Triangle configurations, and Beilinson's conjecture for $K_{1}^{(2)}$ of the product of a curve with itself

ZIGMOND, ROBIN,JAMES

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Robin Zigmond

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Pure Mathematics Group
Department of Mathematical Sciences
University of Durham
England

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Abstract

The aim of this thesis is to look into Beilinson's conjecture on the rank of the integral part of certain algebraic K-groups of varieties over number fields, as applied to $K_1^{(2)}(C \times C)$ where C is a (smooth projective) curve. In particular, it examines whether non-zero integral elements can be obtained from linear combinations of certain special types of elements which I refer to as "triangle" configurations. Most of the thesis examines the special case where C is an elliptic curve.

The main result is that whenever any rational linear combination of such triangle configurations lies in the integral part of $K_1^{(2)}(E \times E)$, then its image under the Beilinson regulator map is the same as that of a "decomposable" integral element, which is to say, one consisting only of constant functions along various curves. Hence, if Beilinson's conjecture is correct and the regulator is injective on the integral part, then no previously unknown integral elements can be produced from these triangle constructions.

I will also examine the same question for some slighly more general elements of $K_1^{(2)}(E \times E)$, and will show that (subject to one conjecture, which seems highly likely to be true, although I have been unable to prove it rigorously) the same result holds, provided that we restrict ourselves to an individual "triangle", as opposed to arbitrary linear combinations. This will follow from conditions on such a triangle which are both necessary (always) and sufficient (at least for certain special types of elliptic curve) for integrality.

Declaration

The work in this thesis is based on research carried out at the Pure Mathematics Group, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

1.1 Background to the Beilinson Conjectures

The work I have done for this thesis has been an attempt to look at one very special case of just one of the celebrated conjectures made by Alexander Beilinson in 1985. Most of the material introducing the conjectures in this section and the next is taken from the excellent article [18] by Peter Schneider.

The Beilinson Conjectures were introduced by Beilinson in his paper "Higher regulators and values of L-functions" [2]. They are perhaps the most general of many conjectures which link special values of L-functions of algebraic varieties over number fields to arithmetic invariants of those varieties – in this case, certain pieces of their algebraic K-groups, which are closely related to motivic cohomology groups.

These ideas go back to the classical Dirichlet regulator, and the analytic class number formula. The Dirichlet regulator map for a number field K maps $\mathcal{O}_K^*$, the group of units in the ring of integers $\mathcal{O}_K$ of K, into a real vector space of dimension $r_1 + r_2 - 1$ (where r_1 and r_2 are, respectively, the number of real embeddings and the number of conjugate pairs of complex embeddings of K) – Dirichlet proved that this map is injective modulo torsion, and that its image is a full lattice in the target space, or in other words that the regulator map becomes an isomorphism when extended to $\mathcal{O}_K^* \otimes \mathbb{R}$. Further, the covolume of this lattice, a real number which is also known, somewhat confusingly, as the regulator, turns out to be an important invariant of the number field. Of course, it is only invariant up to multiplication by a non-zero element of $\mathbb{Q}$, as we first have to choose a $\mathbb{Q}$-basis for $\mathcal{O}_K^* \otimes \mathbb{Q}$. But the (Dirichlet) regulator R_K is often given explicitly as the determinant of a certain matrix (formed using a particular set of generators for the free part of $\mathcal{O}_K^*$), and for this particular value, there is the famous analytic class number formula, which states that the residue of the Dedekind zeta function at $s = 1$ (where it has a simple pole) is equal to

$$\frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{\omega_K \sqrt{|D_K|}},$$

where h_K is the class number of K, D_K the discriminant and ω_K the number of roots of unity. Although often considered to be a tool for computing the class number, if we read this formula modulo non-zero rationals, it gives an expression for the transcendental part of the leading coefficient at $s = 1$ of the Dedekind zeta function, in which the regulator R_K is the most conspicuous element. It is in this sense that the Beilinson conjectures attempt to generalise this formula.

In 1974, Armand Borel was able to compute the ranks of the algebraic K-groups of $\mathcal{O}_K$ [4] – thus generalising Dirichlet's result on the rank of $\mathcal{O}_K^*$, as $K_1(\mathcal{O}_K)$ is precisely $\mathcal{O}_K^*$. He also defined the first so-called "higher regulators", the Borel regulator maps from these algebraic K-groups of $\mathcal{O}_K$ to certain real vector spaces. As for the Dirichlet regulator, the images of the Borel regulator maps are full lattices, whose covolumes are known as the Borel regulators for those K-groups. Borel also later showed [5] that an analogue of the analytic class number formula holds for the Borel regulators – in other words, the leading coefficient of the Dedekind zeta function of the number field at certain integer values is closely related to the values of the Borel regulator (different integer points correspond to different K-groups).

Among several attempts to generalise these results to the K-groups of algebraic varieties over number fields, the Beilinson conjectures are perhaps the most ambitious and far-reaching. There are basically two conjectures, corresponding to the above two results of Borel – one giving information about the ranks of the K-groups, and one relating a quantity called the (Beilinson) regulator to special values of L-functions. It is the first of these two aspects which I have focused on for this thesis – there will be no discussion of L-functions in the main body of the work. But, by way of introduction, I am now going to state both of the conjectures – this will require me to briefly cover some technical definitions which are involved.

1.1.1 $K_i^{(j)}(X)$

We will be dealing throughout with the algebraic K-groups of an algebraic variety X, $K_i(X)$ for i a non-negative integer. For a good introduction to the algebraic K-theory of schemes and varieties, see the book by Srinivas [23].

There are certain natural maps, called Adams operations, on the K-groups of X, denoted by ψ^k – they exist for all positive integers k (see [20]). We define the weight j Adams eigenspace of $K_i(X)$ (strictly speaking, of $K_i(X) \otimes \mathbb{Q}$) as follows:

Definition 1.1

$$K_i^{(j)}(X) := \{z \in K_i(X) \otimes \mathbb{Q} | \psi^k(x) = k^j(x) \quad \forall k \geq 1\}.$$

Note that $K_i^{(j)}(X)$ is by definition a rational vector space. The most important property of these Adams eigenspaces is that they give a decomposition of $K_i(X) \otimes \mathbb{Q}$:

Proposition 1.2 For any algebraic variety X and any $i \geq 0$, we have the decomposition:

$$K_i(X) \otimes \mathbb{Q} = \bigoplus_{j \geq 0} K_i^{(j)}(X).$$

These Adams eigenspaces of the algebraic K-groups of a variety turn out to be very important arithmetic invariants. For example, under certain natural conditions they are isomorphic to both Spencer Bloch's higher Chow groups (after the latter have been tensored with $\mathbb{Q}$), and also to Voevodsky's motivic cohomology groups: $K_i^{(j)}(X) \cong CH^j(X, i) \otimes \mathbb{Q} \cong H^{2j-i}(X, \mathbb{Z}(j))$ [14]. $K_i^{(j)}(X)$ is sometimes referred to as the "weight j part" of $K_i(X)$, and I shall use this terminology below from time to time.

1.1.2 Integral elements in K-theory

Beilinson's conjectures do not in fact concern the whole of $K_i^{(j)}(X)$, but only a certain subspace of so-called "integral" elements, which I shall denote by $K_i^{(j)}(X)_{\mathbb{Z}}$. I will not give the definition of this subspace here, but I shall return to it in Chapter 3.

1.1.3 Deligne Cohomology

Beilinson's regulator map goes from a motivic cohomology group to the corresponding real Deligne cohomology group – that is, from $K_i^{(j)}(X)$ to $H_{\mathcal{D}}^{2j-i}(X, \mathbb{R}(j))$. Here, I shall define the target group. In fact, Beilinson uses a more general form of Deligne cohomology of his own invention, known as Deligne-Beilinson cohomology, but it agrees with the more usual Deligne cohomology for smooth varieties (see [11]), so I will stick to this in the following brief outline.

Let X be a complex manifold, p a positive integer, and A any subring of the complex numbers. Then the Deligne cohomology groups $H^i_{\mathcal{D}}(X, A(p))$ are defined as the hypercohomology of the following complex of sheaves on X:

$$A(p)_{\mathcal{D}} : 0 \to A(p) \to \mathcal{O}_X \to \Omega^1_X \to \ldots \to \Omega^{p-1}_X \to 0.$$

Here, as usual, $A(p)$ denotes the "twisted" constants $(2\pi i)^p A$, $\mathcal{O}_X$ is the sheaf of holomorphic functions on X, and Ω^j_X is the sheaf of holomorphic j-forms, the maps being the usual exterior derivatives, and inclusion of constant functions for the map from $A(p)$. Note that $A(p)$ is in degree 0, so that Ω^j_X is in degree $j+1$, and not degree j. It is clear there is a short exact sequence of complexes $0 \to \Omega^{<p}_X[-1] \to A(p)_{\mathcal{D}} \to A(p) \to 0$ (where the complex $\Omega^{<p}_X$ denotes the usual complex Ω_X of differential forms but replaced by 0 in degrees $\geq p$, and the $[-1]$ means that the degree is shifted upwards by 1), giving rise to a long exact cohomology sequence:

$$\begin{aligned} \ldots \to \frac{H^{i-1}(X, \mathbb{C})}{F^p H^{i-1}(X, \mathbb{C})} &\to H^i_{\mathcal{D}}(X, A(p)) \to H^i(X, A(p)) \\ \to \frac{H^i(X, \mathbb{C})}{F^p H^i(X, \mathbb{C})} &\to \ldots. \end{aligned}$$

(Here, and elsewhere, $F^\bullet H^i(X, \mathbb{C})$ denotes the Hodge filtration.)

The Beilinson regulator map concerns the *real* Deligne cohomology, $H^i_{\mathcal{D}}(X, \mathbb{R}(p))$. Here, providing $i < 2p$, it is easy to see that the map $H^i(X, \mathbb{R}(p)) \to \frac{H^i(X,\mathbb{C})}{F^p H^i(X,\mathbb{C})}$ is injective (this, like most of the material in this subsection, is in [18]), and therefore the above long exact sequence breaks up into a series of short exact sequences:

$$0 \to H^{i-1}(X, \mathbb{R}(p)) \to \frac{H^{i-1}(X, \mathbb{C})}{F^p H^{i-1}(X, \mathbb{C})} \to H^i_{\mathcal{D}}(X, \mathbb{R}^p) \to 0.$$

That is, $H^i_{\mathcal{D}}(X, \mathbb{R}(p)) \cong \frac{H^{i-1}(X,\mathbb{C})}{F^pH^{i-1}(X,\mathbb{C})+H^{i-1}(X,\mathbb{R}(p))}$, which in turn is isomorphic to $\frac{H^{i-1}(X,\mathbb{R}(p-1))}{\pi_1(F^pH^{i-1}(X,\mathbb{C}))}$, where $\pi_1 : H^{i-1}(X, \mathbb{C}) \to H^{i-1}(X, \mathbb{R}(1))$ is induced by the natural projection $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}(1) \to \mathbb{R}(1)$. Since, when $i < 2p$, π_1 is injective when restricted to $F^pH^{i-1}(X, \mathbb{C})$ (by the same injectivity argument as the one referenced above), we see that we have an alternative short exact sequence:

$$0 \to F^pH^{i-1}(X, \mathbb{C}) \to H^{i-1}(X, \mathbb{R}(p-1)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(p)) \to 0.$$

Next, one uses that, since X is a complex manifold, complex conjugation acts on it, and hence on all the cohomology groups in the above exact sequence. There is also an action of complex conjugation on the coefficients of the complex vector space $H^{i-1}(X, \mathbb{C})$, and hence also on all three of the spaces in the short exact sequence (as all can be thought of as subspaces of $H^{i-1}(X, \mathbb{C})$) – it can be seen that both of the maps above are compatible with the action resulting from the composition of the two just mentioned. Therefore, if we pass to the invariants of each space under the combined action of conjugation on X followed by conjugation on coefficients, we still have a short exact sequence (here the "+" superscripts denote these invariants):

$$0 \to F^pH^{i-1}(X, \mathbb{C})^+ \to H^{i-1}(X, \mathbb{R}(p-1))^+ \to H^i_{\mathcal{D}}(X, \mathbb{R}(p))^+ \to 0.$$

And this short exact sequence induces an isomorphism among the highest non-zero exterior powers:

$$\wedge^{\max}\left(F^pH^{i-1}(X, \mathbb{C})^+\right) \otimes \wedge^{\max}\left(H^i_{\mathcal{D}}(X, \mathbb{R}(p))^+\right) \to \wedge^{\max}\left(H^{i-1}(X, \mathbb{R}(p-1))^+\right). \tag{1.1}$$

This, of course, is an isomorphism between *real* vector spaces. But the first and third of the three terms involved here both have a natural $\mathbb{Q}$-structure – the first by the algebraic de Rham cohomology on $X/\mathbb{Q}$, and the third by singular cohomology with rational co-efficients. What the Beilinson regulator map does is provide a way of defining a (rather less obvious) $\mathbb{Q}$-structure on the Deligne cohomology group in the middle, which we can then compare with the other two $\mathbb{Q}$-structures, using the above isomorphism.

1.2 Statement of the conjectures

Beilinson's regulator map, which I shall denote in this thesis by "reg", goes from $K_i^{(j)}(X)$ to the Deligne group $H_{\mathcal{D}}^{2j-i}(X_{\mathbb{C}}, \mathbb{R}(j))^+$, for any algebraic variety X defined over a number field K, and any non-negative integers i and j. $X_{\mathbb{C}}$ here denotes the complex manifold associated to the complex variety $X \times_K \mathbb{C}$. The construction of this map in general is very technical, and we shall not need to know the details – the important thing is that the map exists. The first of Beilinson's conjectures is simply that the regulator map, when restricted to the subspace of integral elements $K_i^{(j)}(X)_{\mathbb{Z}}$, gives rise to a $\mathbb{Q}$-structure on the target space:

Conjecture 1.3 If $i > 1$, the Beilinson regulator map induces an isomorphism $K_i^{(j)}(X)_{\mathbb{Z}} \otimes \mathbb{R} \to H_{\mathcal{D}}^{2j-i}(X, \mathbb{R}(j))^+$ of real vector spaces.

When $i = 1$, which it will be in the case I consider in this thesis, then the conjecture is slightly different. One needs also to consider the group $N^{j-1}(X)$ of algebraic cycles of codimension $j-1$ on X, defined over the basefield K, modulo homological equivalence. There is the usual cycle class map $N^{j-1}(X) \to H^{2j-2}(X, \mathbb{C})$, given via Poincaré duality by integration along the given cycle, whose image lies in $H^{2j-2}(X, \mathbb{R}(j-1))$. If we compose this with the map $H^{2j-2}(X, \mathbb{R}(j-1)) \to H_{\mathcal{D}}^{2j-1}(X, \mathbb{R}(j))$ contained in the exact sequences of the previous section, we therefore obtain a map $N^{j-1}(X) \to H_{\mathcal{D}}^{2j-1}(X, \mathbb{R}(j))$. Its image lies in the "plus" space, and it can also be shown, using our exact sequences, that this map is injective.

Putting this map together with the Beilinson regulator on $K_1^{(j)}(X)$, we obtain an induced map $(K_1^{(j)}(X)_{\mathbb{Z}} \oplus N^{j-1}(X)) \otimes \mathbb{R} \to H_{\mathcal{D}}^{2j-1}(X, \mathbb{R}(j))^+$, which I shall also refer to as simply the (Beilinson) regulator map. Then the conjecture is:

Conjecture 1.4 The above Beilinson regulator map for K_1, reg : $(K_1^{(j)}(X)_{\mathbb{Z}} \oplus N^{j-1}(X)) \otimes \mathbb{R} \to H_{\mathcal{D}}^{2j-1}(X, \mathbb{R}(j))^+$, is an isomorphism.

