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Abstract Characterization of a Conditional Expectation Operator on the Space of Measurable Sections

(Pencirian Abstrak Bagi Pengendali Jangkaan Bersyarat di Ruang Bahagian yang Boleh Diukur)

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ABSTRACT

A conditional expectation operator plays an important role in geometry of Banach spaces. However, the main issue is with regards to the existence of a conditional expectation operator that permits other objects to be considered such as martingales and martingale convergence theorems. Thus, the purpose of this study is to provide an abstract characterization of a conditional expectation operator on a space of measurable sections.

Keywords: Abstract characterization; conditional expectation operator; measurable section

ABSTRAK

Pengendalian jangkaan bersyarat memainkan peranan yang penting di dalam geometri ruang Banach. Walau bagaimanapun, isu utama adalah berkaitan dengan kewujudan pengendali jangkaan bersyarat yang membenarkan objek lain yang perlu dipertimbangkan seperti teori penumpuan martingale dan martingale. Dengan itu, tujuan kajian ini adalah untuk memberikan pencirian abstrak bagi pengendali jangkaan bersyarat di ruang bahagian yang boleh diukur.

Kata kunci: Bahagian yang boleh diukur; pencirian abstrak; pengendali jangkaan bersyarat

INTRODUCTION

One of the important problems of operators theory is an abstract characterization of the conditionally expectation operators in function spaces.

In (Rao 1965) gave a characterization of the conditional probability measures as subclasses of vector measures on a general Banach function spaces and proved the following result.

Theorem 1.1. (Rao 1965) Let (Ω, Σ, μ) be a measurable space with a finite measure μ.

If T: $L_p(\mu) \to L_p(\mu)$, $(1 \leq p < \infty)$ is a positive contractive projection with $T1 = 1$, then $Tf = M^F(f), f \in L_p(\mu)$, for a unique σ-algebra $F \in \Sigma$, where $M^F(\cdot)$ is a conditionally expectation operator relative to F. In (Rao 1976) proved this theorem for Orlicz spaces.

In (Douglas 1965) obtained the necessary and sufficient conditions for T: $L_1(\mu) \to L_1(\mu)$ to be conditionally expectation operator relative to F.

We recall that in the theory of Banach bundles L_0-valued Banach spaces are considered, and such spaces are called *Banach–Kantorovich spaces*. In (Kusraev 1985),(Gutman 1995),(Kusraev 2000) developed the theory of Banach–Kantorovich spaces. To investigate the properties of Banach–Kantorovich spaces, it is natural to use measurable bundles of such spaces. Since, the theory of measurable bundles of Banach lattices is sufficiently well developed (Ganiev 2006), it has become an effective tool that provides an opportunity to obtain various properties of Banach–Kantorovich spaces well. A conditional expectation operator plays an important role in the geometry of Banach spaces and the main concern is the existence of a conditional expectation operators, allows further objects such as martingales and martingale convergence theorems to be considered. The existence of a conditional expectation operator and further properties for a Banach valued measurable functions are well given in (Vakhania et al. 1987) and (Diestel et al. 1977). Some further properties of conditional expectation operators have also been studied.

In (Landers et al 1981) characterized conditional expectation operators for Banach –valued functions.

In (Ganiev et al. 2013) introduced Bochner integral for measurable sections and proved the properties of such integrals.

In (Ganiev et al. 2015) proved the existence of conditional expectation operator on a space of integrable sections and studied the basic properties of conditional expectation operators.

Therefore, this study aims to provide a abstract characterization of a conditional expectation operator in a space measurable sections.

PRELIMINARIES

This section recalls the Bochner integral for measurable sections and the conditional expectation operator in a space of measurable sections.

Let $(\Omega, \Sigma, \lambda)$ be the space with finite measure, $L_0 = L_0(\Omega)$ be the algebra of classes of measurable functions on $(\Omega, \Sigma, \lambda)$ and $L_p(\Omega)$ be a Banach space of measurable functions integrable with degree p, $p \geq 1$, with the norm $\|f\|_p = \left(\int_\Omega |f(\omega)|^p \, d\lambda\right)^{\frac{1}{p}}$.

