Mathematical and Numerical Analysis of a Pair of Coupled Cahn-Hilliard Equations with a Logarithmic Potential

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Dedicated to

my parents and my wife

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Abstract

Mathematical and numerical analysis has been undertaken for a pair of coupled Cahn-Hilliard equations with a logarithmic potential and with homogeneous Neumann boundary conditions. This pair of coupled equations arises in a phase separation model of thin film of binary liquid mixture. Global existence and uniqueness of a weak solution to the problem is proved using Faedo-Galerkin method. Higher regularity results of the weak solution are established under further regular requirements on the initial data. Further, continuous dependence on the initial data is presented.

Numerically, semi-discrete and fully-discrete piecewise linear finite element approximations to the continuous problem are proposed for which existence, uniqueness and various stability estimates of the approximate solutions are proved. Semi-discrete and fully-discrete error bounds are derived where the time discretisation error is optimal. An iterative method for solving the resulting nonlinear algebraic system is introduced and linear stability analysis in one space dimension is studied. Finally, numerical experiments illustrating some of the theoretical results are performed in one and two space dimensions.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, Department of Mathematical Sciences, University of Durham, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

1.1 Motivating the problem statement

Considerable attention has been paid to variants of the Cahn-Hilliard equations in recent years. These equations have gained in importance due to its wide application in diverse fields such as modelling alloys, glasses and polymers for instance see [32] and [46]. The Cahn-Hilliard model was first introduced by Cahn and Hilliard, see [29], to describe the dynamics of separation of a binary mixture into two different phases. This classical model has been successfully applied to modeling the so-called spinodal decomposition or phase separation phenomena and for qualitative studies on this topic we refer to [23], [27] and [56] and the references therein.

The classical Cahn-Hilliard equation is a fourth order time dependent nonlinear partial differential equation and has the following general form:

$$\frac{\partial u}{\partial t} - \Delta w = 0 \qquad \text{in } \ \Omega_T := \Omega \times (0,T), T > 0, \tag{1.1.1a}$$

$$w = -\gamma \Delta u + \Psi'(u) \qquad \text{in } \ \Omega_T, \tag{1.1.1b}$$

supplemented by an appropriate initial condition

$$u(x,0) = u^0(x) \qquad \text{in } \ \Omega, \tag{1.1.2}$$

and boundary conditions, here we consider Neumann,

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.1.3}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with Lipschitz boundary $\partial\Omega$ and ν is the outward unit normal to Ω. The variable u is the concentration of the two components and w is the chemical potential which is defined as the variational derivative of the Ginzburg-Landua free energy functional

$$\Lambda(u) := \int_\Omega [\frac{\gamma}{2}|\nabla u|^2 + \Psi(u)]dx. \tag{1.1.4}$$

Cahn and Hilliard included the gradient term, $\frac{\gamma}{2}|\nabla u|^2$, in the free energy functional Λ in order to model the surface energy separating the phases where γ is a positive constant relating to the surface tension..

The function Ψ in (1.1.4) represents the homogeneous potential which typically has a symmetric double well-form. In order to simplify the mathematical work, Ψ is often taken as a quartic polynomial in the following form

$$\Psi(u) = au^4 - bu^2 + c \qquad a, b > 0,\ c \in \mathbb{R}. \tag{1.1.5}$$

When the quenching temperature, θ, is close to a critical temperature ω, this quartic polynomial potential can be understood as an approximation of the following thermodynamic logarithmic potential, where $0 < \theta < \omega$,

$$\Psi(u) = \frac{\theta}{2}\big[(1+u)\ln(1+u) + (1-u)\ln(1-u)\big] + \frac{\omega}{2}(1-u^2) \quad -1 \leq u \leq 1. \tag{1.1.6}$$

The quartic Taylor polynomial of this logarithmic potential is given by

$$\Psi(u) \approx \frac{\theta}{12}u^4 - \frac{(\omega-\theta)}{2}u^2 + \frac{\omega}{2},$$

which is consistent with the form (1.1.5). The logarithmic form of the potential was suggested by Cahn and Hilliard, see [29]. We remark that Ψ in this logarithmic form has the required double well-form with two minima at α and $-\alpha$, i.e. α is the positive root of, $\Psi'(\alpha) = 0$,

$$\ln\Big(\frac{1+\alpha}{1-\alpha}\Big) = \frac{2\alpha\,\omega}{\theta}.$$

If we consider the case $\theta \to 0$, α tends to 1 and the logarithmic potential in this case can be replaced by the following obstacle potential

$$\Psi(u) = \begin{cases} \frac{\omega}{2}(1-u^2) & \text{if } |u| \leq 1, \\ \infty & \text{if } |u| > 1, \end{cases} \tag{1.1.7}$$

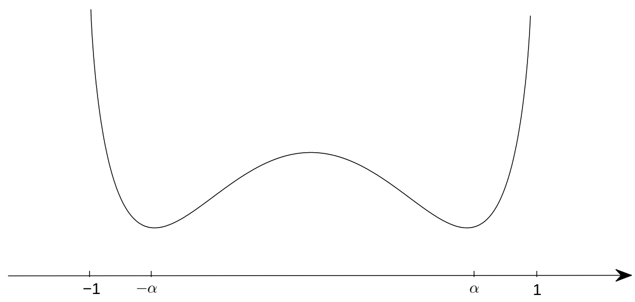


Figure 1.1: A homogeneous logarithmic potential

where this form of the potential was first proposed by Oono and Puri [60].

For mathematical and numerical studies on the classical Cahn-Hilliard equation with different forms of the the free energy we refer to [26], [16], [11], [35] and the references cited therein.

In this thesis we consider two coupled Cahn-Hilliard equations arising in the phase separation process on a thin film of a binary liquid mixture coating a substrate, which is wet by one component denoted by A, the other component is denoted by B, see [17] for further details. We begin by briefly describing their model:

Find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} - \Delta w_1 = 0 \qquad \text{in } \Omega_T, \tag{1.1.8a}$$

$$\frac{\partial u_2}{\partial t} - \Delta w_2 = 0 \qquad \text{in } \Omega_T, \tag{1.1.8b}$$

$$w_1 = \frac{\delta \Lambda(u_1, u_2)}{\delta u_1} \qquad \text{in } \Omega_T, \tag{1.1.8c}$$

$$w_2 = \frac{\delta \Lambda(u_1, u_2)}{\delta u_2} \qquad \text{in } \Omega_T, \tag{1.1.8d}$$

where

$$\Lambda(u_1,u_2) := \int_\Omega \Psi_1(u_1) + \frac{\gamma_1}{2}|\nabla u_1|^2 + \Psi_2(u_2) + \frac{\gamma_2}{2}|\nabla u_2|^2 + D(u_1+\alpha_1)^2(u_2+\alpha_2)^2, \tag{1.1.9}$$

with initial conditions

$$u_1(x,0) = u_1^0(x), \qquad u_2(x,0) = u_2^0(x) \qquad \text{in } \Omega \tag{1.1.10}$$

and boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{1.1.11}$$

In the above, $\frac{\delta\Lambda(u_1,u_2)}{\delta u_i}$, $i = 1,2$, denotes the variational derivative of the free energy functional Λ with respect to u_i. The variable u_1 provides information on the local concentration of A or B and u_2 indicates the presence of a liquid or a vapor phase. The positive constant γ_i, $i = 1,2$, relates to the surface tension of u_i and the coupling constant D is a positive prescribed constant. α_i, $i = 1,2$, is the positive constant where the minimum of a double well potential Ψ_i is achieved.

In the case where Ψ_i is a double well quartic polynomial potential, considered in [17], [44], it can be written as

$$\Psi_i(u_i) = a_i u_i^4 - b_i u_i^2 + c_i \qquad i = 1,2 \text{ and } a_i, b_i > 0,\ c_i \in \mathbb{R}. \tag{1.1.12}$$

In this case the minima of Ψ_i are $\pm\sqrt{\frac{b_i}{2a_i}}$, i.e. $\alpha_i = \sqrt{\frac{b_i}{2a_i}}$, where the coefficient b_i is proportional to $\theta_i - \theta$ and θ_i, $i = 1,2$, is the critical temperature of the A-B phase separation and the liquid-vapor phase separation, respectively. Thus, there are two equilibrium phases for each field corresponding to $u_1 = \pm\alpha_1$ and $u_2 = \pm\alpha_2$ denoted by u_1^+, u_2^+, u_1^- and u_2^-, respectively. The D-coupling term energetically inhibits the existence of the phase denoted (u_1^+, u_2^+). Hence we have three phase systems: liquid A corresponding to (u_1^-, u_2^-) region, liquid B to (u_1^+, u_2^-) region and the vapor phase to (u_1^-, u_2^+) region.

In the case where Ψ_i is an obstacle double well potential and the D-term is replaced by a bilinear term we have the three phase systems, considered in [30].

In the absence of the D-coupling term, i.e. $D = 0$, in the free energy functional, Λ, the above model problem simply becomes two classical single Cahn-Hilliard equations, which has been studied in the mathematical literature, e.g. see [3] and [19].

Now, by considering the logarithmic potential (1.1.7) and for simplicity taking $\gamma := \gamma_1 = \gamma_2$ we are led to the following problem which will be the focus of our interest in this thesis:

(**P**) Find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} - \Delta w_1 = 0 \qquad \text{in } \Omega_T, \tag{1.1.13a}$$

$$\frac{\partial u_2}{\partial t} - \Delta w_2 = 0 \qquad \text{in } \Omega_T, \tag{1.1.13b}$$

$$w_1 = -\gamma \Delta u_1 + \Psi_1'(u_1) + f_D^{(1)}(u_1, u_2) \qquad \text{in } \Omega_T, \tag{1.1.13c}$$

$$w_2 = -\gamma \Delta u_2 + \Psi_2'(u_2) + f_D^{(2)}(u_1, u_2) \qquad \text{in } \Omega_T, \tag{1.1.13d}$$

subject to the initial conditions

$$u_1(x,0) = u_1^0(x), \qquad u_2(x,0) = u_2^0(x) \qquad \text{in } \Omega, \tag{1.1.13e}$$

and boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.1.13f}$$

where

$$\Psi_i(r) = \psi(r) + \frac{\theta_i}{2}(1 - r^2) \quad i = 1,2,\ -1 \le r \le 1,\ 0 < \theta < \theta_i, \tag{1.1.14}$$

$$\psi(r) := \frac{\theta}{2}\big[(1+r)\ln(1+r) + (1-r)\ln(1-r)\big], \tag{1.1.15}$$

$$f_D(r_1, r_2) := D(r_1 + \alpha_1)^2 (r_2 + \alpha_2)^2, \tag{1.1.16}$$

$$f_D^{(i)}(r_1, r_2) := \frac{\partial f_D(r_1, r_2)}{\partial r_i} = 2D(r_i + \alpha_i)(r_j + \alpha_j)^2 \qquad i,j = 1,2 \text{ with } i \ne j, \tag{1.1.17}$$

where, as described earlier, γ, D, θ, θ_i and α_i are positive constants with $\theta < \theta_i$ and $\Psi_i'(\alpha_i) = 0$. Note that (i) since Ψ_i takes its minimum at $\pm\alpha_i$, we have $0 < \alpha_i < 1$,

(ii) Ψ_i is defined at $r = \pm 1$ as $\Psi_i(\pm 1) = \lim_{r \to \pm 1} \Psi_i(r) = \theta \ln 2$.

On introducing $\Phi \in C[0, \infty)$ such that

$$\Phi(r) := \frac{\theta}{2}\, r \ln r, \tag{1.1.18}$$

one can rewrite ψ as

$$\psi(r) = \Phi(1 + r) + \Phi(1 - r). \tag{1.1.19}$$

For the purposes of analysis we define the monotone function $\phi : (-1, 1) \longrightarrow \mathbb{R}$ to be

$$\phi(r) := \psi'(r) = \Phi'(1 + r) - \Phi'(1 - r) = \frac{\theta}{2}\big[\ln(1 + r) - \ln(1 - r)\big]. \tag{1.1.20}$$

To establish a weak formulation we multiply by a test function $\eta \in H^1(\Omega)$ and apply the Green's identity. Further, by a weak solution to the system (1.1.13a)-(1.1.17) we mean that there exists $\{u_1, u_2, w_1, w_2\}$ satisfying $u_1, u_2 \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))')$, $w_1, w_2 \in L^2(0, T; H^1(\Omega))$ and solving the weak formulation:

(**P**) Find $\{u_1, u_2, w_1, w_2\} \in [H^1(\Omega)]^4$ such that for *a.e.* $t \in (0, T)$, for $i = 1, 2$ and for all $\eta \in H^1(\Omega)$

$$\langle \partial_t u_i, \eta \rangle + (\nabla w_i, \nabla \eta) = 0, \tag{1.1.21}$$

$$\gamma(\nabla u_i, \nabla \eta) + (\Psi_i'(u_i), \eta) + (f_D^{(i)}(u_1, u_2), \eta) = (w_i, \eta), \tag{1.1.22}$$

$$u_i(x, 0) = u_i^0, \tag{1.1.23}$$

where $\partial_t u_i$ stands for $\frac{\partial u_i}{\partial t}$.

1.2 Research objectives and outline

The thesis highlights three principle objectives: the classical analysis of the system (1.1.13a)-(1.1.17), the numerical analysis of this system and some numerical experiments and simulations. With the aid of Faedo-Galerkin method of Lions [24] and compactness arguments we achieve the first goal. The second goal is achieved with a finite element method where a semi-discrete and fully-discrete approximation are applied to the system (1.1.13a)-(1.1.17). For the final objective we use Fortran and Matlab programming languages to implement numerical simulations in one and two space dimensions which verify the expected theoretical and physical behaviour of the solution.

In our work we analyse the problem (**P**) classically and numerically under two set of assumptions ($\mathbf{A_1}$) and ($\mathbf{A_2}$), stated in the pages 18 and 34 respectively, on the initial data u_1^0 and u_2^0. Due to the singular nature of the potential Ψ_i, $i = 1, 2$, we study the problem (**P**) by introducing a regularized version, say ($\mathbf{P}_\epsilon$), and then taking the limit as $\epsilon \to 0$. This approach was first used by Elliott and Luckhaus [49] to study a single Cahn-Hilliard equation and later applied in the mathematical literature with many variants of Cahn-Hilliard equations with non-smooth free energy, e.g. [16] and [3]. Numerically, we propose a symmetric coupled, in time, fully-discrete finite element approximation to (**P**) where we prove existence of approximate solutions using Schauder's fixed point theorem. Further, we introduce a semi-discrete approximation to (**P**) which will be necessary to prove an optimal error bound in time for the proposed fully-discrete approximation. In fact, we can analyse the error between the continuous solution and fully-discrete approximation directly but this will not lead to an optimal error bound in time. Our approach to the numerical analysis of the problem (**P**) uses the piecewise-linear finite element method. For studies that use this approach or employ similar arguments and tools to our own, see [7], [6], [4], [50], [10], [5], [11], [15].

We now describe briefly each chapter of the thesis:

In Chapter 2 we introduce a regularized problem to (**P**) and establish some necessary results that help dealing with the terms arising from the nonlinearities involved. We also present equivalent weak formulations to (**P**) and its corresponding regularized version ($\mathbf{P}_\varepsilon$). Existence and uniqueness of solutions to (**P**) and ($\mathbf{P}_\varepsilon$) under set of assumptions ($\mathbf{A_1}$) on the initial data is proved using Faedo-Galerkin method and compactness arguments.

In Chapter 3 we make further regularity requirements on the initial data, assumptions $(\mathbf{A_2})$, and on the boundary of the domain to prove more regularity for the weak solutions obtained in the previous chapter. Then, continuous dependence on the initial data is proved. Finally, an error bound between the solutions of $(\mathbf{P})$ and its regularized version $(\mathbf{P}_\varepsilon)$ is given which will be required in the subsequent analysis.

In Chapter 4 we begin by presenting some tools and results about the piecewise linear finite element space. We then establish some key lemmata that will be necessary to deal with technical problems caused by the nonlinearities (the logarithmic and D-coupling terms) throughout the treatment of the semi-discrete and fully-discrete problems. Then a semi-discrete finite element approximation to the continuous problem $(\mathbf{P})$ is constructed. We employ the same regularization approach used in the continuous problem to prove existence of a solution to the semi-discrete problem where we first consider a semi-discrete regularized problem and then pass to the limit in ε. Further, we derive some stability estimates under the assumptions $(\mathbf{A_1})$ and more regular estimates under the assumptions $(\mathbf{A_2})$ which will be required in the error bound analysis. We finally estimate an error bound between the solutions of the continuous and semi-discrete problems.

In Chapter 5 we formulate a symmetric coupled, in time, fully-discrete finite element approximation to the continuous problem where we discretise in time using backward Euler method. We study the fully-discrete problem by considering a regularized fully-discrete problem where existence of a solution to this problem is established using Schauder's fixed point theorem with no restrictions on the mesh parameter or on the time step. Uniqueness of the fully-discrete approximation is proved under some restrictions on the time step. Furthermore, various of estimates for the solution of the fully-discrete problem, under the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, is given which will be essential for the fully-discrete error bound analysis. Finally, by employing the framework in Nochetto [50] we prove an optimal error bound in time between the continuous solution and the fully-discrete approximation.

Chapter 6 is devoted to the numerical experiments where we write some programs and verify some theoretical and physical results. We first present a practical algorithm for computing the system of algebraic equations arising from the fully-discrete problem at each time step. We then present numerical simulations in one and two space dimensions.

Chapter 2

Weak solutions

In Section 2.1 we mention the basic notation adopted in the thesis, regarding the Sobolev spaces, and recall and show some auxiliary results. In Section 2.2 we introduce a regularized version of the continuous problem (**P**). Then we rewrite the problem (**P**) and its regularized version in equivalent forms. The global existence and uniqueness of the weak solutions are discussed in Section 2.3 where the existence proof relies on the Faedo-Galerkin method and compactness arguments.

2.1 Notation and auxiliary results

Throughout this study Ω denotes a bounded domain in $\mathbb{R}^d, d \leq 3$, with a Lipschitz boundary $\partial\Omega$. We use the usual Sobolev spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $p \in [1, \infty]$ with the associated norms and semi-norms, denoted by $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ respectively. In particular, for $p = 2$, $W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$ and if $m = 0$, $W^{0,2}(\Omega) = L^2(\Omega)$. The $L^2(\Omega)$ inner product over Ω with norm $\|\cdot\|_0 = |\cdot|_0$ is denoted by $(\cdot, \cdot)$.

In addition, $\langle\, .\,, .\,\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ where $(H^1(\Omega))'$ is the dual space of $H^1(\Omega)$. A norm on $(H^1(\Omega))'$ is given by

$$\|f\|_{(H^1(\Omega))'} := \sup_{v \neq 0} \frac{|\langle f, v\rangle|}{\|v\|_1} \equiv \sup_{\|v\|_1 = 1} |\langle f, v\rangle|. \tag{2.1.1}$$

We also introduce the function spaces depending on time and space $L^p(0,T;X)$ $(1 \leq p \leq \infty)$ where X is a Banach space, consisting of all functions u such that for $a.e.$ $t \in (0,T)$ $u \in X$ and the following norm is finite

$$\|u\|_{L^p(0,T;X)} := \Big(\int_0^T \|u(t)\|_X^p \, dt \Big)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty,$$
$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X \qquad \text{if } p = \infty.$$

We also define $L^2(\Omega_T) := L^2(0,T;L^2(\Omega))$.

We also recall the following well-known Sobolev results

$$H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow (H^1(\Omega))', \tag{2.1.2}$$
$$\langle f, \eta \rangle = (f, \eta) \qquad \forall f \in L^2(\Omega) \,\text{and}\, \eta \in H^1(\Omega). \tag{2.1.3}$$

Further, the inclusions[1] (2.1.2) are dense.

For later use we recall the Sobolev interpolation result, see e.g. Adams [2]: let $p \in [1, \infty]$, $m \geq 1$ and $v \in W^{m,p}(\Omega)$. Then there are constants C and $\sigma = \frac{d}{m}(\frac{1}{p} - \frac{1}{r})$ such that the inequality

$$|v|_{0,r} \leq C |v|_{0,p}^{1-\sigma} \, \|v\|_{m,p}^{\sigma} \qquad \text{holds for} \qquad r \in \begin{cases} [p, \infty] & \text{if } m - \frac{d}{p} > 0, \\ [p, \infty) & \text{if } m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m-(d/p)}] & \text{if } m - \frac{d}{p} < 0. \end{cases} \tag{2.1.4}$$

In particular, taking $m = 1$ and $p = 2$ in (2.1.4) we have after noting $|v|_0 \leq \|v\|_1$ that $H^1(\Omega) \hookrightarrow L^r(\Omega)$, where $r \in [2, \infty]$ for $d = 1$, $r \in [2, \infty)$ for $d = 2$, and $r \in [2, 6]$ for $d = 3$.

It is convenient to introduce "the inverse Laplacian Green's operator" $\mathcal{G} : \mathcal{F}_0 \to V_0$ such that

$$(\nabla \mathcal{G} f, \nabla \eta) = \langle f, \eta \rangle \qquad \forall \eta \in H^1(\Omega), \tag{2.1.5}$$

[1] We use "$\hookrightarrow$" to denote continuous injection and "$\overset{c}{\hookrightarrow}$" to denote compact injection.

where $\mathcal{F}_0 := \{f \in (H^1(\Omega))' : \langle f, 1\rangle = 0\}$ and $V_0 := \{\eta \in H^1(\Omega) : (\eta, 1) = 0\}$. The well posedness of $\mathcal{G}$ can be obtained from the Lax-Milgram theorem and the following Poincaré inequality, see e.g. [43],

$$|\eta|_0 \leq C_p(|\eta|_1 + |(\eta, 1)|) \qquad \forall \eta \in H^1(\Omega). \tag{2.1.6}$$

The norm defined in (2.1.1) on $(H^1(\Omega))'$ is also a norm on $\mathcal{F}_0$ and for convenience one can define an equivalent norm on $\mathcal{F}_0$, see the proof in Lemma 2.1.1 below, as

$$\|f\|_{-1} := |\mathcal{G}f|_1 \equiv \langle f, \mathcal{G}f\rangle^{\frac{1}{2}} \qquad \forall f \in \mathcal{F}_0. \tag{2.1.7}$$

It follows from (2.1.3) and (2.1.6) for any $f \in L^2(\Omega) \cap \mathcal{F}_0$ that

$$\|f\|_{-1}^2 = \langle f, \mathcal{G}f\rangle = (f, \mathcal{G}f) \leq |f|_0 |\mathcal{G}f|_0 \leq C_p |f|_0 |\mathcal{G}f|_1 = C_p |f|_0 \|f\|_{-1}$$

which implies that

$$\|f\|_{-1} \leq C_p |f|_0 \quad \forall\, f \in L^2(\Omega) \cap \mathcal{F}_0. \tag{2.1.8}$$

We shall frequently need the following simple version of Young's inequality

$$ab \leq \beta a^2 + \frac{1}{4\beta} b^2 \qquad \forall\, a, b \geq 0,\ \beta > 0 \tag{2.1.9}$$

from which we obtain after noting (2.1.5) and (2.1.7)

$$\langle f, \eta\rangle = (\nabla \mathcal{G}f, \nabla \eta) \leq \|f\|_{-1} |\eta|_1 \leq \beta |\eta|_1^2 + \frac{1}{4\beta}\|f\|_{-1}^2 \quad \forall f \in \mathcal{F}_0,\ \eta \in H^1(\Omega). \tag{2.1.10}$$

This result with (2.1.3) yields for future reference that

$$|v|_0^2 \leq \|v\|_{-1} |v|_1 \leq \beta |v|_1^2 + \frac{1}{4\beta}\|v\|_{-1}^2 \quad \forall v \in V_0. \tag{2.1.11}$$

We also require a σ-version of Young's inequality (see e.g. Malek [51], p.26)

$$ab \leq \sigma a^p + C(\sigma^{-1}) b^q, \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \qquad \forall a, b \geq 0,\ \sigma, p, q > 0. \tag{2.1.12}$$

For later purpose we mention the Hölder's inequality (see e.g. [1], p.23): For $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$|fg|_{0,1} = \int_\Omega |fg| dx \leq \Big(\int_\Omega |f|^p dx \Big)^{\frac{1}{p}} \Big(\int_\Omega |g|^q dx \Big)^{\frac{1}{q}} = |f|_{0,p}\, |g|_{0,q}. \tag{2.1.13}$$

One can generalise this inequality by applying it for example twice to yield

$$|fgh|_{0,1} \leq |f|_{0,p}\, |g|_{0,q}\, |f|_{0,r}, \qquad \text{where } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1. \tag{2.1.14}$$

For later reference we define the mean integral as

$$\fint \eta := \frac{1}{|\Omega|}(\eta, 1) \qquad \forall\, \eta \in L^1(\Omega). \tag{2.1.15}$$

and it is easily seen that

$$\eta - \fint \eta \in V_0 \qquad \forall\, \eta \in H^1(\Omega). \tag{2.1.16}$$

Lemma 2.1.1 The norms (2.1.1) and (2.1.7) are equivalent on $\mathcal{F}_0$.

Proof. Let $0 \neq f \in \mathcal{F}_0$. From (2.1.1) and (2.1.5) we have that

$$\|f\|_{(H^1(\Omega))'} = \sup_{\|v\|_1 = 1} |\langle f, v \rangle| = \sup_{\|v\|_1 = 1} |(\nabla \mathcal{G} f, \nabla v)| \leq \sup_{\|v\|_1 = 1} \|f\|_{-1}\, |v|_1 \leq \|f\|_{-1}.$$

Now by taking $v = \frac{\mathcal{G} f}{\|\mathcal{G} f\|_1} \in H^1(\Omega)$ we deduce using (2.1.7) that

$$\|f\|_{(H^1(\Omega))'} \geq |\langle f, v \rangle| = \frac{|\langle f, \mathcal{G} f \rangle|}{\|\mathcal{G} f\|_1} = \frac{\|f\|_{-1}^2}{\|\mathcal{G} f\|_1} \geq C \frac{\|f\|_{-1}^2}{|\mathcal{G} f|_1} = C\|f\|_{-1},$$

where we have applied (2.1.6) to give $\|\mathcal{G} f\|_1^2 = |\mathcal{G} f|_0^2 + |\mathcal{G} f|_1^2 \leq (C_p^2 + 1)|\mathcal{G} f|_1^2$ and hence $\|\mathcal{G} f\|_1 \leq C |\mathcal{G} f|_1$. □

Throughout the thesis C stands for a generic bounded positive constant, not necessarily the same at different occurrences, which is independent of the regularization parameter ε , the spatial parameter h and the time step Δt, and possibly depending on T, Ω, u_1^0, u_2^0 and δ_0. Furthermore, the symbol $C(\beta)$ denotes a constant depending on the argument β such that $C(\beta) \leq C$ if $\beta \leq C$.

2.2 The regularization and equivalent weak formulations

We adapt a regularization procedure similar to that employed in Elliott and Luckhaus [49]. This procedure is based on introducing a twice continuously differentiable function $\Phi_\varepsilon \in C^2(\mathbb{R})$ such that $\varepsilon \in (0,1)$ and

$$\Phi_\varepsilon(r) = \begin{cases} \frac{\theta}{4\varepsilon} r^2 + \frac{\theta}{2} r \ln \varepsilon - \frac{\theta\varepsilon}{4} & \text{if } r \le \varepsilon, \\ \Phi(r) \equiv \frac{\theta}{2} r \ln r & \text{if } r \ge \varepsilon. \end{cases} \tag{2.2.1}$$

We then define $\psi_\varepsilon \in C^2(\mathbb{R})$ to be

$$\psi_\varepsilon(r) = \Phi_\varepsilon(1+r) + \Phi_\varepsilon(1-r) = \begin{cases} \Phi(1+r) + \Phi_\varepsilon(1-r) & \text{if } r \ge 1-\varepsilon, \\ \psi(r) \equiv \Phi(1+r) + \Phi(1-r) & \text{if } |r| \le 1-\varepsilon, \\ \Phi_\varepsilon(1+r) + \Phi(1-r) & \text{if } r \le -1+\varepsilon. \end{cases} \tag{2.2.2}$$

Thus, for $i = 1, 2$ we regularize the potential Ψ_i by introducing $\Psi_{\varepsilon,i} \in C^2(\mathbb{R})$ such that

$$\Psi_{\varepsilon,i}(r) = \psi_\varepsilon(r) + \frac{\theta_i}{2}(1 - r^2). \tag{2.2.3}$$

We also introduce the monotone odd function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$

$$\phi_\varepsilon(r) := \psi_\varepsilon'(r) = \begin{cases} \Phi'(1+r) - \Phi_\varepsilon'(1-r) & \text{if } r \ge 1-\varepsilon, \\ \psi'(r) \equiv \phi(r) \equiv \Phi'(1+r) - \Phi'(1-r) & \text{if } |r| \le 1-\varepsilon, \\ \Phi_\varepsilon'(1+r) - \Phi'(1-r) & \text{if } r \le -1+\varepsilon. \end{cases} \tag{2.2.4}$$

Below we report some properties of the above functions that we need throughout the thesis:

For all $\varepsilon \in (0,1)$

$$\phi_\varepsilon(r) \le \phi(r) \qquad \forall\, r \in [1-\varepsilon, 1) \quad \text{and} \quad \phi(r) \le \phi_\varepsilon(r) \qquad \forall\, r \in (-1, -1+\varepsilon]. \tag{2.2.5}$$

For $i = 1, 2$ and for all r, s

$$\begin{aligned} \Psi_{\varepsilon,i}'(r)(s-r) = \psi_\epsilon'(r)(s-r) - \theta_i r(s-r) &\le \psi_\varepsilon(s) - \psi_\varepsilon(r) + \theta_i r(r-s) \\ &= \Psi_{\varepsilon,i}(s) - \Psi_{\varepsilon,i}(r) + \frac{\theta_i}{2}(s-r)^2, \end{aligned} \tag{2.2.6}$$

where we have noted the Taylor expansion, the fact that $\psi''_\varepsilon \equiv \phi'_\varepsilon > 0$ and the identity

$$2a(a-b) = a^2 - b^2 + (a-b)^2. \tag{2.2.7}$$

For $\varepsilon \le \frac{1}{2}$ and for all r, s

$$\theta \le \phi'_\varepsilon(r) \le \frac{\theta}{\varepsilon}, \tag{2.2.8}$$

$$\theta(s-r)^2 \le (\phi_\varepsilon(s) - \phi_\varepsilon(r))(s-r), \tag{2.2.9}$$

$$(\phi_\varepsilon(s) - \phi_\varepsilon(r))^2 \le \frac{\theta}{\varepsilon}(\phi_\varepsilon(s) - \phi_\varepsilon(r))(s-r). \tag{2.2.10}$$

Note that (2.2.10) implies, using the monotonicity of ϕ_ε, that

$$|\phi_\varepsilon(s) - \phi_\varepsilon(r)| \le \frac{\theta}{\varepsilon}|s-r|, \tag{2.2.11}$$

which means that ϕ_ε is a Lipschitz continuous with Lipschitz constant $\frac{\theta}{\varepsilon}$. In addition, if $r, s > 1-\varepsilon$ or $r, s < -1+\varepsilon$, then

$$\frac{\theta}{2\varepsilon}(s-r)^2 \le (\phi_\varepsilon(s) - \phi_\varepsilon(r))(s-r). \tag{2.2.12}$$

We also have for any $r \in [a,b] \subset [-1+\varepsilon, 1-\varepsilon]$ that

$$\phi'(r) = \phi'_\varepsilon(r) \le \phi'_\varepsilon(\max\{|a|,|b|\}) = \phi'(\max\{|a|,|b|\}). \tag{2.2.13}$$

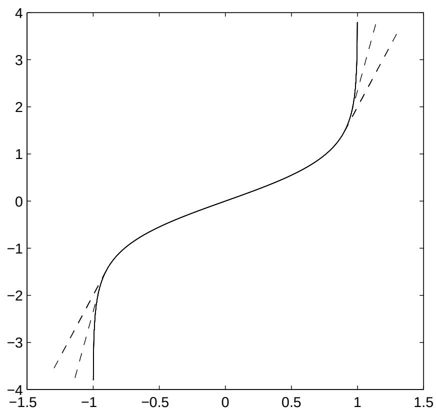


Figure 2.1: The monotone functions ϕ, denoted —, and ϕ_ε with two values of ε, denoted - - -.

For later purpose we mention properties of the monotone functions $\phi_\varepsilon^{-1}:\mathbb{R}\to\mathbb{R}$ and $\phi^{-1}:\mathbb{R}\to(-1,1)$. It follows from (2.2.5) that

$$\phi_\varepsilon^{-1}(r)\geq\phi^{-1}(r)\qquad\forall\, r\geq\phi(1-\varepsilon)=\phi_\varepsilon(1-\varepsilon),\tag{2.2.14}$$

$$\phi^{-1}(r)\geq\phi_\varepsilon^{-1}(r)\qquad\forall\, r\leq\phi(-1+\varepsilon)=\phi_\varepsilon(-1+\varepsilon),\tag{2.2.15}$$

and from(2.2.9) we obtain for all s, r

$$|\phi_\varepsilon^{-1}(s)-\phi_\varepsilon^{-1}(r)|\leq\theta^{-1}|s-r|.\tag{2.2.16}$$

The next lemma shows important results about $\Psi_{\varepsilon,i}$, ϕ and ϕ_ε

Lemma 2.2.1

$$\text{(i)}\ \forall\,\varepsilon\leq\varepsilon_0:=\min\{\frac{\theta}{4\theta_1},\frac{\theta}{4\theta_2}\},\ \Psi_{\varepsilon,i}(r)\geq-\frac{8\theta_i^2+\theta^2}{16\theta_i}=:-C_0\qquad i=1,2\text{ and }r\in\mathbb{R},\tag{2.2.17}$$

$$\text{(ii)}|\phi_\varepsilon^{-1}(r)-\phi^{-1}(r)|\leq\frac{2\varepsilon}{\theta}\Big([r-\phi(1-\varepsilon)]_++[-r-\phi(1-\varepsilon)]_+\Big)\ r\in\mathbb{R},\tag{2.2.18}$$

where $[\cdot]_+:=\max\{\cdot,0\}$.

Proof. To prove (2.2.17) we note from (2.2.3) and (2.2.2) that for $r\in[0,1]$ and $i=1,2$

$$\Psi_{\varepsilon,i}(r)\geq\psi_\varepsilon(r)\geq\psi_\varepsilon(0)=0\geq-\frac{8\theta_i^2+\theta^2}{16\theta_i}.$$

Again using (2.2.3) and (2.2.2) with the aid of the Young inequality we obtain under the stated assumption on ε that for $r>1$ and $i=1,2$

$$\begin{aligned}\Psi_{\varepsilon,i}(r)&\geq\frac{\theta}{4\varepsilon}(r-1)^2-\frac{\theta\varepsilon}{4}+\frac{\theta_i}{2}(1-r^2)=(\frac{\theta}{4\varepsilon}-\frac{\theta_i}{2})(r-1)^2-\theta_i(r-1)-\frac{\theta\varepsilon}{4}\\&\geq(\frac{\theta}{4\varepsilon}-\theta_i)(r-1)^2-\frac{\theta_i}{2}-\frac{\theta\varepsilon}{4}\geq-\frac{\theta_i}{2}-\frac{\theta\varepsilon}{4}\geq-\frac{8\theta_i^2+\theta^2}{16\theta_i}.\end{aligned}$$

Utilizing the fact that $\Psi_{\varepsilon,i}$ is even function, the desired result (2.2.17) therefore follows immediately.

We now turn to proving (2.2.18). Since $\phi_\varepsilon^{-1}(r)=\phi^{-1}(r)$ for $|r|\leq\phi(1-\varepsilon)$, (2.2.18) holds for $|r|\leq\phi(1-\varepsilon)$. For $r>\phi(1-\varepsilon)=\phi_\varepsilon(1-\varepsilon)$ we have by the monotonicity of ϕ_ε^{-1} and ϕ^{-1} that $\phi_\varepsilon^{-1}(r), \phi^{-1}(r) > 1-\varepsilon$. Hence, using (2.2.14), (2.2.12) with $r = \phi^{-1}(r)$ and $s = \phi_\varepsilon^{-1}(r)$ we obtain after noting monotonicity of ϕ_ε that

$$\begin{aligned}|\phi_\varepsilon^{-1}(r) - \phi^{-1}(r)| = \phi_\varepsilon^{-1}(r) - \phi^{-1}(r) &\leq \frac{2\varepsilon}{\theta}(r - \phi_\varepsilon(\phi^{-1}(r)))\\ &\leq \frac{2\varepsilon}{\theta}(r - \phi_\varepsilon(1-\varepsilon)) = \frac{2\varepsilon}{\theta}(r - \phi(1-\varepsilon)).\end{aligned} \tag{2.2.19}$$

Noting that ϕ_ε^{-1} and ϕ^{-1} are odd functions we deduce from (2.2.19) that for $r < -\phi(1-\varepsilon)$

$$|\phi_\varepsilon^{-1}(r) - \phi^{-1}(r)| \leq \frac{2\varepsilon}{\theta}(-r - \phi(1-\varepsilon)). \tag{2.2.20}$$

This result together with (2.2.19) gives the required inequality (2.2.18). □

Remark. The previous lemma shows that $\Psi_{\varepsilon,i}$ is bounded below for sufficiently small ε and it also shows that $|\phi_\varepsilon^{-1}(r) - \phi^{-1}(r)| \to 0$ as $\varepsilon \to 0$.

Now we introduce a regularized version $(\mathbf{P}_\varepsilon)$ of $(\mathbf{P})$:

$(\mathbf{P}_\varepsilon)$ Find $\{u_{\varepsilon,1}, u_{\varepsilon,2}, w_{\varepsilon,1}, w_{\varepsilon,2}\} \in [H^1(\Omega)]^4$ such that for $i = 1,2$ $u_{\varepsilon,i}(0) = u_i^0$ and for *a.e.* $t \in (0,T)$ and all $\eta \in H^1(\Omega)$

$$\langle \partial_t u_{\varepsilon,i}, \eta\rangle \ + (\nabla w_{\varepsilon,i}, \nabla\eta) = 0, \tag{2.2.21a}$$

$$\gamma(\nabla u_{\varepsilon,i}, \nabla\eta) + (\Psi'_{\varepsilon,i}(u_{\varepsilon,i}), \eta) + (f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}), \eta) = (w_{\varepsilon,i}, \eta). \tag{2.2.21b}$$

For convenience we shall give an equivalent form to $(\mathbf{P}_\varepsilon)$. Choosing $\eta = 1$ in (2.2.21a) leads to $\partial_t u_{\varepsilon,i} \in \mathcal{F}_0$ and $(u_{\varepsilon,i}(t), 1) = (u_i^0, 1)$ *a.e* $t \in (0,T)$, $i = 1,2$.

From the definition of $\mathcal{G}$, (2.1.5), and (2.2.21a) we deduce for $i = 1,2$ that

$$(\nabla(\mathcal{G}\partial_t u_{\varepsilon,i} + w_{\varepsilon,i}), \nabla\eta) = 0 \qquad a.e.\ t \in (0,T) \text{ and } \forall\ \eta \in H^1(\Omega).$$

Hence, by taking $\eta = \mathcal{G}\partial_t u_{\varepsilon,i} + w_{\varepsilon,i}$ we obtain

$$|\mathcal{G}\partial_t u_{\varepsilon,i} + w_{\varepsilon,i} - \fint w_{\varepsilon,i}|_1 = |\mathcal{G}\partial_t u_{\varepsilon,i} + w_{\varepsilon,i}|_1 = 0.$$

Thus, with the use of the Poincaré inequality and (2.1.16) it follows that

$$|\mathcal{G}\partial_t u_{\varepsilon,i} + w_{\varepsilon,i} - \fint w_{\varepsilon,i}|_0 \leq C_p|\mathcal{G}\partial_t u_{\varepsilon,i} + w_{\varepsilon,i} - \fint w_{\varepsilon,i}|_1 = 0.$$

We therefore have for $i = 1, 2$ and *a.e.* $t \in (0, T)$

$$w_{\varepsilon,i} = -\mathcal{G}\partial_t u_{\varepsilon,i} + \fint w_{\varepsilon,i}. \tag{2.2.22}$$

In addition, from (2.2.21b) we find

$$\fint w_{\varepsilon,i} = \fint \big(\Psi'_{\varepsilon,i}(u_{\varepsilon,i}) + f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2})\big). \tag{2.2.23}$$

Therefore, $(\mathbf{P}_\varepsilon)$ can be restated equivalently as:

$(\mathbf{P}_\varepsilon)$ Find $\{u_{\varepsilon,1}, u_{\varepsilon,2}\} \in [H^1(\Omega)]^2$ such that $u_{\varepsilon,i}(0) = u_i^0$, $i = 1, 2$, and for *a.e.* $t \in (0, T)$ $(u_{\varepsilon,i}(t), 1) = (u_i^0, 1)$ and

$$\gamma(\nabla u_{\varepsilon,i}, \nabla\eta) + (\Psi'_{\varepsilon,i}(u_{\varepsilon,i}), \eta - \fint \eta) + (f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}), \eta - \fint \eta) + (\mathcal{G}\partial_t u_{\varepsilon,i}, \eta) = 0 \tag{2.2.24}$$

for all $\eta \in H^1(\Omega)$.

Similarly, one can rewrite $(\mathbf{P})$ equivalently as:

$(\mathbf{P})$ Find $\{u_1, u_2\} \in [H^1(\Omega)]^2$ such that $u_i(0) = u_i^0$, $i = 1, 2$, and for *a.e.* $t \in (0, T)$ $(u_i(t), 1) = (u_i^0, 1)$ and

$$\gamma(\nabla u_i, \nabla\eta) + (\Psi'_i(u_i), \eta - \fint \eta) + (f_D^{(i)}(u_1, u_2), \eta - \fint \eta) + (\mathcal{G}\partial_t u_i, \eta) = 0 \tag{2.2.25}$$

for all $\eta \in H^1(\Omega)$.

2.3 Existence and uniqueness

In this section we prove existence and uniqueness of a solution to the continuous problem $(\mathbf{P})$ under the following assumptions on u_1^0 and u_2^0:

$(\mathbf{A_1})$ Let $\{u_1^0, u_2^0\} \in H^1(\Omega) \times H^1(\Omega)$ such that $\max\{|u_1^0|_{0,\infty},\ |u_2^0|_{0,\infty}\} \le 1$ and for some given $\delta_0 \in (0, 1)$, $\max\{|m_1| := |\fint u_1^0|, |m_2| := |\fint u_2^0|\} \le 1 - \delta_0$.

We will prove the existence relying on the classical Faedo-Galerkin method of Lions [24]. Let $\{z_j\}_{j=1}^\infty$ be an orthogonal basis for $H^1(\Omega)$ and orthonormal basis for $L^2(\Omega)$, consisting of the eigenfunctions of the elliptic eigenvalue problem

$$-\Delta z_j + z_j = \mu_j z_j \qquad \text{in } \Omega, \qquad \frac{\partial z_j}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{2.3.1}$$

It is well-known that (e.g. [10], [34]) z_1 is constant and the sequence $\{\mu_j\}_{j=1}^{\infty}$ is nondecreasing where $\mu_1 = 1$. We observe that the weak form of (2.3.1), using $(z_i, z_j) = \delta_{ij}$, implies

$$(\nabla z_i, \nabla z_j) = (\mu_i - 1)\delta_{ij}.$$

For $k \geq 1$ we consider V^k to be the finite dimensional subspace spanned by $\{z_j\}_{j=1}^{k}$. Let $P^k v$ be the projection of $v \in L^2(\Omega)$ onto V^k such that

$$P^k v := \sum_{j=1}^{k} (v, z_j) z_j. \tag{2.3.2}$$

Obviously this definition still makes sense for any $v \in H^1(\Omega) \subset L^2(\Omega)$. From (2.3.2) one can easily deduce the following properties of the projection P^k

$$(P^k v, \chi^k) = (v, \chi^k) \qquad \forall \chi^k \in V^k,\ v \in L^2(\Omega), \tag{2.3.3a}$$

$$(\nabla P^k v, \nabla \chi^k) = (\nabla v, \nabla \chi^k) \qquad \forall \chi^k \in V^k,\ v \in H^1(\Omega) \tag{2.3.3b}$$

and it is easily seen from (2.3.3a) and (2.3.3b) that

$$|P^k v|_m \leq |v|_m, \tag{2.3.4a}$$

$$|P^k v - v|_m \leq |\chi^k - v|_m \qquad \forall \chi^k \in V^k, \tag{2.3.4b}$$

where $m = 0$ if $v \in L^2(\Omega)$ and $m = 0, 1$ if $v \in H^1(\Omega)$.

Using the result (2.3.4b) together with the fact that $\{V^k : k \geq 1\}$ is dense in $L^2(\Omega)$ and $H^1(\Omega)$ we have that

$$P^k v \to v \qquad \text{in } L^2(\Omega) \text{ and } H^1(\Omega), \tag{2.3.5}$$

where "$\to$" represents the strong convergence.

We require the following lemma to facilitate dealing with the nonlinearity D-coupling term.

Lemma 2.3.1 Let $v \in V_0$. Then there are constants $\sigma = d(\frac{1}{2} - \frac{1}{r})$ and C such that for all $\beta > 0$

$$|v|_{0,r}^2 \leq C\|v\|_{-1}^{1-\sigma}|v|_1^{1+\sigma} \leq \beta|v|_1^2 + C(\beta^{-1})\|v\|_{-1}^2 \text{ holds for } r \in \begin{cases} [2,\infty] & \text{if } d=1, \\ [2,\infty) & \text{if } d=2, \\ [2,6) & \text{if } d=3. \end{cases} \tag{2.3.6}$$

Proof. Using Poincaré's inequality gives $\|v\|_1 \leq C|v|_1$. Thus by (2.1.4) and the first inequality in (2.1.11) we obtain

$$|v|_{0,r}^2 \leq C\big(|v|_0^2\big)^{1-\sigma}\|v\|_1^{2\sigma} \leq C\|v\|_{-1}^{1-\sigma}|v|_1^{1-\sigma}|v|_1^{2\sigma} = C\|v\|_{-1}^{1-\sigma}|v|_1^{1+\sigma}. \tag{2.3.7}$$

Finally, the second inequality follows as a consequence of applying the Young inequality with $p = \frac{2}{1-\sigma}$ and $q = \frac{2}{1+\sigma}$. □

Theorem 2.3.2 Let the assumptions $(\mathbf{A_1})$ hold. Then for all $\varepsilon \leq \varepsilon_0$, $(\mathbf{P}_\varepsilon)$ possesses a unique solution $\{u_{\varepsilon,1}, u_{\varepsilon,2}, w_{\varepsilon,1}, w_{\varepsilon,2}\}$ such that for $i = 1,2$ the following estimates hold independently of ε

$$\|u_{\varepsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} + \|u_{\varepsilon,i}\|_{H^1(0,T;(H^1(\Omega))')} \leq C, \tag{2.3.8a}$$

$$\|w_{\varepsilon,i}\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{2.3.8b}$$

$$\|\phi_\varepsilon(u_{\varepsilon,i})\|_{L^2(\Omega_T)} \leq C, \tag{2.3.8c}$$

$$\|f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2})\|_{L^\infty(0.T;L^2(\Omega))} \leq C. \tag{2.3.8d}$$

Further, the unique solution satisfies for $i = 1,2$

$$\theta^{-1}\varepsilon\|\nabla\phi_\varepsilon(u_{\varepsilon,i})\|_{L^2(\Omega_T)}^2 \leq \int_0^T (\nabla u_{\varepsilon,i}, \nabla\phi_\varepsilon(u_{\varepsilon,i}))\, dt \leq C. \tag{2.3.9}$$

Proof. For $k \geq 1$ we seek the Galerkin approximations $\{u_{\varepsilon,1}^k, u_{\varepsilon,2}^k, w_{\varepsilon,1}^k, w_{\varepsilon,2}^k\} \in \big(V^k\big)^4$ solving for $i = 1,2$, $t \in [0,T]$ and for all $\chi^k \in V^k$

$$(\partial_t u_{\varepsilon,i}^k, \chi^k) \ + (\nabla w_{\varepsilon,i}^k, \nabla\chi^k) = 0, \tag{2.3.10a}$$

$$\gamma(\nabla u_{\varepsilon,i}^k, \nabla\chi^k) + (\Psi'_{\varepsilon,i}(u_{\varepsilon,i}^k), \chi^k) + (f_D^{(i)}(u_{\varepsilon,1}^k, u_{\varepsilon,2}^k), \chi^k) = (w_{\varepsilon,i}^k, \chi^k), \tag{2.3.10b}$$

$$u_{\varepsilon,i}^k(0) = P^k u_i^0. \tag{2.3.10c}$$

The Galerkin approximations can be represented as

$$u^k_{\varepsilon,i}(x,t) = \sum_{n=1}^{k} a^k_{in}(t) z_n(x), \qquad w^k_{\varepsilon,i}(x,t) = \sum_{n=1}^{k} b^k_{in}(t) z_n(x) \qquad i = 1,2. \quad (2.3.11)$$

We first establish the local existence of the Galerkin approximations. To this aim, we insert (2.3.11) into (2.3.10a-b) and take $\chi^k = z_j$ to yield a system of $2k$ ODEs in a^k_{1j} and a^k_{2j} for $j = 1, 2, \ldots, k$ as follows

$$\begin{aligned}
\frac{da^k_{1j}(t)}{dt} &= (1-\mu_j) b^k_{1j}(t), \\
\frac{da^k_{2j}(t)}{dt} &= (1-\mu_j) b^k_{2j}(t), \\
b^k_{1j} &= \gamma(\mu_j - 1) a^k_{1j}(t) + (H(a^k_1))_j + (G_1(a^k_1, a^k_2))_j, \\
b^k_{2j} &= \gamma(\mu_j - 1) a^k_{2j}(t) + (H(a^k_2))_j + (G_2(a^k_1, a^k_2))_j,
\end{aligned}$$

with initial conditions

$$\begin{aligned}
a^k_{1j}(0) &= (P^k u^0_1, z_j) = (u^0_1, z_j), \\
a^k_{2j}(0) &= (P^k u^0_2, z_j) = (u^0_2, z_j),
\end{aligned}$$

where

$$\begin{aligned}
a^k_i &= (a^k_{i1}, a^k_{i2}, \ldots, a^k_{ik})^T, \ b^k_i = (b^k_{i1},\ b^k_{i2}, \ldots, b^k_{ik})^T \qquad i = 1,2, \\
(H(a^k_i))_j &= (\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}), z_j), \ (G_i(a^k_1, a^k_2))_j = (f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), z_j) \qquad i = 1,2.
\end{aligned}$$

Letting $\hat{a} = (a^k_1, a^k_2)^T$, the above system can be written as $\frac{d\hat{a}}{dt} = \hat{F}(\hat{a})$ where $\hat{a}(0) = (a^k_1(0), a^k_2(0))^T$ and $\hat{F}$ is locally Lipschitz as $\Psi'_{\varepsilon,i}$ and $f^{(i)}_D$ are locally Lipschitz. Thus from standard existence theory for a system of ODEs, one concludes that the system has a unique solution on some finite time interval $(0, t_k)$, $t_k > 0$.

Now, we prove the global existence of the Galerkin approximations by deriving *a priori* estimates bounding $\{u^k_{\varepsilon,i}, w^k_{\varepsilon,i}\}_{i=1,2}$ independently of k in various Banach spaces.

Testing (2.3.10a) with $\chi^k = 1 \in V^k$ gives for $i = 1,2$ and for all $t \in (0,T)$ that $\frac{\partial u^k_{\varepsilon,i}}{\partial t} \in V_0$ and

$$(u^k_{\varepsilon,i}(t),1) = (u^k_{\varepsilon,i}(0),1) = (P^k u^0_i,1) = (u^0_i,1) = m_i|\Omega|, \tag{2.3.12}$$

where we have also noted the P^k projection property (2.3.3a).

For $i = 1,2$ we take $\chi^k = P^k(\mathcal{G}\partial_t u^k_{\varepsilon,i}) \in V^k$ in (2.3.10a) and we use (2.3.3a-b), the $\|\cdot\|_{-1}$ definition, (2.1.7), and the $\mathcal{G}$ definition, (2.1.5), to result in

$$0 = (\partial_t u^k_{\varepsilon,i}, \mathcal{G}\partial_t u^k_{\varepsilon,i}) + (\nabla w^k_{\varepsilon,i}, \nabla\mathcal{G}\partial_t u^k_{\varepsilon,i}) = \|\partial_t u^k_{\varepsilon,i}\|^2_{-1} + (w^k_{\varepsilon,i}, \partial_t u^k_{\varepsilon,i}). \tag{2.3.13}$$

Choosing $\chi^k = \partial_t u^k_{\varepsilon,i}$ in (2.3.10b) and combining the resulting equation with (2.3.13) yields after summing over $i = 1,2$ that

$$\begin{aligned}\frac{\gamma}{2}\frac{d}{dt}\big[|u^k_{\varepsilon,1}|^2_1 + |u^k_{\varepsilon,2}|^2_1\big] &+ \big[(\Psi'_{\varepsilon,1}(u^k_{\varepsilon,1}), \partial_t u^k_{\varepsilon,1}) + (\Psi'_{\varepsilon,2}(u^k_{\varepsilon,2}), \partial_t u^k_{\varepsilon,2})\big] \\ &+ \big[(f_D^{(1)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \partial_t u^k_{\varepsilon,1}) + (f_D^{(2)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \partial_t u^k_{\varepsilon,2})\big] \\ &+ \big[\|\partial_t u^k_{\varepsilon,1}\|^2_{-1} + \|\partial_t u^k_{\varepsilon,2}\|^2_{-1}\big] = 0, \end{aligned} \tag{2.3.14}$$

where we have also noted $(\nabla u^k_{\varepsilon,i}, \nabla\partial_t u^k_{\varepsilon,i}) = \frac{1}{2}\frac{d}{dt}|u^k_{\varepsilon,i}|^2_1$.

By noting first that for $t \in (0,T)$, $f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}) := \partial_{u^k_{\varepsilon,i}} f_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})$,

$$\begin{aligned}\int_0^t \big[(f_D^{(1)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \partial_s u^k_{\varepsilon,1}) &+ (f_D^{(2)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \partial_s u^k_{\varepsilon,2})\big]ds \\ &= \int_\Omega \int_0^t \frac{d}{ds}(f_D(u^k_{\varepsilon,1}(s), u^k_{\varepsilon,2}(s)))ds \\ &= (f_D(u^k_{\varepsilon,1}(t), u^k_{\varepsilon,2}(t)),1) - (f_D(u^k_{\varepsilon,1}(0), u^k_{\varepsilon,2}(0)),1)\end{aligned}$$

and then integrating (2.3.14) over $t \in (0,T]$ we obtain

$$\begin{aligned}\Lambda_\varepsilon(u^k_{\varepsilon,1}(t), u^k_{\varepsilon,2}(t)) &+ \int_0^t \big[\|\partial_s u^k_{\varepsilon,1}\|^2_{-1} + \|\partial_s u^k_{\varepsilon,2}\|^2_{-1}\big]ds \\ &= \Lambda_\varepsilon(u^k_{\varepsilon,1}(0), u^k_{\varepsilon,2}(0)) = \Lambda_\varepsilon(P^k u^0_1, P^k u^0_2),\end{aligned} \tag{2.3.15}$$

where

$$\Lambda_\varepsilon(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}) := \frac{\gamma}{2}[|u^k_{\varepsilon,1}|^2_1 + |u^k_{\varepsilon,2}|^2_1] + (\Psi_{\varepsilon,1}(u^k_{\varepsilon,1}) + \Psi_{\varepsilon,2}(u^k_{\varepsilon,2}),1) + (f_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}),1). \tag{2.3.16}$$

Our goal now is to prove that $\Lambda_\varepsilon(P^k u^0_1, P^k u^0_2)$ is bounded for sufficiently large k.

Recalling (2.3.4a), a generalised Hölder's inequality and, by (2.1.4), $H^1(\Omega) \hookrightarrow L^4(\Omega)$ yields after noting the assumptions ($\mathbf{A_1}$)

$$\begin{aligned}
&\frac{\gamma}{2}[|P^k u_1^0|_1^2 + |P^k u_2^0|_1^2] + (f_D(P^k u_1^0, P^k u_1^0), 1) \\
&\qquad \le \frac{\gamma}{2}\big[|u_1^0|_1^2 + |u_2^0|_1^2\big] + 2D\big|(P^k u_1^0 + \alpha_1)^2 (P^k u_2^0 + \alpha_2)^2\big|_{0,1} \\
&\qquad = \frac{\gamma}{2}\big[|u_1^0|_1^2 + |u_2^0|_1^2\big] + 2D|P^k u_1^0 + \alpha_1|_{0,4}^2 |P^k u_2^0 + \alpha_2|_{0,4}^2 \\
&\qquad \le \frac{\gamma}{2}\big[|u_1^0|_1^2 + |u_2^0|_1^2\big] + C\|P^k u_1^0 + \alpha_1\|_1^2 \|P^k u_2^0 + \alpha_2\|_1^2 \\
&\qquad \le \frac{\gamma}{2}\big[|u_1^0|_1^2 + |u_2^0|_1^2\big] + C\big(\|u_1^0\|_1^2 + 1\big)\big(\|u_2^0\|_1^2 + 1\big) \le C. \qquad (2.3.17)
\end{aligned}$$

On setting $s = u_i^0,\ r = P^k u_i^0,\ i = 1, 2$, in (2.2.6) and on noting $\Psi'_{\varepsilon,i}(r) = \phi_\varepsilon(r) - \theta_i r$, the Lipschitz continuity of ϕ_ε, (2.2.11), and (2.3.4a) we have

$$\begin{aligned}
(\Psi_{\varepsilon,i}(P^k u_i^0), 1) &= (\Psi_{\varepsilon,i}(P^k u_i^0) - \Psi_{\varepsilon,i}(u_i^0), 1) + (\Psi_{\varepsilon,i}(u_i^0), 1) \\
&\le (\Psi'_{\varepsilon,i}(P^k u_i^0), P^k u_i^0 - u_i^0) + \frac{\theta_i}{2}((P^k u_i^0 - u_i^0)^2, 1) + (\Psi_{\varepsilon,i}(u_i^0), 1) \\
&\le \big[|\phi_\varepsilon(P^k u_i^0)|_0 + \theta_i |P^k u_i^0|_0\big] |P^k u_i^0 - u_i^0|_0 + \frac{\theta_i}{2}|P^k u_i^0 - u_i^0|_0^2 + (\Psi_{\varepsilon,i}(u_i^0), 1) \\
&\le \big[\frac{\theta}{\varepsilon} + \theta_i\big] |u_i^0|_0 |P^k u_i^0 - u_i^0|_0 + \frac{\theta_i}{2}|P^k u_i^0 - u_i^0|_0^2 + (\Psi_{\varepsilon,i}(u_i^0), 1).
\end{aligned} \qquad (2.3.18)$$

Thus, by the strong convergence of $P^k u_i^0 \to u_i^0$ in $L^2(\Omega)$, the assumptions ($\mathbf{A_1}$), (2.2.3) and the fact that $\psi_\varepsilon(r) \le \psi_\varepsilon(1)\ \forall\, r \in [-1, 1]$ it follows that

$$\limsup_{k\to\infty} (\Psi_{\varepsilon,i}(P^k u_i^0), 1) \le (\Psi_{\varepsilon,i}(u_i^0), 1) \le (\psi_\varepsilon(1) + \frac{\theta_i}{2}, 1) \le (\theta \ln 2 + \frac{\theta_i}{2})|\Omega|. \qquad (2.3.19)$$

Combining (2.3.17), (2.3.19) and (2.3.15) gives thus for k sufficiently large

$$\Lambda_\varepsilon(u_{\varepsilon,1}^k(t), u_{\varepsilon,2}^k(t)) + \int_0^t \big[\|\partial_s u_{\varepsilon,1}^k\|_{-1}^2 + \|\partial_s u_{\varepsilon,2}^k\|_{-1}^2\big] ds = \Lambda_\varepsilon(P^k u_1^0, P^k u_2^0) \le C. \qquad (2.3.20)$$

Recalling, by Lemma 2.2.1, that for $\varepsilon \le \varepsilon_0$ $\Psi_{\varepsilon,i}(\cdot),\ i = 1, 2$, is bounded below and that $f_D(r, s) \ge 0$ we obtain from (2.3.20) and (2.3.16) that for all $t \in (0, T]$

$$\frac{\gamma}{2}\big[|u_{\varepsilon,1}^k(t)|_1^2 + |u_{\varepsilon,2}^k(t)|_1^2\big] + \int_0^t \big[\|\partial_s u_{\varepsilon,1}^k\|_{-1}^2 + \|\partial_s u_{\varepsilon,2}^k\|_{-1}^2\big] ds \le C. \qquad (2.3.21)$$

With the aid of the Poincaré inequality and (2.3.12) we find after ignoring the non-negative integral of (2.3.21)

$$\|u_{\varepsilon,1}^k(t)\|_1 + \|u_{\varepsilon,2}^k(t)\|_1 \le C, \qquad (2.3.22)$$

which implies

$$\|u^k_{\varepsilon,1}\|_{L^\infty(0,T;H^1(\Omega))} + \|u^k_{\varepsilon,2}\|_{L^\infty(0,T;H^1(\Omega))} \leq C. \tag{2.3.23}$$

This time we ignore the H^1-semi norms from (2.3.21) to yield

$$\int_0^T \left[\|\partial_t u^k_{\varepsilon,1}\|^2_{-1} + \|\partial_t u^k_{\varepsilon,2}\|^2_{-1}\right] dt \leq C. \tag{2.3.24}$$

Thus we have, noting Lemma 2.1.1,

$$\|\partial_t u^k_{\varepsilon,1}\|_{L^2(0,T;(H^1(\Omega))')} + \|\partial_t u^k_{\varepsilon,2}\|_{L^2(0,T;(H^1(\Omega))')} \leq C. \tag{2.3.25}$$

Since $H^1(\Omega) \hookrightarrow (H^1(\Omega))'$, (2.3.22) gives $\|u^k_{\varepsilon,i}\|_{L^\infty(0,T;(H^1(\Omega))')} \leq C,\ i = 1, 2$. We then use this result with the fact that $L^\infty(0,T;(H^1(\Omega))') \hookrightarrow L^2(0,T;(H^1(\Omega))')$ to obtain

$$\|u^k_{\varepsilon,i}\|_{L^2(0,T;(H^1(\Omega))')} \leq C. \tag{2.3.26}$$

Therefore, (2.3.25) and (2.3.26) imply for $i = 1, 2$ that

$$\|u^k_{\varepsilon,i}\|_{H^1(0,T;(H^1(\Omega))')} \leq C. \tag{2.3.27}$$

From (2.3.10a), (2.1.5), (2.3.3a-b) and (2.1.7) it follows for $i = 1, 2$ that

$$\begin{aligned} |w^k_{\varepsilon,i}|^2_1 &= -(\partial_t u^k_{\varepsilon,i}, w^k_{\varepsilon,i}) = -(\nabla \mathcal{G}\partial_t u^k_{\varepsilon,i}, \nabla w^k_{\varepsilon,i}) = -(\nabla P^k \mathcal{G}\partial_t u^k_{\varepsilon,i}, \nabla w^k_{\varepsilon,i}) \\ &= (P^k \mathcal{G}\partial_t u^k_{\varepsilon,i}, \partial_t u^k_{\varepsilon,i}) = (\mathcal{G}\partial_t u^k_{\varepsilon,i}, \partial_t u^k_{\varepsilon,i}) = \|\partial_t u^k_{\varepsilon,i}\|^2_{-1}, \end{aligned} \tag{2.3.28}$$

and hence, owing to (2.3.24), we have

$$\int_0^T |w^k_{\varepsilon,i} - ⨍ w^k_{\varepsilon,i}|^2_1 dt = \int_0^T |w^k_{\varepsilon,i}|^2_1 dt = \int_0^T \|\partial_t u^k_{\varepsilon,i}\|^2_{-1} dt \leq C. \tag{2.3.29}$$

We apply the Poincaré inequality with $\eta = w^k_{\varepsilon,i} - ⨍ w^k_{\varepsilon,i} \in V_0$ and use (2.3.29) to give for $i = 1, 2$

$$\left\| w^k_{\varepsilon,i} - ⨍ w^k_{\varepsilon,i} \right\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{2.3.30}$$

To show $w^k_{\varepsilon,i}$ is bounded in $L^2(0,T;H^1(\Omega))$, it suffices to show that $⨍ w^k_{\varepsilon,i}$ is bounded in $L^2(0,T;H^1(\Omega))$.

By (2.3.10b) we first remark for $i = 1, 2$ that

$$⨍ w^k_{\varepsilon,i} = ⨍ \left[\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})\right]. \tag{2.3.31}$$

On setting $\chi^k = u^k_{\varepsilon,i} - \fint u^k_{\varepsilon,i} = u^k_{\varepsilon,i} - m_i,\ i = 1,2$, in (2.3.10b) and adding for any $\beta \in \mathbb{R}$, $(\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \beta)$ to the both sides yields after rearranging that

$$\begin{aligned}
&(\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \beta - m_i) = \\
&= -\gamma|u^k_{\varepsilon,i}|_1^2 + (w^k_{\varepsilon,i}, u^k_{\varepsilon,i} - m_i) + (\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}), \beta - u^k_{\varepsilon,i}) + (f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \beta - u^k_{\varepsilon,i}) \\
&\leq (\nabla w^k_{\varepsilon,i}, \nabla \mathcal{G}(u^k_{\varepsilon,i} - m_i)) + (\Psi_{\varepsilon,i}(\beta) - \Psi_{\varepsilon,i}(u^k_{\varepsilon,i}), 1) + \frac{\theta_i}{2}|\beta - u^k_{\varepsilon,i}|_0^2 \\
&\quad + |f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})|_0 |\beta - u^k_{\varepsilon,i}|_0 \\
&\leq |w^k_{\varepsilon,i}|_1 \|u^k_{\varepsilon,i} - m_i\|_{-1} + (\Psi_{\varepsilon,i}(\beta) - \Psi_{\varepsilon,i}(u^k_{\varepsilon,i}), 1) + \frac{\theta_i + 1}{2}|\beta - u^k_{\varepsilon,i}|_0^2 + \frac{1}{2}|f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})|_0^2 \\
&\leq C|w^k_{\varepsilon,i}|_1 |u^k_{\varepsilon,i} - m_i|_0 + C\big[1 + (\Psi_{\varepsilon,i}(\beta), 1) + |\beta - u^k_{\varepsilon,i}|_0^2 + |f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})|_0^2\big] \\
&\leq C\big[1 + |w^k_{\varepsilon,i}|_1 + (\Psi_{\varepsilon,i}(\beta), 1) + |\beta - u^k_{\varepsilon,i}|_0^2 + |f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})|_0^2\big], \qquad (2.3.32)
\end{aligned}$$

where we have used in turn: (2.1.5) and (2.2.6) with $r = u^k_{\varepsilon,i}$ and $s = \beta$, followed by (2.1.7), Young's inequality, (2.1.8), Lemma 2.2.1(i) and the bound (2.3.22).

Using a generalised Hölder's inequality and, by (2.1.4), $H^1(\Omega) \hookrightarrow L^6(\Omega)$ we have for $i, j = 1, 2$ with $i \neq j$

$$\begin{aligned}
\big|f_D^{(i)}(r_1, r_2)\big|_0^2 &= \big|2D(r_i + \alpha_i)(r_j + \alpha_j)^2\big|_0^2 = 4D^2\big|(r_i + \alpha_i)^2(r_j + \alpha_j)^4\big|_{0,1} \\
&\leq 4D^2|r_i + \alpha_i|_{0,6}^2 |r_j + \alpha_j|_{0,6}^4 \leq C\|r_i + \alpha_i\|_1^2 \|r_j + \alpha_j\|_1^4. \qquad (2.3.33)
\end{aligned}$$

This result with the aid of the bound (2.3.22) we obtain for $i, j = 1, 2$ with $i \neq j$

$$|f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})|_0^2 \leq C\|u^k_{\varepsilon,i} + \alpha_i\|_1^2 \|u^k_{\varepsilon,j} + \alpha_j\|_1^4 \leq C. \qquad (2.3.34)$$

Choosing $\beta = \pm 1 \mp \frac{\delta_0}{2}$ in (2.3.32) and noting $\Psi_{\varepsilon,i}(r) \leq \theta \ln 2 + \frac{\theta_i}{2}\ \forall r \in [-1, 1]$ and the bounds (2.3.22) and (2.3.34) leads to for $i = 1, 2$

$$(\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), 1 - \frac{\delta_0}{2} - m_i) \leq C\big[1 + |w_{\varepsilon,i}|_1\big]$$

and

$$(\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), 1 - \frac{\delta_0}{2} + m_i) \geq -C\big[1 + |w_{\varepsilon,i}|_1\big].$$

Dividing the above inequalities by $|\Omega|(1 - \frac{\delta_0}{2} - m_i)$ and $|\Omega|(1 - \frac{\delta_0}{2} + m_i)$ respectively we obtain after recalling the assumptions $(\mathbf{A_1})$, particularly $|m_i| \leq 1 - \delta_0$,

$$\left| \fint \big[\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})\big] \right| \leq C\big[1 + |w^k_{\varepsilon,i}|_1\big]. \qquad (2.3.35)$$

We square (2.3.35) and then integrate over $(0,T)$ and note (2.3.29) to obtain for $i=1,2$ that

$$\Big| \fint \big[\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f^{(i)}_D(u^k_{\varepsilon,1},u^k_{\varepsilon,2})\big]\Big|^2_{L^2(0,T)} \le C\big(T+\int_0^T |w_{\varepsilon,i}|_1^2 dt\big) \le C, \qquad (2.3.36)$$

which implies by (2.3.31) that

$$\begin{aligned}\Big\| \fint w^k_{\varepsilon,i}\Big\|^2_{L^2(0,T;H^1(\Omega))} &= \Big\| \fint \big[\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f^{(i)}_D(u^k_{\varepsilon,1},u^k_{\varepsilon,2})\big]\Big\|^2_{L^2(0,T;H^1(\Omega))} \\ &= |\Omega| \Big\| \fint \big[\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f^{(i)}_D(u^k_{\varepsilon,1},u^k_{\varepsilon,2})\big]\Big\|^2_{L^2(0,T)} \\ &\le C. \end{aligned} \qquad (2.3.37)$$

Therefore, we conclude from (2.3.30) and (2.3.37) that for $i=1,2$

$$\|w^k_{\varepsilon,i}\|_{L^2(0,T;H^1(\Omega))} \le C. \qquad (2.3.38)$$

Before moving onto the passage to the limit step of the proof we recall that $L^\infty(0,T;H^1(\Omega))$ is the dual space of $L^1(0,T;(H^1(\Omega))')$, which is a separable Banach space but not reflexive, while the Banach spaces $L^2(0,T;H^1(\Omega))$, $L^2(0,T;(H^1(\Omega))')$ and $L^2(\Omega_T)$ are reflexive. Thus, by compactness arguments (see Appendix A, Theorem A.0.15 and Theorem A.0.16) and the bounds (2.3.23), (2.3.27) and (2.3.38) we can extract subsequences, still denoted $\{u^k_{\varepsilon,i}\}, \{w^k_{\varepsilon,i}\}$, such that for $i=1,2$ and as $k\to\infty$

$$u^k_{\varepsilon,i} \rightharpoonup u_{\varepsilon,i} \quad \text{in } L^2(0,T;H^1(\Omega))\cap H^1(0,T;(H^1(\Omega))'), \qquad (2.3.39a)$$

$$u^k_{\varepsilon,i} \overset{*}{\rightharpoonup} u_{\varepsilon,i} \quad \text{in } L^\infty(0,T;H^1(\Omega)), \qquad (2.3.39b)$$

$$w^k_{\varepsilon,i} \rightharpoonup w_{\varepsilon,i} \quad \text{in } L^2(0,T;H^1(\Omega)), \qquad (2.3.39c)$$

where "$\rightharpoonup$" and "$\overset{*}{\rightharpoonup}$" denotes weak and weak-star convergence respectively.

From an application of the Lions-Aubin theorem (see appendix A, Theorem A.0.18) with $X_0 = H^1(\Omega)$, $X = L^2(\Omega)$, $X_1 = (H^1(\Omega))'$ and $p_0 = p_1 = 2$ we can extract subsequences, still denoted $\{u^k_{\varepsilon,i}\}$, such that for $i=1,2$

$$u^k_{\varepsilon,i} \to u_{\varepsilon,i} \quad \text{in } L^2(\Omega_T), \qquad (2.3.40)$$

where "$\to$" denotes strong convergence.

We now pass to the limit in the finite weak form (2.3.10a-b). For this purpose we consider an arbitrary function $\xi \in L^2(0,T;H^1(\Omega))$ and set $\chi^k = P^k\xi$ in (2.3.10a-b) to obtain after integration over $(0,T)$

$$\int_0^T (\partial_t u^k_{\varepsilon,i}, P^k\xi) \; + (\nabla w^k_{\varepsilon,i}, \nabla P^k\xi)\; dt = 0, \tag{2.3.41a}$$

$$\int_0^T \gamma(\nabla u^k_{\varepsilon,i}, \nabla P^k\xi) + (\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}), P^k\xi) + (f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), P^k\xi)\; dt = \int_0^T (w^k_{\varepsilon,i}, P^k\xi)\; dt. \tag{2.3.41b}$$

Since the passage to the limit for linear terms is easily shown using the convergence properties of $u^k_{\varepsilon,i},\; w^k_{\varepsilon,i},\; i=1,2$, and P^k properties, we only show convergence of the nonlinear terms.

Using (2.2.11) and the strong convergences (2.3.5) and (2.3.40) yields for $i=1,2$

$$\begin{aligned}
&\Big|\int_0^T (\phi_\varepsilon(u^k_{\varepsilon,i}), P^k\xi) - (\phi_\varepsilon(u_{\varepsilon,i}), \xi) dt\Big| \\
&\quad\le \|\phi_\varepsilon(u^k_{\varepsilon,i})\|_{L^2(\Omega_T)}\|P^k\xi - \xi\|_{L^2(\Omega_T)} + \|\phi_\varepsilon(u^k_{\varepsilon,i}) - \phi_\varepsilon(u_{\varepsilon,i})\|_{L^2(\Omega_T)}\|\xi\|_{L^2(\Omega_T)} \\
&\quad\le \frac{\theta}{\varepsilon}\|u^k_{\varepsilon,i}\|_{L^2(\Omega_T)}\|P^k\xi - \xi\|_{L^2(\Omega_T)} + \frac{\theta}{\varepsilon}\|u^k_{\varepsilon,i} - u_{\varepsilon,i}\|_{L^2(\Omega_T)}\|\xi\|_{L^2(\Omega_T)} \to 0
\end{aligned} \tag{2.3.42}$$

from which we obtain, on noting that $\Psi'_{\varepsilon,i}(r) = \phi_\varepsilon(r) - \theta_i r$, for $i=1,2$

$$\int_0^T (\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}), P^k\xi) dt \to \int_0^T (\Psi'_{\varepsilon,i}(u_{\varepsilon,i}), \xi) dt. \tag{2.3.43}$$

To deal with the D-coupling term we split for $i=1,2$ as

$$\begin{aligned}
&\Big|\int_0^T (f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), P^k\xi) - (f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2}), \xi) dt\Big| \\
&\quad\le \int_0^T \big|(f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), P^k\xi - \xi)\big| dt + \int_0^T \big|(f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}) - f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2}), \xi)\big| dt \\
&\quad\equiv T^k_1 + T^k_2.
\end{aligned} \tag{2.3.44}$$

From the bound (2.3.34), the strong convergence of $P^k\xi$ to ξ in $L^2(\Omega)$ and the Dominated Convergence Theorem (e.g. [14], p.22) it follows that

$$T^k_1 \le \int_0^T |f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2})|_0 |P^k\xi - \xi|_0 dt \le C\int_0^T |P^k\xi - \xi|_0 dt \to 0 \text{ as } k\to\infty. \tag{2.3.45}$$

We note that for any $r_1, r_2, s_1, s_2 \in \mathbb{R}$ and for $i,j = 1,2$ with $i \ne j$

$$\begin{aligned}
f^{(i)}_D(r_1, r_2) - f^{(i)}_D(s_1, s_2) &= 2D\big[(r_i+\alpha_i)(r_j+\alpha_j)^2 - (s_i+\alpha_i)(s_j+\alpha_j)^2\big] \\
&= 2D(r_j+\alpha_j)^2(r_i - s_i) + 2D(s_i+\alpha_i)(r_j+s_j+2\alpha_j)(r_j - s_j).
\end{aligned} \tag{2.3.46}$$

Using this with $r_i = u^k_{\varepsilon,i}$ and $s_i = u_{\varepsilon,i},\ i = 1,2$, the generalised Hölder inequality, the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, the bounds (2.3.22) and (2.3.8a) and the strong convergence (2.3.40) we obtain for $i,j = 1,2$ with $i \neq j$ that

$$
\begin{aligned}
T^k_2 &= 2D\int_0^T \left|\big((u^k_{\varepsilon,j}+\alpha_j)^2\xi, u^k_{\varepsilon,i}-u_{\varepsilon,i}\big) + \big((u_{\varepsilon,i}+\alpha_i)(u^k_{\varepsilon,j}+u_{\varepsilon,j}+2\alpha_j)\xi, u^k_{\varepsilon,j}-u_{\varepsilon,j}\big)\right| dt \\
&\leq 2D\int_0^T \left|(u^k_{\varepsilon,j}+\alpha_j)^2\xi(u^k_{\varepsilon,i}-u_{\varepsilon,i})\right|_{0,1} + \left|(u_{\varepsilon,i}+\alpha_i)(u^k_{\varepsilon,j}+u_{\varepsilon,j}+2\alpha_j)\xi(u^k_{\varepsilon,j}-u_{\varepsilon,j})\right|_{0,1} dt \\
&\leq 2D\int_0^T |u^k_{\varepsilon,j}+\alpha_j|^2_{0,6}\ |\xi|_{0,6}\ |u^k_{\varepsilon,i}-u_{\varepsilon,i}|_0\ dt \\
&\quad + 2D\int_0^T |u^k_{\varepsilon,i}+\alpha_i|_{0,6}\ |u^k_{\varepsilon,j}+u_{\varepsilon,j}+2\alpha_j|_{0,6}\ |\xi|_{0,6}\ |u^k_{\varepsilon,j}-u_{\varepsilon,j}|_0\ dt \\
&\leq C\int_0^T \|u^k_{\varepsilon,j}+\alpha_j\|^2_1\ \|\xi\|_1\ |u^k_{\varepsilon,i}-u_{\varepsilon,i}|_0\ dt \\
&\quad + C\int_0^T \|u^k_{\varepsilon,i}+\alpha_i\|_1\ \|u^k_{\varepsilon,j}+u_{\varepsilon,j}+2\alpha_j\|_1\ \|\xi\|_1\ |u^k_{\varepsilon,j}-u_{\varepsilon,j}|_0\ dt \\
&\leq C\|u^k_{\varepsilon,i}-u_{\varepsilon,i}\|_{L^2(\Omega_T)}\|\xi\|_{L^2(0,T,H^1(\Omega))} + C\|u^k_{\varepsilon,j}-u_{\varepsilon,j}\|_{L^2(\Omega_T)}\|\xi\|_{L^2(0,T,H^1(\Omega))} \to 0
\end{aligned}
$$

as $k \to \infty$. (2.3.47)

Thus, from (2.3.44), (2.3.45) and (2.3.47) it follows that as $k \to \infty$

$$\int_0^T (f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), P^k\xi)dt \to \int_0^T (f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2}), \xi)dt. \tag{2.3.48}$$

We now can pass to the limit as $k \to \infty$ in the finite weak form (2.3.41a-b) to obtain

$$\int_0^T \langle \partial_t u_{\varepsilon,i}, \xi\rangle\ + (\nabla w_{\varepsilon,i}, \nabla\xi)\ dt = 0, \tag{2.3.49a}$$

$$\int_0^T \gamma(\nabla u_{\varepsilon,i}, \nabla\xi) + (\Psi'_{\varepsilon,i}(u_{\varepsilon,i}), \xi) + (f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2}), \xi)\ dt = \int_0^T (w_{\varepsilon,i}, \xi)\ dt. \tag{2.3.49b}$$

To conclude with the required variational equations (2.2.21a-b) of $(\mathbf{P}_\varepsilon)$ we argue as [43] (Theorem 43.3, p. 308). Let g be the characteristic function on the arbitrary time interval $(0,t),\ t \leq T$ and set $\xi = \eta g$ in (2.3.49a-b) where $\eta \in H^1(\Omega)$ to yield

$$\int_0^t \big[\langle \partial_s u_{\varepsilon,i}, \eta\rangle + (\nabla w_{\varepsilon,i}, \nabla\eta)\big] ds = 0, \tag{2.3.50a}$$

$$\int_0^t \big[\gamma(\nabla u_{\varepsilon,i}, \nabla\eta) + (\Psi'_{\varepsilon,i}(u_{\varepsilon,i}), \eta) + (f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2}), \eta)\big] ds = \int_0^t (w_{\varepsilon,i}, \eta) ds, \tag{2.3.50b}$$

and hence the variational equations (2.2.21a-b) of $(\mathbf{P}_\varepsilon)$ is now a consequence of Theorem A.0.12 (see appendix A).