The above conjectures, if true, would allow a relatively easy way to compute the rank of the K-groups concerned, as the Deligne cohomology groups, being analytic rather than algebraic, allow one to compute their dimension relatively straightforwardly. We shall see an example of this in the next chapter, when we shall work out what Beilinson's conjecture says the rank of $K_1^{(2)}(C \times C)$ should be, for C an algebraic curve. These two conjectures are, of course, the analogue of Dirichlet's result that the image of the Dirichlet regulator map is a full lattice in the target space.

Incidentally, Beilinson originally made the above conjectures for $K_i^{(j)}(X)$, not just the subspace $K_i^{(j)}(X)_{\mathbb{Z}}$. The conjecture was modified after Bloch and Grayson [3] found that the whole K-group was too big, for $K_2^{(2)}$ of elliptic curves. But without the restriction, the conjecture in fact even fails for the Dirichlet regulator – it would give that K^*, instead of $\mathcal{O}_K^*$, should have rank $r_1 + r_2 - 1$, which is obviously absurd.

The last of Beilinson's conjectures is the analogue of the second part of Borel's theorem. I observed in the last section that two of the three terms in the isomorphism (1.1) had natural $\mathbb{Q}$-structures. If the previous two conjectures are true, then we also have a $\mathbb{Q}$-structure on the remaining term, namely the image of the relevant K-group under the Beilinson regulator. Assuming this, we can compute the isomorphism (1.1) relative to the $\mathbb{Q}$-structures which we now have on each term. It is of course only well-defined up to a non-zero rational multiple. This quantity is called the *regulator* for $K_i^{(j)}(X)$ (which should not be confused with the regulator map!). Beilinson's final conjecture is:

Conjecture 1.5 The regulator for $K_i^{(j)}(X)$ is, up to a non-zero rational multiple, equal to the leading co-efficient in the Taylor expansion, at the point $s = j - i$, of the L-function of X associated to $H^{2j-i-1}(X)$.

1.3 Some previous work, and the aims of this project

The Beilinson conjectures have not been proven in any cases except for when X is a point (the spectrum of a number field), when Beilinson's conjectures agree with Borel's theorem – even there, it was by no means easy show that Beilinson's and Borel's regulators actually agree up to rational multiples (Beilinson only gave an outline, the first complete proof was in [17]). A proof of any of the conjectures, for varieties of dimension greater than 0, even in very specific cases, seems likely to be unattainable with the current state of knowledge.

However, there are some partial results known. One thing which can be looked at is the rank that Beilinson's conjecture (the first and/or second of the three above) predicts for the integral part of $K_i^{(j)}(X)$. It would be possible to show that the rank is *at least* that predicted by Beilinson if one can find a set of concrete integral elements, of the desired size, and show that they are linearly independent. For some of the lower K-groups – principally Adams eigenspaces of K_1 and K_2 – it is possible to construct quite explicit elements; then one hopes to find that some of these elements are integral, and that one can find enough independent ones to generate the whole group, if the predicted rank is correct. One way to show independence is to use the Beilinson regulator map itself – while Beilinson's conjecture that it is an isomorphism is not known to be true, nor is it even known to be injective, it is at least known to be a homomorphism, by construction. Hence, one way to show that elements of the K-group are linearly independent is to show that their regulator images are linearly independent. One can do this in practice by computing the actual Beilinson regulator numerically – the real number, that is, obtained by taking the determinant of a matrix whose rows or columns are the regulator images of the elements concerned. If this regulator is non-zero, then the original set of elements must be linearly independent.

This has been done, for $K_2^{(2)}$ of certain hyperelliptic curves, by Rob de Jeu, Tim Dokchitser and Don Zagier [9]. They constructed the predicted number of elements, and showed their independence by computing the regulator as the determinant of a certain matrix, and taking the limit of this determinant over a continuous family of curves – the limit was non-zero, so that eventually the regulator must not vanish, and thus the elements are independent. (They also used these elements to numerically verify the second conjecture, about the L-value.) De Jeu later wrote a follow-up paper with similar arguments [8].

Rob de Jeu was my supervisor when I started this research project, and he suggested that I should try to carry out a similar argument for the case of $K_1^{(2)}$ of the product of a curve with itself – he had already come up with a construction of seemingly non-trivial elements in $K_1^{(2)}(C \times C)$ (the "triangle" configurations which I shall introduce in the next chapter), and wondered if it was possible to produce enough integral elements from this construction to fill out the predicted rank. If so, then he hoped that it would be possible to use a similar limiting argument to show their independence, and thereby establish that $K_1^{(2)}(C \times C)$ has rank at least as big as that expected if Beilinson's conjecture is true.

However, as we shall see, I eventually discovered that this would not be possible – there are no integral triangle configurations whose regulator does not vanish. The following chapters will explain all the necessary background material, and give statements and proofs of the results I have obtained.

Chapter 2

Beilinson's Conjecture on the rank of $K_1^{(2)}(C \times C)$

2.1 Description of $K_1^{(2)}(X)$

Having briefly stated Beilinson's conjectures in general, I shall in this chapter have a look at what it says about the dimension of the rational vector space $K_1^{(2)}(C \times C)$, for C a curve over an arbitrary number field K. First, I shall give a concrete description of this vector space, and describe what its elements are.

Let X be any non-singular variety over an arbitrary field – in fact, any regular scheme will do. Then there is a so-called Brown-Gersten-Quillen (or BGQ) spectral sequence, of cohomological type (see [23], Theorem 5.20):

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(k(x)) \Rightarrow K_{-p-q}(X).$$

(Strictly speaking, it actually converges to $K'_{-p-q}(X)$, but with the condition that X is regular, this is naturally isomorphic to the corresponding K-group.)

To explain the notation, $X^{(p)}$ refers to the set of subschemes of X of codimension p, and $k(x)$, for such a subscheme x, denotes its function field (ie. the residue field at the generic point). It is also known that, after tensoring with $\mathbb{Q}$, this spectral sequence "decomposes" into several, one for each Adams weight [24], as follows:

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}^{(j-p)}(k(x)) \otimes \mathbb{Q} \Rightarrow K_{-p-q}^{(j)}(X).$$

We shall see that, for low values of j, this spectral sequence allows us to give a very concrete description of the K-groups concerned. We shall concentrate on the case where $j = 2$, and also assume that X is irreducible. First, we note that $E_1^{p,q} = 0$ whenever $p < 0$ (as then $X^{(p)}$ is empty) or whenever $p + q > 0$ (as fields have trivial negative K-groups). In particular, it is a 4th quadrant spectral sequence, with non-zero entries only below (or on) the antidiagonal $p + q = 0$. Now let us compute some E_1 terms (for ease of notation, we shall write $\mathbb{Q}$ as a subscript to denote tensoring with $\mathbb{Q}$): $E_1^{0,0} = \oplus_{x\in X^{(0)}} K_0^{(2)}(k(x)) = 0$, as K_0 of a field is pure of weight 0; $E_1^{0,-1} = \oplus_{x\in X^{(0)}} K_1^{(2)}(k(x)) = 0$ (K_1 of a field is pure of weight 1); $E_1^{1,-1} = \oplus_{x\in X^{(1)}} K_0^{(1)}(k(x)) = 0$ – so non-zero entries can only occur when $q \leq -2$. $E_1^{0,-2} = \oplus_{x\in X^{(0)}} K_2^{(2)}(k(x)) = K_2^{(2)}(k(X)) = K_2(k(X))_{\mathbb{Q}}$, as K_2 of a field is likewise pure of weight 2. (NB. a similar "purity" result does not hold for K_n with $n > 2$; for a proof when $n = 0$, 1 or 2, note that the i^{th} Milnor K-group of a field is isomorphic to $K_i^{(i)}(K)$ – see Theorem 5.1 of [14] – and that when $i = 0$, 1 or 2, the i^{th} Milnor K-group is the same as the Quillen K-group; this is trivial for $i = 0$ or 1, and Matsumoto's famous theorem for $i = 2$.) Matsumoto's Theorem (see [15], or the first chapter of [23]) tells us that $K_2(K)$, for K a field, is the free abelian group on symbols $\{f, g\}$ with $f, g \in K^*$, subject to bilinearity ($\{f, gh\} = \{f, g\} + \{f, h\}$, $\{fg, h\} = \{f, h\} + \{g, h\}$) and the relation $\{f, 1-f\} = 0$ for all $f \in K\backslash\{0, 1\}$. Meanwhile, $E_1^{1,-2} = \oplus_{x\in X^{(1)}} K_1^{(1)}(k(x)) = \oplus_{x\in X^{(1)}} k(x)^*_{\mathbb{Q}}$, and $E_1^{2,-2} = \oplus_{x\in X^{(2)}} K_0^{(0)}(k(x)) = \oplus_{x\in X^{(2)}} \mathbb{Q}$, so that the row $q = -2$ of the E_1 page of our spectral sequence is:

$$K_2(k(X))_{\mathbb{Q}} \xrightarrow{T} \bigoplus_{x\in X^{(1)}} k(x)^*_{\mathbb{Q}} \xrightarrow{\text{div}} \bigoplus_{x\in X^{(2)}} \mathbb{Q}.$$

The two maps in the above sequence are also easily described – it turns out that the second (labelled "div") is the sum of the usual divisor maps on each codimension 1 subscheme, while the first, labelled T, is the tame symbol, which sends the symbol $\{f, g\}$ to $\left\{(-1)^{\text{ord}_x(f)\text{ord}_x(g)}\left(\frac{f^{\text{ord}_x(g)}}{g^{\text{ord}_x(f)}}\right)|_x\right\}_{x\in X^{(1)}}$ (where ord_x, for x a regular codimension 1 subscheme, as usual refers to the normalised discrete valuation corresponding to x, and the vertical line denotes the restriction of a function). (The proof that the second map is the divisor map is in [23], lemma 5.28, while the second being the tame symbol is a formal consequence of this and the existence of products in K-theory – it is a bit long to put here, but I have included it in a subsection below, as it does not seem to be in the literature.)

We shall also need the $q = -3$ row. Here, $E_1^{0,-3} = \oplus_{x\in X^{(0)}} K_3^{(2)}(k(x)) = K_3^{(2)}(k(X))$, while $E_1^{1,-3} = \oplus_{x\in X^{(1)}} K_2^{(1)}(k(x)) = 0$, and similarly $E_1^{2,-3} = E_1^{3,-3} = 0$. Hence, at the E_2 stage, we find that the $q = -2$ and $q = -3$ rows are:

$$\begin{array}{ccc} \ker T & \frac{\ker(\text{div})}{\text{im }(T)} & \text{coker}(\text{div}) \\ ? & 0 & 0, \end{array}$$

from which we can conclude (since all the other terms on the antidiagonal $p+q = -1$ in the E_2 page must vanish), that $K_1^{(2)}(X) = \frac{\ker \text{div}}{\text{im } T}$. In other words, an element of $K_1^{(2)}(X)$ can be represented by a formal sum of the form $\sum_i (V_i, f_i)$, where the V_i are codimension 1 subschemes, and the f_i are functions on the V_i such that $\sum_i (f_i) = 0$ as a codimension 2 cycle on X (for a function f, (f) will denote its divisor). Such a cycle is zero in $K_1^{(2)}$ if and only if it is in the image of the tame symbol from $K_2(k(X))_{\mathbb{Q}}$. Note that, of course, $(V, f)+(V, g) = (V, fg)$, and hence that $(V, 1) = 0$ for any V, and $(V, f^{-1}) = -(V, f)$ for any V and f.

2.1.1 The Tame Symbol

As promised, I shall now give a proof that the map T mentioned above is in fact the tame symbol map. As I mentioned, I shall use the existence of products in K-theory – for which see pp. 29-34, and Remark 5.7, p. 58, of [23]. In particular, for any ring R, there is a natural product $K_i(R) \otimes K_j(R) \to K_{i+j}(R)$, induced by the tensor product of modules. We can localise and deal with a regular local ring R, with fraction field K and residue field k; then we need to show that the boundary map $K_2(K) \to K_1(k)$ in the relevant localisation sequence is the tame symbol map.

Because all the rings involved here are regular, and thus their K-theory is canonically isomorphic to their K'-theory, we also obtain products $K'_i(R) \otimes K'_j(K) \to K'_{i+j}(K)$ and $K'_i(R) \otimes K'_j(k) \to K'_{i+j}(k)$, since the tensor product of any R-module with one which is divisible by the maximal ideal (respectively, annihilated by the maximal ideal) will have the same property. By the naturality of the product, and of the localisation sequence, the square

$$\begin{tikzcd} K_1(R) \otimes K_1(K) \arrow[r] \arrow[d] & K_1(R) \otimes K_0(k) \arrow[d] \\ K_2(K) \arrow[r] & K_1(k) \end{tikzcd}$$

commutes, where the horizontal maps are the boundary maps in the respective localisation sequences (tensored with $K_1(R)$ on the top one), and the vertical maps are the products. We have already noted that the upper horizontal map is the product of valuation map $K^* \to \mathbb{Z}$ with R^*. We will now use the fact – [23] pp. 15-16 and p. 34 – that the product $K_1(R) \otimes K_1(R) \to K_2(R)$ is the maps which sends $u \otimes v$ to $\{u, v\}$, $u, v \in R^*$. (The elements $\{u, v\}$ exist in $K_2(R)$ for any commutative ring R, even though they may not generate it if R is not a field.) Since the symbols $\{u, v\}$ are multiplicative and skew-symmetric, it will suffice to check that the lower horizontal arrow is the tame symbol for symbols of the form $\{u, v\}$ with $u, v \in R^*$ and $\{u, \pi\}$, where $u \in R^*$ and π is a generator of the maximal ideal of R. But, by the commutativity of the diagram, and the remark above about the product $K_1(R) \otimes K_1(R) \to K_2(R)$, we see that $\{u, v\} \mapsto 1$ and $\{u, \pi\} \mapsto u$, both of which agree with the tame symbol. This completes the proof.

2.2 Elements of $K_1^{(2)}(C \times C)$

When X is the product of a curve with itself, then there are several ways in which we can try to produce (representatives for) elements of $K_1^{(2)}(C \times C)$, for a wide range of curves C. Let us suppose that we can find a function f on C whose divisor has the form

$$(f) = d(P) - d(Q),$$

for some positive integer d and 2 distinct points P, Q on C. (The existence of such a function for a particular d requires, and for an elliptic curve is equivalent to, the existence of non-trivial d-torsion points on the Jacobian of C – for higher genus curves a bit more is needed.) Given this, we can instantly write down the following element of $K_1^{(2)}(C \times C)$, which we may call a "rectangle configuration": $(C \times \{P\}, f) + (C \times \{Q\}, f^{-1}) + (\{P\} \times C, f^{-1}) + (\{Q\} \times C, f)$ (f is defined on C, and can therefore be considered as a function on each of these four curves, which are all isomorphic to C in a natural way). Notice that the zeros and poles cancel out at each "corner": at (P, P) there is a zero from f on $C \times \{P\}$ and a pole from f^{-1} on $\{P\} \times C$, both of order d, and similarly at each of the other three corners. So we do have an element in the kernel of the divisor map, and hence a class in $K_1^{(2)}(C \times C)$. Unfortunately, it is zero, as can be seen by computing the tame symbol of the element $\{f \circ \pi_1, f \circ \pi_2\}$ in $K_2(k(C \times C))$, where π_1 and π_2 are the two natural projections $C \times C \to C$; it is d times the element we just wrote down, and hence the "rectangle configuration" is zero in $K_1^{(2)}(C \times C)$ (recall that we have tensored with $\mathbb{Q}$). There is also a generalisation of this construction, with two functions on C with divisors $(f_1) = d_1(P_1) - d_1(Q_1)$, $(f_2) = d_2(P_2) - d_2(Q_2)$; then $(C \times \{P_1\}, f_2^{d_1}) + (C \times \{Q_1\}, f_2^{-d_1}) + (\{P_2\} \times C, f_1^{-d_2}) + (\{Q_2\} \times C, f_1^{d_2})$ is in $K_1^{(2)}(C \times C)$, but is seen to be zero by considering $T(\{f_2 \circ \pi_1, f_1 \circ \pi_2\})$.