Let F be a vector space over a field of real numbers $\mathbb{R}$.

Definition 2.1. (Kusraev 2000) A map $\| \cdot \|: F \to L_0$ is called an L_0– valued norm on F, if for any $x, y \in F, \lambda \in \mathbb{R}$ it satisfies the following conditions:

1. $\| x \| \geq 0$; $\| x \| = 0 \Leftrightarrow x = 0$;
2. $\| \lambda x \| = |\lambda| \, \| x \|$;
3. $\| x + y \| \leq \| x \| + \| y \|$.

A pair $(F, \| \cdot \|)$ is called *a lattice-normed space* (LNS) over L_0. A LNS F is said to be *d-decomposable*, if for any $x \in y$ and for any decomposition of $\| x \| = f + g$ to a sum of disjunct elements such that there exists $y, z \in F, x = y + z$, $\| x \| = f, \| z \| = g$.

A net$\{x_\alpha\}$ in F is called *(bo)-convergent* to $x \in F$, if the net $\{\| x_\alpha - x \|\}$ is (o)-convergent to zero in L_0.

A lattice normed space is called (*bo*)-*complete* if every (*bo*)-fundamental net is (*bo*)-convergent in it. A Banach–Kantorovich space (BKS) over L_0 is a (*bo*)-complete *d*-decomposable lattice normed space over L_0. It is well known (Kusraev 1985) that every Banach–Kantorovich space F over L_0 admits an L_0-module structure such $\| \lambda x \| = | \lambda | \| x \|$ that for every $x \in F, \lambda L_0$.

Let X be a mapping, which maps every point $\omega \in \Omega$ to some Banach space $(X(\omega), \| \cdot \|_{X(\omega)})$. In what follows, we assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function u is said to be a section of X, if it is defined almost everywhere in Ω and takes its values on $X(\omega)$ ($u(\omega) \in X(\omega)$) for $\omega \in dom(u)$, where *dom*(*u*)is the domain of u.

Let L be some set of sections.

Definition 2.2. (Gutman 1995). A pair (X, L) is said to be a *measurable bundle of Banach spaces* over Ω if

1. $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2$: $\omega \in dom(c_1) \cap dom(c_2) \to \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;
2. the function $\|c\|$: $\omega \in dom(c) \to \|c(\omega)\|_{X(\omega)}$ is measurable for all $c \in L$;
3. for every $\omega \in \Omega$ the set $\{c(\omega)$: $c \in L, \omega \in dom(c)\}$ is dense in $X(\omega)$.

A section s is said to be *a step section*, if there are $c_i \in L, A_i \in \Sigma, i = \overline{1,n}$ such that

$$s(\omega) = \sum\nolimits_{i=1}^n \chi_{A_i}(\omega) c_i(\omega),$$

for almost all $\omega \in \Omega$.

A section u is called *measurable* if there is a sequence $\{s_n\}$ of step sections such that $s_n\{\omega\} \to u(\omega)$ almost everywhere on Ω. The set of all measurable sections is denoted by $M(\Omega, X)$ and $L_0(\Omega, X)$ denotes the factorization of this set with respect to equality everywhere. We denote by $\hat{u}$ the class from $L_0(\Omega, X)$ containing a section $u \in M(\Omega, X)$, and by $\| \hat{u} \|$ the element of L_0 containing the function $\| u(\omega)\|_{X(\omega)}$. It is known (Gutman 1995) that $L_0(\Omega, X)$ is a BKS over L_0.

