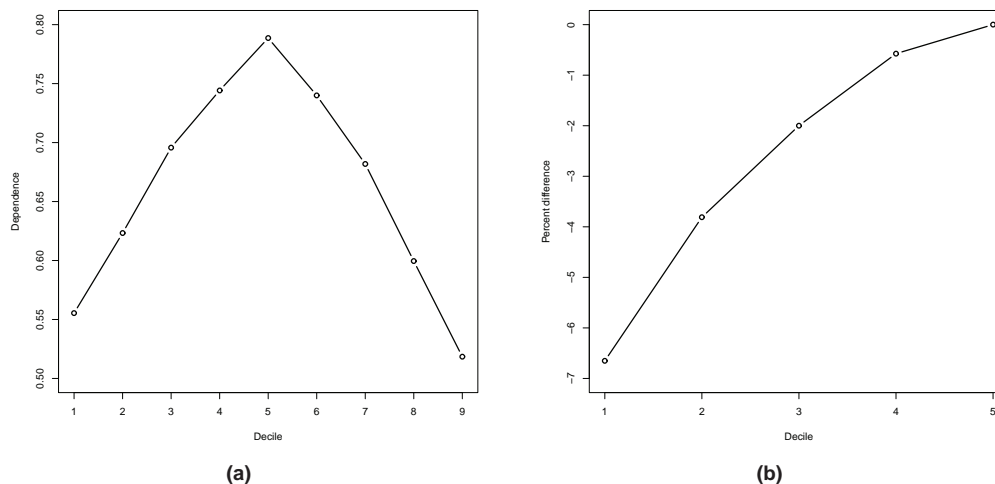


**Figure 6** Distribution across quintile groups of total consumption and net disposable income for the 2014 wave of the SHIW

small) values of consumption appear with large (or small) values of income.

Tail dependence in the income-consumption data for the 2014 wave of the SHIW can be seen in Figure 6, where we show the cross-tabulation of the quintile groups of both resource variables.<sup>18</sup> The bars denote the proportion of households found in the quintile groups of both the income and consumption distributions. The most striking feature is that for households in the top quintile group of disposable income

<sup>18</sup>The plots for the other waves resemble to Figure 6, therefore they are not shown here—but available from the authors upon request.



**Figure 7** Asymmetric tail dependence for the 2000 wave of the SHIW: (a) estimated quantile dependence between income and consumption data; (b) percent difference between corresponding upper and lower quantile dependence estimates

(the five stacked bars that are furthest to the right in the figure) the probability of being in a particular quintile group of total consumption increases very quickly—that is, it is more likely that a household in the top-income quintile group is also in the top consumption group rather than in one of the other four. Likewise, the probability that a lowest-income quintile group household is in the lowest consumption group is higher than such a household being in the second to fifth groups of consumption (see the five stacked bars that are furthest to the left). Thus total consumption and disposable income are highly associated, particularly at the top and bottom of the distribution.

In the SHIW income-consumption data we also find evidence of *asymmetric* tail dependence, in that observations in the lower tail of the bi-variate distribution are somewhat more dependent than observations in the upper tail. This can be seen from Figure 7, where panel (a) presents the estimated quantile dependence plot and panel (b) displays the difference between the upper and lower portions of this plot, respectively.<sup>19</sup> “Quantile dependence” measures the strength of the dependence between two variables in the joint lower or upper tails of their support—see [68, p. 909] for a more formal definition. By estimating the strength of the dependence between two variables as we move from the center to the tails, and by comparing the left tail to the right

<sup>19</sup>The results depicted in the figure focus on the 2000 wave of the SHIW and use deciles as the quantiles of choice—hence the differences are calculated as  $\frac{q_{u-l}}{q_l} \times 100$ , for  $\{(u, l)\} = \{(90, 10), (80, 20), (70, 30), (60, 40), (50, 50)\}$ . The results for the other waves (not shown) are pretty similar and can be obtained on request.



tail, we are thus provided with a richer description of the dependence structure of the two variables. The figure clearly shows that income-consumption pairs at the bottom of the joint distribution are more dependent than observations in the upper tail, with the relative difference between corresponding quantile dependence estimates being as high as nearly 7%.

The above evidence compels us to be flexible in selecting the copula function to use in our empirical analysis. In particular, it should allow for asymmetric positive dependence in either direction. Since some of the copulas presented in the statistics literature impose zero tail dependence in one or both of the tails (e.g. normal, Frank, Clayton and Gumbel) while other copulas such as the Student’s  $t$  allow for positive and symmetric dependence in both tails, these functions are not considered in this chapter. Rather, our choice falls on the “symmetrized Joe-Clayton” copula used in [67]. The symmetrized Joe-Clayton (SJC) copula is given by

$$C_{\text{SJC}}(u_c, u_i; \tau^U, \tau^L) = \frac{1}{2} \times [C_{\text{JC}}(u_c, u_i; \tau^U, \tau^L) + C_{\text{JC}}(1 - u_c, 1 - u_i; \tau^L, \tau^U) + u_c + u_i - 1], \quad (10)$$

where  $C_{\text{JC}}(u_c, u_i; \tau^U, \tau^L)$  is the Joe-Clayton copula defined as

$$C_{\text{JC}}(u_c, u_i; \tau^U, \tau^L) = 1 - \left( 1 - \left\{ [1 - (1 - u_c)^k]^{-r} + [1 - (1 - u_i)^k]^{-r} - 1 \right\}^{-\frac{1}{r}} \right)^{\frac{1}{k}} \quad (11)$$

with  $k = \frac{1}{\log_2(2 - \tau^U)}$  and  $r = -\frac{1}{\log_2(\tau^L)}$  [38].

The copula functional form (10) has two parameters,  $\tau^U$  and  $\tau^L$ , which are measures of tail dependence. The SJC copula exhibits lower tail dependence if  $\tau^L \in (0, 1]$  and no lower tail dependence if  $\tau^L = 0$ ; similarly, it exhibits upper tail dependence if  $\tau^U \in (0, 1]$  and no upper tail dependence if  $\tau^U = 0$ . By construction, the SJC copula also nests symmetry as a special case, which occurs when  $\tau^U = \tau^L$ . From an empirical perspective, the fact that this copula is flexible enough to allow for both upper- and lower-tail asymmetric dependence—with symmetric dependence as a special case—makes it a more interesting specification than many other copulas.<sup>20</sup>

A commonly used procedure will be adopted for estimating the parameters  $\tau^U$  and  $\tau^L$  of the SJC copula. The method is called “inference functions for margins” (IFM) and was introduced by [41]. It consists of two steps: the parameters of the marginal distributions are estimated separately in the first step and then, given these, the pro-

<sup>20</sup>Unfortunately, there is no simple closed-form expression for Kendall’s  $\tau$  and Spearman’s  $\rho_s$  in terms of the SJC copula parameters. In the following, the accuracy of the chosen copula will be thus assessed by comparing the actual measures of association to their values computed from observations drawn at random from the SJC-copula-based joint distribution of income and consumption (see Section 3.2).