Applying Theorem A.0.19 (see appendix A) we obtain, after noting $u_{\varepsilon,i}, u^k_{\varepsilon,i} \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, that $u_{\varepsilon,i}, u^k_{\varepsilon,i} \in C([0,T];L^2(\Omega))$. This result together with the strong convergence of $P^k u^0_i$ to u^0_i in $L^2(\Omega)$ and the strong convergence (2.3.40) one may conclude that $u_{\varepsilon,i}(0) = u^0_i,\ i = 1,2$.

Before showing the uniqueness, we prove the remaining stability estimates. Since $u_{\varepsilon,i} \in L^2(0,T;H^1(\Omega))$ and ϕ_ε is Lipschitz continuous and its first derivative is bounded, see (2.2.8), we are allowed to test (2.2.21b) with $\eta = \phi_\varepsilon(u_{\varepsilon,i}) \in H^1(\Omega)$ to yield for $i = 1,2$ and $a.e.\ t \in (0,T)$

$$\begin{aligned}\gamma(\nabla u_{\varepsilon,i}, \nabla\phi_\varepsilon(u_{\varepsilon,i})) &+ |\phi_\varepsilon(u_{\varepsilon,i})|_0^2 \\ &= (w_{\varepsilon,i}, \phi_\varepsilon(u_{\varepsilon,i})) + \theta_i(u_{\varepsilon,i}, \phi_\varepsilon(u_{\varepsilon,i})) - (f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}), \phi_\varepsilon(u_{\varepsilon,i})) \\ &\le \frac{1}{2}|\phi_\varepsilon(u_{\varepsilon,i})|_0^2 + C\big[|w_{\varepsilon,i}|_0^2 + |u_{\varepsilon,i}|_0^2 + |f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2})|_0^2\big], \quad (2.3.51)\end{aligned}$$

where we have also used Young's inequality. Note that as $\phi'_\varepsilon > 0$, the first term of (2.3.51) is positive.

From (2.3.33) and the bound (2.3.8a) we easily deduce for $i = 1,2$ that

$$\|f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2})\|_{L^\infty(0,T,L^2(\Omega))} \le C. \quad (2.3.52)$$

Thus, integrating (2.3.51) over $(0,T)$ and using the estimates (2.3.8a), (2.3.8b) and (2.3.52) leads to the estimate (2.3.8c) and the second inequality in (2.3.9) while the first one follows from the property (2.2.8) of ϕ_ε.

Finally, it remains to prove the uniqueness. To this aim, assume that $S = \{u_{\varepsilon,i}, w_{\varepsilon,i}\}_{i=1,2}$ and $S^* = \{u^*_{\varepsilon,i} w^*_{\varepsilon,i}\}_{i=1,2}$ are two solutions of $(\mathbf{P}_\varepsilon)$. For $i = 1,2$ define $\bar{u}_{\varepsilon,i} := u_{\varepsilon,i} - u^*_{\varepsilon,i} \in V_0$. Subtract (2.2.24) when S is the solution from (2.2.24) when S^* is the solution and test the resulting variational equation with $\eta = \bar{u}_{\varepsilon,i}$ to yield for $a.e\ t \in (0,T)$

$$\begin{aligned}\gamma|\bar{u}_{\varepsilon,i}|_1^2 &+ (\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^*_{\varepsilon,i}), \bar{u}_{\varepsilon,i}) + (\mathcal{G}\partial_t\bar{u}_{\varepsilon,i}, \bar{u}_{\varepsilon,i}) \\ &= \theta_i|\bar{u}_{\varepsilon,i}|_0^2 - (f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}) - f_D^{(i)}(u^*_{\varepsilon,1}, u^*_{\varepsilon,2}), \bar{u}_{\varepsilon,i}). \quad (2.3.53)\end{aligned}$$

From the definition of $\mathcal{G}$ given by (2.1.5) we note that

$$\frac{d}{dt}\|\bar{u}_{\varepsilon,i}\|_{-1}^2 = \frac{d}{dt}(\nabla\mathcal{G}\bar{u}_{\varepsilon,i}, \nabla\mathcal{G}\bar{u}_{\varepsilon,i}) = 2(\nabla\mathcal{G}\partial_t\bar{u}_{\varepsilon,i}, \nabla\mathcal{G}\bar{u}_{\varepsilon,i}) = 2(\mathcal{G}\partial_t\bar{u}_{\varepsilon,i}, \bar{u}_{\varepsilon,i}), \quad (2.3.54)$$

and hence, by the monotonicity of ϕ_ε and (2.1.11),

$$\gamma|\bar{u}_{\varepsilon,i}|_1^2 + \frac{1}{2}\frac{d}{dt}\|\bar{u}_{\varepsilon,i}\|_{-1}^2 \le \theta_i|\bar{u}_{\varepsilon,i}|_0^2 - (f_D^{(i)}(u_{\varepsilon,1},u_{\varepsilon,2}) - f_D^{(i)}(u^*_{\varepsilon,1},u^*_{\varepsilon,2}),\bar{u}_{\varepsilon,i})$$
$$\le \frac{\gamma}{4}|\bar{u}_{\varepsilon,i}|_1^2 + C\|\bar{u}_{\varepsilon,i}\|_{-1}^2 + \Big|\big(f_D^{(i)}(u_{\varepsilon,1},u_{\varepsilon,2}) - f_D^{(i)}(u^*_{\varepsilon,1},u^*_{\varepsilon,2}),\bar{u}_{\varepsilon,i}\big)\Big|. \tag{2.3.55}$$

Using (2.3.46) with $r_i = u_{\varepsilon,i},\ s_i = u^*_{\varepsilon,i}, i = 1,2$, the Young inequality and a generalised Hölder's inequality and noting, by (2.1.4), $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and the estimate (2.3.8a) yields for $i,j = 1,2$ with $j \neq i$ and *a.e.* $t \in (0,T)$

$$\begin{aligned}
&\Big|\big(f_D^{(i)}(u_{\varepsilon,1},u_{\varepsilon,2}) - f_D^{(i)}(u^*_{\varepsilon,1},u^*_{\varepsilon,2}),\bar{u}_{\varepsilon,i}\big)\Big| \\
&\le 2D\Big|\big((u_{\varepsilon,j}+\alpha_j)^2,\bar{u}^2_{\varepsilon,i}\big) + 2D\big((u^*_{\varepsilon,i}+\alpha_i)(u_{\varepsilon,j}+u^*_{\varepsilon,j}+2\alpha_j),\bar{u}_{\varepsilon,i}\,\bar{u}_{\varepsilon,j}\big)\Big| \\
&\le 2D\big((u_{\varepsilon,j}+\alpha_j)^2,\bar{u}^2_{\varepsilon,i}\big) + 2D\big(|u^*_{\varepsilon,i}+\alpha_i||u_{\varepsilon,j}+u^*_{\varepsilon,j}+2\alpha_j|,|\bar{u}_{\varepsilon,i}||\bar{u}_{\varepsilon,j}|\big) \\
&\le 2D\big((u_{\varepsilon,j}+\alpha_j)^2,\bar{u}^2_{\varepsilon,i}\big) + D\big(|u^*_{\varepsilon,i}+\alpha_i||u_{\varepsilon,j}+u^*_{\varepsilon,j}+2\alpha_j|,\bar{u}^2_{\varepsilon,i}+\bar{u}^2_{\varepsilon,j}\big) \\
&= 2D|(u_{\varepsilon,j}+\alpha_j)^2\bar{u}^2_{\varepsilon,i}\big|_{0,1} + D\big|(u^*_{\varepsilon,i}+\alpha_i)(u_{\varepsilon,j}+u^*_{\varepsilon,j}+2\alpha_j)(\bar{u}^2_{\varepsilon,i}+\bar{u}^2_{\varepsilon,j})\big|_{0,1} \\
&\le 2D|u_{\varepsilon,j}+\alpha_j|^2_{0,4}|\bar{u}_{\varepsilon,i}|^2_{0,4} + D|u^*_{\varepsilon,i}+\alpha_i|_{0,4}|u_{\varepsilon,j}+u^*_{\varepsilon,j}+2\alpha_j|_{0,4}\big[|\bar{u}_{\varepsilon,i}|^2_{0,4}+|\bar{u}_{\varepsilon,j}|^2_{0,4}\big] \\
&\le C\|u_{\varepsilon,j}+\alpha_j\|_1^2|\bar{u}_{\varepsilon,i}|^2_{0,4} + C\|u^*_{\varepsilon,i}+\alpha_i\|_1\|u_{\varepsilon,j}+u^*_{\varepsilon,j}+2\alpha_j\|_1\big[|\bar{u}_{\varepsilon,i}|^2_{0,4}+|\bar{u}_{\varepsilon,j}|^2_{0,4}\big] \\
&\le C\big[|\bar{u}_{\varepsilon,i}|^2_{0,4}+|\bar{u}_{\varepsilon,j}|^2_{0,4}\big] \le \frac{\gamma}{8}\big[|\bar{u}_{\varepsilon,i}|_1^2+|\bar{u}_{\varepsilon,j}|_1^2\big] + C\big[\|\bar{u}_{\varepsilon,i}\|_{-1}^2+\|\bar{u}_{\varepsilon,j}\|_{-1}^2\big],
\end{aligned} \tag{2.3.56}$$

where we also have applied Lemma 2.3.1 to obtain the last inequality.

We thus can rewrite (2.3.55) for $i,j = 1,2$ with $i \neq j$ as

$$\gamma|\bar{u}_{\varepsilon,i}|_1^2 + \frac{1}{2}\frac{d}{dt}\|\bar{u}_{\varepsilon,i}\|_{-1}^2 \le \frac{3\gamma}{8}|\bar{u}_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|\bar{u}_{\varepsilon,j}|_1^2 + C\big[\|\bar{u}_{\varepsilon,i}\|_{-1}^2+\|\bar{u}_{\varepsilon,j}\|_{-1}^2\big]. \tag{2.3.57}$$

Summing this inequality over $i = 1,2$ and rearranging the terms yields

$$\frac{\gamma}{2}\big[|\bar{u}_{\varepsilon,1}|_1^2+|\bar{u}_{\varepsilon,2}|_1^2\big] + \frac{1}{2}\frac{d}{dt}\big[\|\bar{u}_{\varepsilon,1}\|_{-1}^2+\|\bar{u}_{\varepsilon,2}\|_{-1}^2\big] \le C\big[\|\bar{u}_{\varepsilon,1}\|_{-1}^2+\|\bar{u}_{\varepsilon,2}\|_{-1}^2\big]. \tag{2.3.58}$$

Applying a Gronwall lemma (see Appendix A, Theorem A.0.5) implies for *a.e* $t \in (0,T)$ that

$$\gamma\int_0^t \big[|\bar{u}_{\varepsilon,1}|_1^2+|\bar{u}_{\varepsilon,2}|_1^2\big]ds + \big[\|\bar{u}_{\varepsilon,1}(t)\|_{-1}^2+\|\bar{u}_{\varepsilon,2}(t)\|_{-1}^2\big] \le e^{Ct}\big[\|\bar{u}_{\varepsilon,1}(0)\|_{-1}^2+\|\bar{u}_{\varepsilon,2}(0)\|_{-1}^2\big]$$
$$= 0, \tag{2.3.59}$$

from which we conclude, on noting (2.1.11), that $\bar{u}_{\varepsilon,i}(t) = 0,\ i = 1,2$, and hence the uniqueness result of u_i. The uniqueness of $w_{\varepsilon,i}$ follows from (2.2.22) and (2.2.23). □

Theorem 2.3.3 Let the assumptions ($\mathbf{A_1}$) hold. Then there exists a unique solution $\{u_1, u_2, w_1, w_2\}$ to ($\mathbf{P}$) such that

$$u_1, u_2 \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{2.3.60a}$$

$$w_1, w_2 \in L^2(0,T;H^1(\Omega)), \tag{2.3.60b}$$

$$\phi(u_1), \phi(u_2) \in L^2(\Omega_T), \tag{2.3.60c}$$

$$f_D^{(1)}(u_1,u_2), f_D^{(2)}(u_1,u_2) \in L^\infty(0,T;L^2(\Omega)), \tag{2.3.60d}$$

$$\max\{|u_1|,|u_2|\} < 1 \quad a.e. \text{ in } \Omega_T. \tag{2.3.60e}$$

Proof. We first observe that the bounds (2.3.8a-c) are independent of ε. Then for $i = 1, 2$ from compactness arguments we can extract subsequences, still denoted $\{u_{\varepsilon,i}\}$ and $\{w_{\varepsilon,i}\}$, such that

$$u_{\varepsilon,i} \rightharpoonup u_i \quad \text{in } L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{2.3.61a}$$

$$u_{\varepsilon,i} \overset{*}{\rightharpoonup} u_i \quad \text{in } L^\infty(0,T;H^1(\Omega)), \tag{2.3.61b}$$

$$w_{\varepsilon,i} \rightharpoonup w_i \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{2.3.61c}$$

$$\phi_\varepsilon(u_{\varepsilon,i}) \rightharpoonup \dot{\eta}_i \quad \text{in } L^2(\Omega_T). \tag{2.3.61d}$$

Furthermore, a similar argument to that used in Theorem 2.3.2 shows

$$u_{\varepsilon,i} \to u_i \quad \text{in } L^2(\Omega_T). \tag{2.3.62}$$

Now our goal is to prove that $\dot{\eta}_i = \phi(u_i)$ for $i = 1, 2$. We remark that if we show $u_i = \phi^{-1}(\dot{\eta}_i)$ *a.e.* in Ω_T, then we immediately achieve our goal and we also obtain $|u_i| < 1$ *a.e.* in Ω_T as $\phi^{-1}(r) \in (-1, 1)$ for all $r \in \mathbb{R}$. To see this we firstly show that

$$I_i(\xi) := \int_0^T (u_i - \phi^{-1}(\xi), \dot{\eta}_i - \xi)dt \geq 0 \quad \forall \xi \in L^2(\Omega_T). \tag{2.3.63}$$

Choosing $s = u_{\varepsilon,i}$ and $r = \phi_\varepsilon^{-1}(\xi)$ in (2.2.9) yields for $i = 1, 2$ and *a.e.* $t \in (0, T)$

$$(u_{\varepsilon,i} - \phi_\varepsilon^{-1}(\xi), \phi_\varepsilon(u_{\varepsilon,i}) - \xi) \geq \theta|u_{\varepsilon,i} - \phi_\varepsilon^{-1}(\xi)|_0^2 \geq 0,$$

and hence

$$I_{\varepsilon,i}(\xi) := \int_0^T (u_{\varepsilon,i} - \phi_\varepsilon^{-1}(\xi), \phi_\varepsilon(u_{\varepsilon,i}) - \xi)dt \geq 0 \quad \forall \xi \in L^2(\Omega_T). \tag{2.3.64}$$

To show this integral is well-defined we use (2.2.16) with $s = \phi_\varepsilon(u_{\varepsilon,i})$ and $r = \xi$ and recall the estimate (2.3.8c) to yield for $i = 1, 2$

$$I_{\varepsilon,i}(\xi) \leq \int_0^T |u_{\varepsilon,i} - \phi_\varepsilon^{-1}(\xi)|_0 |\phi_\varepsilon(u_{\varepsilon,i}) - \xi|_0 dt \leq \theta^{-1} \int_0^T |\phi_\varepsilon(u_{\varepsilon,i}) - \xi|_0^2 dt$$
$$= \theta^{-1} \|\phi_\varepsilon(u_{\varepsilon,i}) - \xi\|_{L^2(\Omega_T)} < \infty.$$

Now, to obtain the result (2.3.63) it is sufficient to show that $I_{\varepsilon,i} \to I_i$ as $\varepsilon \to 0$. From Lemma 2.2.1 we note that $\phi_\varepsilon^{-1}(r) \to \phi^{-1}(r)\ \forall\, r$ as $\varepsilon \to 0$ and hence with the aid of the strong convergence (2.3.62), the bound (2.3.8c) on $\phi_\varepsilon(u_{\varepsilon,i})$ and the weak convergence (2.3.61d) we obtain for any $\xi \in L^2(\Omega_T)$ and $i = 1, 2$ that

$$\begin{aligned}
|I_{\varepsilon,i}(\xi) - I_i(\xi)| &= \Big| \int_0^T (u_{\varepsilon,i} - \phi_\varepsilon^{-1}(\xi), \phi_\varepsilon(u_{\varepsilon,i}) - \xi) - (u_i - \phi^{-1}(\xi), \dot{\eta}_i - \xi) dt \Big| \\
&\leq \Big| \int_0^T (u_{\varepsilon,i} - u_i, \phi_\varepsilon(u_{\varepsilon,i}) - \xi) dt \Big| + \Big| \int_0^T (\phi^{-1}(\xi) - \phi_\varepsilon^{-1}(\xi), \phi_\varepsilon(u_{\varepsilon,i}) - \xi) dt \Big| \\
&\quad + \Big| \int_0^T (u_i - \phi^{-1}(\xi), \phi_\varepsilon(u_{\varepsilon,i}) - \dot{\eta}_i) dt \Big| \\
&\leq \|u_{\varepsilon,i} - u_i\|_{L^2(\Omega_T)} \|\phi_\varepsilon(u_{\varepsilon,i}) - \xi\|_{L^2(\Omega_T)} + \|\phi^{-1}(\xi) - \phi_\varepsilon^{-1}(\xi)\|_{L^2(\Omega_T)} \|\phi_\varepsilon(u_{\varepsilon,i}) - \xi\|_{L^2(\Omega_T)} \\
&\quad + \Big| \int_0^T (u_i - \phi^{-1}(\xi), \phi_\varepsilon(u_{\varepsilon,i}) - \dot{\eta}_i) dt \Big| \to 0 \text{ as } \varepsilon \to 0. \qquad (2.3.65)
\end{aligned}$$

Thus, for $i = 1, 2$,

$$I_i(\xi) = \lim_{\varepsilon \to 0} I_{\varepsilon,i}(\xi) \geq 0 \qquad \forall\, \xi \in L^2(\Omega_T),$$

as required.

For any $\beta \in \mathbb{R}_{>0}$ and any $\xi \in L^2(\Omega_T)$ we substitute $\dot{\eta}_i \pm \beta\xi \in L^2(\Omega_T)$ into I_i to obtain by (2.3.63) that

$$\int_0^T (u_i - \phi^{-1}(\dot{\eta}_i + \beta\xi), -\beta\xi) dt \geq 0 \quad \text{and} \quad \int_0^T (u_i - \phi^{-1}(\dot{\eta}_i - \beta\xi), \beta\xi) dt \geq 0.$$

Dividing the first inequality by $-\beta$ and the second by β gives

$$\int_0^T (u_i - \phi^{-1}(\dot{\eta}_i + \beta\xi), \xi) dt \leq 0 \quad \text{and} \quad \int_0^T (u_i - \phi^{-1}(\dot{\eta}_i - \beta\xi), \xi) dt \geq 0$$

and then taking the limit as $\beta \to 0$ yields after noting the continuity of ϕ^{-1} that

$$\int_0^T (u_i - \phi^{-1}(\dot{\eta}_i), \xi) dt \leq 0 \quad \text{and} \quad \int_0^T (u_i - \phi^{-1}(\dot{\eta}_i), \xi) dt \geq 0,$$

which implies for $i = 1, 2$ that

$$\int_0^T (u_i - \phi^{-1}(\dot{\eta}_i), \xi) dt = 0 \qquad \forall \xi \in L^2(\Omega_T). \tag{2.3.66}$$

We choose $\xi = u_i - \phi^{-1}(\dot{\eta}) \in L^2(\Omega_T)$ in (2.3.66) to give for $i = 1, 2$ that

$$\|u_i - \phi^{-1}(\dot{\eta}_i)\|^2_{L^2(\Omega_T)} = \int_0^T |u_i - \phi^{-1}(\dot{\eta}_i)|_0^2 dt = 0,$$

leading to $u_i = \phi^{-1}(\dot{\eta}_i)$ *a.e.* in Ω_T. Therefore, for $i = 1, 2$

$$|u_i| < 1 \text{ } a.e. \text{ in } \Omega_T \quad \text{and} \quad \dot{\eta}_i = \phi(u_i).$$

Similarly to Theorem 2.3.2, we can pass to the limit in $(\mathbf{P}_\varepsilon)$ as $\varepsilon \to 0$ to obtain that $\{u_1, u_2, w_1, w_2\}$ solving $(\mathbf{P})$. More precisely, convergence follows immediately from the weak convergence (2.3.61a,c,d) with the exception of

$$\int_0^T (f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}), \eta) \, dt \to \int_0^T (f_D^{(i)}(u_1, u_2), \eta) \, dt \quad \text{as } \varepsilon \to 0,$$

which is immediate on noting a similar inequality to (2.3.47) and the strong convergence (2.3.62).

Finally, to prove uniqueness of a solution to $(\mathbf{P})$ we argue as for $(\mathbf{P}_\varepsilon)$ in Theorem 2.3.2. □

Chapter 3

Regularity results and Continuous dependence

In this chapter we show how increasing the regularity of the boundary of the domain Ω and the initial data u_1^0 and u_2^0 leads to more regular solution to the problem $(\mathbf{P})$. In Section 3.1 we show that the solution of the problem $(\mathbf{P})$ is in higher order Sobolev spaces under further assumptions on Ω and the initial data. In Section 3.2 we show the continuous dependence on the initial data and finally we prove an error bound for the regularization procedure.

3.1 Regularity results

We shall study the problem $(\mathbf{P})$ under the following stronger assumptions on $\{u_1^0, u_2^0\}$:

$(\mathbf{A_2})$ Let $\{u_1^0, u_2^0\} \in H^2(\Omega) \times H^2(\Omega)$, $|\Delta u_1^0|_1 + |\Delta u_2^0|_1 \leq C$, $\frac{\partial u_1^0}{\partial \nu} = \frac{\partial u_2^0}{\partial \nu} = 0$ on $\partial\Omega$ and $\max\{|u_1^0|_{0,\infty}, |u_2^0|_{0,\infty}\} \leq 1 - \delta_0$ for some given $\delta_0 \in (0, 1)$.

We recall that if $u \in H^1(\Omega)$ is a solution of the variational equation

$$(\nabla u, \nabla \eta) + (u, \eta) = (f, \eta) \qquad \forall \eta \in H^1(\Omega),$$

where $f \in L^2(\Omega)$ and if Ω is convex polygonal or $\partial\Omega \in C^2$, then from the standard regularity theory of elliptic problems (see Grisvard [18]) $u \in H^2(\Omega)$ and

$$\|u\|_2 \leq C|f|_0$$

Hence, by the weak form of (2.3.1), we have $z_j \in H^2(\Omega)$ $1 \leq j \leq k$, (k fixed and finite) and thus $V^k \subset H^2(\Omega)$. For the purposes of the analysis, we need the following lemma.

Lemma 3.1.1 If $v \in H^2(\Omega)$ and $d \leq 3$. Then there are constants $\sigma = d(\frac{1}{2} - \frac{1}{r})$ and C such that

$$|\nabla v|_{0,r} \leq C|v|_1^{1-\sigma} \, \|v\|_2^{\sigma} \leq C\|v\|_2 \qquad \text{holds for} \qquad r \in \begin{cases} [2,\infty] & \text{if } d = 1, \\ [2,\infty) & \text{if } d = 2, \\ [2,6] & \text{if } d = 3. \end{cases} \tag{3.1.1}$$

Proof. An application of the Soblev interpolation result (2.1.4) with simple calculations gives

$$\begin{aligned} |\nabla v|_{0,r}^r &= \int_\Omega \Big(\sum_{i=1}^d \Big| \frac{\partial v}{\partial x_i} \Big|^2 \Big)^{\frac{r}{2}} dx \leq C \int_\Omega \Big(\sum_{i=1}^d \Big| \frac{\partial v}{\partial x_i} \Big|^r \Big) dx = C \sum_{i=1}^d \Big| \frac{\partial v}{\partial x_i} \Big|_{0,r}^r \\ &\leq C \sum_{i=1}^d \Big| \frac{\partial v}{\partial x_i} \Big|_0^{r(1-\sigma)} \Big\| \frac{\partial v}{\partial x_i} \Big\|_1^{r\sigma} \leq C \Big(\sum_{i=1}^d \Big| \frac{\partial v}{\partial x_i} \Big|_0^2 \Big)^{\frac{r}{2}(1-\sigma)} \|v\|_2^{r\sigma} = C|v|_1^{r(1-\sigma)} \|v\|_2^{r\sigma}, \end{aligned}$$

and the second inequality follows directly from the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$. □

Theorem 3.1.2 Let the assumptions ($\mathbf{A_1}$) hold. Let Ω be a convex polygonal domain or $\partial\Omega \in C^2$. Then the unique solution of (**P**) is such that the following additional regularity results hold

$$u_1, u_2 \in L^2(0,T; H^2(\Omega)), \tag{3.1.2a}$$

$$f_D^{(1)}(u_1,u_2), f_D^{(2)}(u_1,u_2) \in L^2(0,T; H^1(\Omega)). \tag{3.1.2b}$$

Proof. From Theorem 2.3.3 we have for $i = 1,2$ and *a.e.* $\in (0,T)$ that $u_i \in H^1(\Omega)$ is a solution of the elliptic variational equation

$$\gamma(\nabla u_i, \nabla \eta) + (\phi(u_i) - \theta_i u_i + f_D^{(i)}(u_1,u_2) - w_i, \eta) = 0 \quad \forall \eta \in H^1(\Omega).$$

Thus, by the standard regularity theory of elliptic problems with the aid of the estimates obtained in Theorem 2.3.3 we have for *a.e.* $t \in (0,T)$ that $u_i \in H^2(\Omega)$ and

$$\|u_i\|_2 \leq C|w_i - \phi(u_i) + \theta_i u_i - f_D^{(i)}(u_1,u_2) + u_i|_0.$$

Therefore, by squaring this inequality and integrating over $(0,T)$ we obtain

$$\|u_i\|^2_{L^2(0,T;H^2(\Omega))} \leq C\|w_i - \phi(u_i) + \theta_i u_i - f_D^{(i)}(u_1,u_2) + u_i\|^2_{L^2(\Omega_T)} \leq C. \tag{3.1.3}$$

To obtain the estimate (3.1.2b), we first note that using the generalised Hölder inequality, $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and Lemma 3.1.1 yields for $v_1, v_2 \in H^2(\Omega)$ and for $i,j = 1,2$ with $i \neq j$ that

$$\begin{aligned} |f_D^{(i)}(v_1,v_2)|_1^2 = |\nabla f_D^{(i)}(v_1,v_2)|_0^2 &= 4D^2|\nabla(v_i+\alpha_i)(v_j+\alpha_j)^2|_0^2 = \\ &= 4D^2|(v_j+\alpha_j)^2\nabla v_i + 2(v_i+\alpha_i)(v_j+\alpha_j)\nabla v_j|_0^2 \\ &\leq 8D^2\big|(v_j+\alpha_j)^4|\nabla v_i|^2\big|_{0,1} + 32D^2\big|(v_i+\alpha_i)^2(v_j+\alpha_j)^2|\nabla v_j|^2\big|_{0,1} \\ &\leq 8D^2|v_j+\alpha_j|_{0,6}^4|\nabla v_i|_{0,6}^2 + 32D^2|v_i+\alpha_i|_{0,6}^2|v_j+\alpha_j|_{0,6}^2|\nabla v_j|_{0,6}^2 \\ &\leq C\|v_j+\alpha_j\|_1^4\|v_i\|_2^2 + C\|v_i+\alpha_i\|_1^2\|v_j+\alpha_j\|_1^2\|v_j\|_2^2. \end{aligned} \tag{3.1.4}$$

Thus, by integration over $(0,T)$ and noting the estimates (2.3.60a) and (3.1.3) we have for $i,j = 1,2$ with $i \neq j$

$$\int_0^T |f_D^{(i)}(u_1,u_2)|_1^2 dt \leq C\int_0^T \|u_i\|_2^2 dt + C\int_0^T \|u_j\|_2^2 dt \leq C. \tag{3.1.5}$$

Hence with this estimate and (2.3.60d) we may conclude the desired result (3.1.2b) . □

Theorem 3.1.3 Let the assumptions $(\mathbf{A_2})$. Let Ω be a convex polygonal domain or $\partial\Omega \in C^2$. Then for all $\varepsilon \leq \min\{\varepsilon_0, \frac{\delta_0}{2}\}$ the unique solution of $(\mathbf{P}_\varepsilon)$ is such that for $i = 1,2$ the following additional stability estimates hold independently of ε

$$\|\partial_t u_{\varepsilon,i}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t u_{\varepsilon,i}\|_{L^\infty(0,T;(H^1(\Omega))')} + \|w_{\varepsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{3.1.6a}$$

$$\|\phi_\varepsilon(u_{\varepsilon,i})\|_{L^\infty(0,T;L^2(\Omega))} + \|u_{\varepsilon,i}\|_{L^\infty(0,T;H^2(\Omega))} + \|w_{\varepsilon,i}\|_{L^2(0,T;H^2(\Omega))} \leq C, \tag{3.1.6b}$$

$$\|f_D^{(i)}(u_{\varepsilon,1},u_{\varepsilon,2})\|_{L^\infty(0,T;H^1(\Omega))} \leq C \text{ and } u_{\varepsilon,i} \in C([0,T];H^1(\Omega)). \tag{3.1.6c}$$

Furthermore, we have for $i = 1,2$ that $\frac{\partial u_{\varepsilon,i}}{\partial \nu} = 0$ *a.e.* on $\partial\Omega \times (0,T)$ and

$$\theta^{-1}\varepsilon\|\nabla\phi_\varepsilon(u_{\varepsilon,i})\|_{L^\infty(0,T;L^2(\Omega)))} \leq \|(\nabla u_{\varepsilon,i}, \nabla\phi_\varepsilon(u_{\varepsilon,i}))\|_{L^\infty(0,T)} \leq C. \tag{3.1.7}$$

Proof. Differentiating the finite variational equality (2.3.10b) with respect to time and taking $\chi^k = \partial_t u^k_{\varepsilon,i} \in V^k \cap V_0$ yields for $i = 1,2$

$$\begin{aligned} \gamma|\partial_t u^k_{\varepsilon,i}|_1^2 + (\phi'_\varepsilon(u^k_{\varepsilon,i})\partial_t u^k_{\varepsilon,i}, \partial_t u^k_{\varepsilon,i}) - \theta_i|\partial_t u^k_{\varepsilon,i}|_0^2 + (\partial_t f_D^{(i)}(u^k_{\varepsilon,1},u^k_{\varepsilon,2}), \partial_t u^k_{\varepsilon,i}) \\ = (\partial_t w^k_{\varepsilon,i}, \partial_t u^k_{\varepsilon,i}) = -(\nabla w^k_{\varepsilon,i}, \nabla\partial_t w^k_{\varepsilon,i}) = -\frac{1}{2}\frac{d}{dt}|w^k_{\varepsilon,i}|_1^2, \end{aligned}$$

where we have also noted (2.3.10a) with $\chi^k = \partial_t w^k_{\varepsilon,i}$ to obtain the second equality. On noting that for $i, j = 1, 2$ with $i \neq j$

$$
\begin{aligned}
(\partial_t f_D^{(i)}(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}), \partial u^k_{\varepsilon,i}) &= 2D((u^k_{\varepsilon,j} + \alpha_j)^2, (\partial_t u^k_{\varepsilon,i})^2) \\
&\quad + 4D((u^k_{\varepsilon,i} + \alpha_i)(u^k_{\varepsilon,j} + \alpha_j)\partial_t u^k_{\varepsilon,j}, \partial_t u^k_{\varepsilon,i}) \\
&\geq 4D((u^k_{\varepsilon,i} + \alpha_i)(u^k_{\varepsilon,j} + \alpha_j), \partial_t u^k_{\varepsilon,i}\partial_t u^k_{\varepsilon,j})
\end{aligned}
$$

and recalling, by (2.2.8) and (2.3.28), that $\phi'_\varepsilon(r) > 0$ and $|w^k_{\varepsilon,i}|_1 = \|\partial_t u^k_{\varepsilon,i}\|_{-1}$ we have after noting (2.1.11) for $i, j = 1, 2$ with $i \neq j$ that

$$
\begin{aligned}
\gamma|\partial_t u^k_{\varepsilon,i}|_1^2 &+ \frac{1}{2}\frac{d}{dt}\|\partial_t u^k_{\varepsilon,i}\|_{-1}^2 \\
&\leq \theta_i|\partial_t u^k_{\varepsilon,i}|_0^2 - 4D((u^k_{\varepsilon,i} + \alpha_i)(u^k_{\varepsilon,j} + \alpha_j), \partial_t u^k_{\varepsilon,i}\partial_t u^k_{\varepsilon,j}) \\
&\leq \frac{\gamma}{4}|\partial_t u^k_{\varepsilon,i}|_1^2 + C\|\partial_t u^k_{\varepsilon,i}\|_{-1}^2 + 4D\big|((u^k_{\varepsilon,i} + \alpha_i)(u^k_{\varepsilon,j} + \alpha_j), \partial_t u^k_{\varepsilon,i}\partial_t u^k_{\varepsilon,j})\big|.
\end{aligned}
\tag{3.1.8}
$$

From a generalised Hölder's inequality, $H^1(\Omega) \hookrightarrow L^4(\Omega)$, the bound (2.3.8a), a Young's inequality and Lemma 2.3.1 it follows for $i = 1, 2$ and $t \in (0, T)$ that

$$
\begin{aligned}
\big|((u^k_{\varepsilon,i} + \alpha_i)(u_{\varepsilon,j} + \alpha_j), \partial_t u^k_{\varepsilon,i}\partial_t u^k_{\varepsilon,j})\big| &\leq |u^k_{\varepsilon,i} + \alpha_i|_{0,4}|u^k_{\varepsilon,j} + \alpha_j|_{0,4}|\partial_t u^k_{\varepsilon,i}|_{0,4}|\partial_t u^k_{\varepsilon,j}|_{0,4} \\
&\leq C\|u^k_{\varepsilon,i} + \alpha_i\|_1\|u^k_{\varepsilon,j} + \alpha_j\|_1|\partial_t u^k_{\varepsilon,i}|_{0,4}|\partial_t u^k_{\varepsilon,j}|_{0,4} \\
&\leq C|\partial_t u^k_{\varepsilon,i}|_{0,4}|\partial_t u^k_{\varepsilon,j}|_{0,4} \leq C\big[|\partial_t u^k_{\varepsilon,i}|_{0,4}^2 + |\partial_t u^k_{\varepsilon,j}|_{0,4}^2\big] \\
&\leq \frac{\gamma}{32D}\big[|\partial_t u^k_{\varepsilon,i}|_1^2 + |\partial_t u^k_{\varepsilon,j}|_1^2\big] + C\big[\|\partial_t u^k_{\varepsilon,i}\|_{-1}^2 + \|\partial_t u^k_{\varepsilon,j}\|_{-1}^2\big].
\end{aligned}
\tag{3.1.9}
$$

We insert (3.1.9) into (3.1.8) and rearrange to give for $i, j = 1, 2$ with $i \neq j$

$$
\gamma|\partial_t u^k_{\varepsilon,i}|_1^2 + \frac{1}{2}\frac{d}{dt}\|\partial_t u^k_{\varepsilon,i}\|_{-1}^2 \leq \frac{3\gamma}{8}|\partial_t u^k_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|\partial_t u^k_{\varepsilon,j}|_1^2 + C\big[\|\partial_t u^k_{\varepsilon,i}\|_{-1}^2 + \|\partial_t u^k_{\varepsilon,j}\|_{-1}^2\big].
\tag{3.1.10}
$$

Summing this differential inequality over $i = 1, 2$ and rearranging the terms yields

$$
\frac{\gamma}{2}\big[|\partial_t u^k_{\varepsilon,1}|_1^2 + |\partial_t u^k_{\varepsilon,2}|_1^2\big] + \frac{1}{2}\frac{d}{dt}\big[\|\partial_t u^k_{\varepsilon,1}\|_{-1}^2 + \|\partial_t u^k_{\varepsilon,2}\|_{-1}^2\big] \leq C\big[\|\partial_t u^k_{\varepsilon,1}\|_{-1}^2 + \|\partial_t u^k_{\varepsilon,2}\|_{-1}^2\big],
\tag{3.1.11}
$$

from which we infer, by application of the Gronwall lemma, for $t \in (0, T]$ that

$$
\begin{aligned}
\gamma\int_0^t &\big[|\partial_s u^k_{\varepsilon,1}|_1^2 + |\partial_s u^k_{\varepsilon,2}|_1^2\big]ds + \big[\|\partial_s u^k_{\varepsilon,1}(t)\|_{-1}^2 + \|\partial_s u^k_{\varepsilon,2}(t)\|_{-1}^2\big] \\
&\leq C\big[\|\partial_s u^k_{\varepsilon,1}(0)\|_{-1}^2 + \|\partial_t u^k_{\varepsilon,2}(0)\|_{-1}^2\big] = C\big[|w^k_{\varepsilon,1}(0)|_1^2 + |w^k_{\varepsilon,2}(0)|_1^2\big].
\end{aligned}
\tag{3.1.12}
$$

Our goal now is to bound the right hand side of (3.1.12) independently of ε and k. To accomplish this, we integrate the first term of finite weak form (2.3.10b) by parts and use the P^k projection properties (2.3.3a-b) to obtain for all $\chi^k \in V^k$ and $i = 1, 2$

$$(w^k_{\varepsilon,i}(0) + \gamma\Delta u^k_{\varepsilon,i}(0) - P^k\phi_\varepsilon(u^k_{\varepsilon,i}(0)) + \theta_i u^k_{\varepsilon,i}(0) - P^k f^{(i)}_D(u^k_{\varepsilon,1}(0), u^k_{\varepsilon,2}(0)), \chi^k) = 0,$$

which implies

$$w^k_{\varepsilon,i}(0) = -\gamma\Delta u^k_{\varepsilon,i}(0) + P^k\phi_\varepsilon(u^k_{\varepsilon,i}(0)) - \theta_i u^k_{\varepsilon,i}(0) + P^k f^{(i)}_D(u^k_{\varepsilon,1}(0), u^k_{\varepsilon,2}(0)),$$

and hence, recalling for $i = 1, 2$ that $u^k_{\varepsilon,i}(0) = P^k u^0_i$,

$$|w^k_{\varepsilon,i}(0)|_1 \leq \left[\gamma|\Delta P^k u^0_i|_1 + \theta_i|P^k u^0_i|_1\right] + |P^k\phi_\varepsilon(P^k u^0_i)|_1 + |P^k f^{(i)}_D(P^k u^0_1, P^k u^0_2)|_1. \tag{3.1.13}$$

To deal with the Laplacian term we need to prove that $\Delta P^k u^0_i = P^k\Delta u^0_i$. This can be seen by the P^k properties (2.3.3a-b), integration by parts, the assumptions ($\mathbf{A_2}$) and (2.3.1)

$$(P^k\Delta u^0_i, \chi^k) = (\Delta u^0_i, \chi^k) = -(\nabla u^0_i, \nabla\chi^k) = -(\nabla P^k u^0_i, \nabla\chi^k) = (\Delta P^k u^0_i, \chi^k)\ \forall\chi^k \in V^k,$$

we thus have, by taking $\chi^k = P^k\Delta u^0_i - \Delta P^k u^0_i \in V^k$, that $P^k\Delta u^0_i = \Delta P^k u^0_i$ *a.e.* in Ω.

With the aid of (2.3.4a) and the assumptions ($\mathbf{A_2}$) this result leads to

$$\gamma|\Delta P^k u^0_i|_1 + \theta_i|P^k u^0_i|_1 \leq \gamma|P^k\Delta u^0_i|_1 + \theta_i|u^0_i|_1 \leq \gamma|\Delta u^0_i|_1 + \theta_i|u^0_i|_1 \leq C. \tag{3.1.14}$$

Now we treat the logarithmic term. We have

$$\begin{aligned}|P^k\phi_\varepsilon(P^k u^0_i)|_1 &\leq |\phi_\varepsilon(P^k u^0_i)|_1 = |\nabla\phi_\varepsilon(P^k u^0_i)|_0 \\ &= |\phi'_\varepsilon(P^k u^0_i)\nabla P^k u^0_i|_0 \leq |\phi'_\varepsilon(P^k u^0_i)|_{0,\infty}|P^k u^0_i|_1.\end{aligned} \tag{3.1.15}$$

As $P^k u^0_i \to u^0_i$ in $L^2(\Omega)$, we have from Theorem A.0.17 (see Appendix A) $P^k u^0_i \to u^0_i$ ('pointwise') *a.e.* in Ω and hence for $i = 1, 2$ and sufficiently large k

$$|P^k u^0_i - u^0_i| \leq \frac{\delta_0}{2} \qquad a.e. \text{ in } \Omega.$$

Since, by the assumptions ($\mathbf{A_2}$), $|u_i^0| \leq 1 - \delta_0$ *a.e.* in Ω, it follows for sufficiently large k and for $\varepsilon \leq \frac{\delta_0}{2}$ that

$$|P^k u_i^0| \leq |P^k u_i^0 - u_i^0| + |u_i^0| \leq 1 - \frac{\delta_0}{2} \leq 1 - \varepsilon \quad a.e. \text{ in } \Omega.$$

Thus, from the property (2.2.13) of ϕ'_ε we find for *a.e.* in Ω and $i = 1, 2$ that

$$|\phi'_\varepsilon(P^k u_i^0)| = \phi'(P^k u_i^0) \leq \phi'(1 - \frac{\delta_0}{2}) = C(\delta_0) := \frac{\theta}{1 - (1 - \frac{\delta_0}{2})^2},$$

which implies that $|\phi'_\varepsilon(P^k u_i^0)|_{0,\infty} \leq C$ and hence together with (3.1.15) we conclude, after noting (2.3.4a) and the assumptions ($\mathbf{A_2}$), for sufficiently large k, $\varepsilon \leq \frac{\delta_0}{2}$ and $i = 1, 2$ that

$$|P^k \phi_\varepsilon(P^k u_i^0)|_1 \leq C|u_i^0|_1 \leq C. \tag{3.1.16}$$

Finally, to bound the D-coupling term we first note, using integration by parts and (2.3.1), that $(\nabla P^k u_i^0, \nabla \eta) = (-\Delta P^k u_i^0, \eta)$ $\forall \eta \in H^1(\Omega)$ which leads with the aid of the standard elliptic regularity of elliptic problems to

$$\begin{aligned}\|P^k u_i^0\|_2 &\leq C|-\Delta P^k u_i^0 + P^k u_i^0|_0 \leq C|P^k \Delta u_i^0|_0 + C|P^k u_i^0|_0 \\ &\leq C\big[|\Delta u_i^0|_0 + |u_i^0|_0\big] \leq C\|u_i^0\|_2 \quad i = 1, 2,\end{aligned} \tag{3.1.17}$$

where we have also noted that $P^k \Delta u_i^0 = \Delta P^k u_i^0$ and (2.3.4a).

Hence, using (2.3.4a), (3.1.4) with $v_i = P^k(u_i^0)$ and (3.1.17) and noting again (2.3.4a) and the assumptions ($\mathbf{A_2}$) we obtain for $i, j = 1, 2$ with $i \neq j$

$$\begin{aligned}|P^k f_D^{(i)}(P^k u_1^0, P^k u_2^0)|_1 &\leq |f_D^{(i)}(P^k u_1^0, P^k u_2^0)|_1 \\ &\leq C\|P^k u_j^0 + \alpha_j\|_1^2 \|P^k u_i^0\|_2 + C\|P^k u_i^0 + \alpha_i\|_1 \|P^k u_j^0 + \alpha_j\|_1 \|P^k u_j^0\|_2 \\ &\leq C\|P^k u_j^0 + \alpha_j\|_1^2 \|u_i^0\|_2 + C\|P^k u_i^0 + \alpha_i\|_1 \|P^k u_j^0 + \alpha_j\|_1 \|u_j^0\|_2 \\ &\leq C\big(\|u_j^0\|_1^2 + 1\big)\|u_i^0\|_2 + C\big(\|u_i^0\|_1 + 1\big)\big(\|u_j^0\|_1 + 1\big)\|u_j^0\|_2 \\ &\leq C.\end{aligned} \tag{3.1.18}$$

Combining (3.1.12), (3.1.13), (3.1.14), (3.1.16), and (3.1.18) yields for $t \in (0, T]$ that

$$\begin{aligned}\gamma \int_0^t \big[|\partial_s u_{\varepsilon,1}^k|_1^2 + |\partial_s u_{\varepsilon,2}^k|_1^2\big] ds + \big[\|\partial_s u_{\varepsilon,1}^k(t)\|_{-1}^2 + \|\partial_s u_{\varepsilon,2}^k(t)\|_{-1}^2\big] \\ \leq C\big[|w_{\varepsilon,1}^k(0)|_1^2 + |w_{\varepsilon,2}^k(0)|_1^2\big] \leq C.\end{aligned} \tag{3.1.19}$$

Hence one finds, after ignoring the non-negative integrals, that

$$\|\partial_t u^k_{\varepsilon,1}\|_{L^\infty(0,T;(H^1(\Omega))')} + \|\partial_t u^k_{\varepsilon,2}\|_{L^\infty(0,T;(H^1(\Omega))')} \leq C, \tag{3.1.20}$$

and with the aid of the Poincaré inequality one can also deduce, on ignoring the dual terms, that

$$\|\partial_t u^k_{\varepsilon,1}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t u^k_{\varepsilon,2}\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{3.1.21}$$

Since, by (2.3.28), $|w^k_{\varepsilon,i}|_1 = \|\partial_t u^k_{\varepsilon,i}\|_{-1}$, we have from the bound (3.1.19) for $i = 1, 2$

$$\Big| w^k_{\varepsilon,i} - \fint w^k_{\varepsilon,i} \Big|_1 = |w^k_{\varepsilon,i}|_1 \leq C, \tag{3.1.22}$$

so that together with the Poincaré inequality it follows for $i = 1, 2$ that

$$\Big\| w^k_{\varepsilon,i} - \fint w^k_{\varepsilon,i} \Big\|_{L^\infty(0,T;H^1(\Omega))} \leq C. \tag{3.1.23}$$

From (2.3.31), (2.3.35) and (3.1.22) it follows that

$$\Big| \fint w^k_{\varepsilon,i} \Big| = \Big| \fint \big[\Psi'_{\varepsilon,i}(u^k_{\varepsilon,i}) + f^{(i)}_D(u^k_{\varepsilon,1}, u^k_{\varepsilon,2}) \big] \Big| \leq C\big[1 + |w^k_{\varepsilon,i}|_1\big] \leq C. \tag{3.1.24}$$

Thus we have for $i = 1, 2$ that

$$\Big\| \fint w^k_{\varepsilon,i} \Big\|_{L^\infty(0,T;H^1(\Omega))} = |\Omega|^{\frac{1}{2}} \Big\| \fint w^k_{\varepsilon,i} \Big\|_{L^\infty(0,T)} \leq C. \tag{3.1.25}$$

Hence (3.1.23) and (3.1.25) imply for $i = 1, 2$ that

$$\|w^k_{\varepsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} \leq C \tag{3.1.26}$$

Therefore, from the bounds (3.1.20), (3.1.21) and (3.1.26) the desired bounds in (3.1.6a) follows by the usual compactness arguments.

From (2.3.51) we have for $i = 1, 2$ that

$$\gamma(\nabla u_{\varepsilon,i}, \nabla \phi_\varepsilon(u_{\varepsilon,i})) + \frac{1}{2}|\phi_\varepsilon(u_{\varepsilon,i})|_0^2 \leq C\big[|w_{\varepsilon,i}|_0^2 + |u_{\varepsilon,i}|_0^2 + |f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2})|_0^2\big]. \tag{3.1.27}$$

Thus, from the estimates (3.1.6a), (2.3.8a) and (2.3.8d) we obtain the desired estimate (3.1.6b) on $\phi_\varepsilon(u_{\varepsilon,i})$ and we also have the second inequality in (3.1.7). The first inequality follows directly from (2.2.8).

Using the variational equality (2.2.21b) and the standard regularity theory of elliptic problems it follows that for $i = 1, 2$

$$\|u_{\varepsilon,i}\|_2 \leq C|w_{\varepsilon,i} - \phi_\varepsilon(u_{\varepsilon,i}) + \theta_i u_{\varepsilon,i} - f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}) + u_{\varepsilon,i}|_0, \tag{3.1.28}$$

which leads to the second estimate in (3.1.6b) on noting the third bound in (3.1.6a), the first bound in (3.1.6b), (2.3.8a) and (2.3.8d).

We again use the standard regularity theory of elliptic problems with variational equality (2.2.21a) to result in for $i = 1, 2$ that

$$\|w_{\varepsilon,i}\|_2 \leq C| - \partial_t u_{\varepsilon,i} + w_{\varepsilon,i}|_0, \tag{3.1.29}$$

we thus obtain, by the first and the third bounds of (3.1.6a), the third estimate in (3.1.6b). Applying (3.1.4) with $v_i = u_{\varepsilon,i}$ and noting the bounds (2.3.8a) and (3.1.6b) it follows for *a.e.* $t \in (0, T)$ and $i, j = 1, 2$ with $i \neq j$ that

$$|f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2})|_1 \leq C\|u_{\varepsilon,j} + \alpha_j\|_1^2 \|u_{\varepsilon,i}\|_2 + C\|u_{\varepsilon,i} + \alpha_i\|_1 \|u_{\varepsilon,j} + \alpha_j\|_1 \|u_{\varepsilon,j}\|_2 \leq C, \tag{3.1.30}$$

which together with (2.3.8d) we obtain the desired estimate (3.1.6c).

Furthermore, application of the classical result stated in Theorem A.0.20 (see appendix A) yields, after noting $u_{\varepsilon,i} \in L^\infty(0, T; H^2(\Omega)) \hookrightarrow L^2(0, T; H^2(\Omega))$ and $\partial_t u_{\varepsilon,i} \in L^2(0, T; H^1(\Omega)) \hookrightarrow L^2(\Omega_T)$, for $i = 1, 2$ that $u_{\varepsilon,i} \in C([0, T], H^1(\Omega))$.

Finally, to prove $\frac{\partial u_{\varepsilon,i}}{\partial \nu} = 0$ we argue as in (Thomée [33], p.20). Since $u_{\varepsilon,i} \in H^2(\Omega)$ *a.e.* $t \in (0, T)$, we have on integrating the first term of (2.2.21b) in space by parts that

$$(-\gamma \Delta u_{\varepsilon,i} + \Psi'_{\varepsilon,i}(u_{\varepsilon,i}) + f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}) - w_{\varepsilon,i}, \eta) + \int_{\partial\Omega} \frac{\partial u_{\varepsilon,i}}{\partial \nu} \eta \, ds = 0, \quad \forall \eta \in H^1(\Omega), \tag{3.1.31}$$

which implies $\frac{\partial u_{\varepsilon,i}}{\partial \nu} = 0$ *a.e.* on $\partial\Omega \times (0, T)$, since η is arbitrary. This completes the proof. □

Corollary 3.1.4 Let the assumptions of Theorem 3.1.3 hold. Then the unique solution of $(\mathbf{P})$ is such that the following further regularity results hold

$$\partial_t u_1, \partial_t u_2 \in L^\infty(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)), \tag{3.1.32a}$$

$$w_1, w_2 \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{3.1.32b}$$

$$u_1, u_2 \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T],H^1(\Omega)), \tag{3.1.32c}$$

$$\phi(u_1), \phi(u_2) \in L^\infty(0,T;L^2(\Omega)), \tag{3.1.32d}$$

$$f_D^{(1)}(u_1,u_2), f_D^{(2)}(u_1,u_2) \in L^\infty(0,T;H^1(\Omega)). \tag{3.1.32e}$$

Proof. With the aid of the uniform bounds (3.1.6a-c) in ε and the compactness arguments, one can repeat the same treatment used in Theorem 3.1.3 to obtain the above regularity. □

3.2 Continuous dependence and a regularization error bound

Theorem 3.2.1 For $m_1, m_2 \in (-1,1)$ let

$$X_{m_1,m_2} = \{(v_1,v_2) \in H^1(\Omega) \times H^1(\Omega) : \text{for } i = 1,2, \textstyle{\fint} v_i = m_i \text{ and } |v_i|_{0,\infty} \leq 1\}.$$

Then the mapping $X_{m_1,m_2} \ni (u_1^0, u_2^0) \longmapsto (u_1(t), u_2(t)) \in X_{m_1,m_2}$ is continuous with respect to $(H^1(\Omega))' \times (H^1(\Omega))'$ norm.

Proof. Assume that (u_1,u_2) and (v_1,v_2) satisfy the weak form $(\mathbf{P})$ with initial conditions $(u_1^0,u_2^0), (v_1^0,v_2^0) \in X_{m_1,m_2}$ such that $(u_1^0,u_2^0) \neq (v_1^0,v_2^0)$. By arguing similarly to the uniqueness proof of $(\mathbf{P}_\varepsilon)$ we obtain for *a.e.* $t \in (0,T)$ that

$$\|u_1(t)-v_1(t)\|_{-1}^2 + \|u_2(t)-v_2(t)\|_{-1}^2 \leq e^{ct}\big[\|u_1^0-v_1^0\|_{-1}^2 + \|u_2^0-v_2^0\|_{-1}^2\big]. \tag{3.2.1}$$

Therefore, we have, by Lemma 2.1.1, the required continuity result. □

We now turn to prove an error estimate between the solutions of $(\mathbf{P}_\varepsilon)$ and $(\mathbf{P})$ where we adapt the argument in [11]. This error bound is crucial to derive our fully-discrete error bound as will be seen in the next chapters.

Theorem 3.2.2 Let $\hat{e}_{\varepsilon,1} := u_1 - u_{\varepsilon,1}$ and $\hat{e}_{\varepsilon,2} := u_2 - u_{\varepsilon,2}$. Then, we have that

$$\|\hat{e}_{\varepsilon,1}\|^2_{L^2(0,T;H^1(\Omega))}+\|\hat{e}_{\varepsilon,2}\|^2_{L^2(0,T;H^1(\Omega))}+\|\hat{e}_{\varepsilon,1}\|^2_{L^\infty(0,T;(H^1(\Omega))')}+\|\hat{e}_{\varepsilon,2}\|^2_{L^\infty(0,T;(H^1(\Omega))')} \leq C\varepsilon. \tag{3.2.2}$$

Proof. We first note that $\hat{e}_{\varepsilon,1}, \hat{e}_{\varepsilon,2} \in V_0$ *a.e.* $t \in (0,T)$. We test (**P**), (2.2.25), and the corresponding regularized version $(\mathbf{P}_\varepsilon)$, (2.2.24), with $\eta = \hat{e}_{\varepsilon,i}$ and then subtract to yield for *a.e.* $t \in (0,T)$ and $i = 1,2$

$$\gamma|\hat{e}_{\varepsilon,i}|_1^2+(\phi(u_i)-\phi_\varepsilon(u_{\varepsilon,i}),\hat{e}_{\varepsilon,i})+(f_D^{(i)}(u_1,u_2)-f_D^{(i)}(u_{\varepsilon,1},u_{\varepsilon,2}),\hat{e}_{\varepsilon,i})+(\mathcal{G}\partial_t\hat{e}_{\varepsilon,i},\hat{e}_{\varepsilon,i}) = \theta_i|\hat{e}_{\varepsilon,i}|_0^2. \tag{3.2.3}$$

We deal with the D-coupling term in the same way as for (2.3.56) to result in for *a.e.* $t \in (0,T)$ with $i,j = 1,2$ and $i \neq j$

$$\left|(f_D^{(i)}(u_1,u_2) - f_D^{(i)}(u_{\varepsilon,1},u_{\varepsilon,2}),\hat{e}_{\varepsilon,i})\right| \leq \frac{\gamma}{8}\big[|\hat{e}_{\varepsilon,i}|_1^2 + |\hat{e}_{\varepsilon,j}|_1^2\big] + C\big[\|\hat{e}_{\varepsilon,i}\|_{-1}^2 + \|\hat{e}_{\varepsilon,j}\|_{-1}^2\big]. \tag{3.2.4}$$

Recalling that $\frac{1}{2}\frac{d}{dt}\|\hat{e}_{\varepsilon,i}\|_{-1}^2 = (\mathcal{G}\partial_t\hat{e}_{\varepsilon,i},\hat{e}_{\varepsilon,i})$ and (2.1.11) one can rewrite (3.2.3) as

$$\begin{aligned}\gamma|\hat{e}_{\varepsilon,i}|_1^2 + (\phi(u_i) - \phi_\varepsilon(u_{\varepsilon,i}),\hat{e}_{\varepsilon,i}) + \frac{1}{2}\frac{d}{dt}\|\hat{e}_{\varepsilon,i}\|_{-1}^2 \\ \leq \frac{3\gamma}{8}|\hat{e}_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|\hat{e}_{\varepsilon,j}|_1^2 + C\big[\|\hat{e}_{\varepsilon,i}\|_{-1}^2 + \|\hat{e}_{\varepsilon,j}\|_{-1}^2\big].\end{aligned} \tag{3.2.5}$$

To treat the logarithmic term we define for $i = 1,2$ and *a.e.* $t \in (0,T)$

$$\begin{aligned}\Omega^+_{\varepsilon,i}(t) &:= \{x \in \Omega : 1-\varepsilon \leq u_i(x,t) \leq u_{\varepsilon,i}(x,t)\},\\ \Omega^-_{\varepsilon,i}(t) &:= \{x \in \Omega : u_{\varepsilon,i}(x,t) \leq u_i(x,t) \leq -1+\varepsilon\},\\ \hat{\Omega}_{\varepsilon.i}(t) &:= \Omega^+_{\varepsilon,i}(t) \cup \Omega^-_{\varepsilon,i}(t).\end{aligned}$$

By the monotonicity of ϕ_ε and (2.2.12) it follows for $i = 1,2$ and *a.e.* $t \in (0,T)$ that

$$\begin{aligned}(\phi(u_i) - \phi_\varepsilon(u_{\varepsilon,i}),\hat{e}_{\varepsilon,i}) &= (\phi(u_i) - \phi_\varepsilon(u_i),\hat{e}_{\varepsilon,i}) + (\phi_\varepsilon(u_i) - \phi_\varepsilon(u_{\varepsilon,i}),\hat{e}_{\varepsilon,i})\\ &\geq (\phi(u_i) - \phi_\varepsilon(u_i),\hat{e}_{\varepsilon,i}) + (\phi_\varepsilon(u_i) - \phi_\varepsilon(u_{\varepsilon,i}),\hat{e}_{\varepsilon,i})_{\hat{\Omega}_{\varepsilon,i}(t)}\\ &\geq (\phi(u_i) - \phi_\varepsilon(u_i),\hat{e}_{\varepsilon,i}) + \frac{\theta}{2\varepsilon}|\hat{e}_{\varepsilon,i}|^2_{0,\hat{\Omega}_{\varepsilon,i}(t)}\,[1].\end{aligned} \tag{3.2.6}$$

---

[1] $(u,v)_{\hat{\Omega}_{\varepsilon,i}(t)} := \int_{\hat{\Omega}_{\varepsilon,i}(t)} uv\,dx$ and $|u|^2_{0,\hat{\Omega}_{\varepsilon,i}(t)} := (u,u)_{0,\hat{\Omega}_{\varepsilon,i}(t)}$.

Noting $\phi_\varepsilon(r) = \phi(r)\ \forall r \in [-1+\varepsilon, 1-\varepsilon]$, (2.2.5) and the fact that $\phi_\varepsilon(r) \geq 0\ \forall r \geq 0$ and $\phi_\varepsilon(r) \leq 0\ \forall r \leq 0$ it is a simple matter to see that $(\phi(u_i) - \phi_\varepsilon(u_i))\hat{e}_{\varepsilon,i}$ is non-negative in $\Omega \setminus \hat{\Omega}_{\varepsilon,i}(t)$ and that $\phi_\varepsilon(u_i)\hat{e}_{\varepsilon,i}$ is non positive in $\hat{\Omega}_{\varepsilon,i}(t)$ which implies for $i = 1, 2$ and *a.e.* $t \in (0, T)$ that

$$(\phi(u_i) - \phi_\varepsilon(u_i), \hat{e}_{\varepsilon,i}) \geq (\phi(u_i) - \phi_\varepsilon(u_i), \hat{e}_{\varepsilon,i})_{\hat{\Omega}_{\varepsilon,i}(t)} \geq (\phi(u_i), \hat{e}_{\varepsilon,i})_{\hat{\Omega}_{\varepsilon,i}(t)}. \tag{3.2.7}$$

Hence, combining (3.2.5)-(3.2.7) yields for *a.e.* $t \in (0, T)$ and $i, j = 1, 2$ with $i \neq j$ that

$$\begin{aligned}\gamma|\hat{e}_{\varepsilon,i}|_1^2 &+ \frac{\theta}{2\varepsilon}|\hat{e}_{\varepsilon,i}|^2_{0,\hat{\Omega}_{\varepsilon.i}(t)} + \frac{1}{2}\frac{d}{dt}\|\hat{e}_{\varepsilon,i}\|_{-1}^2 \\ &\leq \frac{3\gamma}{8}|\hat{e}_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|\hat{e}_{\varepsilon,j}|_1^2 + C\big[\|\hat{e}_{\varepsilon,i}\|_{-1}^2 + \|\hat{e}_{\varepsilon,j}\|_{-1}^2\big] - (\phi(u_i), \hat{e}_{\varepsilon,i})_{\hat{\Omega}_{\varepsilon,i}(t)} \\ &\leq \frac{3\gamma}{8}|\hat{e}_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|\hat{e}_{\varepsilon,j}|_1^2 + C\big[\|\hat{e}_{\varepsilon,i}\|_{-1}^2 + \|\hat{e}_{\varepsilon,j}\|_{-1}^2\big] + \frac{\theta}{4\varepsilon}|\hat{e}_{\varepsilon,i}|^2_{0,\hat{\Omega}_{\varepsilon,i}(t)} + C\varepsilon|\phi(u_i)|_0^2,\end{aligned} \tag{3.2.8}$$

where we have also used the Young inequality and $|\cdot|_{0,\hat{\Omega}_{\varepsilon.i}(t)} \leq |\cdot|_0$.

Summing (3.2.8) over $i = 1, 2$ and rearranging gives for *a.e.* $t \in (0, T)$ that

$$\begin{aligned}\frac{\gamma}{2}\big[|\hat{e}_{\varepsilon,1}|_1^2 + |\hat{e}_{\varepsilon,2}|_1^2\big] + \frac{\theta}{4\varepsilon}\big[|\hat{e}_{\varepsilon,1}|^2_{0,\hat{\Omega}_{\varepsilon.i}(t)} + |\hat{e}_{\varepsilon,2}|^2_{0,\hat{\Omega}_{\varepsilon,i}(t)}\big] + \frac{1}{2}\frac{d}{dt}\big[\|\hat{e}_{\varepsilon,1}\|_{-1}^2 + \|\hat{e}_{\varepsilon,2}\|_{-1}^2\big] \\ \leq C\varepsilon\big[|\phi(u_1)|_0^2 + |\phi(u_2)|_0^2\big] + C\big[\|\hat{e}_{\varepsilon,1}\|_{-1}^2 + \|\hat{e}_{\varepsilon,2}\|_{-1}^2\big].\end{aligned} \tag{3.2.9}$$

Applying the Gronwall lemma and noting $\hat{e}_{\varepsilon,1}(0) = \hat{e}_{\varepsilon,2}(0) = 0$ gives for *a.e.* $t \in (0, T]$

$$\gamma\int_0^t \big[|\hat{e}_{\varepsilon,1}|_1^2 + |\hat{e}_{\varepsilon,2}|_1^2\big]ds + \big[\|\hat{e}_{\varepsilon,1}\|_{-1}^2 + \|\hat{e}_{\varepsilon,2}\|_{-1}^2\big] \leq Ce^{ct}\varepsilon\big[\|\phi(u_1)\|^2_{L^2(\Omega_T)} + \|\phi(u_2)\|^2_{L^2(\Omega_T)}\big] \tag{3.2.10}$$

Finally, using the Poincaré inequality and (2.3.60c) we have the desired result (3.2.2) . □

Chapter 4

The finite element space and a semi-discrete approximation

In this chapter we formulate a semi-discrete approximation to the solution of the continuous problem (**P**) where we discretise in the spatial variable using a finite element method.

In Section 4.1 we introduce the finite element method and some basic notation that will be used throughout the rest of the thesis. We also define some necessary operators and mention briefly their associated properties. In Section 4.2 we prove some technical lemmata which are necessary for performing the analytic study. We prove in Section 4.3 the existence and uniqueness of the proposed semi-discrete approximation. Finally, in Section 4.4 we prove an error estimate between the solutions of the continuous and semi-discrete problems.

4.1 Notation and preliminaries

In the remaining chapters of the thesis we shall study semi-discrete and fully-discrete finite element approximations of the problem (**P**) under the following assumptions on the mesh

($\mathbf{A^h}$) Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be a convex polygonal or polyhedral domain if $d = 2$ or $d = 3$. Let $\mathcal{T}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices[1] τ with $h_\tau :=$diam τ and $h := \max_{\tau\in\mathcal{T}^h} h_\tau$, so that $\bar{\Omega} = \cup_{\tau\in\mathcal{T}^h}\bar{\tau}$. In addition, it is assumed that $\mathcal{T}^h$ is a weakly acute (Barrett and Blowey [5]); that is for (i) $d = 2$ the sum of the opposite angles relative to any side does not exceed π and for (ii) $d = 3$ the angle between any two faces of the tetrahedron does not exceed $\frac{\pi}{2}$.

Associated with $\mathcal{T}^h$ we define the standard finite element space consisting of the continuous piecewise linear functions

$$S^h := \{\chi \in C(\bar{\Omega}) : \chi|_\tau \text{ is linear } \forall \tau \in \mathcal{T}^h\} \subset H^1(\Omega). \tag{4.1.1}$$

Recalling that $m_i := ⨍\, u_i^0$ it is also convenient to introduce for $i = 1, 2$

$$S^h_{m_i} := \{\, \chi \in S^h : ⨍\, \chi = m_i\}. \tag{4.1.2}$$

Let $\{\varphi_j\}_{j=1}^J$ be the standard basis functions for S^h satisfying $\varphi_j(x_i) = \delta_{ij}\ \forall\, i, j = 0, 1, ..., J$ where $\{x_j\}_{j=0}^J$ is the set of the nodes of $\mathcal{T}^h$. Let $\pi^h : C(\bar{\Omega}) \longrightarrow S^h$ denote the interpolation operator defined by $\pi^h(\chi(x_j)) = \chi(x_j)\ \forall j = 0, 1, ..., J$. In addition, we define a discrete inner (semi-inner) product on S^h $(C(\bar{\Omega}))$ as

$$(\chi, v)^h = \int_\Omega \pi^h(\chi(x)\, v(x))\, dx \equiv \sum_{j=0}^{J} M_{jj}\, \chi(x_j) v(x_j), \tag{4.1.3}$$

where $M_{jj} = (1, \varphi_j) = (\varphi_j, \varphi_j)^h > 0$.

Below we mention some well-known results concerning the finite element space S^h: By the definition of π^h and $(\cdot,\cdot)^h$ we can easily deduce that

$$(\chi, v)^h = (\pi^h\chi, v)^h\ \forall\, \chi, v \in C(\bar{\Omega}) \text{ and } (\chi, 1)^h = (\chi, 1)\ \forall\, \chi \in S^h. \tag{4.1.4}$$

The discrete inner product induces a norm on S^h given by

$$|\chi|_h := \sqrt{(\chi,\chi)^h} \qquad \forall\, \chi \in S^h. \tag{4.1.5}$$

It is well-known that this norm is equivalent to $|\cdot|_0$ (e.g. Raviart [53]) via

$$|\chi|_0 \le |\chi|_h \le C|\chi|_0 \quad \forall\, \chi \in S^h. \tag{4.1.6}$$

[1]We recall that a simplex τ is (i) an interval if $d = 1$, (ii) a triangle if $d = 2$, (iii) a tetrahedron if $d = 3$.

We also recall the following useful result (e.g. Ciavaldini [36])

$$\left|(\chi, v) - (\chi, v)^h\right| \leq C h^{m+1} \, |\chi|_m |v|_1 \quad \forall \, \chi, v \in S^h, \; m = 0, 1. \tag{4.1.7}$$

For later purpose we introduce the following inverse inequalities which follow from the quasi-uniform condition (see Theorem 3.2.6, in Ciarlet [22])

$$|\chi|_{m,q} \leq C h^{d(1/q-1/p)} |\chi|_{m,p} \quad 1 \leq p \leq q \leq \infty, \; m = 0, 1, \; \forall \, \chi \in S^h, \tag{4.1.8a}$$

$$|\chi|_1 \leq \frac{C}{h} |\chi|_h \qquad \forall \, \chi \in S^h. \tag{4.1.8b}$$

In addition, the following interpolation error estimates (Theorem 5, in Ciarlet and Raviart [41]) holds

$$\left|(I - \pi^h)\eta\right|_{0,1} \leq C h^2 |\eta|_{2,1} \quad \forall \, \eta \in W^{2,1}(\Omega), \tag{4.1.9a}$$

$$\left|(I - \pi^h)\eta\right|_0 + h\left|(I - \pi^h)\eta\right|_1 \leq C h^2 |\eta|_2 \quad \forall \, \eta \in H^2(\Omega). \tag{4.1.9b}$$

In order to improve on the error bound between the solutions of the continuous and semi-discrete problems in the case $d = 1, 2$ we need the discrete result (e.g. Thomeé [33], p.68)

$$|\chi|_{0,\infty} \leq C \big(\ln(1/h)\big)^{d-1} \|\chi\|_1 \quad \forall \, \chi \in S^h, \; \forall \, h \leq h_0. \tag{4.1.10}$$

Similarly to (2.1.5), the discrete Green's operator $\hat{\mathcal{G}}^h : \mathcal{F}_0^{c,h} \longrightarrow V_0^h$ is defined by

$$(\nabla \hat{\mathcal{G}}^h f^c, \nabla \chi) = (f^c, \chi)^h \quad \forall \, \chi \in S^h, \tag{4.1.11}$$

where $\mathcal{F}_0^{c,h} := \{f^c \in C(\bar{\Omega}) : (f^c, 1)^h = 0\}$ and $V_0^h := \{\chi \in S^h : (\chi, 1)^h = 0\}$.

Observe that $V_0^h \subset V_0 \subset \mathcal{F}_0$. In the same way as for (2.1.7) one can define

$$\|f^c\|_{-h} := |\hat{\mathcal{G}}^h f^c|_1 = \sqrt{(f^c, \hat{\mathcal{G}}^h f^c)^h} \quad \forall \, f^c \in \mathcal{F}_0^{c,h}. \tag{4.1.12}$$

From (4.1.12), the equivalent result (4.1.6) and the Poincaré inequality we have

$$\|f^c\|_{-h}^2 = (f^c, \hat{\mathcal{G}}^h f^c)^h \leq C |f^c|_h |\hat{\mathcal{G}}^h f^c|_0 \leq C |f^c|_h |\hat{\mathcal{G}}^h f^c|_1 = C |f^c|_h \|f^c\|_{-h},$$

which leads us to the discrete analogue to (2.1.10), that is,

$$\|f^c\|_{-h} \leq C |f^c|_h \quad \forall \, f^c \in \mathcal{F}_0^{c,h}. \tag{4.1.13}$$

By (4.1.11), (4.1.12) and a Young's inequality we have for any $\beta > 0$

$$(f^c, \chi)^h = (\nabla \hat{\mathcal{G}}^h f^c, \nabla \chi) \leq \|f^c\|_{-h} |\chi|_1 \leq \beta |\chi|_1^2 + \frac{1}{4\beta} \|f^c\|_{-h}^2 \ \forall f^c \in \mathcal{F}_0^{c,h}, \ \chi \in S^h, \tag{4.1.14}$$

which implies, by choosing $f^c = \chi = v^h \in V_0^h$,

$$|v^h|_h^2 \leq \|v^h\|_{-h} |v^h|_1 \leq \beta |v^h|_1^2 + \frac{1}{4\beta} \|v^h\|_{-h}^2 \quad \forall v^h \in V_0^h, \ \beta > 0. \tag{4.1.15}$$

Noting the first inequality in (4.1.15) and the inverse inequality (4.1.8b) we have

$$|v^h|_h \leq \frac{C}{h} \|v^h\|_{-h} \quad \forall v^h \in V_0^h. \tag{4.1.16}$$

For later purpose we recall the following essential results concerning the Green's operators $\mathcal{G}$ and $\hat{\mathcal{G}}^h$:

$$C_1 \|v^h\|_{-h} \leq \|v^h\|_{-1} \leq C_2 \|v^h\|_{-h} \quad \forall v^h \in V_0^h \subset V_0, \tag{4.1.17}$$

$$|\mathcal{G} v^h - \hat{\mathcal{G}}^h v^h|_0 \leq C h^2 \|v^h\|_1 \quad \forall v^h \in V_0^h. \tag{4.1.18}$$

(see Barrett and Blowey [19], pp.642-643).

For dealing with the initial data of the semi-discrete and fully-discrete approximations we introduce the weighted H^1-projection (e.g. Barett and Blowey [15]) $P_\gamma^h : H^1(\Omega) \to S^h$ defined by

$$\gamma(\nabla(I - P_\gamma^h)\eta, \nabla \chi) + ((I - P_\gamma^h)\eta, \chi) = 0 \quad \forall \chi \in S^h, \tag{4.1.19}$$

and we also recall the discrete $L^2(\Omega)$-projection (see e.g. [6], [7]) $P^h : L^2(\Omega) \to S^h$ given by

$$(P^h \eta, \chi)^h = (\eta, \chi) \quad \forall \chi \in S^h. \tag{4.1.20}$$

The above projections satisfy the following important results (e.g. [6], [15])

$$\left|(I - P^h)\eta\right|_m \leq C h^{1-m} |\eta|_1 \qquad m = 0, 1, \ \forall \eta \in H^1(\Omega), \tag{4.1.21}$$

$$\left|(I - P_\gamma^h)\eta\right|_{m,p} \leq C h^{2-m-d(1/2-1/p)} |\eta|_2 \ m = 0, 1, \ p \in [2, \infty], \ \forall \eta \in H^2(\Omega), \tag{4.1.22}$$

$$|P^h \eta|_{0,\infty} \leq |\eta|_{0,\infty} \qquad \forall \eta \in L^\infty(\Omega). \tag{4.1.23}$$

It is also easily established from (4.1.19) and a Young's inequality that

$$\|P_\gamma^h \eta\|_1 \leq C \|\eta\|_1 \quad \forall \eta \in H^1(\Omega). \tag{4.1.24}$$

We remark for later use that (4.1.19) gives $P^h_\gamma \eta - \eta \in V_0 \subset \mathcal{F}_0\ \forall \eta \in H^1(\Omega)$ which together with (2.1.8) and (4.1.22) lead to

$$\|P^h_\gamma \eta - \eta\|^2_{-1} \le C|P^h_\gamma \eta - \eta|^2_0 \le Ch^4|\eta|_2 \quad \forall\, \eta \in H^2(\Omega). \tag{4.1.25}$$

For future reference we define the stiffness matrix A and lumped matrix M via

$$A_{ij} = (\nabla \varphi_i, \nabla \varphi_j), \quad M_{ij} = (\varphi_i, \varphi_j)^h. \tag{4.1.26}$$

The matrix A is positive definite and the matrix M is diagonal with positive entries (see e.g. Thomeé [33], p.239). Further, due to the fact that partitioning is weakly acute we have (see [54], p.49)

$$A_{ij} \le 0 \quad \forall\, i \ne j, \tag{4.1.27}$$

and from the the fact that $\sum_{j=0}^J \varphi_j(x) = 1$ we obtain

$$\sum_{j=0}^J A_{ij} = (\nabla \varphi_i, \nabla \sum_{j=0}^J \varphi_j) = 0 \quad 0 \le i \le J. \tag{4.1.28}$$

These two results are important for the first technical lemma which follows.

4.2 Some technical lemmata

In this section we prove some technical lemmata that are necessary to deal with the nonlinearities, the logarithmic and D-coupling terms, throughout the treatment of the semi-discrete and fully-discrete problems.

In the first two lemmata we show results regarding the monotone logarithmic function ϕ_ε that will be important in deriving some stability estimates. To show the next lemma we employ the ideas in Nochetto [54] and Garvie [40] that have been used to prove similar results.