Let s be a step section and $m_i = \sup_{\omega \in dom)c_i)} \| c_i(\omega) \|_{X(\omega)} < \infty$ for any $i = 1, 2, \ldots, n$ then we define the integral of step section by a measure λ with equality

$$\int_\Omega s(\omega)\, d\lambda = \sum_{i=1}^n c_i(\omega) \lambda(A_i).$$

Definition 2.3. (Ganiev et al. 2013) The measurable section u is said to be *integrable by Bochner*, if there exists a sequence step sections s_n such that $\| s_n(\omega) - u(\omega) \|_{X(\omega)} \to 0$ for almost all $\omega \in \Omega$ and

$$\lim_{n \to \infty} \int_\Omega \left\| s_n(\omega) - u(\omega) \right\|_{X(\omega)} d\lambda = 0.$$

In this case the integral $\int_A u \, d\lambda$ for every $A \in \Sigma$ is defined with equality

$$\int_A u \, d\lambda = \lim_{n \to \infty} \int_A s_n \, d\lambda. \tag{1}$$

By analogy of Banach valued case, it can be proven, that the definition is correct, i.e. (1) is independent from choosing the sequence step sections. We need the following properties of Bochner integral

Theorem 2.4. (Ganiev et al. 2013) If a section is integrable by Bochner, then

1. $\left\| \int_A u(\omega)\, d\lambda \right\|_{X(\omega)} \leq \int_A \left\| u(\omega) \right\|_{X(\omega)} d\lambda$ for all $A \in \Sigma$;
2. $\lim_{\lambda(A) \to 0} \int_A u(\omega)\, d\lambda = 0$;
3. If $c \in L, f \in L_1(\Omega)$ and $\sup_{\omega \in dom(c)} \| c(\omega) \|_{X(\omega)} < \infty$ then cf is integrable by Bochner and

$$\int_\Omega c(\omega)\, f(\omega)\, d\lambda = c(\omega) \int_\Omega f(\omega)\, d\lambda.$$

By $L_1(\Omega, \Sigma, X)$, we denote that the class of measurable sections for which

$$\int_\Omega \| u(\omega) \|_{X(\omega)} \, d\lambda < \infty.$$

Then $L_1(\Omega, \Sigma, X)$ is a Banach space with respect to the mixed norm

$$\| u \|_1 = \| \| u \| \| = \int_\Omega \| u(\omega) \|_{X(\omega)} \, d\lambda,$$

that is

$$L_1(\Omega, \Sigma, X) = \{u \in L_0(\Omega, X): \| u \| \in L_1(\Omega)\}.$$

Let $p \geq 1$,

$$L_1(\Omega, \Sigma, X) = \{u \in L_0(\Omega, X): \| u \|^p \in L_1(\Omega)\}.$$

Then $L_p(\Omega, \Sigma, X)$ is a Banach space with respect to the following mixed norm

$$\| u \|_p = \| \| u \| \| = \left(\int_\Omega \| u(\omega) \|_{X(\omega)}^p \, d\lambda \right)^{\frac{1}{p}}.$$

For abbreviation, the space $L_p(\Omega, \Sigma, X)$ is denoted by $L_p(\Sigma, X)$,

Let $\boldsymbol{A}_1 \subset \Sigma$ be some sub- σ – algebra and λ_1 be the restriction of λ to $\boldsymbol{A}_1$. Then $L_p(A_1, X)$ is a closed subspace of $L_p(\Sigma, X)$.

Theorem 2. 5. (Ganiev et al. 2015) There exists a linear continuous operator

$$\boldsymbol{M}^{A_1}: L_1(\Sigma, X) \to L_1(\boldsymbol{A}_1, X),$$

such that for any $B \in \boldsymbol{A}_1$, one has

$$\int_B \left(\boldsymbol{M}^{A_1} u \right) d\lambda_1 = \int_B u \, d\lambda.$$

Definition 2.6. The linear operator

$$\boldsymbol{M}^{A_1}: L_1(\Sigma, X) \to L_1(\boldsymbol{A}_1, X),$$

is said to be *a conditional expectation operator* with respect to sub-σ – algebra $\boldsymbol{A}_1$.

If is a step section

$$s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) c_i(\omega),$$

then $\boldsymbol{M}^{A_1} s(\omega) = \sum_{i=1}^{n} c_i(\omega) \boldsymbol{M}^{A_1} \left(\chi_{A_i}(\omega) \right)$, where $\boldsymbol{M}^{A_1}$ is a conditional expectation operator on $L_1(\Omega)$.