cedure calculates the estimate of the association parameters of the copula function. Here, this means that the MGF estimates  $\hat{\gamma}_c$  and  $\hat{\gamma}_i$  of the  $\kappa$ -generalized distributions for consumption and income margins are provided in the first step. They are then plugged into the log-likelihood function

$$l(u_c, u_i; \tau^U, \tau^L) = \sum_{j=1}^n \ln [c_{\text{SJC}}(u_c^j, u_i^j; \tau^U, \tau^L)] \quad (12)$$

that is maximized with respect to  $\tau^U$  and  $\tau^L$ , where

$$\begin{aligned} c_{\text{SJC}}(u_c, u_i; \tau^U, \tau^L) &= \frac{\partial^2 C_{\text{SJC}}(u_c, u_i; \tau^U, \tau^L)}{\partial u_c \partial u_i} \\ &= \frac{1}{2} \left[ \frac{\partial^2 C_{\text{JC}}(u_c, u_i; \tau^U, \tau^L)}{\partial u_c \partial u_i} + \frac{\partial^2 C_{\text{JC}}(1 - u_c, 1 - u_i; \tau^U, \tau^L)}{\partial (1 - u_c) \partial (1 - u_i)} \right] \end{aligned} \quad (13)$$

is the density of the SJC copula (10) and  $u_c = F_c(x_c; \hat{\gamma}_c)$ ,  $u_i = F_i(x_i; \hat{\gamma}_i)$  are the estimated  $\kappa$ -generalized cumulative probabilities of consumption and income, respectively.<sup>21</sup> [39] showed that the traditional asymptotic properties of the maximum likelihood estimates still hold for the IFM estimates.

### 3. Results

#### 3.1. Parametric marginal distributions of income and consumption

Estimates by MGF of the parameters of the two marginal distributions for each wave are shown in Table 1. Also displayed are the estimated standard errors, obtained by numerically evaluating the Hessian of the negative log-likelihood under both the data

<sup>21</sup>In Equation (13), the expression for  $\frac{\partial^2 C_{\text{JC}}(1 - u_c, 1 - u_i; \tau^U, \tau^L)}{\partial (1 - u_c) \partial (1 - u_i)}$  is the same as

$$\begin{aligned} \frac{\partial^2 C_{\text{JC}}(u_c, u_i; \tau^U, \tau^L)}{\partial u_c \partial u_i} &= (AB)^{-r-1} (1 - u_c)^{k-1} (1 - u_i)^{k-1} \left\{ \left[ 1 - (A^{-r} + B^{-r} - 1)^{-\frac{1}{r}} \right]^{-1 + \frac{1}{k}} \right. \\ &\quad \left. (A^{-r} + B^{-r} - 1)^{-2 - \frac{1}{r}} (1 + r) k + \left[ 1 - (A^{-r} + B^{-r} - 1)^{-\frac{1}{r}} \right]^{-2 + \frac{1}{k}} \right. \\ &\quad \left. (A^{-r} + B^{-r} - 1)^{-2 - \frac{2}{r}} (k - 1) \right\}, \end{aligned}$$

where  $A = 1 - (1 - u_c)^k$  and  $B = 1 - (1 - u_i)^k$ , but we substitute  $u_c$  and  $u_i$  in the latter with  $1 - u_c$  and  $1 - u_i$  to get the former. Also note that  $k = \frac{1}{\log_2(2 - \tau^L)}$  and  $r = -\frac{1}{\log_2(\tau^U)}$  for the former.

Table 1 Estimated  $\kappa$ -generalized distributions of income and consumption for SHIW data, 1987–2014<sup>a</sup>

Wave	Total consumption			Net disposable income		
	$\hat{\alpha}_c$	$\hat{\beta}_c$	$\hat{\kappa}_c$	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\hat{\kappa}_i$
1987	2.34 (0.03)	13,403 (98)	0.63 (0.03)	2.20 (0.05)	16,892 (93)	0.61 (0.03)
1989	2.89 (0.04)	13,766 (87)	0.84 (0.03)	2.45 (0.06)	18,098 (87)	0.74 (0.03)
1991	2.54 (0.03)	13,669 (92)	0.72 (0.03)	2.41 (0.06)	17,987 (89)	0.68 (0.03)
1993	2.85 (0.04)	12,838 (88)	0.93 (0.03)	2.10 (0.06)	17,674 (91)	0.65 (0.03)
1995	2.65 (0.04)	13,306 (87)	0.75 (0.03)	2.32 (0.06)	16,690 (89)	0.79 (0.03)
1998	2.76 (0.04)	13,021 (100)	0.91 (0.03)	2.52 (0.07)	17,823 (96)	0.91 (0.03)
2000	2.49 (0.03)	13,950 (95)	0.69 (0.03)	2.46 (0.06)	18,569 (93)	0.81 (0.03)
2002	2.59 (0.04)	13,951 (96)	0.76 (0.03)	2.46 (0.06)	18,679 (92)	0.77 (0.03)
2004	2.63 (0.04)	14,913 (100)	0.76 (0.03)	2.78 (0.05)	18,706 (94)	0.99 (0.03)
2006	2.78 (0.04)	15,308 (99)	0.78 (0.03)	2.53 (0.06)	20,123 (98)	0.72 (0.03)
2008	2.90 (0.04)	14,884 (90)	0.76 (0.03)	2.64 (0.08)	19,489 (93)	0.82 (0.03)
2010	2.85 (0.04)	15,333 (99)	0.83 (0.03)	2.52 (0.06)	19,915 (103)	0.79 (0.03)
2012	2.61 (0.04)	15,190 (96)	0.67 (0.03)	2.35 (0.05)	18,426 (97)	0.71 (0.02)
2014	2.79 (0.04)	13,725 (86)	0.76 (0.03)	2.39 (0.08)	18,661 (91)	0.67 (0.03)

<sup>a</sup> Numbers in parentheses: estimated standard errors

and the estimated  $\kappa$ -generalized parameters.<sup>22</sup> Convergence was achieved easily within a few iterations.

The model fit varied slightly across years but was generally excellent, as indicated by the small value of the errors. This is demonstrated by the plots shown in Figures 8 and 9 for the most recent data available (for brevity, we do not report plots for each year but they are available from the authors on request). In fact, the fitted cumulative function well approximates the empirical curve in panels (a); the  $\kappa$ -generalized and empirical densities match appropriately in panels (b), and the points in the Q-Q plots (c) comparing sample quantiles with the theoretical quantiles computed from the model lie extremely close to the 45° ray from the origin—except for a few extreme values—and much closer than is typically observed in plots of this type. Moreover, the double-logarithmic plots in panels (d) show how the  $\kappa$ -generalized performs particularly well in the top part of the empirical distributions.

Thus, the overall fit of the  $\kappa$ -generalized distribution is extremely satisfactory.

### 3.2. The joint distribution of income and consumption

Table 2 presents the estimated parameters of the SJC copula along with asymptotic standard errors.<sup>23</sup> Given parameter estimates, the joint cumulative distribution of income and consumption is easily derived from Equation (7) as

$$H(x_c, x_i; \hat{\gamma}_c, \hat{\gamma}_i, \tau^U, \tau^L) = C(F_c(x_c; \hat{\gamma}_c), F_i(x_i; \hat{\gamma}_i); \tau^U, \tau^L), \quad (14)$$

i.e. by coupling together the  $\kappa$ -generalized marginal distributions of income and consumption via the SJC copula estimation.