Lemma 4.2.1 Assume that $\mathcal{T}^h$ is weakly acute partitioning and $\varepsilon \le 1/2$. Then

$$(i)\ |\pi^h \phi_\varepsilon(\chi)|^2_1 \equiv |\nabla \pi^h \phi_\varepsilon(\chi)|^2_0 \le \frac{\theta}{\varepsilon}(\nabla \chi, \nabla \pi^h \phi_\varepsilon(\chi)) \quad \forall \chi \in S^h. \tag{4.2.1}$$

(ii) Further, if $|\chi|_{0,\infty} \le 1 - \varepsilon$ then

$$|\nabla \pi^h \phi_\varepsilon(\chi)|^2_0 \le \phi'(|\chi|_{0,\infty})(\nabla \chi, \nabla \pi^h \phi_\varepsilon(\chi)). \tag{4.2.2}$$

Proof. Let $\pi^h(\phi_\varepsilon(\chi)) = \sum_{j=0}^{J} \phi_\varepsilon(\chi_j)\varphi_j$ where $\chi_j = \chi(x_j)$. Since by (4.1.28) we have $A_{ii} = -\sum_{\substack{j=0\\ j\neq i}}^{J} A_{ij}$ it then follows from (4.1.26)

$$
\begin{aligned}
(\nabla\pi^h\phi_\varepsilon(\chi), \nabla\pi^h\phi_\varepsilon(\chi)) &= \sum_{i=0}^{J}\sum_{j=0}^{J} \phi_\varepsilon(\chi_i)\phi_\varepsilon(\chi_j)A_{ij} \\
&= \sum_{i=0}^{J}\Big[\sum_{\substack{j=0\\ j\neq i}}^{J}\big(\phi_\varepsilon(\chi_i)\phi_\varepsilon(\chi_j)A_{ij}\big) + \phi_\varepsilon(\chi_i)\phi_\varepsilon(\chi_i)A_{ii}\Big] \\
&= \sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J}\Big[\phi_\varepsilon(\chi_i)\phi_\varepsilon(\chi_j)A_{ij} - (\phi_\varepsilon(\chi_i))^2 A_{ij}\Big] \\
&= \sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J} A_{ij}\phi_\varepsilon(\chi_i)\big[\phi_\varepsilon(\chi_j) - \phi_\varepsilon(\chi_i)\big]. \qquad (4.2.3)
\end{aligned}
$$

Using the fact that $\sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J}(\cdot) = \sum_{j=0}^{J}\sum_{\substack{i=0\\ i\neq j}}^{J}(\cdot)$, swapping the indices i and j and noting that $A_{ij} = A_{ji}$ we may rewrite the right hand side of (4.2.3) as

$$
\sum_{j=0}^{J}\sum_{\substack{i=0\\ i\neq j}}^{J} A_{ij}\phi_\varepsilon(\chi_i)\big[\phi_\varepsilon(\chi_j) - \phi_\varepsilon(\chi_i)\big] = \sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J} A_{ij}\phi_\varepsilon(\chi_j)\big[\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\big]. \quad (4.2.4)
$$

Thus summing (4.2.3) twice gives that

$$
\begin{aligned}
2(\nabla\pi^h\phi_\varepsilon(\chi), \nabla\pi^h\phi_\varepsilon(\chi)) &= \sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J} -A_{ij}\big[\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\big]\big[\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\big] \\
&= \sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J} -A_{ij}\phi'_\varepsilon(\xi_{ij})\big[\chi_i - \chi_j\big]\big[\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\big], \quad (4.2.5)
\end{aligned}
$$

where ξ_{ij} between χ_j and χ_i, which implies $-|\chi|_{0,\infty} \leq \xi_{ij} \leq |\chi|_{0,\infty} = \max_{0\leq j\leq J}|\chi_j|$. Thus from (2.2.8) we have that $0 < \phi'_\varepsilon(\xi_{ij}) \leq \frac{\theta}{\varepsilon}\ \forall\,\chi \in S^h$ and if $|\chi|_{0,\infty} \leq 1-\varepsilon$, we have by (2.2.13) and (2.2.8) that $\phi'_\varepsilon(\xi_{ij}) \leq \phi'_\varepsilon(|\chi|_{0,\infty}) \leq \frac{\theta}{\varepsilon}$. Letting

$$
L := \begin{cases} \frac{\theta}{\varepsilon} & \text{if } |\chi|_{0,\infty} > 1-\varepsilon, \\ \phi'_\varepsilon(|\chi|_{0,\infty}) = \phi'(|\chi|_{0,\infty}) & \text{if } |\chi|_{0,\infty} \leq 1-\varepsilon. \end{cases} \qquad (4.2.6)
$$

We then have on noting (4.2.5), (4.1.27) and the monotonicity of ϕ_ε that

$$\begin{aligned}2(\nabla\pi^h\phi_\varepsilon(\chi),\nabla\pi^h\phi_\varepsilon(\chi)) &\le L\sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J} -A_{ij}\left[\chi_i-\chi_j\right]\left[\phi_\varepsilon(\chi_i)-\phi_\varepsilon(\chi_j)\right]\\ &= L\sum_{i=0}^{J}\sum_{j=0}^{J} -A_{ij}\left[\chi_i-\chi_j\right]\left[\phi_\varepsilon(\chi_i)-\phi_\varepsilon(\chi_j)\right]\\ &= 2L\sum_{i=0}^{J}\sum_{j=0}^{J} A_{ij}\chi_i\phi_\varepsilon(\chi_j)\\ &= 2L(\nabla\chi,\nabla\pi^h\phi_\varepsilon(\chi)),\end{aligned} \tag{4.2.7}$$

as by (4.1.28) $\sum_{i=0}^{J}\sum_{j=0}^{J}(-A_{ij})\chi_i\phi_\varepsilon(\chi_i) = \sum_{i=0}^{J}\big(-\chi_i\phi_\varepsilon(\chi_i)\sum_{j=0}^{J}A_{ij}\big) = 0.$ □

Lemma 4.2.2 For all $\chi \in S^h$, the monotone function ϕ_ε satisfies

$$\big|(I-\pi^h)\phi_\varepsilon(\chi)\big|_0 \le Ch|\nabla\pi^h\phi_\varepsilon(\chi)|_0 \tag{4.2.8}$$

Proof. We refer to Elliott [38] pp.68-69. □

We now prove some technical results concerning the D-coupling term. These results will be necessary for: deriving stability estimates, proving uniqueness and deriving error bounds for the semi-discrete and fully-discrete approximations.

Lemma 4.2.3 Let $v^h \in V_0^h$. Then there are constants $\sigma = d(\frac{1}{2}-\frac{1}{r})$ and C such that for all $\beta > 0$

$$|v^h|_{0,r}^2 \le C\|v^h\|_{-h}^{1-\sigma}|v^h|_1^{1+\sigma} \le \beta|v^h|_1^2 + C(\beta^{-1})\|v^h\|_{-h}^2 \text{ holds for } r \in \begin{cases}[2,\infty] & \text{if } d=1,\\ [2,\infty) & \text{if } d=2,\\ [2,6) & \text{if } d=3.\end{cases} \tag{4.2.9}$$

Proof. The proof is a simple modification the proof of Lemma 2.3.1 where this time we use the equivalent result (4.1.6) and note (4.1.15) instead of (2.1.11). □

Lemma 4.2.4 For any $\chi, v \in C(\bar\Omega)$ we have

$$(\chi,v)^h \equiv \int_\Omega \pi^h(\chi v)dx \le \Big(\int_\Omega \pi^h(|\chi|^p)dx\Big)^{1/p}\Big(\int_\Omega \pi^h(|v|^q)dx\Big)^{1/q}, \tag{4.2.10}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ p,q\ge 1$.

Proof. From the definition of $(\cdot,\cdot)^h$, (4.1.3), and the standard discrete Hölder inequality (see Appendix A) we have

$$
\begin{aligned}
(\chi, v)^h &\le \sum_{j=0}^{J} M_{jj}|\chi(x_j)||v(x_j)| = \sum_{j=0}^{J} M_{jj}^{1/p}|\chi(x_j)|M_{jj}^{1/q}|v(x_j)| \\
&\le \Big(\sum_{j=0}^{J} M_{jj}|\chi(x_j)|^p\Big)^{1/p}\Big(\sum_{j=0}^{J} M_{jj}|v(x_j)|^q\Big)^{1/q} \\
&= \Big(\int_\Omega \pi^h(|\chi|^p)dx\Big)^{1/p}\Big(\int_\Omega \pi^h(|v|^q)dx\Big)^{1/q}. \qquad (4.2.11)
\end{aligned}
$$

□

Lemma 4.2.5 Let $\chi, v \in S^h$. Then

$$
\big|\nabla(I-\pi^h)(\chi v^2)\big|_0 \le Ch|v|_{1,6}\big[|v|_{0,6}|\chi|_{1,6} + |\chi|_{0,6}|v|_{1,6}\big]. \qquad (4.2.12)
$$

Proof. Let τ be a fixed simplex. It follows from (4.1.9b) that

$$
\big|\nabla(I-\pi^h)(\chi v^2)\big|_{0,\tau}^2 = \big|(I-\pi^h)(\chi v^2)\big|_{1,\tau}^2 \le Ch_\tau^2 \sum_{i,j=1}^{d}\int_\tau \Big|\frac{\partial^2}{\partial x_i \partial x_j}(\chi v^2)\Big|^2 dx, \quad (4.2.13)
$$

where $|\cdot|_{1,\tau} := |\cdot|_{H^1(\tau)}$. Recalling that χ and v are linear functions on τ we have

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j}(\chi v^2) &= \frac{\partial}{\partial x_i}\big(v^2\frac{\partial \chi}{\partial x_j} + 2v\chi\frac{\partial v}{\partial x_j}\big) \\
&= 2v\frac{\partial v}{\partial x_i}\frac{\partial \chi}{\partial x_j} + 2\chi\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + 2v\frac{\partial \chi}{\partial x_i}\frac{\partial v}{\partial x_j}.
\end{aligned}
$$

Note also that $\sum_{i,j=1}^{d}\int_\tau \big|v\frac{\partial \chi}{\partial x_i}\frac{\partial v}{\partial x_j}\big|^2 dx = \sum_{i,j=1}^{d}\int_\tau \big|v\frac{\partial \chi}{\partial x_j}\frac{\partial v}{\partial x_i}\big|^2 dx$. Thus on inserting these results into (4.2.13) and using a generalised Hölder's inequality we obtain

$$
\begin{aligned}
\big|(I-\pi^h)(\chi v^2)\big|_{1,\tau}^2 &\le Ch_\tau^2 \sum_{i,j=1}^{d}\int_\tau |v|^2\Big|\frac{\partial v}{\partial x_i}\Big|^2\Big|\frac{\partial \chi}{\partial x_j}\Big|^2 + |\chi|^2\Big|\frac{\partial v}{\partial x_i}\Big|^2\Big|\frac{\partial v}{\partial x_j}\Big|^2 dx \\
&\le Ch_\tau^2 \sum_{i,j=1}^{d} |v|_{0,6,\tau}^2\Big|\frac{\partial v}{\partial x_i}\Big|_{0,6,\tau}^2\Big|\frac{\partial \chi}{\partial x_j}\Big|_{0,6,\tau}^2 + |\chi|_{0,6,\tau}^2\Big|\frac{\partial v}{\partial x_i}\Big|_{0,6,\tau}^2\Big|\frac{\partial v}{\partial x_j}\Big|_{0,6,\tau}^2 \\
&\le Ch_\tau^2\Big[|v|_{0,6,\tau}^2|v|_{1,6,\tau}^2|\chi|_{1,6,\tau}^2 + |\chi|_{0,6,\tau}^2|v|_{1,6,\tau}^4\Big], \qquad (4.2.14)
\end{aligned}
$$

where in the last step we have noted that $\big|\frac{\partial \eta}{\partial x_i}\big|_{0,6,\tau} \le |\eta|_{1,6,\tau}\ \forall \eta \in W^{1,6}(\tau)$.

We finally add all contributions from all simplices to yield that

$$
\begin{aligned}
\big|(I-\pi^h)(\chi v^2)\big|_1^2 &= \sum_{\tau\in\mathcal{T}^h} \big|(I-\pi^h)(\chi v^2)\big|_{1,\tau}^2 \\
&\le Ch^2 \sum_{\tau\in\mathcal{T}^h} \Big[|v|_{0,6,\tau}^2 |v|_{1,6,\tau}^2 |\chi|_{1,6,\tau}^2 + |\chi|_{0,6,\tau}^2 |v|_{1,6,\tau}^4\Big] \\
&\le Ch^2 \Big[\Big(\sum_{\tau\in\mathcal{T}^h} |v|_{0,6,\tau}^6\Big)^{\frac{1}{3}} \Big(\sum_{\tau\in\mathcal{T}^h} |v|_{1,6,\tau}^6\Big)^{\frac{1}{3}} \Big(\sum_{\tau\in\mathcal{T}^h} |\chi|_{0,6,\tau}^6\Big)^{\frac{1}{3}} \\
&\qquad + \Big(\sum_{\tau\in\mathcal{T}^h} |\chi|_{0,6,\tau}^6\Big)^{\frac{1}{3}} \Big(\sum_{\tau\in\mathcal{T}^h} |v|_{1,6,\tau}^6\Big)^{\frac{2}{3}}\Big] \\
&= Ch^2 \big[|v|_{0,6}^2 |v|_{1,6}^2 |\chi|_{1,6}^2 + |\chi|_{0,6}^2 |v|_{1,6}^4\big],
\end{aligned}
\tag{4.2.15}
$$

leading to (4.2.12), as required. □

Lemma 4.2.6 Let $\chi, \eta, v \in S^h$. Then we have

$$
\big|(I-\pi^h)(\chi\eta v^2)\big|_{0,1} \le \begin{cases} Ch^2 \|\chi\|_1 \|\eta\|_1 \|v\|_1^2 & \text{if } d=1, \\ Ch^{2(1-s)} \|\chi\|_1 \|\eta\|_1 \|v\|_1^2 & \text{if } d=2, \text{ where } s\in(0,1], \\ Ch \|\chi\|_1 \|\eta\|_1 \|v\|_1^2 & \text{if } d=3. \end{cases}
\tag{4.2.16}
$$

Proof. Using (4.1.9a) we obtain for an arbitrary simplex τ that

$$
\big|(I-\pi^h)(\chi\eta v^2)\big|_{0,1,\tau} \le Ch_\tau^2 \sum_{i,j=1}^{d} \int_\tau \Big|\frac{\partial^2}{\partial x_i \partial x_j}(\chi\eta v^2)\Big| dx.
\tag{4.2.17}
$$

Since χ, η and v are linear on the simplex τ we have

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j}(\chi\eta v^2) &= \frac{\partial\chi}{\partial x_j}\frac{\partial\eta}{\partial x_i} v^2 + 2\frac{\partial\chi}{\partial x_j}\eta v \frac{\partial v}{\partial x_i} + \frac{\partial\chi}{\partial x_i}\frac{\partial\eta}{\partial x_j} v^2 + 2\chi\frac{\partial\eta}{\partial x_j} v \frac{\partial v}{\partial x_i} \\
&\quad + 2\frac{\partial\chi}{\partial x_i}\eta v \frac{\partial v}{\partial x_j} + 2\chi\frac{\partial\eta}{\partial x_i} v \frac{\partial v}{\partial x_j} + 2\chi\eta\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}.
\end{aligned}
$$

By Hölder's inequality we have on the simplex τ that

$$
\begin{aligned}
\int_\tau \Big|\frac{\partial\chi}{\partial x_j}\frac{\partial\eta}{\partial x_i} v^2\Big| dx &\le \Big|\frac{\partial\chi}{\partial x_j}\Big|_{0,\tau} \Big|\frac{\partial\eta}{\partial x_i}\Big|_{0,\tau} |v|_{0,\infty,\tau}^2 \\
&\le |\chi|_{1,\tau}\, |\eta|_{1,\tau}\, |v|_{0,\infty,\tau}^2.
\end{aligned}
$$

Hence, by applying the same treatment on the remaining terms we conclude from (4.2.17) that

$$\big|(I-\pi^h)(\chi\eta v^2)\big|_{0,1,\tau} \leq Ch_\tau^2\Big[|\chi|_{1,\tau}|\eta|_{1,\tau}|v|_{0,\infty,\tau}^2 + |\chi|_{1,\tau}|v|_{1,\tau}|\eta|_{0,\infty,\tau}|v|_{0,\infty,\tau} + |\eta|_{1,\tau}|v|_{1,\tau}|\chi|_{0,\infty,\tau}|v|_{0,\infty,\tau} + |v|_{1,\tau}^2|\chi|_{0,\infty,\tau}|\eta|_{0,\infty,\tau}\Big].$$

We now add all contributions from all simplices to obtain that

$$\begin{aligned}
\big|(I-\pi^h)(\chi\eta v^2)\big|_{0,1} &= \sum_{\tau\in\mathcal{T}^h} |(I-\pi^h)(\chi\eta v^2)|_{0,1,\tau} \\
&\leq Ch^2\Big[|v|_{0,\infty}^2 \sum_{\tau\in\mathcal{T}^h} |\chi|_{1,\tau}|\eta|_{1,\tau} + |\eta|_{0,\infty}|v|_{0,\infty} \sum_{\tau\in\mathcal{T}^h} |\chi|_{1,\tau}|v|_{1,\tau} \\
&\qquad + |\chi|_{0,\infty}|v|_{0,\infty} \sum_{\tau\in\mathcal{T}^h} |\eta|_{1,\tau}|v|_{1,\tau} + |\chi|_{0,\infty}|\eta|_{0,\infty} \sum_{\tau\in\mathcal{T}^h} |v|_{1,\tau}^2\Big] \\
&\leq Ch^2\Big[|v|_{0,\infty}^2\Big(\sum_{\tau\in\mathcal{T}^h}|\chi|_{1,\tau}^2\Big)^{\frac{1}{2}}\Big(\sum_{\tau\in\mathcal{T}^h}|\eta|_{1,\tau}^2\Big)^{\frac{1}{2}} \\
&\qquad + |\eta|_{0,\infty}|v|_{0,\infty}\Big(\sum_{\tau\in\mathcal{T}^h}|\chi|_{1,\tau}^2\Big)^{\frac{1}{2}}\Big(\sum_{\tau\in\mathcal{T}^h}|v|_{1,\tau}^2\Big)^{\frac{1}{2}} \\
&\qquad + |\chi|_{0,\infty}|v|_{0,\infty}\Big(\sum_{\tau\in\mathcal{T}^h}|\eta|_{1,\tau}^2\Big)^{\frac{1}{2}}\Big(\sum_{\tau\in\mathcal{T}^h}|v|_{1,\tau}^2\Big)^{\frac{1}{2}} \\
&\qquad + |\chi|_{0,\infty}|\eta|_{0,\infty}\sum_{\tau\in\mathcal{T}^h}|v|_{1,\tau}^2\Big] \\
&= Ch^2\Big[|v|_{0,\infty}^2|\chi|_1|\eta|_1 + |\eta|_{0,\infty}|v|_{0,\infty}|\chi|_1|v|_1 \\
&\qquad + |\chi|_{0,\infty}|v|_{0,\infty}|\eta|_1|v|_1 + |\chi|_{0,\infty}|\eta|_{0,\infty}|v|_1^2\Big]. \qquad (4.2.18)
\end{aligned}$$

For $d=1$ we have, by (2.1.4), $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$. For $d=2$ we have from the inverse inequality (4.1.8a) and the Sobolev embedding $H^1(\Omega)\hookrightarrow L^p(\Omega)$ that for any $\chi\in S^h$, $|\chi|_{0,\infty}\leq Ch^{-2/p}|\chi|_{0,p}\leq Ch^{-2/p}\|\chi\|_1\ \forall\, p\in[2,\infty)$ which means that $|\chi|_{0,\infty}\leq Ch^{-s}\|\chi\|_1\ \forall\, s\in(0,1]$. For the case $d=3$, again the inverse inequality and $H^1(\Omega)\hookrightarrow L^p(\Omega)$ give for any $\chi\in S^h$ that $|\chi|_{0,\infty}\leq Ch^{-3/p}|\chi|_{0,p}\leq Ch^{-3/p}\|\chi\|_1\ \forall\, p\in[2,6]$ which leads, in particular, to $|\chi|_{0,\infty}\leq Ch^{-1/2}\|\chi\|_1$.

Then inserting the above estimates of $|\chi|_{0,\infty}$ into (4.2.18) results in the desired result (4.2.16). □

Lemma 4.2.7 Let $\chi \in S^h$ and $r \geq 2$. Then

$$\left|(I-\pi^h)(\chi^r)\right|_{0,1} \leq \begin{cases} Ch^2\|\chi\|_1^r & \text{if } d=1, \\ Ch^{2-s(r-2)}\|\chi\|_1^r & \text{if } d=2, \text{ where } s\in(0,1], \\ Ch^{3-\frac{r}{2}}\|\chi\|_1^r & \text{if } d=3. \end{cases} \tag{4.2.19}$$

Proof. Simple refinement in the proof of Lemma 4.2.6 and noting that

$$\frac{\partial^2}{\partial x_i \partial x_j}(\chi^r) = r(r-1)\chi^{r-2}\frac{\partial \chi}{\partial x_i}\frac{\partial \chi}{\partial x_j}$$

it follows from (4.1.9a) and the Hölder inequality that

$$\left|(I-\pi^h)(\chi^r)\right|_{0,1} \leq Ch^2|\chi|_{0,\infty}^{r-2}|\chi|_1|\chi|_1 \equiv Ch^2|\chi|_{0,\infty}^{r-2}|\chi|_1^2. \tag{4.2.20}$$

Finally, we use the estimates of $|\chi|_{0,\infty}$ derived in the proof of Lemma 4.2.6 to conclude that (4.2.19) is satisfied. □

Lemma 4.2.8 Let $\chi \in S^h$ and $r \geq 2$. Then we have

$$(\chi^r, 1)^h \equiv \int_\Omega \pi^h(\chi^r)dx \leq C\|\chi\|_1^r \text{ holds for } r \in \begin{cases} [2,\infty) & \text{if } d=1,2 \\ [2,6] & \text{if } d=3. \end{cases} \tag{4.2.21}$$

Proof. We split the integrand $\pi^h(\chi^r)$ via

$$\begin{aligned} \int_\Omega \pi^h(\chi^r)dx \leq \left|\int_\Omega \pi^h(\chi^r)dx\right| &\leq \left|\int_\Omega (I-\pi^h)(\chi^r)dx\right| + \left|\int_\Omega \chi^r dx\right| \\ &\leq \left|(I-\pi^h)(\chi^r)\right|_{0,1} + |\chi|_{0,r}^r. \end{aligned} \tag{4.2.22}$$

Applying Lemma 4.2.7 to (4.2.22) and using $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for r given by (2.1.4) it follows that

$$\int_\Omega \pi^h(\chi^r)dx \leq \begin{cases} C\big(h^2+1\big)\|\chi\|_1^r & \text{if } d=1, \\ C\big(h^{2-s(r-2)}+1\big)\|\chi\|_1^r & \text{if } d=2, \text{ where } s\in(0,1], \\ C\big(h^{3-\frac{r}{2}}+1\big)\|\chi\|_1^r & \text{if } d=3,\ r\leq 6. \end{cases} \tag{4.2.23}$$

This inequality proves (4.2.21) after noting that $h \leq |\Omega|$. □

We are now ready to introduce semi-discrete and fully-discrete approximations for the solution of (**P**). The fully-discrete approximation will be introduced in the next chapter but in the remaining of this chapter we will be concerned with a semi-discrete approximation.

4.3 A semi-discrete approximation

4.3.1 Statement of the semi-discrete problem

We consider the following semi-discrete finite element approximations to the problems ($\mathbf{P}$) and ($\mathbf{P}_\varepsilon$) respectively:

($\mathbf{P^h}$) Find $\{u_1^h, u_2^h, w_1^h, w_2^h\} \in S_{m_1}^h \times S_{m_2}^h \times S^h \times S^h$ such that for $i = 1,2$ $u_i^h(0) = u_i^{h,0}$ and for *a.e.* $t \in (0,T)$ and all $\chi \in S^h$

$$(\partial_t u_i^h, \chi)^h + (\nabla w_i^h, \nabla \chi) = 0, \tag{4.3.1a}$$

$$\gamma(\nabla u_i^h, \nabla\chi) + (\Psi_i'(u_i^h), \chi)^h + (f_D^{(i)}(u_1^h, u_2^h), \chi)^h = (w_i^h, \chi)^h, \tag{4.3.1b}$$

where $u_i^{h,0}$ is an appropriate approximation of u_i^0 in $S_{m_i}^h$. For instance, $P^h u_i^0 \in S_{m_i}^h$ or $P_\gamma^h u_i^0 \in S_{m_i}^h$, $i = 1,2$.

($\mathbf{P_\varepsilon^h}$) Find $\{u_{\varepsilon,1}^h, u_{\varepsilon,2}^h, w_{\varepsilon,1}^h, w_{\varepsilon,2}^h\} \in S_{m_1}^h \times S_{m_2}^h \times S^h \times S^h$ such that for $i = 1,2$ $u_{\varepsilon,i}^h(0) = u_i^{h,0}$ and for *a.e.* $t \in (0,T)$ and all $\chi \in S^h$

$$(\partial_t u_{\varepsilon,i}^h, \chi)^h + (\nabla w_{\varepsilon,i}^h, \nabla\chi) = 0, \tag{4.3.2a}$$

$$\gamma(\nabla u_{\varepsilon,i}^h, \nabla\chi) + (\Psi_{\varepsilon,i}'(u_{\varepsilon,i}^h), \chi)^h + (f_D^{(i)}(u_{\varepsilon,1}^h, u_{\varepsilon,2}^h), \chi)^h = (w_{\varepsilon,i}^h, \chi)^h. \tag{4.3.2b}$$

Similarly to the continuous problem, it will be convenient to establish equivalent forms to ($\mathbf{P^h}$) and ($\mathbf{P_\varepsilon^h}$). For this purpose we take $\chi = 1$ in (4.3.2a) to yield for $i = 1,2$ and $t \in (0,T)$ that $\partial_t u_{\varepsilon,i}^h \in V_0^h$ and, by (4.1.4) and (4.1.19), if $u_i^{h,0} = P_\gamma^h u_i^0$

$$(u_{\varepsilon,i}^h(t), 1) \equiv (u_{\varepsilon,i}^h(t), 1)^h = (u_{\varepsilon,i}^h(0), 1)^h = (u_{\varepsilon,i}^h(0), 1) = (P_\gamma^h u_i^0, 1) = (u_i^0, 1) = m_i|\Omega|. \tag{4.3.3}$$

Likewise, in the case where $u_i^{h,0} = P^h u_i^0$ we have by (4.1.20) that

$$(u_{\varepsilon,i}^h(t), 1) \equiv (u_{\varepsilon,i}^h(t), 1)^h = (P^h u_i^0, 1)^h = (u_i^0, 1) = m_i|\Omega|. \tag{4.3.4}$$

Using the definition of $\hat{\mathcal{G}}^h$, (4.1.11), one can rewrite (4.3.2a) as

$$(\nabla(\hat{\mathcal{G}}^h \partial_t u_{\varepsilon,i}^h + w_{\varepsilon,i}^h), \nabla\chi) = 0 \qquad \forall \chi \in S^h,$$

which gives, by choosing $\chi = \hat{\mathcal{G}}^h \partial_t u_{\varepsilon,i}^h + w_{\varepsilon,i}^h$ followed by the Poincaré inequality, that

$$\left|\hat{\mathcal{G}}^h \partial_t u_{\varepsilon,i}^h + (w_{\varepsilon,i}^h - \fint w_{\varepsilon,i}^h)\right|_0 \le C_p \left|\hat{\mathcal{G}}^h \partial_t u_{\varepsilon,i}^h + (w_{\varepsilon,i}^h - \fint w_{\varepsilon,i}^h)\right|_1 = C_p \left|\hat{\mathcal{G}}^h \partial_t u_{\varepsilon,i}^h + w_{\varepsilon,i}^h\right|_1 = 0.$$

Hence, for $i = 1, 2$

$$w^h_{\varepsilon,i} = -\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i} + \fint w^h_{\varepsilon,i}, \tag{4.3.5}$$

where, by (4.3.2b) and (4.1.4),

$$\fint w^h_{\varepsilon,i} = \fint \left[\pi^h \Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + \pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\right]. \tag{4.3.6}$$

According to above $(\mathbf{P^h_\varepsilon})$ can be rewritten equivalently as:

$(\mathbf{P^h_\varepsilon})$ Find $\{u^h_{\varepsilon,1}, u^h_{\varepsilon,2}\} \in S^h_{m_1} \times S^h_{m_2}$ such that for $i = 1, 2$ $u^h_{\varepsilon,i}(0) = u^{h,0}_i$ and for $a.e.$ $t \in (0, T)$ and all $\chi \in S^h$

$$\gamma(\nabla u^h_{\varepsilon,i}, \nabla\chi) + (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}), \chi - \fint \chi)^h + (f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \chi - \fint \chi)^h + (\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}, \chi)^h = 0. \tag{4.3.7}$$

In a similar treatment, one can write an equivalent form to $(\mathbf{P^h})$ as:

$(\mathbf{P^h})$ Find $\{u^h_1, u^h_2\} \in S^h_{m_1} \times S^h_{m_2}$ such that for $i = 1, 2$ $u^h_i(0) = u^{h,0}_i$ and for $a.e.$ $t \in (0, T)$ and all $\chi \in S^h$

$$\gamma(\nabla u^h_i, \nabla\chi) + (\Psi'_i(u^h_i), \chi - \fint \chi)^h + (f^{(i)}_D(u^h_1, u^h_2), \chi - \fint \chi)^h + (\hat{\mathcal{G}}^h \partial_t u^h_i, \chi)^h = 0. \tag{4.3.8}$$

4.3.2 Existence and uniqueness of the approximation

This section is devoted to proof of existence and uniqueness of a solution to the proposed semi-discrete problem $(\mathbf{P^h})$ under the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$. Indeed, we employ the same approach used in the continuous problem. We shall first consider the semi-discrete regularized version $(\mathbf{P^h_\varepsilon})$ and then we pass to the limit as $\varepsilon \to 0$.

Theorem 4.3.1 Let the assumptions $(\mathbf{A_1})$ and $(\mathbf{A^h})$ hold. Let $u^{h,0}_i = P^h u^0_i$. Then for all $\varepsilon \leq \varepsilon_0$ and all $h > 0$, $(\mathbf{P^h_\varepsilon})$ possesses a unique solution $\{u^h_{\varepsilon,1}, u^h_{\varepsilon,2}, w^h_{\varepsilon,1}, w^h_{\varepsilon,2}\}$ such that for $i = 1, 2$ the following stability estimates hold independently of the parameters ε and h

$$\|u^h_{\varepsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} + \|u^h_{\varepsilon,i}\|_{H^1(0,T;(H^1(\Omega))')} \leq C, \tag{4.3.9a}$$

$$\|w^h_{\varepsilon,i}\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{4.3.9b}$$

$$\|\pi^h \phi_\varepsilon(u^h_{\varepsilon,i})\|_{L^2(\Omega_T)} \leq C, \tag{4.3.9c}$$

$$\|\pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\|_{L^\infty(0.T;L^2(\Omega))} \leq C. \tag{4.3.9d}$$

Proof. We first represent $u^h_{\varepsilon,i}$ and $w^h_{\varepsilon,i}$ in terms of the basis functions $\{\varphi_j\}_{j=0}^J$ as

$$u^h_{\varepsilon,i}(x,t) = \sum_{j=0}^{J} a_{ij}(t)\varphi_j(x), \quad w^h_{\varepsilon,i}(x,t) = \sum_{j=0}^{J} b_{ij}(t)\varphi_j(x) \quad i = 1,2, \tag{4.3.10}$$

where a_{ij} and b_{ij} to be determined. Replacing $u^h_{\varepsilon,i}$ and $w^h_{\varepsilon,i}$ in (4.3.2a) and (4.3.2b) by their above presentations and taking $\chi = \varphi_k$, $k = 0, 1, \ldots, J$ yields after noting (4.1.26) that for $i = 1, 2$

$$\sum_{j=0}^{J} \frac{da_{ij}}{dt} M_{jk} + \sum_{j=0}^{J} b_{ij}(t) A_{jk} = 0,$$

$$\gamma \sum_{j=0}^{J} a_{ij}(t) A_{jk} + (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}), \varphi_k)^h + (f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \varphi_k)^h = \sum_{j=0}^{J} b_{ij}(t) M_{jk},$$

$$\sum_{j=0}^{J} a_{ij}(0)\varphi_j(x) = u^{h,0}_i.$$

Note that the last equation implies that $a_{ik}(0) = \frac{1}{M_{kk}}(u^{h,0}_i, \varphi_k)^h$ for $k = 0, 1, ..., J$. The above system can be written in the matrix notation as

$$M\frac{da_i}{dt} = -Ab_i, \tag{4.3.11a}$$

$$Mb_i = \gamma A a_i + g_1(a_i) + g^i_2(a_1, a_2), \tag{4.3.11b}$$

$$Ma_i(0) = a^0_i, \tag{4.3.11c}$$

where for $i = 1, 2$ and $k = 0, 1, \ldots, J$

$$\big(g_1(a_i)\big)_k = (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}), \varphi_k)^h, \quad \big(g^i_2(a_1, a_2)\big)_k = (f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \varphi_k)^h$$
$$\big(a^0_i\big)_k = (u^{h,0}_i, \varphi_k)^h.$$

Since the lumped matrix, M, is invertible, we have for $i = 1, 2$ that

$$\frac{da_i}{dt} = -\gamma M^{-1}AM^{-1}Aa_i - M^{-1}AM^{-1}g_1(a_i) - M^{-1}AM^{-1}g^i_2(a_1, a_2) := F_i(a_1, a_2),$$
$$a_i(0) = M^{-1}a^0_i.$$

Letting $\hat{a} = (a_1, a_2)^T$, $\hat{F}(\hat{a}) = (F_1(a_1, a_2), F_2(a_1, a_2))^T$ and $\hat{a}^0 = (M^{-1}a^0_1, M^{-1}a^0_2)^T$ we can rewrite the above system of ODEs in the form

$$\frac{d\hat{a}}{dt} = \hat{F}(\hat{a}),$$
$$\hat{a}(0) = \hat{a}^0.$$

Since for $i = 1, 2,\ \Psi'_{\varepsilon,i}$ and $f_D^{(i)}$ are locally Lipschitz, $\hat{F}$ is locally Lipschitz and hence the standard existence theory of a system of ODEs asserts that the above system has a unique solution on some finite time interval $(0, t_h),\ t_h > 0$. Therefore, we have for $i = 1, 2$ the local existence of $u^h_{\varepsilon,i}$. By (4.3.11b) we obtain the local existence of $w^h_{\varepsilon,i}$.

To obtain the global existence we derive *a priori* estimates bounding the semi-discrete approximations $u^h_{\varepsilon,i}$ and $w^h_{\varepsilon,i}$.

Taking $\chi = \hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}$ in (4.3.2a) and noting (4.1.12) and (4.1.11) yields for $i = 1, 2$ and $t \in (0, T)$ that

$$(\partial_t u^h_{\varepsilon,i}, \hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i})^h + (\nabla w^h_{\varepsilon,i}, \nabla \hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}) = \|\partial_t u^h_{\varepsilon,i}\|^2_{-h} + (w^h_{\varepsilon,i}, \partial_t u^h_{\varepsilon,i})^h = 0. \quad (4.3.12)$$

Now we test (4.3.2b) with $\chi = \partial_t u^h_{\varepsilon,i} \in V^h_0$ and then we sum over $i = 1, 2$ to obtain, after noting (4.3.12) and $(\nabla u^h_{\varepsilon,i}, \nabla \partial_t u^h_{\varepsilon,i}) = \frac{1}{2}\frac{d}{dt}|u^h_{\varepsilon,i}|^2_1$,

$$\begin{aligned}\frac{\gamma}{2}\frac{d}{dt}\big[|u^h_{\varepsilon,1}|^2_1 + |u^h_{\varepsilon,2}|^2_1\big] &+ \big[(\Psi'_{\varepsilon,1}(u^h_{\varepsilon,1}), \partial_t u^h_{\varepsilon,1})^h + (\Psi'_{\varepsilon,2}(u^h_{\varepsilon,2}), \partial_t u^h_{\varepsilon,2})^h\big] \\ &+ \big[(f_D^{(1)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \partial_t u^h_{\varepsilon,1})^h + (f_D^{(2)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \partial_t u^h_{\varepsilon,2})^h\big] \\ &+ \big[\|\partial_t u^h_{\varepsilon,1}\|^2_{-h} + \|\partial_t u^h_{\varepsilon,2}\|^2_{-h}\big] = 0. \end{aligned} \quad (4.3.13)$$

Using the definition of $(\cdot,\cdot)^h$, (4.1.3), we have for $t \in (0, T]$ and $i = 1, 2$ that

$$\begin{aligned}\int_0^t (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}), \partial_s u^h_{\varepsilon,i})^h ds &= \int_\Omega \pi^h\Big(\int_0^t \Psi'_{\varepsilon,i}(u^h_{\varepsilon,i})\partial_s u^h_{\varepsilon,i} ds\Big) dx \\ &= \int_\Omega \big[\pi^h(\Psi_{\varepsilon,i}(u^h_{\varepsilon,i}(t))) - \pi^h(\Psi_{\varepsilon,i}(u^h_{\varepsilon,i}(0)))\big] dx \\ &= (\Psi_{\varepsilon,i}(u^h_{\varepsilon,i}(t)), 1)^h - (\Psi_{\varepsilon,i}(u^h_{\varepsilon,i}(0)), 1)^h. \end{aligned} \quad (4.3.14)$$

Recalling that $f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}) := \partial_{u^h_{\varepsilon,i}} f_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})$ and again using the definition (4.1.3) we have for $t \in (0, T]$ and $i = 1, 2$ that

$$\begin{aligned}&\int_0^t \big[(f_D^{(1)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \partial_s u^h_{\varepsilon,1})^h + (f_D^{(2)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \partial_s u^h_{\varepsilon,2})^h\big] ds \\ &\quad = \int_\Omega \pi^h\Big(\int_0^t \big[f_D^{(1)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\partial_s u^h_{\varepsilon,1} + f_D^{(2)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\partial_s u^h_{\varepsilon,2}\big] ds\Big) dx \\ &\quad = \int_\Omega \pi^h\Big(\int_0^t \frac{d}{ds}\big[f_D(u^h_{\varepsilon,1}(s), u^h_{\varepsilon,2}(s))\big] ds\Big) dx \\ &\quad = \int_\Omega \Big[\pi^h(f_D(u^h_{\varepsilon,1}(t), u^h_{\varepsilon,2}(t))) - \pi^h(f_D(u^h_{\varepsilon,1}(0), u^h_{\varepsilon,2}(0))\Big] dx \\ &\quad = \big(f_D(u^h_{\varepsilon,1}(t), u^h_{\varepsilon,2}(t)), 1\big)^h - \big(f_D(u^h_{\varepsilon,1}(0), u^h_{\varepsilon,2}(0)), 1\big)^h. \end{aligned} \quad (4.3.15)$$

Hence, integrating (4.3.13) and using (4.3.14) and (4.3.15) it follows for $t \in (0,T]$ and $i = 1,2$ that

$$\Lambda_\varepsilon^h(u_{\varepsilon,1}^h(t), u_{\varepsilon,2}^h(t)) + \int_0^t \left[\|\partial_s u_{\varepsilon,1}^h\|_{-h}^2 + \|\partial_s u_{\varepsilon,2}^h\|_{-h}^2\right] ds = \Lambda_\varepsilon^h(u_{\varepsilon,1}^h(0), u_{\varepsilon,2}^h(0)) = \Lambda_\varepsilon^h(P^h u_1^0, P^h u_2^0), \tag{4.3.16}$$

where

$$\Lambda_\varepsilon^h(u_{\varepsilon,1}^h, u_{\varepsilon,2}^h) := \frac{\gamma}{2}\left[|u_{\varepsilon,1}^h|_1^2 + |u_{\varepsilon,2}^h|_1^2\right] + (\Psi_{\varepsilon,1}(u_{\varepsilon,1}^h), 1)^h + (\Psi_{\varepsilon,2}(u_{\varepsilon,2}^h), 1)^h + (f_D(u_{\varepsilon,1}^h, u_{\varepsilon,2}^h), 1)^h. \tag{4.3.17}$$

Now we bound the terms of $\Lambda_\varepsilon^h(P^h u_1^0, P^h u_2^0)$ in turn. From (4.1.21) and the assumptions $(\mathbf{A_1})$ we have for $i = 1,2$ that

$$\|P^h u_i^0\|_1 \le C\|u_i^0\|_1 \le C. \tag{4.3.18}$$

We also have by (4.1.23) and the assumptions $(\mathbf{A_1})$ that $|P^h u_i^0|_{0,\infty} \le |u_i^0|_{0,\infty} \le 1$. Together with fact that $\psi_\varepsilon(r) \le \psi_\varepsilon(1)\ \forall\, r \in [-1,1]$ this shows for $i = 1,2$ that

$$(\Psi_{\varepsilon,i}(P^h u_i^0), 1)^h \le (\psi_\varepsilon(1) + \tfrac{\theta_i}{2}, 1)^h \le (\theta \ln 2 + \tfrac{\theta_i}{2})|\Omega|. \tag{4.3.19}$$

We employ Lemma 4.2.4 and Lemma 4.2.8 to bound $f_D(P^h u_1^0, P^h u_2^0)$ as follows

$$\begin{aligned}
(f_D(P^h u_1^0, P^h u_2^0), 1)^h &= D\int_\Omega \pi^h\big((P^h u_1^0 + \alpha_1)^2 (P^h u_2^0 + \alpha_2)^2\big)dx \\
&\le D\Big(\int_\Omega \pi^h\big((P^h u_1^0 + \alpha_1)^4\big)dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \pi^h\big((P^h u_2^0 + \alpha_2)^4\big)dx\Big)^{\frac{1}{2}} \\
&\le C\|P^h u_1^0 + \alpha_1\|_1^2 \|P^h u_2^0 + \alpha_2\|_1^2 \\
&\le C\left[\|u_1^0\|_1^2 + 1\right]\left[\|u_2^0\|_1^2 + 1\right] \le C,
\end{aligned} \tag{4.3.20}$$

where we have also used (4.2.21) and the assumptions $(\mathbf{A_1})$.

Collecting the estimates (4.3.18)-(4.3.20) together with (4.3.17) yields that $\Lambda_\varepsilon^h(P^h u_1^0, P^h u_2^0) \le C$ and hence (4.3.16) becomes

$$\Lambda_\varepsilon^h(u_{\varepsilon,1}^h(t), u_{\varepsilon,2}^h(t)) + \int_0^t \left[\|\partial_s u_{\varepsilon,1}^h\|_{-h}^2 + \|\partial_s\, u_{\varepsilon,2}^h\|_{-h}^2\right] ds = \Lambda_\varepsilon^h(P^h u_1^0, P^h u_2^0) \le C. \tag{4.3.21}$$

Noting that, by Lemma 2.2.1, $\Psi_{\varepsilon,i}(\cdot) \geq -C_0$ and $f_D(\cdot,\cdot) \geq 0$ it follows from (4.3.17) and (4.3.21) that for $t \in (0,T]$

$$\frac{\gamma}{2}\big[|u^h_{\varepsilon,1}(t)|_1^2 + |u^h_{\varepsilon,2}(t)|_1^2\big] + \int_0^t \big[\|\partial_s u^h_{\varepsilon,1}\|_{-h}^2 + \|\partial_s u^h_{\varepsilon,2}\|_{-h}^2\big] ds \leq C. \tag{4.3.22}$$

Thus, using the Poincaré inequality and (4.3.3) we have from (4.3.22) that for $i = 1,2$

$$\|u^h_{\varepsilon,1}(t)\|_1 + \|u^h_{\varepsilon,2}(t)\|_1 \leq C, \tag{4.3.23}$$

which gives the first required estimate in (4.3.9a).

In addition, (4.3.22) with the aid of the equivalence result (4.1.17) and Lemma 2.1.1 implies for $i = 1,2$ that

$$\|\partial_t u^h_{\varepsilon,i}\|^2_{L^2(0,T;(H^1(\Omega))')} \leq C \int_0^T \|\partial_t u^h_{\varepsilon,i}\|_{-h}^2 dt \leq C. \tag{4.3.24}$$

From the Sobolev embedding result $L^\infty(0,T;H^1(\Omega)) \hookrightarrow L^\infty(0,T;(H^1(\Omega))') \hookrightarrow L^2(0,T;(H^1(\Omega))')$ and the first estimate in (4.3.9a) we find for $i = 1,2$ that

$$\|u^h_{\varepsilon,i}\|_{L^2(0,T;(H^1(\Omega))')} \leq C. \tag{4.3.25}$$

We therefore obtain, by (4.3.24) and (4.3.25), the second estimate in (4.3.9a).

Now we turn to show the estimate (4.3.9b) on $w^h_{\varepsilon,i},\ i = 1,2$. To see this we first note from (4.3.5) and (4.1.12) that

$$|w^h_{\varepsilon,i}|_1^2 = |-\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i} + ⨍ w^h_{\varepsilon,i}|_1^2 = |\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}|_1^2 = \|\partial_t u^h_{\varepsilon,i}\|_{-h}^2. \tag{4.3.26}$$

Hence, by (4.3.24),

$$\int_0^T |w^h_{\varepsilon,i} - ⨍ w^h_{\varepsilon,i}|_1^2 dt = \int_0^T |w^h_{\varepsilon,i}|_1^2 dt = \int_0^T \|\partial_t u^h_{\varepsilon,i}\|_{-h}^2 dt \leq C. \tag{4.3.27}$$

Together with the Poincaré inequality this shows after noting $w^h_{\varepsilon,i} - ⨍ w^h_{\varepsilon,i} \in V^h_0$ that for $i = 1,2$

$$\Big\|w^h_{\varepsilon,i} - ⨍ w^h_{\varepsilon,i}\Big\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{4.3.28}$$

To achieve our aim it remains now, in view of (4.3.28), to prove $⨍ w^h_{\varepsilon,i}$ is bounded in $L^2(0,T;H^1(\Omega))$.

For this purpose we use (4.3.2b) with $\chi = u^h_{\varepsilon,i} - \fint u^h_{\varepsilon,i} = u^h_{\varepsilon,i} - m_i \in V^h_0$ and rearrange the terms after adding $(\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \beta)^h$ to the both sides where $\beta \in \mathbb{R}$. Then noting in turn: (4.1.11), the inequality (2.2.6) with $r = u^h_{\varepsilon,i}$ and $s = \beta$, Young's inequality, (4.1.13) and, by Lemma 2.2.1, $-\Psi_{\varepsilon,i}(\cdot) \le C_0 \; \forall\, \varepsilon \le \varepsilon_0$ we have for $i = 1, 2$

$$
\begin{aligned}
&(\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \beta - m_i)^h \\
&\quad = (w^h_{\varepsilon,i}, u^h_{\varepsilon,i} - m_i)^h - \gamma |u^h_{\varepsilon,i}|^2_1 + (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}), \beta - u^h_{\varepsilon,i})^h + (f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \beta - u^h_{\varepsilon,i})^h \\
&\quad \le (\nabla w^h_{\varepsilon,i}, \nabla \hat{\mathcal{G}}^h(u^h_{\varepsilon,i} - m_i)) + (\Psi_{\varepsilon,i}(\beta) - \Psi_{\varepsilon,i}(u^h_{\varepsilon,i}), 1)^h + \frac{\theta_i}{2}|\beta - u^h_{\varepsilon,i}|_h \\
&\qquad + |\pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})|_h |\beta - u^h_{\varepsilon,i}|_h \\
&\quad \le |w^h_{\varepsilon,i}|_1 \|u^h_{\varepsilon,i} - m_i\|_{-h} + (\Psi_{\varepsilon,i}(\beta) - \Psi_{\varepsilon,i}(u^h_{\varepsilon,i}), 1)^h + C|\beta - u^h_{\varepsilon,i}|^2_h + \frac{1}{2}|f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})|_h \\
&\quad \le C|w^h_{\varepsilon,i}|_1 |u^h_{\varepsilon,i} - m_i|_h + C\big[1 + (\Psi_{\varepsilon,i}(\beta), 1)^h + |\beta - u^h_{\varepsilon,i}|^2_h + |f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})|^2_h\big] \\
&\quad \le C\big[1 + |w^h_{\varepsilon,i}|_1 + (\Psi_{\varepsilon,i}(\beta), 1)^h + |\beta - u^h_{\varepsilon,i}|^2_h + |f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})|^2_h\big],
\end{aligned}
\tag{4.3.29}
$$

where in the last inequality we have noted, by the bound (4.3.23) and (4.1.6), that $|u^h_{\varepsilon,i} - m_i|_h \le C$.

Using Lemma 4.2.4, Lemma 4.2.8 and the bound (4.3.23) yields for $i, j = 1, 2$ with $i \ne j$ that

$$
\begin{aligned}
\big|f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\big|^2_h &= 4D^2 \int_\Omega \pi^h\big((u^h_{\varepsilon,i} + \alpha_i)^2 (u^h_{\varepsilon,j} + \alpha_j)^4\big) dx \\
&\le 4D^2 \Big(\int_\Omega \pi^h\big((u^h_{\varepsilon,i} + \alpha_i)^6\big) dx\Big)^{\frac{1}{3}} \Big(\int_\Omega \pi^h\big((u^h_{\varepsilon,j} + \alpha_j)^6\big) dx\Big)^{\frac{2}{3}} \\
&\le C\|u^h_{\varepsilon,i} + \alpha_i\|^2_1 \|u^h_{\varepsilon,j} + \alpha_j\|^4_1 \le C.
\end{aligned}
\tag{4.3.30}
$$

We take $\beta = \pm 1 \mp \frac{\delta_0}{2}$ in (4.3.29) to give, on noting $\Psi_{\varepsilon,i}(r) \le \theta \ln 2 + \frac{\theta_i}{2} \; \forall\, r \in [-1, 1]$, (4.3.23), (4.1.6) and (4.3.30), that

$$
\begin{aligned}
&(\pi^h \Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + \pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), 1 - \frac{\delta_0}{2} - m_i) \\
&\qquad = (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), 1 - \frac{\delta_0}{2} - m_i)^h \le C\big[1 + |w^h_{\varepsilon,i}|_1\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&(\pi^h \Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + \pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), 1 - \frac{\delta_0}{2} + m_i) \\
&\qquad = (\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), 1 - \frac{\delta_0}{2} + m_i)^h \ge -C\big[1 + |w^h_{\varepsilon,i}|_1\big].
\end{aligned}
$$

By assumptions $(\mathbf{A_1})$ we have $1-\frac{\delta_0}{2}-m_i>0$. Hence, division of the first inequality by $|\Omega|(1-\frac{\delta_0}{2}-m_i)$ and the second one by $|\Omega|(1-\frac{\delta_0}{2}+m_i)$ yields for $i=1,2$ that

$$\Big| \fint \big[\pi^h\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i})+\pi^h f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2})\big]\Big| \le C\big[1+|w^h_{\varepsilon,i}|_1\big]. \tag{4.3.31}$$

Squaring this inequality and integrating over $(0,T)$ it follows after noting (4.3.27) that for $i=1,2$

$$\Big\| \fint \big[\pi^h\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i})+\pi^h f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2})\big]\Big\|^2_{L^2(0,T)} \le C\big[T+\int_0^T |w^h_{\varepsilon,i}|^2_1 dt\big]\le C. \tag{4.3.32}$$

For $i=1,2$ this result together with (4.3.6) leads to

$$\begin{aligned}\Big\|\fint w^h_{\varepsilon,i}\Big\|_{L^2(0,T;H^1(\Omega))} &= |\Omega|^{\frac{1}{2}}\Big\|\fint w^h_{\varepsilon,i}\Big\|_{L^2(0,T)}\\ &= |\Omega|^{\frac{1}{2}}\Big\|\fint \big[\pi^h\Psi'_{\varepsilon,i}(u^h_{\varepsilon,i})+\pi^h f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2})\big]\Big\|_{L^2(0,T)} \le C.\end{aligned} \tag{4.3.33}$$

Thus the desired estimate (4.3.9b) follows from (4.3.28) and (4.3.33). Furthermore, using (4.3.30), the fact that $|f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2})|^2_h = |\pi^h f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2})|^2_h$ and the equivalence result (4.1.6) we obtain (4.3.9d).

Testing (4.3.2b) with $\chi=\pi^h\phi_\varepsilon(u^h_{\varepsilon,i})\in S^h$ and noting (4.1.4) and a Young's inequality we arrive at

$$\begin{aligned}\gamma(\nabla u^h_{\varepsilon,i},&\nabla\pi^h\phi_\varepsilon(u^h_{\varepsilon,i}))+|\pi^h\phi_\varepsilon(u^h_{\varepsilon,i})|^2_h\\ &= (w^h_{\varepsilon,i},\pi^h\phi_\varepsilon(u^h_{\varepsilon,i}))^h+\theta_i(u^h_{\varepsilon,i},\pi^h\phi_\varepsilon(u^h_{\varepsilon,i}))^h-(f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2}),\pi^h\phi_\varepsilon(u^h_{\varepsilon,i}))^h\\ &\le \frac{1}{2}|\pi^h\phi_\varepsilon(u^h_{\varepsilon,i})|^2_h + C\big[|w^h_{\varepsilon,i}|^2_h+|u^h_{\varepsilon,i}|^2_h+|\pi^h f^{(i)}_D(u^h_{\varepsilon,1},u^h_{\varepsilon,2})|^2_h\big].\end{aligned} \tag{4.3.34}$$

By Lemma 4.2.1 the first term on the left hand side of (4.3.34) is non-negative. Hence we deduce the estimate (4.3.9c) after integrating the above over $(0,T)$ and noting the bounds (4.3.9a), (4.3.9b) and (4.3.9d).

To show the uniqueness we set $\bar{u}^h_{\varepsilon,i}:=u^h_{\varepsilon,i}-u^{h*}_{\varepsilon,i}$ where $B_h=\{u^h_{\varepsilon,i},w^h_{\varepsilon,i}\}_{i=1,2}$ and $B^*_h=\{u^{h*}_{\varepsilon,i},w^{h*}_{\varepsilon,i}\}_{i=1,2}$ are two solutions of the problem $(\mathbf{P^h_\varepsilon})$. Taking $\chi=\bar{u}^h_{\varepsilon,i}\in V^h_0$ in (4.3.7) when B_h is the solution and again taking $\chi=\bar{u}^h_{\varepsilon,i}$ in (4.3.7) when B^*_h is the solution. Then we subtract the resulting equations and rearrange to have for $i = 1, 2$

$$\gamma|\bar{u}^h_{\varepsilon,i}|_1^2 + (\phi_\varepsilon(u^h_{\varepsilon,i}) - \phi_\varepsilon(u^{h*}_{\varepsilon,i}), \bar{u}^h_{\varepsilon,i})^h + (\hat{\mathcal{G}}^h \partial_t \bar{u}^h_{\varepsilon,i}, \bar{u}^h_{\varepsilon,i})^h$$
$$= \theta_i |\bar{u}^h_{\varepsilon,i}|_h^2 - (f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}) - f_D^{(i)}(u^{h*}_{\varepsilon,1}, u^{h*}_{\varepsilon,1}), \bar{u}^h_{\varepsilon,i})^h. \qquad (4.3.35)$$

By (4.1.11) and (4.1.12) we have

$$\frac{d}{dt}\|\bar{u}^h_{\varepsilon,i}\|_{-h}^2 = 2(\hat{\mathcal{G}}^h \partial_t \bar{u}^h_{\varepsilon,i}, \bar{u}^h_{\varepsilon,i})^h. \qquad (4.3.36)$$

On using this result, the monotonicity of ϕ_ε and (4.1.15) one can rewrite (4.3.35) as

$$\gamma|\bar{u}^h_{\varepsilon,i}|_1^2 + \frac{1}{2}\frac{d}{dt}\|\bar{u}^h_{\varepsilon,i}\|_{-h}^2 \le \frac{\gamma}{4}|\bar{u}^h_{\varepsilon,i}|_1^2 + C\|\bar{u}^h_{\varepsilon,i}\|_{-h}^2 - (f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}) - f_D^{(i)}(u^{h*}_{\varepsilon,1}, u^{h*}_{\varepsilon,1}), \bar{u}^h_{\varepsilon,i})^h. \qquad (4.3.37)$$

Bounding the D-coupling term is more technical. To do so, we first use (2.3.46) with $r_i = u^h_{\varepsilon,i}$ and $s_i = u^{h*}_{\varepsilon,i}$ and then apply Lemma 4.2.4 and Lemma 4.2.8 to yield after noting the estimate (4.3.23) and a Young's inequality that for $i, j = 1, 2$ with $i \ne j$

$$\begin{aligned}
&\big|(f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}) - f_D^{(i)}(u^{h*}_{\varepsilon,1}, u^{h*}_{\varepsilon,1}), \bar{u}^h_{\varepsilon,i})^h\big| \\
&\le 2D\big|\big((u^h_{\varepsilon,j}+\alpha_j)^2, (\bar{u}^h_{\varepsilon,i})^2\big)^h\big| + 2D\big|\big((u^{h*}_{\varepsilon,i}+\alpha_i)(u^h_{\varepsilon,j}+u^{h*}_{\varepsilon,j}+2\alpha_j), \bar{u}^h_{\varepsilon,i}\bar{u}^h_{\varepsilon,j}\big)^h\big| \\
&\le 2D\big((u^h_{\varepsilon,j}+\alpha_j)^2, (\bar{u}^h_{\varepsilon,i})^2\big)^h + 2D\big(|u^{h*}_{\varepsilon,i}+\alpha_i||u^h_{\varepsilon,j}+u^{h*}_{\varepsilon,j}+2\alpha_j|, |\bar{u}^h_{\varepsilon,i}||\bar{u}^h_{\varepsilon,j}|\big)^h \\
&\le 2D\Big(\int_\Omega \pi^h\big((u^h_{\varepsilon,j}+\alpha_j)^4\big)dx\Big)^{\frac12}\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12} \\
&\quad + 2D\Big[\Big(\int_\Omega \pi^h\big((u^{h*}_{\varepsilon,i}+\alpha_i)^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((u^h_{\varepsilon,j}+u^{h*}_{\varepsilon,j}+2\alpha_j)^4\big)dx\Big)^{\frac14} \\
&\qquad \Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,j})^4\big)dx\Big)^{\frac14}\Big] \\
&\le C\|u^h_{\varepsilon,j}+\alpha_j\|_1^2\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12} \\
&\quad + \Big[C\|u^{h*}_{\varepsilon,i}+\alpha_i\|_1\|u^h_{\varepsilon,j}+u^{h*}_{\varepsilon,j}+2\alpha_j\|_1\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,j})^4\big)dx\Big)^{\frac14}\Big] \\
&\le C\Big[\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12} + \Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,j})^4\big)dx\Big)^{\frac14}\Big] \\
&\le C\Big[\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12} + \Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,j})^4\big)dx\Big)^{\frac12}\Big] \equiv T_1 + T_2.
\end{aligned} \qquad (4.3.38)$$

Now we bound the right hand side of (4.3.38). For $i = 1, 2$ we split T_i as

$$\begin{aligned}
T_i := C\Big(\int_\Omega \pi^h\big((\bar{u}^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12} &\le C\Big(\big|(I-\pi^h)\big((\bar{u}^h_{\varepsilon,i})^4\big)\big|_{0,1} + \big|\bar{u}^h_{\varepsilon,i}\big|_{0,4}^4\Big)^{\frac12} \\
&\le C\big|(I-\pi^h)\big((\bar{u}^h_{\varepsilon,i})^4\big)\big|_{0,1}^{\frac12} + C\big|\bar{u}^h_{\varepsilon,i}\big|_{0,4}^2 \equiv T_{i,1} + T_{i,2}.
\end{aligned} \qquad (4.3.39)$$

We bound each term on the right hand side of (4.3.39) separately. Applying Lemma 4.2.7 with $r = 4$, where for $d = 1$ we note that $h^2 \le |\Omega| h$ and for $d = 2$ we take $s = \frac{1}{2}$, followed by the Poincaré inequality, the inverse inequality (4.1.8b), the first inequality in (4.1.15) and finally the Young inequality with $p = 8$ and $q = \frac{8}{7}$ we obtain for $i = 1, 2$ that

$$\begin{aligned} T_{i,1} &= C\big|\big(I - \pi^h\big)\big((\bar{u}^h_{\varepsilon,i})^4\big)\big|^{\frac{1}{2}}_{0,1} \le Ch^{\frac{1}{2}}\|\bar{u}^h_{\varepsilon,i}\|^2_1 \\ &\le Ch^{\frac{1}{2}}|\bar{u}^h_{\varepsilon,i}|^2_1 = Ch^{\frac{1}{2}}|\bar{u}^h_{\varepsilon,i}|^{\frac{1}{2}}_1|\bar{u}^h_{\varepsilon,i}|^{\frac{3}{2}}_1 \\ &\le C|\bar{u}^h_{\varepsilon,i}|^{\frac{1}{2}}_h|\bar{u}^h_{\varepsilon,i}|^{\frac{3}{2}}_1 \le C\|\bar{u}^h_{\varepsilon,i}\|^{\frac{1}{4}}_{-h}|\bar{u}^h_{\varepsilon,i}|^{\frac{1}{4}}_1|\bar{u}^h_{\varepsilon,i}|^{\frac{3}{2}}_1 \\ &= C\|\bar{u}^h_{\varepsilon,i}\|^{\frac{1}{4}}_{-h}|\bar{u}^h_{\varepsilon,i}|^{\frac{7}{4}}_1 \le \frac{\gamma}{16}|\bar{u}^h_{\varepsilon,i}|^2_1 + C\|\bar{u}^h_{\varepsilon,i}\|^2_{-h}. \end{aligned} \tag{4.3.40}$$

By Lemma 4.2.3 we have for $i = 1, 2$ that

$$T_{i,2} = C\big|\bar{u}^h_{\varepsilon,i}\big|^2_{0,4} \le \frac{\gamma}{16}|\bar{u}^h_{\varepsilon,i}|^2_1 + C\|\bar{u}^h_{\varepsilon,i}\|^2_{-h}. \tag{4.3.41}$$

Combining (4.3.38)-(4.3.41) together yields for $i, j = 1, 2$ with $i \ne j$ that

$$\big|(f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}) - f^{(i)}_D(u^{h*}_{\varepsilon,1}, u^{h*}_{\varepsilon,1}), \bar{u}^h_{\varepsilon,i})^h\big| \le \frac{\gamma}{8}\big[|\bar{u}^h_{\varepsilon,i}|^2_1 + |\bar{u}^h_{\varepsilon,j}|^2_1\big] + C\big[\|u^h_{\varepsilon,i}\|^2_{-h} + \|\bar{u}^h_{\varepsilon,j}\|^2_{-h}\big], \tag{4.3.42}$$

and hence (4.3.37) becomes for $i, j = 1, 2$ with $i \ne j$

$$\gamma|\bar{u}^h_{\varepsilon,i}|^2_1 + \frac{1}{2}\frac{d}{dt}\|\bar{u}^h_{\varepsilon,i}\|^2_{-h} \le \frac{3\gamma}{8}|\bar{u}^h_{\varepsilon,i}|^2_1 + \frac{\gamma}{8}|\bar{u}^h_{\varepsilon,j}|^2_1 + C\big[\|\bar{u}^h_{\varepsilon,i}\|^2_{-h} + \|\bar{u}^h_{\varepsilon,i}\|^2_{-h}\big]. \tag{4.3.43}$$

We sum the above differential inequality over $i = 1, 2$ and simplify to have

$$\frac{\gamma}{2}\big[|\bar{u}^h_{\varepsilon,1}|^2_1 + |\bar{u}^h_{\varepsilon,2}|^2_1\big] + \frac{1}{2}\frac{d}{dt}\big[\|\bar{u}^h_{\varepsilon,1}\|^2_{-h} + \|\bar{u}^h_{\varepsilon,2}\|^2_{-h}\big] \le C\big[\|\bar{u}^h_{\varepsilon,1}\|^2_{-h} + \|\bar{u}^h_{\varepsilon,2}\|^2_{-h}\big]. \tag{4.3.44}$$

We then use the Gronwall lemma to obtain for $t \in (0, T]$ that

$$\begin{aligned} \int_0^t \gamma\big[|\bar{u}^h_{\varepsilon,1}|^2_1 + |\bar{u}^h_{\varepsilon,2}|^2_1\big]ds + \big[\|\bar{u}^h_{\varepsilon,1}(t)\|^2_{-h} + \|\bar{u}^h_{\varepsilon,2}(t)\|^2_{-h}\big] &\le e^{Ct}\big[\|\bar{u}^h_{\varepsilon,1}(0)\|^2_{-h} + \|\bar{u}^h_{\varepsilon,2}(0)\|^2_{-h}\big] \\ &= 0. \end{aligned} \tag{4.3.45}$$

We therefore have, by (4.1.16), the uniqueness of $u^h_{\varepsilon,i}$, $i = 1, 2$. The uniqueness of $w^h_{\varepsilon,i}$ is achieved immediately from (4.3.5) and (4.3.6). □

Theorem 4.3.2 Let the assumptions of Theorem 4.3.1 hold. Then there exists a unique solution $\{u_1^h, u_2^h, w_1^h, w_2^h\}$ to ($\mathbf{P^h}$) such that the following stability estimates hold independently of h:

$$u_1^h, u_2^h \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{4.3.46a}$$

$$w_1^h, w_2^h \in L^2(0,T;H^1(\Omega)), \tag{4.3.46b}$$

$$\pi^h\phi(u_1^h), \pi^h\phi(u_2^h) \in L^2(\Omega_T), \tag{4.3.46c}$$

$$\pi^h f_D^{(1)}(u_1^h, u_2^h), \pi^h f_D^{(2)}(u_1^h, u_2^h) \in L^\infty(0,T;L^2(\Omega)). \tag{4.3.46d}$$

Furthermore, the unique solution satisfies

$$\max\{|u_1^h|, |u_2^h|\} < 1 \quad a.e. \text{ in } (0,T). \tag{4.3.47}$$

Proof. As the bounds (4.3.9a)-(4.3.9c) are independent of ε it follows that, by the compactness arguments, there exist subsequences of $u_{\varepsilon,i}^h$, $w_{\varepsilon,i}^h$, $\pi^h\phi_\varepsilon(u_{\varepsilon,i}^h)$ such that for $i = 1, 2$

$$u_{\varepsilon,i}^h \rightharpoonup u_i^h \quad \text{in } L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{4.3.48a}$$

$$u_{\varepsilon,i}^h \overset{*}{\rightharpoonup} u_i^h \quad \text{in } L^\infty(0,T;H^1(\Omega)), \tag{4.3.48b}$$

$$w_{\varepsilon,i}^h \rightharpoonup w_i^h \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{4.3.48c}$$

$$\pi^h\phi_\varepsilon(u_{\varepsilon,i}^h) \rightharpoonup \dot{\chi}_i^h \quad \text{in } L^2(\Omega_T). \tag{4.3.48d}$$

In addition, in the same way as (2.3.40) in Theorem 2.3.2 one can see for $i = 1, 2$

$$u_{\varepsilon,i}^h \to u_i^h \quad \text{in } L^2(\Omega_T). \tag{4.3.49}$$

Next we show that $\dot{\chi}_i^h = \pi^h\phi(u_i^h),\ i = 1, 2$. For this purpose we first prove that

$$I_i^h(v) := \int_0^T (u_i^h - \phi^{-1}(v), \dot{\chi}_i^h - v)^h dt \geq 0 \quad \forall\, v \in L^2(0,T;S^h). \tag{4.3.50}$$

In order to obtain (4.3.50) we introduce for $i = 1, 2$ and any $v \in L^2(0,T;S^h)$

$$I_{\varepsilon,i}^h(v) := \int_0^T (u_{\varepsilon,i}^h - \phi_\varepsilon^{-1}(v), \phi_\varepsilon(u_{\varepsilon,i}^h) - v)^h dt. \tag{4.3.51}$$

From (2.2.9) with $s = u_{\varepsilon,i}^h(x_j,t),\ r = \phi_\varepsilon^{-1}(v(x_j,t)),\ j = 0, 1, ..., J$ it follows that

$$I_{\varepsilon,i}^h(v) = \int_0^T \sum_{j=0}^J M_{jj}[u_{\varepsilon,i}^h(x_j,t) - \phi_\varepsilon^{-1}(v(x_j,t))][\phi_\varepsilon(u_{\varepsilon,i}^h(x_j,t)) - v(x_j,t)]dt \geq 0.$$

This integral is well-defined and to see this we use (2.2.16) with $s = \phi_\varepsilon(u^h_{\varepsilon,i}(x_j,t))$ $r = v(x_j,t),\ j = 0,1,....,J$ and note the bound (4.3.9c) to yield

$$\begin{aligned}
I^h_{\varepsilon,i}(v) &= \int_0^T \sum_{j=0}^J M_{jj}[u^h_{\varepsilon,i}(x_j,t) - \phi_\varepsilon^{-1}(v(x_j,t))][\phi_\varepsilon(u^h_{\varepsilon,i}(x_j,t)) - v(x_j,t)]dt \\
&\le \int_0^T \sum_{j=0}^J M_{jj}|u^h_{\varepsilon,i}(x_j,t) - \phi_\varepsilon^{-1}(v(x_j,t))||\phi_\varepsilon(u^h_{\varepsilon,i}(x_j,t)) - v(x_j,t)|dt \\
&\le \int_0^T \sum_{j=0}^J M_{jj}\theta^{-1}|\phi_\varepsilon(u^h_{\varepsilon,i}(x_j,t)) - v(x_j,t)|^2 \\
&= \theta^{-1}\int_0^T \left|\pi^h\phi_\varepsilon(u^h_{\varepsilon,i}) - v\right|_h^2 = \theta^{-1}\|\pi^h\phi_\varepsilon(u^h_{\varepsilon,i}) - v\|_{L^2(0,T;S^h)} < \infty.
\end{aligned}$$

For any $v \in L^2(0,T;S^h)$ we split the difference $I^h_{\varepsilon,i}(v) - I^h_i(v)$ as

$$\begin{aligned}
&\left|I^h_{\varepsilon,i}(v) - I^h_i(v)\right| = \left|\int_0^T (u^h_{\varepsilon,i} - \phi_\varepsilon^{-1}(v), \phi_\varepsilon(u^h_{\varepsilon,i}) - v)^h - (u^h_i - \phi^{-1}(v), \dot{\chi}^h_i - v)^h dt\right| \\
&\le \left|\int_0^T (u^h_{\varepsilon,i} - u^h_i, \phi_\varepsilon(u^h_{\varepsilon,i}) - v)^h dt\right| + \left|\int_0^T (\phi^{-1}(v) - \phi_\varepsilon^{-1}(v), \phi_\varepsilon(u^h_{\varepsilon,i}) - v)^h dt\right| \\
&\quad + \left|\int_0^T (u^h_i - \phi^{-1}(v), \phi_\varepsilon(u^h_{\varepsilon,i}) - \dot{\chi}^h_i)^h dt\right| \\
&\le \left[\|u^h_{\varepsilon,i} - u^h_i\|_{L^2(0,T;S^h)} + \|\phi^{-1}(v) - \phi_\varepsilon^{-1}(v)\|_{L^2(0,T;S^h)}\right]\|\phi_\varepsilon(u^h_{\varepsilon,i}) - v\|_{L^2(0,T;S^h)} \\
&\quad + \left|\int_0^T (u^h_i - \phi^{-1}(v), \pi^h\phi_\varepsilon(u^h_{\varepsilon,i}) - \dot{\chi}^h_i)^h dt\right| \to 0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

on noting the strong convergence (4.3.49), the convergence, using Lemma 2.2.1, $\phi_\varepsilon^{-1}(r) \to \phi^{-1}(r)\ \forall\, r$ and the weak convergence (4.3.48d).

Therefore, we have for $i = 1,2$ and $v \in L^2(0,T;S^h)$ $I^h_i(v) \ge 0$ as $I^h_{\varepsilon,i}(v) \ge 0$. Now we compute $I^h_i(\dot{\chi}^h_i + \beta v)$ and $I^h_i(\dot{\chi}^h_i - \beta v)$ where $v \in L^2(0,T;S^h)$ and $\beta \in \mathbb{R}_{>0}$ to obtain that

$$\int_0^T (u^h_i - \phi^{-1}(\dot{\chi}^h_i + \beta v), -\beta v)^h dt \ge 0 \text{ and } \int_0^T (u^h_i - \phi^{-1}(\dot{\chi}^h_i - \beta v), \beta v)^h dt \ge 0.$$

Thus, by division by $-\beta$ and β respectively it follows that

$$\int_0^T (u^h_i - \phi^{-1}(\dot{\chi}^h_i + \beta v), v)^h dt \le 0 \text{ and } \int_0^T (u^h_i - \phi^{-1}(\dot{\chi}^h_i + \beta v), v)^h dt \ge 0,$$

and hence by passage to the limit as $\beta \to 0$, on noting the continuity of ϕ^{-1},

$$\int_0^T (u^h_i - \phi^{-1}(\dot{\chi}^h_i), v)^h = \int_0^T \sum_{j=0}^J M_{jj}[u^h_i - \phi^{-1}(\dot{\chi}^h_i)](x_j,t)v(x_j,t) = 0. \qquad (4.3.52)$$

Taking $v = u_i^h - \pi^h\phi^{-1}(\dot{\chi}_i^h) \in L^2(0,T;S^h)$ we find for $i = 1,2$ that

$$\int_0^T \left[u_i^h - \phi^{-1}(\dot{\chi}_i^h)\right]^2(x_j,t)dt = 0, \quad j = 0,1,....,J,$$

from which we deduce that $u_i^h(x_j,t) = \phi^{-1}(\dot{\chi}_i^h(x_j,t))$ *a.e.* $t \in (0,T)$. Since $\phi^{-1}(r) \in (-1,1)$ for all r, we have for *a.e.* $t \in (0,T)$, $i = 1,2$ and for $j = 0,1,....,J$ that

$$|u_i^h(x_j,t)| < 1, \qquad \phi(u_i^h(x_j,t)) = \dot{\chi}_i^h(x_j,t).$$

We therefore conclude *a.e.* $t \in (0,T)$ that $|u_i^h| < 1$ and $\pi^h\phi(u_i^h) = \dot{\chi}_i^h$, as required.

With the aid of the above convergence properties one can pass to the limit in $(\mathbf{P}_\varepsilon^\mathbf{h})$ to obtain that $\{u_1^h, u_2^h, w_1^h, w_2^h\}$ is a solution for $(\mathbf{P}^\mathbf{h})$, where this step is a simple modification of the passage to the limit proof in Theorem 2.3.3. Finally, one can prove uniqueness of a solution to $(\mathbf{P}^\mathbf{h})$ by adapting a similar argument to that used for $(\mathbf{P}_\varepsilon^\mathbf{h})$ in Theorem 4.3.1. □

In Theorem 4.3.4 we prove further stability estimates of the semi-discrete approximations that will be essential for the error bound analysis. For this purpose we require the assumptions $(\mathbf{A_2})$ on the initial data and we also need the weighted $H^1(\Omega)$ projection P_γ^h, given by (4.1.19), instead of the discrete $L^2(\Omega)$ projection P^h. We remark by (4.1.23) and assumptions $(\mathbf{A_1})$ that P^h satisfies $|P^h u_i^0|_{0,\infty} \leq 1 \; \forall h > 0$ which is a crucial property in the proof of Theorem 4.3.1 which does not hold automatically for the P_γ^h projection. However, under the assumptions $(\mathbf{A_2})$ we will be able to prove that P_γ^h satisfies a similar result for sufficiently small h, see Lemma 4.3.3 which follows.

Lemma 4.3.3 Let the assumptions $(\mathbf{A_2})$ hold. Then there exists $h_* > 0$ such that for all $h \leq h_*$ and for $i = 1,2$

$$|P_\gamma^h u_i^0|_{0,\infty} \leq 1 - \frac{\delta_0}{2}. \tag{4.3.53}$$

Proof. Using (4.1.22) and recalling the assumptions $(\mathbf{A_2})$ we have

$$|P_\gamma^h u_i^0|_{0,\infty} \leq \left|(I - P_\gamma^h)u_i^0\right|_{0,\infty} + |u_i^0|_{0,\infty} \leq Ch^{2-\frac{d}{2}}|u_i^0|_2 + 1 - \delta_0.$$

We now choose h_* small enough such that $Ch_*^{2-\frac{d}{2}}|u_i^0|_2 \leq \frac{\delta_0}{2}$ for $d = 1, 2, 3$. Thus, we obtain for all $h \leq h_*$ the desired result (4.3.53). □

Remark. The results of Theorem 4.3.1 and Theorem 4.3.2 can be obtained for the choice $u_i^{h,0} = P_\gamma^h u_i^0$, $i = 1, 2$, under the assumptions $(\mathbf{A_2})$ and the restriction stated in Lemma 4.3.3 on the spatial parameter. Indeed, the proof remains the same with the only changes that we replace $P^h u_i^0$ by $P_\gamma^h u_i^0$ and we use (4.1.24) and (4.3.53) in place of (4.1.21) and (4.1.23), respectively.

Theorem 4.3.4 Let the assumptions $(\mathbf{A_2})$ and $(\mathbf{A^h})$ hold. Let $u_i^{h,0} = P_\gamma^h u_i^0$, $i = 1, 2$. Then for all $\varepsilon \leq \min\{\varepsilon_0, \frac{\delta_0}{2}\}$ and for all $h \leq h_*$ the unique solution of $(\mathbf{P_\varepsilon^h})$ is such that the following further stability estimates hold independently of the parameters ε and h:

$$\|\partial_t u_{\varepsilon,i}^h\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t u_{\varepsilon,i}^h\|_{L^\infty(0,T;(H^1(\Omega))')} \leq C, \tag{4.3.54a}$$

$$\|w_{\varepsilon,i}^h\|_{L^\infty(0,T;H^1(\Omega))} + \|\pi^h \phi_\varepsilon(u_{\varepsilon,i}^h)\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{4.3.54b}$$

Proof. We differentiate (4.3.2b) with respect to t and then set $\chi = \partial_t u_{\varepsilon,i}^h$ to have after noting (4.3.2a) with $\chi = \partial_t w_{\varepsilon,i}^h$ that for $i = 1, 2$

$$\begin{aligned}\gamma|\partial_t u_{\varepsilon,i}^h|_1^2 + (\phi_\varepsilon'(u_{\varepsilon,i}^h), (\partial_t u_{\varepsilon,i}^h)^2)^h - \theta_i|\partial_t u_{\varepsilon,i}^h|_h^2 + (\partial_t f_D^{(i)}(u_{\varepsilon,1}^h, u_{\varepsilon,2}^h), \partial_t u_{\varepsilon,i}^h)^h \\ = (\partial_t w_{\varepsilon,i}^h, \partial_t u_{\varepsilon,i}^h)^h = -(\nabla w_{\varepsilon,i}^h, \nabla \partial_t w_{\varepsilon,i}^h) = -\frac{1}{2}\frac{d}{dt}|w_{\varepsilon,i}^h|_1^2.\end{aligned} \tag{4.3.55}$$

Since $\phi_\varepsilon'(\cdot) > 0$, $\|\partial_t u_{\varepsilon,i}^h\|_{-h} = |w_{\varepsilon,i}^h|_1$ (by (4.3.26)) and

$$\begin{aligned}(\partial_t f_D^{(i)}(u_{\varepsilon,1}^h, u_{\varepsilon,2}^h), \partial_t u_{\varepsilon,i}^h)^h &= 2D((u_{\varepsilon,j}^h + \alpha_j)^2, (\partial_t u_{\varepsilon,i}^h)^2)^h \\ &\quad + 4D((u_{\varepsilon,i}^h + \alpha_i)(u_{\varepsilon,j}^h + \alpha_j), \partial_t u_{\varepsilon,i}^h \partial_t u_{\varepsilon,j}^h)^h \\ &\geq 4D((u_{\varepsilon,i}^h + \alpha_i)(u_{\varepsilon,j}^h + \alpha_j), \partial_t u_{\varepsilon,i}^h \partial_t u_{\varepsilon,j}^h)^h,\end{aligned}$$

we may rewrite (4.3.55), for $i, j = 1, 2$ with $i \neq j$, as

$$\begin{aligned}\gamma|\partial_t u_{\varepsilon,i}^h|_1^2 + \frac{1}{2}\frac{d}{dt}\|\partial_t u_{\varepsilon,i}^h\|_{-h}^2 &\leq \theta_i|\partial_t u_{\varepsilon,i}^h|_h^2 - 4D((u_{\varepsilon,i}^h + \alpha_i)(u_{\varepsilon,j}^h + \alpha_j), \partial_t u_{\varepsilon,i}^h \partial_t u_{\varepsilon,j}^h)^h \\ &\leq \frac{\gamma}{4}|\partial_t u_{\varepsilon,i}^h|_1^2 + C\|\partial_t u_{\varepsilon,i}^h\|_{-h}^2 \\ &\quad + 4D\big|((u_{\varepsilon,i}^h + \alpha_i)(u_{\varepsilon,j} + \alpha_j), \partial_t u_{\varepsilon,i}^h \partial_t u_{\varepsilon,j}^h)^h\big|,\end{aligned} \tag{4.3.56}$$

where in the last step we have used (4.1.15).