However, we can also consider the "triangle configuration", which, given a function f as above, is

$$\alpha_f := (C \times \{P\}, f) + (\{Q\} \times C, f) + (\Delta, f^{-1}),$$

where here Δ denotes the diagonal. One may suspect, or worry, that a similar, if less obvious, computation to those above might prove elements of this form to be always zero as well, but this turns out not to be the case – while they may be zero in certain special circumstances, they are not universally 0, in contrast to the rectangle configurations above. I will delay a proof of this fact until I have introduced the regulator map for $K_1^{(2)}$, in a later section.

One issue that needs to be mentioned here – and it will prove crucial in our computations of integrality, and of the regulator images of these elements – is that the above-mentioned elements remain in the kernel of the divisor map if we replace each copy of the functions concerned by any suitable constant multiple. This means that a "rectangle" configuration, in a slightly more general sense than that mentioned above, need not be zero, for the tame symbol of $\{f_2 \circ \pi_1, f_1 \circ \pi_2\}$ gives functions which are the exact inverse of each other on opposite sides of the rectangle, but we are free to choose a different constant on each curve. However, the fact that the prototypical rectangle is zero means that any element derived from it by replacing the functions by constant multiples is equal, in $K_1^{(2)}$, to a similar rectangle with just constant functions along each of the four sides. And such elements have little interest for us – as we shall see in the last section of this chapter, there will, over all number fields except for $\mathbb{Q}$ and imaginary quadratic fields, always be such elements in the integral part of $K_1^{(2)}(C \times C)$, but they are not expected to generate all of it (except in special cases).

But similar considerations apply for the triangles too. The main aim of this research project was to find examples of integral elements in $K_1^{(2)}(C \times C)$ which are linear combinations of such triangles, and for this we shall have to bear in mind that, on each of the three curves, we may use any constant multiple of f in the construction, and can choose these three constants independently. This will be taken account of in all future computations – it clearly allows much more scope for finding integral elements.

We can also state this more formally. We shall first recall a definition:

Definition 2.1 An element of $K_1^{(2)}(X)$, for X any variety, is called *decomposable* if it is of the form $\sum_i (V_i, c_i)$, where the c_i are all constant functions.

We will be working essentially in the quotient of $K_1^{(2)}(X)$ with the subspace generated by all decomposable elements. Given that, by the remarks above, our "triangle" configurations in this extended sense may not be zero, the question arises of how many linearly independent such elements we can expect to find for a given curve C. For we shall now see that there can certainly be relations among such elements.

Let us suppose that there exists a finite set $\{P_1, \ldots, P_n\}$ of points on C (defined over the base field K), such that all the differences $(P_i) - (P_j)$ are torsion in the Jacobian of C (the existence of any function f as above gives us such a set with $n = 2$, but larger such sets may exist in principle). This means that we can find, for each ordered pair (i, j), a function f_{ij} such that $(f_{ij}) = d_{ij}(P_i) - d_{ij}(P_j)$. We shall make the assumption that C is geometrically irreducible (which will be the case for all curves considered later), so that the f_{ij} are uniquely determined up to constant multiples. Each pair (i, j), together with a choice of f_{ij}, gives us a "triangle" configuration $\alpha_{f_{ij}}$, constructed as above, in $K_1^{(2)}(C \times C)$, which I shall denote by α_{ij}. With this setup, we find that the following situation holds in $K_1^{(2)}(C \times C)$:

Lemma 2.2 Let $\{P_1, \ldots, P_n\}$ be a set of points on C such that the differences $P_i - P_j$ are all torsion divisors, and define α_{ij} as above for each ordered pair (i, j). Then, given a suitable choice of functions f_{ij}, we have:

1. $\alpha_{ij} + \alpha_{ji} = 0$

2. For any triple (i, j, k) of indices, let L_{ijk} be the common value of $\mathrm{lcm}(d_{ij}, d_{ik})$, $\mathrm{lcm}(d_{ij}, d_{jk})$ and $\mathrm{lcm}(d_{ik}, d_{jk})$. (These values will be the same provided that we choose our f_{ij} so that the d_{ij} are as small as possible.) Then

$$\frac{L_{ijk}}{d_{ij}}\alpha_{ij} + \frac{L_{ijk}}{d_{jk}}\alpha_{jk} = \frac{L_{ijk}}{d_{ik}}\alpha_{ik}.$$

Proof 1. We have that $\alpha_{ij} = (C \times \{P_i\}, f_{ij}) + (\{P_j\} \times C, f_{ij}) + (\Delta, f_{ij}^{-1})$, while $\alpha_{ji} = (C \times \{P_j\}, f_{ji}) + (\{P_i\} \times C, f_{ji}) + (\Delta, f_{ji}^{-1})$. But clearly, we can ensure that $f_{ji} = f_{ij}^{-1}$ by choosing the functions appropriately, and then the sum will be the "rectangle" configuration $(C \times \{P_i\}, f_{ij}) + (\{P_j\} \times C, f_{ij}) + (C \times \{P_j\}, f_{ij}^{-1}) + (\{P_i\} \times C, f_{ij}^{-1})$, which we earlier observed to be zero.

2. To see that all of the three least common multiples are equal, under the assumption that the d_{ij} are as small as possible, note that any common multiple of two of d_{ij}, d_{ik} and d_{jk} must also be a multiple of the third, since any integer which annihilates, say, $(P_i) - (P_j)$ and $(P_j) - (P_k)$ in the divisor class group (or Picard group) of C must also annihilate $(P_i) - (P_k)$, which means that it must be a multiple of d_{ik} since we chose this to be as small as possible (ie. the order of $(P_i) - (P_k)$ in the divisor class group).

   For the relation, we first write out $\frac{L_{ijk}}{d_{ij}}\alpha_{ij} + \frac{L_{ijk}}{d_{jk}}\alpha_{jk} - \frac{L_{ijk}}{d_{ik}}\alpha_{ik}$ in detail – it has components on 2 horizontal and 2 vertical curves, as well as the diagonal, on which the function is $f_{ij}^{-\frac{L_{ijk}}{d_{ij}}} f_{jk}^{-\frac{L_{ijk}}{d_{jk}}} f_{ik}^{\frac{L_{ijk}}{d_{ik}}}$, whose divisor is easily seen to be 0. Hence we can assume that this function is 1 if we choose the component functions appropriately. Having done so, we can use this relation to simplify the various functions involved, to find that the difference between the two sides of our proposed relation is:

$$(C \times \{P_i\}, f_{jk}^{-\frac{L_{ijk}}{d_{jk}}}) + (\{P_j\} \times C, f_{ij}^{\frac{L_{ijk}}{d_{ij}}}) + (C \times \{P_j\}, f_{jk}^{\frac{L_{ijk}}{d_{jk}}})$$
$$+(\{P_k\} \times C, f_{ij}^{-\frac{L_{ijk}}{d_{ij}}}).$$

This is simply $-\frac{L_{ijk}}{d_{ij}d_{jk}}$ times the generalised rectangle configuration which was also mentioned on page 14, with f_{ij} and f_{jk} playing the roles of f_1 and f_2 respectively, and so it is zero.

□

It is clear from these relations that, given a set $\{P_1, \dots, P_n\}$ of such points on C, the group generated by the $n(n-1)$ elements α_{ij}, $i, j \in \{1, \dots, n\}, i \neq j$ has rank at most $n-1$, as all α_{ij} can be expressed as linear combinations of, for example, the α_{in} with $1 \leq i \leq n-1$. In fact, it is not hard to see that the subgroup of the free abelian group on the α_{ij} generated by the relations given in the lemma can be generated by just those where $j = n$ (by using a relation of the type given in part 2 to express any α_{ij} in terms of α_{in} and α_{jn}), and further that these relations, of which there are $(n-1) + (n-1)(n-2) = (n-1)^2$, are linearly independent. (For each α_{ij} with neither of i, j equal to n occurs in only one of the relations, and all of those of the second type occur in this way. Thus any linear dependence among these relations must involve only those relations of the first type, which is clearly impossible as they each involve different α_{in}.) Hence, provided that there are no more relations among the α_{ij} than the ones listed in the above lemma, a set of points with the desired properties gives us a subgroup of $K_1^{(2)}(C \times C)$ of rank $n-1$.

(Of course, there is no guarantee that further relations may not exist in certain cases.)

Lemma 2.2 might not appear to really answer the question of relationships among different α_{ij} in general. For if we have two functions of the desired type, say $(f_i) = d_i(P_i) - d_i(Q_i)$ for $i = 1, 2$, then it is not clear that there need be any function with divisor of the form $d(P_1) - d(P_2)$, for example, for any positive integer d. Thus it will not be the case that all of our triangle configurations come from a single set of points $\{P_1, \ldots, P_n\}$, as was assumed in the lemma. But in the remainder of this thesis, I am going to concentrate on the case when C is an elliptic curve E, in which case the group structure allows us to simplify the situation. In particular, by replacing f with $f \circ \tau_Q$, where Q is the pole of f and τ_Q is the "translation-by-Q" map $R \mapsto R+Q$, we can ensure that $Q = O$, the identity for the group structure on E, in all cases, and then the set of all points of the form $P_i - Q_i$, where the P_i are the zeros of the given functions and the Q_i the poles, along with O, is just such a set of points as dealt with in the lemma. Of course, the new elements with f replaced by translations of f need not be the same – but in a later chapter I shall show that they have the same image under the regulator map, meaning that, if Beilinson's conjecture is true, they coincide whenever they happen to be integral. In any case, in examining these elements for integrality in the subsequent chapters, I will make no assumption that all the elements in a given linear combination share the same zero or pole.

Notice that, for an elliptic curve E with torsion subgroup of order t, the above lemma tells us that there are (ignoring translations as just discussed) precisely $t-1$ independent triangles to be found in $K_1^{(2)}(E \times E)$ – providing, of course, that there are no relations among them other than the ones given in the lemma. (It is unlikely that there are no such relations though – later we shall see that even some individual triangles are almost certainly zero, and for non-trivial reasons.) And we can certainly say that there will be *at most* $t-1$ independent "triangle" elements.

Finally, and staying for the moment in the case of elliptic curves, we can also adapt the triangle constructions to get some slightly more general elements of $K_1^{(2)}(E \times E)$. Again, we shall utilise the group structure on E, with basepoint O. For any $a \in \mathbb{Z}$, we can consider the curve consisting of all points of the form $\{(P, aP) | P \in E\}$ on $E \times E$. Clearly it is isomorphic to E (the obvious set-theoretic bijection between the two is also an isomorphism of curves, since the addition law on E is a morphism, and its inverse is just projection to the first component) – we shall denote this curve by Δ_a. Now suppose that, as before, we are given a function $f \in K(E)$ whose divisor has the form $d(P) - d(Q)$. This means that $dP = dQ$ in the group structure on E, which in turn means that $daP = daQ$, and hence that there exists a function f_a with divisor $f_a = d(aP) - d(aQ)$ [21]. And

$$\alpha_{a,f} := (E \times \{aP\}, f) + (\{Q\} \times E, f_a) + (\Delta_a, f^{-1})$$

satisfies the condition to be (a representative for) an element of $K_1^{(2)}(E \times E)$ – the zeros of the three given functions are at (P, aP), (Q, aP) and (Q, aQ) respectively, while the respective poles are at (Q, aP), (Q, aQ) and (P, aP), and they all cancel out since all have order d. Of course, the previous construction, when restricted to elliptic curves, is just the case $a = 1$ of this one. (And when $a = 0$ we get an element of the type we are not interested in, with only constant functions, since f_0 is constant and $\Delta_a = E \times \{aP\} = E \times \{0\}$.) Note also that, under the conditions of the previous lemma, if we replace each α_{ij} with the corresponding element for any fixed a, the same results hold – the proof is exactly the same as it was for $a = 1$.

I can now explain what the main result of this thesis is. Originally, the hope was to find certain linear combinations of triangles which would lie in the integral part of $K_1^{(2)}(C \times C)$, enough of them to fill out the rank of this group predicted by Beilinson's conjecture (which we shall compute in the next section), and then, by using the regulator map, to show that these elements are linearly independent, and hence that the rank must be at least that conjectured. Unfortunately, this turns out to be impossible, in the case of an elliptic curve – for as we shall see, any linear combination of such elements, if integral, must have vanishing regulator. Hence either all integral triangles are zero, or the Beilinson regulator fails to be injective (of which he first option seems the most plausible). I have been able to prove this result if we restrict to the "original" triangles with $a = 1$. I have also tried to generalise it to cases where a is allowed to vary, but I have been unable to attain such a general result here (although I still expect it to be true). However, I have been able to show certain partial results, including that any *individual* triangle, with certain restrictions on a, cannot be both integral and have non-vanishing regulator. The proofs of these facts will take up the remainder of the thesis.

2.3 The Deligne Cohomology group

While $K_1^{(2)}(C \times C)$ is the group in which my direct interest lies, in order to understand what the Beilinson conjecture says about it, or to make use of the Beilinson regulator, we need to know something about the Deligne cohomology group which is the target space of the regulator map, which in this case is $H^3_{\mathcal{D}}((C \times C)_{\mathbb{C}}, \mathbb{R}(2))$. Note that, since C is assumed to be smooth, so is $C \times C$, and thus we can use the "ordinary" Deligne cohomology, rather than the more general Deligne-Beilinson cohomology.

The general definition of Deligne cohomology, for an arbitrary complex manifold X, was given in the introduction. Recall that it is related to ordinary (de Rham) cohomology, and the Hodge filtration on it, by the following long exact sequence:

$$\begin{aligned} \ldots \to \frac{H^{i-1}(X, \mathbb{C})}{F^p H^{i-1}(X, \mathbb{C})} \to H^i_{\mathcal{D}}(X, A(p)) \to H^i(X, A(p)) \\ \to \frac{H^i(X, \mathbb{C})}{F^p H^i(X, \mathbb{C})} \to \ldots . \end{aligned}$$

The particular Deligne cohomology group which I am interested in, as the target of the Beilinson regulator from $K_1^{(2)}$, is $H^3_{\mathcal{D}}(X, \mathbb{R}(2))$, where X will be the complex manifold associated to the surface $C \times C$ with C a curve defined over a number field. However, for now we are going to use the above exact sequence to analyse $H^3_{\mathcal{D}}(X, \mathbb{R}(2))$ for an arbitrary X. We have the following exact sequence:

$$\begin{aligned} \ldots \to H^2(X, \mathbb{R}) \to \frac{H^2(X, \mathbb{C})}{F^2 H^2(X, \mathbb{C})} \to H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \to H^3(X, \mathbb{R}) \\ \to \frac{H^3(X, \mathbb{C})}{F^2 H^3(X, \mathbb{C})} \to \ldots \end{aligned}$$

It can be shown that the maps $H^i(X, \mathbb{R}) \to \frac{H^i(X,\mathbb{C})}{F^p H^i(X,\mathbb{C})}$ here are nothing other than the composition of the natural inclusion and projection maps, so the kernel of this map is $H^i(X, \mathbb{R}) \cap F^p H^i(X, \mathbb{C})$. But, when $2p > i$, which is the case when $p = 2$ and $i = 2$ or 3, then this intersection is trivial, as complex conjugation takes a form of type (a, b) to one of type (b, a), so any non-trivial form of type (a, b) in the intersection must not only have $a \geq p$, but also $b \geq p$ since, being in $H^i(X, \mathbb{R})$, it must be equal to its own conjugation; so $i = a + b \geq 2p > i$, which is a contradiction. Therefore, the first and last maps in the part of the sequence shown above are injective, and so we reduce to a short exact sequence, which tells us that

$$H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \cong \frac{H^2(X, \mathbb{C})}{F^2H^2(X, \mathbb{C}) + H^2(X, \mathbb{R})}.$$

We can also give an alternative description, which will be slightly easier to work with. Firstly, if we apply the projection $\pi_1 : H^2(X, \mathbb{C}) = H^2(X, \mathbb{R}(1)) \oplus H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R}(1))$ (note that this has nothing to do with the projection map $\pi_1 : C \times C \to C$ used in the previous section; this should cause no confusion!), we can also write the above group as $H^2(X, \mathbb{R}(1))/\pi_1(F^2H^2(X, \mathbb{C}))$. Next, we have the following simple computation among subgroups of $H^2(X, \mathbb{C})$:

Lemma 2.3 Let $U = H^2(X, \mathbb{R}(1)) \cap F^1H^2(X, \mathbb{C})$ and $V = \pi_1(F^2H^2(X, \mathbb{C}))$. Then

1. $U \cap V = 0$
2. $U + V = H^2(X, \mathbb{R}(1))$

Proof We will use the Hodge decomposition of $H^2(X, \mathbb{C})$ as $H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$, and write $x \in H^2(X, \mathbb{C})$ as (α, β, γ) accordingly. Clearly, for such x, its conjugate $\overline{x}$ is $(\overline{\gamma}, \overline{\beta}, \overline{\alpha})$. So, if x is in $F^2H^2(X, \mathbb{C})$, meaning that $\alpha = \beta = 0$, then $\pi_1(x) = \frac{1}{2}(x - \overline{x}) = \frac{1}{2}(-\overline{\gamma}, 0, \gamma)$. In other words, V is the set of all x for which $\beta = 0$ and $\gamma = -\overline{\alpha}$. And $H^2(X, \mathbb{R}(1))$, the image of the π_1 map, consists of all elements with $\gamma = -\overline{\alpha}$ and $\beta = -\overline{\beta}$, while lying in $F^1H^2(X, \mathbb{C})$ means that $\alpha = 0$. So U is the set of all x for which $\alpha = \gamma = 0$ and $\beta = -\overline{\beta}$. The two statements are now both obvious. $\square$

Now, a direct application of the Second Isomorphism theorem gives us the following result:

Proposition 2.4 For any complex manifold X, $H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \cong H^2(X, \mathbb{R}(1)) \cap F^1H^2(X, \mathbb{C})$.