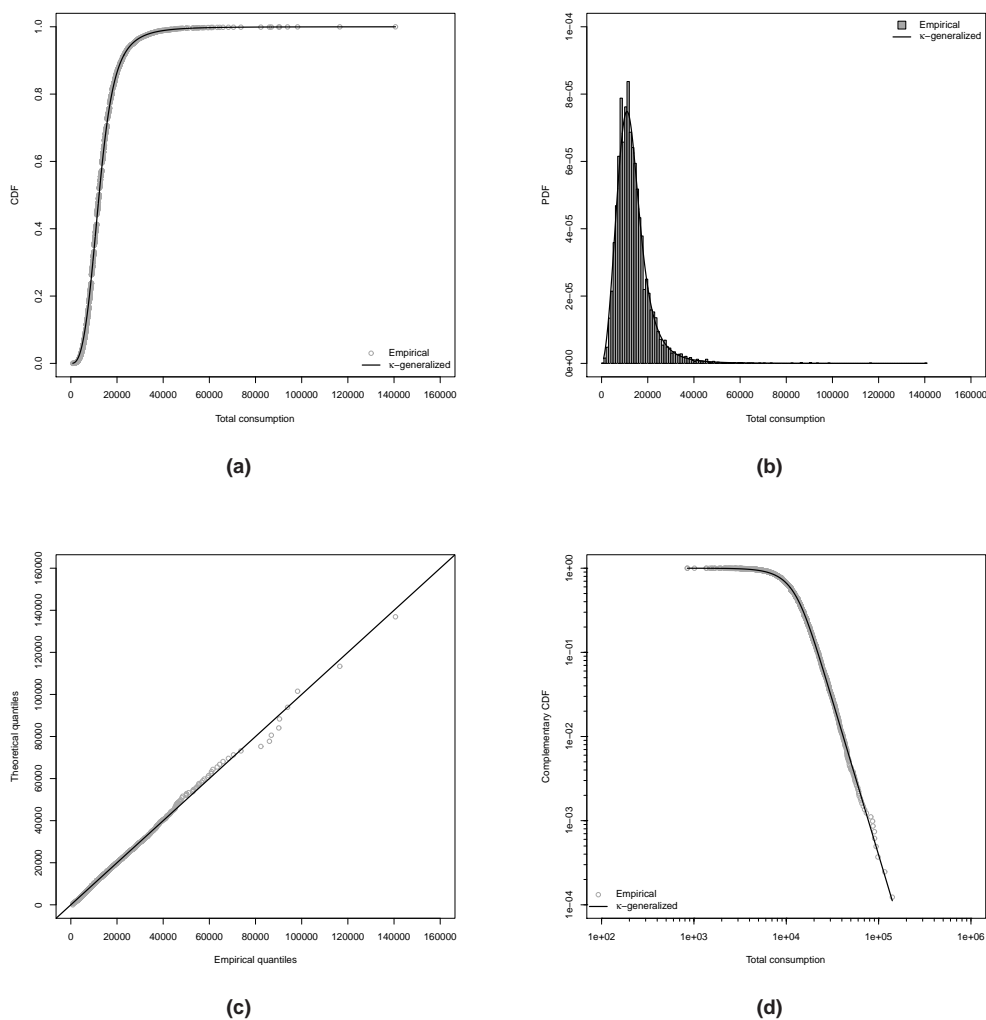
The overall goodness of fit of the bi-variate model (14) can be gauged in two ways.

<sup>22</sup>The  $\kappa$ -generalized log-likelihood for a complete random sample of size  $n$  is

$$l(x; \alpha, \beta, \kappa) = n \ln(\alpha) - n\alpha \ln(\beta) + (\alpha - 1) \sum_{j=1}^n \ln(x_j) + \frac{1}{\kappa} \sum_{j=1}^n \ln \left[ \sqrt{1 + \kappa^2 \left(\frac{x_j}{\beta}\right)^{2\alpha}} - \kappa \left(\frac{x_j}{\beta}\right)^\alpha \right] - \frac{1}{2} \sum_{j=1}^n \ln \left[ 1 + \kappa^2 \left(\frac{x_j}{\beta}\right)^{2\alpha} \right],$$

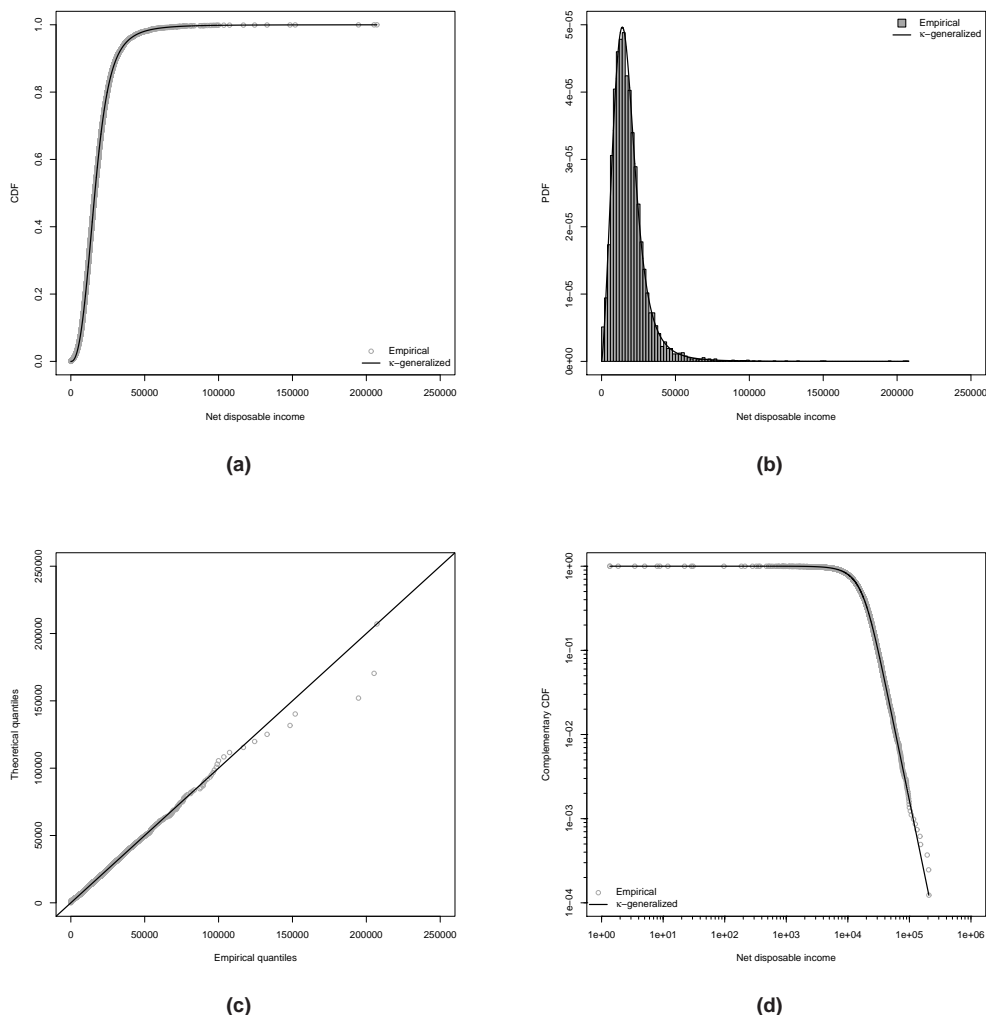
where the consumption and income subscripts have been omitted for notational convenience. By numerically evaluating the Hessian  $H(\hat{\gamma})$  of the negative of  $l(\mathbf{x}; \hat{\gamma})$  under both the data  $\mathbf{x} = \{x_1, \dots, x_n\}$  and the estimated  $\kappa$ -generalized parameters  $\hat{\gamma} = \{\hat{\alpha}, \hat{\beta}, \hat{\kappa}\}$ , the sampling covariance of the MGF estimates has been estimated from the Fisher information as  $V_{\hat{\gamma}} = H^{-1}(\hat{\gamma})$ . The standard errors for each of the unknown  $\gamma = \{\alpha, \beta, \kappa\}$  have been finally obtained as the square roots of the off-diagonal elements of  $V_{\hat{\gamma}}$ . To calculate numerical approximations to the Hessian matrix  $H(\hat{\gamma})$  at the estimated parameter values, we use here the R function `hessian` from the library `numDeriv` [34].