Theorem 2.7. (Ganiev et al. 2015) (1) If $u \in L_1(\boldsymbol{A}_1, X)$, then $\boldsymbol{M}^{A_1} u = u$ and $\| \boldsymbol{M}^{A_1} \| = 1$; (2) If $u \in L_p(\Sigma, X)$, then $\boldsymbol{M}^{A_1} u \in L_p(\Sigma, X)$ and $\| \boldsymbol{M}^{A_1} \| = 1$.

Proposition 2.8. Let $c \in L$ and $sup_{\omega \in dom(c)} \| c(\omega) \|_{X(\omega)} < \infty$ then

1. $\boldsymbol{M}^{A_1} (c(\omega)) = c(\omega)$.
2. If $f \in L_1(\Omega)$, $\boldsymbol{M}^{A_1} (c(\omega) f(\omega)) = c(\omega) \boldsymbol{M}^{A_1} (f(\omega))$.

Proof. 1) Follows from the definition of conditionally expectation operator. 2) As $\int_B \boldsymbol{M}^{A_1} (c(\omega) f(\omega)) \, d\lambda_1 = \int_B c(\omega) f(\omega) \, d\lambda$ using Theorem 2.3 3) we get $\int_B \boldsymbol{M}^{A_1} (c(\omega) f(\omega)) \, d\lambda_1 = c(\omega) \int_B f(\omega) \, d\lambda = c(\omega) \; M^{A_1}$ for any $B \in \boldsymbol{A}_1$. Hence $\boldsymbol{M}^{A_1} (c(\omega) f(\omega)) = c(\omega) M^{A_1} (f(\omega))$.

Theorem 2.9. (Ganiev et al. 2015) Let $u \in L_1(\Sigma, X)$. Then $\| (\boldsymbol{M}^{A_1} u(\omega)) \|_{X(\omega)} \leq M^{A_1} (\| u(\omega) \|_{X(\omega)})$ for almost all $\omega \in \Omega$.

Theorem 2.10. (Ganiev et al. 2015) Let be a sequence of sections, each of which is Bochner integrable and there exist a section u and an integrable function g such that

1. $\lim_{n \to \infty} u_n(\omega) = u(\omega)$ for almost all $\omega \in \Omega$;
2. $\| u_n(\omega) \|_{X(\omega)} \leq |g(\omega)|$ for almost all $\omega \in \Omega$.

Then

$$\boldsymbol{M}^{A_1} (u_n(\omega)) \to \boldsymbol{M}^{A_1} (u(\omega)),$$

a.e. on Ω.

AN ABSTRACT CHARACTERIZATION OF CONDITIONAL EXPECTATION OPERATORS

This section proves the theorem of abstract characterization of a conditional expectation operators in a space of measurable sections.

A Banach space $(V, \| \, \|)$ is called strictly convex if $x \neq 0$ and $y \neq 0$ and $\| x + y \| = \| x \| + \| y \|$ together imply that $x = cy$ for some constant $c > 0$.

Lemma 3. 1. Let $T{:}L_1(\Sigma, X) \to L_1(\Sigma, X)$ be a linear contraction and for almost all $\omega \in \Omega$ the Banach space $X(\omega)$ be a strictly convex. If $T^2 = T$ and $T(c(\omega)) = c(\omega)$ for $c \in L$ such that $\sup_{\omega \in dom(c)} \|c(\omega)\|_{X(\omega)} < \infty$, then there exist $f_A \in L_1(\Omega)$, such that

$$T(\chi_A(\omega) c(\omega) = f_A(\omega) c(\omega),$$

for all $A \in \Sigma$.

Proof. If = 0, it is obvious.