We shall now specialise to the case we are interested in, and compute the dimension of the real vector space $H^3_{\mathcal{D}}(X, \mathbb{R}(2))^+$ when X is the complex manifold associated to the surface $C \times_K C$ defined over a number field K, C being a (projective) curve. This manifold will consist of d disconnected copies of the product of a Riemann surface S with itself, where d is the degree of the number field. So the desired Deligne cohomology group, $H^3_{\mathcal{D}}(X, \mathbb{R}(2))$ (I shall ignore the "+" superscript for the moment) is the direct sum of d copies of $H^3_{\mathcal{D}}(S \times S, \mathbb{R}(2))$, which as we have just seen is isomorphic to $H^2(S \times S, \mathbb{R}(1)) \cap F^1 H^2(S \times S, \mathbb{C})$. As we just saw in the proof of Lemma 2.3, this is contained within $H^{1,1}(S \times S)$, being the part which is anti-invariant under conjugation. The Künneth formula tells us that, as a *complex* vector space, $H^{1,1}(S \times S) \cong (H^{0,0}(S) \otimes H^{1,1}(S)) \oplus (H^{1,0}(S) \otimes H^{0,1}(S)) \oplus (H^{0,1}(S) \otimes H^{1,0}(S)) \oplus (H^{1,1}(S) \otimes H^{0,0}(S))$, whose dimension over $\mathbb{C}$ is therefore $1 \times 1 + g \times g + g \times g + 1 \times 1 = 2g^2 + 2$, using the well-known basic results about the cohomology of Riemann surfaces – here g is the genus of the surface (and hence that of the algebraic curve from which it came). This description also gives us a basis to use – if we let $\omega_1, \ldots, \omega_g$ be holomorphic 1-forms whose classes form a basis for $H^{1,0}(S)$, and if $\pi_1, \pi_2 : S \times S \to S$ are the natural projections, then one possible basis is:

- $\pi_1^*(\omega_1 \wedge \overline{\omega_1})$
- $\pi_1^*(\omega_k) \wedge \pi_2^*(\overline{\omega_l})$, $k, l \in \{1, \ldots, g\}$
- $\pi_1^*(\overline{\omega_k}) \wedge \pi_2^*(\omega_l)$, $k, l \in \{1, \ldots, g\}$
- $\pi_2^*(\omega_1 \wedge \overline{\omega_1})$.

A basis for $H^{1,1}(S \times S)$ as a *real* vector space, of course, is twice as big, and can consist of the preceding elements along with those same elements multiplied by i. Then the real vector space of elements anti-invariant under conjugation, which clearly must have (real) dimension $2g^2 + 2$, has the following as a basis over $\mathbb{R}$:

- $\pi_1^*(\omega_1 \wedge \overline{\omega_1})$
- $\pi_1^*(\omega_k) \wedge \pi_2^*(\overline{\omega_l}) - \pi_1^*(\overline{\omega_k}) \wedge \pi_2^*(\omega_l)$, $k, l \in \{1, \ldots, g\}$
- $i\pi_1^*(\omega_k) \wedge \pi_2^*(\overline{\omega_l}) + i\pi_1^*(\overline{\omega_k}) \wedge \pi_2^*(\omega_l)$, $k, l \in \{1, \ldots, g\}$
- $\pi_2^*(\omega_1 \wedge \overline{\omega_1})$.

In particular, we see at this point that $H^3_{\mathcal{D}}(X, \mathbb{R}(2))$, when X comes from a product of a curve with itself over a number field of degree d, has dimension $2d(g^2+1)$. But the target group for the Beilinson regulator from $K_1^{(2)}(C \times C)$ is not this whole group, but rather the subgroup $H^3_{\mathcal{D}}(X, \mathbb{R}(2))^+$ consisting of all the elements which are invariant under the action induced by complex conjugation on X, followed by conjugation on all coefficients.

So, in the case where the base field is $\mathbb{Q}$, meaning that there is just one component, on which the above $2g^2+2$ elements are a complete basis, we see that those in the first, second and fourth lines all remain invariant under the action concerned, as its overall effect on such an expression is simply to replace i with $-i$ (a form ω goes to $\overline{\omega}$ under conjugation on the manifold, and then back to ω after conjugating coefficients), while those in the third line are anti-invariant. Therefore, on any specific self-product of Riemann surfaces, $H^3_{\mathcal{D}}(S \times S, \mathbb{R}(2))^+$ has dimension g^2+2.

And in the case of a general number field K of degree d, where we have d different copies of $S \times S$, the combined action will have the effect of acting *on* (in other words, leaving fixed, as a set, although not pointwise) the copy of $\mathbb{R}^{2g^2+2}$ corresponding to any *real* embedding of K, but exchanging the remaining ones in complex conjugate pairs. So, to find an element which is invariant, we need its components corresponding to any real embeddings to be invariant under the corresponding action – over which, as we have just seen, we have g^2+2 degrees of freedom – and *any* choice at all for each remaining *pair* (as whatever we choose on one of the pair determines uniquely what needs to happen on the other). So, if we write r_1 and r_2 respectively for the number of real embeddings and the number of pairs of complex embeddings, we see that the dimension over $\mathbb{R}$ of $H^3_{\mathcal{D}}((C \times C)_{\mathbb{C}}, \mathbb{R}(2))^+$ is $r_1(g^2+2) + r_2(2g^2+2) = dg^2 + 2(r_1+r_2)$.

We are now in a position to find out what Beilinson's conjecture says about the dimension of $K_1^{(2)}(C \times C)_{\mathbb{Z}}$, the integral part of $K_1^{(2)}(C \times C)$. (I have not defined this group yet, but will do so at the start of Chapter 3 – its definition is not important for now.) Recall, from Chapter 1, that because we are dealing with K_1, in our case the Beilinson regulator will be a map

$$\mathrm{reg} : K_1^{(2)}(C \times C)_{\mathbb{Z}} \oplus N^1(C \times C) \to H^3_{\mathcal{D}}((C \times C)_{\mathbb{C}}, \mathbb{R}(2))^+,$$

and is conjectured to be an isomorphism. We now know the dimension of the space on the right – what about $N^1(C \times C)$, also known as the Néron-Severi group, $NS(C \times C)$? Clearly it contains the "horizontal" and "vertical" curves as independent elements, and it is known that (for $g > 0$) once we quotient out the subgroup generated by these two elements, we obtain a group isomorphic to the additive group of the endomorphism ring of the Jacobian of C (each endomorphism corresponding to its graph). This typically has rank 1 (the multiplication by n maps), but can of course be bigger for some specific curves – for example, elliptic curves with complex multiplication.

In fact, from here on we shall restrict to the case where C is an elliptic curve (which we shall therefore usually denote by E, rather than C). Then we can say precisely that $NS(E \times E)$ has rank 4 if E has CM *defined over the base field* K, and has rank 3 otherwise. (Recall that the group $N^i(X)$ was defined to consist only of those algebraic cycles of codimension i which are defined over the base field.) Hence, Beilinson's conjecture in our case predicts:

Conjecture 2.5 Let E/K be an elliptic curve defined over the number field K of degree $d = r_1 + 2r_2$. Then $K_1^{(2)}(E \times E)_{\mathbb{Z}}$ has dimension $d + 2(r_1 + r_2) - 4 = 3r_1 + 4r_2 - 4$ if E has CM over K, and $3r_1 + 4r_2 - 3$ otherwise.

In particular, for $K = \mathbb{Q}$ ($r_1 = d = 1, r_2 = 0$, and no CM possible), the above conjecture says that $K_1^{(2)}(E \times E)_{\mathbb{Z}}$ is trivial – ie. that there are *no* non-trivial integral elements in $K_1^{(2)}(E \times E)$. The main result of this thesis can be seen as supporting evidence for this conjecture – except that the result is independent of the base field, so does not identify anything special about $\mathbb{Q}$. Moving on to quadratic fields, in the real quadratic case the predicted dimension is 3. However, in real quadratic fields we have a non-trivial unit u in the ring of integers $\mathcal{O}_K$ – and then the elements of $K_1^{(2)}(E \times E)$ given by u as a constant function on any curve within $E \times E$ can be easily seen to be integral (I shall return to this when we discuss integrality in the next chapter), and furthermore, the three elements (V, u), for V a horizontal, vertical or diagonal curve, are all independent (as we shall see in the next section when we compute their regulators). Hence there are already 3 independent elements, but these are decomposable, and the conjecture predicts no more. However, for imaginary quadratic fields there are no units (roots of unity give trivial elements of course, as we have tensored with $\mathbb{Q}$), so we do not find such decomposables – but the conjecture now predicts a rank of 1 in the non-CM case (and 0 in the CM case). So we do expect to find non-trivial (that is, indecomposable) integral elements for these fields. (However, as already mentioned, we will prove that they cannot be made up of triangle configurations.)

In general, of course, there will be r_1+r_2-1 independent units, and hence $3(r_1+r_2-1)$ decomposable integral elements. So the expected number of independent nondecomposables is $3r_1+4r_2-3-3(r_1+r_2-1)=r_2$, in the non-$CM$ case – a pleasingly neat prediction. In the CM case, on the other hand, there are four curves to play with, and hence $4(r_1+r_2-1)$ (independent) decomposables – which is r_1 *more* than the total predicted rank! But of course, any complex multiplication cannot be defined over any field which is not totally imaginary, so this is no contradiction – and in the CM case, we expect no elements at all other than the decomposables, whatever the (totally imaginary) base field.

2.4 The regulator map

In this section, I shall complete the description of the ingredients of the Beilinson conjecture in our case, by giving an explicit description of the Beilinson regulator map $\mathrm{reg} : (K_1^{(2)}(X) \oplus NS(X)) \otimes \mathbb{R} \to H^3_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(2))^+$. This uses a slightly different description of the relevant Deligne cohomology group from the one used in the previous section, namely $H^2(X,\mathbb{R}(1)) \cap F^1H^2(X,\mathbb{C})$. If X has (complex) dimension d, then Poincaré duality tells us that $H^2(X,\mathbb{C})$ is the dual vector space to $H^{2d-2}(X,\mathbb{C})$, via the pairing $< \omega, \omega' >= \int_X \omega \wedge \omega'$. Clearly, if ω is of type $(1,1)$ or $(2,0)$, then ω' must be of type $(d-1,d-1)$ or $(d-2,d)$ respectively, as the only non-zero top-dimensional forms are of type (d,d), and so it follows that the dual space of $F^1H^2(X,\mathbb{C})$ is $\overline{F^{d-1}H^{2d-2}(X,\mathbb{C})}$. But also, if ω is in $H^2(X,\mathbb{R}(1))$, meaning that $\overline{\omega}=-\omega$, then we find that, for any form ω' of the required type, $< \omega, \overline{\omega'} >= \overline{< \omega, \omega' >} =< \overline{\omega}, \omega' >=< -\omega, \omega' >=< \omega, -\omega' >$, from which it follows that the pairing also identifies $H^2(X, \mathbb{R}(1))$ with the dual space of $H^{2d-2}(X, \mathbb{R}(d-1))$ (as real vector spaces, now). Putting these two observations together, we can conclude that for any complex manifold X, $H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \cong H^2(X, \mathbb{R}(1)) \cap F^1H^2(X, \mathbb{C})$ is isomorphic to the dual of $H^{2d-2}(X, \mathbb{R}(d-1)) \cap \overline{F^{d-1}H^{2d-2}(X, \mathbb{C})}$ in a natural way – and this space, by the same simple argument as we used in the proof of Lemma 2.3, is the same as $H^{2d-2}(X, \mathbb{R}(d-1)) \cap F^{d-1}H^{2d-2}(X, \mathbb{C})$, as both consist simply of all the forms of type $(d-1, d-1)$ which are either invariant (of d is odd) anti-invariant (if d is even) under conjugation. (Note that, for the purposes of describing and computing the regulator, we do not need to worry about whether the "plus spaces" of each cohomology group coincide under the duality pairing or not – for the description of the regulator map we can simply map into the whole space; the only reason for restricting to the plus space in the previous section was because that was the subspace whose dimension we wished to determine.)

So, returning to our special case, in which $d = 2$, and letting X be an algebraic surface over a number field, we can also consider $H^3_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(2))$ as the dual space of $H^2(X_{\mathbb{C}}, \mathbb{R}(1)) \cap F^1H^2(X_{\mathbb{C}}, \mathbb{C})$, as well as the space itself. Using this, we can now write down an explicit description of the regulator in our case. Given an element α in $K_1^{(2)}(X)$, which we can represent by a cycle $\sum_k (V_k, f_k)$, then ([13], p.349) the regulator image of α is the current which sends an $\mathbb{R}(1)$-valued form ω of type $(1, 1)$ to the real number:

$$\frac{1}{2\pi i} \sum_k \int_{(V_k)_{\mathbb{C}} \setminus \mathrm{sing}((V_k)_{\mathbb{C}})} \log |f_k| \wedge \omega,$$

where $\mathrm{sing}(V)$ denotes the set of singular points on V. (Note that this current, as it is written here, is *not* a closed current, ie. it does not vanish on all exact forms ω. However, there is another current, more complicated to write down, which is closed, and which coincides with the above on all forms of the desired type.)

And, on $NS(X)$, the regulator is just the most natural map possible, consisting of the cycle map to ordinary cohomology, followed by the natural map to Deligne cohomology (the one which occurs in the long exact sequence used in the previous section), which, it is easy to see, just sends a cycle to the current given by integration along it (divided by $2\pi i$ as usual).