<sup>23</sup>The SJC copula for all waves of the SHIW was estimated using MATLAB code provided by Andrew Patton to replicate the results presented in Patton [68]. The code is freely available at: <http://public.econ.duke.edu/~ap172/code.html>.



**Figure 8** Adequacy of the  $\kappa$ -generalized distribution for the SHIW consumption data, 2014: (a) empirical and fitted CDFs; (b) empirical and fitted PDFs; (c) Q-Q plot; (d) empirical and fitted complementary CDFs

First, we compare measures of association derived from the parameter estimates to the statistics computed from the raw data. Since no closed-form expression exists for deriving the various association measures from the SJC copula parameters, our estimation is based on Monte Carlo sampling using the parametric models and their estimated parameters. That is, we simulate pseudo-samples of income and consumption pairs for each wave of the SHIW based on the inverse sampling method



**Figure 9** Adequacy of the  $\kappa$ -generalized distribution for the SHIW income data, 2014: (a) empirical and fitted CDFs; (b) empirical and fitted PDFs; (c) Q-Q plot; (d) empirical and fitted complementary CDFs

(e.g. [59]): we first draw  $n$  correlated pairs of uniformly distributed variates  $(u_c, u_i)$ , where  $n$  equals the original sample size and the correlation is determined by the SJC copula parameters, and then we generate the consumption and income pairs as  $(x_c = F_c^{-1}(u_c; \hat{\gamma}_c), x_i = F_i^{-1}(u_i; \hat{\gamma}_i))$ , i.e. the  $x_c$ -th and  $x_i$ -th theoretical quantiles implied by the parameter estimates of the marginal distributions. Model-based measures of association are finally obtained by performing standard calculations on the pseudo-

**Table 2** Parameter estimates for the SJC copula, 1987–2014<sup>a</sup>

Wave	$\tau^U$	$\tau^L$
1987	0.64 (0.01)	0.78 (0.00)
1989	0.63 (0.01)	0.79 (0.00)
1991	0.60 (0.01)	0.79 (0.00)
1993	0.57 (0.01)	0.62 (0.01)
1995	0.58 (0.01)	0.68 (0.01)
1998	0.57 (0.01)	0.45 (0.01)
2000	0.56 (0.01)	0.55 (0.01)
2002	0.56 (0.01)	0.55 (0.01)
2004	0.59 (0.01)	0.55 (0.01)
2006	0.57 (0.01)	0.63 (0.01)
2008	0.56 (0.01)	0.64 (0.01)
2010	0.65 (0.01)	0.61 (0.01)
2012	0.66 (0.01)	0.65 (0.01)
2014	0.65 (0.01)	0.57 (0.01)

<sup>a</sup> Numbers in parentheses: estimated standard errors

samples.

Figure 10 shows model-based predictions of the Spearman’s and Kendall’s correlation coefficients. As can be seen, measures of association computed from simulated data reproduce well the time-varying profile of dependence observed in Figure 1, confirming that the SJC copula can give an adequate description of the dependence structure in the Italian income-consumption data.

A second approach to assessing whether our model really conform with data consists in generating a probability plot of the theoretical joint CDF given by Equation (14) against the empirical copula,<sup>24</sup> as shown in Figure 11 for the 2014 wave.<sup>25</sup> The 45° line from (0, 0) to (1, 1) is the comparison line: the cumulative distributions are equal if the plot falls approximately on this line, whereas any deviation from it indi-

<sup>24</sup>Similar to the empirical distribution, the empirical copula can be defined for multivariate data after a transform to ranks. Suppose data are realizations from a continuous bi-variate distribution of size  $n$ . The empirical copula is then the empirical distribution function corresponding to [59, p. 219]

$$C_n\left(\frac{j}{n}, \frac{k}{n}\right) = \frac{\text{number of pairs } (x_c, x_i) \text{ in the sample with } x_c \leq x_c^{(j)} \text{ and } x_i \leq x_i^{(k)}}{n}, \quad 1 \leq j, k \leq n,$$

where  $x_c^{(j)}$  and  $x_i^{(k)}$  denote order statistics from the sample.

<sup>25</sup>Plots for the other waves of the SHIW are similar and can be obtained on request.

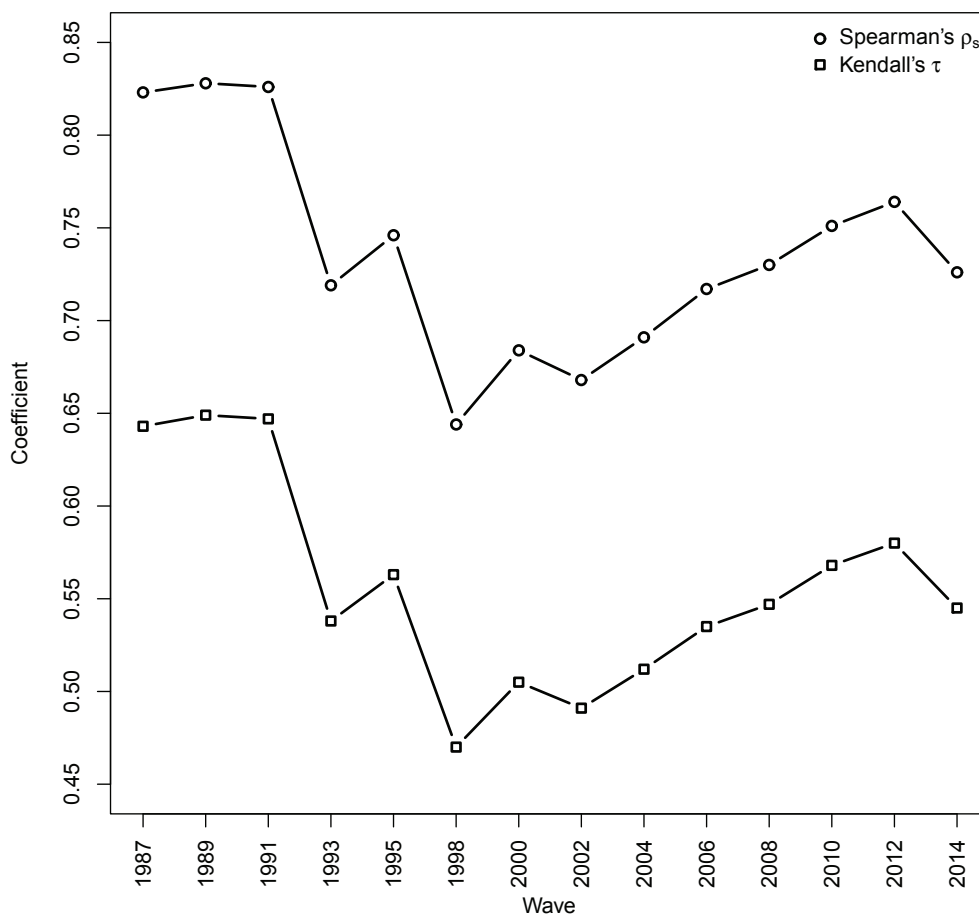


Figure 10 Model-based predictions of the Spearman's and Kendall's correlation coefficients, 1987–2014