Let $c \neq 0$. Since $c(\omega) = T(c(\omega)) = T(\chi_A(\omega)c(\omega) + \chi_{CA}(\omega)c(\omega)) = T(\chi_A(\omega)c(\omega)) + T(\chi_{CA}(\omega)c(\omega))$ and T is a contraction

$$\|c(\omega)\|_{X(\omega)} = \|T(c(\omega))\|_{X(\omega)} 1 = \|T(c(\omega))\|_{X(\omega)}$$

$$\int d\lambda = \int \|T(c(\omega))|_{X(\omega)} \, d\lambda$$

$$\leq \int \| T(\chi_A(\omega)c(\omega)) \|_{X(\omega)} \, d\lambda + \int \| T(\chi_{CA}(\omega)c(\omega)) \|_{X(\omega)} \, d\lambda$$

$$\leq \int \| \chi_A(\omega)c(\omega) \|_{X(\omega)} \, d\lambda + \int \|\chi_{CA}(\omega)c(\omega) \|_{X(\omega)} \, d\lambda$$

$$= \int (\chi_A(\omega) + \chi_{CA}(\omega)) \, \| (c(\omega)) \|_{X(\omega)} \, d\lambda = \| c(\omega) \|_{X(\omega)}.$$

Hence

$$\int \| T(c(\omega)) \|_{X(\omega)} \, d\lambda = \int \| (T(\chi_A(\omega)c(\omega)) \|_{X(\omega)} + \| T(\chi_{CA}(\omega)) |_{X(\omega)}) \, d\lambda$$

and

$$\| T(c(\omega)) \|_{X(\omega)} = \|T(\chi_A(\omega)c(\omega) + \chi_{CA}(\omega)c(\omega))\|_{X(\omega)}$$

$$= \| T(\chi_A(\omega)c(\omega) \|_{X(\omega)} + \| T(\chi_{CA}(\omega)c(\omega)) \|_{X(\omega)},$$

for almost all $\omega \in \Omega$. Since $X(\omega)$ is strictly convex for almost all $\omega \in \Omega$ there exists number $d_\omega > 0$ such that

$$T(\chi_{CA}(\omega)c(\omega)) = d_\omega T(\chi_A(\omega)c(\omega)).$$

As $d_\omega = \frac{\|T(\chi_{CA}(\omega)c(\omega))\|_{X(\omega)}}{\|T(\chi_A(\omega)c(\omega))\|_{X(\omega)}}$ we get that $\omega \to d_\omega$ is a measurable function.

Let $f_{A,c(\omega)}(\omega) = \frac{1}{1+d_\omega}$.

Then from

$$c(\omega) = (1 + d_\omega)T(\chi_A(\omega)c(\omega)),$$

we obtain $T(\chi_A(\omega)c(\omega)) = f_{A,c(\omega)}(\omega)c(\omega)$ for almost all $\omega \in \Omega$.

Let $c_0 \in L$, $c_0 \neq 0$ and $f_A := f_{A,c_0(\omega)}$. Then we prove that $T(\chi_A(\omega)c(\omega)) = f_A(\omega)c(\omega)$ for almost all $\omega \in \Omega$ and for any $c \in L$, such that $\sup_{\omega \in dom(c)} \| c(\omega) \|_{X(\omega)} < \infty$.

1. Let $c(\omega) = \lambda c_0(\omega)$ for $\lambda \in \mathbb{R}$. Then $T(\chi_A(\omega)c(\omega)) = \lambda T(\chi_A(\omega)c(\omega)) = \lambda f_A(\omega)c_0(\omega) = f_A(\omega)c(\omega)$.
2. Let $c(\omega)$ and $c_0(\omega)$ be linearly independent for almost all $\omega \in \Omega$. Then $T(\chi_A(\omega)(c(\omega) + c_0(\omega)) = f_{A,c(\omega)+c_0(\omega)}(\omega)(c(\omega) + c_0(\omega))$ and $T(\chi_A(\omega)(c(\omega) + c_0(\omega)) = T(\chi_A(\omega)c(\omega)) + T(\chi_A(\omega)c_0(\omega)) = f_{A,c(\omega)}(\omega)c(\omega) + f_A(\omega)c_0(\omega)$.