To bound the last term in the right hand side of (4.3.56) we apply Lemma 4.2.4 and Lemma 4.2.8, the bound (4.3.23) and a Young's inequality to yield for $i,j=1,2$ with $i\neq j$ that

$$\begin{aligned}
&\big|((u^h_{\varepsilon,i}+\alpha_i)(u^h_{\varepsilon,j}+\alpha_j),\partial_t u^h_{\varepsilon,i}\partial_t u^h_{\varepsilon,j})^h\big| \\
&\quad\le \Big[\Big(\int_\Omega \pi^h\big((u^h_{\varepsilon,i}+\alpha_i)^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((u^h_{\varepsilon,j}+\alpha_j)^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,i})^4\big)dx\Big)^{\frac14} \\
&\qquad\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,j})^4\big)dx\Big)^{\frac14}\Big] \\
&\quad\le C\|u^h_{\varepsilon,i}+\alpha_i\|_1\|u^h_{\varepsilon,j}+\alpha_j\|_1\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,i})^4\big)dx\Big)^{\frac14}\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,j})^4\big)dx\Big)^{\frac14} \\
&\quad\le C\Big[\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12}+\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,j})^4\big)dx\Big)^{\frac12}\Big]. \qquad (4.3.57)
\end{aligned}$$

Applying the same technique used in treating the right hand side of (4.3.38) of the uniqueness proof one can obtain for $i=1,2$ that

$$\begin{aligned}
C\Big(\int_\Omega \pi^h\big((\partial_t u^h_{\varepsilon,i})^4\big)dx\Big)^{\frac12} &\le C\big|(I-\pi^h)\big((\partial_t u^h_{\varepsilon,i})^4\big)\big|^{\frac12}_{0,1}+C\big|\partial_t u^h_{\varepsilon,i}\big|^2_{0,4} \\
&\le \frac{\gamma}{32D}|\partial_t u^h_{\varepsilon,i}|_1^2+C\|\partial_t u^h_{\varepsilon,i}\|^2_{-h}. \qquad (4.3.58)
\end{aligned}$$

Hence, by (4.3.57) and (4.3.58), (4.3.56) becomes for $i,j=1,2$ with $i\neq j$

$$\gamma|\partial_t u^h_{\varepsilon,i}|_1^2+\frac12\frac{d}{dt}\|\partial_t u^h_{\varepsilon,i}\|^2_{-h}\le\frac{3\gamma}{8}|\partial_t u^h_{\varepsilon,i}|_1^2+\frac{\gamma}{8}|\partial_t u^h_{\varepsilon,j}|_1^2+C\big[\|\partial_t u^h_{\varepsilon,i}\|^2_{-h}+\|\partial_t u^h_{\varepsilon,j}\|^2_{-h}\big]. \qquad (4.3.59)$$

Next we sum (4.3.59) over $i=1,2$ and simplify to have

$$\frac{\gamma}{2}\big[|\partial_t u^h_{\varepsilon,1}|_1^2+|\partial_t u^h_{\varepsilon,2}|_1^2\big]+\frac12\frac{d}{dt}\big[\|\partial_t u^h_{\varepsilon,1}\|^2_{-h}+\|\partial_t u^h_{\varepsilon,2}\|^2_{-h}\big]\le C\big[\|\partial_t u^h_{\varepsilon,1}\|^2_{-h}+\|\partial_t u^h_{\varepsilon,2}\|^2_{-h}\big]. \qquad (4.3.60)$$

With the aid of the Gronwall lemma and (4.3.26) we have for $t\in(0,T]$ that

$$\begin{aligned}
&\gamma\int_0^t\big[|\partial_s u^h_{\varepsilon,1}|_1^2+|\partial_s u^h_{\varepsilon,2}|_1^2\big]ds+\big[\|\partial_t u^h_{\varepsilon,1}(t)\|^2_{-h}+\|\partial_t u^h_{\varepsilon,2}(t)\|^2_{-h}\big] \\
&\qquad\le C\big[\|\partial_t u^h_{\varepsilon,1}(0)\|^2_{-h}+\|\partial_t u^h_{\varepsilon,2}(0)\|^2_{-h}\big]=C\big[|w^h_{\varepsilon,1}(0)|_1^2+|w^h_{\varepsilon,2}(0)|_1^2\big]. \qquad (4.3.61)
\end{aligned}$$

We now bound $|w^h_{\varepsilon,i}(0)|_1$, $i = 1,2$, independently of ε and h. To this aim, we note from the definition of P^h_γ given by (4.1.19), integration by parts , owing to assumptions $(\mathbf{A_2})$ and the definition (4.1.20) of P^h that for any $\chi \in S^h$

$$\begin{aligned}\gamma(\nabla u^h_{\varepsilon,i}(0), \nabla\chi) &= \gamma(\nabla P^h_\gamma u^0_i, \nabla\chi) = \gamma(\nabla u^0_i, \nabla\chi) + ((I - P^h_\gamma)u^0_i, \chi)\\ &= ((I - P^h_\gamma)u^0_i, \chi) - \gamma(\Delta u^0_i, \chi) = \left(P^h[(I - P^h_\gamma)u^0_i - \gamma\Delta u^0_i], \chi\right)^h\end{aligned} \tag{4.3.62}$$

Substituting this into (4.3.2b) which makes sense with $t = 0$ we obtain for $i = 1,2$ and any $\chi \in S^h$ that

$$(w^h_{\varepsilon,i}(0) - P^h[(I - P^h_\gamma)u^0_i - \gamma\Delta u^0_i] - \phi_\varepsilon(P^h_\gamma u^0_i) + \theta_i P^h_\gamma u^0_i - f^{(i)}_D(P^h_\gamma u^0_1, P^h_\gamma u^0_2), \chi)^h = 0,$$

and hence

$$w^h_{\varepsilon,i}(0) = P^h[(I - P^h_\gamma)u^0_i - \gamma\Delta u^0_i] + \pi^h\phi_\varepsilon(P^h_\gamma u^0_i) - \theta_i P^h_\gamma u^0_i + \pi^h f^{(i)}_D(P^h_\gamma u^0_1, P^h_\gamma u^0_2). \tag{4.3.63}$$

Therefore,

$$\begin{aligned}|w^h_{\varepsilon,i}(0)|_1 &\le \left|P^h[(I - P^h_\gamma)u^0_i - \gamma\Delta u^0_i] - \theta_i P^h_\gamma u^0_i\right|_1 + \left|\pi^h\phi_\varepsilon(P^h_\gamma u^0_i)\right|_1\\ &\quad + \left|\pi^h f^{(i)}_D(P^h_\gamma u^0_1, P^h_\gamma u^0_2)\right|_1\\ &\equiv T_1 + T_2 + T_3.\end{aligned} \tag{4.3.64}$$

Thus, it remains to find a bound independently of h and ε for all terms involving u^0_i. Owing to (4.1.21) and (4.1.24) and recalling the assumptions $(\mathbf{A_2})$ we have for $i = 1,2$ that

$$\begin{aligned}T_1 &\le \left|P^h[(I - P^h_\gamma)u^0_i - \gamma\Delta u^0_i]\right|_1 + \left|\theta_i P^h_\gamma u^0_i\right|_1 \le C\left[|(I - P^h_\gamma)u^0_i|_1 + |\Delta u^0_i|_1 + |P^h_\gamma u^0_i|_1\right]\\ &\le C\left[\|u^0_i\|_1 + |\Delta u^0_i|_1\right] \le C.\end{aligned} \tag{4.3.65}$$

Using Lemma 4.3.3 we have for all $\varepsilon \le \frac{\delta_0}{2}$, $h \le h_*$ and $i = 1,2$ that

$$|P^h_\gamma u^0_i|_{0,\infty} \le 1 - \frac{\delta_0}{2} \le 1 - \varepsilon.$$

This result with the aid of Lemma 4.2.1 (ii), (2.2.13) and (4.1.24) gives for all $\varepsilon \le \frac{\delta_0}{2}$ and $h \le h_*$

$$T_2 = \left|\pi^h\phi_\varepsilon(P^h_\gamma u^0_i)\right|_1 \le \phi'(|P^h_\gamma u^0_i|_{0,\infty})|P^h_\gamma u^0_i|_1 \le C\phi'(1 - \frac{\delta_0}{2})\|u^0_i\|_1 \le C. \tag{4.3.66}$$

Bounding the third term is more technical. We first split this term via

$$T_3 = \left|\pi^h f_D^{(i)}(P_\gamma^h u_1^0, P_\gamma^h u_2^0)\right|_1 \leq \left|(I-\pi^h) f_D^{(i)}(P_\gamma^h u_1^0, P_\gamma^h u_2^0)\right|_1 + \left|f_D^{(i)}(P_\gamma^h u_1^0, P_\gamma^h u_2^0)\right|_1$$
$$\equiv T_{3,1} + T_{3,2}. \tag{4.3.67}$$

Next we employ Lemma 4.2.5 to bound $T_{3,1}$. This lemma with $\chi = P_\gamma^h u_i^0 + \alpha_i$ and $v = P_\gamma^h u_j^0 + \alpha_j$ shows for $i, j = 1, 2$ with $i \neq j$ that

$$T_{3,1} = 2D\left|(I-\pi^h)[(P_\gamma^h u_i^0 + \alpha_i)(P_\gamma^h u_j^0 + \alpha_j)^2]\right|_1$$
$$\leq Ch|P_\gamma^h u_j^0|_{1,6}\left[|P_\gamma^h u_j^0 + \alpha_j|_{0,6}|P_\gamma^h u_i^0|_{1,6} + |P_\gamma^h u_i^0 + \alpha_i|_{0,6}|P_\gamma^h u_j^0|_{1,6}.\right. \tag{4.3.68}$$

From (4.1.22), the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the assumptions ($\mathbf{A_2}$) it follows for $i = 1, 2$ and $d = 1, 2, 3$ that

$$|P_\gamma^h u_i^0 + \alpha_i|_{0,6} \leq \left|(I - P_\gamma^h)u_i^0\right|_{0,6} + |u_i^0 + \alpha_i|_{0,6}$$
$$\leq Ch^{2-d/3}|u_i^0|_2 + C[\|u_i^0\|_1 + 1] \leq C[h^{2-d/3} + 1] \leq C \tag{4.3.69}$$

and, this time we also note $|\eta|_{1,6} \leq |\nabla\eta|_{0,6}$ and Lemma 3.1.1,

$$|P_\gamma^h u_i^0|_{1,6} \leq \left|(I - P_\gamma^h)u_i^0\right|_{1,6} + |u_i^0|_{1,6}$$
$$\leq Ch^{1-d/3}|u_i^0|_2 + C\|u_i^0\|_2 \leq C[h^{1-d/3} + 1] \leq C. \tag{4.3.70}$$

Using the second inequality of (3.1.4) with $v_1 = P_\gamma^h u_1^0,\ v_2 = P_\gamma^h u_2^0$ and noting that $|\nabla\eta|_{0,6} \leq C|\eta|_{1,6}$ yields for $i, j = 1, 2$ with $i \neq j$ that

$$T_{3,2} \leq C\left[|P_\gamma^h u_j^0 + \alpha_j|_{0,6}^2 |P_\gamma^h u_i^0|_{1,6} + |P_\gamma^h u_i^0 + \alpha_i|_{0,6}|P_\gamma^h u_j^0 + \alpha_j|_{0,6}|P_\gamma^h u_j^0|_{1,6}\right] \leq C, \tag{4.3.71}$$

where we have noted (4.3.69) and (4.3.70) to obtain the last inequality.

We thus have, by combining (4.3.64)-(4.3.71), for $i = 1, 2$, for all $\varepsilon \leq \frac{\delta_0}{2}$ and for all $h \leq h_*$ that $|w_{\varepsilon,i}^h(0)|_1 \leq C$. Therefore, we conclude from (4.3.61) for $t \in (0, T]$ that

$$\gamma \int_0^t \left[|\partial_s u_{\varepsilon,1}^h|_1^2 + |\partial_s u_{\varepsilon,2}^h|_1^2\right] ds + \left[\|\partial_t u_{\varepsilon,1}^h(t)\|_{-h}^2 + \|\partial_t u_{\varepsilon,2}^h(t)\|_{-h}^2\right] \leq C. \tag{4.3.72}$$

In particular we have from (4.3.72) that

$$\int_0^T \left[|\partial_t u_{\varepsilon,1}^h|_1^2 + |\partial_t u_{\varepsilon,2}^h|_1^2\right] dt \leq C, \tag{4.3.73}$$

which together with the Poincaré inequality shows the first estimate in (4.3.54a).

Ignoring the integral term from (4.3.72), recalling the equivalence result (4.1.17) and owing to Lemma 2.1.1 we obtain the second estimate in (4.3.54a).

Recalling (4.3.26) and (4.3.72) results in for $i = 1, 2$ that

$$\Big|w^h_{\varepsilon,i} - \fint w^h_{\varepsilon,i}\Big|_1 = \big|w^h_{\varepsilon,i}\big|_1 = \big\|\partial_t u^h_{\varepsilon,i}\big\|_{-h} \leq C. \tag{4.3.74}$$

With the aid of the Poincaré inequality we find for $i = 1, 2$ that

$$\Big\|w^h_{\varepsilon,i} - \fint w^h_{\varepsilon,i}\Big\|_{L^\infty(0,T;H^1(\Omega))} \leq C. \tag{4.3.75}$$

It follows from (4.3.6), (4.3.31) and (4.3.74) that

$$\Big|\fint w^h_{\varepsilon,i}\Big| = \Big|\fint \big[\pi^h \Psi'_{\varepsilon,i}(u^h_{\varepsilon,i}) + \pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\big]\Big| \leq C\big[1 + |w^h_{\varepsilon,i}|_1\big] \leq C, \tag{4.3.76}$$

from which we have

$$\Big\|\fint w^h_{\varepsilon,i}\Big\|_{L^\infty(0,T;H^1(\Omega))} \leq C. \tag{4.3.77}$$

Hence, the first estimate in (4.3.54b) follows directly from (4.3.75) and (4.3.77). By (4.3.34) we have

$$|\pi^h \phi_\varepsilon(u^h_{\varepsilon,i})|^2_h \leq C\big[|w^h_{\varepsilon,i}|^2_h + |u^h_{\varepsilon,i}|^2_h + |\pi^h f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})|^2_h\big]. \tag{4.3.78}$$

This result implies the second desired estimate in (4.3.54b) after noting the equivalents result (4.1.6), the bounds (4.3.9a) and (4.3.9d) and the first estimate in (4.3.54b). □

Theorem 4.3.5 Let the assumptions of Theorem 4.3.4 hold. Then for all $h \leq h_*$ the unique solution of $(\mathbf{P^h})$ is such that the following additional estimates hold independently of h:

$$\partial_t u^h_1, \partial_t u^h_2 \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;(H^1(\Omega))'), \tag{4.3.79a}$$

$$w^h_1, w^h_2 \in L^\infty(0,T;H^1(\Omega)), \tag{4.3.79b}$$

$$\pi^h \phi(u^h_1), \pi^h \phi(u^h_2) \in L^\infty(0,T;L^2(\Omega)). \tag{4.3.79c}$$

Proof. Since the bounds (4.3.54a-b) are independent of ε, the above results are direct consequences of the usual compactness arguments. □

4.4 A semi-discrete error bound

We estimate an error bound between the continuous solution of the problem (**P**) and the semi-discrete solution of ($\mathbf{P^h}$). We firstly prove an error estimate between ($\mathbf{P^h}$) and its regularized version ($\mathbf{P}^{\mathbf{h}}_{\varepsilon}$). Then we estimate an error bound between ($\mathbf{P}^{\mathbf{h}}_{\varepsilon}$) and ($\mathbf{P}_{\varepsilon}$). Finally, the semi-discrete error bound is achieved by combining these error bounds with the regularization error bound of the continuous problem (**P**) derived in Theorem 3.2.2.

Lemma 4.4.1 Let $\hat{e}^h_{\varepsilon,i} := u^h_1 - u^h_{\varepsilon,1}$, $\hat{e}^h_{\varepsilon,i} := u^h_2 - u^h_{\varepsilon,2}$. Then

$$\|\hat{e}^h_{\varepsilon,1}\|^2_{L^2(0,T;H^1(\Omega))}+\|\hat{e}^h_{\varepsilon,2}\|^2_{L^2(0,T;H^1(\Omega))}+\|\hat{e}^h_{\varepsilon,1}\|^2_{L^\infty(0,T;(H^1(\Omega))')}+\|\hat{e}^h_{\varepsilon,2}\|^2_{L^\infty(0,T;(H^1(\Omega))')} \le C\varepsilon. \tag{4.4.1}$$

Proof. The proof is a discrete analogue of the proof of Theorem 3.2.2. Subtracting the regularized version (4.3.7) from (4.3.8) and testing the resulting variational equation with $\chi = \hat{e}^h_{\varepsilon,i} \in V^h_0$ it follows for $i = 1, 2$ and *a.e.* $t \in (0, T)$ that

$$\gamma|\hat{e}^h_{\varepsilon,i}|^2_1 + (\phi(u^h_i) - \phi_\varepsilon(u^h_{\varepsilon,i}), \hat{e}^h_{\varepsilon,i})^h + (f^{(i)}_D(u^h_1, u^h_2) - f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \hat{e}^h_{\varepsilon,i})^h + (\hat{\mathcal{G}}^h\partial_t\hat{e}^h_{\varepsilon,i}, \hat{e}^h_{\varepsilon,i})^h = \theta_i|\hat{e}^h_{\varepsilon,i}|^2_h. \tag{4.4.2}$$

The D-coupling term can be treated in the same way as for (4.3.38)-(4.3.42) in the uniqueness proof of Theorem 4.3.1 to obtain for $i, j = 1, 2$ with $i \neq j$ and *a.e.* $t \in (0, T)$ that

$$\left|(f^{(i)}_D(u^h_1, u^h_2) - f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), \hat{e}^h_{\varepsilon,i})^h\right| \le \frac{\gamma}{8}\left[|\hat{e}^h_{\varepsilon,i}|^2_1 + |\hat{e}^h_{\varepsilon,j}|^2_1\right] + C\left[\|\hat{e}^h_{\varepsilon,i}\|^2_{-h} + \|\hat{e}^h_{\varepsilon,j}\|^2_{-h}\right]. \tag{4.4.3}$$

Now, we insert the above estimate of the D-coupling term into (4.4.2), note that $\frac{1}{2}\frac{d}{dt}\|\hat{e}^h_{\varepsilon,i}\|^2_{-h} = (\hat{\mathcal{G}}^h\partial_t\hat{e}^h_{\varepsilon,i}, \hat{e}^h_{\varepsilon,i})^h$ and use (4.1.15) to have for $i, j = 1, 2$ with $i \neq j$

$$\gamma|\hat{e}^h_{\varepsilon,i}|^2_1+(\phi(u^h_i)-\phi_\varepsilon(u^h_{\varepsilon,i}), \hat{e}^h_{\varepsilon,i})^h+\frac{1}{2}\frac{d}{dt}\|\hat{e}^h_{\varepsilon,i}\|^2_{-h} \le \frac{3\gamma}{8}|\hat{e}^h_{\varepsilon,i}|^2_1+\frac{\gamma}{8}|\hat{e}^h_{\varepsilon,j}|^2_1+C\left[\|\hat{e}^h_{\varepsilon,i}\|^2_{-1}+\|\hat{e}^h_{\varepsilon,j}\|^2_{-h}\right]. \tag{4.4.4}$$

For $i = 1, 2$ and $t \in (0, T)$ we define the following sets

$$\Omega^{+,h}_{\varepsilon,i}(t) := \{j : 1 - \varepsilon \le u^h_i(x_j, t) \le u^h_{\varepsilon,i}(x_j, t)\},$$
$$\Omega^{-,h}_{\varepsilon,i}(t) := \{j : u_{\varepsilon,i}(x_j, t) \le u^h_i(x_j, t) \le -1 + \varepsilon\},$$
$$\hat{\Omega}^h_{\varepsilon,i}(t) := \Omega^{+,h}_{\varepsilon,i}(t) \cup \Omega^{-,h}_{\varepsilon,i}(t),$$

and we also define for any $\chi, v \in S^h$

$$(\chi, v)^h_{\hat{\Omega}^h_{\varepsilon,i}(t)} := \sum_{j \in \hat{\Omega}^h_{\varepsilon,i}(t)} M_{jj}\chi(x_j)v(x_j),$$

$$|\chi|^2_{h,\hat{\Omega}^h_{\varepsilon,i}(t)} := (\chi, \chi)^h_{\hat{\Omega}^h_{\varepsilon,i}(t)}.$$

Using the monotonicity of ϕ_ε and owing to (2.2.12) we find that

$$\begin{aligned}(\phi(u^h_i) - \phi_\varepsilon(u^h_{\varepsilon,i}), \hat{e}^h_{\varepsilon,i})^h &= (\phi(u^h_i) - \phi_\varepsilon(u^h_i), \hat{e}^h_{\varepsilon,i})^h + (\phi_\varepsilon(u^h_i) - \phi_\varepsilon(u^h_{\varepsilon,i}), \hat{e}^h_{\varepsilon,i})^h \\ &\geq (\phi(u^h_i) - \phi_\varepsilon(u^h_i), \hat{e}^h_{\varepsilon,i})^h + (\phi_\varepsilon(u^h_i) - \phi_\varepsilon(u^h_{\varepsilon,i}), \hat{e}^h_{\varepsilon,i})^h_{\hat{\Omega}^h_{\varepsilon,i}(t)} \\ &\geq (\phi(u^h_i) - \phi_\varepsilon(u^h_i), \hat{e}^h_{\varepsilon,i})^h + \frac{\theta}{2\varepsilon}|\hat{e}^h_{\varepsilon,i}|^2_{h,\hat{\Omega}^h_{\varepsilon,i}(t)}. \end{aligned} \tag{4.4.5}$$

Using the fact that $(\phi(u^h_i(x_j,t)) - \phi_\varepsilon(u^h_i(x_j,t))\hat{e}^h_{\varepsilon,i}(x_j,t)$ is non-negative for all $j \notin \hat{\Omega}^h_{\varepsilon,i}(t)$ and that $\phi_\varepsilon(u^h_i(x_j,t))\hat{e}^h_{\varepsilon,i}(x_j,t)$ is non-positive for all $j \in \hat{\Omega}^h_{\varepsilon,i}(t)$ we obtain for $i = 1,2$ and $a.e\ t \in (0,T)$

$$(\phi(u^h_i) - \phi_\varepsilon(u^h_i), \hat{e}^h_{\varepsilon,i})^h \geq (\phi(u^h_i) - \phi_\varepsilon(u^h_i), \hat{e}^h_{\varepsilon,i})^h_{\hat{\Omega}^h_{\varepsilon,i}(t)} \geq (\phi(u^h_i), \hat{e}^h_{\varepsilon,i})^h_{\hat{\Omega}^h_{\varepsilon,i}(t)}. \tag{4.4.6}$$

From (4.4.4)- (4.4.6) and the Young inequality it follows after noting $|\cdot|_{h,\hat{\Omega}^h_{\varepsilon,i}(t)} \leq |\cdot|_h$ that for $i,j = 1,2$ with $i \neq j$ and for $a.e\ t \in (0,T)$

$$\begin{aligned}&\gamma|\hat{e}^h_{\varepsilon,i}|^2_1 + \frac{\theta}{2\varepsilon}|\hat{e}^h_{\varepsilon,i}|^2_{h,\hat{\Omega}^h_{\varepsilon,i}(t)} + \frac{1}{2}\frac{d}{dt}\|\hat{e}^h_{\varepsilon,i}\|^2_{-h} \\ &\leq \frac{3\gamma}{8}|\hat{e}^h_{\varepsilon,i}|^2_1 + \frac{\gamma}{8}|\hat{e}^h_{\varepsilon,j}|^2_1 + C\big[\|\hat{e}^h_{\varepsilon,i}\|^2_{-h} + \|\hat{e}^h_{\varepsilon,j}\|^2_{-h}\big] - (\phi(u^h_i), \hat{e}^h_{\varepsilon,i})^h_{\hat{\Omega}^h_{\varepsilon,i}(t)} \\ &\leq \frac{3\gamma}{8}|\hat{e}^h_{\varepsilon,i}|^2_1 + \frac{\gamma}{8}|\hat{e}^h_{\varepsilon,j}|^2_1 + C\big[\|\hat{e}^h_{\varepsilon,i}\|^2_{-h} + \|\hat{e}^h_{\varepsilon,j}\|^2_{-h}\big] + \frac{\theta}{4\varepsilon}|\hat{e}^h_{\varepsilon,i}|^2_{h,\hat{\Omega}^h_{\varepsilon,i}(t)} + C\varepsilon|\pi^h\phi(u^h_i)|^2_h. \end{aligned} \tag{4.4.7}$$

Next we sum the above differential inequality over $i = 1,2$ and simplify to yield for $a.e.\ t \in (0,T)$

$$\begin{aligned}\frac{\gamma}{2}\big[|\hat{e}^h_{\varepsilon,1}|^2_1 + |\hat{e}^h_{\varepsilon,2}|^2_1\big] + \frac{\theta}{4\varepsilon}\big[|\hat{e}^h_{\varepsilon,1}|^2_{h,\hat{\Omega}^h_{\varepsilon,1}(t)} + |\hat{e}^h_{\varepsilon,2}|^2_{h,\hat{\Omega}_{\varepsilon,2}(t)}\big] + \frac{1}{2}\frac{d}{dt}\big[\|\hat{e}^h_{\varepsilon,1}\|^2_{-h} + \|\hat{e}^h_{\varepsilon,2}\|^2_{-h}\big] \\ \leq C\varepsilon\big[|\pi^h\phi(u^h_1)|^2_h + |\pi^h\phi(u^h_2)|^2_h\big] + C\big[\|\hat{e}^h_{\varepsilon,1}\|^2_{-h} + \|\hat{e}^h_{\varepsilon,2}\|^2_{-h}\big]. \end{aligned} \tag{4.4.8}$$

By the Gronwall lemma, the equivalence result (4.1.6) and $\hat{e}^h_{\varepsilon,1}(0) = \hat{e}^h_{\varepsilon,2}(0) = 0$ we have for $a.e.\ t \in (0,T)$ that

$$\begin{aligned}\gamma\int_0^t \big[|\hat{e}^h_{\varepsilon,1}|^2_1 + |\hat{e}^h_{\varepsilon,2}|^2_1\big]ds + \big[\|\hat{e}^h_{\varepsilon,1}\|^2_{-h} + \|\hat{e}^h_{\varepsilon,2}\|^2_{-h}\big] \\ \leq Ce^{ct}\varepsilon\big[\|\pi^h\phi(u^h_1)\|^2_{L^2(\Omega_T)} + \|\pi^h\phi(u^h_2)\|^2_{L^2(\Omega_T)}\big]. \end{aligned} \tag{4.4.9}$$

With the aid of the Poincaré inequality, the equivalence (4.1.17) of $\|\cdot\|_{-h}$ and $\|\cdot\|_{-1}$ norms, Lemma 2.1.1 and the estimate (4.3.79c) we conclude that (4.4.1) holds as required. □

In the next theorem we prove an error estimate between the solutions of ($\mathbf{P}_\varepsilon^{\mathbf{h}}$) and ($\mathbf{P}_\varepsilon$).

Theorem 4.4.2 Let the assumptions of Theorem 4.3.4 hold. Then for all $h \leq h_1$ and for all $\varepsilon \leq \min\{\varepsilon_0, \frac{\delta_0}{2}\}$

$$\|e_{\varepsilon,1}\|^2_{L^2(0,T;H^1(\Omega))} + \|e_{\varepsilon,2}\|^2_{L^2(0,T;H^1(\Omega))} + \|e_{\varepsilon,1}\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|e_{\varepsilon,2}\|^2_{L^\infty(0,T;(H^1(\Omega))')}$$
$$\leq \begin{cases} C\big[h^2 + \varepsilon^{-1}h^4 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big] & \text{if } d = 1,2, \\ C\big[h^2 + \varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big] & \text{if } d = 3, \end{cases} \tag{4.4.10}$$

where $e_{\varepsilon,1} := u_{\varepsilon,1} - u^h_{\varepsilon,1}$, $e_{\varepsilon,2} := u_{\varepsilon,2} - u^h_{\varepsilon,2}$ and $h_1 := \begin{cases} h_* & \text{if } d = 1,3, \\ \min\{h_*, h_0\} & \text{if } d = 2. \end{cases}$

Proof. For $i = 1,2$ and *a.e.* $t \in (0,T)$ we define[2]

$$e^A_{\varepsilon,i} := u_{\varepsilon,i} - \pi^h u_{\varepsilon,i}, \qquad e^h_{\varepsilon,i} := \pi^h u_{\varepsilon,i} - u^h_{\varepsilon,i}. \tag{4.4.11}$$

From the above definitions we observe for $i = 1,2$ that

$$e^A_{\varepsilon,i} + e^h_{\varepsilon,i} = e_{\varepsilon,i} \in V_0, \qquad ⨍ e^h_{\varepsilon,i} = -⨍ e^A_{\varepsilon,i} \tag{4.4.12}$$

Further, for later use one requires the following results which can be easily verified by (4.1.9b)

$$|e^h_{\varepsilon,i}|^2_0 \leq 2|e_{\varepsilon,i}|^2_0 + Ch^4|u_{\varepsilon,i}|^2_2, \tag{4.4.13a}$$
$$|e^h_{\varepsilon,i}|^2_1 \leq 2|e_{\varepsilon,i}|^2_1 + Ch^2|u_{\varepsilon,i}|^2_2. \tag{4.4.13b}$$

[2]Note that $\pi^h u_{\varepsilon,i}$, $i = 1,2$ is well-defined since $u_{\varepsilon,i} \in H^2(\Omega)$ for *a.e.* $t \in (0,T)$ (see Theorem 3.1.3) and $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ for $d \leq 3$ (see [22], p.114).

We note also for future reference that using (4.1.3) and the fact that $|\eta - \fint \eta|_0^2 = (\eta - \fint \eta, \eta)$ yields

$$(\chi, e^A_{\varepsilon,i})^h = 0 \quad \forall\, \chi \in C(\bar{\Omega}), \tag{4.4.14a}$$

$$\Big|\fint \eta\Big|_0 \le |\eta|_0, \quad \Big|\eta - \fint \eta\Big|_0 \le |\eta|_0 \quad \forall\, \eta \in L^2(\Omega). \tag{4.4.14b}$$

Choosing $\eta = e^h_{\varepsilon,i}$ in the continuous regularized version $(\mathbf{P}_\varepsilon)$ given by (2.2.24), taking $\chi = e^h_{\varepsilon,i}$ in the corresponding semi-discrete regularized version $(\mathbf{P}^{\mathbf{h}}_\varepsilon)$ given by (4.3.7) and then subtracting the resulting equations and adding the terms $(\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h$ and $(-\mathcal{G}\partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})$ to both sides yields after rearranging for $i = 1, 2$ and *a.e.* $t \in (0, T)$

$$\begin{aligned}
\gamma(\nabla e_{\varepsilon,i}, \nabla e^h_{\varepsilon,i}) &+ (\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h + (\mathcal{G}\partial_t e_{\varepsilon,i}, e^h_{\varepsilon,i}) \\
&= \theta_i\big[(u_{\varepsilon,i}, e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}) - (u^h_{\varepsilon,i}, e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h\big] \\
&\quad + \big[(\hat{\mathcal{G}}^h\partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})^h - (\mathcal{G}\partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})\big] \\
&\quad + \big[(f^{(i)}_D(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h - (f^{(i)}_D(u_{\varepsilon,1}, u_{\varepsilon,2}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})\big] \\
&\quad + \big[(\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h - (\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})\big].
\end{aligned} \tag{4.4.15}$$

Owing to (4.4.12) and (4.4.14a), using that $\frac{1}{2}\frac{d}{dt}\|e_{\varepsilon,i}\|^2_{-1} = (\mathcal{G}\partial_t e_{\varepsilon,i}, e_{\varepsilon,i})$ and noting (2.2.10) one can rewrite the left hand side of (4.4.15) as

$$\begin{aligned}
\gamma(\nabla e_{\varepsilon,i}, \nabla e^h_{\varepsilon,i}) &+ (\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h + (\mathcal{G}\partial_t e_{\varepsilon,i}, e^h_{\varepsilon,i}) \\
&= \gamma|e_{\varepsilon,i}|_1^2 - \gamma(\nabla e_{\varepsilon,i}, \nabla e^A_{\varepsilon,i}) + (\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i}), e_{\varepsilon,i} + \fint e^A_{\varepsilon,i})^h \\
&\quad + \frac{1}{2}\frac{d}{dt}\|e_{\varepsilon,i}\|^2_{-1} - (\mathcal{G}\partial_t e_{\varepsilon,i}, e^A_{\varepsilon,i}) \\
&\ge \gamma|e_{\varepsilon,i}|_1^2 - \gamma(\nabla e_{\varepsilon,i}, \nabla e^A_{\varepsilon,i}) + \frac{\varepsilon}{\theta}|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i})|_h^2 \\
&\quad + (\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i}), \fint e^A_{\varepsilon,i})^h + \frac{1}{2}\frac{d}{dt}\|e_{\varepsilon,i}\|^2_{-1} - (\mathcal{G}\partial_t e_{\varepsilon,i}, e^A_{\varepsilon,i}).
\end{aligned} \tag{4.4.16}$$

Hence, for $i = 1, 2$ and $a.e.\ t \in (0, T)$ (4.4.15) becomes

$$\begin{aligned}
&\gamma|e_{\varepsilon,i}|_1^2 + \frac{\varepsilon}{\theta}|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i})|_h^2 + \frac{1}{2}\frac{d}{dt}\|e_{\varepsilon,i}\|_{-1}^2 \\
&\quad\leq \gamma(\nabla e_{\varepsilon,i}, \nabla e^A_{\varepsilon,i}) + (\mathcal{G}\partial_t e_{\varepsilon,i}, e^A_{\varepsilon,i}) + (\phi_\varepsilon(u^h_{\varepsilon,i}) - \phi_\varepsilon(u_{\varepsilon,i}), \textstyle\fint e^A_{\varepsilon,i})^h \\
&\qquad + \theta_i\big[(u_{\varepsilon,i}, e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i}) - (u^h_{\varepsilon,i}, e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i})^h\big] \\
&\qquad + \big[(\hat{\mathcal{G}}^h\partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})^h - (\mathcal{G}\partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})\big] \\
&\qquad + \big[(f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i})^h - (f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}), e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i})\big] \\
&\qquad + \big[(\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i})^h - (\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i})\big] \\
&\quad =: \sum_{k=1}^{7} T_k. \qquad (4.4.17)
\end{aligned}$$

We bound each term on the right hand side of (4.4.17) separately. By (4.1.9b), a Young's inequality and the second bound in (3.1.6b) we have for $i = 1, 2$ and $a.e.\ t \in (0, T)$ that

$$T_1 \leq \gamma|e_{\varepsilon,i}|_1|e^A_{\varepsilon,i}|_1 \leq Ch|e_{\varepsilon,i}|_1|u_{\varepsilon,i}|_2 \leq \frac{\gamma}{8}|e_{\varepsilon,i}|_1^2 + Ch^2|u_{\varepsilon,i}|_2^2 \leq \frac{\gamma}{8}|e_{\varepsilon,i}|_1^2 + Ch^2. \quad (4.4.18)$$

From the Poincaré inequality, again (4.1.9b) and the bounds (4.3.54a) and (3.1.6a-b) it follows, taking Lemma 2.2.1 into account, for $i = 1, 2$ and $a.e.\ t \in (0, T)$ that

$$\begin{aligned}
T_2 &\leq C|\mathcal{G}\partial_t e_{\varepsilon,i}|_1|e^A_{\varepsilon,i}|_0 \leq Ch^2\big[|\mathcal{G}\partial_t u_{\varepsilon,i}|_1 + |\mathcal{G}\partial_t u^h_{\varepsilon,i}|_1\big]|u_{\varepsilon,i}|_2 \\
&= Ch^2\big[\|\partial_t u_{\varepsilon,i}\|_{-1} + \|\partial_t u^h_{\varepsilon,i}\|_{-1}\big]|u_{\varepsilon,i}|_2 \leq Ch^2. \qquad (4.4.19)
\end{aligned}$$

Noting again (4.1.9b), (4.4.14b) and the bound (3.1.6b) and applying a Young's inequality yields for $i = 1, 2$ and $a.e.\ t \in (0, T)$ that

$$\begin{aligned}
T_3 &\leq C|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i})|_h|e^A_{\varepsilon,i}|_0 \leq Ch^2|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i})|_h|u_{\varepsilon,i}|_2 \\
&\leq \frac{\varepsilon}{2\theta}|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u^h_{\varepsilon,i})|_h^2 + C\varepsilon^{-1}h^4. \qquad (4.4.20)
\end{aligned}$$

To bound the fourth term we split it as

$$\begin{aligned}
T_4 &= \theta_i(e_{\varepsilon,i}, e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i}) + \theta_i\big[(u^h_{\varepsilon,i}, e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i}) - (u^h_{\varepsilon,i}, e^h_{\varepsilon,i} - \textstyle\fint e^h_{\varepsilon,i})^h\big] \\
&=: T_{4,1} + T_{4,2}. \qquad (4.4.21)
\end{aligned}$$

With the aid of (4.4.14b), a Young's inequality, (4.4.13a), the bound (3.1.6a) and (2.1.11) we have for $i = 1, 2$ and for *a.e.* $t \in (0, T)$ that

$$\begin{aligned} T_{4,1} &\leq \theta_i |e_{\varepsilon,i}|_0 \Big| e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i} \Big|_0 \leq \theta_i |e_{\varepsilon,i}|_0 |e^h_{\varepsilon,i}|_0 \leq \frac{\theta_i}{2} |e_{\varepsilon,i}|_0^2 + \frac{\theta_i}{2} |e^h_{\varepsilon,i}|_0^2 \\ &\leq \frac{3\theta_i}{2} |e_{\varepsilon,i}|_0^2 + Ch^4 |u_{\varepsilon,i}|_2^2 \leq \frac{\gamma}{16} |e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + Ch^4. \end{aligned} \tag{4.4.22}$$

Using (4.1.7), a Young's inequality, (4.4.13b) and noting the bounds (3.1.6b) and (4.3.9a) we find for $i = 1, 2$ and for *a.e.* $t \in (0, T)$ that

$$\begin{aligned} T_{4,2} &\leq Ch^2 |u^h_{\varepsilon,i}|_1 |e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}|_1 \leq \frac{\gamma}{32} |e^h_{\varepsilon,i}|_1^2 + Ch^4 |u^h_{\varepsilon,i}|_1^2 \\ &\leq \frac{\gamma}{16} |e_{\varepsilon,i}|_1^2 + Ch^2 |u_{\varepsilon,i}|_2^2 + Ch^4 |u^h_{\varepsilon,i}|_1^2 \leq \frac{\gamma}{16} |e_{\varepsilon,i}|_1^2 + Ch^2. \end{aligned} \tag{4.4.23}$$

The fifth term can be expressed, by subtracting and adding $(\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})$, as

$$\begin{aligned} T_5 &= \big[(\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})^h - (\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i})\big] + (\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i} - \mathcal{G} \partial_t u^h_{\varepsilon,i}, e^h_{\varepsilon,i}) \\ &=: T_{5,1} + T_{5,2}. \end{aligned} \tag{4.4.24}$$

From (4.1.7), (4.1.12), a Young's inequality, (4.4.13b), the equivalence result (4.1.17) and the bounds (3.1.6b) and (4.3.54a) we have, taking Lemma 2.2.1 into account, that for $i = 1, 2$ and for *a.e.* $t \in (0, T)$

$$\begin{aligned} T_{5,1} &\leq Ch^2 |\hat{\mathcal{G}}^h \partial_t u^h_{\varepsilon,i}|_1 |e^h_{\varepsilon,i}|_1 = Ch^2 \|\partial_t u^h_{\varepsilon,i}\|_{-h} |e^h_{\varepsilon,i}|_1 \leq \frac{\gamma}{32} |e^h_{\varepsilon,i}|_1^2 + Ch^4 \|\partial_t u^h_{\varepsilon,i}\|_{-h}^2 \\ &\leq \frac{\gamma}{16} |e_{\varepsilon,i}|_1^2 + Ch^2 |u_{\varepsilon,i}|_2^2 + Ch^4 \|\partial_t u^h_{\varepsilon,i}\|_{-1}^2 \leq \frac{\gamma}{16} |e_{\varepsilon,i}|_1^2 + Ch^2. \end{aligned} \tag{4.4.25}$$

It follows from (4.1.18), the Young inequality, (4.4.13a), (2.1.11) and the bound (3.1.6b) that for $i = 1, 2$ and for *a.e.* $t \in (0, T)$

$$\begin{aligned} T_{5,2} &\leq |(\hat{\mathcal{G}}^h - \mathcal{G}) \partial_t u^h_{\varepsilon,i}|_0 |e^h_{\varepsilon,i}|_0 \leq Ch^2 \|\partial_t u^h_{\varepsilon,i}\|_1 |e^h_{\varepsilon,i}|_0 \\ &\leq \frac{1}{2} |e^h_{\varepsilon,i}|_0^2 + Ch^4 \|\partial_t u^h_{\varepsilon,i}\|_1^2 \leq |e_{\varepsilon,i}|_0^2 + Ch^4 |u_{\varepsilon,i}|_2^2 + Ch^4 \|\partial_t u^h_{\varepsilon,i}\|_1^2 \\ &\leq \frac{\gamma}{16} |e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + Ch^4 \|\partial_t u^h_{\varepsilon,i}\|_1^2 + Ch^4. \end{aligned} \tag{4.4.26}$$

Bounding the sixth and seventh terms is more technical. To bound the sixth term we first rewrite it as

$$\begin{aligned} T_6 &= \big[(f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h - (f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})\big] \\ &\quad + (f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2}) - f_D^{(i)}(u_{\varepsilon,1}, u_{\varepsilon,2}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}) =: T_{6,1} + T_{6,2}. \end{aligned} \tag{4.4.27}$$

We employ Lemma 4.2.6 to estimate $T_{6,1}$. This lemma, where for $d=1$ note $h^2 \leq |\Omega| h$ and for $d=2$ take $s=\frac{1}{2}$, together with the bound (4.3.9a), (4.4.14b), the Young inequality, (4.4.13a-b), the bound (3.1.6b) and (2.1.11) shows for $i=1,2$, for *a.e.* $t \in (0,T)$ and $d=1,2,3$ that

$$
\begin{aligned}
T_{6,1} &\leq \left|(I-\pi^h)[f_D^{(i)}(u^h_{\varepsilon,1}, u^h_{\varepsilon,2})\, e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}]\right|_{0,1} \\
&= 2D\left|(I-\pi^h)[(u^h_{\varepsilon,i}+\alpha_i)(u^h_{\varepsilon,j}+\alpha_j)^2(e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})]\right|_{0,1} \\
&\leq Ch\|u^h_{\varepsilon,i}+\alpha_i\|_1\, \|u^h_{\varepsilon,j}+\alpha_j\|_1^2\, \|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\|_1 \\
&\leq Ch\|e^h_{\varepsilon,i}\|_1 \leq \frac{\gamma}{64}\|e^h_{\varepsilon,i}\|_1^2 + Ch^2 \\
&\leq \frac{\gamma}{32}|e_{\varepsilon,i}|_0^2 + \frac{\gamma}{32}|e_{\varepsilon,i}|_1^2 + Ch^4|u_{\varepsilon,i}|_2^2 + Ch^2|u_{\varepsilon,i}|_2^2 + Ch^2 \\
&\leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + Ch^2.
\end{aligned} \tag{4.4.28}
$$

Using (2.3.46) with $r_i = u^h_{\varepsilon,i}$ and $s_i = u_{\varepsilon,i}$, a generalised Hölder's inequality, $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and (2.3.8a) and noting again (4.3.9a),(4.4.14b), the Young inequality, (4.4.13a-b), (3.1.6b) and (2.1.11) gives for $i,j=1,2$ with $i \neq j$ and *a.e.* $t \in (0,T)$

$$
\begin{aligned}
T_{6,2} &\leq 2D|u^h_{\varepsilon,j}+\alpha_j|_{0,6}^2\, |u^h_{\varepsilon,i} - u_{\varepsilon,i}|_0\, \left|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\right|_{0,6} \\
&\quad + 2D|u_{\varepsilon,i}+\alpha_i|_{0,6}\, |u^h_{\varepsilon,j}+u_{\varepsilon,j}+2\alpha_j|_{0,6}\, |u^h_{\varepsilon,j}-u_{\varepsilon,j}|_0\, \left|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\right|_{0,6} \\
&\leq C\left[\|u^h_{\varepsilon,j}+\alpha_j\|_1^2\, |e_{\varepsilon,i}|_0 + \|u_{\varepsilon,i}+\alpha_i\|_1\, \|u^h_{\varepsilon,j}+u_{\varepsilon,j}+2\alpha_j\|_1\, |e_{\varepsilon,j}|_0\right] \left\|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\right\|_1 \\
&\leq C\left[|e_{\varepsilon,i}|_0 + |e_{\varepsilon,j}|_0\right] \|e^h_{\varepsilon,i}\|_1 \leq \frac{\gamma}{64}\|e^h_{\varepsilon,i}\|_1^2 + C\left[|e_{\varepsilon,i}|_0^2 + |e_{\varepsilon,j}|_0^2\right] \\
&\leq \frac{\gamma}{32}|e_{\varepsilon,i}|_1^2 + Ch^4|u_{\varepsilon,i}|_2^2 + Ch^2|u_{\varepsilon,i}|_2^2 + C\left[|e_{\varepsilon,i}|_0^2 + |e_{\varepsilon,j}|_0^2\right] \\
&\leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|e_{\varepsilon,j}|_1^2 + C\left[\|e_{\varepsilon,i}\|_{-1}^2 + \|e_{\varepsilon,j}\|_{-1}^2\right] + Ch^2.
\end{aligned} \tag{4.4.29}
$$

Now we turn to estimate the seventh term. To accomplish this, we split this term via

$$
\begin{aligned}
T_7 &= \left[(\pi^h\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})^h - (\pi^h\phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i})\right] \\
&\quad + (\pi^h\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u_{\varepsilon,i}), e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}) =: T_{7,1} + T_{7,2}.
\end{aligned} \tag{4.4.30}
$$

Noting (4.1.7), the Young inequality and (4.4.13b) results in for $i = 1,2$ and for *a.e.* $t \in (0,T)$ that

$$T_{7,1} \leq Ch^2\big|\pi^h\phi_\varepsilon(u_{\varepsilon,i})\big|_1\big|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\big|_1 \leq \frac{\gamma}{32}|e^h_{\varepsilon,i}|_1^2 + Ch^4|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2$$
$$\leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + Ch^2|u|_2^2 + Ch^4|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2. \tag{4.4.31}$$

Next we estimate the third term on the right hand side of (4.4.31). An application of Lemma 4.2.1 (i) with $\chi = \pi^h u_{\varepsilon,i}$ and integration by parts yields after noting (3.1.7) that for $i = 1,2$ and for *a.e.* $t \in (0,T)$

$$\frac{\varepsilon}{\theta}|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2 = \frac{\varepsilon}{\theta}|\nabla\pi^h\phi_\varepsilon(\pi^h u_{\varepsilon,i})|_0^2 \leq (\nabla\pi^h u_{\varepsilon,i}, \nabla\pi^h\phi_\varepsilon(u_{\varepsilon,i}))$$
$$= (\nabla u_{\varepsilon,i}, \nabla\phi_\varepsilon(u_{\varepsilon,i})) - (\nabla u_{\varepsilon,i}, \nabla(I-\pi^h)\phi_\varepsilon(u_{\varepsilon,i}))$$
$$- (\nabla(I-\pi^h)u_{\varepsilon,i}, \nabla\pi^h\phi_\varepsilon(u_{\varepsilon,i}))$$
$$\leq C + (\Delta u_{\varepsilon,i}, (I-\pi^h)\phi_\varepsilon(u_{\varepsilon,i})) - (\nabla(I-\pi^h)u_{\varepsilon,i}, \nabla\pi^h\phi_\varepsilon(u_{\varepsilon,i})). \tag{4.4.32}$$

We use (2.2.11), Lemma 4.2.2, (4.1.9b), a Young's inequality and the bound (3.1.6b) to obtain for $i = 1,2$ and for *a.e.* $t \in (0,T)$ that

$$(\Delta u_{\varepsilon,i}, (I-\pi^h)\phi_\varepsilon(u_{\varepsilon,i})) \leq |\Delta u_{\varepsilon,i}|_0\big[|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(\pi^h u_{\varepsilon,i})|_0 + |(I-\pi^h)\phi_\varepsilon(\pi^h u_{\varepsilon,i})|_0\big]$$
$$\leq C\varepsilon^{-1}|u_{\varepsilon,i}|_2|u_{\varepsilon,i} - \pi^h u_{\varepsilon,i}|_0 + Ch|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1|u_{\varepsilon,i}|_2$$
$$\leq C\varepsilon^{-1}h^2|u_{\varepsilon,i}|_2^2 + \frac{\varepsilon}{4\theta}|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2$$
$$\leq C\varepsilon^{-1}h^2 + \frac{\varepsilon}{4\theta}|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2. \tag{4.4.33}$$

We also have by (4.1.9b), the Young inequality and (3.1.6b) that

$$\big|(\nabla(I-\pi^h)u_{\varepsilon,i}, \nabla\pi^h\phi_\varepsilon(u_{\varepsilon,i}))\big| \leq |(I-\pi^h)u_{\varepsilon,i}|_1|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1 \leq Ch|u_{\varepsilon,i}|_2|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1$$
$$\leq C\varepsilon^{-1}h^2|u_{\varepsilon,i}|_2^2 + \frac{\varepsilon}{4\theta}|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2$$
$$\leq C\varepsilon^{-1}h^2 + \frac{\varepsilon}{4\theta}|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2. \tag{4.4.34}$$

Thus, from (4.4.32)-(4.4.34) we conclude that

$$|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2 \leq C\varepsilon^{-1}\big[1 + \varepsilon^{-1}h^2\big], \tag{4.4.35}$$

and hence for $i = 1,2$ and for $a.e.\ t \in (0,T)$ the term $T_{7,1}$ can be estimated , owing to (4.4.31) and (3.1.6b), as

$$T_{7,1} \leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + Ch^2 + C\big[\varepsilon^{-1}h^4 + \varepsilon^{-2}h^6\big]. \tag{4.4.36}$$

In order to treat the term $T_{7,2}$ we consider two cases. For the case $d = 3$ we use (4.4.14b), a Young's inequality, (4.4.13a), (2.1.11), bound (3.1.6b), Lipschitz continuity (2.2.11), Lemma 4.2.2, (4.1.9b) and (4.4.35) to give for $i = 1,2$ and for $a.e.\ t \in (0,T)$ that

$$\begin{aligned} T_{7,2} &\leq \big|(I-\pi^h)\phi_\varepsilon(u_{\varepsilon,i})\big|_0 \, \Big|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\Big|_0 \leq \frac{1}{2}|e^h_{\varepsilon,i}|_0^2 + \frac{1}{2}\big|(I-\pi^h)\phi_\varepsilon(u_{\varepsilon,i})\big|_0^2 \\ &\leq |e_{\varepsilon,i}|_0^2 + Ch^4|u_{\varepsilon,i}|_2^2 + |\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(\pi^h u_{\varepsilon,i})|_0^2 + \big|(I-\pi^h)\phi_\varepsilon(\pi^h u_{\varepsilon,i})\big|_0^2 \\ &\leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + Ch^4 + \theta^2\varepsilon^{-2}|u_{\varepsilon,i} - \pi^h u_{\varepsilon,i}|_0^2 + Ch^2|\pi^h\phi_\varepsilon(u_{\varepsilon,i})|_1^2 \\ &\leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + Ch^4 + C\varepsilon^{-2}h^4 + C\varepsilon^{-1}h^2\big[1+\varepsilon^{-1}h^2\big] \\ &\leq \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + C\big[\varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big]. \end{aligned} \tag{4.4.37}$$

Note that the above estimate of $T_{7,2}$ is still valid for the case $d = 1,2$. However, when $d = 1,2$ we improve this estimate of $T_{7,2}$ by adapting an argument used in Barrett and Knabner [25]. From Hölder's inequality, (4.1.9a) and (4.1.10) it follows for $d = 1,2$, $h \leq h_0$ and $i = 1,2$ that

$$\begin{aligned} T_{7,2} &\leq \big|(I-\pi^h)[\phi_\varepsilon(u_{\varepsilon,i})\big|_{0,1}\, \Big|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\Big|_{0,\infty} \\ &\leq Ch^2\big(\ln(1/h)\big)^{d-1}\, |\phi_\varepsilon(u_{\varepsilon,i})|_{2,1}\, \Big\|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\Big\|_1. \end{aligned} \tag{4.4.38}$$

By the definitions of Φ_ε and ϕ_ε given by (2.2.1) and (2.2.4) we have that

$$|\phi_\varepsilon(u_{\varepsilon,i})|_{2,1} \leq |\Phi'_\varepsilon(1+u_{\varepsilon,i})|_{2,1} + |\Phi'_\varepsilon(1-u_{\varepsilon,i})|_{2,1}. \tag{4.4.39}$$

Noting Theorem A.0.14 (see Appendix A) in Gilbarg and Trudinger ([42], pp.153-154) we have

$$\frac{\partial}{\partial x_j}\Phi''_\varepsilon(1 \pm u_{\varepsilon,i}) = \begin{cases} \pm\,\Phi'''_\varepsilon(1\pm u_{\varepsilon,i})\frac{\partial u_{\varepsilon,i}}{\partial x_j} & \text{if } u_{\varepsilon,i} \neq \mp 1 \pm \varepsilon, \\ 0 & \text{if } u_{\varepsilon,i} = \mp 1 \pm \varepsilon. \end{cases} \tag{4.4.40}$$

Letting for $i = 1,2$

$$\Omega_i^+ := \{x \in \Omega : u_{\varepsilon,i}(x,t) = -1+\varepsilon\}, \quad \Omega_i^- := \{x \in \Omega : u_{\varepsilon,i}(x,t) = 1-\varepsilon\}.$$

Thus

$$
\begin{aligned}
|\Phi'_\varepsilon(1+u_{\varepsilon,i})|_{2,1} &= \sum_{k,j=1}^{d} \int_\Omega \Big|\frac{\partial^2}{\partial x_j \partial x_k}[\Phi'_\varepsilon(1+u_{\varepsilon,i})]\Big| \\
&\le \sum_{k,j=1}^{d} \int_\Omega \Big|\Phi''_\varepsilon(1+u_{\varepsilon,i})\frac{\partial^2 u_{\varepsilon,i}}{\partial x_j \partial x_k}\Big| + \int_\Omega \Big|\frac{\partial}{\partial x_j}\Phi''_\varepsilon(1+u_{\varepsilon,i})\frac{\partial u_{\varepsilon,i}}{\partial x_k}\Big| dx \\
&=: I_1 + I_2. \qquad (4.4.41)
\end{aligned}
$$

Using the fact that $0 < \Phi''_\varepsilon(r) \le \frac{\theta}{2\varepsilon}\ \forall\, r \in \mathbb{R}$ yields for $i = 1,2$ that

$$
I_1 \le \frac{\theta}{2\varepsilon}\int_\Omega \sum_{k,j=1}^{d} \Big|\frac{\partial^2 u_{\varepsilon,i}}{\partial x_j \partial x_k}\Big| dx = \frac{\theta}{2\varepsilon}|u_{\varepsilon,i}|_{2,1}. \qquad (4.4.42)
$$

Since $\Phi'''_\varepsilon(r) \le 0,\ \forall\, r \in \mathbb{R} - \{\varepsilon\}$, we obtain after noting (4.4.40) and (3.1.7) that for $i = 1,2$

$$
\begin{aligned}
I_2 &= \sum_{k,j=1}^{d} \int_{\Omega\setminus\Omega_i^+} -\Phi'''_\varepsilon(1+u_{\varepsilon,i})\Big|\frac{\partial u_{\varepsilon,i}}{\partial x_k}\frac{\partial u_{\varepsilon,i}}{\partial x_j}\Big| dx \le \int_{\Omega\setminus\Omega_i^+} -\Phi'''_\varepsilon(1+u_{\varepsilon,i})|\nabla u_{\varepsilon,i}|^2 dx \\
&= -(\nabla \Phi''_\varepsilon(1+u_{\varepsilon,i}), \nabla u_{\varepsilon,i}) = (\Phi''_\varepsilon(1+u_{\varepsilon,i}), \Delta u_{\varepsilon,i}) \\
&\le \frac{\theta}{2\varepsilon}|\Delta u_{\varepsilon,i}|_{0,1} \le \frac{\theta}{2\varepsilon}|u_{\varepsilon,i}|_{2,1}. \qquad (4.4.43)
\end{aligned}
$$

We therefore conclude from (4.4.41)-(4.4.43) that

$$
|\Phi'_\varepsilon(1+u_{\varepsilon,i})|_{2,1} \le \frac{\theta}{\varepsilon}|u_{\varepsilon,i}|_{2,1}. \qquad (4.4.44)
$$

Similarly, one can show for $i = 1,2$ that

$$
|\Phi'_\varepsilon(1-u_{\varepsilon,i})|_{2,1} \le \frac{\theta}{\varepsilon}|u_{\varepsilon,i}|_{2,1}. \qquad (4.4.45)
$$

Combining (4.4.38), (4.4.39), (4.4.44) and (4.4.45) and then noting (4.4.14b), the Young inequality, (4.4.13a-b), (2.1.11) and the bound (3.1.6b) it follows for $i = 1,2$ and $a.e.\ t \in (0,T)$ that

$$
\begin{aligned}
T_{7,2} &\le C\varepsilon^{-1}h^2\big(\ln(1/h)\big)^{d-1}\, |u_{\varepsilon,i}|_{2,1}\, \Big\|e^h_{\varepsilon,i} - \fint e^h_{\varepsilon,i}\Big\|_1 \\
&\le C\varepsilon^{-1}h^2\big(\ln(1/h)\big)^{d-1}\, \|u_{\varepsilon,i}\|_2\, \|e^h_{\varepsilon,i}\|_1 \\
&\le \frac{\gamma}{64}\|e^h_{\varepsilon,i}\|_1^2 + C\varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\, \|u_{\varepsilon,i}\|_2^2 \\
&\le \frac{\gamma}{32}|e_{\varepsilon,i}|_0^2 + \frac{\gamma}{32}|e_{\varepsilon,i}|_1^2 + Ch^4|u_{\varepsilon,i}|_2^2 + Ch^2|u_{\varepsilon,i}|_2^2 + C\varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\, \|u_{\varepsilon,i}\|_2^2 \\
&\le \frac{\gamma}{16}|e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + C\big[h^2 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big]. \qquad (4.4.46)
\end{aligned}
$$

Therefore, from (4.4.30), (4.4.36), (4.4.37) and (4.4.46) we obtain for $i = 1, 2$ and *a.e.* $t \in (0, T)$ that

$$T_7 \le \frac{\gamma}{8}|e_{\varepsilon,i}|_1^2 + C\|e_{\varepsilon,i}\|_{-1}^2 + Ch^2 + \begin{cases} C\big[\varepsilon^{-1}h^4 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big] & \text{if } d = 1, 2, \\ C\big[\varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big] & \text{if } d = 3. \end{cases} \tag{4.4.47}$$

Combining (4.4.17)-(4.4.29) with (4.4.47) yields for $i, j = 1, 2$ with $i \neq j$ and for *a.e.* $t \in (0, T)$ that

$$\begin{aligned} \gamma|e_{\varepsilon,i}|_1^2 &+ \frac{\varepsilon}{2\theta}|\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon(u_{\varepsilon,i}^h)|_h^2 + \frac{1}{2}\frac{d}{dt}\|e_{\varepsilon,i}\|_{-1}^2 \\ &\le \frac{5\gamma}{8}|e_{\varepsilon,i}|_1^2 + \frac{\gamma}{8}|e_{\varepsilon,j}|_1^2 + C\big[\|e_{\varepsilon,i}\|_{-1}^2 + \|e_{\varepsilon,j}\|_{-1}^2\big] + Ch^4\|\partial_t u_{\varepsilon,i}^h\|_1^2 \\ &\quad + Ch^2 + \begin{cases} C\big[\varepsilon^{-1}h^4 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big] & \text{if } d = 1, 2, \\ C\big[\varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big] & \text{if } d = 3. \end{cases} \end{aligned} \tag{4.4.48}$$

We sum the above differential inequality over $i = 1, 2$ and simplify to have for *a.e.* $t \in (0, T)$

$$\begin{aligned} \frac{\gamma}{4}\big[|e_{\varepsilon,1}|_1^2 + |e_{\varepsilon,2}|_1^2\big] &+ \frac{1}{2}\frac{d}{dt}\big[\|e_{\varepsilon,1}\|_{-1}^2 + \|e_{\varepsilon,2}\|_{-1}^2\big] \\ &\le C\big[\|e_{\varepsilon,1}\|_{-1}^2 + \|e_{\varepsilon,2}\|_{-1}^2\big] + Ch^4\big[\|\partial_t u_{\varepsilon,1}^h\|_1^2 + \|\partial_t u_{\varepsilon,2}^h\|_1^2\big] \\ &\quad + Ch^2 + \begin{cases} C\big[\varepsilon^{-1}h^4 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big] & \text{if } d = 1, 2, \\ C\big[\varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big] & \text{if } d = 3. \end{cases} \end{aligned} \tag{4.4.49}$$

Applying the Gronwall lemma, recalling the bound (4.3.54a) and noting (4.1.25) we find for *a.e.* $t \in (0, T]$ that

$$\begin{aligned} \frac{\gamma}{2}\int_0^t \big[|e_{\varepsilon,1}|_1^2 + |e_{\varepsilon,2}|_1^2\big]ds &+ \big[\|e_{\varepsilon,1}(t)\|_{-1}^2 + \|e_{\varepsilon,2}(t)\|_{-1}^2\big] \\ &\le Ch^2 + \begin{cases} C\big[\varepsilon^{-1}h^4 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big] & \text{if } d = 1, 2, \\ C\big[\varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big] & \text{if } d = 3. \end{cases} \end{aligned} \tag{4.4.50}$$

By Poincaré's inequality we finally conclude that (4.4.10) holds as required. □

We are now in a position to introduce an error bound between the solutions of the continuous problem $(\mathbf{P})$ and the semi-discrete problem $(\mathbf{P^h})$ which we state in the next theorem.

Theorem 4.4.3 Let the assumptions of Theorem 4.3.5 hold. Then for all $h \leq h_1$

$$\|e_1\|^2_{L^2(0,T;H^1(\Omega))} + \|e_2\|^2_{L^2(0,T;H^1(\Omega))} + \|e_1\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|e_2\|^2_{L^\infty(0,T;(H^1(\Omega))')}$$
$$\leq \begin{cases} Ch^{\frac{4}{3}}\big(\ln(1/h)\big)^{\frac{2(d-1)}{3}} & \text{if } d = 1, 2, \\ Ch & \text{if } d = 3. \end{cases} \tag{4.4.51}$$

where $e_1 := u_1 - u_1^h$ and $e_2 := u_2 - u_2^h$.

Proof. Splitting the error for $i = 1, 2$ via

$$e_i = u_i - u_i^h = (u_i - u_{\varepsilon,i}) + (u_{\varepsilon,i} - u_{\varepsilon,i}^h) + (u_{\varepsilon,i}^h - u_i^h) = \hat{e}_{\varepsilon,i} + e_{\varepsilon,i} + \hat{e}_{\varepsilon,i}^h. \tag{4.4.52}$$

Therefore, combining the errors derived in Theorem 3.2.2, Lemma 4.4.1 and Theorem 4.4.2 yields that

$$\|e_1\|^2_{L^2(0,T;H^1(\Omega))} + \|e_2\|^2_{L^2(0,T;H^1(\Omega))} + \|e_1\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|e_2\|^2_{L^\infty(0,T;(H^1(\Omega))')}$$
$$\leq C\varepsilon + Ch^2 + \begin{cases} C\big[\varepsilon^{-1}h^4 + \varepsilon^{-2}h^4\big(\ln(1/h)\big)^{2(d-1)}\big] & \text{if } d = 1, 2, \\ C\big[\varepsilon^{-1}h^2 + \varepsilon^{-2}h^4\big] & \text{if } d = 3. \end{cases} \tag{4.4.53}$$

On choosing $\varepsilon = Ch^{\frac{4}{3}}\big(\ln(1/h)\big)^{\frac{2(d-1)}{3}} \leq \min\{\varepsilon_0, \frac{\delta_0}{2}\}$ if $d = 1, 2$ and $\varepsilon = Ch \leq \min\{\varepsilon_0, \frac{\delta_0}{2}\}$ if $d = 3$ we obtain the desired result (4.4.51). □

Remark. As a result of the semi-discrete error bound in Theorem 4.4.3, we have convergence of the semi-discrete approximation to the solution of the continuous problem

$$u_1^h, u_2^h \to u_1, u_2 \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;(H^1(\Omega))'),$$

as $h \to 0$.

Chapter 5

A fully-discrete approximation

In this chapter we discretise the continuous problem (**P**) in space using a finite element method and in time using a finite difference method.

In Section 5.1 we present a symmetric coupled in time fully practical finite element approximation of (**P**) and we also introduce the corresponding regularized version. In Section 5.2 we prove existence of a solution to the coupled regularized version. We establish in Section 5.3 stability estimates for the fully-discrete approximations and conclude the section with the uniqueness proof. We then prove further stability estimates that will be essential for the subsequent error bound analysis. Finally, in Section 5.4 we employ the ideas in Nochetto [50] to analyse the error bound.

5.1 Statement of the proposed coupled fully-discrete problem

We define the time step to be $\Delta t := \frac{T}{N}$, where N is a given positive integer. For our fully finite element approximation we discretise the nonlinearities Ψ_i and $f_D^{(i)}$, $i = 1, 2$, at the level time $t = t_n := n\Delta t$, $n = 1, ..., N$ as functions of U_i^n and U_i^{n-1}, where U_i^n is an approximation of the continuous solution u_i at the time $t = t_n$. This discretisation for the nonlinearities is:

For $i=1,2$ we approximate the logarithmic term in $(\mathbf{P})$, $\Psi_i'(u_i)=\phi(u_i)-\theta_i u_i$, as

$$\phi(U_i^n)-\mu\theta_i U_i^n-(1-\mu)\theta_i U_i^{n-1} \quad \mu\in[0,\frac{1}{2}], \tag{5.1.1}$$

and we approximate the D-coupling term, $f_D^{(i)}(u_1,u_2)=2D(u_i+\alpha_i)(u_j+\alpha_j)^2$, as

$$D(U_i^n+\alpha_i)\big[(U_j^n+\alpha_j)^2+(U_j^{n-1}+\alpha_j)^2\big] \quad i,j=1,2 \text{ with } i\neq j. \tag{5.1.2}$$

For notational convenience we introduce $\bar{f}_{n,n-1}^{(i)}$ defined by

$$\bar{f}_{n,n-1}^{(i)}:=2D(U_i^n+\alpha_i)(U_j^{n-1}+\alpha_j)^2 \quad i,j=1,2 \text{ with } i\neq j, \tag{5.1.3}$$

i.e.

$$\bar{f}_{n,n-1}^{(1)}=f_D^{(1)}(U_1^n,U_2^{n-1}), \quad \bar{f}_{n,n-1}^{(2)}=f_D^{(2)}(U_1^{n-1},U_2^n). \tag{5.1.4}$$

From (5.1.2) and (5.1.3) one can represent the D-coupling term as

$$\frac{1}{2}\big[f_D^{(i)}(U_1^n,U_2^n)+\bar{f}_{n,n-1}^{(i)}\big] \quad i=1,2. \tag{5.1.5}$$

Therefore, for given $\mu\in[0,\frac{1}{2}]$ and $u_i^{h,0}\in S_{m_i}^h$ we consider the following coupled fully-discrete finite element approximation of $(\mathbf{P})$:

$(\mathbf{P}_\mu^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ For $n=1,....,N$ find $\{U_1^n,U_2^n,W_1^n,W_2^n\}\in S_{m_1}^h\times S_{m_2}^h\times S^h\times S^h$ such that $U_i^0=u_i^{h,0}$, $i=1,2$, and for all $\chi\in S^h$

$$\Big(\frac{U_i^n-U_i^{n-1}}{\Delta t},\chi\Big)^h+(\nabla W_i^n,\nabla\chi)=0, \tag{5.1.6a}$$

$$\gamma(\nabla U_i^n,\nabla\chi)+\big(\phi(U_i^n)-\mu\theta_i U_i^n-(1-\mu)\theta_i U_i^{n-1},\chi\big)^h + \frac{1}{2}\big(f_D^{(i)}(U_1^n,U_2^n)+\bar{f}_{n,n-1}^{(i)},\chi\big)^h=(W_i^n,\chi)^h. \tag{5.1.6b}$$

The corresponding regularized version of $(\mathbf{P}_\mu^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$, for given $\mu\in[0,\frac{1}{2}]$ and $u_i^{h,0}\in S_{m_i}^h$, is

$(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ For $n=1,....,N$ find $\{U_{\varepsilon,1}^n,U_{\varepsilon,2}^n,W_{\varepsilon,1}^n,W_{\varepsilon,2}^n\}\in S_{m_1}^h\times S_{m_2}^h\times S^h\times S^h$ such that $U_{\varepsilon,i}^0=u_i^{h,0}$, $i=1,2$, and for all $\chi\in S^h$

$$\Big(\frac{U_{\varepsilon,i}^n-U_{\varepsilon,i}^{n-1}}{\Delta t},\chi\Big)^h+(\nabla W_{\varepsilon,i}^n,\nabla\chi)=0, \tag{5.1.7a}$$

$$\gamma(\nabla U_{\varepsilon,i}^n,\nabla\chi)+\big(\phi_\varepsilon(U_{\varepsilon,i}^n)-\mu\theta_i U_{\varepsilon,i}^n-(1-\mu)\theta_i U_{\varepsilon,i}^{n-1},\chi\big)^h + \frac{1}{2}\big(f_D^{(i)}(U_{\varepsilon,1}^n,U_{\varepsilon,2}^n)+\bar{f}_{\varepsilon,n,n-1}^{(i)},\chi\big)^h=(W_{\varepsilon,i}^n,\chi)^h, \tag{5.1.7b}$$

where

$$\bar{f}_{\varepsilon,n,n-1}^{(1)}:=f_D^{(1)}(U_{\varepsilon,1}^n,U_{\varepsilon,2}^{n-1}), \quad \bar{f}_{\varepsilon,n,n-1}^{(2)}:=f_D^{(2)}(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^n). \tag{5.1.8}$$

Similarly to the semi-discrete problems (4.3.7) and (4.3.8), using the discrete Green's operator $\hat{\mathcal{G}}^h$, one can restate $(\mathbf{P}_{\mu}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ and $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ equivalently as:

$(\mathbf{P}_{\mu}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ For $n = 1, ..., N$ find $\{U_1^n, U_2^n\} \in S_{m_1}^h \times S_{m_2}^h$ such that $U_i^0 = u_i^{h,0}$, $i = 1, 2$, and for all $\chi \in S^h$

$$\gamma(\nabla U_i^n, \nabla \chi) + \big(\phi(U_i^n) - \mu\theta_i U_i^n - (1-\mu)\theta_i U_i^{n-1}, \chi - \fint \chi\big)^h \\ + \frac{1}{2}\big(f_D^{(i)}(U_1^n, U_2^n) + \bar{f}_{n,n-1}^{(i)}, \chi - \fint \chi\big)^h + \Big(\hat{\mathcal{G}}^h\big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\big), \chi\Big)^h = 0, \tag{5.1.9}$$

where

$$W_i^n = -\hat{\mathcal{G}}^h\big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\big) + \fint W_i^n, \tag{5.1.10}$$

$$\fint W_i^n = \fint \Big[\pi^h\phi(U_i^n) + \frac{1}{2}(\pi^h f_D^{(i)}(U_1^n, U_2^n) + \pi^h \bar{f}_{n,n-1}^{(i)})\Big] - \theta_i\, m_i. \tag{5.1.11}$$

$(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ For $n = 1, ..., N$ find $\{U_{\varepsilon,1}^n, U_{\varepsilon,2}^n\} \in S_{m_1}^h \times S_{m_2}^h$ such that $U_{\varepsilon,i}^0 = u_i^{h,0}$, $i = 1, 2$, and for all $\chi \in S^h$

$$\gamma(\nabla U_{\varepsilon,i}^n, \nabla \chi) + \big(\phi_\varepsilon(U_{\varepsilon,i}^n) - \mu\theta_i U_{\varepsilon,i}^n - (1-\mu)\theta_i U_{\varepsilon,i}^{n-1}, \chi - \fint \chi\big)^h \\ + \frac{1}{2}\big(f_D^{(i)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n) + \bar{f}_{\varepsilon,n,n-1}^{(i)}, \chi - \fint \chi\big)^h + \Big(\hat{\mathcal{G}}^h\big(\frac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\Delta t}\big), \chi\Big)^h = 0, \tag{5.1.12}$$

where

$$W_{\varepsilon,i}^n = -\hat{\mathcal{G}}^h\big(\frac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\Delta t}\big) + \fint W_{\varepsilon,i}^n, \tag{5.1.13}$$

$$\fint W_{\varepsilon,i}^n = \fint \Big[\pi^h\phi_\varepsilon(U_{\varepsilon,i}^n) + \frac{1}{2}(\pi^h f_D^{(i)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n) + \pi^h \bar{f}_{\varepsilon,n,n-1}^{(i)})\Big] - \theta_i\, m_i. \tag{5.1.14}$$

5.2 Existence of a regularized approximation

In this section we establish existence of a solution to the problem $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ by adapting a similar approach applied in [4] to prove existence of a finite element approximation of a cross diffusion equation. The approach relies on constructing a contradiction to the Schauder fixed point theorem (see Theorem A.0.4 in Appendix A).

Theorem 5.2.1 Let the assumptions of Theorem 4.3.1 hold with $u_i^{h,0} = P^h u_i^0$ or $u_i^{h,0} = P_\gamma^h u_i^0$, $i = 1,2$. Then for all $\mu \in [0, \frac{1}{2}]$, for all $\varepsilon \leq \varepsilon_0$, for all $h > 0$ and for all $\Delta t > 0$ there exists a solution $\{U_{\varepsilon,1}^n, U_{\varepsilon,2}^n, W_{\varepsilon,1}^n, W_{\varepsilon,2}^n\} \in S_{m_1}^h \times S_{m_2}^h \times S^h \times S^h$ to $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\mathbf{\Delta t}})$ for $n = 1, ..., N$.

Proof. We use an inductive proof. We have from (4.3.3) and (4.3.4) that $\{U_{\varepsilon,1}^0, U_{\varepsilon,2}^0\} \in S_{m_1}^h \times S_{m_2}^h$ for the above choices of $u_i^{h,0}$. For fixed $n \geq 1$ assume that $\{U_{\varepsilon,1}^{n-1}, U_{\varepsilon,2}^{n-1}\} \in S_{m_1}^h \times S_{m_2}^h$ exists and we shall prove existence of a solution to $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\mathbf{\Delta t}})$ at the next time level $t = t_n$ (the n-th step). For $i = 1,2$ define $A_i : S_{m_1}^h \times S_{m_2}^h \to V_0^h$ is such that for all $\chi \in S^h$

$$(A_i(U_1,U_2),\chi)^h = \gamma(\nabla U_i, \nabla\chi) + \big(\phi_\varepsilon(U_i) - \mu\theta_i U_i - (1-\mu)\theta_i U_{\varepsilon,i}^{n-1}, \chi - \textstyle{\fint} \chi\big)^h + \frac{1}{2}\big(f_D^{(i)}(U_1,U_2) + \bar{f}_{\varepsilon,n-1}^{(i)}, \chi - \textstyle{\fint}\chi\big)^h + \Big(\hat{\mathcal{G}}^h\big(\frac{U_i - U_{\varepsilon,i}^{n-1}}{\Delta t}\big), \chi\Big)^h, \tag{5.2.1}$$

where

$$\bar{f}_{\varepsilon,n-1}^{(i)} = 2D(U_i + \alpha_i)(U_{\varepsilon,j}^{n-1} + \alpha_j)^2 \quad i,j = 1,2 \text{ with } i \neq j. \tag{5.2.2}$$

$A_i(U_1,U_2) \in S^h$ is well-defined by setting $\chi = \phi_j$, $j = 0,1,...,J$. It can be easily seen for $i = 1,2$ that $(A_i(U_1,U_2),1) = 0$.