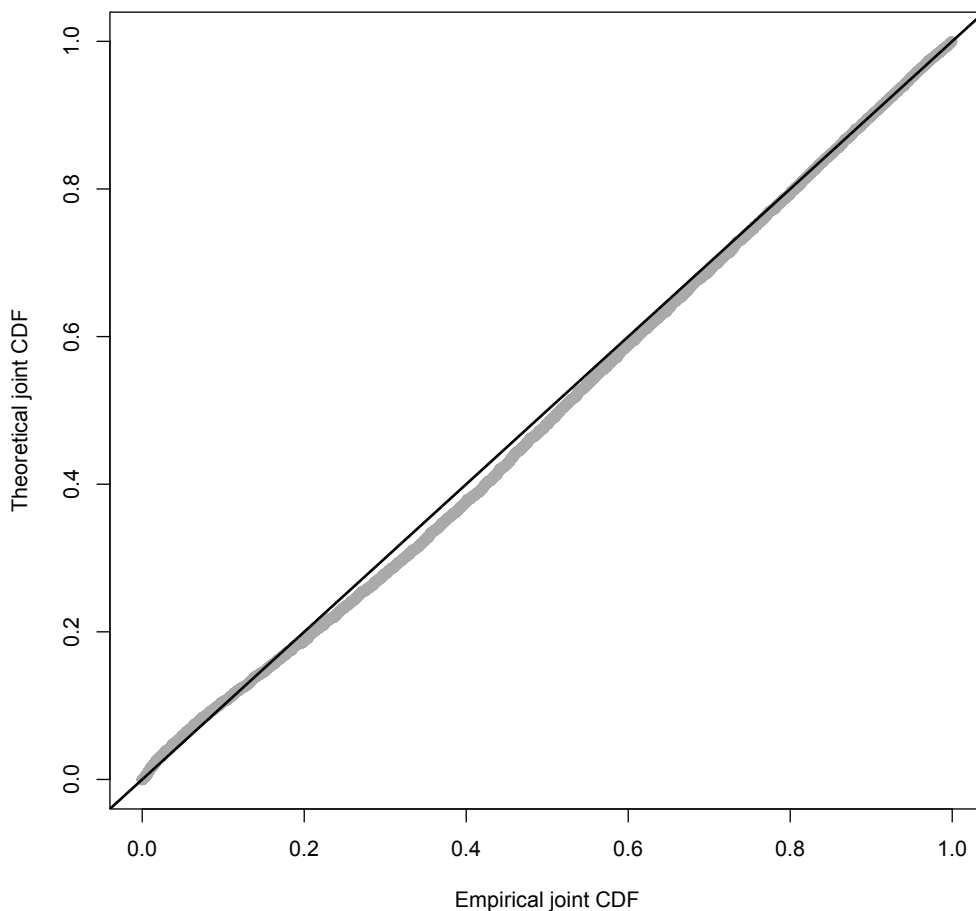
cates a difference between the theoretical and empirical joint distributions of income and consumption. As can be seen, the points almost coincides with the comparison line and the majority of the probability plot is linear. Hence, the hypothesis that income and consumption can be modeled as non-identically  $\kappa$ -generalized distributed variables, and their dependence by a SJC copula, is not rejected for the examined data.

The estimated joint PDF of income and consumption

$$h(x_c, x_i; \hat{\gamma}_c, \hat{\gamma}_i, \tau^U, \tau^L) = c(F_c(x_c; \hat{\gamma}_c), F_i(x_i; \hat{\gamma}_i); \tau^U, \tau^L) \times f_c(x_c; \hat{\gamma}_c) \times f_i(x_i; \hat{\gamma}_i), \quad (15)$$

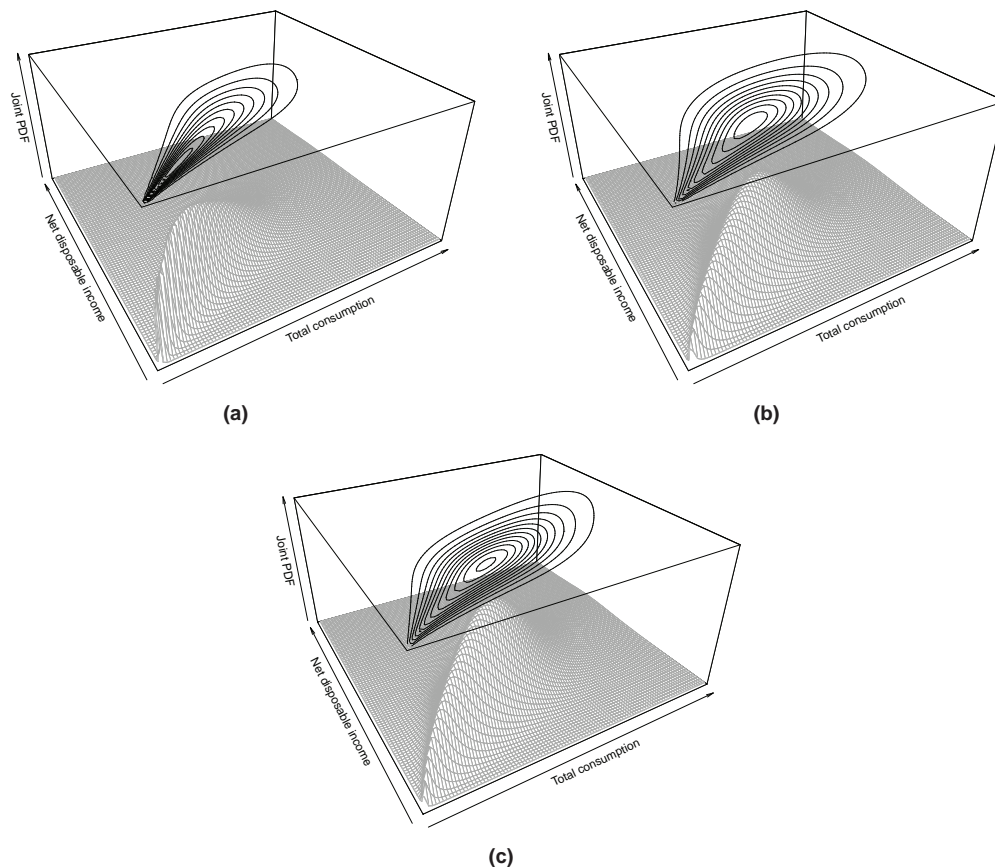
obtained as the product of the SJC copula density (13) with the  $\kappa$ -generalized distri-





**Figure 11** Adequacy of the SJC-copula-based joint distribution of income and consumption for the 2014 wave of the SHIW