Therefore, $f_{A,c(\omega)}(\omega)c(\omega) + f_A(\omega)c_0(\omega) = f_{A,c(\omega)+c_0(\omega)}c(\omega) + f_{A,c(\omega)+c_0(\omega)}c_0(\omega))$ for almost all $\omega \in \Omega$. Following this, we have

$$(f_{A,c(\omega)}(\omega) - f_{A,c(\omega)+c_0(\omega)c(\omega)})c(\omega) = (f_{A,c(\omega)+c_0(\omega)} - f_{A,c(\omega)})c_0(\omega).$$

Since $c(\omega)$ and $c_0(\omega)$ are linearly independent for almost all $\omega \in \Omega$ we get

$$f_{A,c(\omega)}(\omega) = f_{A,c(\omega)+c_0(\omega)}$$

$$f_{A,c(\omega)+c_0(\omega)} = f_A(\omega) \text{ or } f_{A,c(\omega)}(\omega) = f_A(\omega),$$

for almost all $\omega \in \Omega$ and for all $c \in L$ such that $\sup_{\omega \in dom(c)} \| c(\omega) \|_{X(\omega)} < \infty$.

Therefore,

$$T(\chi_A(\omega)c(\omega)) = f_A(\omega)c(\omega),$$

for all $A \in \Sigma$, $c \in L$, such that $\sup_{\omega \in dom(c)} \| c(\omega) \|_{X(\omega)} < \infty$ and for some $f_A \in L_1(\Omega)$.

Using linearity of T we get

Corollary. $f_{A \cup B} = f_A + f_B$ for any $A, B \in \Sigma$.

Theorem 3.2. Let $T:L_1(\Sigma, X) \to L_1(\Sigma, X)$ be a linear contraction and for almost all $\omega \in \Omega$ the Banach $X(\omega)$ be a strictly convex. If $T^2 = T$ and $T(c(\omega)) = c(\omega)$ for $c \in L$, such that $\sup_{\omega \in dom(c)} \| c(\omega) \|_{X(\omega)} < \infty$ then there exists a σ–subalgebra $\boldsymbol{A}_1 \subset \Sigma$ such that

$$T(f) = \boldsymbol{M}^{\boldsymbol{A}_1}(f),$$

for all $f \in L_1(\Sigma, X)$.

Proof. We will show that

$$T(g(\omega)c(\omega)) = S(g(\omega))c(\omega),$$

for all $c \in L$ such that supω∈dom(c) ‖ c(ω) < ∞, g ∈ L1(Ω) and for some linear operator S:L1(Ω) → L1(Ω).

Let $g(\omega) = \lambda\chi_A(\omega)$. Then $T(g(\omega)c(\omega)) = \lambda T(\chi_A(\omega)c(\omega))$ and by Lemma 3.1 $T(g(\omega)c(\omega)) = \lambda f_A(\omega)c(\omega)$.

Let g be a simple function from $L_1(\Omega)$, i.e. $g = \sum_{i=1}^{n} \lambda_i \chi_{A_i}$. Then $T(g(\omega)c(\omega)) = \sum_{i=1}^{n} \lambda_i f_A(\omega)c(\omega) = S'(g(\omega)c(\omega)$, where

$$S'(g(\omega) = \sum_{i=1}^{n} \lambda_i f_A(\omega).$$

We shall show that S' well defined. For simplicity we will consider next two realization of g: $g = \lambda_1\chi_{A_1} + \lambda_2\chi_{A_2}$ and $g = \lambda_1\chi_{B_1} + \lambda_1\chi_{B_2} + \lambda_2\chi_{B_3} + \lambda_2\chi_{B_4}$, where $A_1 = B_1 \cup B_2$, $A_2 = B_3 \cup B_4$ and $B_i \cap B_j = \varnothing$ for $i \neq j$, $(i, j = 1, 2)$.