Therefore, from (5.2.1) we have that (5.1.12) at the n-th step is equivalent to the problem:

Find $\{U_{\varepsilon,1}^n, U_{\varepsilon,2}^n\} \in S_{m_1}^h \times S_{m_2}^h$ such that for $i = 1,2$ and for all $\chi \in S^h$

$$(A_i(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n),\chi)^h = 0. \tag{5.2.3}$$

By a contradiction for a given $R \in \mathbb{R}_{>0}$ sufficiently large we prove existence of at least one solution to (5.2.3). For this purpose, we assume that for all $R \in \mathbb{R}_{>0}$ there does not exist $\{U_1,U_2\} \in \big[S_{m_1}^h \times S_{m_2}^h\big]_R$ with $A_i(U_1,U_2) = 0$, where $\big[S_{m_1}^h \times S_{m_2}^h\big]_R := \{(\chi_1,\chi_2) \in S_{m_1}^h \times S_{m_2}^h : |\chi_1 - U_{\varepsilon,1}^{n-1}|_h^2 + |\chi_2 - U_{\varepsilon,2}^{n-1}|_h^2 \leq R^2\}$. It can be easily seen that A_i is continuous on $\big[S_{m_1}^h \times S_{m_2}^h\big]_R$ and hence one can define a continuous function $B \equiv (B_1,B_2) : \big[S_{m_1}^h \times S_{m_2}^h\big]_R \to \big[S_{m_1}^h \times S_{m_2}^h\big]_R$ where

$$B_i(U_1,U_2) = \frac{-R\,A_i(U_1,U_2)}{\sqrt{\sum_{i=1}^2 |A_i(U_1,U_2)|_h^2}} + U_{\varepsilon,i}^{n-1} \quad i = 1,2, \tag{5.2.4}$$

which is well-defined.

Since $\left[S^h_{m_1} \times S^h_{m_2}\right]_R$ is a convex and compact subset of the finite dimensional space $S^h \times S^h$, the Schauder fixed point theorem shows that there exists a pair $\{U_1^*, U_2^*\} \in \left[S^h_{m_1} \times S^h_{m_2}\right]_R$ such that

$$B_i(U_1^*, U_2^*) = U_i^* \quad i = 1, 2. \tag{5.2.5}$$

Hence, it follows from (5.2.4) that

$$|U_1^* - U_{\varepsilon,1}^{n-1}|_h^2 + |U_2^* - U_{\varepsilon,2}^{n-1}|_h^2 = R^2. \tag{5.2.6}$$

Recalling that $\Psi'_{\varepsilon,i}(r) = \phi_\varepsilon(r) - \theta_i r$ one can write

$$\phi_\varepsilon(U_i^*) - \mu\theta_i U_i^* - (1-\mu)\theta_i U_{\varepsilon,i}^{n-1} = \Psi'_{\varepsilon,i}(U_i^*) + (1-\mu)\theta_i(U_i^* - U_{\varepsilon,i}^{n-1}). \tag{5.2.7}$$

Choosing $\chi = U_i^* - U_{\varepsilon,i}^{n-1} \in V_0^h$ in (5.2.1) yields for $i = 1, 2$ on noting the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$, (5.2.7) and (4.1.12) that

$$\begin{aligned}(A_i(U_1^*, U_2^*), U_i^* - U_{\varepsilon,i}^{n-1})^h = {} & \frac{\gamma}{2}\left[|U_i^*|_1^2 - |U_{\varepsilon,i}^{n-1}|_1^2 + |U_i^* - U_{\varepsilon,i}^{n-1}|_1^2\right] \\ & + \left(\Psi'_{\varepsilon,i}(U_i^*) + (1-\mu)\theta_i(U_i^* - U_{\varepsilon,i}^{n-1}), U_i^* - U_{\varepsilon,i}^{n-1}\right)^h \\ & + \frac{1}{2}\left(f_D^{(i)}(U_1^*, U_2^*) + \bar{f}^{(i)}_{\varepsilon,n-1}, U_i^* - U_{\varepsilon,i}^{n-1}\right)^h \\ & + \Delta t \left\| \frac{U_i^* - U_{\varepsilon,i}^{n-1}}{\Delta t} \right\|_{-h}^2. \end{aligned} \tag{5.2.8}$$

From (2.2.6) with $r = U_i^*$ and $s = U_{\varepsilon,i}^{n-1}$ and Lemma 2.2.1 (i) it follows for $i = 1, 2$ and for all $\varepsilon \le \varepsilon_0$ that

$$\begin{aligned}&\left(\Psi'_{\varepsilon,i}(U_i^*) + (1-\mu)\theta_i(U_i^* - U_{\varepsilon,i}^{n-1}), U_i^* - U_{\varepsilon,i}^{n-1}\right)^h \\ &\qquad \ge (\Psi_{\varepsilon,i}(U_i^*) - \Psi_{\varepsilon,i}(U_{\varepsilon,i}^{n-1}), 1)^h + \theta_i(\frac{1}{2} - \mu)|U_i^* - U_{\varepsilon,i}^{n-1}|_h^2 \\ &\qquad \ge -C_0|\Omega| - (\Psi_{\varepsilon,i}(U_{\varepsilon,i}^{n-1}), 1)^h + \theta_i(\frac{1}{2} - \mu)|U_i^* - U_{\varepsilon,i}^{n-1}|_h^2 \\ &\qquad := -C(U_{\varepsilon,i}^{n-1}) + \theta_i(\frac{1}{2} - \mu)|U_i^* - U_{\varepsilon,i}^{n-1}|_h^2. \end{aligned} \tag{5.2.9}$$

Summing (5.2.8) over $i = 1, 2$ and noting (5.2.9), (5.2.6), Poincaré's inequality and (4.1.6) gives after recalling that $\mu \in [0, \frac{1}{2}]$

$$\begin{aligned}
\sum_{i=1}^{2}(A_i(U_1^*, U_2^*), U_i^* - U_{\varepsilon,i}^{n-1})^h \geq\ & \frac{\gamma}{2}\sum_{i=1}^{2}\big[- |U_{\varepsilon,i}^{n-1}|_1^2 + |U_i^* - U_{\varepsilon,i}^{n-1}|_1^2\big] \\
&+ \sum_{i=1}^{2}\big[- C(U_{\varepsilon,i}^{n-1}) + \theta_i(\frac{1}{2} - \mu)|U_i^* - U_{\varepsilon,i}^{n-1}|_h^2\big] \\
&+ \frac{1}{2}\sum_{i=1}^{2}\big(f_D^{(i)}(U_1^*, U_2^*) + \bar{f}_{\varepsilon,n-1}^{(i)}, U_i^* - U_{\varepsilon,i}^{n-1}\big)^h \\
\geq\ & \frac{\gamma}{2C}R^2 + \hat{\theta}(\frac{1}{2} - \mu)R^2 - C_1(U_{\varepsilon,1}^{n-1}, U_{\varepsilon,2}^{n-1}) \\
&+ \frac{1}{2}\sum_{i=1}^{2}\big(f_D^{(i)}(U_1^*, U_2^*) + \bar{f}_{\varepsilon,n-1}^{(i)}, U_i^* - U_{\varepsilon,i}^{n-1}\big)^h, \quad (5.2.10)
\end{aligned}$$

where $\hat{\theta} := \min\{\theta_1, \theta_2\}$ and $C_1(U_{\varepsilon,1}^{n-1}, U_{\varepsilon,2}^{n-1}) := \sum_{i=1}^{2}\big[C(U_{\varepsilon,i}^{n-1}) + \frac{\gamma}{2}|U_{\varepsilon,i}^{n-1}|_1^2\big]$.

The treatment of the last term is more technical and in order to simplify the calculations we introduce the following notation

$$\xi_{i,*} := U_i^* + \alpha_i, \quad \xi_{i,n-1} := U_{\varepsilon,i}^{n-1} + \alpha_i \quad i = 1, 2. \tag{5.2.11}$$

Thus $\xi_{i,*} - \xi_{i,n-1} = U_i^* - U_{\varepsilon,i}^{n-1}$ and hence we have with the aid of the Young inequality that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\big(f_D^{(i)}(U_1^*, U_2^*) + \bar{f}_{\varepsilon,n-1}^{(i)}, U_i^* - U_{\varepsilon,i}^{n-1}\big)^h \\
&\quad = D(\xi_{1,*}\xi_{2,*}^2 + \xi_{1,*}\xi_{2,n-1}^2, \xi_{1,*} - \xi_{1,n-1})^h + D(\xi_{2,*}\xi_{1,*}^2 + \xi_{2,*}\xi_{1,n-1}^2, \xi_{2,*} - \xi_{2,n-1})^h \\
&\quad = D\big[(\xi_{1,*}^2, \xi_{2,*}^2)^h - (\xi_{1,*}\xi_{1,n-1}, \xi_{2,*}^2)^h + (\xi_{1,*}^2, \xi_{2,n-1}^2)^h - (\xi_{1,*}\xi_{1,n-1}, \xi_{2,n-1}^2)^h\big] \\
&\qquad + D\big[(\xi_{2,*}^2, \xi_{1,*}^2)^h - (\xi_{2,*}\xi_{2,n-1}, \xi_{1,*}^2)^h + (\xi_{2,*}^2, \xi_{1,n-1}^2)^h - (\xi_{2,*}\xi_{2,n-1}, \xi_{1,n-1}^2)^h\big] \\
&\quad \geq D\big[2(\xi_{1,*}^2, \xi_{2,*}^2)^h + (\xi_{1,*}^2, \xi_{2,n-1}^2)^h + (\xi_{2,*}^2, \xi_{1,n-1}^2)^h - \frac{1}{2}(\xi_{1,*}^2 + \xi_{1,n-1}^2, \xi_{2,*}^2 + \xi_{2,n-1}^2)^h \\
&\qquad - \frac{1}{2}(\xi_{2,*}^2 + \xi_{2,n-1}^2, \xi_{1,*}^2 + \xi_{1,n-1}^2)^h\big] \\
&\quad = D\big[(\xi_{1,*}^2, \xi_{2,*}^2)^h - (\xi_{1,n-1}^2, \xi_{2,n-1}^2)^h\big] \\
&\quad = D\big((U_1^* + \alpha_1)^2, (U_2^* + \alpha_2)^2\big)^h - D\big((U_{\varepsilon,1}^{n-1} + \alpha_1)^2, (U_{\varepsilon,2}^{n-1} + \alpha_2)^2\big)^h \\
&\quad \geq -D\big((U_{\varepsilon,1}^{n-1} + \alpha_1)^2, (U_{\varepsilon,2}^{n-1} + \alpha_2)^2\big)^h := -C_2(U_{\varepsilon,1}^{n-1}, U_{\varepsilon,2}^{n-1}). \qquad (5.2.12)
\end{aligned}$$

Inserting (5.2.12) into (5.2.10) yields

$$\sum_{i=1}^{2}(A_i(U_1^*,U_2^*),U_i^*-U_{\varepsilon,i}^{n-1})^h \geq \frac{\gamma}{2C}R^2+\hat{\theta}(\frac{1}{2}-\mu)R^2-C_1(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^{n-1}) - C_2(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^{n-1})>0, \tag{5.2.13}$$

which will be positive for R sufficiently large.

On the contrary, from (5.2.4), (5.2.5) and (5.2.6) we obtain that for all $R\in\mathbb{R}_{>0}$

$$\begin{aligned}\sum_{i=1}^{2}&(A_i(U_1^*,U_2^*),U_i^*-U_{\varepsilon,i}^{n-1})^h \\ &= \frac{\sqrt{\sum_{i=1}^{2}|A_i(U_1^*,U_2^*)|_h^2}}{-R}\sum_{i=1}^{2}\big(B_i(U_1^*,U_2^*)-U_{\varepsilon,i}^{n-1},U_i^*-U_{\varepsilon,i}^{n-1}\big)^h \\ &= \frac{\sqrt{\sum_{i=1}^{2}|A_i(U_1^*,U_2^*)|_h^2}}{-R}\sum_{i=1}^{2}|U_i^*-U_{\varepsilon,i}^{n-1}|_h^2 \\ &= -R\sqrt{\sum_{i=1}^{2}|A_i(U_1^*,U_2^*)|_h^2}<0. \end{aligned}\tag{5.2.14}$$

Therefore, this contradiction guarantees existence of $\{U_{\varepsilon,1}^n,U_{\varepsilon,2}^n\}\in S_{m_1}^h\times S_{m_2}^h$ solving (5.2.3) and hence ($\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\mathbf{\Delta t}}$) at the n-th time step. Existence of $W_{\varepsilon,1}^n$ and $W_{\varepsilon,2}^n$ follows directly from (5.1.13) and (5.1.14). This completes the proof. □

Remark. We note that, in view of (5.2.13), the restriction $\mu\in[0,\frac{1}{2}]$ is essential for existence proof, otherwise we need to impose a restriction on the physical parameters γ, θ_1 and θ_2 which is not desirable. For the case $D=0$ (which is not of interest in this thesis) the existence can be achieved for $\mu\in[0,1]$ by an alternative technique (e.g. Barrett and Blowey [12], Barrett and Blowey [5]) which can not be applied to our coupled fully-discrete problem.

5.3 Stability estimates and uniqueness

In this section we first derive stability estimates for the regularized approximations $U_{\varepsilon,i}^n, W_{\varepsilon,i}^n,\ i=1,2$ which enable us to prove existence and uniqueness of a solution to ($\mathbf{P}_{\mu}^{\mathbf{h},\mathbf{\Delta t}}$). We then establish further stability estimates under the assumptions ($\mathbf{A_2}$) which will be needed in the subsequent section.

Theorem 5.3.1 Let the assumptions of Theorem 5.2.1 hold with $u_i^{h,0} = P^h u_i^0$. Then for all $\mu \in [0, \frac{1}{2}]$, for all $\varepsilon \le \varepsilon_0$, for all $h > 0$ and for all $\Delta t > 0$ a solution $\{U_{\varepsilon,1}^n, U_{\varepsilon,2}^n, W_{\varepsilon,1}^n, W_{\varepsilon,2}^n\}$ to the n-th step of $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ is such that

$$\max_{n=1\to N} \left[\|U_{\varepsilon,1}^n\|_1^2 + \|U_{\varepsilon,2}^n\|_1^2\right] + \sum_{n=1}^{N} \left[\|U_{\varepsilon,1}^n - U_{\varepsilon,1}^{n-1}\|_1^2 + \|U_{\varepsilon,2}^n - U_{\varepsilon,2}^{n-1}\|_1^2\right] \le C, \quad (5.3.1a)$$

$$\Delta t \sum_{n=1}^{N} \left[\left\|\frac{U_{\varepsilon,1}^n - U_{\varepsilon,1}^{n-1}}{\Delta t}\right\|_{-h}^2 + \left\|\frac{U_{\varepsilon,2}^n - U_{\varepsilon,2}^{n-1}}{\Delta t}\right\|_{-h}^2\right] \le C, \quad (5.3.1b)$$

$$\Delta t \sum_{n=1}^{N} \left[\|W_{\varepsilon,1}^n\|_1^2 + \|W_{\varepsilon,2}^n\|_1^2\right] + \Delta t \sum_{n=1}^{N} \left[|\pi^h \phi_\varepsilon(U_{\varepsilon,1}^n)|_0^2 + |\pi^h \phi_\varepsilon(U_{\varepsilon,2}^n)|_0^2\right] \le C, \quad (5.3.1c)$$

$$\begin{aligned} \max_{n=1\to N} \Big[&|\pi^h f_D^{(1)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n)|_0^2 + |\pi^h f_D^{(2)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n)|_0^2 \\ &+ |\pi^h f_D^{(1)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^{n-1})|_0^2 + |\pi^h f_D^{(2)}(U_{\varepsilon,1}^{n-1}, U_{\varepsilon,2}^n)|_0^2\Big] \le C. \end{aligned} \quad (5.3.1d)$$

Proof. Testing (5.1.7a) with $\chi = \hat{\mathcal{G}}^h(U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}),\ i = 1, 2$, we obtain on noting (4.1.12) and (4.1.11) that

$$\begin{aligned} 0 &= \Delta t \left\|\frac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\Delta t}\right\|_{-h}^2 + (\nabla W_{\varepsilon,i}^n, \nabla \hat{\mathcal{G}}^h(U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1})) \\ &= \Delta t \left\|\frac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\Delta t}\right\|_{-h}^2 + (W_{\varepsilon,i}^n, U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1})^h \\ &= \Delta t \left\|\frac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\Delta t}\right\|_{-h}^2 + \gamma(\nabla U_{\varepsilon,i}^n, \nabla U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}) + (\Psi_{\varepsilon,i}'(U_{\varepsilon,i}^n), U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1})^h \\ &\quad + \theta_i(1-\mu)|U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}|_h^2 + \frac{1}{2}(f_D^{(i)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n) + \bar{f}_{\varepsilon,n,n-1}^{(i)}, U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1})^h, \end{aligned} \quad (5.3.2)$$

where we have also noted (5.1.7b) with $\chi = U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}$ and (5.2.7) to obtain the last equality.

We use the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and (2.2.6) to yield for $i = 1, 2$

$$\begin{aligned} \frac{\gamma}{2}|U_{\varepsilon,i}^n|_1^2 + \frac{\gamma}{2}|U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}|_1^2 + (\Psi_{\varepsilon,i}(U_{\varepsilon,i}^n), 1)^h + \theta_i(\frac{1}{2} - \mu)|U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}|_h^2 \\ + \Delta t \left\|\frac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\Delta t}\right\|_{-h}^2 \\ + \frac{1}{2}(f_D^{(i)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n) + \bar{f}_{\varepsilon,n,n-1}^{(i)}, U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1})^h \\ \le \frac{\gamma}{2}|U_{\varepsilon,i}^{n-1}|_1^2 + (\Psi_{\varepsilon,i}(U_{\varepsilon,i}^{n-1}), 1)^h. \end{aligned} \quad (5.3.3)$$

By arguing as for (5.2.12) in the proof of Theorem 5.2.1 one can show that

$$\frac{1}{2}\sum_{i=1}^{2}(f_D^{(i)}(U^n_{\varepsilon,1},U^n_{\varepsilon,2})+\bar{f}^{(i)}_{\varepsilon,n,n-1},U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i})^h$$
$$\geq D\big((U^n_{\varepsilon,1}+\alpha_1)^2,(U^n_{\varepsilon,2}+\alpha_2)^2\big)^h - D\big((U^{n-1}_{\varepsilon,1}+\alpha_1)^2,(U^{n-1}_{\varepsilon,2}+\alpha_2)^2\big)^h$$
$$=(f_D(U^n_{\varepsilon,1},U^n_{\varepsilon,2}),1)^h-(f_D(U^{n-1}_{\varepsilon,1},U^{n-1}_{\varepsilon,2}),1)^h. \tag{5.3.4}$$

Next we sum (5.3.3) over $i=1,2$ and then $\forall\, m\leq N$ we sum the resulting inequality from $n=1\to m$, note (5.3.4) and rearrange to result in

$$\frac{\gamma}{2}\big[|U^m_{\varepsilon,1}|_1^2+|U^m_{\varepsilon,2}|_1^2\big]+\frac{\gamma}{2}\sum_{n=1}^{m}\big[|U^n_{\varepsilon,1}-U^{n-1}_{\varepsilon,1}|_1^2+|U^n_{\varepsilon,2}-U^{n-1}_{\varepsilon,2}|_1^2\big]$$
$$+\big[(\Psi_{\varepsilon,1}(U^m_{\varepsilon,1}),1)^h+(\Psi_{\varepsilon,2}(U^m_{\varepsilon,2}),1)^h\big]+(\frac{1}{2}-\mu)\sum_{n=1}^{m}\big[\theta_1|U^n_{\varepsilon,1}-U^{n-1}_{\varepsilon,1}|_h^2+\theta_2|U^n_{\varepsilon,2}-U^{n-1}_{\varepsilon,2}|_h^2\big]$$
$$+\Delta t\sum_{n=1}^{m}\Big[\Big\|\frac{U^n_{\varepsilon,1}-U^{n-1}_{\varepsilon,1}}{\Delta t}\Big\|_{-h}^2+\Big\|\frac{U^n_{\varepsilon,2}-U^{n-1}_{\varepsilon,2}}{\Delta t}\Big\|_{-h}^2\Big]+(f_D(U^m_{\varepsilon,1},U^m_{\varepsilon,2}),1)^h$$
$$\leq\frac{\gamma}{2}\big[|U^0_{\varepsilon,1}|_1^2+|U^0_{\varepsilon,2}|_1^2\big]+\big[(\Psi_{\varepsilon,1}(U^0_{\varepsilon,1}),1)^h+(\Psi_{\varepsilon,2}(U^0_{\varepsilon,2}),1)^h\big]+(f_D(U^0_{\varepsilon,1},U^0_{\varepsilon,2}),1)^h$$
$$\leq C, \tag{5.3.5}$$

where we have noted the bounds (4.3.18), (4.3.19) and (4.3.20) to obtain the last inequality.

Recalling that $\mu\in[0,\frac{1}{2}]$ and $f_D(\cdot,\cdot)\geq 0$, using Lemma 2.2.1 (i) and noting Poincaré's inequality we obtain from (5.3.5) the desired estimate (5.3.1a). In addition, (5.3.5) gives directly the estimate (5.3.1b).

It follows from (5.1.13) and (4.1.12) that for $i=1,2$ and $n=1\to N$

$$|W^n_{\varepsilon,i}|_1^2=\Big|-\hat{\mathcal{G}}^h\big(\frac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t}\big)+\fint W^n_{\varepsilon,i}\Big|_1^2=\Big\|\frac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t}\Big\|_{-h}^2. \tag{5.3.6}$$

This result with Poincaré's inequality shows that for $i=1,2$

$$\Delta t\sum_{n=1}^{N}\|W^n_{\varepsilon,i}-\fint W^n_{\varepsilon,i}\|_1^2\leq C\Delta t\sum_{n=1}^{N}|W^n_{\varepsilon,i}-\fint W^n_{\varepsilon,i}|_1^2$$
$$=C\Delta t\sum_{n=1}^{N}\Big\|\frac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t}\Big\|_{-h}^2\leq C, \tag{5.3.7}$$

where we have also noted the estimate (5.3.1b) in the last step.

To obtain the first estimate in (5.3.1c), it remains to show $\Delta t \sum_{n=1}^{N} \|\fint W^n_{\varepsilon,i}\|_1^2$ is bounded for $i = 1, 2$. To this aim, we note first, using Lemma 2.2.1 (i) and (5.3.5), that

$$\big|(\psi_\varepsilon(U^n_{\varepsilon,i}), 1)^h\big| \le \big|(\Psi_{\varepsilon,i}(U^n_{\varepsilon,i}), 1)^h\big| + \frac{\theta_i}{2}\big|(1 - (U^n_{\varepsilon,i})^2, 1)^h\big| \le C. \tag{5.3.8}$$

Next we choose $\chi = U^n_{\varepsilon,i} - \fint U^n_{\varepsilon,i} = U^n_{\varepsilon,i} - m_i$ in (5.1.7b) and add for any $\beta \in \mathbb{R}$ the term $\big(\phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}), \beta\big)^h$ to the both sides to give after rearranging that

$$\begin{aligned}
\big(\phi_\varepsilon(U^n_{\varepsilon,i}) &+ \frac{1}{2}(f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}), \beta - m_i\big)^h = \\
&= (W^n_{\varepsilon,i}, U^n_{\varepsilon,i} - m_i)^h - \gamma|U^n_{\varepsilon,i}|_1^2 + (\mu\theta_i U^n_{\varepsilon,i} + (1-\mu)\theta_i U^{n-1}_{\varepsilon,i}, U^n_{\varepsilon,i} - m_i)^h \\
&\quad + (\phi_\varepsilon(U^n_{\varepsilon,i}), \beta - U^n_{\varepsilon,i})^h + \frac{1}{2}\big(f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}, \beta - U^n_{\varepsilon,i}\big)^h \\
&\le (\nabla W^n_{\varepsilon,i}, \nabla \hat{\mathcal{G}}^h(U^n_{\varepsilon,i} - m_i)) + \theta_i\big[|U^n_{\varepsilon,i}|_h + |U^{n-1}_{\varepsilon,i}|_h\big]|U^n_{\varepsilon,i} - m_i|_h \\
&\quad + (\psi_\varepsilon(\beta), 1)^h - (\psi_\varepsilon(U^n_{\varepsilon,i}), 1)^h + \frac{1}{2}|f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}|_h|\beta - U^n_{\varepsilon,i}|_h \\
&\le C\big[1 + |W^n_{\varepsilon,i}|_1\|U^n_{\varepsilon,i} - m_i\|_{-h} + (\psi_\varepsilon(\beta), 1)^h \\
&\qquad + |f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}|_h\,|\beta - U^n_{\varepsilon,i}|_h\big] \\
&\le C\big[1 + |W^n_{\varepsilon,i}|_1 + (\psi_\varepsilon(\beta), 1)^h + |f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}|_h\,|\beta - U^n_{\varepsilon,i}|_h\big],
\end{aligned} \tag{5.3.9}$$

where we have also used (4.1.11), (2.2.6), (4.1.12), the bound (5.3.1a) and (5.3.8) followed by (4.1.13) and again the bound (5.3.1a) to obtain the last inequality.

Applying Lemma 4.2.4 with $p = 3$ and $q = \frac{3}{2}$ and Lemma 4.2.8 and noting the bounds (5.3.1a) and (4.3.18) yields for $i, j = 1, 2$ with $i \neq j$ and for $n = 1 \to N$ that

$$\begin{aligned}
|\bar{f}^{(i)}_{\varepsilon,n,n-1}|_h^2 &= 4D^2 \int_\Omega \pi^h\big((U^n_{\varepsilon,i} + \alpha_i)^2 (U^{n-1}_{\varepsilon,j} + \alpha_j)^4\big)\, dx \\
&\le 4D^2 \Big(\int_\Omega \pi^h\big((U^n_{\varepsilon,i} + \alpha_i)^6\big) dx\Big)^{\frac{1}{3}} \Big(\int_\Omega \pi^h\big((U^{n-1}_{\varepsilon,j} + \alpha_j)^6\big) dx\Big)^{\frac{2}{3}} \\
&\le C\|U^n_{\varepsilon,i} + \alpha_i\|_1^2 \|U^{n-1}_{\varepsilon,j} - \alpha_j\|_1^4 \le C,
\end{aligned} \tag{5.3.10}$$

and similarly one can show

$$|f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2})|_h^2 \le C\|U^n_{\varepsilon,i} + \alpha_i\|_1^2 \|U^n_{\varepsilon,j} - \alpha_j\|_1^4 \le C. \tag{5.3.11}$$

On choosing $\beta = \pm 1 \mp \frac{\delta_0}{2}$ in (5.3.9) and noting $\psi_\varepsilon(r) \le \theta \ln 2 \; \forall\, r \in [-1,1]$, (5.3.10), (5.3.11) and (5.3.1a) leads to the following inequalities

$$\begin{aligned}
&\big(\pi^h \phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(\pi^h f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \pi^h \bar{f}^{(i)}_{\varepsilon,n,n-1}), 1 - \frac{\delta_0}{2} - m_i\big) \\
&\qquad = \big(\phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}), 1 - \frac{\delta_0}{2} - m_i\big)^h \\
&\qquad \le C\big[1 + |W^n_{\varepsilon,i}|_1\big] \qquad (5.3.12)
\end{aligned}$$

and

$$\begin{aligned}
&\big(\pi^h \phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(\pi^h f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \pi^h \bar{f}^{(i)}_{\varepsilon,n,n-1}), 1 - \frac{\delta_0}{2} + m_i\big) \\
&\qquad = \big(\phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}), 1 - \frac{\delta_0}{2} + m_i\big)^h \\
&\qquad \ge -C\big[1 + |W^n_{\varepsilon,i}|_1\big] \qquad i = 1,2. \qquad (5.3.13)
\end{aligned}$$

Dividing (5.3.12) and (5.3.13) by $|\Omega|(1 - \frac{\delta_0}{2} - m_i)$ and $|\Omega|(1 - \frac{\delta_0}{2} + m_i)$ respectively and noting that $|m_i| \le 1 - \delta_0$ yields that

$$\begin{aligned}
\Big| \fint \big[\pi^h \phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(\pi^h f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \pi^h \bar{f}^{(i)}_{\varepsilon,n,n-1})\big] \Big| &\le C\big[1 + |W^n_{\varepsilon,i}|_1\big] \\
&= C\Big[1 + \Big\| \frac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\Delta t} \Big\|_{-h}\Big], \qquad (5.3.14)
\end{aligned}$$

where in the last step we have noted (5.3.6).

By squaring the above inequality and summing from $n = 1 \to N$ we have after multiplying by Δt and noting the bound (5.3.1b) that for $i = 1,2$

$$\begin{aligned}
\Delta t \sum_{n=1}^{N} \Big| \fint \big[\pi^h \phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(\pi^h f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \pi^h \bar{f}^{(i)}_{\varepsilon,n,n-1})\big] \Big|^2 \\
\le C\Big[\Delta t N + \Delta t \sum_{n=1}^{N} \Big\| \frac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\Delta t} \Big\|^2_{-h}\Big] \\
\le C, \qquad (5.3.15)
\end{aligned}$$

since $N\Delta t = T$.

Thus, from (5.1.14) and (5.3.15) one finds for $i = 1, 2$ that

$$\begin{aligned}\Delta t \sum_{n=1}^{N} \Big\| \fint W^n_{\varepsilon,i} \Big\|_1^2 &= |\Omega| \Delta t \sum_{n=1}^{N} \Big| \fint W^n_{\varepsilon,i} \Big|^2 \\ &\leq 2|\Omega|\Delta t \sum_{n=1}^{N} \Big| \fint \big[\pi^h \phi_\varepsilon(U^n_{\varepsilon,i}) + \frac{1}{2}(\pi^h f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \pi^h \bar{f}^{(i)}_{\varepsilon,n,n-1})\big] \Big|^2 \\ &\quad + 2|\Omega|\Delta t N \theta_i^2 m_i^2 \\ &\leq C, \end{aligned} \tag{5.3.16}$$

and hence together with (5.3.7) this shows the first estimate in (5.3.1c), as required.

We now turn to prove the second estimate in (5.3.1c). To obtain this we test (5.1.7b) with $\chi = \pi^h \phi_\varepsilon(U^n_{\varepsilon,i})$ and then apply a Young's inequality to result in for $i = 1, 2$ that

$$\begin{aligned}\gamma(\nabla U^n_{\varepsilon,i}, \nabla \pi^h \phi_\varepsilon(U^n_{\varepsilon,i})) + |\pi^h \phi_\varepsilon(U^n_{\varepsilon,i})|_h^2 \qquad & \\ = (W^n_{\varepsilon,i}, \pi^h \phi_\varepsilon(U^n_{\varepsilon,i}))^h &+ \theta_i\big(\mu U^n_{\varepsilon,i} + (1-\mu)U^{n-1}_{\varepsilon,i}, \pi^h \phi_\varepsilon(U^n_{\varepsilon,i})\big)^h \\ &- \frac{1}{2}(f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) + \bar{f}^{(i)}_{\varepsilon,n,n-1}, \pi^h \phi_\varepsilon(U^n_{\varepsilon,i}))^h \\ \leq \frac{1}{2}|\pi^h \phi_\varepsilon(U^n_{\varepsilon,i})|_h^2 &+ C\Big[|W^n_{\varepsilon,i}|_h^2 + |U^n_{\varepsilon,i}|_h^2 + |U^{n-1}_{\varepsilon,i}|_h^2 \\ &\qquad + |f_D^{(i)}(U^n_{\varepsilon,1}, U^n_{\varepsilon,2})|_h^2 + |\bar{f}^{(i)}_{\varepsilon,n,n-1}|_h^2\Big] \\ \leq \frac{1}{2}|\pi^h \phi_\varepsilon(U^n_{\varepsilon,i})|_h^2 &+ C\big[1 + |W^n_{\varepsilon,i}|_h^2\big]. \end{aligned} \tag{5.3.17}$$

where to obtain the last inequality we have noted (5.3.1a), (5.3.10) and (5.3.11).

Using Lemma 4.2.1 (i) we have the first term in (5.3.17) is non-negative and hence by summing the both sides from $n = 1 \to N$ and recalling the first bound in (5.3.1c) we conclude for $i = 1, 2$ that

$$\Delta t \sum_{n=1}^{N} |\pi^h \phi_\varepsilon(U^n_{\varepsilon,i})|_h^2 \leq C\Big[T + \Delta t \sum_{n=1}^{N} |W^n_{\varepsilon,i}|_h^2\Big] \leq C, \tag{5.3.18}$$

which, on noting the equivalence result (4.1.6), leads to the second desired estimate in (5.3.1c).

Finally, noting for $i = 1, 2$ that $|\pi^h f_D^{(i)}(\chi, v)|_h = |f_D^{(i)}(\chi, v)|_h \;\forall\, \chi, v \in S^h$, recalling that $\bar{f}^{(1)}_{\varepsilon,n,n-1} = f_D^{(1)}(U^n_{\varepsilon,1}, U^{n-1}_{\varepsilon,2})$ and $\bar{f}^{(2)}_{\varepsilon,n,n-1} = f_D^{(2)}(U^{n-1}_{\varepsilon,1}, U^n_{\varepsilon,2})$ and using (5.3.10), (5.3.11) and the equivalence result (4.1.6) we obtain the desired estimate (5.3.1d), which completes the proof. □

Theorem 5.3.2 Let the assumptions of Theorem 5.2.1 hold with $u_i^{h,0} = P^h u_i^0$. Then for all $\mu \in [0, \frac{1}{2}]$, for all $h > 0$, for all $\Delta t > 0$ and for all $n = 1 \to N$ there exists a solution $\{U_1^n, U_2^n, W_1^n, W_2^n\} \in S^h_{m_1} \times S^h_{m_2} \times S^h \times S^h$ to $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_\mu)$ such that

$$\max_{n=1\to N} \left[\|U_1^n\|_1^2 + \|U_2^n\|_1^2\right] + \sum_{n=1}^{N} \left[\|U_1^n - U_1^{n-1}\|_1^2 + \|U_2^n - U_2^{n-1}\|_1^2\right] \le C, \tag{5.3.19a}$$

$$\Delta t \sum_{n=1}^{N} \left[\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_{-h}^2 + \left\|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right] \le C, \tag{5.3.19b}$$

$$\Delta t \sum_{n=1}^{N} \left[\|W_1^n\|_1^2 + \|W_2^n\|_1^2\right] + \Delta t \sum_{n=1}^{N} \left[|\pi^h \phi(U_1^n)|_0^2 + |\pi^h \phi(U_2^n)|_0^2\right] \le C, \tag{5.3.19c}$$

$$\max_{n=1\to N} \left[|\pi^h f_D^{(1)}(U_1^n, U_2^n)|_0^2 + |\pi^h f_D^{(2)}(U_1^n, U_2^n)|_0^2 \right.$$
$$\left. + |\pi^h f_D^{(1)}(U_1^n, U_2^{n-1})|_0^2 + |\pi^h f_D^{(2)}(U_1^{n-1}, U_2^n)|_0^2\right] \le C, \tag{5.3.19d}$$

$$\max\{|U_1^n|, |U_2^n|\} < 1 \quad \text{for all } x \in \bar{\Omega} \text{ and } n = 1 \to N. \tag{5.3.19e}$$

Furthermore, the solution is uniquely defined for all $\Delta t > 0$ if $\theta \ge 8D + \mu\theta_*$ and for all $\Delta t < \frac{4\gamma}{(8D + \mu\theta_* - \theta)^2}$ if $\theta < 8D + \mu\theta_*$ where $\theta_* = \max\{\theta_1, \theta_2\}$.

Proof. From the bounds (5.3.1a) and (5.3.1c) we have for $i = 1, 2$ that $|U^n_{\varepsilon,i}|_h$, $|W^n_{\varepsilon,i}|_h$ and $|\pi^h \phi_\varepsilon(U^n_{\varepsilon,i})|_h$ are bounded independently of ε. Hence, one can extract subsequences, still denoted $\{U^n_{\varepsilon,i}\}$, $\{W^n_{\varepsilon,i}\}$ and $\{\pi^h \phi_\varepsilon(U^n_{\varepsilon,i})\}$, such that for $i = 1, 2$ and $n = 1 \to N$

$$U^n_{\varepsilon,i} \to U^n_i \quad \text{in } S^h, \tag{5.3.20a}$$

$$W^n_{\varepsilon,i} \to W^n_i \quad \text{in } S^h, \tag{5.3.20b}$$

$$\pi^h \phi_\varepsilon(U^n_{\varepsilon,i}) \to \chi_i^{h,n} \quad \text{in } S^h. \tag{5.3.20c}$$

We now prove for $i = 1, 2$ and $n = 1 \to N$ that $\chi_i^{h,n} = \pi^h \phi(U_i^n)$. For $i = 1, 2$ and $n = 1 \to N$ we define for any $\chi \in S^h$

$$I_i^{h,n}(\chi) := (U_i^n - \phi^{-1}(\chi), \chi_i^{h,n} - \chi)^h, \tag{5.3.21}$$

$$I_{\varepsilon,i}^{h,n}(\chi) := (U_{\varepsilon,i}^n - \phi_\varepsilon^{-1}(\chi), \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi)^h. \tag{5.3.22}$$

The above quantities are well-defined as

$$\left|I_i^{h,n}(\chi)\right| \leq |U_i^n - \phi^{-1}(\chi)|_h |\chi_i^{h,n} - \chi|_h < \infty,$$
$$\left|I_{\varepsilon,i}^{h,n}(\chi)\right| \leq |U_{\varepsilon,i}^n - \phi_\varepsilon^{-1}(\chi)|_h |\phi_\varepsilon(U_{\varepsilon,i}^n) - \chi|_h \leq \theta^{-1} |\phi_\varepsilon(U_{\varepsilon,i}^n) - \chi|_h^2 < \infty,$$

where we have used the fact that $|\phi^{-1}(\cdot)| < 1$, (2.2.16), (4.1.6) and the bounds (5.3.1a) and (5.3.1c).

Using (2.2.9) with $s = U_{\varepsilon,i}^n(x_j)$ and $r = \phi_\varepsilon^{-1}(\chi(x_j))$, $j = 0, 1, ..., J$ it follows for $i = 1, 2$ that

$$I_{\varepsilon,i}^{h,n}(\chi) \geq \theta \sum_{j=0}^{J} M_{jj} (U_{\varepsilon,i}^n(x_j) - \phi_\varepsilon^{-1}(\chi(x_j)))^2 \geq 0 \quad \forall \chi \in S^h.$$

From the strong convergences (5.3.20a) and (5.3.20c) and the strong convergence $\phi_\varepsilon^{-1}(r) \to \phi^{-1}(r)\ \forall\, r \in \mathbb{R}$ (see Lemma 2.2.1 (ii)) we have

$$\begin{aligned}\left|I_{\varepsilon,i}^{h,n}(\chi) - I_i^{h,n}(\chi)\right| &\leq \left|(U_{\varepsilon,i}^n - U_i^n, \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi)^h\right| + \left|(\phi^{-1}(\chi) - \phi_\varepsilon^{-1}(\chi), \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi)^h\right| \\ &\quad + \left|(U_i^n - \phi^{-1}(\chi), \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi_i^{h,n})^h\right| \\ &\leq |U_{\varepsilon,i}^n - U_i^n|_h |\pi^h \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi|_h + |\phi^{-1}(\chi) - \phi_\varepsilon^{-1}(\chi)|_h |\pi^h \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi|_h \\ &\quad + |U_i^n - \phi^{-1}(\chi)|_h |\pi^h \phi_\varepsilon(U_{\varepsilon,i}^n) - \chi_i^{h,n}|_h \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{5.3.23}$$

We therefore have for $i = 1, 2$ and for any $\chi \in S^h$ that

$$I_i^{h,n}(\chi) = \lim_{\varepsilon \to 0} I_{\varepsilon,i}^{h,n}(\chi) \geq 0. \tag{5.3.24}$$

Now, for any $\beta \in \mathbb{R}_{>0}$ and any $v \in S^h$ we take $\chi = \chi_i^{h,n} \pm \beta v \in S^h$ in (5.3.21) to give, on noting (5.3.24), that for $i = 1, 2$ and $n = 1 \to N$

$$(U_i^n - \phi^{-1}(\chi_i^{h,n} + \beta v), -\beta v)^h \geq 0 \text{ and } (U_i^n - \phi^{-1}(\chi_i^{h,n} - \beta v), \beta v)^h \geq 0,$$

and hence, dividing by $-\beta$ and β respectively,

$$(U_i^n - \phi^{-1}(\chi_i^{h,n} + \beta v), v)^h \leq 0 \text{ and } (U_i^n - \phi^{-1}(\chi_i^{h,n} - \beta v), v)^h \geq 0,$$

which leads by taking the limit as $\beta \to 0$ and noting the continuity of ϕ^{-1} to

$$(U_i^n - \phi^{-1}(\chi_i^{h,n}), v)^h = \sum_{j=0}^{J} M_{jj}[U_i^n(x_j) - \phi^{-1}(\chi_i^{h,n}(x_j))]v(x_j) = 0. \qquad (5.3.25)$$

Then, we choose $v = U_i^n - \pi^h \phi^{-1}(\chi_i^{h,n}) \in S^h$ to yield for $i = 1, 2$ and $n = 1 \to N$ that $U_i^n(x_j) = \phi^{-1}(\chi_i^{h,n}(x_j))$ $j = 0, 1, ..., J$. This result gives directly for $i = 1, 2$ that $\chi_i^{h,n} = \pi^h \phi(U_i^n)$, as required. Further, recalling that $\phi^{-1}(r) \in (-1, 1)$ we deduce for $i = 1, 2$ and $n = 1 \to N$ that

$$U_i^n(x_j) = \phi^{-1}(\chi^{h,n}(x_j)) \in (-1, 1) \quad j = 0, 1, ..., J,$$

which is the desired result (5.3.19e).

In order to pass to the limit in the regularized version $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\mathbf{\Delta t}})$ we first note using the definition of $(\cdot, \cdot)^h$ that the strong convergence (5.3.20a) means

$$U_{\varepsilon,i}^n(x_j) \to U_i^n(x_j) \quad \text{as } \varepsilon \to 0, \; j = 0, 1, ..., J,$$

which implies, as $\varepsilon \to 0$ and for $j = 0, 1, ..., J$,

$$f_D^{(1)}(U_{\varepsilon,1}^n(x_j), U_{\varepsilon,2}^{n-1}(x_j)) \to f_D^{(1)}(U_1^n(x_j), U_2^{n-1}(x_j)),$$
$$f_D^{(2)}(U_{\varepsilon,1}^{n-1}(x_j), U_{\varepsilon,2}^n(x_j)) \to f_D^{(1)}(U_1^{n-1}(x_j), U_2^n(x_j)),$$

i.e., for $i = 1, 2$,

$$\pi^h \bar{f}_{\varepsilon,n,n-1}^{(i)} \to \pi^h \bar{f}_{n,n-1}^{(i)} \quad \text{in } S^h, \qquad (5.3.26)$$

and similarly we have for $i = 1, 2$

$$\pi^h f_D^{(i)}(U_{\varepsilon,1}^n, U_{\varepsilon,2}^n) \to \pi^h f_D^{(i)}(U_1^n, U_2^n) \quad \text{in } S^h. \qquad (5.3.27)$$

Now, from the strong convergences (5.3.20a)-(5.3.20c), (5.3.26) and (5.3.27) one can immediately pass to the limit as $\varepsilon \to 0$ in the regularized version $(\mathbf{P}_{\mu,\varepsilon}^{\mathbf{h},\mathbf{\Delta t}})$ (5.1.7a-b) to find that $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ is a solution to $(\mathbf{P}_{\mu}^{\mathbf{h},\mathbf{\Delta t}})$ at the n-th step. In addition, using the same strong convergences and the fact that all norms on a finite dimensional space are equivalent one can take the limit as $\varepsilon \to 0$ in the estimates (5.3.1a-d) derived in Theorem 5.3.1 to obtain the corresponding desired estimates (5.3.19a-d).

We now finish the proof by showing the uniqueness of the fully-discrete approximation using induction under the above stated conditions on Δt. Since we have uniqueness at time level $t = t_0 = 0$ one can assume uniqueness of the approximation at time level $t = t_{n-1}, n \geq 1$. Now, let $B_h^n = \{U_i^n, W_i^n\}_{i=1,2}$ and $B_h^{n*} = \{U_i^{n*}, W_i^{n*}\}_{i=1,2}$ be two fully-discrete solutions to $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ at time level $t = t_n$. Setting $\chi = \bar{U}_i^n := U_i^n - U_i^{n*} \in V_0^h$ in (5.1.9) and subtracting the approximations yields for $i = 1, 2$ on noting the definition (4.1.12) of $\|\cdot\|_{-h}$ that

$$\gamma|\bar{U}_i^n|_1^2 + (\phi(U_i^n) - \phi(U_i^{n*}), \bar{U}_i^n)^h + \frac{1}{\Delta t}\|\bar{U}_i^n\|_{-h}^2 = \mu\,\theta_i|\bar{U}_i^n|_h^2 + \frac{1}{2}\big(f_D^{(i)}(U_1^{n*}, U_2^{n*}) - f_D^{(i)}(U_1^n, U_2^n), \bar{U}_i^n\big)^h + \frac{1}{2}\big(\bar{f}_{n,n-1}^{(i)*} - \bar{f}_{n,n-1}^{(i)}, \bar{U}_i^n\big)^h, \tag{5.3.28}$$

where $\bar{f}_{n,n-1}^{(i)*} := 2D(U_i^{n*} + \alpha_i)(U_j^{n-1} + \alpha_j)^2$.

Using (2.3.46) with $r_i = U_i^{n*}$ and $s_i = U_i^n$, owing to (5.3.19e) and noting that $\alpha_i \in (-1, 1)$ it follows for $i, j = 1, 2$ with $i \neq j$ that

$$\begin{aligned}\frac{1}{2}(f_D^{(i)}(U_1^{n*}, U_2^{n*}) &- f_D^{(i)}(U_1^n, U_2^n), \bar{U}_i^n)^h \\ &= D\big((U_j^{n*} + \alpha_j)^2(-\bar{U}_i^n) + (U_i^n + \alpha_i)(U_j^n + U_j^{n*} + 2\alpha_j)(-\bar{U}_j^n), \bar{U}_i^n\big)^h \\ &\leq \big((U_i^n + \alpha_i)(U_j^n + U_j^{n*} + 2\alpha_j)(-\bar{U}_j^n), \bar{U}_i^n\big)^h \leq 8D(|\bar{U}_j^n|, |\bar{U}_i^n|)^h.\end{aligned} \tag{5.3.29}$$

We also have from (5.3.19e) for $i, j = 1, 2$ with $i \neq j$ that

$$\begin{aligned}\frac{1}{2}(\bar{f}_{n,n-1}^{(i)*} - \bar{f}_{n,n-1}^{(i)}, \bar{U}_i^n)^h &= D\big((U_i^{n*} + \alpha_i)(U_j^{n-1} + \alpha_j)^2 - (U_i^n + \alpha_i)(U_j^{n-1} + \alpha_j)^2, \bar{U}_i^n\big)^h \\ &= D\big((U_j^{n-1} + \alpha_j)^2(-\bar{U}_i^n), \bar{U}_i^n\big)^h \leq 0.\end{aligned} \tag{5.3.30}$$

Inserting (5.3.29) and (5.3.30) into (5.3.28), using the fact that

$$(\phi(s) - \phi(r))(s - r) \geq \theta(s - r)^2 \quad \forall\, r, s \in (-1, 1)$$

and rearranging we obtain for $i, j = 1, 2$ with $i \neq j$ that

$$\gamma|\bar{U}_i^n|_1^2 + \frac{1}{\Delta t}\|\bar{U}_i^n\|_{-h}^2 \leq (\mu\theta_i - \theta)|\bar{U}_i^n|_h^2 + 8D(|\bar{U}_j^n|, |\bar{U}_i^n|)^h. \tag{5.3.31}$$

We set $\theta_* := \max\{\theta_1, \theta_2\}$ and then sum (5.3.31) over $i = 1, 2$ and use a Young's inequality to yield

$$\gamma\big[|\bar{U}_1^n|_1^2 + |\bar{U}_2^n|_1^2\big] + \frac{1}{\Delta t}\big[\|\bar{U}_1^n\|_{-h}^2 + \|\bar{U}_2^n\|_{-h}^2\big] \\ \leq (\mu\theta_* - \theta)\big[|\bar{U}_1^n|_h^2 + |\bar{U}_2^n|_h^2\big] + 16D(|\bar{U}_1^n|, |\bar{U}_2^n|)^h \\ \leq (8D + \mu\theta_* - \theta)\big[|\bar{U}_1^n|_h^2 + |\bar{U}_2^n|_h^2\big]. \tag{5.3.32}$$

Clearly, if $\theta \geq 8D + \mu\theta_*$ we then have for all $\Delta t > 0$ that

$$\|\bar{U}_1^n\|_{-h}^2 + \|\bar{U}_2^n\|_{-h}^2 \leq 0, \tag{5.3.33}$$

which implies, by (4.1.16), uniqueness of $U_i^n,\ i = 1, 2$, for all $\Delta t > 0$.

If $\theta < 8D + \mu\theta_*$ we can treat the right hand side of (5.3.32) with the aid of (4.1.15) as follows

$$(8D + \mu\theta_* - \theta)\big[|\bar{U}_1^n|_h^2 + |\bar{U}_2^n|_h^2\big] \leq \gamma\big[|\bar{U}_1^n|_1^2 + |\bar{U}_2^n|_1^2\big] \\ + \frac{(8D + \mu\theta_* - \theta)^2}{4\gamma}\big[\|\bar{U}_1^n\|_{-h}^2 + \|\bar{U}_2^n\|_{-h}^2\big]. \tag{5.3.34}$$

Hence, by substituting (5.3.34) into (5.3.32) and simplifying we find that (5.3.33) holds for all $\Delta t < \frac{4\gamma}{(8D+\mu\theta_*-\theta)^2}$. We conclude thus that $U_i^n,\ i = 1, 2$, is unique. Finally, we obtain uniqueness of $W_i^n,\ i = 1, 2$, by (5.1.10) and (5.1.11). □

Remark. For the same reasons discussed earlier in the comment after the proof of Theorem 4.3.2 we find that the results of Theorem 5.3.2 hold for the initial choice $u_i^{h,0} = P_\gamma^h u_i^0,\ i = 1, 2$ (i.e. $U_i^0 = P_\gamma^h u_i^0$) under the assumptions ($\mathbf{A_2}$) and the stated condition in Lemma 4.3.3 on the mesh parameter h. Furthermore, one advantage of this choice is that we shall obtain stronger stability estimates than those derived in Theorem 5.3.2 (see Theorem 5.3.3 below) which will be required to prove an optimal error bound in time for the discretization ($\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}}$).

For the purposes of the analysis we introduce $\{W_1^0, W_2^0\} \in S^h \times S^h$ defined by

$$(W_i^0, \chi)^h = \gamma(\nabla U_i^0, \nabla\chi) + (\phi(U_i^0) - \theta_i U_i^0, \chi)^h + (f_D^{(i)}(U_1^0, U_2^0), \chi)^h \quad i = 1, 2. \tag{5.3.35}$$

Adapting the same argument applied in derivation of (4.3.63) we have for $i = 1, 2$

$$W_i^0 = P^h\big[(I - P_\gamma^h)u_i^0 - \gamma\Delta u_i^0\big] + \pi^h\phi(P_\gamma^h u_i^0) - \theta_i P_\gamma^h u_i^0 + \pi^h f_D^{(i)}(P_\gamma^h u_1^0, P_\gamma^h u_2^0). \quad (5.3.36)$$

Before moving onto next theorem, we state below some properties of the convex part ψ of the potential $\Psi_i,\ i = 1, 2$, and $\phi \equiv \psi'$ which will be needed for the forthcoming analysis

(i) For any $r, s \in (-1, 1)$ we have by Taylor's theorem and the fact that $\phi'(r) \geq \theta > 0$

$$\phi(r)(s - r) \leq \psi(s) - \psi(r). \quad (5.3.37)$$

(ii) For any $\chi \in S^h$ with $|\chi|_{0,\infty} < 1$ we have

$$|\nabla\pi^h\phi(\chi)|_0^2 \leq \phi'(|\chi|_{0,\infty})(\nabla\chi, \nabla\pi^h\phi(\chi)). \quad (5.3.38)$$

To see (ii) we consider an arbitrary $\chi \in S^h$ with $|\chi|_{0,\infty} < 1$. Then, we choose $\varepsilon = \min\{\frac{1}{2}, 1 - |\chi|_{0,\infty}\}$ to obtain that $|\chi|_{0,\infty} \leq 1 - \varepsilon$. Therefore, (5.3.38) is an immediate consequence form Lemma 4.2.1 (ii) and the fact that $\phi_\varepsilon(r) = \phi(r)$ for all $|r| \leq 1 - \varepsilon$.

Theorem 5.3.3 Let the assumptions of Theorem 4.3.4 hold with $u_i^{h,0} = P_\gamma^h u_i^0$. Then for all $\mu \in [0, \frac{1}{2}]$, for all $h \leq h_*$, for all $\Delta t > 0$ and for $n = 1 \to N$ a solution $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ to the n-th step of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ is such that

$$\Delta t \sum_{n=1}^{N} \left[\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_1^2 + \left\|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right\|_1^2\right] + \sum_{n=1}^{N} \left[|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\right]$$
$$+ \max_{n=1\to N} \left[\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_{-h}^2 + \left\|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right] \leq C, \quad (5.3.39a)$$

$$\max_{n=1\to N} \left[\|W_1^n\|_1^2 + \|W_2^n\|_1^2\right] + \max_{n=1\to N} \left[|\pi^h\phi(U_1^n)|_0^2 + |\pi^h\phi(U_2^n)|_0^2\right] \leq C. \quad (5.3.39b)$$

Proof. For future reference we begin the proof with establishing a bound for $|W_1^0|_1$ and $|W_2^0|_1$. It follows from (5.3.36) that

$$|W_i^0|_1 \leq \big|P^h\big[(I - P_\gamma^h)u_i^0 - \gamma\Delta u_i^0\big] - \theta_i P_\gamma^h u_i^0\big|_1 + |\pi^h\phi(P_\gamma^h u_i^0)|_1 + |\pi^h f_D^{(i)}(P_\gamma^h u_1^0, P_\gamma^h u_2^0)|_1. \quad (5.3.40)$$

The first and the third terms on on the right hand side of (5.3.40) are bounded by (4.3.65) and (4.3.67)-(4.3.71). Whereas the second term can be treated using

(5.3.38), Lemma 4.3.3, (2.2.13) and (4.1.24) to yield for all $h \le h_*$ and for $i = 1, 2$ that

$$|\pi^h \phi(P^h_\gamma u^0_i)|_1 \le \phi'(|P^h_\gamma u^0_i|_{0,\infty})|P^h_\gamma u^0_i|_1 \le C\phi'(1 - \frac{\delta_0}{2})\|u^0_i\|_1 \le C. \tag{5.3.41}$$

Thus we conclude that

$$|W^0_i|_1 \le C \quad i = 1, 2. \tag{5.3.42}$$

For fixed $n \ge 1$ we subtract (5.1.6b) at the step $n-1$ from (5.1.6b) at the subsequent step n and rearrange to have on noting (5.3.35) with step $n = 1$ that for $i = 1, 2$

$$\begin{aligned}(W^n_i - W^{n-1}_i, \chi)^h = {} & \gamma(\nabla U^n_i - U^{n-1}_i, \nabla\chi) + (\phi(U^n_i) - \phi(U^{n-1}_i), \chi)^h \\ & - \mu\theta_i(U^n_i - U^{n-1}_i, \chi)^h + \frac{1}{2}\big(f^{(i)}_D(U^n_1, U^n_2) - f^{(i)}_D(U^{n-1}_1, U^{n-1}_2), \chi\big)^h \\ & + \begin{cases} \frac{1}{2}\big(\bar{f}^{(i)}_{1,0} - f^{(i)}_D(U^0_1, U^0_2), \chi\big)^h & n = 1, \\ -(1-\mu)\theta_i(U^{n-1}_i - U^{n-2}_i, \chi)^h + \frac{1}{2}(\bar{f}^{(i)}_{n,n-1} - \bar{f}^{(i)}_{n-1,n-2}, \chi)^h & n \ge 2. \end{cases}\end{aligned} \tag{5.3.43}$$

Testing (5.1.6a) with $\chi = W^n_i - W^{n-1}_i$ and noting the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ yields for $i = 1, 2$ and $n = 1 \to N$

$$\begin{aligned}\Big(\frac{U^n_i - U^{n-1}_i}{\Delta t}, W^n_i - W^{n-1}_i\Big)^h &= -(\nabla W^n_i, \nabla W^n_i - W^{n-1}_i) \\ &= -\frac{1}{2}|W^n_i|^2_1 + \frac{1}{2}|W^{n-1}_i|^2_1 - \frac{1}{2}|W^n_i - W^{n-1}_i|^2_1.\end{aligned} \tag{5.3.44}$$

For convenience we continue the proof using the following notation

$$Z^n_i := \frac{U^n_i - U^{n-1}_i}{\Delta t} \quad i = 1, 2,\ n = 1 \to N. \tag{5.3.45}$$

Now, for $i = 1, 2$ we take $\chi = Z^n_i$ in (5.3.43) and use (5.3.44) which gives upon rearranging for $n = 1 \to N$ that

$$\begin{aligned}&\gamma\Delta t|Z^n_i|^2_1 + (\phi(U^n_i) - \phi(U^{n-1}_i), Z^n_i)^h + \frac{1}{2}|W^n_i|^2_1 + \frac{1}{2}|W^n_i - W^{n-1}_i|^2_1 \\ &\quad = \frac{1}{2}|W^{n-1}_i|^2_1 + \mu\theta_i\Delta t|Z^n_i|^2_h - \frac{1}{2}\big(f^{(i)}_D(U^n_1, U^n_2) - f^{(i)}_D(U^{n-1}_1, U^{n-1}_2), Z^n_i\big)^h \\ &\quad + \begin{cases} -\frac{1}{2}\big(\bar{f}^{(i)}_{1,0} - f^{(i)}_D(U^0_1, U^0_2), Z^1_i\big)^h & n = 1, \\ (1-\mu)\theta_i\Delta t(Z^{n-1}_i, Z^n_i)^h - \frac{1}{2}(\bar{f}^{(i)}_{n,n-1} - \bar{f}^{(i)}_{n-1,n-2}, Z^n_i)^h & n \ge 2. \end{cases}\end{aligned} \tag{5.3.46}$$

Next we estimate the terms on the right hand side of (5.3.46). From (4.1.15) it follows for $i = 1, 2$ and $n = 1 \to N$ that

$$\mu\theta_i\Delta t|Z_i^n|_h^2 \leq \frac{\gamma\Delta t}{8}|Z_i^n|_1^2 + C\Delta t\|Z_i^n\|_{-h}^2. \tag{5.3.47}$$

Setting $r_i = U_i^n$, $s_i = U_i^{n-1}$, $i = 1, 2$, in (2.3.46), using the result (5.3.19e) and noting $|\alpha_i| < 1$ we find for $i, j = 1, 2$ with $i \neq j$ and $n = 1 \to N$ that

$$\begin{aligned}
&\left|\frac{1}{2}\big(f_D^{(i)}(U_1^n, U_2^n) - f_D^{(i)}(U_1^{n-1}, U_2^{n-1}), Z_i^n\big)^h\right| \\
&\quad = D\left|\big((U_j^n + \alpha_j)^2(U_i^n - U_i^{n-1}) + (U_i^{n-1} + \alpha_i)(U_j^n + U_j^{n-1} + 2\alpha_j)(U_j^n - U_j^{n-1}), Z_i^n\big)^h\right| \\
&\quad = D\Delta t\left|\big((U_j^n + \alpha_j)^2 Z_i^n + (U_i^{n-1} + \alpha_i)(U_j^n + U_j^{n-1} + 2\alpha_j) Z_j^n, Z_i^n\big)^h\right| \\
&\quad \leq 4D\Delta t|Z_i^n|_h^2 + 8D\Delta t(|Z_j^n|, |Z_i^n|)^h \\
&\quad \leq 8D\Delta t|Z_i^n|_h^2 + 4D\Delta t|Z_j^n|_h^2 \\
&\quad \leq \frac{\gamma\Delta t}{8}\big[|Z_i^n|_1^2 + |Z_j^n|_1^2\big] + C\Delta t\big[\|Z_i^n\|_{-h}^2 + \|Z_j^n\|_{-h}^2\big],
\end{aligned} \tag{5.3.48}$$

where we have also noted a Young's inequality and (4.1.15).

Again (5.3.19e) and (4.1.15) show for $i, j = 1, 2$ with $i \neq j$ that

$$\begin{aligned}
\frac{1}{2}&\left|\big(\bar{f}_{1,0}^{(i)} - f_D^{(i)}(U_1^0, U_2^0), Z_i^1\big)^h\right| \\
&= D\left|\big((U_i^1 + \alpha_i)(U_j^0 + \alpha_j)^2 - (U_i^0 + \alpha_i)(U_j^0 + \alpha_j)^2, Z_i^1\big)^h\right| \\
&= D\Delta t\left|\big((U_j^0 + \alpha_j)^2 Z_i^1, Z_i^1\big)\right| \\
&\leq 4D\Delta t|Z_i^1|_h^2 \\
&\leq \frac{\gamma\Delta t}{4}|Z_i^1|_1^2 + C\Delta t\|Z_i^1\|_{-h}^2.
\end{aligned} \tag{5.3.49}$$

For $n \geq 2$ we have by (4.1.14) that

$$(1-\mu)\theta_i\Delta t(Z_i^{n-1}, Z_i^n)^h \leq \frac{\gamma\Delta t}{8}|Z_i^n|_1^2 + C\Delta t\|Z_i^{n-1}\|_{-h}^2. \tag{5.3.50}$$

Subtracting and adding $(U_i^{n-1} + \alpha_i)(U_j^{n-1} + \alpha_j)^2$, $i, j = 1, 2$ with $i \neq j$, noting (5.3.19e), recalling that $|\alpha_i| < 1$ and using a Young's inequality and (4.1.15) we have for $n \geq 2$ that

$$\begin{aligned}
&\left|\frac{1}{2}(\bar{f}^{(i)}_{n,n-1} - \bar{f}^{(i)}_{n-1,n-2}, Z_i^n)^h\right| \\
&= D\left|\big((U_i^n + \alpha_i)(U_j^{n-1} + \alpha_j)^2 - (U_i^{n-1} + \alpha_i)(U_j^{n-2} + \alpha_j)^2, Z_i^n\big)^h\right| \\
&= D\left|\big((U_j^{n-1} + \alpha_j)^2(U_i^n - U_i^{n-1}) + (U_i^{n-1} + \alpha_i)(U_j^{n-1} + U_j^{n-2} + 2\alpha_j)(U_j^{n-1} - U_j^{n-2}), Z_i^n\big)^h\right| \\
&= D\Delta t\left|\big((U_j^{n-1} + \alpha_j)^2 Z_i^n + (U_i^{n-1} + \alpha_i)(U_j^{n-1} + U_j^{n-2} + 2\alpha_j)Z_j^{n-1}, Z_i^n\big)^h\right| \\
&\le 4D\Delta t|Z_i^n|_h^2 + 8D\Delta t(|Z_j^{n-1}|, |Z_i^n|)^h \\
&\le 8D\Delta t|Z_i^n|_h^2 + 4D\Delta t|Z_j^{n-1}|_h^2 \\
&\le \frac{\gamma\Delta t}{8}\big[|Z_i^n|_1^2 + |Z_j^{n-1}|_1^2\big] + C\Delta t\big[\|Z_i^n\|_{-h}^2 + \|Z_j^{n-1}\|_{-h}^2\big]. \qquad (5.3.51)
\end{aligned}$$

Combining (5.3.47)-(5.3.51) with (5.3.46) and noting the monotonicity of ϕ yields for $i, j = 1, 2$ with $i \neq j$ and $n = 1 \to N$ that

$$\begin{aligned}
\gamma\Delta t|Z_i^n|_1^2 + \frac{1}{2}|W_i^n|_1^2 + \frac{1}{2}|W_i^n - W_i^{n-1}|_1^2& \\
\le \frac{1}{2}|W_i^{n-1}|_1^2 + \frac{\gamma\Delta t}{2}|Z_i^n|_1^2 + \frac{\gamma\Delta t}{8}|Z_j^n|_1^2 &+ C\Delta t\big[\|Z_i^n\|_{-h}^2 + \|Z_j^n\|_{-h}^2\big] \\
&+ \begin{cases} 0 & n = 1, \\ \frac{\gamma\Delta t}{8}|Z_j^{n-1}|_1^2 + C\Delta t\big[\|Z_i^{n-1}\|_{-h}^2 + \|Z_j^{n-1}\|_{-h}^2\big] & n \ge 2. \end{cases} \qquad (5.3.52)
\end{aligned}$$

We now aim to prove for $m = 1 \to N$

$$\begin{aligned}
\frac{\gamma\Delta t}{4}\sum_{n=1}^{m}\big[|Z_1^n|_1^2 + |Z_2^n|_1^2\big] + \frac{1}{2}\big[|W_1^m|_1^2 + |W_2^m|_1^2\big]& \\
+ \frac{1}{2}\sum_{n=1}^{m}\big[|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\big] &\le C \qquad (5.3.53)
\end{aligned}$$

For $n = 1$ we sum (5.3.52) over $i = 1, 2$ and use the bounds (5.3.42) and (5.3.19b) to result in after simplifying that

$$\begin{aligned}
\frac{3\gamma\Delta t}{8}\big[|Z_1^1|_1^2 + |Z_2^1|_1^2\big] + \frac{1}{2}\big[|W_1^1|_1^2 + |W_2^1|_1^2\big] + \frac{1}{2}\big[|W_1^1 - W_1^0|_1^2 + |W_2^1 - W_2^0|_1^2\big]& \\
\le \frac{1}{2}\big[|W_1^0|_1^2 + |W_2^0|_1^2\big] + C\Delta t\big[\|Z_1^1\|_{-h}^2 + \|Z_2^1\|_{-h}^2\big] &\le C, \qquad (5.3.54)
\end{aligned}$$

which is sufficient to prove (5.3.53) with $m = 1$.

For $n \geq 2$ we sum (5.3.52) over $i = 1, 2$ and simplify to obtain

$$\frac{3\gamma\Delta t}{8}\big[|Z_1^n|_1^2 + |Z_2^n|_1^2\big] + \frac{1}{2}\big[|W_1^n|_1^2 + |W_2^n|_1^2\big] + \frac{1}{2}\big[|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\big]$$
$$\leq \frac{1}{2}\big[|W_1^{n-1}|_1^2 + |W_2^{n-1}|_1^2\big] + \frac{\gamma\Delta t}{8}\big[|Z_1^{n-1}|_1^2 + |Z_2^{n-1}|_1^2\big]$$
$$+ C\Delta t\big[\|Z_1^n\|_{-h}^2 + \|Z_2^n\|_{-h}^2 + \|Z_1^{n-1}\|_{-h}^2 + \|Z_2^{n-1}\|_{-h}^2\big]. \tag{5.3.55}$$

Noting first for all $m = 2 \to N$ and $i = 1, 2$ that

$$\frac{3\gamma\Delta t}{8}\sum_{n=2}^{m}|Z_i^n|_1^2 - \frac{\gamma\Delta t}{8}\sum_{n=2}^{m}|Z_i^{n-1}|_1^2 = \frac{\gamma\Delta t}{4}\sum_{n=2}^{m}|Z_i^n|_1^2 + \frac{\gamma\Delta t}{8}|Z_i^m|_1^2 - \frac{\gamma\Delta t}{8}|Z_i^1|_1^2$$
$$\geq \frac{\gamma\Delta t}{4}\sum_{n=2}^{m}|Z_i^n|_1^2 - \frac{\gamma\Delta t}{8}|Z_i^1|_1^2, \tag{5.3.56}$$

and then by summing (5.3.55) from $n = 2 \to m$ we have after rearranging for all $2 \leq m \leq N$ that

$$\frac{\gamma\Delta t}{4}\sum_{n=2}^{m}\big[|Z_1^n|_1^2 + |Z_2^n|_1^2\big] + \frac{1}{2}\big[|W_1^m|_1^2 + |W_2^m|_1^2\big]$$
$$+ \frac{1}{2}\sum_{n=2}^{m}\big[|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\big]$$
$$\leq \frac{1}{2}\big[|W_1^1|_1^2 + |W_2^1|_1^2\big] + \frac{\gamma\Delta t}{8}\big[|Z_1^1|_1^2 + |Z_2^1|_1^2\big]$$
$$+ C\Delta t\sum_{n=2}^{m}\big[\|Z_1^n\|_{-h}^2 + \|Z_2^n\|_{-h}^2 + \|Z_1^{n-1}\|_{-h}^2 + \|Z_2^{n-1}\|_{-h}^2\big]$$
$$\leq C, \tag{5.3.57}$$

where we have noted (5.3.54) and (5.3.19b) to obtain the last inequality.

Finally, adding (5.3.54) with $m = 1$ to (5.3.57) $\forall m \geq 2$ yields (5.3.53) . Therefore, the second estimate in (5.3.39a) follows immediately from (5.3.53). The first estimate in (5.3.39a) follows also from (5.3.53) with the aid of the Poincaré inequality. In addition, (5.3.53) together with the fact that $|W_i^n|_1 = \|\frac{U_i^n - U_i^{n-1}}{\Delta t}\|_{-h}$, see (5.3.6), shows the third estimate in (5.3.39a).

To obtain the first estimate in (5.3.39b) we first note from Poincré's inequality and (5.3.53) that for $i = 1, 2$ and $n = 1 \to N$

$$\Big\| W_i^n - \fint W_i^n \Big\|_1^2 \leq C \Big| W_i^n - \fint W_i^n \Big|_1^2 = C |W_i^n|_1^2 \leq C. \tag{5.3.58}$$

Repeating the same technique used in Theorem 5.3.1 for deriving the inequality (5.3.9) where this time we use (5.3.37), the bound (5.3.19a) and that $|\psi(r)| \leq \psi(1) = \theta \ln 2 \;\forall r \in [-1, 1]$ in place of (2.2.6), the bound (5.3.1a) and (5.3.8) respectively to conclude for any $\beta \in (-1, 1)$ that

$$\begin{aligned} \big(\phi(U_i^n) + \frac{1}{2}(f_D^{(i)}(U_1^n, U_2^n) + \bar{f}_{n,n-1}^{(i)}), \beta - m_i\big)^h &= \\ &\leq C\big[1 + |W_i^n|_1 + (\psi(\beta), 1)^h + |f_D^{(i)}(U_1^n, U_2^n) + \bar{f}_{n,n-1}^{(i)}|_h \, |\beta - U_i^n|_h\big] \\ &\leq C, \end{aligned} \tag{5.3.59}$$

where to obtain the last inequality we note the equivalence result (4.1.6) and the bounds (5.3.53), (5.3.19a) and (5.3.19d).

Thus, on choosing $\beta = \pm 1 \mp \frac{\delta_0}{2}$ we conclude, similarly to (5.3.14), for $i = 1, 2$ and $n = 1 \to N$ that

$$\Big| \fint \big[\pi^h \phi(U_i^n) + \frac{1}{2}(\pi^h f_D^{(i)}(U_1^n, U_2^n) + \pi^h \bar{f}_{n,n-1}^{(i)})\big] \Big| \leq C. \tag{5.3.60}$$

We therefore have by (5.1.11) that for $i = 1, 2$ and $n = 1 \to N$

$$\begin{aligned} \Big\| \fint W_i^n \Big\|_1^2 = |\Omega| \Big| \fint W_i^n \Big|^2 &\leq 2|\Omega| \Big| \fint \big[\pi^h \phi(U_i^n) + \frac{1}{2}(\pi^h f_D^{(i)}(U_1^n, U_2^n) + \pi^h \bar{f}_{n,n-1}^{(i)})\big] \Big|^2 \\ &+ 2|\Omega| m_i^2 \theta_i^2 \leq C, \end{aligned} \tag{5.3.61}$$

from which together with (5.3.58) we deduce the first estimate in (5.3.39b).

Finally, for $i = 1, 2$ and $n = 1 \to N$ we take $\chi = \pi^h \phi(U_i^n)$ in the unregularized version (5.1.7b) to obtain the following analogue of (5.3.17)

$$\begin{aligned} \gamma(\nabla U_i^n, \nabla \pi^h \phi(U_i^n)) + |\pi^h \phi(U_i^n)|_h^2 &\leq \frac{1}{2} |\pi^h \phi(U_i^n)|_h^2 + C\Big[|W_i^n|_h^2 + |U_i^n|_h^2 + |U_i^{n-1}|_h^2 \\ &+ |\pi^h f_D^{(i)}(U_1^n, U_2^n)|_h^2 + |\pi^h \bar{f}_{n,n-1}^{(i)}|_h^2 \Big]. \end{aligned} \tag{5.3.62}$$

With the aid of (5.3.38), the first bound in (5.3.39b) and the bounds (5.3.19a) and (5.3.19d) this shows the second bound in (5.3.39b). □

5.4 A fully-discrete error bound

In this section we prove an error bound between the solutions of the continuous problem ($\mathbf{P}$) and the symmetric coupled fully-discrete problem ($\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}}$). This error bound is derived via the error bound between ($\mathbf{P^h}$) and ($\mathbf{P}$) derived in the previous chapter (see Theorem 4.4.3), and an error bound between ($\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}}$) and ($\mathbf{P^h}$). In fact, by applying the framework in Nochetto [50] for analysing the discretization error in the backward Euler method, we shall prove an optimal error bound in time between ($\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}}$) and ($\mathbf{P^h}$). For the error bound analysis we first consider the following definitions:

$$\ell(t) := \frac{t_n - t}{\Delta t}, \quad t \in (t_{n-1}, t_n],\ n \geq 1. \tag{5.4.1}$$

For $i = 1, 2$ we also define

$$\begin{aligned} U_i(t) &:= \left(\frac{t - t_{n-1}}{\Delta t}\right) U_i^n + \left(\frac{t_n - t}{\Delta t}\right) U_i^{n-1} \\ &= (1 - \ell(t))\, U_i^n + \ell(t)\, U_i^{n-1}, \quad t \in [t_{n-1}, t_n],\ n \geq 1 \end{aligned} \tag{5.4.2}$$

and

$$U_i^+(t) := U_i^n, \qquad U_i^-(t) := U_i^{n-1}, \quad t \in (t_{n-1}, t_n],\ n \geq 1. \tag{5.4.3}$$

Using the above definitions one can easily see for $i = 1, 2$ that

$$\partial_t U_i = \frac{U_i^+ - U_i^-}{\Delta t} = \frac{U_i - U_i^+}{-\ell\, \Delta t} = \frac{U_i - U_i^-}{(1-\ell)\Delta t},\ t \in (t_{n-1}, t_n),\ n \geq 1. \tag{5.4.4}$$

The Nochetto's method is based on exploiting the convex part of the potential and defining quantities satisfying some properties (see Lemma 5.4.1 below). This will lead us to a differential error inequality (see Lemma 5.4.2 below) from which the time discretisation error can be bounded by non-negative quantities with optimal order as will be proved in Theorem 5.4.3.

For notational convenience we also introduce the subspace $S^h_{[-1,1]} \subset S^h$

$$S^h_{[-1,1]} := \{\chi \in S^h : |\chi| \leq 1\}. \tag{5.4.5}$$

Let $\bar{J}^h : S^h_{[-1,1]} \times S^h_{[-1,1]} \to \mathbb{R}$ be defined by

$$\bar{J}^h(\chi_1, \chi_2) := \frac{\gamma}{2}\left[|\chi_1|_1^2 + |\chi_2|_1^2\right] + (\psi(\chi_1), 1)^h + (\psi(\chi_2), 1)^h. \tag{5.4.6}$$

Note for $i = 1, 2$ that ψ is the convex part of the free energy Ψ_i as $\psi''(r) > 0$ for all $r \in (-1, 1)$.

We introduce for *a.e.* $t \in (0, T)$

$$\begin{aligned}\mathcal{R}(t) := \Big[\Big(&-\mu\theta_1 U_1^+ - (1-\mu)\theta_1 U_1^- + \frac{1}{2}\big(f_D^{(1)}(U_1^+, U_2^+) + f_D^{(1)}(U_1^+, U_2^-)\big),\, U_1 - U_1^+\Big)^h \\ &+ \Big(\hat{\mathcal{G}}^h \partial_t U_1,\, U_1 - U_1^+\Big)^h\Big] \\ &+ \Big[\Big(-\mu\theta_2 U_2^+ - (1-\mu)\theta_2 U_2^- + \frac{1}{2}\big(f_D^{(2)}(U_1^+, U_2^+) + f_D^{(2)}(U_1^-, U_2^+)\big),\, U_2 - U_2^+\Big)^h \\ &+ \Big(\hat{\mathcal{G}}^h \partial_t U_2,\, U_2 - U_2^+\Big)^h\Big] \\ &+ \big[\bar{J}^h(U_1, U_2) - \bar{J}^h(U_1^+, U_2^+)\big]. \qquad (5.4.7)\end{aligned}$$

For $n \geq 1$ we define

$$\begin{aligned}\mathcal{E}^n := \Big[\Big(&\mu\theta_1 U_1^n + (1-\mu)\theta_1 U_1^{n-1} - \frac{1}{2}\big(f_D^{(1)}(U_1^n, U_2^n) + f_D^{(1)}(U_1^n, U_2^{n-1})\big),\, \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\ &- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^n - U_1^{n-1}}{\Delta t}\Big),\, \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h\Big] \\ &+ \Big[\Big(\mu\theta_2 U_2^n + (1-\mu)\theta_2 U_2^{n-1} - \frac{1}{2}\big(f_D^{(2)}(U_1^n, U_2^n) + f_D^{(2)}(U_1^{n-1}, U_2^n)\big),\, \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h \\ &- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^n - U_2^{n-1}}{\Delta t}\Big),\, \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big] \\ &- \frac{1}{\Delta t}\big[\bar{J}^h(U_1^n, U_2^n) - \bar{J}^h(U_1^{n-1}, U_2^{n-1})\big]. \qquad (5.4.8)\end{aligned}$$

For theoretical purposes we introduce $\{U_1^{-1}, U_2^{-1}\} \in S_{m_1}^h \times S_{m_2}^h$ such that for $i = 1, 2$ and for all $\chi \in S^h$

$$\begin{aligned}\gamma(\nabla U_i^0, \nabla \chi) &+ \big(\phi(U_i^0) - \mu\theta_i U_i^0 - (1-\mu)\theta_i U_i^{-1}, \chi - {\textstyle\fint} \chi\big)^h \\ &+ \big(f_D^{(i)}(U_1^0, U_2^0), \chi - {\textstyle\fint} \chi\big)^h + \Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^0 - U_i^{-1}}{\Delta t}\Big), \chi\Big)^h = 0. \qquad (5.4.9)\end{aligned}$$

Now we establish existence and uniqueness of U_1^{-1} and U_2^{-1}. The existence can be proved by considering the following minimization problem

$$\begin{aligned}\min_{\{\chi_1, \chi_2\} \in S_{m_1}^h \times S_{m_2}^h} \Big\{I^h(\chi_1, \chi_2) := &\frac{1}{2\Delta t}\|\chi_1 - U_1^0\|_{-h}^2 + \frac{1}{2\Delta t}\|\chi_2 - U_2^0\|_{-h}^2 \\ &+ \frac{(1-\mu)}{2}\big[\theta_1|\chi_1|_h^2 + \theta_2|\chi_2|_h^2\big] - (g_1^0, \chi_1)^h - (g_2^0, \chi_2)^h\Big\}, \qquad (5.4.10)\end{aligned}$$

where g_1^0 and g_2^0 are given such that for all $\chi \in S^h$ and $i = 1, 2$

$$(g_i^0, \chi)^h = \gamma(\nabla U_i^0, \nabla \chi) + (\phi(U_i^0) - \mu\theta_i U_i^0 + f_D^{(i)}(U_1^0, U_2^0), \chi - \fint \chi)^h. \tag{5.4.11}$$

Using Young's inequality it follows for $i = 1, 2$ that

$$I^h(\chi_1, \chi_2) \geq \frac{(1-\mu)\theta_1}{4}|\chi_1|_h^2 + \frac{(1-\mu)\theta_2}{4}|\chi_2|_h^2 - \frac{1}{(1-\mu)\theta_1}|g_1^0|_h^2 - \frac{1}{(1-\mu)\theta_2}|g_2^0|_h^2. \tag{5.4.12}$$

Thus, I^h is bounded below in $S_{m_1}^h \times S_{m_2}^h$. Let $\rho = \inf_{S_{m_1}^h \times S_{m_2}^h} I^h(\chi_1, \chi_2)$ and $\{\chi_{1,n}, \chi_{2,n}\}$ be a minimizing sequence of I^h in $S_{m_1}^h \times S_{m_2}^h$ (*i.e.* $\lim_{n\to\infty} I^h(\chi_{1,n}, \chi_{2,n}) = \rho$). From the estimate (5.4.12) it follows that $\{\chi_{1,n}\}$ and $\{\chi_{2,n}\}$ are bounded in S^h and hence we can extract subsequences $\{\chi_{1,n}\}$ and $\{\chi_{2,n}\}$ such that

$$\chi_{1,n} \to U_1^{-1} \in S^h, \quad \chi_{2,n} \to U_2^{-1} \in S^h.$$

Since $S_{m_1}^h \times S_{m_2}^h$ is a closed set, we have $\{U_1^{-1}, U_2^{-1}\} \in S_{m_1}^h \times S_{m_2}^h$. By the continuity of I^h we conclude thus that of I^h we conclude thus that

$$I^h(\chi_{1,n}, \chi_{2,n}) \to I^h(U_1^{-1}, U_2^{-1}) \equiv \rho.$$

Therefore, we have that $\{U_1^{-1}, U_2^{-1}\}$ is a solution of the minimization problem (5.4.10). Now we can easily see for $i = 1, 2$ that (5.4.9) is the Euler-Lagrange equations of the minimization problem.