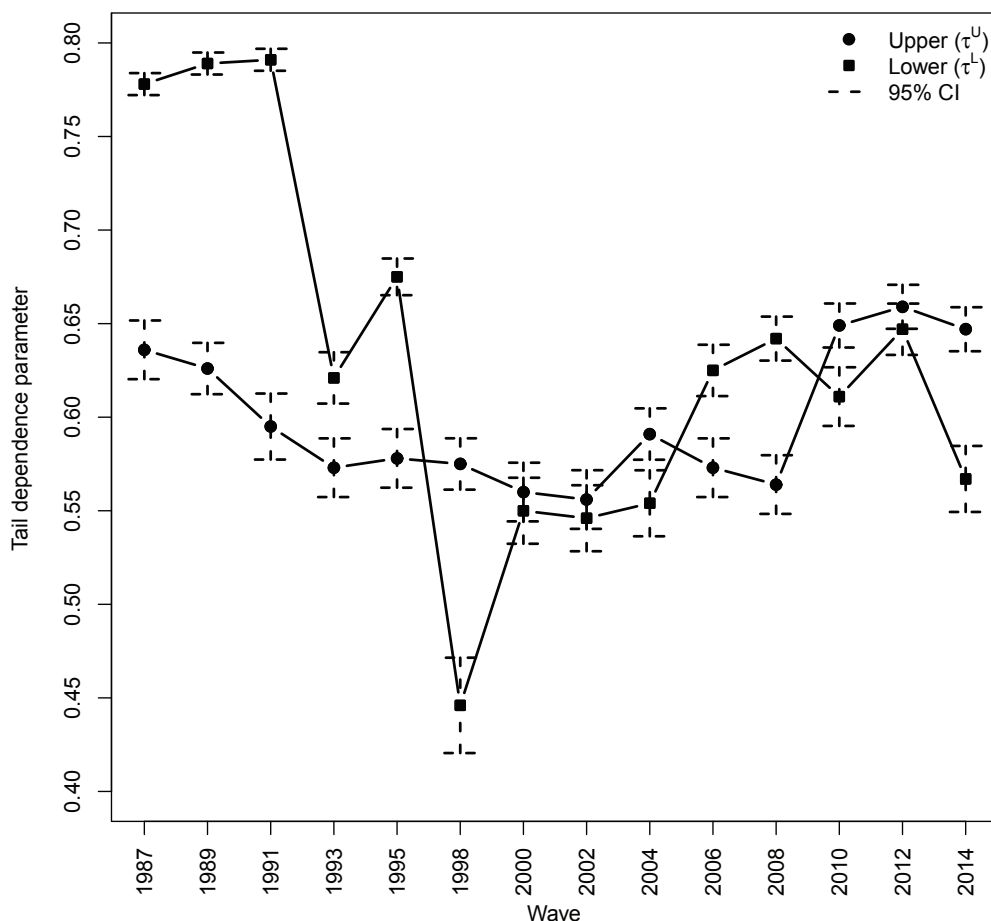
bution for margins given by Equation (5), is charted in Figure 12 for 1987, 2000, and 2014. The contours of the joint densities are also shown at the top to help visualize the overall pattern. There are a number of interesting features revealed by the bi-variate PDF graphs. The narrow profile of the contours of the distribution at the lower end of the income-consumption space suggests a strong positive dependence between the two variates for bottom-ranked households. By contrast, the round profile of the contours of the joint distribution at the upper end suggests a lesser degree of dependence between income and consumption for top-ranked households. Furthermore, there is a gradual evolution of the dependence structure between income and consumption during the sample period, especially seen in the wider contours of the 2000 joint distribu-



**Figure 12** Joint PDF of the bi-variate distribution of income and consumption in Italy: (a) 1987; (b) 2000; (c) 2014

tion: the dependence in the lower tail of the distribution is reduced compared to 1987 and 2014, although the contours remain narrower than those at the top-right corner; at the same time, by comparing contours of the joint distributions, we note that also the dependence in the upper tail is somewhat lower in 2000 than in 1987 and 2014.

The above evidence suggests that the dependence structure between income and consumption in Italy varied *asymmetrically* over the sample period. Figure 13 shows the degree of asymmetry implied by the SJC copula by plotting the upper and lower tail dependence estimates presented in Table 2 along with 95% (point-wise) confidence intervals for these estimates. The plot confirms that the change in dependence also took place in the tails of the joint distribution, with average tail dependence—defined as  $(\tau^U + \tau^L)/2$ —dropping from 0.71 in 1987 to 0.56 in 2000, and then rising to 0.61 in 2014. However, the level and the dynamics of dependence were both substantially



**Figure 13** Upper and lower tail dependence between income and consumption from the SJC copula, 1987–2014. Vertical bars denote 95% point-wise confidence intervals based on the standard normal distribution

different in the two tails of the income–consumption distribution: in the first part of the sample period, from 1987 to 2000, lower tail dependence was on average about 12% greater than upper tail dependence, as well as the average rate of decline for the former dependence was consistently higher than for the latter (respectively, an average decline of more than 2% per annum versus nearly 1%). By contrast, during the 2000s and up to 2014 the asymmetry pattern is reversed and somewhat weaker: upper tail dependence was on average about 1% greater than lower tail dependence and also grew faster than the latter (respectively, an average yearly increase of about 1.3% versus nearly 0.3%). In particular, Figure 13 shows quite clearly that the level of

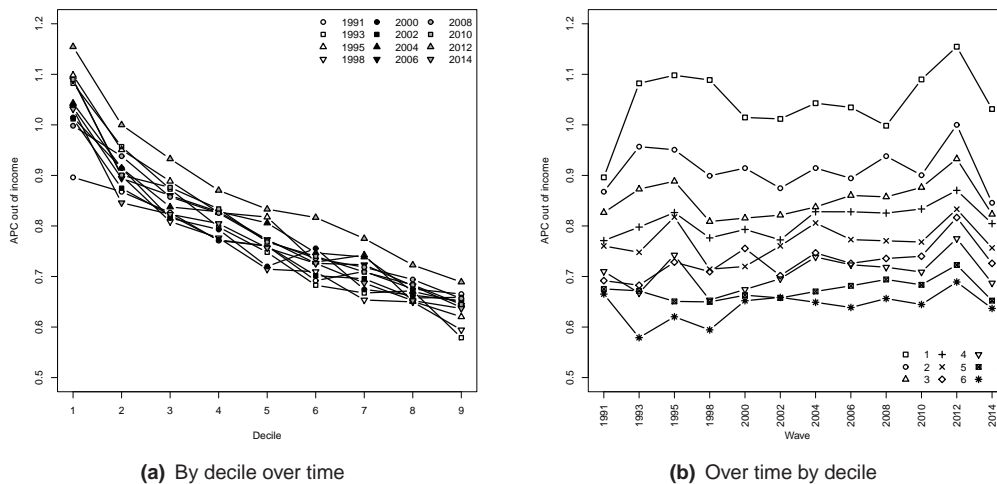


Figure 14 The average propensity to consume (APC) in Italy, 1991–2014

dependence between income and consumption increased markedly following the break that occurred around the Great Recession of 2008, and the dependence structure went from significantly asymmetric in one direction to weakly asymmetric in the opposite direction.

Overall, our results can be deemed to be consistent with compelling evidence that the propensity to consume declines as household income increases (e.g. [8], [51], [57], [13], [30], and [37]). This can be seen in panel (a) of Figure 14, where we plot estimates of the average propensity to consume (APC) of Italian households for the years 1991–2014.<sup>26</sup> The figure shows quite clearly that the APC, which measures the average association between total consumption and net disposable income, declines for all years when moving from the bottom towards the top of the income distribution, meaning that consumption and income are somewhat more dependent in the lower than in the upper tail. Furthermore, from panel (b) of the same figure we note that APC increased substantially for all income deciles around the outbreak of the 2008 crisis, following years of decline during the 1990s and of relative stagnation during the first half of the 2000s. Thus, it appears that consumption became more dependent on income toward the end of the sample period in all deciles of the distribution, thereby reducing the degree of asymmetry in the dependence structure of the Italian income-consumption data.