Then using corollary we obtain $S'(\lambda_1\chi_{A_1} + \lambda_2\chi_{A_2}) = \lambda_1 f_{A_1} + \lambda_2 f_{A_2} = \lambda_1 f_{A_2} = \lambda_1 f_{B_1 \cup B_2} + \lambda_2 f_{B_3 \cup B_4} = \lambda_1 f_{B_1} + \lambda_1 f_{B_2} + \lambda_2 f_{B_3} + \lambda_2 f_{B_4} = S'(\lambda_1\chi_{B_1} + \lambda_1\chi_{B_2} + \lambda_2\chi_{B_3} + \lambda_2\chi_{B_4})$.

It is clear that $S'(g_1(\omega) + g_2(\omega)) = S'(g_1(\omega)) + S'(g_2(\omega))$, $S'(\lambda g(\omega)) = \lambda S'(g(\omega))$ for all simple functions g_1, g_2, g from $L_1(\Omega)$ and a real number λ.

Since T is $\| \cdot \|_{L_1(\Sigma, X)}$ a contraction we get

$$\| c(\omega) \|_{X(\omega)} \int | S'(g(\omega)) | d\lambda = \int \| S'(g(\omega))c(\omega) \|_{X(\omega)} d\lambda$$

$$= \int \| T(g(\omega)c(\omega)) \|_{X(\omega)} d\lambda \leq \int \| g(\omega)c(\omega) \|_{X(\omega)} d\lambda = \|c(\omega)\|_{X(\omega)} \int | g(\omega) | d\lambda,$$

i.e.

$$\| S'(g) \| L1(\Omega) \leq g \| L1(\Omega).$$

Let $g \in L_1(\Omega)$ and (g_n) be a sequence of simple functions from $L_1(\Omega)$ such that $\| g_n - g |_{|L_1(\Omega)} \to 0$. As $\{g_n\}$ is fundamental in $L_1(\Omega)$ from

$$\| S'(g_n) - S'(g_m) \|_{L_1(\Omega)} \leq \| g_n - g_m \|_{L_1(\Omega)},$$

we get a sequence $\{S'(g_n)\}$ also is fundamental in $L_1(\Omega)$.

Put

$$S(g) = \lim_{n \to \infty} S'(g_n).$$

Hence $T(g(\omega)c(\omega)) = \lim_{n\to\infty} T(g_n(\omega)c(\omega)) = \lim_{n\to\infty} S'(g_n(\omega))c(\omega) = S(g(\omega))c(\omega)$. It is clear that S is linear and $L_1(\Omega)$ is contractive. We will show that S is idempotent and constant preserving:

1. As $T^2 = T$, we get $S^2(g(\omega))c(\omega) = T(S(g(\omega))c(\omega)) = T(T(g(\omega)c(\omega))) = T(g(\omega)c(\omega)) = S(g(\omega))c(\omega)$.

2. Let a be a constant. Then , i.e.

According to Theorem 1.1

$$\underline{S}(g) = M^{A_1}(g),$$

for some σ–subalgebra $\boldsymbol{A}_1$ of Σ. Then using Proposition 2.8 2) we get $T(gc) = M^{A_1}(\mathrm{g})\mathrm{c} = \boldsymbol{M}^{A_1}(gc)$

Let s be a step section, i.e. if there are $c_i \in L$, $A_i \in \Sigma$, $i = \overline{1,n}$ such that $s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega)c_i(\omega)$ for almost all $\omega \in \Omega$. Then $T(s(\omega)) = \sum_{i=1}^{n} T(\chi_{A_i}(\omega)c_i(\omega)) = \sum_{i=1}^{n} S(\chi_{A_i}(\omega)) c_i(\omega) = \sum_{i=1}^{n} M^{A_1}(\chi_{A_i}(\omega))c_i(\omega) = \boldsymbol{M}^{A_1}(s(\omega))$.

Let $f \in L_1(\Sigma, X)$ and $\{s_n\}$ be a sequence of step sections such that $\| s_n - f \|_{L_1(\Sigma,\mathrm{X})} \to 0$. Then using Theorem 2.10 we get $T(f(\omega)) = \lim_{n\to\infty} T(s_n(\omega)) = \lim_{n\to\infty} \boldsymbol{M}^{A_1}(s_n(\omega)) = \boldsymbol{M}^{A_1}(f(\omega))$

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