As well as figuring heavily in the actual statement of Beilinson's conjectures (especially with regard to the L-value, which I shall not consider in this thesis), for us the main use of the regulator map will be simply in its role as a homomorphism of vector spaces – to show that certain elements of $K_1^{(2)}(X)$ are non-zero, or are independent of each other, if the same is true of their regulator images. And if an integral element of $K_1^{(2)}(C \times C)$ can be shown to have regulator zero, then although this does not guarantee that the element itself is trivial, it does if we accept Beilinson's conjecture that the regulator map is an isomorphism when restricted to the integral part. As an illustration, I shall now use the regulator to prove some statements which I made earlier in this chapter. Firstly, I will show that, if u is a unit (not a root of unity) in the ring of integers of our number field K, then the three elements (V, u) for V a horizontal, vertical or diagonal curve in $C \times C$, are linearly independent. (In fact, this holds whenever u is in K^* and not a root of unity.) So let ω be any closed, non-constant holomorphic 1-form on C (we assume that the genus of C is at least 1, so that such ω certainly exist). Then, with π_1 and π_2 as projections $C \times C \to C$ as before, consider the three forms $\pi_1^*(\omega \wedge \overline{\omega})$, $\pi_2^*(\omega \wedge \overline{\omega})$ and $\pi_1^*(\omega) \wedge \pi_2^*(\overline{\omega}) - \pi_1^*(\overline{\omega}) \wedge \pi_2^*(\omega)$, all of which are in $H^2((C \times C)_{\mathbb{C}}, \mathbb{R}(1)) \cap H^{1,1}((C \times C)_{\mathbb{C}}, \mathbb{C})$. We have to be a bit careful here over a general number field K, as there, as we have already seen, $(C \times C)_{\mathbb{C}}$ consists of $d = [K : \mathbb{Q}]$ different components, one for each embedding of K in $\mathbb{C}$. If we first assume that our field has at least one real embedding σ, then we know that we can take the form which consists of just one of the three above on that component, and 0 on all other components. Then, we see that the regulator of u on the horizontal, say, kills the second and third of these three forms, but sends the first to $\frac{1}{2\pi i} \int_{C^\sigma} \log|\sigma(u)| \omega^\sigma \wedge \overline{\omega^\sigma}$, which cannot be zero because the integral itself cannot be (as our form is part of a basis for H^2), and nor is $\log|\sigma(u)|$ as u is assumed not to be a root of unity (recall that σ is a *real* embedding, so the only way for $\sigma(u)$ to have absolutely value 1 is for u to be plus or minus 1). The same happens to the vertical, with the first and second of our forms switching roles, while for the diagonal we get non-zero results on all three forms. So we see that these three (integral) elements of $K_1^{(2)}(C \times C)$ have independent regulator images, and hence are independent themselves. A similar argument also holds even if K has no real embeddings; we simply have to take a pair of complex embeddings instead. This establishes:

Proposition 2.6 Let $u \in K^*$, not a root of unity. Then the three elements $(C \times \{P\}, u)$, $(\{Q\} \times C, u)$ and (Δ, u) of $K_1^{(2)}(C \times C)$ are linearly independent, for any points P and Q on C.

We can also now prove that the "triangle configuration" elements are *not* zero in general. For an example of one which isn't, consider the elliptic curve E over $\mathbb{Q}$ given by the affine equation $y^2 = x(x-s)(x+s)$, where $s \in \mathbb{Q}$ is arbitrary (for now). Then we have that $(x) = 2(0,0) - 2(O)$ (where O is the point at infinity on this affine model), so we can use it to form the element $(C \times \{(0,0)\}, x) + (\{O\} \times C, x) + (\Delta, x^{-1})$ in $K_1^{(2)}(E \times E)$. Then we can take the form $\mathrm{d}x/y$ as our ω, as it is a closed, holomorphic and non-exact 1-form. We can take any of the three $\mathbb{R}(1)$-valued forms of type $(1,1)$ associated to this given in the previous paragraph, and it is clear that the result of pairing these with the regulator of the just-mentioned element of $K_1^{(2)}$ is in each case a constant multiple (zero in two of the three cases, and non-zero in the other) of $\int_{E_{\mathbb{C}}} \log|x| \frac{\mathrm{d}x}{y} \wedge \frac{\mathrm{d}\overline{x}}{\overline{y}} = \int_{E_{\mathbb{C}}} \frac{\log|x|}{|x(x-s)(x+s)|} \mathrm{d}x \wedge \mathrm{d}\overline{x}$. Using the fact that $E_{\mathbb{C}}$ is a double cover of $\mathbb{C}$, and that, over $\mathbb{C}$, $\mathrm{d}x \wedge \mathrm{d}\overline{x} = -2\pi i \mathrm{d}(\mathrm{Re}\ x) \wedge \mathrm{d}(\mathrm{Im}\ x)$, and ignoring constant multiples, we see that it will suffice to show the following:

Proposition 2.7 For sufficiently large values of s, the integral

$$\int_{\mathbb{C}} \frac{\log|z|}{|z||z-s||z+s|} \mathrm{d}x\mathrm{d}y$$

does not vanish (where $z = x + iy$, $x, y \in \mathbb{R}$). Therefore, the element $(E \times \{(0,0)\}, x) + (\{\infty\} \times E, x) + (\Delta_E, x^{-1})$ of $K_1^{(2)}(E \times E)$ is non-trivial.

Proof Here, we shall use a very naive argument which gets the job done. Note first that the integrand is negative inside the unit disc, and positive outside. If we assume that $s > 1$ then, inside the unit disc, $|z \pm s| \geq ||z| - s| = s - |z| > s - 1$, so that the absolute value of the negative contribution to the integral is bounded above by $-\frac{1}{(s-1)^2} \int_{|z|<1} \frac{\log|z|}{|z|} \mathrm{d}x\mathrm{d}y = -\frac{2\pi}{(s-1)^2} \int_{R<1} \log R \mathrm{d}R = \frac{2\pi}{(s-1)^2}$. Meanwhile, if we further assume that $s > 2$, then the positive contribution includes that from the disc $|z-s| < 1$. On this disc, $|z| = |(z-s)+s| < s+1$, and so $|z+s| < 2s+1$, while also

$|z| \geq ||z-s|-|s|| = s-|z-s| > s-1$, so $\log|z| > \log(s-1)$. Hence the positive contribution is bounded below by $\frac{\log(s-1)}{(s+1)(2s+1)} \int_{|z-s|<1} \frac{\mathrm{d}x\mathrm{d}y}{|z-s|} = \frac{2\pi \log(s-1)}{(s+1)(2s+1)}$. Comparing the two bounds, we see that the total integral must be greater than zero if we have that $\frac{\log(s-1)}{(s+1)(2s+1)} > \frac{1}{(s-1)^2}$, or that $\log(s-1) > \frac{(s+1)(2s+1)}{(s-1)^2}$. But that must happen whenever s gets large enough, as the left-hand side of this inequality tends to ∞ as $s \to \infty$, while the right-hand side tends to 2. Hence, whenever s is large enough, the integral cannot vanish. That means that this particular "triangle" element of $K_1^{(2)}(E \times E)$ has non-vanishing regulator, and hence is non-zero. $\square$

(Later, we shall take a more algebraic approach to evaluating integrals such as this one, which will tell us that this integral is zero if and only if s lies on the unit disc.)

Note that this does not provide a counter-example to Beilinson's conjecture (which predicts no non-zero integral elements of $K_1^{(2)}(E \times E)$ when E is an elliptic curve defined over $\mathbb{Q}$), as the element in question turns out not to be integral. (Except when $s = \pm 1$, in which case, as remarked above, the corresponding integral is zero in any case.) Nor does it, of course, say that any *particular* "triangle configuration" (other than the one shown here, for s large enough) will not be trivial – as we shall see, we will be able to find integral "triangle" elements for elliptic curves over $\mathbb{Q}$, which do appear to be zero (at least, they have zero regulator), thus not contradicting Beilinson's conjecture. As we have to be working with integral elements in order to say anything about the conjecture itself, in the next chapter I shall look in detail at the integrality condition, and derive some results which can tell us whether our "triangle" elements – or more generally, linear combinations of them – are integral or not.

Chapter 3

The Integrality Condition

We will fix some notation for the rest of the thesis – K denotes a number field, $\mathcal{O}_K$ its ring of integers, and $\mathfrak{p}$ is a prime in Spec $\mathcal{O}_K$.. Later, when we have fixed a prime $\mathfrak{p}$, we will introduce $k = \mathcal{O}_K/\mathfrak{p}$ the residue field, R for the localisation of $\mathcal{O}_K$ at $\mathfrak{p}$, and π for a uniformiser in R.

3.1 Integral elements

As mentioned in the introduction, Beilinson's original conjecture – that $K_i^{(j)}(X)$ for $i > 1$, or $K_1^{(j)}(X) \oplus N^{j-1}(X)$, is isomorphic to $H_{\mathcal{D}}^{2j-i}(X_{\mathbb{C}}, \mathbb{R}(j))^+$ – turned out not to be correct, as was first pointed out by Bloch and Grayson. They found that $K_2^{(2)}$ of certain elliptic curves defined over $\mathbb{Q}$ is bigger than the conjecture predicted. Beilinson responded by modifying the conjecture, to apply only to a certain subspace of $K_i^{(j)}(X)$, consisting of the so-called "integral" elements, which here will be denoted by $K_i^{(j)}(X)_{\mathbb{Z}}$. In hindsight, it should have been obvious that this modification to the original conjecture was needed, as the conjecture as originally formulated does not even work for the most basic classical case, of the Dirichlet regulator for the units in a number field! For if we take $X = \text{Spec } K$, where K is a number field, then the original conjecture says that there is an isomorphism $K_1^{(1)}(K) \otimes \mathbb{R} \to H_{\mathcal{D}}^1(K \otimes \mathbb{C}, \mathbb{R}(1))^+$. It is easy to compute that the right-hand side here has dimension r_1+r_2 (where r_1 and r_2, as in the previous chapter, are the number of real embeddings and number of pairs of complex embeddings, respectively), while $K_1^{(1)}(K)$ is simply

K^* (tensored with $\mathbb{Q}$), which has infinite rank! The (Deligne) regulator map (which coincides with Beilinson's here), as we know from the classical result, becomes an isomorphism (onto its image, a hyperplane in $\mathbb{R}^{r_1+r_2}$) only when restricted to $\mathcal{O}_K^*$ – and this is precisely the group of integral elements in K^*. This example may help to motivate the definition.

To define in general what is meant by $K_i^{(j)}(X)_{\mathbb{Z}}$, we shall suppose that X admits a *regular* proper model $\mathcal{X}$ over the ring of integers $\mathcal{O}_K$ (that is, $\mathcal{X}$ is a regular scheme which is proper over $\mathcal{O}_K$, and has generic fibre isomorphic to X). Then there is a long exact "localisation" sequence in K-theory (see p.68 of [23]), or more accurately, for the K'-groups:

$$\ldots \to \bigoplus_{\mathfrak{p}\in\mathrm{Spec}\ \mathcal{O}_K} K_i'(\mathcal{X}_\mathfrak{p}) \to K_i(\mathcal{X}) \to K_i(X) \xrightarrow{\delta} \bigoplus_{\mathfrak{p}\in\mathrm{Spec}\ \mathcal{O}_K} K_{i-1}'(\mathcal{X}_\mathfrak{p}) \to \ldots$$

– where we have used the fact that, for a regular scheme Y, the natural map $K_i(Y) \to K_i'(Y)$ is an isomorphism [23]. ($\mathcal{X}_\mathfrak{p}$ here denotes the closed fibre of $\mathcal{X}$ at the prime $\mathfrak{p}$.) Noting that the sequence will still be exact when we tensor each term with $\mathbb{Q}$, we then have ([18], p.13):

Definition 3.1 Suppose X has a regular proper model $\mathcal{X}$ over $\mathcal{O}_K$. Then the integral elements of $K_i^{(j)}(X)$ are defined to be those elements of $K_i(X) \otimes \mathbb{Q}$ which lie in the kernel of the boundary map δ (or in the image of the preceding map from K_i of the integral model, hence the "integrality" of the elements – these are equivalent by the exactness of the localisation sequence), as well as in the weight j Adams eigenspace.

Note that this definition is independent of the choice of regular proper model $\mathcal{X}$ ([18] p.68). Further, the maps in the localisation sequence are compatible with the Adams operations, so in particular δ maps $K_i^{(j)}(\mathcal{X})$ into $K_i^{(j)}(X)$.

When X doesn't have such a model, a different definition is needed. Beilinson originally conjectured that, if $\mathcal{X}$ is any proper *flat* model (which always exists), then the kernel of δ in the exact sequence above (now coming from $K_i'(\mathcal{X})$) is independent of the choice of $\mathcal{X}$, but this conjecture was found to be false by Rob de Jeu [7]. But Anthony Scholl, in [19], came up with an alternative definition of integral elements which is well-defined, and has all the desired properties – including coinciding with the above when X has a regular proper model. We shall not need to know the details of Scholl's construction, but will use one consequence of it in the next section.

In almost all cases I shall consider, X will admit a regular proper model $\mathcal{X}$ over $\mathcal{O}_K$, and the integral elements of $K_1^{(2)}(X)$ are those in the kernel of the boundary map from $K_1^{(2)}(\mathcal{X})$ to $\oplus_{\mathfrak{p}\in\mathrm{Spec}\ \mathcal{O}_K} K_0'(\mathcal{X}_\mathfrak{p})$. K_0' of a variety is just the direct sum of its various Chow groups of various codimensions (with coefficients in $\mathbb{Q}$), as can be seen from the BGQ spectral sequence mentioned in the previous chapter – and so can the fact that, when the variety is non-singular, so that we can replace K_0' with K_0, the weight j part is the codimension j Chow group of X. And in our case, the map from $K_1^{(2)}(\mathcal{X})$ to $\oplus_{\mathfrak{p}\in\mathrm{Spec}\ \mathcal{O}_K} K_0'(\mathcal{X}_\mathfrak{p})$ lands in just the codimension 1 part, and is as follows. Given an element $\sum_i (V_i, f_i)$ in $K_1^{(2)}(X)$, first take the (Zariski) closures $\overline{V_i}$ of the V_i in the integral model $\mathcal{X}$ of which X is the generic fibre, together with the corresponding functions via the natural isomorphism of the function fields of V_i and $\overline{V_i}$, which we shall still denote by f_i, with no risk of confusion. $\sum_i (f_i)$, the sum of the divisors of these new f_i, will be a codimension 2 cycle on $\mathcal{X}$. Now, each irreducible codimension 2 subscheme (and indeed any irreducible subscheme) of $\mathcal{X}$ is of one of two types: "horizontal" ones, whose image under the natural map $\mathcal{X} \to \mathrm{Spec}\ \mathcal{O}_K$ is the whole of Spec $\mathcal{O}_K$, and "vertical" ones, whose image is a single closed point $\mathfrak{p}$. These are the only two possiblities, since the image must be an irreducible closed subscheme of Spec $\mathcal{O}_K$. (This terminology comes from viewing Spec $\mathcal{O}_K$ as a horizontal line, with $\mathcal{X}$ above it, with the fibre at each prime $\mathfrak{p}$ lying directly above the point corresponding to $\mathfrak{p}$.) Because all elements $\sum_i (V_i, f_i)$ of $K_1^{(2)}(X)$ satisfy $\sum_i (f_i) = 0$ on X, it follows that all "horizontal" components of our codimension 2 cycle on $\mathcal{X}$ will cancel out, leaving us with only "vertical" ones. And, of course, these algebraic cycles of codimension 2 on $\mathcal{X}$ which are each contained in a single fibre are, in a natural way, elements of $\oplus_{\mathfrak{p}\in\mathrm{Spec}\ \mathcal{O}_K} CH^1(\mathcal{X}_\mathfrak{p})$, the direct sum of the (rational) codimension 1 Chow groups of $\mathcal{X}_\mathfrak{p}$ as $\mathfrak{p}$ runs over all primes, which is a subspace of $\oplus_{\mathfrak{p}\in\mathrm{Spec}\ \mathcal{O}_K} K_0'(\mathcal{X}_\mathfrak{p})$.