<sup>26</sup>Appendix A provides detailed specification and estimates of the consumption function used to gauge the propensity to consume based on the SHIW data.

#### 4. Conclusions and directions for future research

The purpose of this work was to provide a contribution to the estimation of joint distributions where both the variables are dependent and parametrically distributed. Since the independence between household income and consumption variables is not the most appropriate assumption to work with, the approach concerned modeling the bi-variate distribution of income and consumption in Italy with uni-variate margins belonging to a given parametric family, and a copula function which summarizes the existing dependence structure. To do this, we applied the “symmetrized Joe-Clayton” copula to model the dependence between income and consumption margins whose non-identical distributions belong to the  $\kappa$ -generalized family.

The proposed copula-based approach was found to capture well the complex dependence between income and consumption observed in our samples, but more needs to be done.

One clear extension of this work would be to extend the model to account for other measures of economic well-being, such as wealth. There can be indeed little dispute that wealth is a relevant measure of living standard too, and one which is probably able to capture long-term economic resources better than income—as, over and above any income flow, it represents resources that people are able to draw upon to face adverse shocks. Thus, since both income and wealth may be used to finance current consumption, or retained to support future consumption, they can be thought of as alternative means of securing the living standards of individuals, families or households. Information on household wealth holdings is also collected in the Bank of Italy’s SHIW database.

Another aspect of interest is studying whether accounting for income and consumption jointly reveals a different pattern of economic inequality than the traditional “income only” approach.<sup>27</sup> To capture inequality in the joint distribution of income and consumption, we can rely on the bi-variate Gini coefficient [53], which is determined by the degree of inequality in the two marginal distributions as well as by the association among the two variates. This would also allow us to compare overall inequality of income and consumption across time and examine if changes are driven by differences in the association between income and consumption, or by differences in the marginal distributions. In our view, the most straightforward way to attain this goal is to work with parametric estimates of our bi-variate model to design a counter-factual analysis in order to assess the implications of variations in the model parameters on the bi-variate version of the Gini coefficient: we would evaluate these implications by calculating what the bi-variate Gini would be if the dependence structure of income

<sup>27</sup>Clearly, the same is true for what concerns the relationship between consumption and wealth or income and wealth.

and consumption was fixed but the parameter estimates of the marginal distributions changed over time, and by assessing how would bi-variate inequality change if the relationship between income and consumption changed over time but the marginal distributions were fixed.

All these aspects are still open and in need of an in-depth study.

### Appendix A. The propensity to consume in Italy

In order to estimate the marginal propensity to consume at the household level and using cross-sectional information, we follow the empirical approach proposed by [62]. We consider a simple consumption function based on the life cycle model where individuals use income and wealth accumulation to smooth consumption over their life cycle. In this framework, current consumption is proportional to total net disposable income (i.e. the sum of total consumption and saving) and total net wealth (i.e. the sum of real and financial assets minus the financial liabilities).

We start with the following simple consumption function

$$C_{it} = \beta_0 Y_{it} + \beta_1 W_{it}, \quad (\text{A.1})$$

where each period of time  $t$  available in the SHIW survey is considered as a dynasty. Dividing Equation (A.1) by the level of net income, we obtain the expression for estimating the average propensity to consume (APC) with respect to net income and net wealth as

$$\frac{C_{it}}{Y_{it}} = \beta_0 + \beta_1 \frac{W_{it}}{Y_{it}}, \quad (\text{A.2})$$

where  $C_{it}$ ,  $Y_{it}$ , and  $W_{it}$  denote, respectively, total consumption, total income, and net wealth at time  $t$  for a given household  $i$ . In this model,  $\beta_0$  and  $\beta_1$  are the APC out of income (or “income effect”) and the APC out of wealth (or “wealth effect”), respectively.<sup>28</sup>

The results of the micro-based estimates are reported in Table A.1. The results show a strong income effect and a limited wealth effect on consumption in Italy: the estimated APC out of income varies from 0.75 to 0.85, but overall it is increasing in the period 1991–2014, while the impact of wealth on consumption appears to be negligible, about 0.006, meaning that one additional euro of wealth would increase annual consumption by 0.6%.

We consider now a more flexible specification where we allow the APC to vary across the income distribution. We define income categories in which the household income composition is quite homogeneous. We introduce dummy variables account-

<sup>28</sup>Equation (A.2) is estimated taking into account the period 1991–2014, where data on net wealth are available.

**Table A.1** Linear regression estimates, 1991–2014

		Dependent variable: $C/Y$												
		1991	1993	1995	1998	2000	2002	2004	2006	2008	2010	2012	2014	
$Y/Y$	0.754*** (0.003)	0.760*** (0.005)	0.803*** (0.004)	0.754*** (0.005)	0.773*** (0.004)	0.755*** (0.004)	0.793*** (0.004)	0.781*** (0.004)	0.789*** (0.003)	0.809*** (0.004)	0.854*** (0.003)	0.770*** (0.003)		
$W/Y$	0.005*** (0.000)	0.006*** (0.000)	0.003*** (0.000)	0.002** (0.000)	0.001* (0.001)	0.002*** (0.000)	0.000 (0.000)	0.000 (0.000)	-0.001*** (0.000)	-0.001*** (0.000)	0.000 (0.000)	0.000 (0.000)		
Obs	7,763	7,436	7,510	6,593	7,427	7,468	7,467	7,224	7,357	7,327	7,428	7,496		

*Notes:* in order to reduce the impact of outliers, we drop the bottom 0.01% and the top 2%; numbers in parentheses: estimated standard errors; star codes for significance: \* \* \* = 1%, \*\* = 5%, \* = 10%.

ing for the households belonging to the considered income position, which are interacted with the variables in Equation A.2. We consider 9 income groups, defined according to the following income deciles: ~~0.0 to 0.10-th, 0.10-th to 0.20-th, 0.20-th to 0.30-th, 0.30-th to 0.40-th, 0.40-th to 0.50-th, 0.50-th to 0.60-th, 0.60-th to 0.70-th, 0.70-th to 0.80-th, 0.80-th to 0.90th, and 0.90-th to 1-st decile.~~

The results are presented in Table A.2. As expected, the impact of net wealth is still negligible, but remarkable differences of the impact of income among the different percentiles emerge. The APC out of income, that we consider as a proxy of the marginal propensity to consume, shows a decreasing trend when we move forward along the income distributions. Indeed, we obtain an APC decreasing from 0.8-1.1 cents of euro for households in the first decile to about 0.6 cent of euro for households at the top of the income distribution. This imply that the average APC out of income, estimated from the baseline model (A.2), is likely to be biased by the non-linear effects arising along the income distribution.