To prove the uniqueness, we assume that $X^{-1} := \{U_1^{-1}, U_2^{-1}\}$ and $X^{-1*} := \{U_1^{-1*}, U_2^{-1*}\}$ are two solutions of (5.4.9) and define $\bar{U}_i^{-1} := U_i^{-1} - U_i^{-1*} \in V_0^h$, $i = 1, 2$. By (5.4.9) we find for $i = 1, 2$ and for all $\chi \in S^h$ that

$$-\theta_i(1-\mu)(\bar{U}_i^{-1}, \chi - \fint \chi)^h - \frac{1}{\Delta t}\left(\hat{\mathcal{G}}^h \bar{U}_i^{-1}, \chi\right)^h = 0. \tag{5.4.13}$$

Choosing $\chi = \bar{U}_i^{-1}$ in (5.4.13) yields for $i = 1, 2$ that

$$\theta_i(1-\mu)|\bar{U}_i^{-1}|_h^2 + \frac{1}{\Delta t}\|\bar{U}_i^{-1}\|_{-h}^2 = 0, \tag{5.4.14}$$

which implies for $i = 1, 2$ that $\bar{U}_i^{-1} \equiv 0$ and therefore the uniqueness result.

Finally, we introduce

$$\begin{aligned}
\mathcal{D}^1 := \Big[&\Big(\mu\theta_1 U_1^1 + (1-\mu)\theta_1 U_1^0 - \frac{1}{2}\big(f_D^{(1)}(U_1^1, U_2^1) + f_D^{(1)}(U_1^1, U_2^0)\big),\ \frac{U_1^1 - U_1^0}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^1 - U_1^0}{\Delta t}\Big),\ \frac{U_1^1 - U_1^0}{\Delta t}\Big)^h\Big] \\
+ \Big[&\Big(\mu\theta_2 U_2^1 + (1-\mu)\theta_2 U_2^0 - \frac{1}{2}\big(f_D^{(2)}(U_1^1, U_2^1) + f_D^{(2)}(U_1^0, U_2^1)\big),\ \frac{U_2^1 - U_2^0}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^1 - U_2^0}{\Delta t}\Big),\ \frac{U_2^1 - U_2^0}{\Delta t}\Big)^h\Big] \\
- \Big[&\Big(\mu\theta_1 U_1^0 + (1-\mu)\theta_1 U_1^{-1} - f_D^{(1)}(U_1^0, U_2^0),\ \frac{U_1^1 - U_1^0}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^0 - U_1^{-1}}{\Delta t}\Big),\ \frac{U_1^1 - U_1^0}{\Delta t}\Big)^h\Big] \\
- \Big[&\Big(\mu\theta_2 U_2^0 + (1-\mu)\theta_2 U_2^{-1} - f_D^{(2)}(U_1^0, U_2^0),\ \frac{U_2^1 - U_2^0}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^0 - U_2^{-1}}{\Delta t}\Big),\ \frac{U_2^1 - U_2^0}{\Delta t}\Big)^h\Big],
\end{aligned} \tag{5.4.15}$$

and for $n \geq 2$

$$\begin{aligned}
\mathcal{D}^n := \Big[&\Big(\mu\theta_1 U_1^n + (1-\mu)\theta_1 U_1^{n-1} - \frac{1}{2}\big(f_D^{(1)}(U_1^n, U_2^n) + f_D^{(1)}(U_1^n, U_2^{n-1})\big),\ \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^n - U_1^{n-1}}{\Delta t}\Big),\ \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h\Big] \\
+\Big[&\Big(\mu\theta_2 U_2^n + (1-\mu)\theta_2 U_2^{n-1} - \frac{1}{2}\big(f_D^{(2)}(U_1^n, U_2^n) + f_D^{(2)}(U_1^{n-1}, U_2^n)\big),\ \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^n - U_2^{n-1}}{\Delta t}\Big),\ \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big] \\
-\Big[&\Big(\mu\theta_1 U_1^{n-1} + (1-\mu)\theta_1 U_1^{n-2} - \frac{1}{2}\big(f_D^{(1)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(1)}(U_1^{n-1}, U_2^{n-2})\big),\ \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\Big),\ \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h\Big] \\
-\Big[&\Big(\mu\theta_2 U_2^{n-1} + (1-\mu)\theta_2 U_2^{n-2} - \frac{1}{2}\big(f_D^{(2)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(2)}(U_1^{n-2}, U_2^{n-1})\big),\ \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\Big),\ \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big].
\end{aligned} \tag{5.4.16}$$

In the next lemma we prove some essential results concerning the quantities $\mathcal{R}$, $\mathcal{E}^n$ and $\mathcal{D}^n$, defined in (5.4.7), (5.4.8) and (5.4.15-16) respectively.

Lemma 5.4.1 Let the assumptions of Theorem 5.3.3 hold. Then, for $n \geq 1$, $\mathcal{E}^n$ and $\mathcal{D}^n$ satisfy that

$$\begin{aligned} 0 \leq \frac{\gamma}{2}\big[|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\big] &\leq \Delta t \mathcal{E}^n \\ &\leq \Delta t \mathcal{D}^n - \frac{\gamma}{2}\big[|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\big] \\ &\leq \Delta t \mathcal{D}^n. \end{aligned} \tag{5.4.17}$$

For $t \in (t_{n-1}, t_n]$ and $n = 1 \to N$ define

$$\mathcal{E}(t) := \mathcal{E}^n, \qquad \mathcal{D}(t) := \mathcal{D}^n, \tag{5.4.18}$$

then for *a.e.* $t \in (0, T)$ we have that

$$\mathcal{R}(t) \leq \ell(t)\, \Delta t\, \mathcal{E}(t) \leq \ell(t)\, \Delta t\, \mathcal{D}(t). \tag{5.4.19}$$

Furthermore, we have that

$$\sum_{n=1}^{N} \mathcal{E}^n \leq \sum_{n=1}^{N} \mathcal{D}^n \leq C. \tag{5.4.20}$$

Proof. We test (5.1.9) with $\chi = U_i^n - U_i^{n-1} \in V_0^h$ and use the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ to result in for $i = 1, 2$

$$\begin{aligned} &\frac{\gamma}{2}|U_i^n|_1^2 - \frac{\gamma}{2}|U_i^{n-1}|_1^2 + \frac{\gamma}{2}\big[|U_1^n - U_1^{n-1}|_1^2 + (\phi(U_i^n), U_i^n - U_i^{n-1})^h \\ &\quad + \big(-\mu\theta_i U_i^n - (1-\mu)\theta_i U_i^{n-1} + \frac{1}{2}(f_D^{(i)}(U_1^n, U_2^n) + \bar{f}_{n,n-1}^{(i)}), U_i^n - U_i^{n-1}\big)^h \\ &\quad + \Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big), U_i^n - U_i^{n-1}\Big)^h = 0. \end{aligned} \tag{5.4.21}$$

With the aid of (5.3.37) we obtain after rearranging for $i = 1, 2$ that

$$\begin{aligned} \frac{\gamma}{2}\big[|U_i^n - U_i^{n-1}|_1^2 &\leq \frac{\gamma}{2}|U_i^{n-1}|_1^2 - \frac{\gamma}{2}|U_i^n|_1^2 + (\psi(U_i^{n-1}), 1)^h - (\psi(U_i^n), 1)^h \\ &\quad + \big(\mu\theta_i U_i^n + (1-\mu)\theta_i U_i^{n-1} - \frac{1}{2}(f_D^{(i)}(U_1^n, U_2^n) + \bar{f}_{n,n-1}^{(i)}), U_i^n - U_i^{n-1}\big)^h \\ &\quad - \Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big), U_i^n - U_i^{n-1}\Big)^h. \end{aligned} \tag{5.4.22}$$

Then, by summing (5.4.22) over $i = 1, 2$, recalling that $\bar{f}_{n,n-1}^{(1)} = f_D^{(1)}(U_1^n, U_2^{n-1})$ and $\bar{f}_{n,n-1}^{(2)} = f_D^{(2)}(U_1^{n-1}, U_2^n)$ and owing to the definitions (5.4.6) and (5.4.8) of $\bar{J}^h$ and $\mathcal{E}^n$ we conclude for $n = 1 \to N$ that

$$\frac{\gamma}{2}\big[|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\big] \leq \Delta t \mathcal{E}^n, \tag{5.4.23}$$

which is the first inequality in (5.4.17).

To see the second inequality in (5.4.17) we first note from (5.4.8) and (5.4.16) that for $n = 2 \to N$

$$\begin{aligned}
\mathcal{E}^n - \mathcal{D}^n = &-\frac{1}{\Delta t}\big[\bar{J}^h(U_1^n, U_2^n) - \bar{J}^h(U_1^{n-1}, U_2^{n-1})\big] \\
&+\Big(\mu\theta_1 U_1^{n-1} + (1-\mu)\theta_1 U_1^{n-2} - \frac{1}{2}\big(f_D^{(1)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(1)}(U_1^{n-1}, U_2^{n-2})\big), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\
&+\Big(\mu\theta_2 U_2^{n-1} + (1-\mu)\theta_2 U_2^{n-2} - \frac{1}{2}\big(f_D^{(2)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(2)}(U_1^{n-2}, U_2^{n-1})\big), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h \\
&-\Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\Big), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h - \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\Big), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h.
\end{aligned} \tag{5.4.24}$$

On the other hand, rewriting (5.1.9) of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ at time level $t = t_{n-1}$ and then taking $\chi = U_i^n - U_i^{n-1} \in V_0^h$ yields for $i = 1, 2$ and for $n = 2 \to N$ that

$$\begin{aligned}
&\gamma(\nabla U_i^{n-1}, \nabla U_i^n - U_i^{n-1}) + (\phi(U_i^{n-1}), U_i^n - U_i^{n-1})^h \\
&\quad + \big(-\mu\theta_i U_i^{n-1} - (1-\mu)\theta_i U_i^{n-2} + \frac{1}{2}(f_D^{(i)}(U_1^{n-1}, U_2^{n-1}) + \bar{f}_{n-1,n-2}^{(i)}), U_i^n - U_i^{n-1}\big)^h \\
&\quad + \Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big), U_i^n - U_i^{n-1}\Big)^h = 0.
\end{aligned} \tag{5.4.25}$$

Using the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and (5.3.37) and rearranging it follows for $i = 1, 2$ and for $n = 2 \to N$ that

$$\begin{aligned}
-\frac{\gamma}{2}|U_i^n - U_i^{n-1}|_1^2 \geq &\frac{\gamma}{2}|U_i^{n-1}|_1^2 - \frac{\gamma}{2}|U_i^n|_1^2 + (\psi(U_i^{n-1}), 1)^h - (\psi(U_i^n), 1)^h \\
&+ \big(\mu\theta_i U_i^{n-1} + (1-\mu)\theta_i U_i^{n-2} - \frac{1}{2}(f_D^{(i)}(U_1^{n-1}, U_2^{n-1}) + \bar{f}_{n-1,n-2}^{(i)}), U_i^n - U_i^{n-1}\big)^h \\
&- \Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big), U_i^n - U_i^{n-1}\Big)^h.
\end{aligned} \tag{5.4.26}$$

Thus, by summation over $i = 1, 2$ we have, on noting (5.4.6) and recalling that $\bar{f}_{n-1,n-2}^{(1)} = f_D^{(1)}(U_1^{n-1}, U_2^{n-2})$ and $\bar{f}_{n-1,n-2}^{(2)} = f_D^{(2)}(U_1^{n-2}, U_2^{n-1})$, that for $n = 2 \to N$

$$\begin{aligned}
-\frac{\gamma}{2}\big[|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\big] \geq &\big[\bar{J}^h(U_1^{n-1}, U_2^{n-1}) - \bar{J}^h(U_1^n, U_2^n)\big] \\
&+\big(\mu\theta_1 U_1^{n-1} + (1-\mu)\theta_1 U_1^{n-2} - \frac{1}{2}\big(f_D^{(1)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(1)}(U_1^{n-1}, U_2^{n-2})\big), U_1^n - U_1^{n-1}\big)^h \\
&+\big(\mu\theta_2 U_2^{n-1} + (1-\mu)\theta_2 U_2^{n-2} - \frac{1}{2}\big(f_D^{(2)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(2)}(U_1^{n-2}, U_2^{n-1})\big), U_2^n - U_2^{n-1}\big)^h \\
&-\Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\Big), U_1^n - U_1^{n-1}\Big)^h - \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\Big), U_2^n - U_2^{n-1}\Big)^h.
\end{aligned} \tag{5.4.27}$$

Comparing the right hand side of (5.4.27) with (5.4.24) we therefore find for $n = 2 \to N$ that

$$-\frac{\gamma}{2}\big[|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\big] \geq \Delta t\big[\mathcal{E}^n - \mathcal{D}^n\big]. \tag{5.4.28}$$

For the case $n = 1$, we argue as for $n \geq 2$ to obtain on using (5.4.15) in place of (5.4.16) and (5.4.9) in palace of (5.1.9) that the results (5.4.24)-(5.4.27) holds for $n = 1$ with the only change that $\frac{1}{2}(f_D^{(1)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(1)}(U_1^{n-1}, U_2^{n-2}))$ and $\frac{1}{2}(f_D^{(2)}(U_1^{n-1}, U_2^{n-1}) + f_D^{(2)}(U_1^{n-2}, U_2^{n-1}))$ are replaced by $f_D^{(1)}(U_1^0, U_2^0)$ and $f_D^{(2)}(U_1^0, U_2^0)$, respectively. Therefore, for $n = 1 \to N$ (5.4.28) is valid and hence we conclude that the second inequality in (5.4.17) holds as required.

We now turn to prove (5.4.19). Noting (5.4.3) and (5.4.4) and owing to the definitions (5.4.7), (5.4.18), (5.4.8) and (5.4.6) we rewrite for $t \in (t_{n-1}, t_n)$ the difference $\mathcal{R} - \ell\,\Delta t\mathcal{E}$ as

$$\begin{aligned}
\mathcal{R} - \ell\,\Delta t\mathcal{E} &= \bar{J}^h(U_1, U_2) - \bar{J}^h(U_1^+, U_2^+) + \ell\big[\bar{J}^h(U_1^n, U_2^n) - \bar{J}^h(U_1^{n-1}, U_2^{n-1})\big] \\
&= \frac{\gamma}{2}\big[|U_1|_1^2 + |U_2|_1^2 - |U_1^+|_1^2 - |U_2^+|_1^2\big] \\
&\quad + \frac{\gamma}{2}\ell\big[|U_1^n|_1^2 + |U_2^n|_1^2 - |U_1^{n-1}|_1^2 - |U_2^{n-1}|_1^2\big] \\
&\quad + \big[(\psi(U_1), 1)^h + (\psi(U_2), 1)^h - (\psi(U_1^+), 1)^h - (\psi(U_2^+), 1)^h\big] \\
&\quad + \ell\big[(\psi(U_1^n), 1)^h + (\psi(U_2^n), 1)^h - (\psi(U_1^{n-1}), 1)^h - (\psi(U_2^{n-1}), 1)^h\big].
\end{aligned} \tag{5.4.29}$$

We now prove that the right hand side of (5.4.29) is non-positive. To see this, we firstly note from (5.4.3) and (5.4.4) for $i = 1, 2$ and $t \in (t_{n-1}, t_n)$

$$\begin{aligned}
&\frac{\gamma}{2}\big[|U_i|_1^2 - |U_i^+|_1^2\big] + \frac{\gamma}{2}\ell\big[|U_i^n|_1^2 - |U_i^{n-1}|_1^2\big] \\
&\qquad = \frac{\gamma}{2}\big(\nabla(U_i - U_i^+), \nabla(U_i + U_i^+)\big) + \frac{\gamma}{2}\ell\big(\nabla(U_i^n - U_i^{n-1}), \nabla(U_i^n + U_i^{n-1})\big) \\
&\qquad = -\ell\,\Delta t\frac{\gamma}{2}\big(\nabla\partial_t U_i, \nabla(U_i + U_i^+)\big) + \ell\,\Delta t\frac{\gamma}{2}\big(\nabla\partial_t U_i, \nabla(U_i^n + U_i^{n-1})\big) \\
&\qquad = \ell\,\Delta t\frac{\gamma}{2}\big(\nabla\partial_t U_i, \nabla(U_i^{n-1} - U_i)\big) \\
&\qquad = -\frac{\gamma}{2}\ell(1-\ell)(\Delta t)^2|\partial_t U_i|_1^2 \leq 0,
\end{aligned} \tag{5.4.30}$$

on noting $0 \leq \ell \leq 1$.

From (5.4.2) and the convexity of ψ it follows for $i = 1, 2$ and $t \in (t_{n-1}, t_n)$ that

$$\begin{aligned}(\psi(U_i), 1)^h - (\psi(U_i^+), 1)^h + \ell\big[(\psi(U_i^n), 1)^h - (\psi(U_i^{n-1}), 1)^h\big] \\ \leq (1-\ell)(\psi(U_i^n), 1)^h + \ell(\psi(U_i^{n-1}), 1)^h - (\psi(U_i^+), 1)^h \\ + \ell\big[(\psi(U_i^n), 1)^h - (\psi(U_i^{n-1}), 1)^h\big] = 0. \qquad (5.4.31)\end{aligned}$$

Adding (5.4.30) to (5.4.31) and then summing over $i = 1, 2$ yields after substitution into (5.4.29) that for $t \in (t_{n-1}, t_n)$

$$\mathcal{R} - \ell\, \Delta t \mathcal{E} \leq 0, \qquad (5.4.32)$$

from which we obtain the first inequality in (5.4.19). The second inequality follows immediately from (5.4.17).

Next we show that $\sum_{n=1}^{N} \mathcal{D}^n \leq C$. To this aim, we express $\sum_{n=2}^{N} \mathcal{D}^n$ using (5.4.16) as

$$\begin{aligned}\sum_{n=2}^{n} \mathcal{D}^n = \sum_{n=2}^{N} \Big[&\Big(\mu\theta_1(U_1^n - U_1^{n-1}) + (1-\mu)\theta_1(U_1^{n-1} - U_1^{n-2}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\ &+ \Big(\mu\theta_2(U_2^n - U_2^{n-1}) + (1-\mu)\theta_2(U_2^{n-1} - U_2^{n-2}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big] \\ - \frac{1}{2}\sum_{n=2}^{N} \Big[&\Big(f_D^{(1)}(U_1^n, U_2^n) - f_D^{(1)}(U_1^{n-1}, U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\ &+ \Big(f_D^{(2)}(U_1^n, U_2^n) - f_D^{(2)}(U_1^{n-1}, U_2^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big] \\ - \frac{1}{2}\sum_{n=2}^{N} \Big[&\Big(f_D^{(1)}(U_1^n, U_2^{n-1}) - f_D^{(1)}(U_1^{n-1}, U_2^{n-2}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\ &+ \Big(f_D^{(2)}(U_1^{n-1}, U_2^n) - f_D^{(2)}(U_1^{n-2}, U_2^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big] \\ - \sum_{n=2}^{N} \Big[&\Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^n - U_1^{n-1}}{\Delta t}\Big) - \hat{\mathcal{G}}^h\Big(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\Big), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h \\ &+ \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^n - U_2^{n-1}}{\Delta t}\Big) - \hat{\mathcal{G}}^h\Big(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\Big), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big] \\ := T_1 + T_2 + T_3 + T_4. \qquad (5.4.33)\end{aligned}$$

We deal with the terms on the right hand side of (5.4.33) separately. With the aid of the Young inequality and the first bound in (5.3.39a) we find for $i = 1, 2$ that

$$
\begin{aligned}
&\sum_{n=2}^{N}\Big(\mu\theta_i(U_i^n - U_i^{n-1}) + (1-\mu)\theta_i(U_i^{n-1} - U_i^{n-2}), \frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)^h \\
&\quad\le \mu\theta_i\Delta t\sum_{n=2}^{N}\Big|\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big|_h^2 + (1-\mu)\theta_i\Delta t\sum_{n=2}^{N}\Big|\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big|_h\Big|\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big|_h \\
&\quad\le C\Delta t\Big[\sum_{n=2}^{N}\Big|\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big|_h^2 + \Big|\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big|_h^2\Big] \le C, \qquad (5.4.34)
\end{aligned}
$$

and hence by summing over $i = 1, 2$ we have T_1 is bounded.

Using (2.3.46) with $r_i = U_i^n$ and $s_i = U_i^{n-1}$, noting the result (5.3.19e) and that $|\alpha_i| < 1$, applying Young's inequality and noting the first bound in (5.3.39a) yields for $i, j = 1, 2$ with $i \neq j$ that

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{N}\Big|\Big(f_D^{(i)}(U_1^n, U_2^n) - f_D^{(i)}(U_1^{n-1}, U_2^{n-1}), \frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)^h\Big| \\
&\quad\le D\sum_{n=2}^{N}\Big|\Big((U_j^n + \alpha_j)^2(U_i^n - U_i^{n-1}), \frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)^h\Big| \\
&\qquad + D\sum_{n=2}^{N}\Big|\Big((U_i^{n-1} + \alpha_i)(U_j^n + U_j^{n-1} + 2\alpha_j)(U_j^n - U_j^{n-1}), \frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)^h\Big| \\
&\quad\le 4D\Delta t\sum_{n=2}^{N}\Big|\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big|_h^2 + 8D\Delta t\sum_{n=2}^{N}\Big|\frac{U_j^n - U_j^{n-1}}{\Delta t}\Big|_h\Big|\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big|_h \\
&\quad\le 8D\Delta t\sum_{n=2}^{N}\Big|\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big|_h^2 + 4D\Delta t\sum_{n=2}^{N}\Big|\frac{U_j^n - U_j^{n-1}}{\Delta t}\Big|_h^2 \le C, \qquad (5.4.35)
\end{aligned}
$$

which implies, by summation over $i = 1, 2$, that T_2 is bounded. Similarly, one can show that

$$
\begin{aligned}
T_3 &\le \frac{1}{2}\sum_{n=2}^{N}\Big|\Big(f_D^{(1)}(U_1^n, U_2^{n-1}) - f_D^{(1)}(U_1^{n-1}, U_2^{n-2}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\Big)^h\Big| \\
&\quad + \frac{1}{2}\sum_{n=2}^{N}\Big|\Big(f_D^{(2)}(U_1^{n-1}, U_2^n) - f_D^{(2)}(U_1^{n-2}, U_2^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\Big)^h\Big| \\
&\le 8D\Delta t\sum_{n=2}^{N}\Big|\frac{U_1^n - U_1^{n-1}}{\Delta t}\Big|_h^2 + 4D\Delta t\sum_{n=2}^{N}\Big|\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\Big|_h^2 \\
&\quad + 8D\Delta t\sum_{n=2}^{N}\Big|\frac{U_2^n - U_2^{n-1}}{\Delta t}\Big|_h^2 + 4D\Delta t\sum_{n=2}^{N}\Big|\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\Big|_h^2 \le C. \qquad (5.4.36)
\end{aligned}
$$

From the definition (4.1.11) of $\hat{\mathcal{G}}^h$, the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and the third bound in (5.3.39a) it follows for $i = 1, 2$ that

$$
\begin{aligned}
-\sum_{n=2}^{N}\Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big) - \hat{\mathcal{G}}^h\Big(\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big), \frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)^h \\
&= -\sum_{n=2}^{N}\Big(\nabla\hat{\mathcal{G}}^h\Big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big) - \nabla\hat{\mathcal{G}}^h\Big(\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big), \nabla\hat{\mathcal{G}}^h\Big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)\Big) \\
&= \frac{1}{2}\sum_{n=2}^{N}\Big[-\Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^n - U_i^{n-1}}{\Delta t}\Big)\Big|_1^2 + \Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^{n-1} - U_i^{n-2}}{\Delta t}\Big)\Big|_1^2 \\
&\quad - \Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^n - 2U_i^{n-1} + U_i^{n-2}}{\Delta t}\Big)\Big|_1^2\Big] \\
&= \frac{1}{2}\Big[-\Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^N - U_i^{N-1}}{\Delta t}\Big)\Big|_1^2 + \Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^1 - U_i^0}{\Delta t}\Big)\Big|_1^2\Big] \\
&\quad - \frac{1}{2}\sum_{n=2}^{N}\Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^n - 2U_i^{n-1} + U_i^{n-2}}{\Delta t}\Big)\Big|_1^2 \\
&\leq \frac{1}{2}\Big|\hat{\mathcal{G}}^h\Big(\frac{U_i^1 - U_i^0}{\Delta t}\Big)\Big|_1^2 = \frac{1}{2}\Big\|\frac{U_i^1 - U_i^0}{\Delta t}\Big\|_{-h}^2 \leq C,
\end{aligned}
\tag{5.4.37}
$$

from which we deduce by summation over $i = 1, 2$ that T_4 is bounded.

Therefore, inserting (5.4.34)-(5.4.37) into (5.4.33) leads to

$$
\sum_{n=2}^{N}\mathcal{D}^n \leq C. \tag{5.4.38}
$$

It remains now to show that $\mathcal{D}^1 \leq C$. Choosing $\chi = \frac{U_i^1 - U_i^0}{\Delta t}$ in (5.3.35), noting that $\theta_i U_i^0 = \mu\theta_i U_i^0 + (1-\mu)\theta_i U_i^0$ and rearranging gives for $i = 1, 2$ that

$$
\begin{aligned}
\Big((1-\mu)\theta_i U_i^0, \frac{U_i^1 - U_i^0}{\Delta t}\Big)^h &= \gamma\Big(\nabla U_i^0, \nabla\frac{U_i^1 - U_i^0}{\Delta t}\Big) \\
&\quad + \Big(\phi(U_i^0) - \mu\theta_i U_i^0 + f_D^{(i)}(U_1^0, U_2^0) - W_i^0, \frac{U_i^1 - U_i^0}{\Delta t}\Big)^h.
\end{aligned}
\tag{5.4.39}
$$

Then, by taking $\chi = \frac{U_i^1 - U_i^0}{\Delta t}$ in (5.4.9) and rearranging yields for $i = 1, 2$ that

$$
\begin{aligned}
\Big(-\mu\theta_i U_i^0 - (1-\mu)\theta_i U_i^{-1} + f_D^{(i)}(U_1^0, U_2^0), \frac{U_i^1 - U_i^0}{\Delta t}\Big)^h + \Big(\hat{\mathcal{G}}^h\Big(\frac{U_i^0 - U_i^{-1}}{\Delta t}\Big), \frac{U_i^1 - U_i^0}{\Delta t}\Big)^h \\
= -\gamma\Big(\nabla U_i^0, \nabla\frac{U_i^1 - U_i^0}{\Delta t}\Big) - \Big(\phi(U_i^0), \frac{U_i^1 - U_i^0}{\Delta t}\Big)^h.
\end{aligned}
\tag{5.4.40}
$$

Substituting (5.4.39) and (5.4.40) into (5.4.15) and simplifying it follows that

$$
\begin{aligned}
\mathcal{D}^1 &= \Big(\mu\theta_1 U_1^1 - \frac{1}{2}\big(f_D^{(1)}(U_1^1,U_2^1) + f_D^{(1)}(U_1^1,U_2^0)\big), \frac{U_1^1-U_1^0}{\Delta t}\Big)^h + \gamma\Big(\nabla U_1^0, \nabla\frac{U_1^1-U_1^0}{\Delta t}\Big)\\
&\quad + \Big(\phi(U_1^0) - \mu\theta_1 U_1^0 + f_D^{(1)}(U_1^0,U_2^0) - W_1^0, \frac{U_1^1-U_1^0}{\Delta t}\Big)^h - \Big(\hat{\mathcal{G}}^h\Big(\frac{U_1^1-U_1^0}{\Delta t}\Big), \frac{U_1^1-U_1^0}{\Delta t}\Big)^h\\
&\quad + \Big(\mu\theta_2 U_2^1 - \frac{1}{2}\big(f_D^{(2)}(U_1^1,U_2^1) + f_D^{(2)}(U_1^0,U_2^1)\big), \frac{U_1^1-U_1^0}{\Delta t}\Big)^h + \gamma\Big(\nabla U_2^0, \nabla\frac{U_2^1-U_2^0}{\Delta t}\Big)\\
&\quad + \Big(\phi(U_2^0) - \mu\theta_2 U_2^0 + f_D^{(2)}(U_1^0,U_2^0) - W_2^0, \frac{U_2^1-U_2^0}{\Delta t}\Big)^h - \Big(\hat{\mathcal{G}}^h\Big(\frac{U_2^1-U_2^0}{\Delta t}\Big), \frac{U_2^1-U_2^0}{\Delta t}\Big)^h\\
&\quad - \gamma\Big(\nabla U_1^0, \nabla\frac{U_1^1-U_1^0}{\Delta t}\Big) - \Big(\phi(U_1^0), \frac{U_1^1-U_1^0}{\Delta t}\Big)^h\\
&\quad - \gamma\Big(\nabla U_2^0, \nabla\frac{U_2^1-U_2^0}{\Delta t}\Big) - \Big(\phi(U_2^0), \frac{U_2^1-U_2^0}{\Delta t}\Big)^h\\
&= \Big(\mu\theta_1(U_1^1-U_1^0) - W_1^0 - \hat{\mathcal{G}}^h\Big(\frac{U_1^1-U_1^0}{\Delta t}\Big), \frac{U_1^1-U_1^0}{\Delta t}\Big)^h\\
&\quad + \Big(\mu\theta_2(U_2^1-U_2^0) - W_2^0 - \hat{\mathcal{G}}^h\Big(\frac{U_2^1-U_2^0}{\Delta t}\Big), \frac{U_2^1-U_2^0}{\Delta t}\Big)^h\\
&\quad + \frac{1}{2}\Big(f_D^{(1)}(U_1^0,U_2^0) - f_D^{(1)}(U_1^1,U_2^1) + f_D^{(1)}(U_1^0,U_2^0) - f_D^{(1)}(U_1^1,U_2^0), \frac{U_1^1-U_1^0}{\Delta t}\Big)^h\\
&\quad + \frac{1}{2}\Big(f_D^{(2)}(U_1^0,U_2^0) - f_D^{(2)}(U_1^1,U_2^1) + f_D^{(2)}(U_1^0,U_2^0) - f_D^{(2)}(U_1^0,U_2^1), \frac{U_2^1-U_2^0}{\Delta t}\Big)^h\\
&= T_1 + T_2 + T_3 + T_4. \qquad (5.4.41)
\end{aligned}
$$

To bound the first two terms we use the first inequality in (4.1.14), (4.1.12) and the bounds (5.3.39a) and (5.3.42) to have for $i = 1, 2$ that

$$
\begin{aligned}
&\Big(\mu\theta_i(U_i^1-U_i^0) - W_i^0 - \hat{\mathcal{G}}^h\Big(\frac{U_i^1-U_i^0}{\Delta t}\Big), \frac{U_i^1-U_i^0}{\Delta t}\Big)^h\\
&\qquad\qquad \le \mu\theta_i\Delta t\Big|\frac{U_i^1-U_i^0}{\Delta t}\Big|_h^2 + |W_i^0|_1\Big\|\frac{U_i^1-U_i^0}{\Delta t}\Big\|_{-h} - \Big\|\frac{U_i^1-U_i^0}{\Delta t}\Big\|_{-h}^2\\
&\qquad\qquad \le C, \qquad (5.4.42)
\end{aligned}
$$

and then we sum (5.4.42) over $i = 1, 2$ to obtain

$$
T_1 + T_2 \le C. \qquad (5.4.43)
$$

Similarly to (5.4.35), noting (5.3.19e) and $|U_i^0| = |P_\gamma^h u_i^0| < 1\ \forall h \le h_*$ (see Lemma 4.3.3) we have for $i, j = 1, 2$ with $i \ne j$ that

$$
\begin{aligned}
\frac{1}{2}\Big(f_D^{(i)}(U_1^0,U_2^0) - f_D^{(i)}(U_1^1,U_2^1), \frac{U_i^1-U_i^0}{\Delta t}\Big)^h &\le 8D\Delta t\Big|\frac{U_i^1-U_i^0}{\Delta t}\Big|_h^2 + 4D\Delta t\Big|\frac{U_j^1-U_j^0}{\Delta t}\Big|_h^2\\
&\le C. \qquad (5.4.44)
\end{aligned}
$$

We also have that

$$\frac{1}{2}\Big(f_D^{(1)}(U_1^0,U_2^0)-f_D^{(1)}(U_1^1,U_2^0),\frac{U_1^1-U_1^0}{\Delta t}\Big)^h+\frac{1}{2}\Big(f_D^{(2)}(U_1^0,U_2^0)-f_D^{(2)}(U_1^0,U_2^1),\frac{U_2^1-U_2^0}{\Delta t}\Big)^h$$
$$=D\Big((U_2^0+\alpha_2)^2(U_1^0-U_1^1),\frac{U_1^1-U_1^0}{\Delta t}\Big)^h+D\Big((U_1^0+\alpha_1)^2(U_2^0-U_2^1),\frac{U_2^1-U_2^0}{\Delta t}\Big)^h$$
$$=-D\Big((U_2^0+\alpha_2)^2,\frac{(U_1^1-U_1^0)^2}{\Delta t}\Big)^h-D\Big((U_1^0+\alpha_1)^2,\frac{(U_2^1-U_2^0)^2}{\Delta t}\Big)^h\le 0. \tag{5.4.45}$$

Thus, summing (5.4.44) over $i=1,2$ and adding the resulting inequality to (5.4.45) leads to

$$T_3+T_4\le C. \tag{5.4.46}$$

We therefore conclude from (5.4.41), (5.4.43) and (5.4.46) that

$$\mathcal{D}^1\le C. \tag{5.4.47}$$

Finally, the first inequality in (5.4.20) follows directly from (5.4.17). □

Lemma 5.4.2 Let the assumptions of Theorem 5.3.3 hold. Then for *a.e.* $t\in(0,T)$ we have

$$\frac{\gamma}{2}\big(|E_1|^2+|E_2|_1^2+|E_1^+|_1^2+|E_2^+|_1^2\big)+\frac{1}{2}\frac{d}{dt}\big[\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\big]$$
$$\le\big(\mu\theta_1E_1^++(1-\mu)\theta_1E_1^-,E_1\big)^h+\big(\mu\theta_2E_2^++(1-\mu)\theta_2E_2^-,E_2\big)^h$$
$$-D\big([(U_2^++\alpha_2)^2+(U_2^-+\alpha_2)^2]E_1^++(u_1^h+\alpha_1)(U_2^++u_2^h+2\alpha_2)E_2^+,E_1\big)^h$$
$$-D\big([(U_1^++\alpha_1)^2+(U_1^-+\alpha_1)^2]E_2^++(u_2^h+\alpha_2)(U_1^++u_1^h+2\alpha_1)E_1^+,E_2\big)^h$$
$$-D\big((u_1^h+\alpha_1)(U_2^-+u_2^h+2\alpha_2)E_2^-,E_1\big)^h-D\big((u_2^h+\alpha_2)(U_1^-+u_1^h+2\alpha_1)E_1^-,E_2\big)^h$$
$$+\mathcal{R}(t), \tag{5.4.48}$$

where[1]

$$E_1^{(\pm)}:=u_1^h-U_1^{(\pm)},\quad E_2^{(\pm)}:=u_2^h-U_2^{(\pm)}. \tag{5.4.49}$$

Proof. Using (5.4.2) and (5.4.3) one can restate the problem $(\mathbf{P}_\mu^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ as follows:

[1]The notation $E_i^{(\pm)}$ and $U_i^{(\pm)}$ means with and without the superscripts ±.

Find$\{U_1(t), U_2(t)\} \in H^1(0,T;S^h) \times H^1(0,T;S^h)$ such that for $i = 1,2$ $U_i^0 = P_\gamma^h u_i^0$ and for $a.e.$ $t \in (0,T)$ and all $\chi \in S^h$

$$\gamma(\nabla U_i^+, \nabla \chi) + (\phi(U_i^+) - \mu\theta_i U_i^+ - (1-\mu)\theta_i U_i^-, \chi - \fint \chi)^h \\ + \frac{1}{2}\big(f_D^{(i)}(U_1^+, U_2^+) + \bar{f}_{+,-}^{(i)}, \chi - \fint \chi\big)^h + (\hat{\mathcal{G}}^h \partial_t U_i, \chi)^h = 0, \qquad (5.4.50)$$

where

$$\bar{f}_{+,-}^{(1)} := f_D^{(1)}(U_1^+, U_2^-), \quad \bar{f}_{+,-}^{(2)} := f_D^{(2)}(U_1^-, U_2^+). \qquad (5.4.51)$$

We test the semi-discrete problem ($\mathbf{P^h}$) (4.3.8) with $\chi = E_i = u_i^h - U_i \in V_0^h$ to result in for $i = 1,2$ and $a.e.$ $t \in (0,T)$ that

$$\gamma|E_i|_1^2 + \gamma(\nabla U_i, \nabla u_i^h - U_i) + (\phi(u_i^h) - \theta_i u_i^h, u_i^h - U_i)^h \\ + \big(f_D^{(i)}(u_1^h, u_2^h), E_i\big)^h + (\hat{\mathcal{G}}^h \partial_t u_i^h, E_i)^h = 0. \qquad (5.4.52)$$

Noting the identity $2a(b-a) = b^2 - a^2 - (a-b)^2$ and (5.3.37) yields for $i = 1,2$ and $a.e.$ $t \in (0,T)$ that

$$\frac{\gamma}{2}\big[|E_i|_1^2 + |u_i^h|_1^2 - |U_i|_1^2\big] + (\psi(u_i^h) - \psi(U_i), 1)^h + (\hat{\mathcal{G}}^h \partial_t u_i^h, E_i)^h \\ \leq \theta_i(u_i^h, E_i)^h - \big(f_D^{(i)}(u_1^h, u_2^h), E_i\big)^h. \qquad (5.4.53)$$

On the other hand, by choosing $\chi = -E_i^+ = U_i^+ - u_i^h \in V_0^h$ in (5.4.50) we have for $i = 1,2$ and $a.e.$ $t \in (0,T)$ that

$$\gamma|E_i^+|_1^2 + \gamma(\nabla u_i^h, \nabla U_i^+ - u_i^h) + (\phi(U_i^+), U_i^+ - u_i^h)^h + (\mu\theta_i U_i^+ + (1-\mu)\theta_i U_i^-, E_i^+)^h \\ - \frac{1}{2}\big(f_D^{(i)}(U_1^+, U_2^+) + \bar{f}_{+,-}^{(i)}, E_i^+\big)^h - (\hat{\mathcal{G}}^h \partial_t U_i, E_i^+)^h = 0. \qquad (5.4.54)$$

Once again from the identity $2a(b-a) = b^2 - a^2 - (a-b)^2$ and (5.3.37) it follows for $i = 1,2$ and $a.e.$ $t \in (0,T)$ that

$$\frac{\gamma}{2}\big[|E_i^+|_1^2 + |U_i^+|_1^2 - |u_i^h|_1^2\big] + (\psi(U_i^+) - \psi(u_i^h), 1)^h - (\hat{\mathcal{G}}^h \partial_t U_i, E_i^+)^h \\ \leq -(\mu\theta_i U_i^+ + (1-\mu)\theta_i U_i^-, E_i^+)^h + \frac{1}{2}\big(f_D^{(i)}(U_1^+, U_2^+) + \bar{f}_{+,-}^{(i)}, E_i^+\big)^h. \qquad (5.4.55)$$

Adding (5.4.53) to (5.4.55) and rearranging gives for $i = 1, 2$ and $a.e.\ t \in (0, T)$ that

$$\begin{aligned}
\frac{\gamma}{2}\big[|E_i|_1^2 + |E_i^+|_1^2\big] + \frac{\gamma}{2}\big[|U_i^+|_1^2 - |U_i|_1^2\big] + (\psi(U_i^+) - \psi(U_i), 1)^h& \\
+ \big[(\hat{\mathcal{G}}^h \partial_t u_i^h, E_i)^h - (\hat{\mathcal{G}}^h \partial_t U_i, E_i^+)^h\big]& \\
\leq \big[(\theta_i u_i^h, E_i)^h - (\mu\theta_i U_i^+ + (1-\mu)\theta_i U_i^-, E_i^+)^h\big]& \\
+ \frac{1}{2}\big[\big(f_D^{(i)}(U_1^+, U_2^+), E_i^+\big)^h - \big(f_D^{(i)}(u_1^h, u_2^h), E_i\big)^h\big]& \\
+ \frac{1}{2}\big[(\bar{f}_{+,-}^{(i)}, E_i^+)^h - (f_D^{(i)}(u_1^h.u_2^h), E_i)^h\big]& \\
=: T_1 + T_2 + T_3.& \qquad (5.4.56)
\end{aligned}$$

By adding and subtracting $(\hat{\mathcal{G}}^h \partial_t U_i, E_i)^h$, noting (4.1.12) and owing to (5.4.49) we have for $i =, 2$ and $a.e.\ t \in (0, T)$ that

$$\begin{aligned}
(\hat{\mathcal{G}}^h \partial_t u_i^h, E_i)^h - (\hat{\mathcal{G}}^h \partial_t U_i, E_i^+)^h &= (\hat{\mathcal{G}}^h \partial_t E_i, E_i)^h + (\hat{\mathcal{G}}^h \partial_t U_i, E_i - E_i^+)^h \\
&= \frac{1}{2}\frac{d}{dt}\|E_i\|_{-h}^2 + (\hat{\mathcal{G}}^h \partial_t U_i, U_i^+ - U_i)^h. \qquad (5.4.57)
\end{aligned}$$

On noting that $\theta_i u_i^h = \mu\theta_i u_i^h + (1-\mu)\theta_i u_i^h$ and recalling (5.4.49) we alternatively express the term T_1 for $i = 1, 2$ and $a.e.\ t \in (0, T)$ as

$$\begin{aligned}
T_1 &= \mu\theta_i(u_i^h - U_i^+, E_i)^h + \mu\theta_i(U_i^+, E_i - E_i^+)^h + (1-\mu)\theta_i(u_i^h - U_i^-, E_i)^h \\
&\quad + (1-\mu)\theta_i(U_i^-, E_i - E_i^+)^h \\
&= (\mu\theta_i E_i^+ + (1-\mu)\theta_i E_i^-, E_i)^h + (\mu\theta_i U_i^+ + (1-\mu)\theta_i U_i^-, U_i^+ - U_i)^h. \qquad (5.4.58)
\end{aligned}$$

On setting $r_i = U_i^+$ and $s_i = u_i^h$ in (2.3.46) and noting (5.4.49) we can rewrite the term T_2 for $i, j = 1, 2$ with $i \neq j$ and $a.e.\ t \in (0, T)$ as

$$\begin{aligned}
T_2 &= \frac{1}{2}\big(f_D^{(i)}(U_1^+, U_2^+) - f_D^{(i)}(u_1^h, u_2^h), E_i\big)^h + \frac{1}{2}\big(f_D^{(i)}(U_1^+, U_2^+), E_i^+ - E_i\big)^h \\
&= D\big((U_j^+ + \alpha_j)^2(-E_i^+) + (u_i^h + \alpha_i)(U_j^+ + u_j^h + 2\alpha_j)(-E_j^+), E_i\big)^h \\
&\quad + \frac{1}{2}\big(f_D^{(i)}(U_1^+, U_2^+), U_i - U_i^+\big)^h. \qquad (5.4.59)
\end{aligned}$$

Again with the aid of (5.4.49) the term T_3 may be represented for $i,j=1,2$ with $i\neq j$ and *a.e.* $t\in(0,T)$ as

$$\begin{aligned}
T_3 &= \frac{1}{2}(\bar{f}^{(i)}_{+,-} - f^{(i)}_D(u_1^h,u_2^h),E_i)^h + \frac{1}{2}(\bar{f}^{(i)}_{+,-},E_i^+ - E_i)^h \\
&= D\big((U_i^+ + \alpha_i)(U_j^- + \alpha_j)^2 - (u_i^h+\alpha_i)(u_j^h+\alpha_j)^2, E_i\big)^h + \frac{1}{2}(\bar{f}^{(i)}_{+,-},U_i - U_i^+)^h \\
&= D\big((U_j^-+\alpha_j)^2(U_i^+ - u_i^h) + (u_i^h+\alpha_i)(U_j^- + u_j^h + 2\alpha_j)(U_j^- - u_j^h), E_i\big)^h \\
&\quad + \frac{1}{2}(\bar{f}^{(i)}_{+,-},U_i - U_i^+)^h \\
&= D\big((U_j^-+\alpha_j)^2(-E_i^+) + (u_i^h+\alpha_i)(U_j^- + u_j^h + 2\alpha_j)(-E_j^-), E_i\big)^h \\
&\quad + \frac{1}{2}(\bar{f}^{(i)}_{+,-},U_i - U_i^+)^h. \qquad (5.4.60)
\end{aligned}$$

Combining (5.4.57)-(5.4.60) with (5.4.56) and rearranging the terms we thus conclude for $i,j=1,2$ with $i\neq j$ and *a.e.* $t\in(0,T)$ that

$$\begin{aligned}
&\frac{\gamma}{2}\big[|E_i|_1^2 + |E_i^+|_1\big] + \frac{1}{2}\frac{d}{dt}\|E_i\|_{-h}^2 \\
&\qquad \leq \frac{\gamma}{2}\big[|U_i|_1^2 - |U_i^+|_1^2\big] + (\psi(U_i),1)^h - (\psi(U_i^+),1)^h \\
&\qquad + \Big(-\mu\theta_i U_i^+ - (1-\mu)\theta_i U_i^- + \frac{1}{2}\big(f_D^{(i)}(U_1^+,U_2^+) + \bar{f}^{(i)}_{+,-}\big), U_i - U_i^+\Big)^h \\
&\qquad + \big(\hat{\mathcal{G}}^h\partial_t U_i, U_i - U_i^+\big)^h \\
&\qquad + \big(\mu\theta_i E_i^+ + (1-\mu)\theta_i E_i^-, E_i\big)^h \\
&\qquad - D\big((U_j^+ + \alpha_j)^2 E_i^+ + (u_i^h+\alpha_i)(U_j^+ + u_j^h + 2\alpha_j)E_j^+, E_i\big)^h \\
&\qquad - D\big((U_j^- + \alpha_j)^2 E_i^+ + (u_i^h+\alpha_i)(U_j^- + u_j^h + 2\alpha_j)E_j^-, E_i\big)^h. \qquad (5.4.61)
\end{aligned}$$

Finally, we sum (5.4.61) over $i=1,2$ and note the definitions (5.4.51), (5.4.6) and (5.4.7) of $\bar{f}^{(i)}_{+,-}$, $\bar{J}^h$ and $\mathcal{R}$ respectively to obtain immediately the desired error inequality (5.4.48). □

Remark. We observe that the right hand side the differential error inequality (5.4.48) involves $\mathcal{R}$ and D-terms each of which may be non-positive. However, Lemma 5.4.1 shows that $\mathcal{R}$ is bounded above by $\ell\,\Delta t\,\mathcal{E}$ which is in turn a bounded non-negative quantity. Also, the D-terms in (5.4.48) can be bounded by non-negative quantities $|\partial_t U_1|_h$ and $|\partial_t U_2|_h$ which are, in view of (5.3.39a), bounded in $L^2(0,T)$. These key observations will enable us to derive an optimal error bound in time between $(\mathbf{P^h})$ and $(\mathbf{P_\mu^{h,\Delta t}})$ as will be seen in the next theorem.

Theorem 5.4.3 Let the assumptions of Theorem 5.3.3 hold. Then we have

$$\begin{aligned}
&\|E_1^+\|^2_{L^2(0,T;H^1(\Omega))} + \|E_2^+\|^2_{L^2(0,T;H^1(\Omega))} + \|E_1\|^2_{L^2(0,T;H^1(\Omega))} \\
&+ \|E_2\|^2_{L^2(0,T;H^1(\Omega))} + \|E_1\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|E_2\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\
&\quad \leq Ce^{CT}(\Delta t)^3 \Big[\sum_{n=1}^{N} \Big| \frac{U_1^n - U_1^{n-1}}{\Delta t} \Big|_h^2 + \sum_{n=1}^{N} \Big| \frac{U_2^n - U_2^{n-1}}{\Delta t} \Big|_h^2 \Big] + Ce^{CT}(\Delta t)^2 \sum_{n=1}^{N} \mathcal{E}^n \\
&\quad \leq C(\Delta t)^2. \qquad (5.4.62)
\end{aligned}$$

Furthermore,

$$\|E_1^+\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|E_2^+\|^2_{L^\infty(0,T;(H^1(\Omega))')} \leq C(\Delta t)^2. \qquad (5.4.63)$$

Proof. Using the definitions (5.4.49) and (5.4.4) and the fact that $0 \leq \ell \leq 1$ we note for later use that for $i = 1, 2$ and *a.e.* $t \in (0, T)$

$$\begin{aligned}
|E_i^+| &= |(u_i^h - U_i) + (U_i - U_i^+)| \\
&= |E_i - \ell\, \Delta t\, \partial_t U_i| \leq |E_i| + \Delta t |\partial_t U_i|, \qquad (5.4.64a) \\
|E_i^-| &= |(u_i^h - U_i) + (U_i - U_i^-)| \\
&= |E_i + (1-\ell)\Delta t \partial_t U_i| \leq |E_i| + \Delta t |\partial_t U_i|. \qquad (5.4.64b)
\end{aligned}$$

Now we estimate the terms on the right hand side of the error inequality (5.4.48) derived in the previous Lemma 5.4.2. From (4.1.14) we have for $i = 1, 2$ and *a.e.* $t \in (0, T)$ that

$$\mu\theta_i(E_i^+, E_i)^h \leq \frac{\gamma}{4}|E_i^+|_1^2 + C\|E_i\|^2_{-h}. \qquad (5.4.65)$$

Noting (5.4.64b), a Young's inequality and (4.1.15) it follows for $i = 1, 2$ and *a.e.* $t \in (0, T)$ that

$$\begin{aligned}
(1-\mu)\theta_i(E_i^-, E_i)^h &\leq (1-\mu)\theta_i(|E_i^-|, |E_i|)^h \\
&\leq (1-\mu)\theta_i|E_i|_h^2 + (1-\mu)\theta_i\Delta t(|\partial_t U_i|, |E_i|)^h \\
&\leq \frac{3}{2}(1-\mu)\theta_i|E_i|_h^2 + \frac{(\Delta t)^2}{2}(1-\mu)\theta_i|\partial_t U_i|_h^2 \\
&\leq \frac{\gamma}{16}|E_i|_1^2 + C\|E_i\|^2_{-h} + C(\Delta t)^2|\partial_t U_i|_h^2. \qquad (5.4.66)
\end{aligned}$$

With the aid of the results (5.3.19e), (4.3.47) and $|\alpha_i| < 1$ and using (5.4.64a), Young's inequality and (4.1.15) we obtain for $i, j = 1, 2$ with $i \neq j$ and *a.e.* $t \in (0, T)$ that

$$\begin{aligned}
&\Big| - D\big([(U_j^+ + \alpha_j)^2 + (U_j^- + \alpha_j)^2]E_i^+ + (u_i^h + \alpha_i)(U_j^+ + u_j^h + 2\alpha_j)E_j^+, E_i\big)^h\Big| \\
&\qquad \leq 8D(|E_i^+| + |E_j^+|, |E_i|)^h \\
&\qquad \leq 8D(|E_i| + |E_j|, |E_i|)^h + 8D\Delta t(|\partial_t U_i| + |\partial_t U_j|, |E_i|)^h \\
&\qquad \leq 16D|E_i|_h^2 + 4D|E_j|_h^2 + C(\Delta t)^2\big[|\partial_t U_i|_h^2 + |\partial_t U_j|_h^2\big] \\
&\qquad \leq \frac{\gamma}{16}\big[|E_i|_1^2 + |E_j|_1^2\big] + C\big[\|E_i\|_{-h}^2 + \|E_j\|_{-h}^2\big] + C(\Delta t)^2\big[|\partial_t U_i|_h^2 + |\partial_t U_j|_h^2\big].
\end{aligned} \tag{5.4.67}$$

Similarly, we have using (5.4.64b) for $i, j = 1, 2$ with $i \neq j$ and *a.e.* $t \in (0, T)$ that

$$\begin{aligned}
&\Big| - D\big((u_i^h + \alpha_i)(U_j^- + u_j^h + 2\alpha_j)E_j^-, E_i\big)^h\Big| \\
&\qquad \leq 8D(|E_j^-|, |E_i|)^h \\
&\qquad \leq 8D(|E_j|, |E_i|)^h + 8D\Delta t(|\partial_t U_j|, |E_i|)^h \\
&\qquad \leq 8D|E_i|_h^2 + 4D|E_j|_h^2 + C(\Delta t)^2|\partial_t U_j|_h^2 \\
&\qquad \leq \frac{\gamma}{32}\big[|E_i|_1^2 + |E_j|_1^2\big] + C\big[\|E_i\|_{-h}^2 + \|E_j\|_{-h}^2\big] + C(\Delta t)^2|\partial_t U_j|_h^2.
\end{aligned} \tag{5.4.68}$$

Thus, summing (5.4.65)-(5.4.68) over $i = 1, 2$ and then substituting into (5.4.48) and noting (5.4.19) in Lemma 5.4.1 implies for *a.e.* $t \in (0, T)$ that

$$\begin{aligned}
&\frac{\gamma}{4}\big[|E_1|_1^2 + |E_2|_1^2 + |E_1^+|_1^2 + |E_2^+|_1^2\big] + \frac{1}{2}\frac{d}{dt}\big[\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\big] \\
&\qquad \leq C\big[\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\big] + C(\Delta t)^2\big[|\partial_t U_1|_h^2 + |\partial_t U_2|_h^2\big] + C\Delta t\,\mathcal{E}(t).
\end{aligned} \tag{5.4.69}$$

We then apply the Gronwall lemma and note $E_1(0) = E_2(0) = 0$ and $\mathcal{E} \geq 0$ to yield for *a.e.* $t \in (0, T]$ that

$$\begin{aligned}
&\frac{\gamma}{2}\int_0^t \big[|E_1|_1^2 + |E_2|_1^2 + |E_1^+|_1^2 + |E_2^+|_1^2\big]ds + \big[\|E_1(t)\|_{-h}^2 + \|E_2(t)\|_{-h}^2\big] \\
&\qquad \leq Ce^{CT}(\Delta t)^2\int_0^T \big[|\partial_s U_1|_h^2 + |\partial_s U_2|_h^2\big]ds + Ce^{CT}\Delta t\int_0^T \mathcal{E}(s)\,ds.
\end{aligned} \tag{5.4.70}$$

To bound the right hand side of (5.4.70) we note from (5.4.3), (5.4.4) and the first bound in (5.3.39a) for $i = 1, 2$ that

$$\int_0^T |\partial_t U_i|_h^2 dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left|\frac{U_i^n - U_i^{n-1}}{\Delta t}\right|_h^2 = \Delta t \sum_{n=1}^N \left|\frac{U_i^n - U_i^{n-1}}{\Delta t}\right|_h^2 \leq C, \tag{5.4.71}$$

and with the aid of (5.4.18) and the result (5.4.20) derived in Lemma 5.4.1 we have

$$\int_0^T \mathcal{E}(t)\, dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \mathcal{E}^n\, dt = \Delta t \sum_{n=1}^N \mathcal{E}^n \leq C\Delta t. \tag{5.4.72}$$

Hence (5.4.70) becomes

$$\begin{aligned} &\frac{\gamma}{2}\int_0^t \left[|E_1|_1^2 + |E_2|_1^2 + |E_1^+|_1^2 + |E_2^+|_1^2\right] ds + \left[\|E_1(t)\|_{-h}^2 + \|E_2(t)\|_{-h}^2\right] \\ &\qquad \leq C(\Delta t)^3 \sum_{n=1}^N \left[\left|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right|_h^2 + \left|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right|_h^2\right] + C(\Delta t)^2 \sum_{n=1}^N \mathcal{E}^n \\ &\qquad \leq C(\Delta t)^2. \end{aligned} \tag{5.4.73}$$

This result together with Poincaré's inequality, the equivalence result (4.1.17) and Lemma 2.1.1 leads to the desired error result (5.4.62).

By (5.4.4), the equivalence result (4.1.17), Lemma 2.1.1 and the third bound in (5.3.39a) it follows for $i = 1, 2$ that

$$\begin{aligned} \|E_i - E_i^+\|_{L^\infty(0,T;(H^1(\Omega))')} &= \|U_i^+ - U_i\|_{L^\infty(0,T;(H^1(\Omega))')} \leq \Delta t\, \|\partial_t U_i\|_{L^\infty(0,T;(H^1(\Omega))')} \\ &\leq C\Delta t \max_{n=1\to N} \left\|\frac{U_i^n - U_i^{n-1}}{\Delta t}\right\|_{-h} \leq C\Delta t. \end{aligned} \tag{5.4.74}$$

Therefore, from (5.4.62) and (5.4.74) we obtain for $i = 1, 2$

$$\begin{aligned} \|E_i^+\|_{L^\infty(0,T;(H^1(\Omega))')}^2 &\leq 2\|E_i^+ - E_i\|_{L^\infty(0,T;(H^1(\Omega))')}^2 + 2\|E_i\|_{L^\infty(0,T;(H^1(\Omega))')}^2 \\ &\leq C(\Delta t)^2, \end{aligned} \tag{5.4.75}$$

which is the required result (5.4.63). □

Now, we present the main numerical result in the thesis.

Theorem 5.4.4 Let the assumptions ($\mathbf{A_2}$) and ($\mathbf{A^h}$) hold. Then for all $\mu \in [0, \frac{1}{2}]$, for all $h \leq h_1$, for all $\Delta t > 0$ if $\theta \geq 8D + \mu\theta_*$ and for all $\Delta t < \frac{4\gamma}{(8D+\mu\theta_*-\theta)^2}$ if $\theta < 8D + \mu\theta_*$, the unique solution $\{U_1^n, U_2^n\}$ of ($\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}}$) satisfies the error bounds

$$\begin{aligned}
&\Big[\|u_1 - U_1^+\|^2_{L^2(0,T;H^1(\Omega))} + \|u_2 - U_2^+\|^2_{L^2(0,T;H^1(\Omega))} \\
&\qquad\qquad + \|u_1 - U_1^+\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|u_2 - U_2^+\|^2_{L^\infty(0,T;(H^1(\Omega))')}\Big] \\
&+\Big[\|u_1 - U_1\|^2_{L^2(0,T;H^1(\Omega))} + \|u_2 - U_2\|^2_{L^2(0,T;H^1(\Omega))} \\
&\qquad\qquad + \|u_1 - U_1\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|u_2 - U_2\|^2_{L^\infty(0,T;(H^1(\Omega))')}\Big] \\
&\qquad\qquad \leq C(\Delta t)^2 + \begin{cases} Ch^{\frac{4}{3}}\big(\ln(1/h)\big)^{\frac{2(d-1)}{3}} & \text{if } d = 1, 2, \\ Ch & \text{if } d = 3. \end{cases}
\end{aligned} \tag{5.4.76}$$

Proof. Noting for $i = 1, 2$ that

$$u_i - U_i^+ = (u_i - u_i^h) + (u_i^h - U_i^+) = e_i + E_i^+,$$

$$u_i - U_i = (u_i - u_i^h) + (u_i^h - U_i) = e_i + E_i,$$

and recalling the semi-discrete error bound in Theorem 4.4.3 and the time discretisation error bound in Theorem 5.4.3 we obtain the desired result (5.4.76). □

Remark. As a result of the fully-discrete error bound in Theorem 5.4.4, we have the following convergence to the solution of the continuous problem

$$U_1, U_1^+ \to u_1 \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;(H^1(\Omega))'),$$

$$U_2, U_2^+ \to u_2 \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;(H^1(\Omega))'),$$

as $h, \Delta t \to 0$.

Chapter 6

Numerical experiments

In this chapter we shall perform numerical experiments in one and two dimensions which verify the theoretical results derived before and to see the growth behaviour of the solutions. All simulations were run by programs written in Fortran and Matlab programming languages. In Section 6.1 we present a practical algorithm for computing the numerical solution. In Section 6.2 we discuss computational results of the fully-discrete error bound in one dimension. Further, a comparison between the linear stability analysis and the numerical approximation is investigated. Finally, two dimensional simulations are presented in Section 6.3.

6.1 Practical algorithm

In this section we present a practical algorithm for solving the nonlinear algebraic system arising from problem $(\mathbf{P}_{\mu}^{\mathbf{h},\mathbf{\Delta t}})$ for $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ at each time step. In our algorithm we rely on the general splitting algorithm of Lions and Mercier [47], which has been used to solve other variants of Cahn-Hilliard equations e.g. [16] and [13].

For given $\lambda > 0$ and n fixed we define $\{R_1^n, R_2^n\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$(R_i^n, \chi)^h = (U_i^n + \lambda\phi(U_i^n), \chi)^h. \tag{6.1.1}$$

We also define $\{Y_1^n, Y_2^n\} \in S^h \times S^h$ such that all $\chi \in S^h$

$$(Y_1^n, \chi)^h = \frac{\lambda}{2}(f_D^{(1)}(U_1^n, U_2^n) + f_D^{(1)}(U_1^n, U_2^{n-1}), \chi)^h, \tag{6.1.2a}$$

$$(Y_2^n, \chi)^h = \frac{\lambda}{2}(f_D^{(2)}(U_1^n, U_2^n) + f_D^{(2)}(U_1^{n-1}, U_2^n), \chi)^h. \tag{6.1.2b}$$

Multiplying the equation (5.1.6b) of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ by $\lambda > 0$, adding and subtracting $(U_i^n, \chi)^h$ to the left hand side and noting (6.1.1), (6.1.2a-b) and (5.1.4), it follows that $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ satisfy for $i = 1, 2$ and for all $\chi \in S^h$

$$(R_i^n - U_i^n, \chi)^h = -\lambda\big[\gamma(\nabla U_i^n, \nabla\chi) - (\mu\theta_i U_i^n + (1-\mu)\theta_i U_i^{n-1} + W_i^n, \chi)^h\big] - (Y_i^n, \chi)^h, \tag{6.1.3}$$

We also introduce $\{X_1^n, X_2^n\} \in S^h \times S^h$ such that for all $\chi \in S^h$ and for $i = 1, 2$

$$(X_i^n - U_i^n, \chi)^h = \lambda\big[\gamma(\nabla U_i^n, \nabla\chi) - (\mu\theta_i U_i^n + (1-\mu)\theta_i U_i^{n-1} + W_i^n, \chi)^h\big] + (Y_i^n, \chi)^h. \tag{6.1.4}$$

From (6.1.3) and (6.1.4) we note for $i = 1, 2$ that $X_i^n = 2U_i^n - R_i^n$. Now, we introduce our iterative procedure relying on the above splitting of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$.
For fixed $n \geq 1$ set $U_1^{n,0} = U_1^{n-1} \in S_{m_1}^h$ and $U_2^{n,0} = U_2^{n-1} \in S_{m_2}^h$.
For $p \geq 0$ we define $\{Y_1^{n,p}, Y_2^{n,p}\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$(Y_1^{n,p}, \chi)^h = \frac{\lambda}{2}(f_D^{(1)}(U_1^{n,p-1}, U_2^{n,p-1}) + f_D^{(1)}(U_1^{n,p-1}, U_2^{n-1}), \chi)^h, \tag{6.1.5a}$$

$$(Y_2^{n,p}\chi)^h = \frac{\lambda}{2}(f_D^{(2)}(U_1^{n,p-1}, U_2^{n,p-1}) + f_D^{(2)}(U_1^{n-1}, U_2^{n,p-1}), \chi)^h, \tag{6.1.5b}$$

where $U_1^{n,-1} := U_1^{n,0}$ and $U_2^{n,-1} := U_2^{n,0}$ and then we define $\{R_1^{n,p}, R_2^{n,p}\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$(R_i^{n,p} - U_i^{n,p}, \chi)^h = -\lambda\big[\gamma(\nabla U_i^{n,p}, \nabla\chi) - (\mu\theta_i U_i^{n,p} + (1-\mu)\theta_i U_i^{n-1} + W_i^{n,p}, \chi)^h\big] - (Y_i^{n,p}, \chi)^h, \tag{6.1.6}$$

where $\{W_1^{n,0}, W_2^{n,0}\}$ is arbitrary in $S^h \times S^h$.

Next, we find $\{U_1^{n,p+\frac{1}{2}}, U_2^{n,p+\frac{1}{2}}\} \in S^h \times S^h$ such that for all $\chi \in S^h$ and for $i = 1, 2$

$$(R_i^{n,p}, \chi)^h = (U_i^{n,p+\frac{1}{2}} + \lambda\phi(U_i^{n,p+\frac{1}{2}}), \chi)^h \tag{6.1.7}$$

and we find $\{U_1^{n,p+1}, U_2^{n,p+1}, W_1^{n,p+1}, W_2^{n,p+1}\} \in S^h_{m_1} \times S^h_{m_2} \times S^h \times S^h$ such that for all $\chi \in S^h$ and for $i = 1, 2$

$$\Big(\frac{U_i^{n,p+1} - U_i^{n-1}}{\Delta t}, \chi\Big)^h + \big(\nabla W_i^{n,p+1}, \chi\big)^h = 0, \tag{6.1.8a}$$

$$(U_i^{n,p+1}, \chi)^h + \lambda\big[\gamma(\nabla U_i^{n,p+1}, \nabla\chi) - (\mu\theta_i U_i^{n,p+1} + W_i^{n,p+1}, \chi)^h\big] = (X_i^{n,p+1} + \lambda(1-\mu)\theta_i U_i^{n-1} - Y_i^{n,p+1}, \chi)^h, \tag{6.1.8b}$$

where $X_i^{n,p+1} := 2U_i^{n,p+\frac{1}{2}} - R_i^{n,p}, i = 1, 2$. Note that from (6.1.6) and (6.1.8b) one can easily see that $X_i^{n,p+1} = 2U_i^{n,p+1} - R_i^{n,p+1}, i = 1, 2$, for $p \geq 0$.

Existence and uniqueness of $\{U_1^{n,p+\frac{1}{2}}, U_2^{n,p+\frac{1}{2}}\}$ solving (6.1.7) follows from setting $\chi = \varphi_j,\ j = 0, 1, ..., J$ and noting the monotonicity of ϕ. Now, we prove existence and uniqueness of a solution to (6.1.8a-b). To do so, we first rewrite (6.1.8a-b), similarly to (5.1.9)-(5.1.11), in the following equivalent form
Find $\{U_1^{n,p+1}, U_2^{n,p+1}\} \in S^h_{m_1} \times S^h_{m_2}$ such that for all $\chi \in S^h$ and for $i = 1, 2$

$$\begin{aligned}&\Big(U_i^{n,p+1}, \chi - ⨍ \chi\Big)^h \\ &\quad + \lambda\Big[\gamma(\nabla U_i^{n,p+1}, \nabla\chi) - \Big(\mu\theta_i U_i^{n,p+1} - \hat{\mathcal{G}}^h\Big(\frac{U_i^{n,p+1} - U_i^{n-1}}{\Delta t}\Big), \chi - ⨍ \chi\Big)^h\Big] \\ &\quad = \Big(X_i^{n,p+1} + \lambda(1-\mu)\theta_i U_i^{n-1} - Y_i^{n,p+1}, \chi - ⨍ \chi\Big)^h,\end{aligned} \tag{6.1.9}$$

where

$$W_i^{n,p+1} = -\hat{\mathcal{G}}^h\Big(\frac{U_i^{n,p+1} - U_i^{n-1}}{\Delta t}\Big) + ⨍ W_i^{n,p+1}, \tag{6.1.10a}$$

$$⨍ W_i^{n,p+1} = \lambda^{-1} ⨍ \big[-X_i^{n,p+1} + Y_i^{n,p+1}\big] + \lambda^{-1} m_i - \theta_i\, m_i. \tag{6.1.10b}$$

To prove existence of a solution to (6.1.9) we consider the following minimization problem

$$\begin{aligned}\min_{\{\chi_1,\chi_2\}\in S^h_{m_1}\times S^h_{m_2}} \Big\{ I^h(\chi_1,\chi_2) &:= (1-\lambda\mu\theta_1)|\chi_1|_h^2 + (1-\lambda\mu\theta_2)|\chi_2|_h^2 + \lambda\gamma\big[|\chi_1|_1^2 + |\chi_2|_1^2\big] \\ &+ \frac{\lambda}{\Delta t}\big[\|\chi_1 - U_1^{n-1}\|_{-h}^2 + \|\chi_2 - U_2^{n-1}\|_{-h}^2\big] \\ &- 2\big[(L_1^{n,p+1}, \chi_1)^h + (L_2^{n,p+1}, \chi_2)^h\big],\end{aligned} \tag{6.1.11}$$

where, for $i = 1, 2$,

$$L_i^{n,p+1} := X_i^{n,p+1} + \lambda(1-\mu)\theta_i U_i^{n-1} - Y_i^{n,p+1}. \tag{6.1.12}$$

Setting $\theta_* := \max\{\theta_1, \theta_2\}$ we have

$$I^h(\chi_1, \chi_2) \geq (1 - \lambda\,\mu\,\theta_*)\big[|\chi_1|_h^2 + |\chi_2|_h^2\big] + \lambda\gamma\big[|\chi_1|_1^2 + |\chi_2|_1^2\big] \\ + \frac{\lambda}{\Delta t}\big[\|\chi_1 - U_1^{n-1}\|_{-h}^2 + \|\chi_2 - U_2^{n-1}\|_{-h}^2\big] - 2\big[(L_1^{n,p+1}, \chi_1)^h + (L_2^{n,p+1}, \chi_2)^h\big]. \tag{6.1.13}$$

Now if $1 - \lambda\mu\theta_* \geq 0$, it then follows from Poncaré's and Young's inequalities that

$$\begin{aligned} I^h(\chi_1, \chi_2) &\geq (1 - \lambda\mu\theta_* + C)\big[|\chi_1|_h^2 + |\chi_2|_h^2\big] - \lambda\gamma\big[m_1^2|\Omega|^2 + m_2^2|\Omega|^2\big] \\ &\quad - 2\big[(L_1^{n,p+1}, \chi_1)^h + (L_2^{n,p+1}, \chi_2)^h\big] \\ &\geq \frac{(1 - \lambda\mu\theta_* + C)}{2}\big[|\chi_1|_h^2 + |\chi_2|_h^2\big] - C\big[1 + |L_1^{n,p+1}|_h^2 + |L_2^{n,p+1}|_h^2\big]. \end{aligned} \tag{6.1.14}$$

If $1 - \lambda\mu\theta_* < 0$ we first note from (4.1.14) that

$$\begin{aligned} |\chi_i|_h^2 &= (\chi_i - U_i^{n-1}, \chi_i)^h + (U_i^{n-1}, \chi_i)^h \\ &\leq \frac{\lambda}{\Delta t(\lambda\mu\theta_* - 1)}\|\chi_i - U_i^{n-1}\|_{-h}^2 + \frac{\Delta t(\lambda\mu\theta_* - 1)}{4\lambda}|\chi_i|_1^2 + (U_i^{n-1}, \chi_i)^h \quad i = 1, 2, \end{aligned}$$

which leads together with (6.1.13) to

$$I^h(\chi_1, \chi_2) \geq \big(\lambda\gamma - \frac{\Delta t(1 - \lambda\mu\theta_*)^2}{4\lambda}\big)\big[|\chi_1|_1^2 + |\chi_2|_1^2\big] - (2L_1^{n,p+1} - (1 - \lambda\mu\theta_*)U_1^{n-1}, \chi_1)^h \\ - (2L_2^{n,p+1} - (1 - \lambda\mu\theta_*)U_2^{n-1}, \chi_2)^h\big].$$

Thus, for $\Delta t < \frac{4\gamma\lambda^2}{(1-\lambda\mu\theta_*)^2}$ we have, similarly to (6.1.14), by Poincaré's and Young's inequalities that

$$I^h(\chi_1, \chi_2) \geq \frac{1}{2}\big(\lambda\gamma - \frac{\Delta t(1 - \lambda\mu\theta_*)^2}{4\lambda}\big)\, C\big[|\chi_1|_h^2 + |\chi_2|_h^2\big] \\ - C\big[1 + |2L_1^{n,p+1} - (1 - \lambda\mu\theta_*)U_1^{n-1}|_h^2 + |2L_2^{n,p+1} - (1 - \lambda\mu\theta_*)U_2^{n-1}|_h^2\big]. \tag{6.1.15}$$

Therefore, from (6.1.14) and (6.1.15) one can conclude that there exist $\{U_1^{n,p+1}, U_2^{n,p+1}\} \in S_{m_1}^h \times S_{m_2}^h$ solving the above minimization problem. Now we can easily see for $i = 1, 2$ that (6.1.9) is the Euler-Lagrange equations of the minimization problem.

It remains to show the uniqueness result which can be easily established and for completeness we provide the proof. To this aim, let $B_{n,p+1} := \{U_1^{n,p+1}, U_2^{n,p+1}\}$ and $B^*_{n,p+1} := \{U_1^{n,p+1,*}, U_2^{n,p+1,*}\}$ be two solutions to (6.1.9). Substituting χ in (6.1.9) by $\bar{U}_i^{n,p+1} := U_i^{n,p+1} - U_i^{n,p+1,*} \in V_0^h$ and then subtracting yields for $i = 1, 2$ that

$$\lambda\,\gamma|\bar{U}_i^{n,p+1}|_1^2 + \frac{\lambda}{\Delta t}\|\bar{U}_i^{\,n,p+1}\|_{-h}^2 = -(1 - \lambda\,\mu\,\theta_i)|\bar{U}_i^{n,p+1}|_h^2. \qquad (6.1.16)$$

If $1 - \lambda\,\mu\,\theta_i \geq 0$, then the uniqueness result follows immediately from Poincaré's inequality . Whereas if $1 - \lambda\,\mu\,\theta_i < 0$ we apply (4.1.15) to the right hand side of (6.1.16) to give for $i = 1, 2$ that

$$\lambda\,\gamma|\bar{U}_i^{n,p+1}|_1^2 + \frac{\lambda}{\Delta t}\|\bar{U}_i^{\,n,p+1}\|_{-h}^2 \leq \frac{\Delta t\,(\lambda\,\mu\,\theta_i - 1)^2}{4\lambda}|\bar{U}_i^{n,p+1}|_1^2 + \frac{\lambda}{\Delta t}\|\bar{U}_i^{\,n,p+1}\|_{-h}^2 \quad (6.1.17)$$

and hence we obtain the uniqueness result by Poincaré's inequality for all $\Delta t < \frac{4\lambda^2\,\gamma}{(\lambda\,\mu\,\theta_i - 1)^2}$. Finally, existence and uniqueness of $W_1^{n,p+1}$ and $W_2^{n,p+1}$ follows directly from (6.1.10a -b). Therefore, the iterative approach (6.1.5a-b)-(6.1.8a-b) is well-defined for any $\lambda > 0$, for any $\mu \in [0, \frac{1}{2}]$ and for Δt sufficiently small. In fact we were unable to prove the convergence of this iterative procedure, however, we observed good convergence properties in practice. For each $n \geq 1$ we adopted the stopping criteria

$$\max\{|U_1^{n,p} - U_1^{n,p-1}|_{0,\infty}, |U_2^{n,p} - U_2^{n,p-1}|_{0,\infty}\} \leq tol.$$

From the above iteration procedure we observe that at each iteration p we need to solve (i) (6.1.7) for $\{U_1^{n,p+\frac{1}{2}}, U_2^{n,p+\frac{1}{2}}\}$ and (ii) (6.1.9) for $\{U_1^{n,p+1}, U_2^{n,p+1}\}$.

For (i) we set $\chi = \varphi_j,\ j = 0 \rightarrow J$ and then we solve the resulting equations at each node x_j for $\{U_1^{n,p+\frac{1}{2}}(x_j), U_2^{n,p+\frac{1}{2}}(x_j)\}$ using Newton's method. For (ii) we first represent $U_i^{n,p+1}$, U_i^{n-1} and $L_i^{n,p+1}$, $i = 1, 2$ in terms of the basis functions $\{\varphi_j\}_{j=0}^J$ as

$$U_i^{n,p+1} = \sum_{j=0}^{J} U_{i,j}^{n,p+1}\,\varphi_j, \quad U_i^{n-1} = \sum_{j=0}^{J} U_{i,j}^{n-1}\,\varphi_j, \quad L_i^{n,p+1} = \sum_{j=0}^{J} L_{i,j}^{n,p+1}\,\varphi_j. \quad (6.1.18)$$

Using the matrices defined by (4.1.26) we can write (4.1.11) for any $v \in V_0^h$ in the matrix form as: Find $\hat{\mathcal{G}}^h(\underline{v}) \in \mathbb{R}^{J+1}$ such that

$$A\hat{\mathcal{G}}^h(\underline{v}) = M\underline{v}, \tag{6.1.19}$$

where $(\hat{\mathcal{G}}^h(\underline{v}))_j = \hat{\mathcal{G}}^h v(x_j)$ and $(\underline{v})_j = v(x_j),\ j = 0 \rightarrow J.$

Hence, we have

$$M^{-1}\, A\, \hat{\mathcal{G}}^h(\underline{v}) = \underline{v}. \tag{6.1.20}$$

Now, by inserting (6.1.18) into (6.1.9), setting $\chi = \varphi_k$, noting (6.1.12) and (6.1.20) and multiplying by $M^{-1}AM^{-1}$ we can restate (6.1.9) in the vector from as:
Find $\{\underline{U}_1^{n,p+1}, \underline{U}_2^{n,p+1}\} \in \mathbb{R}^{J+1} \times \mathbb{R}^{J+1}$ such that for $i = 1, 2$

$$\mathcal{R}\underline{U}_i^{n,p+1} + \lambda\Big(\gamma\mathcal{R}^2\underline{U}_i^{n,p+1} - \mu\theta_i\mathcal{R}\underline{U}_i^{n,p+1} + \frac{1}{\Delta t}(\underline{U}_i^{n,p+1} - \underline{U}_i^{n-1})\Big) = \mathcal{R}\underline{L}_i^{n,p+1}, \tag{6.1.21}$$

where $\mathcal{R} := M^{-1}A$. Such linear systems can be solved using a discrete cosine transform when we have a uniform partitioning $\mathcal{T}^h$, see e.g. [9] where the same approach was considered for similar system.