With this description of the boundary map, we can confirm my statement in the previous chapter that if $X = C \times C$ and we take a unit u in the ring of integers, and consider it as a constant function along any curve inside $C \times C$, then we get an integral element of $K_1^{(2)}(C \times C)$ – at least in the case where $C \times C$ has a regular proper model. (The result for those elliptic curves C which do not possess such a model will follow from the result of the next section.) This is because the constant function u on a curve V corresponds to the same constant function on $\overline{V}$, and because $u \in \mathcal{O}_K^*$ it will have order zero at each prime.

3.2 Field extensions

In this section, and for the rest of this thesis, we shall restrict to the case where $X = E \times E$ and E is an elliptic curve, defined over the number field K.

The purpose of this section and the next is to show that we can simplify our discussion of the integrality of our "triangle" elements in a couple of ways. Here, we will show that it is essentially unaffected if we extend the base field – but we first need to make this statement precise. In general, we are going to consider a linear combination of triangle elements, of the form $\sum_h e_h \alpha_{f_h}$, where the h come from some finite index set, the e_h are rational numbers, and the α_{f_h} are the triangle elements introduced in the previous chapter – each constructed from a function $f_h \in K(E)$ with $(f_h) = d_h(P_h) - d_h(Q_h)$. (Note that, in this chapter and the next, we will restrict to the simplest form of triangle elements which were introduced first, with $a = 1$; in the final chapter I shall see what happens if we allow other more general triangles as well.) However, as noted then, within each such triangle we have three extra degrees of freedom in trying to obtain an integral element, in that the three copies of f involved in the definition of α_f can each be replaced by any constant multiple and still produce an element of $K_1^{(2)}(E \times E)$. So we are interested in whether or not there are any such constant multiples which make the given linear combination integral. This motivates the following definition:

Definition 3.2 Suppose we have a set of functions f_h on E, for h in some finite index set, allowing us to construct a triangle configuration $\alpha_{f_h} \in K_1^{(2)}(E \times E)$. Further, let us associate to each h an $e_h \in \mathbb{Q}$. If for each h there exists an element $\beta_h \in K_1^{(2)}(E \times E)$ which is constructed from constant functions along the three curves $E \times \{P_h\}$, $\{Q_h\} \times E$ and Δ, and an integer k, such that $\sum_h (ke_h\alpha_{f_h} - \beta_h)$ is integral, we will say that the sum $\sum_h e_h\alpha_{f_h}$ is *potentially* integral.

(The integer k appearing in this definition might appear to be strange at first, but this slightly more general definition of potential integrality allows for a cleaner statement of Proposition 3.3.)

Our goal is now to prove the following:

Proposition 3.3 Let L/K be a finite Galois extension, and let ϕ denote the natural map from $(E \times E)/L$ to $(E \times E)/K$. Then $\alpha \in K_1^{(2)}(E \times E)$ is integral if and only if $\phi^*(\alpha)$ is. More generally, if α is a linear combination of triangle configurations, then α is potentially integral if and only if $\phi^*(\alpha)$ is.

For this, we need to recall some of the functoriality properties of K-theory and the related K'-groups. K-theory is a contravariant functor on the category of schemes, so given a morphism $\phi : X \to Y$ of schemes, there is for each i a pullback map $\phi^* : K_i(Y) \to K_i(X)$. K'-theory is also a contravariant functor when restricted to the subcategory of Noetherian schemes and flat morphisms (see [23]). Finally, if we restrict further to only *proper* morphisms, K'-theory becomes a covariant functor as well – for a proper morphism $\phi : X \to Y$, there is a "proper pushforward" $\phi_* : K'_i(X) \to K'_i(Y)$ (see [24]).

To return to our situation, let L be a finite extension of the base field K – then we can consider the curve E as also being defined over L: $E/L := E \times_K L$. We will also consider $(E \times E)/L$, which is the same as $(E/L) \times_L (E/L)$. There is a natural (projection) map $E/L \to E/K$, and a corresponding map $\phi : (E \times E)/L \to (E \times E)/K$. And if $\mathcal{X}_\mathcal{E}/\mathcal{O}_K$ and $\mathcal{X}_\mathcal{E}/\mathcal{O}_L$ are integral models for $(E \times E)/K$ and $(E \times E)/L$ respectively, then ϕ also induces a map $\mathcal{X}_\mathcal{E}/\mathcal{O}_K \to \mathcal{X}_\mathcal{E}/\mathcal{O}_L$, and hence further induces maps on each fibre of this model – all these maps will also be denoted by ϕ. The thing to note here is that all of these maps ϕ are necessarily flat and proper – they are proper because they all come via basechange from the inclusion maps of K into L, $\mathcal{O}_K$ into $\mathcal{O}_L$, or similarly of their residue fields, all of which are finite and therefore proper, and they are flat because the induced maps on local rings all make the images into free modules over the sources. Further, all the schemes involved are Noetherian, and therefore each of these maps denoted ϕ induces both pullback and pushforward maps in K'-theory. What is more, these maps are compatible with the localisation sequences in K-theory, meaning that the following diagram is commutative:

$$\begin{tikzcd} K_1'\left(\mathcal{X}_{\mathcal{E}}/\mathcal{O}_L\right) \arrow[r] \arrow[d, shift left, "\phi_*"] & K_1((E\times E)/L) \arrow[r, "\delta"] \arrow[d, shift left, "\phi_*"] & \bigoplus_{\mathfrak{p}'\in \mathrm{Spec}\ \mathcal{O}_L} K_0'\left((\mathcal{X}_{\mathcal{E}})_{\mathfrak{p}'}\right) \arrow[d, shift left, "\phi_*"] \\ K_1'\left(\mathcal{X}_{\mathcal{E}}/\mathcal{O}_K\right) \arrow[r] \arrow[u, shift left, "\phi^*"] & K_1((E\times E)/K) \arrow[r, "\delta"] \arrow[u, shift left, "\phi^*"] & \bigoplus_{\mathfrak{p}\in \mathrm{Spec}\ \mathcal{O}_K} K_0'\left((\mathcal{X}_{\mathcal{E}})_{\mathfrak{p}}\right) \arrow[u, shift left, "\phi_*"] \end{tikzcd}$$

– here $(\mathcal{X}_{\mathcal{E}})_{\mathfrak{p}}$ denotes the fibre of $\mathcal{X}_{\mathcal{E}}$ at the prime $\mathfrak{p}$.

We are going to assume first that we can find a model $\mathcal{X}_{\mathcal{E}}$ for $E\times E$ over $\mathcal{O}_K$ which is regular and proper, so that the integral elements in K_1, in both rows, are those in the kernel of δ. Suppose $\alpha = \sum_h e_h \alpha_{f_h}$ is a potentially integral linear combination of triangle configurations, as defined above – so in other words $\delta(k\alpha+\beta) = 0$, for some β consisting of constants on the various copies of E which occur in the expansion of α, and some integer k. Then, using the diagram above, we have that $0 = \phi^*\delta(k\alpha+\beta) = \delta\phi^*(k\alpha+\beta)$ – so $\phi^*(k\alpha+\beta) = k\phi^*(\alpha)+\phi^*(\beta)$ is integral. And $\phi^*(\beta)$ is also made up of constant functions – the constants in β are contained in K, hence also in L, and $\phi^*(\beta)$ consists of the same constants on the same curves. Thus, if α is potentially integral, so is $\phi^*(\alpha)$.

Conversely, suppose $\phi^*(\alpha)$ is potentially integral, so that $\delta(\phi^*(k\alpha)+\beta) = 0$, where β this time comes from constants in L, although not necessarily in K. Then $0 = \phi_*\delta(\phi^*(k\alpha)+\beta) = k\delta\phi_*\phi^*(\alpha)+\delta\phi_*(\beta)$. And the effect of $\phi_*\phi^*$, pullback followed by push-forward, if L/K is a Galois extension (which we can ensure by replacing L with its normal closure), is to multiply the original element by $[L:K]$ (as the pushforward takes an element to the sum of all its Galois conjugates). We therefore find that $0 = k[L:K]\delta(\alpha)+\delta(\phi_*(\beta))$. The element $\phi_*(\beta)$ will also be made up of constant functions on the same curves as β – because ϕ_* is a sum over all Galois conjugates (which corresponds to multiplication on the level of functions, since $(V,f)+(V,g) = (V,fg)$ in $K_1^{(2)}(E\times E)$), each constant in L becomes the constant over K given by the norm map from L to K. Thus we find that, if $\phi^*(\alpha)$ is potentially integral, then so is α.

Next, we want to deal with the case where there might not exist any regular proper model $\mathcal{X}_\mathcal{E}$ for $E \times E$ over $\mathcal{O}_K$. In this case, the definition given in the previous section cannot be used, so we must instead work with the construction of Scholl, which works in all cases. We will not need the whole construction, but merely note the following consequence, which is Corollary 1.3.4 in [19] (I have altered the notation so it matches that used here). Here, an *alteration* is a proper, surjective and generically finite morphism between integral Noetherian schemes.

Proposition 3.4 Let X/K be a smooth and proper variety, and let $\phi : Y \to X$ be an alteration. If Y has a regular proper model $\mathcal{O}_K$, so that we can define the subgroup $K_i^{(j)}(Y)_\mathbb{Z}$ of integral elements of $K_i^{(j)}(Y)$, then $K_i^{(j)}(X)_\mathbb{Z} = \phi_*(K_i^{(j)}(Y)_\mathbb{Z})$.

In the case we are considering, X being $E \times E$ where E is an elliptic curve, there will always exist a finite extension field L such that $(E \times E)/L$ does admit a regular proper model $\mathcal{X}_\mathcal{E}$. This is because, after a suitable extension, we can ensure that E has either good reduction or split multiplicative reduction at each prime (see [21]), and as we shall see in the remainder of this chapter, in both these cases such models do exist. (They are not known to exist in general when E has additive reduction at any prime.)

We are going to take such an L so that $(E \times E)/L$ does admit a regular proper model $\mathcal{X}_\mathcal{E}$ – this will be a scheme over $\mathcal{O}_L$, and hence can also be considered as a scheme over $\mathcal{O}_K$ via the natural inclusion of $\mathcal{O}_K$ in $\mathcal{O}_L$. The natural morphism $\phi : (E \times E)/L \to (E \times E)/K$ satisfies all the properties required to be an alteration – it is clearly generically finite, is proper because it comes from Spec $L \to$ Spec K (which is finite, and hence proper, since L/K is a finite extension) by base-change, and is surjective by the same reason (as surjective morphisms are stable under base-change).

Therefore, by Proposition 3.4, an element of $K_1^{(2)}(E \times E)$ (over K) is integral if and only if it is of the form $\phi_*(\alpha)$ for some integral element α in $K_1^{(2)}((E \times E)/L)$. If so, then if we apply ϕ^* to this element then we obtain the sum of all the Galois conjugates of α, which must be integral since α is. Conversely, if α is in $K_1^{(2)}(E \times E)$ and $\phi^*(\alpha)$ is integral, then so, by the proposition, is $\phi_*\phi^*(\alpha) = [L : K]\alpha$, and therefore so is α.

This completes the proof of Proposition 3.3.

3.3 Local-global behaviour

There is another aspect of the integrality condition which we need to look at, and this is the relationship between "local integrality" at an individual prime $\mathfrak{p}$, and actual integrality as defined above, which we may call "global integrality". We shall assume that $E \times E$ admits a regular proper model over the ring of integers $\mathcal{O}_K$, as the results of the previous section allow us to do so without losing any information about the integrality of our elements. So we know that α is integral if and only if it vanishes under the boundary map to the direct sum, over all primes $\mathfrak{p}$ of $\mathcal{O}_K$, of the codimension one Chow groups of the fibres of the model at $\mathfrak{p}$. Being a direct sum, we can examine this question one prime at a time, for α will be integral if and only if its image in the Chow group vanishes at each individual prime.

However, when we wish to look at potential integrality, rather than actual integrality, we encounter a problem. Clearly, if there is a β made up of constants in K, and an integer k, for which $k\alpha + \beta$ integral, then for each prime $\mathfrak{p}$ there is a $\beta_{\mathfrak{p}}$ (namely β) which makes $k\alpha + \beta_p$ integral locally at $\mathfrak{p}$. But the converse is not obvious – that the existence of a $\beta_{\mathfrak{p}}$ at all $\mathfrak{p}$ guarantees the existence of a β.

But it turns out that we are OK, due to the finiteness of the class number of K, and the fact that we are working in a rational vector space. For suppose a $\beta_{\mathfrak{p}}$ exists at each prime $\mathfrak{p}$. Because it comes from working in the localisation, the only relevant thing about $\beta_{\mathfrak{p}}$ is its order at $\mathfrak{p}$ – if we multiply it by any unit in the discrete valuation ring associated to $\mathfrak{p}$ before adding it to $k\alpha$, the result will still be integral at $\mathfrak{p}$. So we are looking for a $\beta \in K$ which has a specified order at each particular prime $\mathfrak{p}$. In general, of course, this is not possible, although it is when the class number of K is 1. If h is the class number, what we can guarantee is that a β exists with order $h \cdot \mathrm{ord}_{\mathfrak{p}}(\beta_{\mathfrak{p}})$ at each $\mathfrak{p}$, which again makes α potentially integral, according to the definition above. Thus, we have established the following:

Proposition 3.5 If α is a linear combination of triangle configurations which is potentially integral at each prime $\mathfrak{p}$, then it is potentially integral (globally).

3.4 The case of good reduction

Thanks to the results of the previous two sections, we may approach the integrality question one prime at a time, and also may extend the base field first if that will make life easier. Extending the field will be extremely useful, because it is known that, after a suitable finite extension, an elliptic curve E over K attains either good or multiplicative reduction at each prime. After a further extension, if necessary – still finite – we may assume further that the reduction is *split* multiplicative or good at each prime. So we need only look at what happens at each prime individually, in each of these two cases.

The case of good reduction is particularly easy. In this case, there is a regular proper model $\mathcal{E}$ for E, over the localisation of $\mathcal{O}_K$ at $\mathfrak{p}$, whose special fibre is itself an elliptic curve over the residue field. Namely, we take as $\mathcal{E}$ the closure of E inside the projective plane over R. Then $\mathcal{E} \times \mathcal{E}$, which is clearly a proper model for $E \times E$, will also be regular, as $\mathcal{E}$ is smooth over R, and the product of two smooth schemes over the base is itself smooth and therefore regular. Clearly, the closure of, for example, $E \times \{P\}$ inside this model will be $\mathcal{E} \times \overline{\{P\}}$ (the bar denotes Zariski closure), which is naturally isomorphic to $\mathcal{E}$ itself. And if we take the function f, from which we constructed the triangle α_f, on this surface and look at the "vertical" component of its divisor (in other words those components which lie within the special fibre), we find something of the form $\{a(\mathcal{E}_\mathfrak{p} \times P_\mathfrak{p})\}$, where $P_\mathfrak{p}$ denotes the reduction of P mod $\mathfrak{p}$ (in other words the point of intersection of $\overline{\{P\}}$ with the fibre at $\mathfrak{p}$), and a is an integer. If we then pick a constant on $E \times \{P\}$ which has order a at $\mathfrak{p}$, this will clearly have the same image under the boundary map in the localisation sequence (as far as "vertical" components are concerned, which is all that matters in the end as the horizontal ones cancel over the whole triangle). The other curves involved in our triangles – those of the form $\{Q\} \times E$ and the various Δ_a – will similarly give a constant times the equivalent construction in $\mathcal{E}_\mathfrak{p} \times \mathcal{E}_\mathfrak{p}$, and let us find constants on these curves with the right images. Thus, at a prime of good reduction we can always find constants to cancel out the contributions from any individual triangle.

Summing up, then, we have:

Proposition 3.6 If E has good reduction at $\mathfrak{p}$, then any triangle configuration α_f is potentially integral at $\mathfrak{p}$, and thus so is any linear combination of them.

We should also note that this description of the fibre at a prime of good reduction gives us an alternative way of proving Proposition 2.6 in the case of an elliptic curve – for, if we pick any prime at which the curve has good reduction (which is almost all primes), and consider the images of the three elements discussed there under the boundary map at that prime, one can see at once that these are independent.