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Table A.2 Linear regression estimates by income decile, 1991–2014

		Dependent variable: $(C/Y)^d$											
		1991	1993	1995	1998	2000	2002	2004	2006	2008	2010	2012	2014
$(Y/Y)^{d1}$	Obs	789	638	635	517	604	576	540	531	542	549	521	507
		0.896*** (0.006)	1.082*** (0.019)	1.098*** (0.013)	1.089*** (0.018)	1.014*** (0.015)	1.011*** (0.015)	1.043*** (0.015)	1.034*** (0.013)	0.998*** (0.011)	1.089*** (0.014)	1.155*** (0.012)	1.031*** (0.015)
$(W/Y)^{d1}$	Obs	789	638	635	517	604	576	540	531	542	549	521	507
		0.004*** (0.001)	0.008*** (0.002)	0.006*** (0.001)	0.003* (0.002)	0.012*** (0.002)	0.008*** (0.002)	0.003* (0.002)	0.002 (0.002)	0.003* (0.002)	0.003** (0.001)	0.001 (0.001)	0.000 (0.000)
$(Y/Y)^{d2}$	Obs	789	638	635	517	604	576	540	531	542	549	521	507
		0.868*** (0.008)	0.956*** (0.013)	0.950*** (0.012)	0.899*** (0.015)	0.914*** (0.013)	0.874*** (0.012)	0.914*** (0.011)	0.894*** (0.011)	0.938*** (0.011)	0.900*** (0.012)	0.999*** (0.012)	0.846*** (0.013)
$(W/Y)^{d2}$	Obs	789	638	635	517	604	576	540	531	542	549	521	507
		0.000 (0.001)	0.003* (0.002)	0.007*** (0.001)	0.009*** (0.002)	0.004*** (0.001)	0.007*** (0.001)	0.003** (0.001)	0.001 (0.001)	0.001 (0.001)	0.003* (0.001)	0.001 (0.001)	0.007*** (0.002)
$(Y/Y)^{d3}$	Obs	755	740	726	637	654	698	623	650	615	665	602	560
		0.826*** (0.008)	0.872*** (0.013)	0.888*** (0.012)	0.808*** (0.016)	0.816*** (0.013)	0.821*** (0.010)	0.837*** (0.012)	0.860*** (0.012)	0.857*** (0.010)	0.876*** (0.012)	0.933*** (0.011)	0.823*** (0.011)
$(W/Y)^{d3}$	Obs	755	740	726	637	654	698	623	650	615	665	602	560
		0.005*** (0.001)	0.009*** (0.002)	0.008*** (0.002)	0.011*** (0.002)	0.007*** (0.001)	0.001* (0.001)	0.005*** (0.001)	0.002 (0.001)	0.002 (0.001)	0.001 (0.001)	0.002* (0.001)	0.006*** (0.001)

Table A.2 Continued

		Dependent variable: $(C/Y)^d$												
		1991	1993	1995	1998	2000	2002	2004	2006	2008	2010	2012	2014	
$(Y/Y)^{d4}$		0.771*** (0.009)	0.798*** (0.013)	0.826*** (0.011)	0.776*** (0.013)	0.793*** (0.011)	0.772*** (0.010)	0.828*** (0.009)	0.828*** (0.011)	0.825*** (0.011)	0.833*** (0.011)	0.870*** (0.011)	0.804*** (0.010)	
$(W/Y)^{d4}$		0.012*** (0.002)	0.010*** (0.002)	0.010*** (0.001)	0.005*** (0.002)	0.004*** (0.001)	0.003*** (0.001)	0.001 (0.000)	0.002 (0.001)	0.002 (0.001)	0.004*** (0.001)	0.004*** (0.001)	0.001 (0.001)	
Obs		727	728	763	672	744	767	789	760	769	768	746	792	
$(Y/Y)^{d5}$		0.760*** (0.009)	0.747*** (0.012)	0.817*** (0.011)	0.714*** (0.013)	0.719*** (0.012)	0.761*** (0.011)	0.805*** (0.010)	0.773*** (0.011)	0.770*** (0.009)	0.768*** (0.011)	0.833*** (0.011)	0.756*** (0.010)	
$(W/Y)^{d5}$		0.0081*** (0.002)	0.015*** (0.002)	0.007*** (0.001)	0.007*** (0.002)	0.008*** (0.002)	0.005*** (0.001)	0.001 (0.001)	0.001 (0.001)	0.003*** (0.001)	0.003*** (0.001)	0.005*** (0.001)	0.001 (0.001)	
Obs		741	783	782	626	751	809	835	740	791	751	825	803	
$(Y/Y)^{d6}$		0.692*** (0.009)	0.683*** (0.011)	0.728*** (0.012)	0.709*** (0.011)	0.755*** (0.009)	0.701*** (0.012)	0.746*** (0.012)	0.726*** (0.009)	0.735*** (0.011)	0.739*** (0.010)	0.816*** (0.009)	0.726*** (0.010)	
$(W/Y)^{d6}$		0.010*** (0.002)	0.013*** (0.001)	0.007*** (0.002)	0.004*** (0.001)	0.003*** (0.001)	0.005*** (0.001)	0.005*** (0.001)	0.003*** (0.001)	0.003*** (0.001)	0.003*** (0.001)	0.002* (0.001)	0.007*** (0.001)	
Obs		740	787	768	717	781	726	811	776	779	801	738	801	

Table A.2 Continued

		Dependent variable: $(C/Y)^d$												
		1991	1993	1995	1998	2000	2002	2004	2006	2008	2010	2012	2014	
$(Y/Y)^{d7}$		0.710 (0.009)	0.667*** (0.012)	0.742*** (0.012)	0.653*** (0.014)	0.674*** (0.014)	0.695*** (0.012)	0.738*** (0.012)	0.723*** (0.011)	0.718*** (0.008)	0.708*** (0.012)	0.775*** (0.011)	0.687*** (0.011)	
$(W/Y)^{d7}$		0.008*** (0.001)	0.009*** (0.001)	0.009*** (0.002)	0.004*** (0.002)	0.007*** (0.001)	0.005*** (0.001)	0.001 (0.001)	0.001 (0.001)	0.000 (0.001)	0.006*** (0.001)	0.003* (0.001)	0.003*** (0.001)	
Obs		761	709	781	642	764	749	767	779	795	778	807	775	
$(Y/Y)^{d8}$		0.675*** (0.010)	0.672*** (0.014)	0.651*** (0.011)	0.649*** (0.013)	0.663*** (0.012)	0.657*** (0.012)	0.670*** (0.013)	0.682*** (0.011)	0.694*** (0.009)	0.683*** (0.012)	0.722*** (0.011)	0.652*** (0.010)	
$(W/Y)^{d8}$		0.009*** (0.002)	0.005* (0.002)	0.009*** (0.002)	0.004*** (0.006)	0.008*** (0.001)	0.007*** (0.001)	0.006*** (0.002)	0.005*** (0.001)	0.002* (0.001)	0.006*** (0.001)	0.006*** (0.001)	0.004*** (0.001)	
Obs		780	695	747	671	795	827	793	834	831	819	851	806	
$(Y/Y)^{d9}$		0.665*** (0.01)	0.579*** (0.012)	0.620*** (0.012)	0.594*** (0.013)	0.652*** (0.012)	0.658*** (0.012)	0.649*** (0.012)	0.639*** (0.011)	0.656*** (0.007)	0.645*** (0.010)	0.689*** (0.010)	0.636*** (0.009)	
$(W/Y)^{d9}$		0.007*** (0.002)	0.012*** (0.001)	0.008*** (0.001)	0.006*** (0.002)	0.004*** (0.001)	0.002* (0.001)	0.004*** (0.001)	0.005*** (0.001)	0.000 (0.000)	0.005*** (0.001)	0.005*** (0.001)	0.001 (0.001)	
Obs		829	814	757	727	816	812	811	804	776	779	810	851	

Notes: in order to reduce the impact of outliers, we drop the bottom 0.01% and the top 2%; numbers in parentheses: estimated standard errors; star codes for significance: \* = 1%, \*\* = 5%, \*\*\* = 10%.

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