6.2 One-dimensional simulations

6.2.1 Verification of the fully-discrete error bound

In this section we present numerical evidence in one space dimension for the fully-discrete error bound (5.4.76) derived in Theorem 5.4.4. As no exact solution to the continuous problem $(\mathbf{P})$ is known, we made a comparison between the computed solution of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ on a fine mesh and small time step with some computed on a sequence of coarse meshes or larger time steps.

Let $\{\hat{u}_1^n, \hat{u}_2^n\}$ be the computed solutions of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ at the level time n on the uniform fine mesh with space step h_f and the small time step $\Delta t_f = T/N_f$, and $\{U_1^n, U_2^n\}$ be the solution at the level time n on a coarse uniform mesh with space step h or larger time step $\Delta t = T/N$. Now, we define for $i = 1, 2$

$$\hat{u}_i^+(t) := \hat{u}_i^n, \quad t \in (t_{n-1}, t_n], \quad t_n = n\Delta t_f, \quad 1 \leq n \leq N_f, \tag{6.2.1a}$$

$$U_i^+(t) := U_i^n, \quad t \in (t_{n-1}, t_n], \quad t_n = n\Delta t, \quad 1 \leq n \leq N. \tag{6.2.1b}$$

Treating $\hat{u}_i^+$ as the exact solution it follows from (5.4.76) that for $d = 1$

$$\|\hat{u}_1^+ - U_1^+\|^2_{L^2(0,T;H^1(\Omega))} + \|\hat{u}_2^+ - U_2^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C[h^{4/3} + (\Delta t)^2]. \tag{6.2.2}$$

In order to calculate exactly the left hand side of error bound (6.2.2) we shall choose h to be a multiple of h_f and Δt to be a multiple of Δt_f. In other words, the above parameters are subject to the following relations

$$h = p_s h_f, \quad \Delta t = p_t \Delta t_f, \tag{6.2.3}$$

for some $p_s, p_t \in \mathbb{N}$.

We then evaluate the error via the quantities

$$\rho_1(h, \Delta t) := \|\hat{u}_1^+ - U_1^+\|^2_{L^2(0,T;H^1(\Omega))} = \Delta t_f \sum_{n=1}^{N_f} |\hat{u}_1^n - U_1^m|_0^2 + |\hat{u}_1^n - U_1^m|_1^2, \tag{6.2.4}$$

$$\rho_2(h, \Delta t) := \|\hat{u}_2^+ - U_2^+\|^2_{L^2(0,T;H^1(\Omega))} = \Delta t_f \sum_{n=1}^{N_f} |\hat{u}_2^n - U_2^m|_0^2 + |\hat{u}_2^n - U_2^m|_1^2, \tag{6.2.5}$$

$$\rho(h, \Delta t) := \rho_1(h, \Delta t) + \rho_2(h, \Delta t), \tag{6.2.6}$$

where

$$m = \left\lceil \frac{n}{p_t} \right\rceil,$$

and for any $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer greater than or equal to x.

In addition, the H^1-norm in space involved in (6.2.4) and (6.2.5) can be computed exactly, since for any $\chi^{h_f} \in S^{h_f}$ and $v^h \in S^h$

$$|\chi^{h_f} - v^h|_1^2 = h_f \sum_{j=0}^{J_f-1} \left(\frac{1}{h_f}(\chi_{j+1}^{h_f} - \chi_j^{h_f}) - \frac{1}{h}(v_{\ell+1}^h - v_\ell^h) \right)^2, \tag{6.2.7}$$

$$|\chi^{h_f} - v^h|_0^2 = h_f \sum_{j=0}^{J_f-1} F_j, \tag{6.2.8}$$

where, on noting that $\{\hat{x}_j\}_{j=0}^{J_f}$ and $\{x_\ell\}_{\ell=0}^{J}$ are the set of nodes of S^{h_f} and S^h respectively, $\chi_j^{h_f} \equiv \chi^{h_f}(\hat{x}_j)$, $v_\ell^h \equiv v^h(x_\ell)$,

$$\ell = \left\lceil \frac{j+1}{p_s} \right\rceil - 1,$$

and F_j is defined according to the value of $\omega_j := \frac{1}{h_f}(\chi_{j+1}^{h_f} - \chi_j^{h_f}) - \frac{1}{h}(v_{\ell+1}^h - v_\ell^h)$ by

$$F_j = \begin{cases} \Big(\frac{1}{h_f}(\chi_j^{h_f}\hat{x}_{j+1} - \chi_{j+1}^{h_f}\hat{x}_j) - \frac{1}{h}(v_\ell^h x_{\ell+1} - v_{\ell+1}^h x_\ell)\Big)^2 & \text{if } \omega_j = 0, \\ \\ \frac{1}{3\omega_j}\Big(\big(\chi_{j+1}^{h_f} - \frac{1}{h}(v_{\ell+1}^h(\hat{x}_{j+1} - x_\ell) - v_\ell^h(\hat{x}_{j+1} - x_{\ell+1}))\big)^3 & \\ \qquad -\big(\chi_j^{h_f} - \frac{1}{h}(v_{\ell+1}^h(\hat{x}_j - x_\ell) - v_\ell^h(\hat{x}_j - x_{\ell+1}))\big)^3\Big) & \text{if } \omega_j \neq 0. \end{cases} \tag{6.2.9}$$

To verify the error bound (6.2.2) we used the following data in the experiments: $\Omega = (0,1)$, $\gamma = 0.005$, $D = 0.2$, $\mu = 0.5$, $\theta_1 = \theta_2 = 1$, $\theta = 0.25$, $T = 0.5$, $tol = 10^{-7}$ and $\lambda = 0.1$. We computed $\hat{u}_1^n$ and $\hat{u}_2^n$ on uniform fine mesh with fixed space step $h_f = 2^{-11}$ and fixed small time step $\Delta t_f = \frac{1}{3(2^{14})}$. While U_1^n and U_2^n were computed on uniform coarse meshes with $h = 2^{-p}$ where $(p = 5, 6, 7, 8, 9)$ or on larger time steps $\Delta t = \frac{1}{3(2^q)}$ where $(q = 8, 9, 10, 11, 12)$. The initial data u_1^0 and u_2^0 were taken to be the clamped cubic splines generated by the values

$$\{-0.4 \quad 0.5 \quad 0.88 \quad -0.4 \quad -0.3\} \text{ and } \{-0.2 \quad 0.7 \quad -0.5 \quad -0.3 \quad -0.7\}$$

at the points $i/4$, $i = 0, 1, 2, 3, 4$ and we set $\hat{u}_i^0 = U_i^0 = P_\gamma^h u_i^0$, $i = 1, 2$. Note that this choice of initial data satisfies the assumptions $(\mathbf{A_2})$, stated in page 34, rigorously.

Using (6.2.4)-(6.2.6) we computed the following ratios

$$R^h := \frac{\rho(h,\Delta t) - \rho(h/2,\Delta t)}{\rho(h/2,\Delta t) - \rho(h/4,\Delta t)}, \quad R^{\Delta t} := \frac{\rho(h,\Delta t) - \rho(h,\Delta t/2)}{\rho(h,\Delta t/2) - \rho(h,\Delta t/4)} \tag{6.2.10}$$

and the results are displayed in Table 6.1 and Table 6.2.

Assuming that we can write the quantity $\rho(h, \Delta t)$ in the form

$$a_s h^{k_s} + a_t(\Delta t)^{k_t}, \quad a_s, a_t, k_s, k_t \in \mathbb{R}$$

and inserting this form into (6.2.10) yields after simplifying that $R^h = 2^{k_s}$ and $R^{\Delta t} = 2^{k_t}$. The results shown in Table 6.1 and Table 6.2 indicate that the rates of convergence in space and in time are both 4, i.e. $k_s = k_t = 2$. In comparison with the rates of convergence proved in Theorem 5.4.4 (that are, $2^2 = 4$ in time and $2^{4/3} \approx 2.52$ in space), this is consistent with the theoretical result in time but it is practically better in space. Therefore, one concludes that it may be possible to prove an optimal error bound in space for $(\mathbf{P}_{\mu}^{\mathbf{h},\mathbf{\Delta t}})$; that is, $C[h^{4/3} + (\Delta t)^2]$ in the error bound (5.4.76) is replaced by $C[h^2 + (\Delta t)^2]$. On the other hand, our choice of initial data may be flawed in some way.

We performed several experiments with other parameter values which led to similar results. In Figure 6.1 we plot the evolution of the finite element approximations with the above initial data (the cubic splines) at different times where the graph at time $T = 0.5$ represents the stationary solutions[1]. We also found that the numerical approximations, U_1 and U_2, are strictly between -1 and 1 which is consistent with our theoretical result. In fact, this result has been observed with all of experiments in this chapter. In addition, for any choice $\mu \in [0, 1/2]$ we have noticed that the stationary solution is the same.

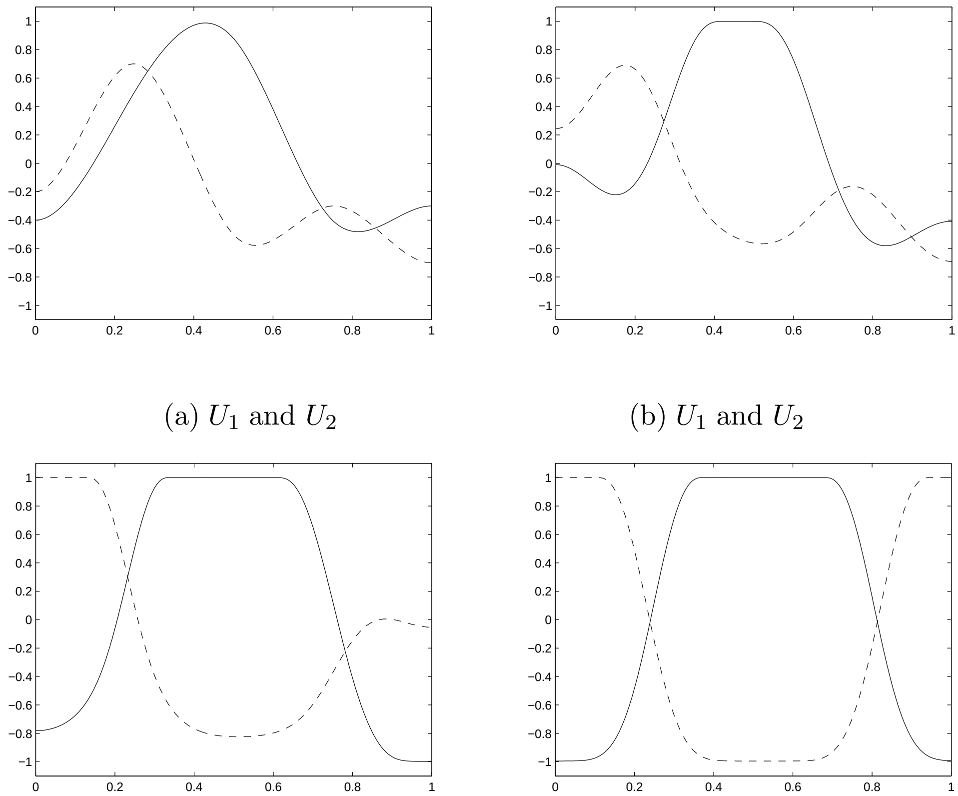
[1]By a stationary solution we mean that the numerical solution does not change from one time level to the next.

h	$\rho_1(h, \Delta t)$	$\rho_2(h, \Delta t)$	$\rho(h, \Delta t)$	R^h
1/32	0.232647985	0.223472506	0.456120491	4.06
1/64	0.0573374517	0.0552525558	0.112590007	4.02
1/128	0.014204694	0.0136775514	0.0278822444	4.03
1/256	0.00347895757	0.00334953098	0.00682848832	
1/512	0.000816646731	0.000786298187	0.00160294492	

Table 6.1: Verification of the error bound in Theorem 5.4.4: $\hat{u}_1^n$ and $\hat{u}_2^n$ were computed with $h_f = 1/2^{11}$ and $\Delta t_f = 1/3(2)^{14}$, and U_1^n and U_2^n were computed with successive $h = 1/2^p,\ p = 5, 6, 7, 8, 9$ and fixed $\Delta t = 49152$.

Δt	$\rho_1(h, \Delta t)$	$\rho_2(h, \Delta t)$	$\rho(h, \Delta t)$	$R^{\Delta t}$
1/768	0.00187837437	0.00300267292	0.00488104718	3.69
1/1536	0.000500014808	0.000793130719	0.00129314547	3.93
1/3072	0.000124316095	0.000195824061	0.000320140156	4.22
1/6144	2.83569134E-005	4.43079516E-005	7.2664865E-005	
1/12288	5.51559924E-006	8.52606081E-006	1.40416596E-005	

Table 6.2: Verification of the error bound in Theorem 5.4.4: $\hat{u}_1^n$ and $\hat{u}_2^n$ were computed with $h_f = 1/2^{11}$ and $\Delta t_f = 1/3(2)^{14}$, and U_1^n and U_2^n were computed with fixed $h = 1/2048$ and successive $\Delta t = 1/3(2)^q,\ q = 8, 9, 10, 11, 12$.



(a) U_1 and U_2 (b) U_1 and U_2

(c) U_1 and U_2 (d) U_1 and U_2

Figure 6.1: Numerical solutions U_1, denoted —, and U_2, denoted - - -, with cubic splines initial data at times (a) $t = 0$ (b) $t = 0.0125$ (c) $t = 0.05$ (d) $t = 0.5$.

6.2.2 A comparison between the linearised solution and the numerical approximation

In this subsection we compare the numerical approximation of $(\mathbf{P}_\mu^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ with the solution of the corresponding linearised problem. We have analysed the linearised problem of $(\mathbf{P})$ and found that a necessary condition to have growth in at least one of the linearised solutions u_1 or u_2 is that $\lambda^-(m_1, m_2) + \gamma\pi^2 < 0$, where

$$\lambda^-(m_1, m_2) = \big(a + c - \sqrt{(a-c)^2 + 4b^2}\big)/2, \quad a = \frac{\theta}{1 - m_1^2} - \theta_1 + 2D(m_2 + \alpha_2)^2,$$
$$b = 4D(m_1 + \alpha_1)(m_2 + \alpha_2), \quad c = \frac{\theta}{1 - m_2^2} - \theta_2 + 2D(m_1 + \alpha_1)^2. \tag{6.2.11}$$

Furthermore, for the case $\theta_1 = \theta_2$ and $m_1 = m_2$ we found that the linearised solution may be written in the form

$$u_1(x,t) = m_1 + \frac{1}{2}\sum_{k=1}^{\infty}\Big[\exp(d_{1,k}t)(Q^0_{1,k} + Q^0_{2,k}) + \exp(d_{2,k}t)(Q^0_{1,k} - Q^0_{2,k})\Big]\cos(k\pi x),$$
$$u_2(x,t) = m_2 + \frac{1}{2}\sum_{k=1}^{\infty}\Big[\exp(d_{1,k}t)(Q^0_{1,k} + Q^0_{2,k}) - \exp(d_{2,k}t)(Q^0_{1,k} - Q^0_{2,k})\Big]\cos(k\pi x),$$
$$Q^0_{i,k} = \int_0^1 u_i^0(x)\,\cos(k\pi x)\,dx, \quad d_{i,k} = (k\pi)^2(-\gamma(k\pi)^2 - (a + (-1)^{i+1}b)). \tag{6.2.12}$$

A comparison with an exact solution

We consider the linearised problem of $(\mathbf{P})$ with the following initial conditions

$$u_i^0(x) = \xi_i(\cos(\pi x) - \cos(3\pi x)) \quad i = 1, 2,$$

where ξ_1 and ξ_2 are small.

Thus we have for $i = 1, 2$ that

$$Q^0_{i,k} = \begin{cases} \xi_i & \text{if } k = 1, \\ -\xi_i & \text{if } k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

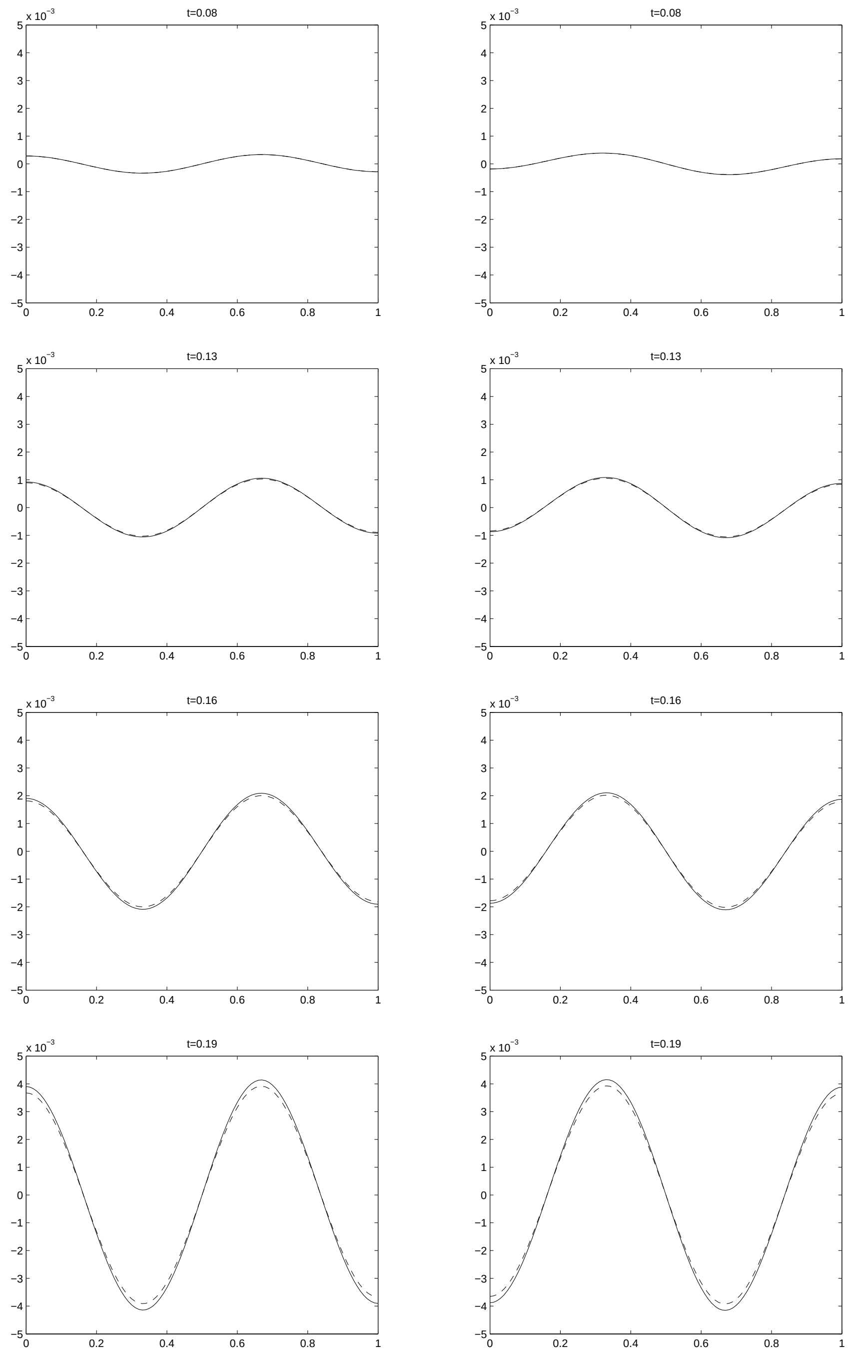
Since $m_1 = m_2 = 0$ for the above initial data, we conclude from (6.2.12) that the linearised solutions for the case $\theta_1 = \theta_2$ are

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{2}(\xi_1 + \xi_2)\big[\exp(d_{1,1}t)\cos(\pi x) - \exp(d_{1,3}t)\cos(3\pi x)\big] \\
&\quad + \frac{1}{2}(\xi_1 - \xi_2)\big[\exp(d_{2,1}t)\cos(\pi x) - \exp(d_{2,3}t)\cos(3\pi x)\big], \\
u_2(x,t) &= \frac{1}{2}(\xi_1 + \xi_2)\big[\exp(d_{1,1}t)\cos(\pi x) - \exp(d_{1,3}t)\cos(3\pi x)\big] \\
&\quad - \frac{1}{2}(\xi_1 - \xi_2)\big[\exp(d_{2,1})\cos(\pi x) - \exp(d_{2,3}t)\cos(3\pi x)\big],
\end{aligned}
$$

We ran four simulations to compare the numerical approximations with the above exact linearised solutions. In each simulation, we take, unless otherwise stated, $h = 0.01$, $\Delta t = h/40$, $\gamma = 0.005$, $\theta_1 = \theta_2 = 1$, $D = 0.5$ and $\mu = 0.5$. We kept the parameters of the iterative algorithm as taken in Section 2.1.1.

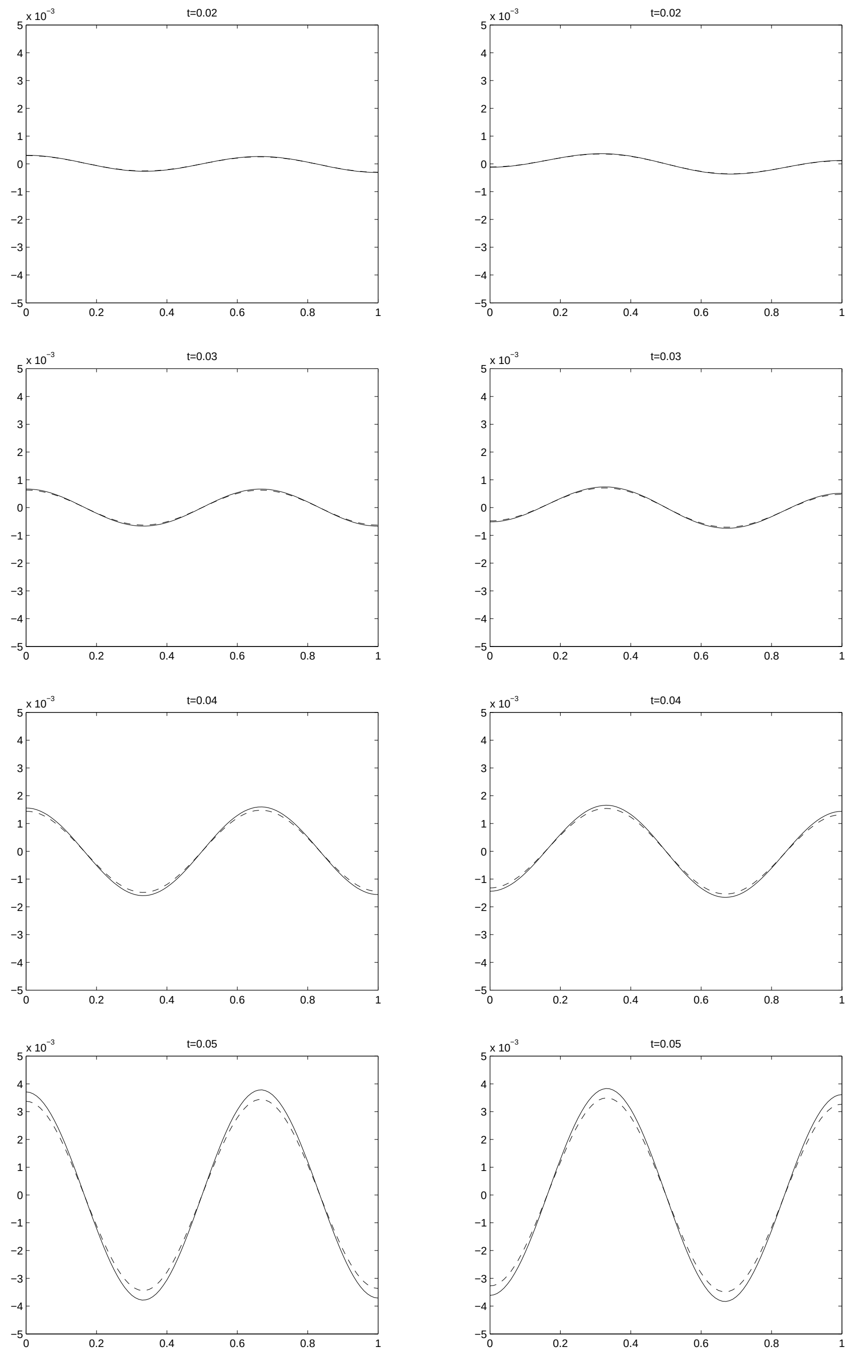
In the first experiment we chose $\xi_1 = 0.0001$, $\xi_2 = 0.0002$ and $\theta = 0.8$. We found that the linearised solutions u_1 and u_2 grow as time increases. The numerical approximations U_1 and U_2 are consistent with this behaviour, where they evolve in time until the stationary solutions are achieved. Similar results were obtained in the second experiment where the data used was the same as before except $\theta = 0.5$. The results of the first two experiments in the early stages of the evolution can be seen in Figure 6.2 and Figure 6.3.

In the third experiment we let $\xi_1 = 0.001$, $\xi_2 = 0.002$ and $\theta = 0.98$. Similarly to the first two experiments the growth behavior occurred in the linearised and numerical solutions as displayed in Figure 6.4. We repeated the third experiment with the same data except $D = 0.2$. This time we found, on the contrary, the linearised solutions decreases to zero as time increases and the numerical solutions behaved in the same manner, see Figure 6.5. In all experiments we found that the behaviour of the numerical approximations, U_1 and U_2, are in agreement with the linearised solutions, u_1 and u_2, behaviour. In addition, we noticed that the solutions evolves significantly faster when θ is far from θ_1 and θ_2.



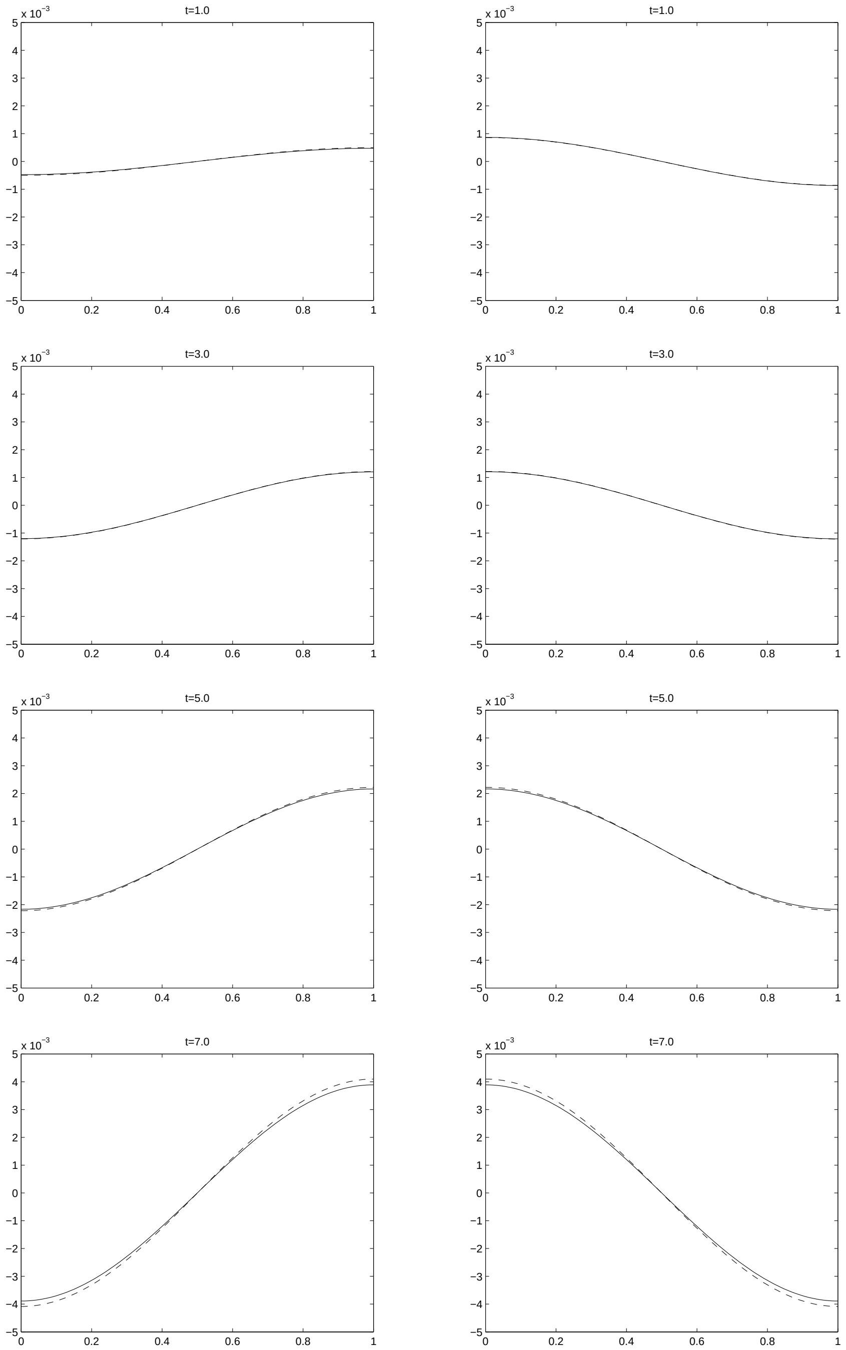
(a) u_1 and U_1. (b) u_2 and U_2.

Figure 6.2: A comparison of the linearised solution u_i, denoted —, and numerical approximation U_i, denoted - -, in time where (a) plots of u_1 and U_1, (b) plots of u_2 and U_2. The parameters values used are: $\theta = 0.8$, $\theta_1 = \theta_2 = 1$, $D = 0.5$, $\xi_1 = 0.0001$, $\xi_2 = 0.0002$.



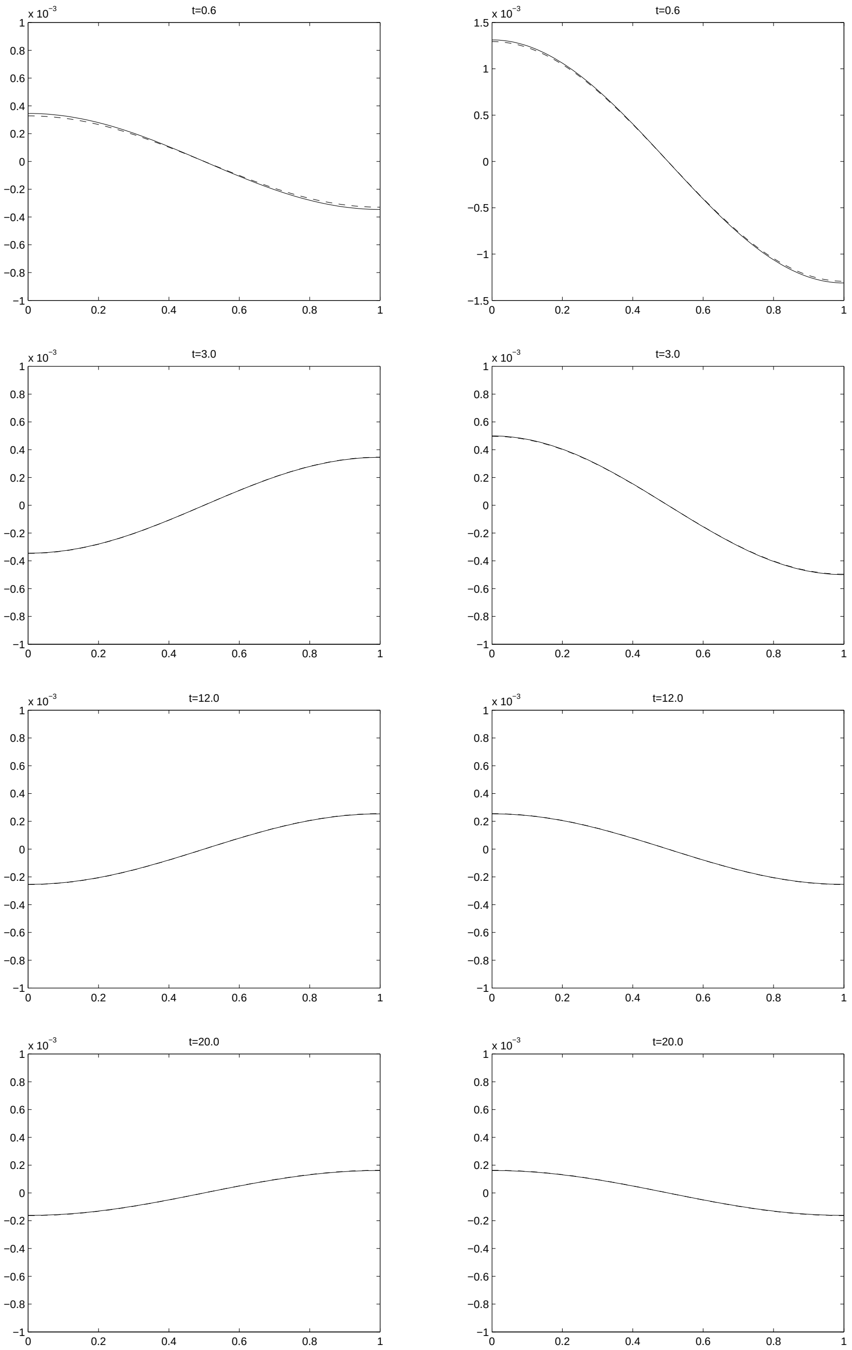
(a) u_1 and U_1. (b) u_2 and U_2.

Figure 6.3: A comparison of the linearised solution u_i, denoted —, and numerical approximation U_i, denoted - -, in time where (a) plots of u_1 and U_1, (b) plots of u_2 and U_2. The parameters values used are: $\theta = 0.5$, $\theta_1 = \theta_2 = 1$, $D = 0.5$, $\xi_1 = 0.0001$, $\xi_2 = 0.0002$.



(a) u_1 and U_1. (b) u_2 and U_2.

Figure 6.4: A comparison of the linearised solution u_i, denoted —, and numerical approximation U_i, denoted - -, where (a) plots of u_1 and U_1, (b) plots of u_2 and U_2. The parameters values used are: $\theta = 0.98$, $\theta_1 = \theta_2 = 1$, $D = 0.5$, $\xi_1 = 0.001$, $\xi_2 = 0.002$.



(a) u_1 and U_1. (b) u_2 and U_2.

Figure 6.5: A comparison of the linearised solution u_i, denoted —, and numerical approximation U_i, denoted - -, in time where (a) plots of u_1 and U_1, (b) plots of u_2 and U_2. The parameters values used are: $\theta = 0.98$, $\theta_1 = \theta_2 = 1$, $D = 0.2$, $\xi_1 = 0.001$, $\xi_2 = 0.002$.

A comparison with no exact solution

In all simulations of this section we take the initial data to be random perturbations of mean values m_1 and m_2 with fluctuation no larger than 0.05 at equally spaced points. Our aim is to investigate growth behaviour of the numerical solutions of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ under the condition $\lambda^-(m_1, m_2) + \gamma\pi^2 < 0$, see (6.2.11). We shall test this condition with the numerical approximations of $(\mathbf{P}_\mu^{\mathbf{h},\mathbf{\Delta t}})$ where we expect that if this condition holds, then growth at least one of the approximations occurs. To this aim, we consider some examples with different values of the parameters θ, θ_1, θ_2, D and γ involved in the explicit formula of $\lambda^-(m_1, m_2)+\gamma\pi^2$. In each example we first find the growth region by solving the equation $\lambda^-(m_1, m_2)+\gamma\pi^2 = 0$ for m_1 and m_2 and then we perform several simulations with different initial data inside and outside the growth region to see the behaviour of the numerical solutions. In all simulations we take $h = 0.01$, $\mu = 0.5$, $\lambda = 0.1$ and $tol = 10^{-7}$.

In the first example we take $\theta_1 = \theta_2 = 1.0$, $\theta = 0.2$, $D = 0.5$ and $\gamma = 0.005$. The growth region of this case is plotted in Figure 6.6(a). In this example we ran four simulations with time step $\Delta t = h/40$. As expected, for the initial data inside the growth region at least one of the numerical solutions grows until the stationary solutions are attained (see Figure 6.7 - Figure 6.9) while for the initial data outside the growth region we found that the numerical solutions are stable about m_1 and m_2 as shown in Figure 6.10.

In Figure 6.6 (b)-(d) we consider other examples of the growth region defined by different values of the parameters θ, θ_1, θ_2, D and γ. In these examples we performed several simulations with different values of m_1 and m_2 inside and outside the corresponding growth region. The results were similar to the first example where we found that the results are consistent with the growth regions. Figure 6.11 - Figure 6.13 show results for the growth region depicted in Figure 6.6 (b) where in this case we use the same parameters in the first example except $D = 0.4$ and $\theta = 0.6$. In Figure 6.14 - Figure 6.16 we test the growth region generated by $\gamma = 0.002$, $\theta_1 = 1.0$, $\theta_2 = 2.0$, $\theta = 0.8$ and $D = 0.5$, depicted in Figure 6.6 (c), with $\Delta t = h^2$.

Finally, the growth region of the parameter values $\gamma = 0.0005$, $\theta_1 = \theta_2 = 1.0$, $\theta = 0.95$ and $D = 0.6$, plotted in Figure 6.6 (d), was tested in Figure 6.17 and Figure 6.18 with $\Delta t = h^2$.

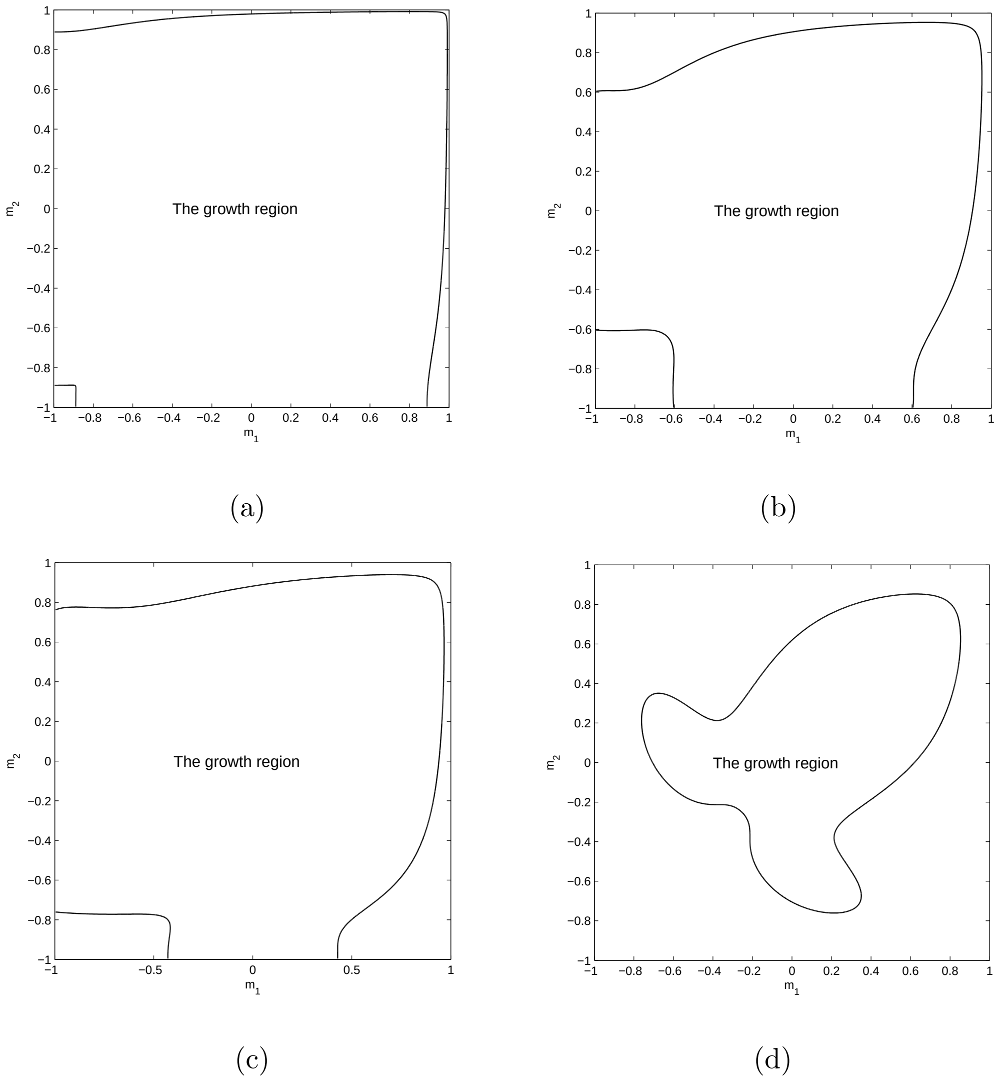
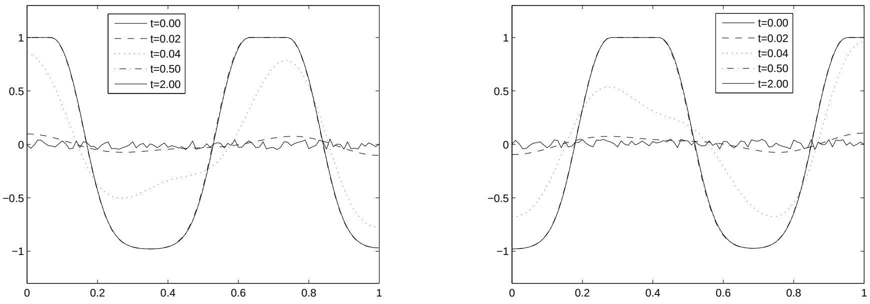
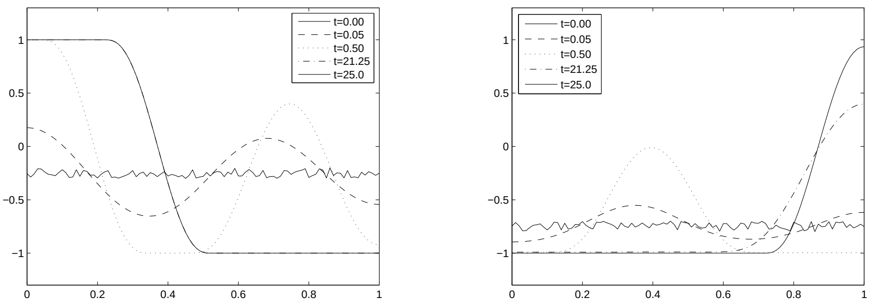


Figure 6.6: Growth region in which $\lambda^{-}(m_1, m_2) + \gamma\pi^2 < 0$ where the parameter values are: (a) $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.2,\ D = 0.5$,
(b) $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.6,\ D = 0.4$,
(c) $\gamma = 0.002,\ \theta_1 = 1,\ \theta_2 = 2,\ \theta = 0.8,\ D = 0.5$,
(d) $\gamma = 0.0005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.95,\ D = 0.6$.



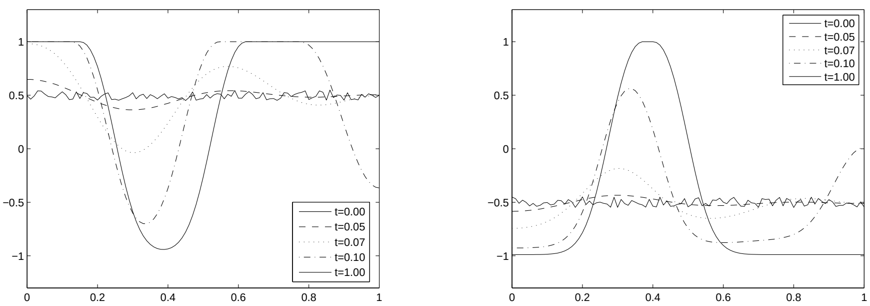
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.7: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (0, 0)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.2$ and $D = 0.5$.



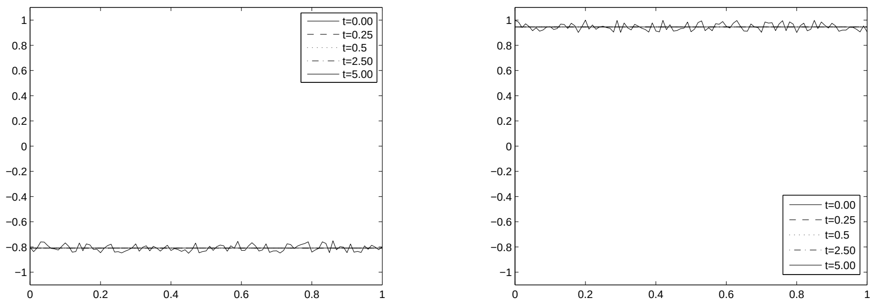
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.8: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.25, -0.75)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.2$ and $D = 0.5$.



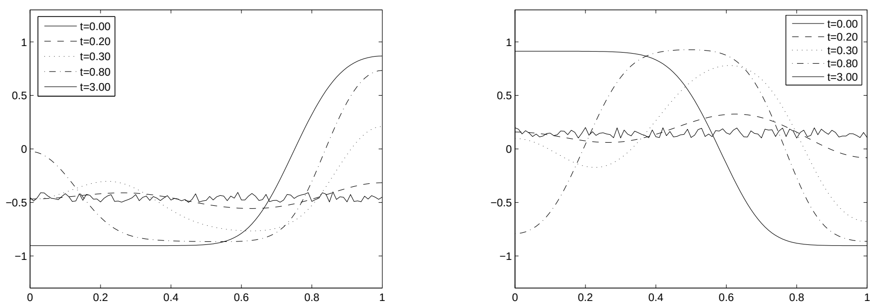
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.9: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (0.5, -0.5)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.2$ and $D = 0.5$.



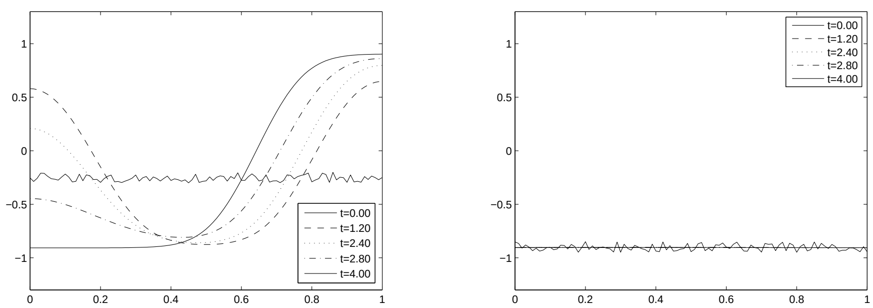
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.10: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.8, 0.95)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.2$ and $D = 0.5$.



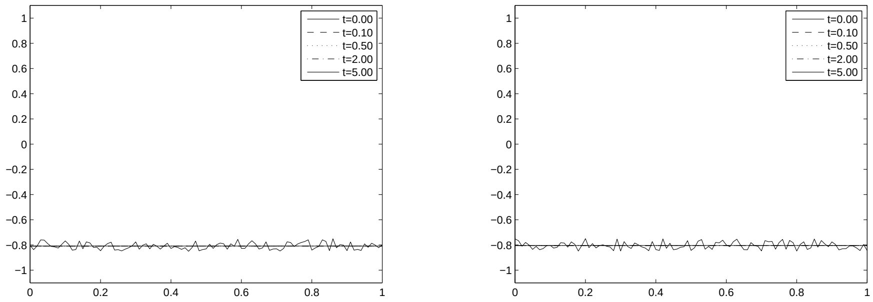
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.11: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.45, 0.15)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.6$ and $D = 0.4$.



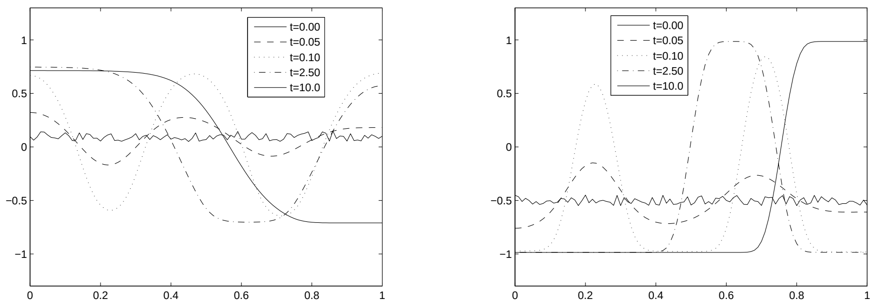
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.12: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.25, -0.9)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.6$ and $D = 0.4$.



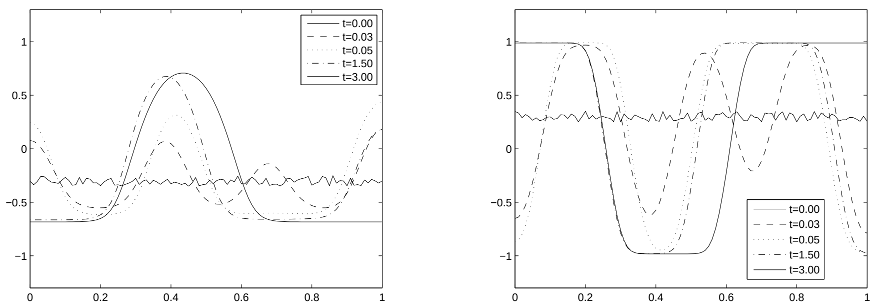
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.13: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.8, -0.8)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.6$ and $D = 0.4$.



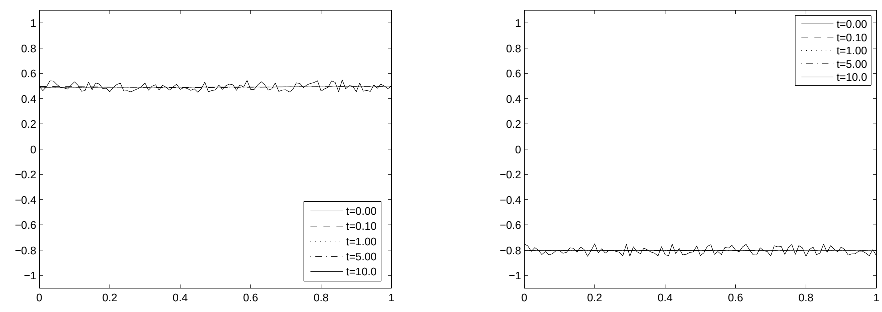
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.14: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (0.1, -0.5)$ and parameter values: $\gamma = 0.002,\ \theta_1 = 1,\ \theta_2 = 2,\ \theta = 0.8$ and $D = 0.5$.



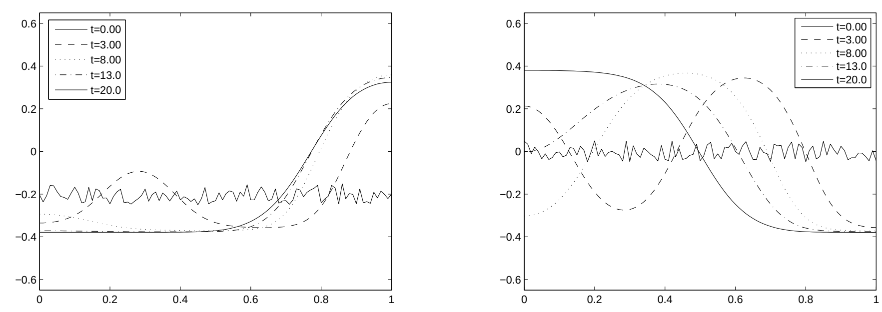
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.15: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.3, 0.3)$ and parameter values: $\gamma = 0.002,\ \theta_1 = 1,\ \theta_2 = 2,\ \theta = 0.8$ and $D = 0.5$.



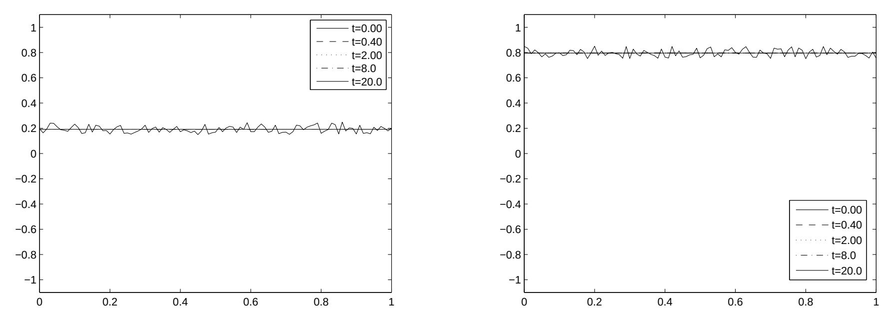
(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.16: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (0.5, -0.8)$ and parameter values: $\gamma = 0.002,\ \theta_1 = 1,\ \theta_2 = 2,\ \theta = 0.8$ and $D = 0.5$.



(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.17: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (-0.2, 0)$ and parameter values: $\gamma = 0.0005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.95$ and $D = 0.6$.



(a) Numerical approximation of u_1 (b) Numerical approximation of u_2

Figure 6.18: Numerical approximation of (u_1, u_2) at various times with $(m_1, m_2) = (0.2, 0.8)$ and parameter values: $\gamma = 0.0005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.95$ and $D = 0.6$.

6.3 Two-dimensional simulations

We take the computational domain to be a square uniform mesh $\Omega = (0,1)\times(0,1)$ with space step $h = 1/J$ in both x and y directions where $J+1$ is the number of the nodes in each direction. Then, we apply a right-angled triangulation on Ω in which each subsquare is bisected by its north-east diagonal (see Figure 6.19).

As explained earlier the D-coupling term involved in the free energy functional $\Lambda(u_1,u_2)$ given by (1.1.9) prevents appearance of region denoted by (u_1^+,u_2^+) in which the numerical solution of (u_1,u_2) is close to the value (α_1,α_2). Thus, the regions likely to appear are only (u_1^-,u_2^-), (u_1^+,u_2^-) and (u_1^-,u_2^+) in which the approximation of (u_1,u_2) takes approximately the values $(-\alpha_1,-\alpha_2)$, $(\alpha_1,-\alpha_2)$ and $(-\alpha_1,\alpha_2)$ respectively. In order to be in touch with the above classification of the regions we represent the numerical approximations U_1 and U_2 graphically on the mesh Ω by employing the RGB colour. We introduce an invertible map that takes the average values of U_1 and U_2 on each subsquare of the mesh Ω into the RGB colour. Let s_1 and s_2 be the average values on the subsquare with vertices $(x_i,y_j) = (ih,jh)$, (x_i,y_{j+1}), (x_{i+1},y_j) and (x_{i+1},y_{j+1}). We then define the RGB colour mapping as

$$(t_1,t_2,t_3) = \Big(\frac{1}{2}(1+\frac{s_1}{\alpha_1}), \frac{1}{2}(1+\frac{s_2}{\alpha_2}), \frac{1}{4}(-\frac{s_1}{\alpha_1}-\frac{s_2}{\alpha_2}+\frac{s_1}{\alpha_1}\frac{s_2}{\alpha_2}+1)\Big).$$

Note that $t_3 = -t_1 - t_2 + t_1\,t_2 + 1$. For $-\alpha_1 \le s_1 \le \alpha_1$ and $-\alpha_2 \le s_2 \le \alpha_2$ this mapping has the property that if (s_1,s_2) are equal to the values $(-\alpha_1,-\alpha_2)$, $(\alpha_1,-\alpha_2)$, $(-\alpha_1,\alpha_2)$ and (α_1,α_2), we then obtain the colours: pure blue, pure red, pure green and pure yellow respectively. The colour key of the rates $-1 \le s_1/\alpha_1 \le 1$ and $-1 \le s_2/\alpha_2 \le 1$ is depicted in Figure 6.20. We shall see that when $D > 0$ the pure yellow colour, which corresponds the region denoted by (u_1^+,u_2^+), does not appear in the experiments and there are only at most three pure colours.

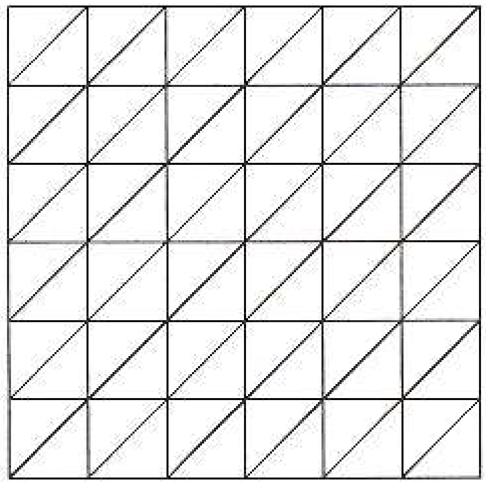


Figure 6.19: Right-angled uniform mesh for two dimensional simulations.

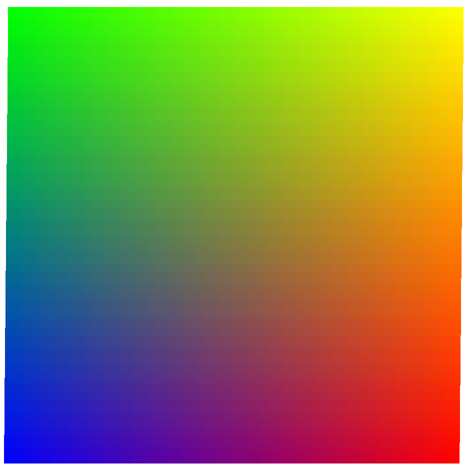


Figure 6.20: The colour key of the rates s_1/α_1 and s_2/α_2 where the x-axis and y-axis represent $-1 \leq s_1/\alpha_1 \leq 1$ and $-1 \leq s_2/\alpha_2 \leq 1$ respectively.

In the two dimensional experiments we consider two types of initial condition. We first use a two-dimensional version of the initial condition taken in [17] which is defined as follows

$$(u_1^0, u_2^0) = \begin{cases} (-\alpha_1, -\alpha_2) & \text{if } 0 \le x \le 1,\ 0 \le y \le \frac{1}{16}, \\ (m_x, -\alpha_2) & \text{if } 0 \le x \le 1,\ \frac{1}{16} < y \le \frac{3}{4}, \\ (-\alpha_1, \alpha_2) & \text{if } 0 \le x \le 1,\ \frac{3}{4} < y \le 1, \end{cases}$$

where m_x is a small random perturbation of the state m_x.

For this initial condition we ran two simulations with $h = 1/J = 1/64$, $\Delta t = 0.0002$, $\mu = 0.5$, $\gamma = 0.001$, $D = 0.25$, $\lambda = 0.1$ and $tol = 10^{-7}$. Note that in each figure of this section we arrange the pictures in a format matrix of three rows and two columns with time increasing to the right in rows, then downwards. In the first experiment we take $m_x = -0.25$ and set $\theta = 0.4$ and $\theta_1 = \theta_2 = 1.0$, which implies that $\alpha_1 = \alpha_2 = 0.986$ to three decimal places. The evolution of the numerical solution (see Figure 6.21) shows that there are only three regions with pure colours: blue, red and green. We observe that throughout the green region is virtually unchanged while below the evolution is from a mixture of lamella and blobs in the early stages which quickly changes into a blob only mixture where upon additional development takes place finally resulting in a quarter red in the lower left hand portion.

In the second experiment all the data remained the same as in the first experiment except we took $\theta = 0.3$, $\theta_1 = 1.0$ and $\theta_2 = 1.5$, that is $\alpha_1 = 0.999$ and $\alpha_2 = 0.997$ to three decimal places. In Figure 6.22 we plot pictures of the evolution of the numerical solution at different times where the last picture represents the stationary solution which has a similar structure to that obtained in the first experiment. However, the main differences are that the strip form of the green region is interfered with before returning to its original form and the lamellar region is kept for larger time. These differences can be explained as we have taken uneven potentials, i.e. $\theta_1 \neq \theta_2$, and θ is smaller.

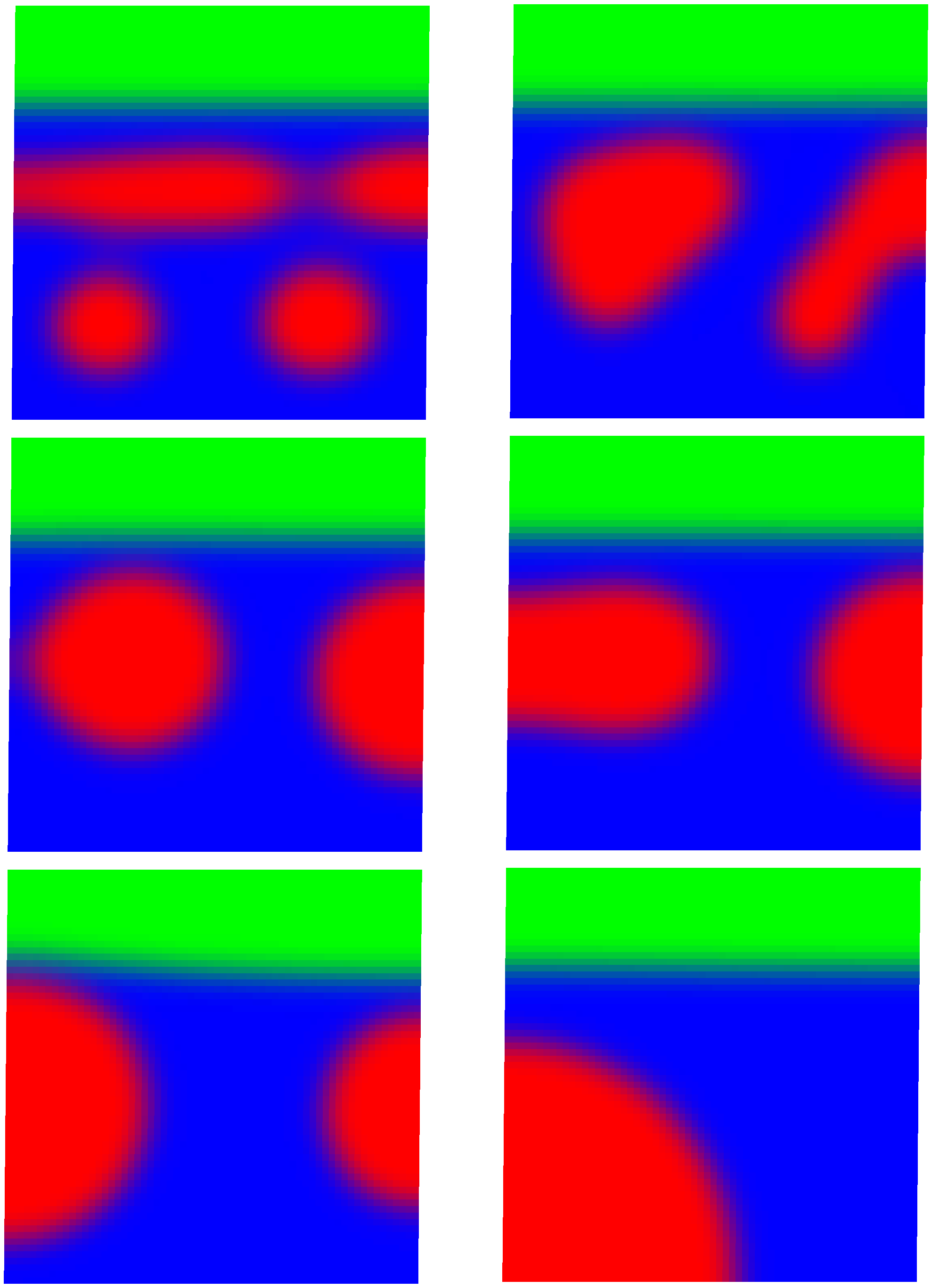


Figure 6.21: The structure of the numerical approximation at times $t = 0.2$, $t = 0.3$, $t = 0.6$, $t = 0.8$, $t = 4.0$, $t = 12.0$ where $m_x = -0.25$, $\gamma = 0.001$, $\theta_1 = \theta_2 = 1.0$, $\theta = 0.4$ and $D = 0.25$.

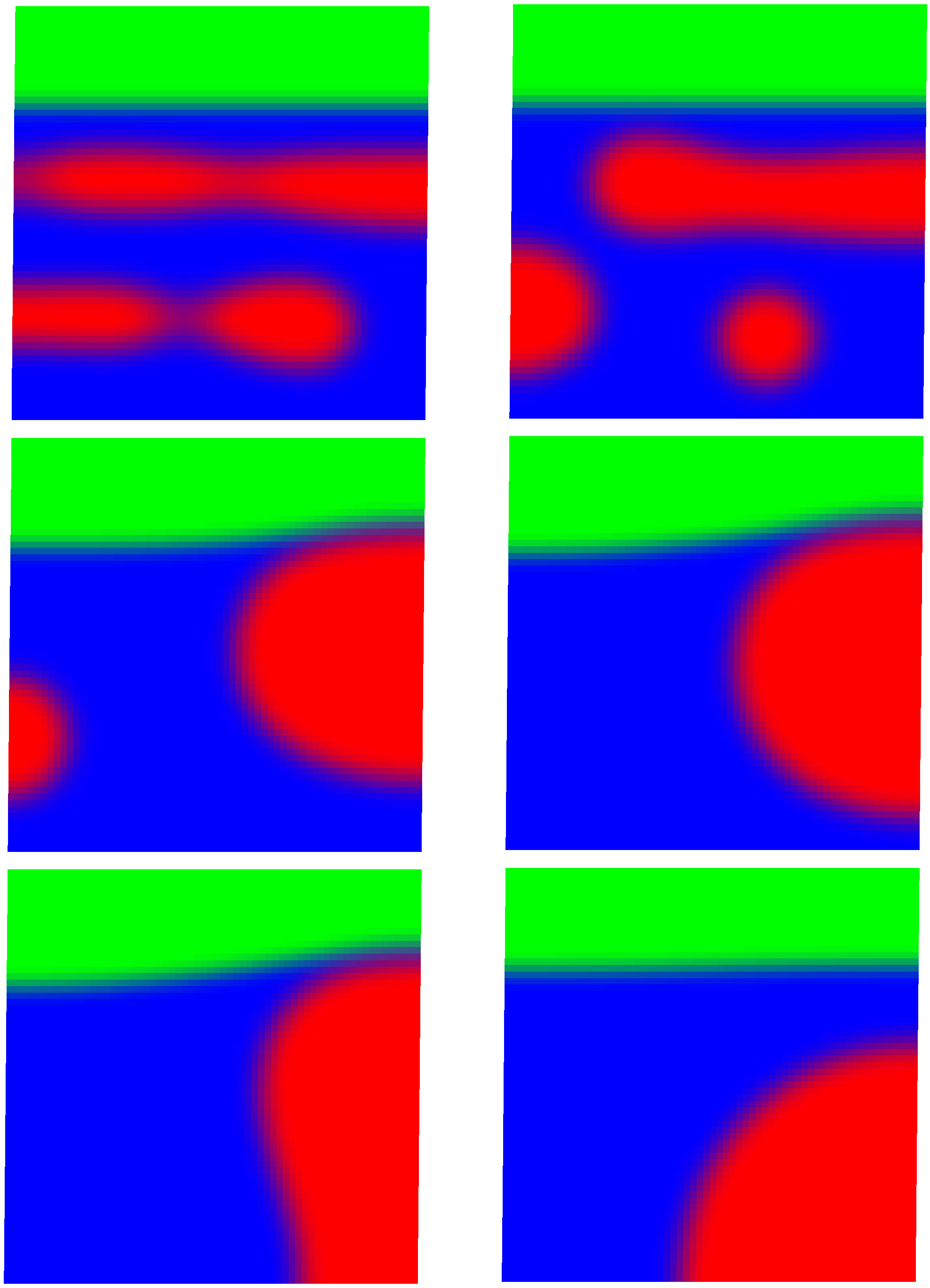


Figure 6.22: The structure of the numerical approximation at times $t = 0.2$, $t = 0.3$, $t = 1.7$, $t = 2.8$, $t = 3.6$, $t = 12.0$ where $m_x = -0.25$, $\gamma = 0.001$, $\theta_1 = 1.0$, $\theta_2 = 1.5$, $\theta = 0.3$ and $D = 0.25$.

Now, we take the initial data (U_1^0, U_2^0) to be random perturbations of the uniform state (m_1, m_2) with fluctuation no larger than 0.05. In each simulation with this type of initial data we set $\gamma = 0.005$, $D = 0.4$, $h = 1/64$ and $\Delta t = 0.0004$. The parameters: μ, λ and tol are kept as for the previous simulations. For this type of initial data we performed four simulations with different values of the parameters θ, θ_1, θ_2, m_1 and m_2.

In the third and fourth experiments of this section we used the data $\theta_1 = \theta_2 = 1.0$, $\theta = 0.2$, that is $\alpha_1 = \alpha_2 \approx 0.999$, and (m_1, m_2) are $(0, 0)$ and $(-0.25, -0.75)$. For the third experiment we found that there are only two pure colours (red and green) where in the early stages of the evolution we noticed a lamellar structure of green and red regions which develop in time to form finally two strip regions as displayed in Figure 6.24. While in the fourth experiment depicted in Figure 6.25 we found that there are three pure colours (red, green and blue) and the structure of the numerical solution is completely different, i.e. not lamella. Circular green and red regions were observed in the early stages which evolve quickly in time into a single central green circle and fewer circular red domains. After more time of the evolution, finally, green and red quarter circles were constructed in the lower left and upper right corners of the domain Ω, representing the stationary structure of the numerical approximation.

For the fifth and sixth experiments we choose the following parameter values: $\theta_1 = \theta_2 = 1.0$, $\theta = 0.6$, *i.e.* $\alpha_1 = \alpha_2 \approx 0.907$, and $(m_1, m_2) = (-0.45, 0.15)$ for the fifth experiment, and $\theta_1 = 1.0$, $\theta_2 = 2.0$, $\theta = 0.8$, *i.e.* $\alpha_1 \approx 0.710$ and $\alpha_2 \approx 0.907$, and $(m_1, m_2) = (0.1, -0.5)$ for the sixth experiment. Figure 6.26 and Figure 6.27 show the structure of the numerical solutions of these experiments at different times. The pictures in each figure again consist of three colours and the last picture in each figure represents the numerical stationary solutions. What is of interest in these two simulations is that the transition between the green and red regions is always wetted by a blue layer which is thin in the early stages and thickens as time increases. The presence of the blue layers can be understood as the energy required to travel directly between the green and red regions is much greater than that required to travel via the blue region. That is ignoring interfacial terms in the potential the geodesic which travels from the minimum $(-\alpha_1, \alpha_2)$ of $F(u_1, u_2) := \Psi_1(u_1) + \Psi_2(u_2) + f_D(u_1, u_2)$ to $(\alpha_1, -\alpha_2)$ stays away from the centre and travels via $(-\alpha_1, -\alpha_2)$, see Figure 6.23.

It is interesting to see the structure of the numerical solutions when $D = 0$. In the seventh and eighth experiments we repeated the second and fifth experiments with $D = 0$ and kept the remaining parameter values the same as before. We found that the structure of the numerical solutions is different to that with $D > 0$. In particular, with $D = 0$ the structure admits a pure yellow colour in its time evolution, compare Figures 6.22, 6.28 and Figures 6.26, 6.29. Therefore, we conclude that in the absence of the D-coupling term, the region denoted by (u_1^+, u_2^+) (the pure yellow region) may occur.

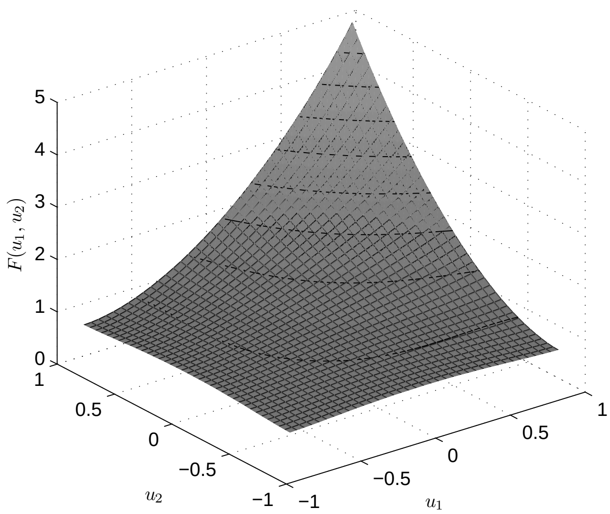


Figure 6.23: The plot of $F(u_1, u_2)$ with $\theta = 0.6$, $\theta_1 = \theta_2 = 1.0$ and $D = 0.4$.

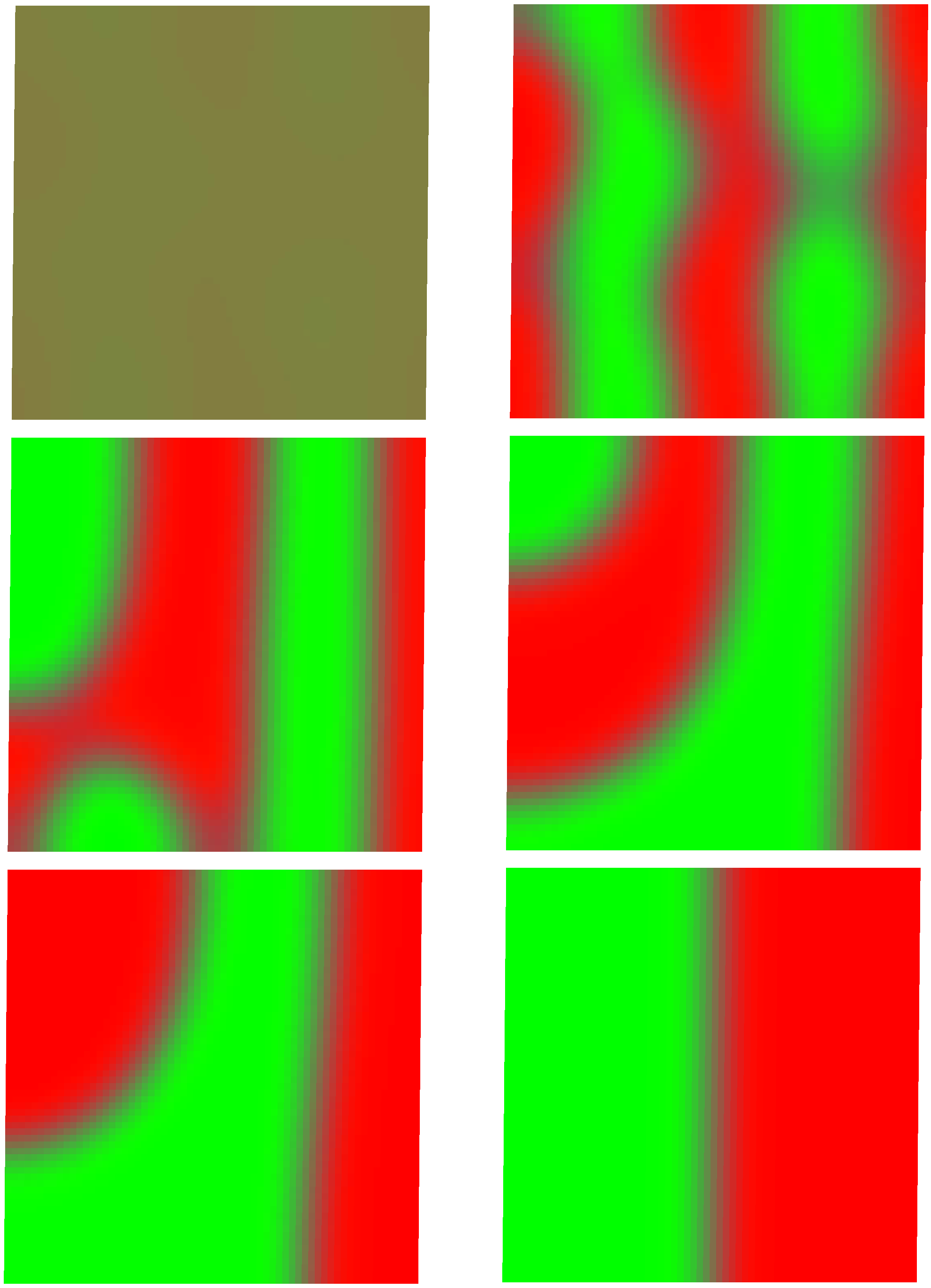


Figure 6.24: The structure of the numerical approximation at times $t = 0.02$, $t = 0.06$, $t = 0.32$, $t = 0.64$, $t = 1.4$, $t = 12.0$ with $(m_1, m_2) = (0, 0)$ and parameter values: $\gamma = 0.005$, $\theta_1 = \theta_2 = 1$, $\theta = 0.2$ and $D = 0.4$.