3.5 The case of split multiplicative reduction

As we have just seen, the case of good reduction is particularly easy, because we have a regular model for $E \times E$ readily to hand. In the case of bad reduction, though, we cannot do this in the same way – the natural model $\mathcal{E}$ for E over $\mathcal{O}_K$ (or its localisation at the relevant prime) is no longer smooth, as it has singular points on the fibres of bad reduction, and this in turn means that $\mathcal{E} \times_{\mathcal{O}_K} \mathcal{E}$ is not necessarily regular; in fact, it never is. So we must blow up any non-regular points that occur as the singular points in the various fibres, and continue if there are any non-regular points on the resulting model. We can continue this process indefinitely, in theory – the difficulty is that it is not known, in general, whether or not this process even comes to an end in a finite number of steps! The problem of resolving the singularities in such a model was studied by Mansour Aghasi in his thesis [1] – he found that the only cases he could treat were those where the reduction type of the elliptic curve is either I_n, III or IV (using the Kodaira symbols for the classification of the reduction types of elliptic curves), leaving the remaining cases (II,II^*,III^* and IV^*) as open problems. Fortunately, it turns out we shall not need to discover regular integral models for all these cases. The only cases that we need to deal with are the I_n's – either good reduction (for $n = 0$) or multiplicative reduction, as we can deal with the other cases by extending the base field. Having found good reduction simple to deal with, it means that it is only multiplicative reduction which we have left to consider – and here, too, a regular integral model is easy to come by (although not quite as easy as in the case of good reduction). What is more, we can concentrate just on the easier case of *split* multiplicative reduction, as once again we can always obtain this after a suitable finite field extension (see [21]). Thus, from now on we will assume that E has split multiplicative reduction at our chosen prime $\mathfrak{p}$.

3.5.1 A regular model

Let us suppose, then, that our curve E has split multiplicative reduction at the prime $\mathfrak{p}$. We will, as usual, localise, and look for a regular, proper model for $E \times E$ over the localisation of $\mathcal{O}_K$ at $\mathfrak{p}$ – this allows us to concentrate just on integrality at the prime $\mathfrak{p}$, which we have seen is enough (Proposition 3.5). As stated at the beginning of this chapter, let us write R for this localisation, and π for any uniformiser of R. Let us take the Néron model for E over R. Because we have assumed split multiplicative reduction, the fibre of the Néron model is a so-called "Néron n-gon" for some n – that is, n copies of $\mathbb{P}^1_k$ (here, and throughout, k will denote the residue field $\mathcal{O}_K/\mathfrak{p} = R/(\pi)$), glued together at their end-points so that ∞ on one corresponds to zero on the next, and with the singular points (the points where these joins occur) removed. In fact, we shall need to keep these singularities in place, so that our model is proper – the result is called the minimal proper regular model for E over R, and we shall denote it by $\mathcal{E}$.

But unfortunately, $\mathcal{E}$ being regular does *not* necessarily mean that $\mathcal{E} \times_R \mathcal{E}$ is also regular, as I have already said. Non-regular points can occur in this case, but only at singular (closed) points on a fibre. And the special fibre, which will be the product of a Néron n-gon with itself, is singular along the $2n$ lines along which one of the two coordinates is one of the singularities of the n-gon. The arithmetic threefold $\mathcal{E} \times_R \mathcal{E}$ is still regular at most points on these lines, but fails to be so at any of these n^2 points of intersection of these lines – ie. at the points (P, Q) where both P and Q are singularities of the fibre of $\mathcal{E}$.

Thus our model is not regular as it stands. In this multiplicative case, however, we can obtain a regular model with just a single blowup at each of the n^2 non-regular points. This fact is well-known (see [1], for example), but it needs to be gone through in detail, as we shall need the details of the blown-up model in order to compute the codimension 1 Chow group of the fibre, and thus to determine the image of our triangle configurations under the boundary map. However, this is a somewhat lengthy, and computationally intensive, process, so to avoid distracting from the flow of this chapter, I have put the relevant computations into Appendix A, and will here merely refer to results in that appendix. I will use the notation $\mathcal{X}_{\mathcal{E}}$ for the result of the n^2 necessary blowups of $\mathcal{E} \times \mathcal{E}$, which will be our regular proper model for $E \times E$.

The first fact which we shall need is what the exceptional divisor of each blowup is. The answer is:

Proposition 3.7 When we blow $\mathcal{E} \times \mathcal{E}$ up at one of its singular points, the exceptional divisor is a copy of $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

This allows us to describe what the fibre of $\mathcal{X}_{\mathcal{E}}$ is. Before the blowup, we had $\mathcal{E}_{\mathfrak{p}} \times \mathcal{E}_{\mathfrak{p}}$, with $\mathcal{E}_{\mathfrak{p}}$ consisting of n copies of $\mathbb{P}^1$ all joined up in a "circle". So this product consisted of a torus, tiled with n^2 copies of $\mathbb{P}^1 \times \mathbb{P}^1$ – there were n^2 different points where the "corners" of these tiles met, and these were the non-regular points of $\mathcal{E} \times \mathcal{E}$ which we had to blow up. We now know that the effect of this blowup was to replace each of these n^2 vertices with a new copy of $\mathbb{P}^1 \times \mathbb{P}^1$ – giving $2n^2$ copies in total.

This doesn't quite give the full details, however, because new "edges" have been introduced to this picture by the blowup, and it is these edges which will prove crucial in computing the Chow group of the fibre. To be more precise, by an "edge" of a $\mathbb{P}^1 \times \mathbb{P}^1$ I shall mean one of the four curves $\{0\} \times \mathbb{P}^1$, $\{\infty\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{0\}$ and $\mathbb{P}^1 \times \{\infty\}$. Note that, before the blowup, the tiles were glued edge-to-edge, in this sense, so that this definition makes sense. As a result of the blowup, at each corner we have a new $\mathbb{P}^1 \times \mathbb{P}^1$, and thus four new edges. Each new $\mathbb{P}^1 \times \mathbb{P}^1$, coming as it did from a corner, touches four of the old tiles, and for reasons of symmetry it is clear that one of the edges of this exceptional divisor must become a new edge for each of these four. So we see that each of the tiles which existed before the blowup now has 8 edges – ie. has become an octagon. Pleasingly, these can be characterised in more familiar terms:

Lemma 3.8 Each of the octagons referred to above is isomorphic (as a variety over k) to a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ which has been blown up at each of its four corners.

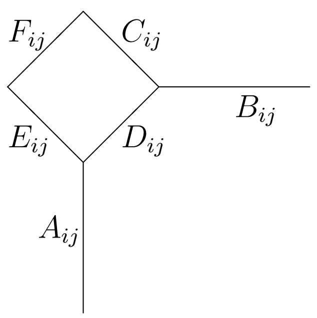
Proof We recall a basic result on blowups: if Y and Z are closed subschemes of the scheme X, and we blow X up along Z, then the strict transform of Y in this blowup is isomorphic to the blowup of Y along $Y \cap Z$ (see, for example, Proposition IV-21 in [10], and the comments following it). We shall apply this to our case, with X being $\mathcal{E} \times \mathcal{E}$, Y one of the copies of $\mathbb{P}^1 \times \mathbb{P}^1$ in the fibre, and Z that one of the singular points at the "corners" of Y at which we are blowing up. This tells us that the strict transform of the $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at this corner.

The octagons which we are considering are precisely the strict transforms of these "tiles" in the fibre of $\mathcal{E} \times \mathcal{E}$. So, by the previous result, if we blow up at just one point, so adding only one more edge to our original tile, the new edge is precisely the exceptional divisor of the original tile when blown up at that corner. Since any one corner is as good as another, this proves the desired result. $\square$

3.5.2 The Chow group of the fibre

The next stage in figuring out the conditions needed to make our triangle configurations integral (at the given prime $\mathfrak{p}$ of split multiplicative reduction) is to compute the target group of the boundary map (or rather its component at the prime $\mathfrak{p}$) – namely the codimension 1 Chow group of the fibre. We shall compute the rank of this group, in terms of the number n of sides of the Néron n-gon which was the fibre of $\mathcal{E}$, as well as an explicit set of generators and relations.

We already noted that, prior to the blowup, the fibre was a torus tiled with n^2 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Pictorially, we can represent this as an n by n grid, in which the pairs of outer edges are identified. After blowing up, each vertex becomes another square, and the edges of these cause the former squares to look like octagons. And each "octagon", as we just saw, is in fact a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ which has been blown up at each of its 4 vertices. So there are $6n^2$ edges in total – we need to choose a consistent system of labelling to keep track of all of them. So let us label the vertices in the form (i, j) with $i, j \in \mathbb{Z}/n\mathbb{Z}$, and then label the 6 edges at the $(i, j)^{th}$ vertex as follows:

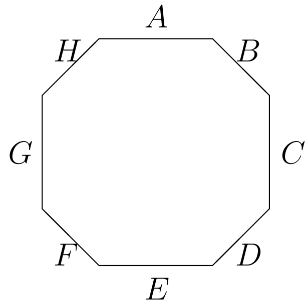


Finding generators and relations for the codimension 1 Chow group of the fibre is greatly simplified when we recall that each of the $2n^2$ separate components of which it is made up is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, whose codimension 1 Chow group is free of rank 2, the generators being any "horizontal" and any "vertical" copy of $\mathbb{P}^1$, all such curves in the same direction being equivalent. So, the Chow group of our fibre can be generated by $4n^2$ elements, one in each direction in each of the $2n^2$ tiles – but these will not be independent, as all the edges belong to more than one of the tiles. In order to compute precisely what the relations are, we will find it easier to work, to begin with, with $6n^2$ generators, namely the ones given in the diagram above – we shall see that all of the edges can be expressed in terms of these, so that these elements do in fact generate the whole group.

One set of relations is immediately apparent from the picture – clearly $C_{ij} = E_{ij}$ and $F_{ij} = D_{ij}$ for each pair (i, j), and this is clearly all that we get from the "squares" (ie. the exceptional divisors of the n^2 blowups which we have performed). So we are back down to $4n^2$ generators (and will write everything in terms of just the A_{ij}, B_{ij}, C_{ij}, and D_{ij} from now on) and need to compute what relations the "octagons" give us.

We saw in the previous section that each octagon is just a $\mathbb{P}^1 \times \mathbb{P}^1$ which has been blown up at each of the four vertices. This allows us to compute what each of the four edges before the blowup becomes, in terms of the edges of the octagon, in the following way: if we use x and y for the two co-ordinate functions on $\mathbb{P}^1 \times \mathbb{P}^1$, then the edges are simply the zeroes and the poles of these two functions. So all we need to do is to figure out the divisors of the functions x and y on the octagon created by blowing up at each of the four corners, and set these equal to zero – these will be all the relations which occur, and also demonstrate that our edges listed above are enough to generate the whole group, as all of the original edges will be expressed in terms of them.

As we are now concentrating on one individual octagon, it will be more convenient to label its sides in a different way – I will do it as A through H, starting at the top and going clockwise, as shown below. Afterwards, we will match these up with the generators given above for the whole group.



Let us first compute the divisor of the function x on the octagon. Clearly, it has a simple zero along G and a simple pole along C, as it had these before the blowup, and the blowup doesn't affect these two sides (or A and E, along both of which x has order 0). What remains to be computed is its order along each of the four exceptional divisors. Let us first compute its order along F, which is the exceptional divisor when $\mathbb{P}^1 \times \mathbb{P}^1$ is blown up at the point $(0,0)$. As the blowup is local, we can consider just the blowup of the affine plane at the origin. This blowup is just Proj $\frac{k[x,y,X,Y]}{(xY-yX)}$, which can be covered by 2 affine co-ordinate charts. The first occurs when $X=1$, and is Spec $\frac{k[x,y,Y]}{(xY-y)} \cong$ Spec $k[x,Y]$, and similarly the second is Spec $k[y,X]$. The exceptional divisor is given by the ideal (x,y) in the global Proj picture, and this becomes principal on each of the affine charts – generated by x on the first and y on the second. So we see that the function x, which is yX on the second chart, has order 1 along the exceptional divisor – ie. along the side we have labelled F. The same computation gives the order of x at the other 4 corners of the octagon – find the order along D, for example, which is the exceptional divisor of the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the point $(\infty, 0)$, we simply replace x by $1/x$, to get order

-1. So we see that the divisor of x is:

$$(x) = F + G + H - B - C - D,$$

and similarly that

$$(y) = D + E + F - H - A - B.$$

Translating these back in terms of the generators A_{ij}, B_{ij}, C_{ij} and D_{ij}, with reference to the picture on page 43, we find the following $2n^2$ relations:

$$\begin{aligned} A_{i,j+1} + C_{ij} + D_{i,j+1} &= A_{i+1,j+1} + C_{i+1,j+1} + D_{i+1,j} \qquad (X_{ij}) \\ B_{ij} + C_{ij} + D_{i+1,j} &= B_{i,j+1} + C_{i+1,j+1} + D_{i,j+1}. \qquad (Y_{ij}) \end{aligned}$$

We can now use these relations to compute the rank (or the dimension, as the target group of the boundary map is really this Chow group tensored with $\mathbb{Q}$) of the codimension 1 Chow group. It is clear that the $2n^2$ relations given above are not independent – adding together all n^2 of each type produces the trivial relation. However, I claim that these are the only two dependencies among the relations. To see this, let us label the relations – that is, the differences between the two sides, linear combinations of the generators which equal zero – as X_{ij} and Y_{ij} respectively (shown above). Then each A_{ij} occurs only in $X_{i,j-1}$ and $X_{i-1,j-1}$, and with opposite signs in each. Hence, in any linear combination of these relations which is trivial, the coefficient of X_{ij} depends only on j. By the same argument with B_{ij}, the coefficient of Y_{ij} depends only on i. Let x_j be the coefficient of any X_{ij}, and y_i that of any Y_{ij}. Then looking at either C_{ij} or D_{ij} will tell us that we must have $x_j + y_i = x_{j-1} + y_{i-1}$ – for each pair (i, j). If we fix j and add these n relations over all i in $\mathbb{Z}/n\mathbb{Z}$, we see that $x_j = x_{j-1}$, so all the x_j are equal. Fixing i and summing over j proves that all the y_i are equal too. So any relation among these relations is a linear combination of the two already noted. Thus, the $2n^2$ relations above generate a subgroup of rank $2n^2 - 2$ of the free group on the $4n^2$ generators, and establish that:

Proposition 3.9 The Chow group of the fibre at $\mathfrak{p}$, for $\mathfrak{p}$ a prime of split multiplicative reduction of type I_n $(n > 0)$, has rank $2n^2 + 2$.

Of course, we are still going to need to be able to determine whether or not a given linear combination of our generators lies in the span of the relations, so merely knowing the rank of the quotient is not enough – we shall need to make careful use of the relations.

3.5.3 Computations on $\mathcal{E}$

Now that we have an explicit description of the target group, we are in a position to compute the image of our triangle configurations under the boundary map, at primes of split multiplicative reduction.

As before, $\mathcal{E}$ will denote the minimal regular proper model of E (over R), or the Néron model with the singular points on the fibre included. The properties of both properness and minimality will be crucial in the following arguments.

Of the three copies of E inside $E \times E$ which occur inside each individual triangle configuration, two of them – namely the horizontal and vertical – do not touch any of the singularities of the fibre when we take their closure inside $\mathcal{E} \times \mathcal{E}$. For if they did, then the closure of either P or Q would go through one of the singularities in the fibre of $\mathcal{E}$, and it is impossible for any point on the generic fibre of a regular arithmetic surface to do this ([22], Proposition 4.3b)). This means that, for these two curves, we are free to work inside $\mathcal{E} \times \mathcal{E}$ before the blowup – and further, their closures are clearly isomorphic to $\mathcal{E}$ itself.