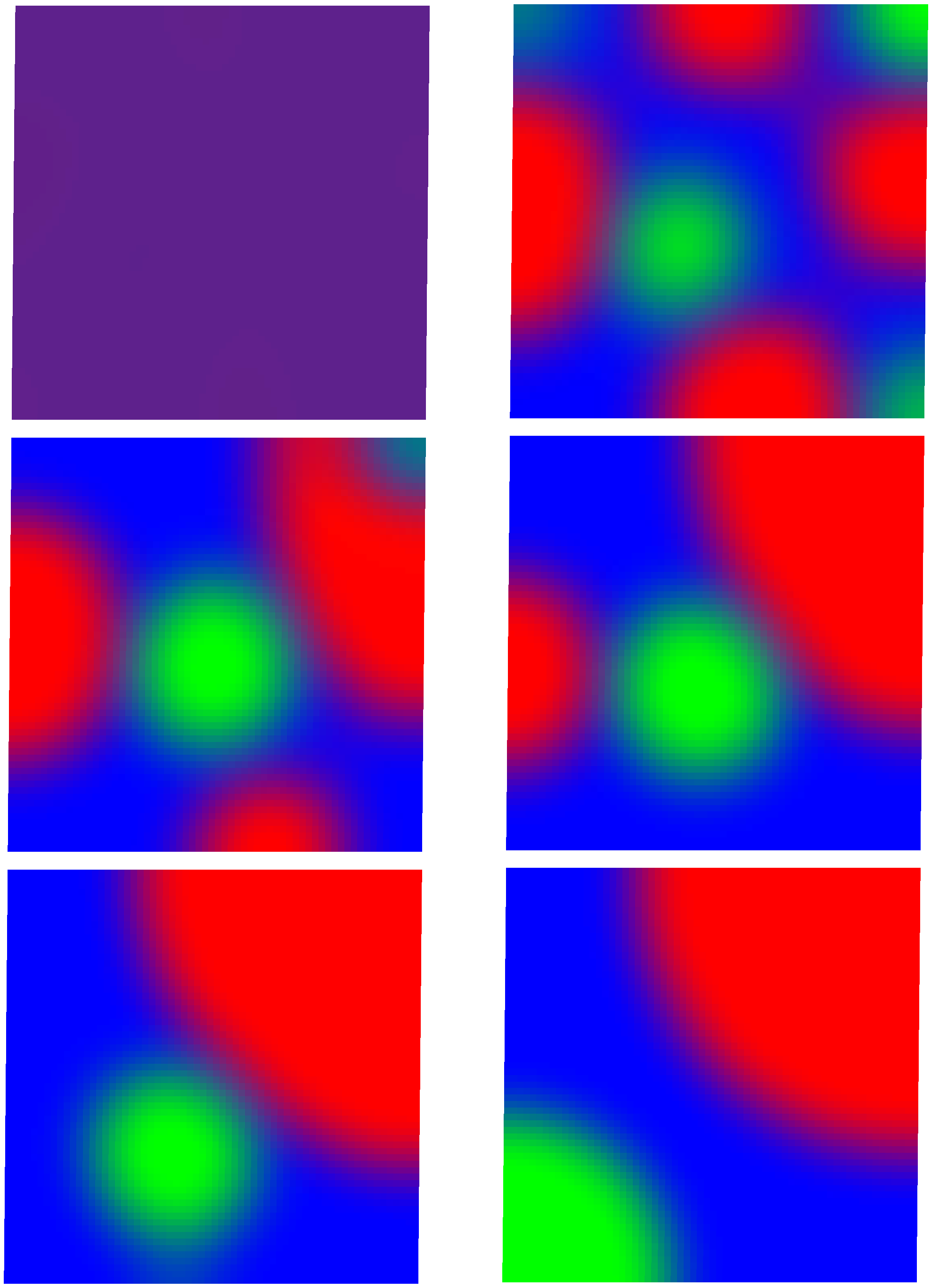


Figure 6.25: The structure of the numerical approximation at times $t = 0.06$, $t = 0.32$, $t = 0.64$, $t = 1.28$, $t = 1.7$, $t = 12.0$ with $(m_1, m_2) = (-0.25, -0.75)$ and parameter values: $\gamma = 0.005$, $\theta_1 = \theta_2 = 1$, $\theta = 0.2$ and $D = 0.4$.

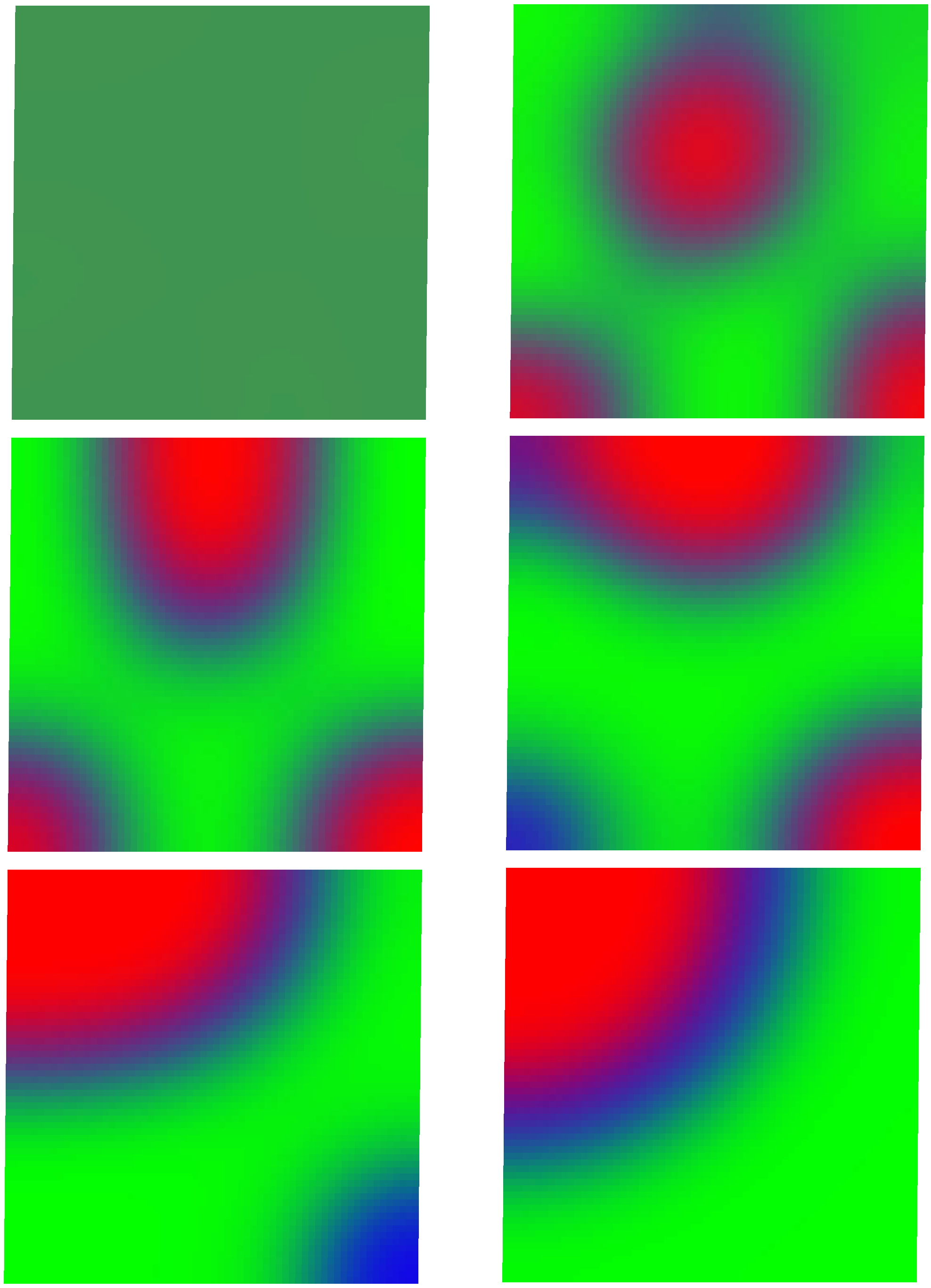


Figure 6.26: The structure of the numerical approximation at times $t = 0.1,\ t = 0.4,\ t = 0.7,\ t = 1.2,\ t = 2.4,\ t = 12.0$ with $(m_1, m_2) = (-0.45, 0.15)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.6$ and $D = 0.4$.

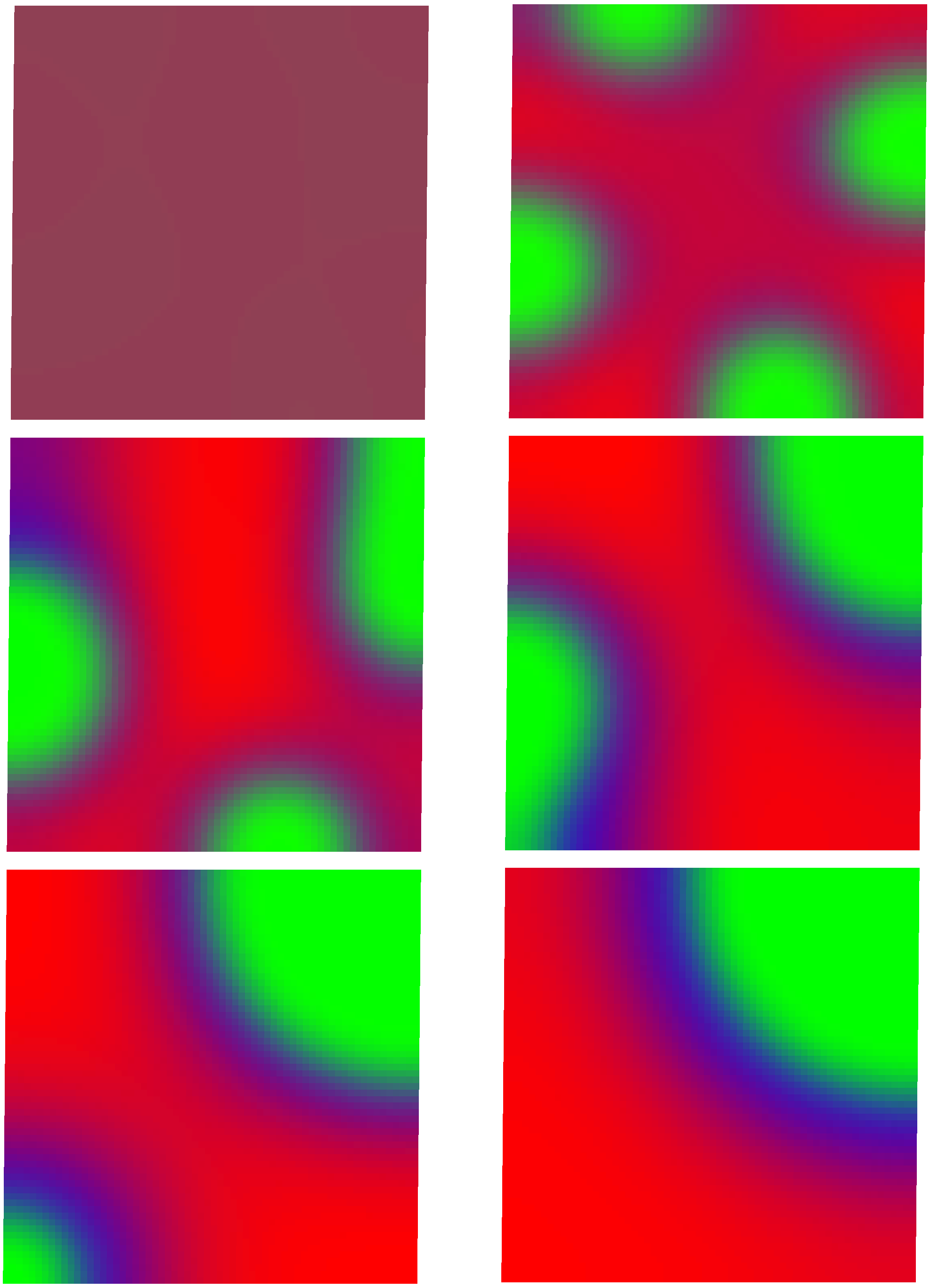


Figure 6.27: The structure of the numerical approximation at times $t = 0.06$, $t = 0.2$, $t = 0.5$, $t = 1.1$, $t = 3.2$, $t = 12.0$ with $(m_1, m_2) = (0.1, -0.5)$ and parameter values: $\gamma = 0.005$, $\theta_1 = 1.0, \theta_2 = 2.0$, $\theta = 0.8$ and $D = 0.4$.

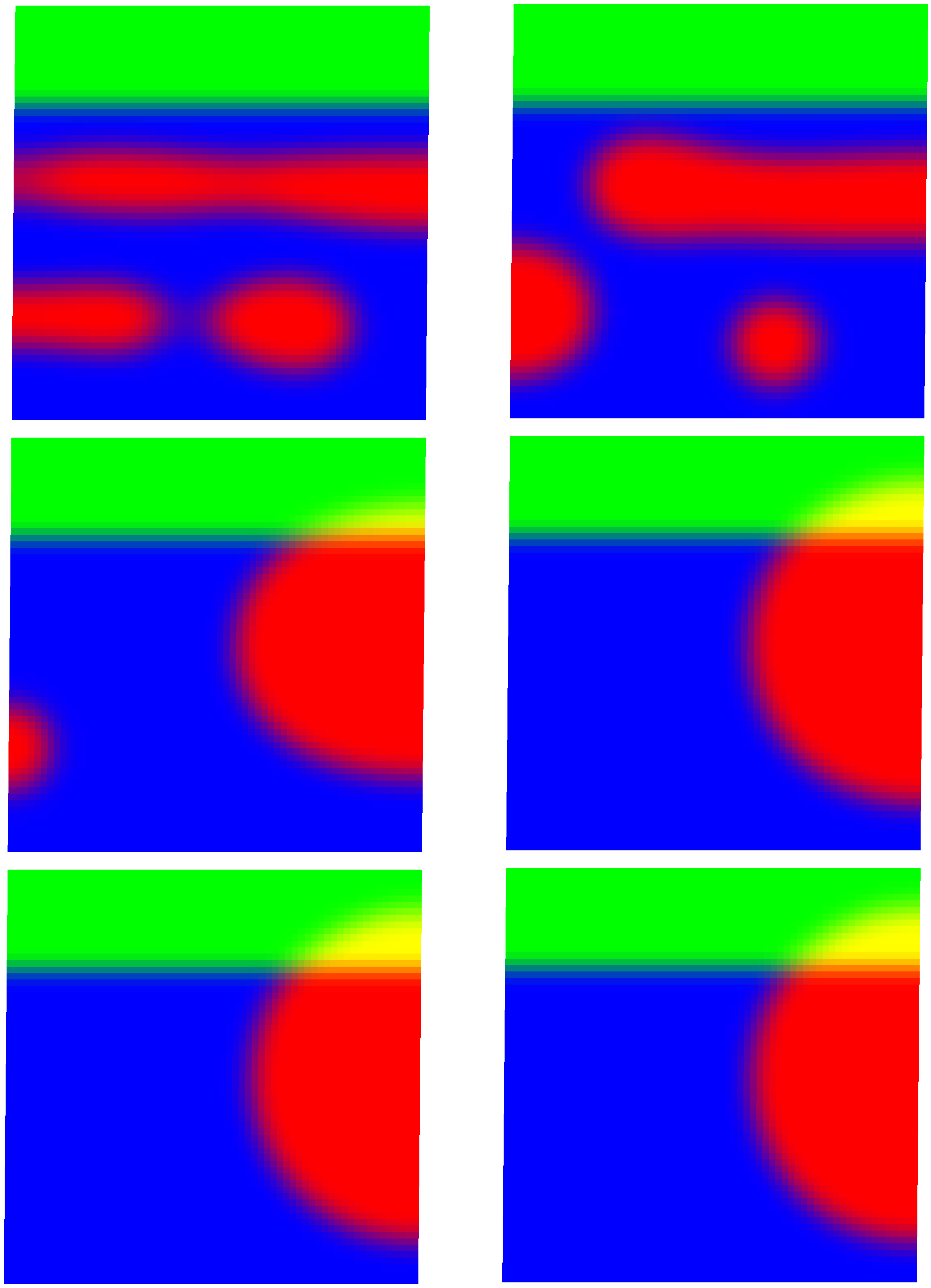


Figure 6.28: The structure of the numerical approximation at times $t = 0.2,\ t = 0.3,\ t = 1.7,\ t = 2.8,\ t = 3.6,\ t = 12.0$ where $m_x = -0.25$, $\gamma = 0.001$, $\theta_1 = 1.0$, $\theta_2 = 1.5$, $\theta = 0.3$ and $D = 0$.

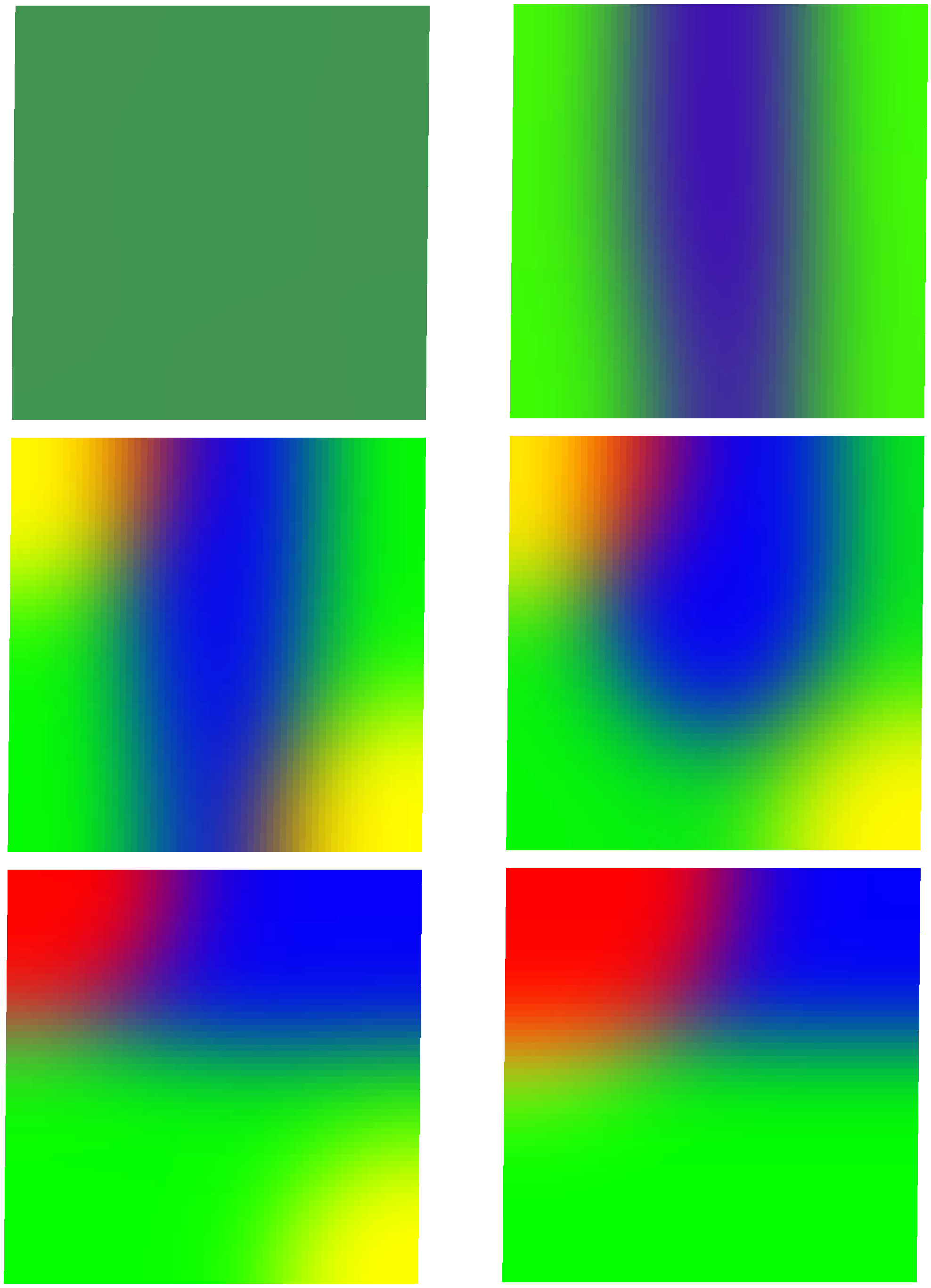


Figure 6.29: The structure of the numerical approximation at times $t = 0.1,\ t = 1.2,\ t = 2.8,\ t = 4.4,\ t = 5.2,\ t = 12.0$ with $(m_1, m_2) = (-0.45, 0.15)$ and parameter values: $\gamma = 0.005,\ \theta_1 = \theta_2 = 1,\ \theta = 0.6$ and $D = 0$.

Chapter 7

Conclusions

In this thesis we studied two coupled Cahn-Hilliard equations with a logarithmic potential and zero Neumann boundary conditions in $d \leq 3$ space dimensions. Under some assumptions $(\mathbf{A_1})$ on the initial data we proved existence, uniqueness and some stability estimates of the weak solution. This was achieved by considering first a smooth replacement of the logarithmic potential to have the regularized problem $(\mathbf{P}_\varepsilon)$ of the continuous problem $(\mathbf{P})$. With the aid of Faedo-Galerkin method and compactness arguments we established existence and uniqueness of a solution to $(\mathbf{P}_\varepsilon)$ and then by passing to the limit in ε we obtained existence of a solution of $(\mathbf{P})$.

Chapter 3 dealt with higher regularity results of the weak solutions of the problems $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$. With the aid of the standard regularity theory of elliptic problems and by imposing further assumptions on the boundary of the domain and the initial data we proved that the weak solutions are in higher order Sobolev spaces. We also proved the continuous dependence of the weak solution on the initial data with respect $(H^1(\Omega))' \times (H^1(\Omega))'$. Finally, we estimated the difference between the solutions of the problems $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$.

The finite element space used in the numerical study and some associated tools and results were given in the beginning of Chapter 4. Then, some key technical lemmata concerning the nonlinearities are proved. The semi-discrete problem

$(\mathbf{P^h})$ of $(\mathbf{P})$ and its regularized version $(\mathbf{P^h_\varepsilon})$ were suggested. The existence, uniqueness, stability estimates under the assumptions $(\mathbf{A_1})$ and additional necessary stability estimates under the assumptions $(\mathbf{A_2})$ of the semi-discrete approximations were proved. The error bound between the solutions of the continuous problem $(\mathbf{P})$ and the semi-discrete problem $(\mathbf{P^h})$ is investigated. This error bound was derived via the error bound between $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$, the error bound between $(\mathbf{P}_\varepsilon)$ and $(\mathbf{P^h_\varepsilon})$ and the error bound between $(\mathbf{P^h_\varepsilon})$ and $(\mathbf{P^h})$. The advantage of analysing the semi-discrete problems is that we could apply the framework in Nochetto [50] to prove an optimal error bound in time between the fully-discrete and semi-discrete approximations.

In Chapter 5 we proposed a symmetric coupled, in time, fully-discrete approximation $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_\mu)$, $\mu \in [0, \frac{1}{2}]$, of $(\mathbf{P})$ by discretising the semi-discrete problem $(\mathbf{P^h})$ in time using the backward Euler method. The corresponding regularized problem $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_{\mu,\varepsilon})$ was also introduced for which we proved existence and stability estimates of a solution using the Schauder fixed point theorem. The existence, uniqueness, stability estimates under the assumptions $(\mathbf{A_1})$ of the solution of $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_\mu)$ were proved. Further, essential stability estimates for the solution of $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_\mu)$ were deduced under the assumptions $(\mathbf{A_2})$. The error bound between the solutions of the continuous problem $(\mathbf{P})$ and fully-discrete problem $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_\mu)$ is proved, which is optimal in Δt. We obtained this error bound by combining the error bound between the solutions of $(\mathbf{P})$ and $(\mathbf{P^h})$ and the optimal error bound in time between the solutions of $(\mathbf{P^h})$ and $(\mathbf{P}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}}_\mu)$.

A practical algorithm for computing the numerical solutions was given at the beginning of Chapter 6. We then performed numerical experiments in one space dimension demonstrating the fully-discrete error bound and the growth behaviour of the numerical approximation. Furthermore, simulations in two space dimensions were performed.

There are still mathematical and numerical work to be done in the future. By considering the system (1.1.13a)-(1.1.17) with a diffusional mobility $M(u_i)$ depending on $u_i,\ i = 1, 2$ we will be led to the following coupled system

$$\frac{\partial u_1}{\partial t} = \nabla.(M(u_1)\nabla w_1),$$
$$\frac{\partial u_2}{\partial t} = \nabla.(M(u_2)\nabla w_2),$$

where $w_1,\ w_2$ and the nonlinearities involved are defined as before in (1.1.13c)-(1.1.17). This type of dependent mobility was suggested by Cahn and Hilliard [23]. It would be possible to mimic our study to analyse the above system with possibly some restrictions on $M(u_i)$ or with a specific reasonable example of $M(u_i)$ such as $M(u_i) = 1 - u_i^2$. Analysing the above system is recommended for future work.

Numerical results in Chapter 6 indicated that the rate of convergence of the fully-discrete approximations in one space dimension is $\mathcal{O}(h + \Delta t)$ while what we were able to prove theoretically is $\mathcal{O}(h^{2/3} + \Delta t)$. One question is "Can we find an example satisfying our theoretical error bound?". We leave this point and additional numerical experiments in higher space dimensions for future work.

Many studies of other variants of Cahn-Hilliard equations are concerned with the asymptotic behaviour of the solution as $\gamma \to 0^+$, for instance Modica [56] . So, it might be possible to study the system in this thesis and we also leave this work for future study.

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Appendix A

Definitions and Auxiliary Results

Definition A.0.1 (Convex functional, Johnson [45], p.249)
Let X be a normed space and let K be a convex subset of X. A functional $F : K \to \mathbb{R}$ is said to be convex if for all $x, y \in K$ and $\lambda \in [0, 1]$, we have

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y).$$

Theorem A.0.2 (Green's formula, Rodrigues [28], p.76)
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with outward unit normal ν. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then

$$\int_\Omega \nabla u \, \nabla v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, ds - \int_\Omega v \, \Delta u \, dx. \tag{A.0.1}$$

Theorem A.0.3 (Lax-Milgram, [21], p.83)
Let V be a Hilbert space. Let a be a bounded bilinear form on $V \times V$ and let $f \in V'$ (i.e. f is a bounded linear functional on V). If a is a coercive, i.e.,

$$\exists \alpha > 0, \ \forall \, u \in V, \quad a(u, v) \geq \alpha \|u\|_V^2.$$

Then, there exists a unique $u \in V$ such that

$$a(u, v) = f(v) \equiv \langle f, v \rangle_{V,V'} \quad \forall \, v \in V.$$

In addition,

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Theorem A.0.4 (Schauder's Theorem, Baiocchi, p.215)
Let X be a normed space and let K be a non-empty convex compact set of X. If $f : K \to K$ is a continuous function then f has at least one fixed point, i.e. $\exists x_0 \in K : \ f(x_0) = x_0$.

Theorem A.0.5 (Gronwall lemma in differential form, see Proposition 2.2 in [52])
Let $E \in W^{1,1}(0,t)$ and $P,\ Q,\ R \in L^1(0,t)$, where all functions are non-negative. Then

$$\frac{dE}{dt} + P(t) \leq R(t)\, E(t) + Q(t) \quad a.e. \text{ in } [0,t]$$

implies

$$E(t) + \int_0^t P(s)\, ds \leq e^{\int_0^t R(s)ds} E(0) + e^{\int_0^t R(s)ds} \int_0^t Q(s)\, ds.$$

Theorem A.0.6 (Some results of Sobolev spaces)
Let m be a positive integer. The Sobolev spaces $W^{m,p}(\Omega)$, equipped with appropriate norms, satisfy
(i) For $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is a Banach space .(Adams [1], p.45)
(ii) For $1 \leq p < \infty$, $W^{m,p}(\Omega)$ is separable.(Adams [1], p.47)
(iii) For $1 < p < \infty$, $W^{m,p}(\Omega)$ is reflexive.(Adams [1], p.47)
(iv) If $m, n \in \mathbb{N} \cup \{0\},\ k \leq m$ and $1 \leq p \leq q \leq \infty$, then $W^{m,q}(\Omega) \hookrightarrow W^{k,p}(\Omega)$.(Berner [61], p.30)

Theorem A.0.7 (Some results of time-dependent spaces)
Let X and Y be Banach spaces. The time-dependent spaces $L^p(0,T;X)$, associated with the norms introduced In Chapter 2, satisfy the following
(i) For $1 \leq p \leq \infty$, $L^p(0,T;X)$ is a Banach space.
(ii) For $1 \leq p < \infty$, $L^p(0,T;X)$ is separable if and only if X is separable.
(iii) For $1 < p < \infty$, $L^p(0,T;X)$ is reflexive if and only if X is reflexive.
(iv) If X is a reflexive or separable Banach space and $1 \leq p < \infty$ then $[L^p(0,T;X)]' \cong L^q(0,T;X')$ where $1/p + 1/q = 1$ (the symbol "$\cong$" means isometrically isomorphic.
(v) If $1 \leq p \leq q \leq \infty$. Then the continuous injection $X \hookrightarrow Y$ implies $L^q(0,T;X) \hookrightarrow L^p(0,T;Y)$. These results are collected in [40] from Kufner [39], pp.113-118 and Zenisek [43], p.40.

Definition A.0.8 (strong convergence)

Le V be a normed vector space. Then $x_n \in V$ converges strongly to $x \in V$, written $x_n \to x$, if and only if

$$\|x_n - x\|_V \to 0.$$

Definition A.0.9 (Weak convergence)

Let X be a Banach space. Then $x_n \in X$ converges weakly to $x \in X$, written $x_n \rightharpoonup x$, if and only if

$$\langle f, x_n \rangle \to \langle f, x \rangle \quad \forall\, f \in X',$$

where we use use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between X and X'.

Definition A.0.10 (Weak-star convergence)

Let X be a Banach space. Then $f_n \in X'$ converges weakly-star to $f \in X'$, written $f_n \overset{*}{\rightharpoonup} f$, if and only if

$$\langle f_n, x \rangle \to \langle f, x \rangle \quad \forall\, x \in X.$$

Theorem A.0.11 (Some results of weak and weak-star convergence)

Let X be Banach space and X' its dual. Then

(i) $x_n \to x$ in X implies $x_n \rightharpoonup x$ in X.(Robinson [14], p.102)

(ii) $x_n \rightharpoonup x$ in X implies $\|x\|_X$ is bounded and $\|x\|_X \leq \liminf \|x_n\|_X$. (Rodrigues, [28], p.55)

(iii)$f_n \overset{*}{\rightharpoonup} f$ in X' implies $\|f\|_{X'}$ is bounded and $\|f\|_{X'} \leq \liminf \|f_n\|_{X'}$. (Rodrigues, [28], p.56)

(iv) Weak (weak-star) convergence has a unique limit. (Robinson [14], p.104).

Theorem A.0.12 (Zenisek [43], p.8)

Let the function f have a finite Lebesgue integral over (a, b). Then the derivative of the indefinite Lebesgue integral

$$F(x) = \int_a^x f(t)\, dt$$

satisfies the relation $F'(x) = f(x),\ a.e.\ x \in (a, b)$.

Theorem A.0.13 (Kufner [39], p.116)
Let X be a Banach space and let $f \in L^\infty(0,T;X)$. Then there exists a set $A \subset (0,T)$ of measure zero such that

$$\|f\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)-A} \|f\|_X.$$

Theorem A.0.14 (Gilbarg [42], pp.153-154)
Let f be a piecewise smooth function on $\mathbb{R}$ (i.e. it is continuous and has piecewise continuous first derivative) with $f' \in L^\infty(\mathbb{R})$. Then if $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$,we have $f \circ u \in W^{1,p}(\Omega)$. Furthermore, letting L denote the set of corner points of f, we have

$$D(f \circ u) = \begin{cases} f'(u)Du & \text{if } u \notin L, \\ 0 & \text{if } u \in L. \end{cases}$$

Theorem A.0.15 (Weak sequential compactness, Dautary [59], p.289)
Let X be a reflexive Banach space and let $\{x_n\}$ be a bounded sequence in X. Then x_n has a subsequence which converges weakly in X.

Theorem A.0.16 (Weak-star sequential compactness, Dautary [59], p.291)
Let X be a separable Banach space and let $\{f_n\}$ be a bounded sequence in X'. Then f_n has a subsequence which converges weakly star in X'.

Theorem A.0.17 (Robinson [14], p.27, Rodrigues [28], p.59)
If $f_n \to f$ in $L^p(\Omega)$, $1 \le p < \infty$, then there exists a subsequence, still denoted f_n, such that

$$f_n(x) \to f(x) \quad a.e.\ x \in \Omega.$$

Theorem A.0.18 (Lions-Aubin Theorem, Temam [48], p.271)
Let X_0, X , X_1 be three Banach spaces such that

$$X_0 \overset{c}{\hookrightarrow} X \hookrightarrow X_1,$$

where X_0 and X_1 are reflexive. Let T be finite and $1 < p_0, p_1 < \infty$, then the space

$$W = \left\{ v : v \in L^{p_0}(0,T;X_0), \quad \frac{dv}{dt} \in L^{p_1}(0,T;X_1) \right\}$$

with the norm

$$\|v\|_W := \|v\|_{L^{p_0}(0,T;X_0)} + \|v\|_{L^{p_1}(0,T;X_1)},$$

is a Banach space and the injection W into $L^{p_0}(0,T;X)$ is compact.

Theorem A.0.19 (Temam [55], p.69)
Let $V,\ H,\ V'$ be three Hilbert spaces, each space included and dense in the following one, V' being the dual of V. If $u \in L^2(0,T;V)$ and $u' \equiv \frac{du}{dt} \in L^2(0,T;V')$, then $u \in C([0,T];H)$ $a.e$ and the following holds in the scalar distribution sense on $(0,T)$

$$\frac{d}{dt}|u|^2 = 2\langle u', u\rangle.$$

Theorem A.0.20 (see Robinson [14], p193)
If $u \in L^2(0,T;H^2(\Omega))$ and $\frac{du}{dt} \in L^2(\Omega_T)$, then $u \in C([0,T];H^1(\Omega))$.

Theorem A.0.21 (Some useful inequalities)
(i) For arbitrary $a, b \geq 0$ and $p > 0$

$$2^{-[p-1]_-}(a^p + b^p) \leq (a+b)^p \leq 2^{[p-1]_+}(a^p + b^p),$$

where $[r]_+ = \max\{r, 0\},\ [r]_- = \max\{-r, 0\}$. (Rodrigues [28], p.54)
(ii) For finite sums or infinite sums (discrete Hölder's inequality)

$$\sum |a_k b_k| \leq \Big(\sum |a_k|^p\Big)^{1/p}\Big(\sum |a_k|^p\Big)^{1/q},$$

where $1/p + 1/q = 1$. (Adams [1], p. 23)

Appendix B

Programs

In the appendix we include some programs we wrote to perform the numerical experiments in the thesis. The first program computes the numerical solutions U_1^n and U_2^n with cubic splines initial data and calculates the error (6.3.2) with a fixed space step and successive refinement of the time step.

```
      Program errodp
      implicit none
      integer nmax
      PARAMETER (nmax=5620)
      double precision u1(0:nmax),u_n1(0:nmax),ukph1(0:nmax),
     .  u10(0:nmax),u2(0:nmax),u_n2(0:nmax),ukph2(0:nmax),u20(0:nmax),
     .  ru1(0:nmax), eig(0:nmax),uk1(0:nmax),cu1(0:nmax),ru2(0:nmax),
     .  uk2(0:nmax),yu2(0:nmax),cu2(0:nmax), xu1(0:nmax),xu2(0:nmax),
     .  yu1(0:nmax),cxu2(0:nmax),unm1(0:nmax),unm2(0:nmax),cxu1(0:nmax),
     .  wsave(0:3*nmax),w1(0:nmax),w2(0:nmax), b1(0:nmax),b2(0:nmax),
     .  c1(0:nmax),c2(0:nmax),lh1(0:nmax),lh2(0:nmax),uc1(0:nmax),
     .  u_nc1(0:nmax),ukphc1(0:nmax),uc2(0:nmax),u_nc2(0:nmax),
     .  ukphc2(0:nmax),ruc1(0:nmax),eigc(0:nmax),ukc1(0:nmax),
     .  ruc2(0:nmax),ukc2(0:nmax),yuc2(0:nmax), xuc1(0:nmax),
     .  xuc2(0:nmax),yuc1(0:nmax),unmc1(0:nmax),unmc2(0:nmax),
     .  cxuc1(0:nmax), za(0:nmax),zb(0:nmax),zc(0:nmax),zd(0:nmax),
     .  cuc1(0:nmax),cuc2(0:nmax),cxuc2(0:nmax),bc1(0:nmax),bc2(0:nmax),
     .  cc1(0:nmax),cc2(0:nmax),lhc1(0:nmax),lhc2(0:nmax),va(0:nmax),
     .  vb(0:nmax), vc(0:nmax),vd(0:nmax),lambda,
     .  hc2,tempc,ra1,ra2,a,c,xmin,len,tau,t,h, h2,pi,gamma,diff,mu,
     .  theta,theta1,D,r,s,m1,m2,theta2,temp,ermu1,ermu2,alpha1,alpha2,
     .  sumu1,sumu2,sumu10,sumu20,time,tol,x,tauc, ras1,ras2, hc
      double precision A1(24576,2049),A2(24576,2049),AC1(3072,2049),
     . AC2(3072,2049)
      integer i,m,n,loopy,loop,k5,nc,step,j,p0,p,q,val,pc,mc,l,f,
     .   nloops,nloops_tot,imax,imaxc,s0,int0
```

```
      character*30 datafile1,datafile2
      character*1 number1
      character*2 number2,lettert,letterw,lettertc,letterwc
      character*3 number3
      character*4 number4
      lettert='h1'
      letterw='h2'
      lettertc='l1'
      letterwc='l2'
C
C       READING THE INTIAL DATA
      open(1,status='old',file='temp120.dat')
        read(1,*) gamma
        read(1,*) D
        read(1,*) lambda
        read(1,*) tol
      close(1)
C     Reading the corfficients of the cubic splines generated by MATLAB
      open(2,status='old',file='coefe55a.dat')
      do 2020 i=0,3
        read(2,*) zd(i),zc(i),zb(i),za(i)
2020    continue
      close(2)
      open(9,status='old',file='coefe52a.dat')
      do 20208 i=0,3
        read(9,*) vd(i),vc(i),vb(i),va(i)
20208    continue
      close(9)
      theta=0.25D0
      theta1=1.0D0
      theta2=1.0D0
C
C This step is to find the positive roots alpha1 and alpha2
      call ROOT_PROG(theta,theta1,r)
      call ROOT_PROG(theta,theta2,s)
      alpha1=r
      alpha2=s
      print*,'alpha1=',alpha1
      print*,'alpha2=',alpha2
C       THE SPACE STEP OF THE FINE MECH
       pi=3.14159265358979323846
       h=1.0/2048.0
       n=2048
       p0=512
C
```

```
C       We intialize our problem
        sumu10=0.0D0
        sumu20=0.0D0
C       calculating P_gamma-H^1 projection of u_1^0 and u_2^0 using
C        discrete cosine transformation
        int0=0
65      int0=int0+1
C       Computing bi(j)=(u_i^0,phi_j),i=1,2,j=0,...,n
        b1(0)=(za(0)*h**2/2+zb(0)*h**3/6+zc(0)*h**4/12+zd(0)*h**5/20)
     .    *2.0/h**2
C
        b1(n)=(za(3)*h**2/2+zb(3)*((0.25)**2/2*h-(0.25)**3/6)
     .    +zc(3)*((0.25)**3/3*h-(0.25)**4/12)
     .    +zd(3)*((0.25)**4/4*h-(0.25)**5/20)
     .   +zb(3)*(real(n-1)*h-0.75)**3/6
     .   +zc(3)*(real(n-1)*h-0.75)**4/12
     .   +zd(3)*(real(n-1)*h-0.75)**5/20)*2.0/h**2
C
        s0=0
        do 122 j=0,3
            do 123 i=s0*p0+1,(s0+1)*p0
        b1(i)=(za(j)*h**2-2.0*zb(j)*(real(i)*h-real(j)*0.25)**3/6
     . -2.0*zc(j)*(real(i)*h-real(j)*0.25)**4/12
     . -2.0*zd(j)*(real(i)*h-real(j)*0.25)**5/20
     . +(zb(j)*(real(i-1)*h-real(j)*0.25)**3/6
     . +zc(j)*(real(i-1)*h-real(j)*0.25)**4/12
     . +zd(j)*(real(i-1)*h-real(j)*0.25)**5/20)
     .  +(zb(j)*(real(i+1)*h-real(j)*0.25)**3/6
     .  +zc(j)*(real(i+1)*h-real(j)*0.25)**4/12
     .  +zd(j)*(real(i+1)*h-real(j)*0.25)**5/20))*1.0/h**2
123         continue
         s0=s0+1
122     continue
C
        do 124 i=1,3
        b1(i*p0)=(za(i-1)*h**2/2
     .  +zb(i-1)*((real(i*p0)*h-real(i-1)*0.25)**2/2*h
     .  -(real(i*p0)*h-real(i-1)*0.25)**3/6)
     .  +zc(i-1)*((real(i*p0)*h-real(i-1)*0.25)**3/3*h
     .  -(real(i*p0)*h-real(i-1)*0.25)**4/12)
     .  +zd(i-1)*((real(i*p0)*h-real(i-1)*0.25)**4/4*h
     .  -(real(i*p0)*h-real(i-1)*0.25)**5/20)
     .  +zb(i-1)*(real(i*p0-1)*h-real(i-1)*0.25)**3/6
     .  +zc(i-1)*(real(i*p0-1)*h-real(i-1)*0.25)**4/12
     .  +zd(i-1)*(real(i*p0-1)*h-real(i-1)*0.25)**5/20
```

```
     . +za(i)*h**2/2
     .  +zb(i)*((real(i*p0)*h-real(i)*0.25)**2/2*(-h)
     .  -(real(i*p0)*h-real(i)*0.25)**3/6)
     .  +zc(i)*((real(i*p0)*h-real(i)*0.25)**3/3*(-h)
     .  -(real(i*p0)*h-real(i)*0.25)**4/12)
     .  +zd(i)*((real(i*p0)*h-real(i)*0.25)**4/4*(-h)
     .  -(real(i*p0)*h-real(i)*0.25)**5/20)
     .  +zb(i)*(real(i*p0+1)*h-real(i)*0.25)**3/6
     .  +zc(i)*(real(i*p0+1)*h-real(i)*0.25)**4/12
     .  +zd(i)*(real(i*p0+1)*h-real(i)*0.25)**5/20)*1.0/h**2
124    continue
C
C      Computing ci(j)=(grad u_i^0, grad phi_j),i=1,2,j=0,...,n
       c1(0)=(-zb(0)*h-zc(0)*h**2-zd(0)*h**3)*2.0/h**2
C
        c1(n)=(zb(3)+zc(3)*(0.25)**2+zd(3)*(0.25)**3
     .   -(zb(3)*real(n-1)*h+zc(3)*(real(n-1)*h-0.75)**2
     .      +zd(3)*(real(n-1)*h-0.75)**3))*2.0/h**2
C
        s0=0
        do 125 j=0,3
           do 126 i=s0*p0+1,(s0+1)*p0
       c1(i)=(2.0*(zb(j)*real(i)*h+zc(j)*(real(i)*h-real(j)*0.25)**2
     .           +zd(j)*(real(i)*h-real(j)*0.25)**3)
     .      -(zb(j)*real(i-1)*h+zc(j)*(real(i-1)*h-real(j)*0.25)**2
     .           +zd(j)*(real(i-1)*h-real(j)*0.25)**3)
     .      -(zb(j)*real(i+1)*h+zc(j)*(real(i+1)*h-real(j)*0.25)**2
     .           +zd(j)*(real(i+1)*h-real(j)*0.25)**3))*1.0/h**2
126          continue
           s0=s0+1
125     continue
C
       do 127 i=1,3
       c1(i*p0)=(zb(i-1)*real(i*p0)*h
     .  +zc(i-1)*(real(i*p0)*h-real(i-1)*0.25)**2
     .           +zd(i-1)*(real(i*p0)*h-real(i-1)*0.25)**3
     .      -(zb(i-1)*real(i*p0-1)*h
     .       +zc(i-1)*(real(i*p0-1)*h-real(i-1)*0.25)**2
     .           +zd(i-1)*(real(i*p0-1)*h-real(i-1)*0.25)**3)
     .       +zb(i)*real(i*p0)*h+zc(i)*(real(i*p0)*h-real(i)*0.25)**2
     .           +zd(i)*(real(i*p0)*h-real(i)*0.25)**3
     .     -(zb(i)*real(i*p0+1)*h+zc(i)*(real(i*p0+1)*h-real(i)*0.25)**2
     .           +zd(i)*(real(i*p0+1)*h-real(i)*0.25)**3))*1.0/h**2
127    continue
C
```

```
       if (int0.eq.1) then
           do 104 i=0,n
            lh1(i)=b1(i)+gamma*c1(i)
104         continue
       else
          do 11 i=0,n
            lh2(i)=b1(i)+gamma*c1(i)
11         continue
       end if
C
      do 13013 i=0,3
       za(i)=va(i)
       zb(i)=vb(i)
       zc(i)=vc(i)
       zd(i)=vd(i)
13013   continue
       if (int0.eq.1) then
        go to 65
       end if
C
C      Using NAG routines to compute the corresponding coefficients of
C      the values of lh1 and lh2 at the nodes
       CALL DCOSTI(n+1,wsave)
       print *, 'Hi';
      CALL C06HBF(n,lh1,wsave)
      CALL C06HBF(n,lh2,wsave)
      h2=h**(2.0D0)
C     The eigenvalues of the matrix R in one dimension
      eig(0)=0.0
      do 150 i=1,n
        eig(i)=(2.0D0-2.0D0*dcos(pi*real(i)/real(n)))/h2
150    continue
C      Computing Fourier coefficents of u10 and u20
       do 16 i=0,n
       u10(i)=lh1(i)/(gamma*eig(i)+1.0D0)
       u20(i)=lh2(i)/(gamma*eig(i)+1.0D0)
16      continue
C      Computing u10 and u20 at the nodes by NAG subroutines
       CALL C06HBF(n,u10,wsave)
       CALL C06HBF(n,u20,wsave)
      open(3,status='old',file='g2.dat')
      do 15  i=0,n
      write(3,*) real(i)*h, u10(i),u20(i)
15     continue
      close(3)
```

```
       do 2 i=0,n
            u1(i)=u10(i)
            u_n1(i)=u1(i)
            unm1(i)=u1(i)
            ru1(i)=u1(i)
            uk1(i)=u1(i)
            sumu10=u1(i)+sumu10
            u2(i)=u20(i)
            u_n2(i)=u2(i)
            unm2(i)=u2(i)
            ru2(i)=u2(i)
            uk2(i)=u2(i)
            sumu20=u2(i)+sumu20
2      continue
C C   The next step is to check the mean value of the intial data
       sumu10=(sumu10-(u1(0)+u1(n))*0.5)*h
       sumu20=(sumu20-(u2(0)+u2(n))*0.5)*h
       print*,'mean value u1^0=',sumu10
       print*,'mean value u2^0=',sumu20
C
       CALL C06HBF(n,u_n1,wsave)
       CALL C06HBF(n,u_n2,wsave)
3      print*,'number of prints'
       read*,k5
       a=-1.0
       c=5.0D-8
       xmin=a+c
       tau=1.0/49152.0
       m=24576
      if (mod(m,k5).ne.0) go to 3
      print *,'tau=',tau
      time = 0.0D0
      step=0
C Calcalating U1^{n,k+1} and U1^{n,k+1} at the level time n
      do 51 loopy=1,k5
        do 52 loop=1,m/k5

        nloops=0
55      nloops=nloops+1
C The next step is to find U1^{n,k+0.5} and U2^{n,k+0.5} at the nodes.We
C also calculate X_i{n,k+1},y_i{n,k+1),i=1,2
          do 113 i=0,n
           CALL LOG_PROJ(ru1(i),u1(i),ermu1,lambda,xmin,theta)
           CALL LOG_PROJ(ru2(i),u2(i),ermu2,lambda,xmin,theta)
```

```
           ukph1(i)=ermu1
           ukph2(i)=ermu2
C
           xu1(i)=2.0*ukph1(i)-ru1(i)
           xu2(i)=2.0*ukph2(i)-ru2(i)
          yu1(i)=(uk1(i)+alpha1)*((uk2(i)+alpha2)**2 +
     .          (unm2(i)+alpha2)**2)
          yu2(i)=(uk2(i)+alpha2)*((uk1(i)+alpha1)**2 +
     .          (unm1(i)+alpha1)**2)
113    continue
C
       do 114 i=0,n
           cxu1(i)=xu1(i)
           cxu2(i)=xu2(i)
114    continue
C
         CALL C06HBF(n,cxu1,wsave)
         CALL C06HBF(n,cxu2,wsave)
         CALL C06HBF(n,yu1,wsave)
         CALL C06HBF(n,yu2,wsave)
C Now we calculate U1^{n,k+1} and U1^{n,k+1} at the nodes where we first
C calculate the corresponding Fourier constants and then we use the DCT
C to obtain the values at the nodes
       mu=0.5D0
          do 80 i=0,n
            temp=eig(i)*tau
         u1(i)=(lambda*(1.0+theta1*temp*(1-mu))*u_n1(i)
     .           +(cxu1(i)-D*lambda*yu1(i))*temp)
     .     /(lambda+temp+lambda*gamma*eig(i)*temp-lambda*mu*theta1*temp)
            cu1(i)=u1(i)
               if (i.ne.0) then
                 w1(i)=(-(u1(i)-u_n1(i)))/temp
               endif
C
         u2(i)=(lambda*(1.0+theta2*temp*(1-mu))*u_n2(i)
     .           +(cxu2(i)-D*lambda*yu2(i))*temp)
     .     /(lambda+temp+lambda*gamma*eig(i)*temp-lambda*mu*theta2*temp)
            cu2(i)=u2(i)
               if (i.ne.0) then
                 w2(i)=(-(u2(i)-u_n2(i)))/temp
               endif
80        continue
C
          CALL C06HBF(n,u1,wsave)
          CALL C06HBF(n,u2,wsave)
```

```
C          Computing the difference |U_i^{n,k+1}-U_i^{n,k}|,i=1,2
          diff=0.0D0
          do 83 i=0,n
             if (max(abs(u1(i)-uk1(i)),abs(u2(i)-uk2(i))).gt.diff) then
                diff=max(abs(u1(i)-uk1(i)),abs(u2(i)-uk2(i)),diff)
                imax=i
              endif
              uk1(i)=u1(i)
              uk2(i)=u2(i)
83        continue
C
          do 34 i=0,n
         ru1(i)=2.0*u1(i)-xu1(i)
         ru2(i)=2.0*u2(i)-xu2(i)
 34       continue

       if (mod(nloops,100).eq.0) print *,loopy, loop, nloops, imax,diff
C      If our stopping criterion holds, we then move onto the next level
C      time. Otherwise, we go to the next iteration.
        if (diff.lt.tol) then
          goto 56
        end if
C
       go to 55
C
C      We update the time
56     time=time+tau
       step=step+1
C
C     Storing the solutions at time level n in n-th row of the matrices
        do 909 j=0,n
           A1(step,j+1)= u1(j)
           A2(step,j+1)= u2(j)
909     continue
C
C      We intialize the next time level and check that the mean-values
C      are conserved
        sumu1=0.0D0
        sumu2=0.0D0
        do 811 i=0,n
          u_n1(i)=cu1(i)
          u_n2(i)=cu2(i)
          unm1(i)=u1(i)
          unm2(i)=u2(i)
          sumu1=sumu1+u1(i)
```

```
          sumu2=sumu2+u2(i)
811    continue
C
        sumu1=(sumu1-(u1(0)+u1(n))*0.5)*h-sumu10
        sumu2=(sumu2-(u2(0)+u2(n))*0.5)*h-sumu20
        print *,loopy, loop, nloops,sumu1,sumu2
        nloops_tot=nloops_tot+nloops
52     continue
C
C      printing results
        if (loopy.le.9) then
          write(number1,901) loopy
          datafile1 =lettert//number1//'.dat'
          datafile2 =letterw//number1//'.dat'
        else
          if (loopy.le.99) then
            write(number2,902) loopy
            datafile1 =lettert//number2//'.dat'
            datafile2 =letterw//number2//'.dat'
          else
            if (loopy.le.999) then
              write(number3,903) loopy
              datafile1 =lettert//number3//'.dat'
              datafile2 =letterw//number3//'.dat'
            else
              write(number4,904) loopy
              datafile1 =lettert//number4//'.dat'
              datafile2 =letterw//number4//'.dat'
            end if
          end if
        endif
C      Write to a data file
        open(1,status='new',file=datafile1)
        open(2,status='new',file=datafile2)
          do 120 i=0,n
            x=real(i)*h
            write(1,*) sngl(x),sngl(u1(i))
            write(2,*) sngl(x),sngl(u2(i))
120       continue
        close(2)
        close(1)

51    continue
C
      print *, nloops_tot
```

```
CCCC   Now we compute the solutions uc1,uc2 on a coarse mesh CCCCCCCCCCCCC
CCCC             or on a larger time step                   CCCCCCCCCCCCC
       hc=1.0/2048.0
       nc=2048
       pc=512
C      since the space step is fixed, we do not need to recompute P_gamma^hc
       sumu10=0.0D0
       sumu20=0.0D0

       do 62 i=0,nc
            uc1(i)=u10(i)
            u_nc1(i)=uc1(i)
            unmc1(i)=uc1(i)
            ruc1(i)=uc1(i)
            ukc1(i)=uc1(i)
            sumu10=uc1(i)+sumu10
            uc2(i)=u20(i)
            u_nc2(i)=uc2(i)
            unmc2(i)=uc2(i)
            ruc2(i)=uc2(i)
            ukc2(i)=uc2(i)
            sumu20=uc2(i)+sumu20
62      continue
      sumu10=(sumu10-(uc1(0)+uc1(nc))*0.5)*hc
      sumu20=(sumu20-(uc2(0)+uc2(nc))*0.5)*hc
C  The next step is to check the mean value of the intial data
      print*,'mean value uc1^0=',sumu10
      print*,'mean value uc2^0=',sumu20
      CALL C06HBF(nc,u_nc1,wsave)
      CALL C06HBF(nc,u_nc2,wsave)

C Calcalating UC1^{n,k+1} and UC1^{n,k+1} at the level time n

      tauc=1.0/12288.0
      mc=6144
       if (mod(m,mc).ne.0)
     .  print*,'fine time step is not a multiple of the large time step'
       step=0
       time=0.0D0
      do 651 loopy=1,k5
        do 652 loop=1,mc/k5

        nloops=0
655      nloops=nloops+1
C The next step is to find UC1^{n,k+0.5} and UC2^{n,k+0.5} at the nodes
```

```
          do 6113 i=0,nc
           CALL LOG_PROJ(ruc1(i),uc1(i),ermu1,lambda,xmin,theta)
           CALL LOG_PROJ(ruc2(i),uc2(i),ermu2,lambda,xmin,theta)
           ukphc1(i)=ermu1
           ukphc2(i)=ermu2
C
           xuc1(i)=2.0*ukphc1(i)-ruc1(i)
           xuc2(i)=2.0*ukphc2(i)-ruc2(i)
          yuc1(i)=(ukc1(i)+alpha1)*((ukc2(i)+alpha2)**2 +
     .          (unmc2(i)+alpha2)**2)
          yuc2(i)=(ukc2(i)+alpha2)*((ukc1(i)+alpha1)**2 +
     .          (unmc1(i)+alpha1)**2)
6113    continue
C
       do 6114 i=0,nc
           cxuc1(i)=xuc1(i)
           cxuc2(i)=xuc2(i)
6114    continue
C
         CALL C06HBF(nc,cxuc1,wsave)
         CALL C06HBF(nc,cxuc2,wsave)
         CALL C06HBF(nc,yuc1,wsave)
         CALL C06HBF(nc,yuc2,wsave)
C Now we calculate UC1^{n,k+1} and UC2^{n,k+1} at the nodes.
          do 680 i=0,nc
           tempc=eig(i)*tauc
         uc1(i)=(lambda*(1.0+theta1*tempc*mu)*u_nc1(i)
     .     +(cxuc1(i)-D*lambda*yuc1(i))*tempc)
     .  /(lambda+tempc+lambda*gamma*eig(i)*tempc-lambda*mu*theta1*tempc)
            cuc1(i)=uc1(i)
CC
          uc2(i)=(lambda*(1.0+theta2*tempc*mu)*u_nc2(i)
     .      +(cxuc2(i)-D*lambda*yuc2(i))*tempc)
     .  /(lambda+tempc+lambda*gamma*eig(i)*tempc-lambda*mu*theta2*tempc)
            cuc2(i)=uc2(i)
680        continue
C
          CALL C06HBF(nc,uc1,wsave)
          CALL C06HBF(nc,uc2,wsave)
          diff=0.0D0

          do 683 i=0,nc
         if (max(abs(uc1(i)-ukc1(i)),abs(uc2(i)-ukc2(i))).gt.diff) then
                diff=max(abs(uc1(i)-ukc1(i)),abs(uc2(i)-ukc2(i)),diff)
                imax=i
```

```
              endif
              ukc1(i)=uc1(i)
              ukc2(i)=uc2(i)
683        continue
C
          do 634 i=0,nc
         ruc1(i)=2.0*uc1(i)-xuc1(i)
         ruc2(i)=2.0*uc2(i)-xuc2(i)
634       continue

       if (mod(nloops,100).eq.0) print *,loopy, loop, nloops, diff

        if (diff.lt.tol) then
          goto 656
        end if
C
       go to 655
C
C     we update the time
656     time=time+tauc
C
        step=step+1
        do 9009 j=1,nc+1
           AC1(step,j)=uc1(j-1)
           AC2(step,j)=uc2(j-1)
9009     continue
        sumu1=0.0D0
        sumu2=0.0D0
        do 6811 i=0,nc
          u_nc1(i)=cuc1(i)
          u_nc2(i)=cuc2(i)
          unmc1(i)=uc1(i)
          unmc2(i)=uc2(i)
          sumu1=sumu1+uc1(i)
          sumu2=sumu2+uc2(i)
6811    continue
C
        sumu1=(sumu1-(uc1(0)+uc1(nc))*0.5)*hc-sumu10
        sumu2=(sumu2-(uc2(0)+uc2(nc))*0.5)*hc-sumu20
        print *,loopy, loop, nloops
        nloops_tot=nloops_tot+nloops
652     continue
C
C     printing the solutions on the coarse mesh or on the larger time step at
C      some time levels
```

```
        if (loopy.le.9) then
          write(number1,901) loopy
          datafile1 =lettertc//number1//'.dat'
          datafile2 =letterwc//number1//'.dat'
        else
          if (loopy.le.99) then
            write(number2,902) loopy
            datafile1 =lettertc//number2//'.dat'
            datafile2 =letterwc//number2//'.dat'
          else
            if (loopy.le.999) then
              write(number3,903) loopy
              datafile1 =lettertc//number3//'.dat'
              datafile2 =letterwc//number3//'.dat'
            else
              write(number4,904) loopy
              datafile1 =lettertc//number4//'.dat'
              datafile2 =letterwc//number4//'.dat'
            end if
          end if
        endif
C      Writing  to a data file
        open(3,status='new',file=datafile1)
        open(4,status='new',file=datafile2)
          do 6120 i=0,nc
            x=real(i)*hc
            write(3,*) sngl(x),sngl(uc1(i))
            write(4,*) sngl(x),sngl(uc2(i))
6120       continue
        close(4)
        close(3)

651    continue
C
      print *, nloops_tot
901   format(i1)
902   format(i2)
903   format(i3)
904   format(i4)
C
C      Calculating the error with fixed space step and successive
C      refinement of tau
       ra1=0.0D0
       ra2=0.0D0
C       Since tauc=p*tau and T=0.5, mc=p*m.h=1/n=hc=1/nc.
```

```
       p=4
        f=0
        do 1003 i=1,mc
         do 2003 l=f*p+1,(f+1)*p
           do 2002 j=1,n
         ra1=ra1+tau*h/3.0*((A1(l,j+1)-AC1(i,j+1))**2+
     .      (A1(l,j+1)-AC1(i,j+1))*(A1(l,j)-AC1(i,j))+
     .      (A1(l,j)-AC1(i,j))**2)
     .  +tau/h*((A1(l,j+1)-AC1(i,j+1))-(A1(l,j)-AC1(i,j)))**2
C
         ra2=ra2+tau*h/3.0*((A2(l,j+1)-AC2(i,j+1))**2+
     .      (A2(l,j+1)-AC2(i,j+1))*(A2(l,j)-AC2(i,j))+
     .      (A2(l,j)-AC2(i,j))**2)
     .  +tau/h*((A2(l,j+1)-AC2(i,j+1))-(A2(l,j)-AC2(i,j)))**2
2002      continue
2003      continue
         f=f+1
1003   continue
C
        open(7,status='old',file='4stepstime.dat')
        write(7,*)'The H^1 error is'
        write(7,*)'tau=',tau,'tauc=',tauc,'p=',p
        write(7,*)'ra1=',sngl(ra1),'______','ra2=',sngl(ra2)
        write(7,*)'Total=',sngl(ra1+ra2)
       stop
       end program errodp
C
       SUBROUTINE ROOT_PROG(theta,thetac,x2)
       double precision theta,thetac,x0,x1,x2,f0,f1,f2
       integer itest,stest
       x0=0.999999999999999D0
       x1=0.000000000000001D0
       itest=0
5      itest=itest+1
       f0=0.5*theta*log((1+x0)/(1-x0))-thetac*x0
       f1=0.5*theta*log((1+x1)/(1-x1))-thetac*x1
       if(f0*f1.gt.0)then
       x0=(x0+1)/2
       x1=x1/2
       go to 5
       end if
       stest=0
10     stest=stest+1
       if(abs(x0-x1).gt.0.1D-9)then
       x2=(x0+x1)/2
```

```
       f2=0.5*theta*log((1+x2)/(1-x2))-thetac*x2
       f0=0.5*theta*log((1+x0)/(1-x0))-thetac*x0
         if(f0*f2.lt.0)then
         x1=x2
         go to 10
         else
         x0=x2
         go to 10
         end if
       end if
       x2=(x0+x1)/2
       end
C
C
      SUBROUTINE LOG_PROJ(b,x0,xo,lambda,xmin,theta)
      double precision theta,lambda,b,nu,f0,s,xo,x0,xmin
      nu=lambda*theta*0.5
      xo=x0
      xo=dmax1(xmin,dmin1(xo,-xmin))

C        tolerance = 5D-08
C        |b|<=1.0
C        theta = 0.2,  lambda = 0.1, 1.17504384986494, 0.96086005180472
C        theta=0.25, lambda=0.1,   1.218804824831179, 0.953324218882453
C        theta = 0.5, lambda= 0.1, 1.437609699662358,0.920413662859303
C        theta = 0.8, lambda = 0.1, 1.700175549459774,0.887280434883490
      if (abs(b).le.1.0) then
        s = 2.0
        do while (abs(s-xo).gt.1.0D-07)
          s = xo
          xo = xo-(xo+nu*log((1+xo)/(1-xo))-b)
     .         /(1-xo*xo+2.0*nu)*(1-xo*xo)
          xo=dmax1(xmin,dmin1(xo,-xmin))
        end do
      else
        if (abs(b).gt.1.218804824831179) then
          if (b.gt.0) then
            xo=-xmin
          else
            xo=xmin
          endif
        else
          if (b.gt.0) then
            xo=0.953324218882453
            x1=-xmin
```

```
          else
            x1=-0.953324218882453
            xo=xmin
          end if
          f0=xo+nu*log((1+xo)/(1-xo))-b
          f1=x1+nu*log((1+x1)/(1-x1))-b
          do while (abs(xo-x1).gt.1.0D-07)
            x2=(xo+x1)*0.5
            f2=x2+nu*log((1+x2)/(1-x2))-b
            if (f0*f2.lt.0) then
              x1=x2
              f1=f2
            else
              xo=x2
              f0=f2
            end if
          end do
          xo = x2
         end if
       end if
       end
```

To compute the error (6.3.2) with fixed time step and successive refinement of the space step, one can modify the above program as follows. Since in this case the space step is not fixed, we need to compute again the H^1-projection of the initial data but this time with the coarse mesh parameter. We also replace the part devoted to the computation of the error with a fixed space step by the following:

```
       ras1=0.0D0
       ras2=0.0D0
C       hc=1/nc=p*h=p*1/n,i.e. n=p*nc
       p=n/nc

C      Here we compute H^1 semi-norm of the error
         f=0
       do 4001 j=1,nc
         do 4007 l=f*p+1,(f+1)*P
           do 4008 i=1,m
         ras1=ras1+tau*h*((A1(i,l)*(-1.0/h)+A1(i,l+1)*(1.0/h))
     .      -(AC1(i,j)*(-1.0/hc)+AC1(i,j+1)*(1.0/hc)))**2
C
         ras2=ras2+tau*h*((A2(i,l)*(-1.0/h)+A2(i,l+1)*(1.0/h))
     .      -(AC2(i,j)*(-1.0/hc)+AC2(i,j+1)*(1.0/hc)))**2
4008          continue
4007        continue
         f=f+1
```

```
4001   continue

C      Here we compute the L^2 norm of the error
       ra1=0.0D0
       ra2=0.0D0
       f=0
       do 5007 j=1,nc
         do 5008  l=f*p+1,(f+1)*P
           do 5006 i=1,m
      if(1/h*(-A1(i,l)+A1(i,l+1))-1/hc*(-AC1(i,j)+AC1(i,j+1)).eq.0.0D0)
     . then
       ra1=ra1+tau*h*((A1(i,l)*real(l+1)-A1(i,l+1)*real(l))
     .            -(AC1(i,j)*real(j+1)-AC1(i,j+1)*real(j)))**2
       else
       ra1=ra1+
     .  tau*1/(1/h*(-A1(i,l)+A1(i,l+1))-1/hc*(-AC1(i,j)+AC1(i,j+1)))
     . *((A1(i,l+1)-(AC1(i,j)*(-1/hc)*(real(l+1)*h-real(j+1)*hc)
     .                 +AC1(i,j+1)*(1/hc)*(real(l+1)*h-real(j)*hc)))**3
     .  -(A1(i,l)-(AC1(i,j)*(-1.0/hc)*(real(l)*h-real(j+1)*hc)
     .               +AC1(i,j+1)*(1.0/hc)*(real(l)*h-real(j)*hc)))**3)
     . *(1.0/3.0)
      end if
CCC
      if(1/h*(-A2(i,l)+A2(i,l+1))-1/hc*(-AC2(i,j)+AC2(i,j+1)).eq.0.0D0)
     . then
       ra2=ra2+tau*h*((A2(i,l)*real(l+1)-A2(i,l+1)*real(l))
     .            -(AC2(i,j)*real(j+1)-AC2(i,j+1)*real(j)))**2
       else
       ra2=ra2+
     .  tau*1/(1/h*(-A2(i,l)+A2(i,l+1))-1/hc*(-AC2(i,j)+AC2(i,j+1)))
     . *((A2(i,l+1)-(AC2(i,j)*(-1/hc)*(real(l+1)*h-real(j+1)*hc)
     .                 +AC2(i,j+1)*(1/hc)*(real(l+1)*h-real(j)*hc)))**3
     .  -(A2(i,l)-(AC2(i,j)*(-1.0/hc)*(real(l)*h-real(j+1)*hc)
     .               +AC2(i,j+1)*(1.0/hc)*(real(l)*h-real(j)*hc)))**3)
     . *(1.0/3.0)
      end if
5006          continue
5008       continue
       f=f+1
5007     continue
C
C      Printing rsults
       open(7,status='old',file='4coarsesteps.dat')
       write(7,*)'The H1 norm of the error with coarse mesh is'
       write(7,*)'ras1+ra1=',sngl(ra1+ras1),'ras2+ra2=',sngl(ra2+ras2)
```

```
       write(7,*)'TOTAL NORM=',sngl(ra1+ras1+ra2+ras2)
      close(7)
```

The next program is for the two-dimensional simulations

```
      PROGRAM tdp
      implicit none
      integer nmax
      PARAMETER (nmax=260)
      double precision u1(0:nmax**2),u_n1(0:nmax**2),ukph1(0:nmax**2),
     .    u2(0:nmax**2),u_n2(0:nmax**2),ukph2(0:nmax**2),ru1(0:nmax**2),
     .    eig(0:nmax),uk1(0:nmax**2), ru2(0:nmax**2),uk2(0:nmax**2),
     .    cu2(0:nmax**2),temp,tol,unm1(0:nmax**2),unm2(0:nmax**2),
     .    yu1(0:nmax**2),yu2(0:nmax**2),w1(0:nmax**2),w2(0:nmax**2),
     .    wsave(0:3*nmax),xu1(0:nmax**2),xu2(0:nmax**2),cxu1(0:nmax**2),
     .    cxu2(0:nmax**2),ccu1(0:nmax**2),ccu2(0:nmax**2),lambda,
     .    len,tau,t,h,h2,pi,gamma,diff,mu,theta,alpha1,alpha2,
     .    a,c,xmin,time,theta1,theta2,ermu1,ermu2,r,s,
     .    y,m1,m2,D,x
      integer i,j,m,n,loopy,loop,k5,nloops,nloops_tot,imax,np1,ntop,ij
      character*30 datafile1,datafile2
      character*1 number1
      character*2 number2,lettert,letterw
      character*3 number3
      character*4 number4
      lettert='h1'
      letterw='h2'
      pi=3.14159265358979323846
C
      open(1,status='old',file='temp2.dat')
        read(1,*) len
        read(1,*) n
        read(1,*) t
        read(1,*) m
        read(1,*) gamma,D
        read(1,*) lambda
        read(1,*) tol
        read(1,*) m1
        read(1,*) m2
      close(1)
C
      np1 = n+1
      ntop = n*n+2*n
       theta=0.3D0
       theta1=1.0D0
       theta2=1.5D0
```

```
      call ROOT_PROG(theta,theta1,r)
      call ROOT_PROG(theta,theta2,s)
      alpha1=r
      alpha2=s
      print*,'alpha1=',alpha1
      print*,'alpha2=',alpha2
C
C      Rading the random perturbations of the state (m_1, m_2). Here we
C     consider the first type of the intial condition. The second type
C     can be covered immediately on stting (alpha_1,alpha_2)=(m_1,m_2).
      open(1,status='old',file='t02.dat')
      do 4 i=0,n
        do 5 j=0,n
            ij=i+np1*j
            read(1,*) u1(ij), u2(ij)
5       continue
4     continue
      close(1)
C
      do 14 i=0,n
       if(i.le.4) then
         do 15 j=0,n
          ij=i+np1*j
          u1(ij)=-alpha1
          u2(ij)=-alpha2
15       continue
       else
           if(i.le.48)then
              do 16 j=0,n
               ij=i+np1*j
               u1(ij)=u2(ij)-0.25
               u2(ij)=-alpha2
16             continue
             else
              do 17 j=0,n
               ij=i+np1*j
               u1(ij)=-alpha1
               u2(ij)=alpha2
17            continue
            end if
       end if
14     continue
      do 18 i=0,n
        do 19 j=0,n
            ij=i+np1*j
```

```
            u_n1(ij)=u1(ij)
            u_n2(ij)=u2(ij)
            unm1(ij)=u1(ij)
            unm2(ij)=u2(ij)
            ru1(ij)=u1(ij)
            ru2(ij)=u2(ij)
            uk1(ij)=u1(ij)
            uk2(ij)=u2(ij)
19        continue
18       continue
      open(1,status='old',file='int.dat')
      do 3 i=0,n
       do 6 j=0,n
       ij=i+np1*j
       write(1,*) u1(ij),u2(ij)
6      continue
3     continue
      close(1)
C
      h=real(len)/real(n)
C
13     print *,'number of prints is a rational number'
      read*, k5
      if (mod(m,k5).ne.0) go to 13
      tau=t/real(m)
      print *,tau
       a=-1.0D0
       c=5.0D-8
       xmin=a+c
      h2=h**(2.0D0)

C       these NAG routines calculate the Cosine Transform
      CALL DCOSTI(n+1,wsave)
      CALL C06HBF(n,u_n1,wsave)
      CALL C06HBF(n,u_n2,wsave)
C      We shall use the following 1-D eigenvalues to compute 2-D
C       eigenvalue  as will be seen below
      eig(0)=0.0D0
      do 50 i=1,n
        eig(i)=(2.0D0-2.0D0*dcos(pi*real(i)/real(n)))/h2
50    continue
      time = 0.0D0
C
      do 51 loopy=1,k5
        do 52 loop=1,m/k5
```

```
      nloops=0
55    nloops=nloops+1

        do 113 i=0,ntop
          CALL LOG_PROJ(ru1(i),u1(i),ermu1,lambda,xmin,theta)
          CALL LOG_PROJ(ru2(i),u2(i),ermu2,lambda,xmin,theta)
         ukph1(i)=ermu1
         ukph2(i)=ermu2
         xu1(i)=2.0*ukph1(i)-ru1(i)
         xu2(i)=2.0*ukph2(i)-ru2(i)
        yu1(i)=(uk1(i)+alpha1)*((uk2(i)+alpha2)**2 +
     .        (unm2(i)+alpha2)**2)
        yu2(i)=(uk2(i)+alpha2)*((uk1(i)+alpha1)**2 +
     .        (unm1(i)+alpha1)**2)
113   continue
C
      do 114 i=0,ntop
          cxu1(i)=xu1(i)
          cxu2(i)=xu2(i)
114   continue
C
        CALL C06HBF(n,cxu1,wsave)
        CALL C06HBF(n,cxu2,wsave)
        CALL C06HBF(n,yu1,wsave)
        CALL C06HBF(n,yu2,wsave)

C     Computing U_i^{n,k+1},i=1,2 at the nodes (ih,jh)
       mu=0.5D0
        do 980 i=0,n
          do 990 j=0,n
            ij=i+np1*j
            temp=(eig(i)+eig(j))*tau
         u1(ij)=(lambda*(1.0+theta1*(1-mu)*temp)*u_n1(ij)
     .    +(cxu1(ij)-D*lambda*yu1(ij))*temp)
     .    /(lambda+temp+lambda*gamma*(eig(i)+eig(j))*temp
     .       -lambda*mu*theta1*temp)
           ccu1(ij)=u1(ij)
             if (i.ne.0) then
               w1(ij)=(-(u1(ij)-u_n1(ij)))/temp
             endif
C
         u2(ij)=(lambda*(1.0+theta2*(1-mu)*temp)*u_n2(ij)
     .    +(cxu2(ij)-D*lambda*yu2(ij))*temp)
     .     /(lambda+temp+lambda*gamma*(eig(i)+eig(j))*temp
     .        -lambda*mu*theta2*temp)
```

```
             ccu2(ij)=u2(ij)
               if (i.ne.0) then
                 w2(ij)=(-(u2(ij)-u_n2(ij)))/temp
               endif
 990       continue
 980       continue
C
          CALL C06HBF(n,u1,wsave)
          CALL C06HBF(n,u2,wsave)
C
          diff=0.0D0
          do 83 i=0,ntop
          if (max(abs(u1(i)-uk1(i)),abs(u2(i)-uk2(i))).gt.diff) then
                diff=max(abs(u1(i)-uk1(i)),abs(u2(i)-uk2(i)),diff)
                imax=i
              endif
              uk1(i)=u1(i)
              uk2(i)=u2(i)
83        continue
           do 34 i=0,ntop
            ru1(i)=2.0*u1(i)-xu1(i)
            ru2(i)=2.0*u2(i)-xu2(i)
 34        continue

        if (mod(nloops,100).eq.0) print *,loopy, loop, nloops, diff,imax
          if (diff.lt.tol) then
            goto 56
          end if
C
          go to 55

C      we update the old time and intialize the next time level
 56     time=time+tau
        do 811 i=0,ntop
          u_n1(i) = ccu1(i)
          u_n2(i) = ccu2(i)
          unm1(i) = u1(i)
          unm2(i) = u2(i)
811    continue
C
        print *,loopy, loop, nloops
        nloops_tot=nloops_tot+nloops
52     continue
C      printing results of U_1(ih,jh),U_2(ih,jh) at some time levels
```

```
        if (loopy.le.9) then
          write(number1,901) loopy
          datafile1 =lettert//number1//'.dat'
          datafile2 =letterw//number1//'.dat'
        else
          if (loopy.le.99) then
            write(number2,902) loopy
            datafile1 =lettert//number2//'.dat'
            datafile2 =letterw//number2//'.dat'
          else
            if (loopy.le.999) then
              write(number3,903) loopy
              datafile1 =lettert//number3//'.dat'
              datafile2 =letterw//number3//'.dat'
            else
              write(number4,904) loopy
              datafile1 =lettert//number4//'.dat'
              datafile2 =letterw//number4//'.dat'
            end if
          end if
        endif

        open(1,status='new',file=datafile1)
C        open(2,status='new',file=datafile2)
      do 1124 i=0,n
        do 1125 j=0,n
            x=real(i)*h
            y=real(j)*h
            ij=i+np1*j
            write(1,*) sngl(u1(ij)),sngl(u2(ij))
C            write(2,*) sngl(x),sngl(y),sngl(u1(ij)),sngl(u2(ij))
 1125    continue
 1124    continue
C        close(2)
        close(1)
51    continue
      print *, nloops_tot
 901  format(i1)
 902  format(i2)
 903  format(i3)
 904  format(i4)
      stop
      end program tdp
```

The following program is to generate the solution in RGB structure

```
      PROGRAM colour_picture
      implicit none
      integer nmax
      PARAMETER (nmax=260)
      doubleprecision u1(0:nmax,0:nmax),s1,t1,t3,u2(0:nmax,0:nmax),
     .       s2,t2,alpha1,alpha2
      character*10 datafile1(1:12)
      integer n,i,j,k
C
      n=64
      alpha1=0.72
      alpha2=0.986
C
C m_1=-0.25, m_2=0.5
      datafile1(3) ='l15.dat'
      datafile1(6) ='l110.dat'
      datafile1(2) ='l125.dat'
      datafile1(5) ='l155.dat'
      datafile1(1) ='l1160.dat'
      datafile1(4) ='l1600.dat'
C
      write(99,'(A)') '%!'
      write(99,'(A)') '%%BoundingBox: 57 60 503 694'
      write(99,'(A)') 'newpath'
      write(99,'(A)') '/square'
      write(99,'(A)') '{newpath'
      write(99,'(A)') '0 0 moveto'
      write(99,'(A)') '8 0 lineto'
      write(99,'(A)') '8 8 lineto'
      write(99,'(A)') '0 8 lineto'
      write(99,'(A)') 'closepath'
      write(99,'(A)') 'fill}def'
      write(99,900) 0.4,0.4,'scale'
      write(99,901) 150.0,150.0,'translate'
C
      do 119 k=1,6
      open(1,status='old',file=datafile1(k))
        do 120 i=0,n
          do 121 j=0,n
            read(1,*) u1(i,j), u2(i,j)
 121      continue
120     continue
        close(1)
C
```

```
        do 130 i=0,n-1
          do 131 j=0,n-1
            s1=(u1(i,j)+u1(i+1,j)+u1(i+1,j+1)+u1(i,j+1))*0.25
            s2=(u2(i,j)+u2(i+1,j)+u2(i+1,j+1)+u2(i,j+1))*0.25
            t1=0.5*(1.0+s1/alpha1)
            t2=0.5*(1.0+s2/alpha2)
            t3=-t1-t2+t1*t2+1
       write(99,903) t1,t2,t3,'setrgbcolor','square',8,0,'translate'
 131     continue
C
          if (i.ne.(n-1))
     .     write(99,901) -511.925,8.0,'translate'
130     continue
C
        if (mod(k,3).ne.0) then
          write(99,901) -511.925,30.0,'translate'
        else
          write(99,901) 90.0,-1570.0,'translate'
        end if
 119  continue
      write(99,'(A)') 'showpage'
C
 900  format(F4.1,1X,F3.1,1X,A5)
 901  format(F10.3,1X,F9.3,1X,A9)
 902  format(F5.2,1X,F5.2,1X,F5.2,1X,A11,1X,I3,1X,I1,1X,A5,1X,A6)
 903  format(F5.2,1X,F5.2,1X,F5.2,1X,A11,1X,A6,1X,I1,1X,I1,1X,A9)
      stop
      end
```