Consider then the function f, whose divisor on E was $d(P) - d(Q)$, extended to $\mathcal{E}$ – as always, its divisor can be split into "horizontal" components, which have points in both the generic and special fibres, and "vertical" ones which lie entirely in the special fibre. The horizontal components here must consist of a zero along $\overline{P}$ and a pole along $\overline{Q}$, both of order d – there can be no others, as if f has a zero or pole along a "horizontal" curve, then its restriction to the generic fibre must also be a zero or pole where it intersects, hence must be P or Q. And the only vertical components here are the n components (copies of $\mathbb{P}^1_k$) of the Néron n-gon, which we shall label so that ∞ on D_i is glued to 0 on D_{i+1}, with the indices in $\mathbb{Z}/n\mathbb{Z}$. So we must have that

$$(f) = d\overline{P} - d\overline{Q} + a_1 D_1 + \ldots + a_n D_n,$$

where the a_i are integers, as yet unknown. In fact, because we are working throughout in rational vector spaces, we will also allow the a_i to be non-integral rational numbers.

We can work out the a_i – or at least the relations between them – by using the fact that the above divisor, being the divisor of a function, has intersection product zero with any of the D_i. (I shall denote the intersection product on the arithmetic surface $\mathcal{E}$ by a dot.) It is clear that $D_i \cdot D_j = 0$ unless $j = i-1, i$ or $i+1$, and that $D_i \cdot D_{i\pm 1} = 1$. Further, it is a well-known fact that, when $\mathcal{E}$ is the *minimal* regular model, as it is here, $D_i^2 = -2$. But we shall recall the proof, as we shall need to use similar reasoning in a moment for the closure of the diagonal, which is not a minimal model of E. The following argument is taken from [22].

First, $(\pi) = D_1 + \ldots + D_n$. This is because π must vanish along all of the D_i, so has order at least one among all, and by symmetry must have the same order along each; if this common order were more than one, then all points of the fibre would be singular, which is untrue. Therefore, $\sum_i D_i$ has zero intersection with any divisor on $\mathcal{C}$ which lies within the fibre, and, in particular, with itself. We shall write $\mathcal{E}_\mathfrak{p}$ for this whole fibre. The adjunction formula ([22], Proposition 7.4a) tells us that $\mathcal{E}_\mathfrak{p} \cdot \mathcal{E}_\mathfrak{p} + K_\mathcal{E} \cdot \mathcal{E}_\mathfrak{p} = 2\rho_a(\mathcal{E}_\mathfrak{p}) - 2$, where ρ_a denotes the arithmetic genus, and $K_\mathcal{E}$ the canonical divisor on $\mathcal{E}$. Since $\mathcal{E}_\mathfrak{p} \cdot \mathcal{E}_\mathfrak{p} = 0$, while the arithmetic genus of the special fibre is the same as that of the generic fibre, namely 1, we find that $K_\mathcal{E} \cdot \mathcal{E}_p = 0$.

Next, we apply the adjunction formula again to D_i individually:

$$D_i^2 + K_\mathcal{E} \cdot D_i = 2\rho_a(D_i) - 2.$$

First, we shall suppose that $K_\mathcal{E} \cdot D_i < 0$. Then $D_i^2 = 2\rho_a(D_i) - 2 - K_\mathcal{E} \cdot D_i > 2\rho_a(D_i) - 2 \geq -2$. But it is also well-known in general that, for any divisor D lying in the special fibre on an arithmetic surface, $D^2 \leq 0$, with equality if and only if D is a multiple of the whole fibre. Provided $n > 1$, the latter is clearly not the case for D_i, so $D_i^2 < 0$. Combining with the inequality just proved, we find $D_i^2 = -1$, and hence $\rho_a(D_i) = 0$ as the left-hand-side of the adjunction formula must be negative. (If $n = 1$ then $D^2 = 0$, because it is the whole fibre, but the same result as we shall obtain later – namely that the part of (f) which lies in the fibre is a multiple of (π) – follows trivially in this case.) But if this were the case, D_i could be blown down (Castelnuovo's criterion), contradicting the minimality of $\mathcal{E}$. Thus we can conclude that $K_\mathcal{E} \cdot D_i \geq 0$ for all i. But we observed above that $\sum_i K_\mathcal{E} \cdot D_i = 0$, so can deduce that $K_{\mathcal{E}} \cdot D_i = 0$ for each i. Now, looking at the adjunction formula above, the left-hand-side is negative, which forces $\rho_a(D_i)$ to be 0, and then $D_i^2 = -2$.

Now, we want to take the intersection product of the above expression for (f) with D_i. The result depends on whether $\overline{P}$ or $\overline{Q}$, or both, have non-zero intersection (necessarily 1) with any particular D_i. So let us now fix some notation. From now on – until the end of the thesis – i and j will refer to *specific* indices, as follows:

Definition 3.10
- Let $i \in \mathbb{Z}/n\mathbb{Z}$ be such that D_i intersects $\overline{P}$ (so that the reduction of P mod $\mathfrak{p}$ lies on component D_i).
- Similarly, let us use j for the fixed index for which the reduction of Q lies on D_j.
- Let Δ be the difference $j - i$.

(Δ will play a large role in the coming computations, but should cause no confusion with the Δ used to represent the diagonal in $E \times E$.)

We will first deal with the case when $\Delta = 0$ – so the reductions of P and Q lie on the same component of the n-gon. Then each D_k intersects either neither or both of $\overline{P}$ and $\overline{Q}$, so that the $d\overline{P} - d\overline{Q}$ part always gives zero, resulting in $0 = a_{k-1} + a_{k+1} - 2a_k$. We can utilise this relation between the a_k to see that, for each k, if we choose an integer representative which is positive, $a_k = (k-1)a_2 - (k-2)a_1$ – this follows by a straightforward induction. But then, recalling that these indices are really elements of $\frac{\mathbb{Z}}{n\mathbb{Z}}$, $a_1 = a_{n+1} = na_2 - (n-1)a_1$, from which we see that $a_1 = a_2$, and hence that all the a_i are identical. As $(\pi) = D_1 + \ldots + D_n$, we see that the part of (f) which lies in the fibre is the same as that of the constant π^{a_k} (for any k), so we can multiply f by a suitable constant in order to make the contribution of the horizontal and vertical curves to the image of the boundary map 0. Note that, because we are only concerned here with *local* integrality, the only thing that matters about the constant we pick is that it has the correct order at $\mathfrak{p}$.

When $\Delta \neq 0$, then we get a different result from intersecting (f) with different

D_k. More precisely, we obtain:

$$\begin{aligned} 2a_i &= d + a_{i-1} + a_{i+1} \\ 2a_j + d &= a_{j-1} + a_{j+1} \\ 2a_k &= a_{k-1} + a_{k+1} \qquad \text{for } k \neq i, j. \end{aligned}$$

One can then solve these equations in terms of just one parameter, just as we did in the $\Delta = 0$ case. (There are n equations, but that they are dependent is easily seen by adding them all together – this is to be expected, as we can alter the a_i by multiplying the function f by a constant.) So far we are considering the indices to lie in $\mathbb{Z}/n\mathbb{Z}$, but it will prove more convenient now to choose integer representatives for them. We shall choose these representatives for i and j so that $\Delta = j - i$ lies between 0 and $n-1$, inclusive. Having done this, we will choose representatives for each k which lie from i to $i+n-1$ (although the following formulae work for $k = i+n$ as well). Then, with these conventions, we find that:

$$a_k = a_i - \frac{(k-i)(n-\Delta)d}{n} \qquad \text{if } k - i \leq \Delta \tag{3.1}$$

$$a_k = a_i - \frac{\Delta(n-(k-i))d}{n} \qquad \text{if } k - i \geq \Delta. \tag{3.2}$$

(These can be derived as we did in the previous case, by writing all the a_k in terms of say a_i and a_{i+1}, and then using $a_i = a_{i+n}$ to write a_{i+1} in terms of a_i. But, having derived these formulae, one can easily confirm that they are correct, just by verifying that they satisfy all of the equations above, and hence give a one-parameter family of solutions – for all the solutions must lie in a one-parameter family.)

From these formulae, we can extract the following result:

Proposition 3.11 Again choose k so that $i \leq k \leq i+n-1$. Then:

$$a_k - a_{k+1} = \begin{cases} \frac{(n-\Delta)d}{n} & \text{if } i \leq k \leq i+\Delta-1 \\ -\frac{\Delta d}{n} & \text{if } i+\Delta \leq k \leq i+n-1 \end{cases}$$

It is worth noting that these last formulae actually apply to the case where $\Delta = 0$ as well – then it is the second line of the above result which always applies, and says that all the a_i are equal.

We will later draw out various consequences of these formulae, but will state and prove one here, as it will be used time and again in the next chapter, and is easy to prove at this stage:

Proposition 3.12 With the a_k as above, we have that

$$\sum_k a_k = n\left(a_i - \frac{\Delta(n-\Delta)d}{2n}\right). \tag{3.3}$$

Proof If $\Delta = 0$, the result is trivial, as we have already seen that all a_k are equal. If $\Delta \neq 0$, the proof is a simple computation using the formulae already given:

$$\begin{aligned}
\sum_k a_k &= na_i - \frac{d}{n}\sum_{k=i}^{i+\Delta}(k-i)(n-\Delta) - \frac{d}{n}\sum_{k=i+\Delta+1}^{i+n-1}\Delta(n-(k-i)) \\
&= na_i - \frac{d}{n}\sum_{k=0}^{\Delta}k(n-\Delta) - \frac{d}{n}\sum_{k=\Delta+1}^{n-1}\Delta(n-k) \\
&= na_i - \frac{(n-\Delta)d}{n}\sum_{k=0}^{\Delta}k - \frac{\Delta d}{n}\sum_{k=1}^{n-\Delta-1}k \\
&= na_i - \frac{(n-\Delta)\Delta(\Delta+1)d}{2n} - \frac{\Delta d(n-\Delta-1)(n-\Delta)}{2n} \\
&= na_i - \frac{\Delta(n-\Delta)d}{2n}\left((\Delta+1)+(n-\Delta-1)\right) \\
&= n\left(a_i - \frac{\Delta(n-\Delta)d}{2n}\right).
\end{aligned}$$

□

So far, we have concentrated on the closures of the horizontal and vertical curves which occur in our triangle configurations – but there will also be a contribution from the diagonal, which we need to examine. We first need to know what the image of the diagonal is in the special fibre – in other words, how the fibre intersects with the closure of the diagonal in our regular model for $E \times E$. It is clear that the image of the diagonal in the special fibre of $\mathcal{E} \times \mathcal{E}$, before blowing up, is itself a diagonal in our picture – at this stage the closure of Δ is still isomorphic to the model $\mathcal{E}$ of the curve. But it runs through n of the singular points at which we blow up, which means that, after blowing up, its intersection with the fibre will have not n but $2n$ components – the original n, plus an extra one at each point where it goes through a singularity of the special fibre. Another way to view this is to note that this intersection will consist of the fibre of $\mathcal{E}$ – the Néron n-gon – after it has been blown up at each of its n singular points. We shall call this blowup of the Néron model $\mathcal{E}'$. If we label the original components D_k as before, with $k \in \frac{\mathbb{Z}}{n\mathbb{Z}}$, and use E_k to denote the new component which came from the point of intersection of D_k and D_{k+1}, then we know that, on $\overline{\Delta}$,

$$(f) = d\overline{P} - d\overline{Q} + a_1 D_1 + \ldots + a_n D_n + b_1 E_1 + \ldots + b_n E_n,$$

for certain (integer) constants a_k and b_k.

We wish to again use the fact that the intersection of (f) with each D_k and each E_k is zero to be able to solve for the a_k and b_k in terms of just one parameter. For this we shall need to know what the self-intersections of the D_k and E_k are on this model. The earlier argument, for the Néron model $\mathcal{E}$, goes through, right up until the point where we ruled out the possibility that $D_k^2 = -1$. (Note that the argument now works for $n = 1$, too, as even here there are 2 components in the fibre, so that neither D_k nor E_k can be a multiple of the whole fibre.) We can no longer rule this out, because the model is no longer minimal. In fact, we *know* that $E_k^2 = -1$, and that $\rho_a(E_k) = 0$, precisely because it is the exceptional divisor of a blowup of a surface at a point (namely, of the minimal model at one of the singularities of its fibre) – this is Castelnuovo's criterion again. Hence, by the adjunction formula for E_k, $K_{\mathcal{E}'} \cdot E_k = -1$. And since we know that each D_k is a $\mathbb{P}^1$, its arithmetic genus is 0, and hence $D_k^2 + K_{\mathcal{E}'} \cdot D_k = -2$. We can now use these in the formula $0 = \mathcal{E}'_{\mathfrak{p}} \cdot K_{\mathcal{E}'}$, for which we need to know what $\mathcal{E}'_{\mathfrak{p}}$ is – that is, (π) – in terms of the components D_k and E_k of the special fibre.

Clearly, π vanishes along each of the D_k and E_k, and further has a simple zero at each D_k, since this was the case before the blowup, and the blowup did not affect the D_k at all. The order at the E_k turns out to be 2; for this computation, again see the appendix.

So, $\mathcal{E}'_{\mathfrak{p}} = D_1 + \ldots + D_n + 2E_1 + \ldots + 2E_n$, and thus it follows that

$$\begin{aligned} 0 &= \sum_k \left(K_{\mathcal{E}'} \cdot D_k + 2K_{\mathcal{E}'} \cdot E_k\right) \\ &= \sum_k \left(-2 - D_k^2 - 2\right), \end{aligned}$$

so $\sum_k D_k^2 = -4n$. Since it is clear that all the D_k^2 must be equal, $D_k^2 = -4$.

We will again divide into two cases, according to whether $\Delta = 0$ or not. If it is, then we obtain:

$$\begin{aligned} 0 &= -4a_k + b_{k-1} + b_k \\ 0 &= a_k + a_{k+1} - b_k, \end{aligned}$$

so that $b_k = a_k + a_{k+1}$, and hence that $0 = -4a_k + (a_{k-1} + a_k) + (a_k + a_{k+1})$, or $2a_k = a_{k-1} + a_{k+1}$, which is exactly the same relation we had among the a_k for the images of the horizontal and vertical curves. So the result will be the same – that all the a_k must be equal, and therefore can be made all zero if we use a suitable constant multiple of f. Hence we already see the following result:

Theorem 3.13 If E has split multiplicative reduction at $\mathfrak{p}$, then any triangle with $\Delta = 0$ is potentially integral – and hence so is any linear combination of such triangles.

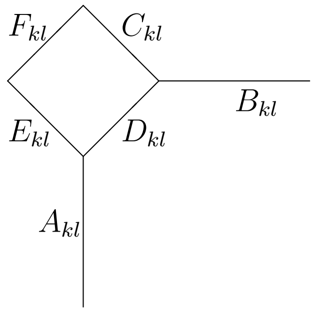
If $\Delta \neq 0$, then we find that:

$$\begin{aligned} 0 &= d - 4a_i + b_{i-1} + b_i \\ 0 &= -d - 4a_j + b_{j-1} + b_j \\ 0 &= -4a_k + b_{k-1} + b_k \qquad \text{for } k \neq i, j \\ 0 &= a_k + a_{k+1} - b_k. \end{aligned}$$

If we use the last of these (for each k) to express everything in terms of the a_k, we find that the resulting relations are identical to the ones which we had among the a_k for the images of the horizontal and vertical. Thus the relations among the a_k here are exactly the same as the ones we saw earlier, which means in particular that the relations (3.1) and (3.2), Proposition 3.11, and the sum formula in Proposition 3.12, all still hold. Further, we know that $b_k = a_k + a_{k+1}$ for each k.

3.5.4 The image of a linear combination of triangles

We are now going to use the results of the previous section to compute the image in the codimension 1 Chow group of the fibre of our regular model $\mathcal{X}_\mathcal{E}$ of $E \times E$, at a prime of split multiplicative reduction, of a linear combination of triangles under the boundary map – in terms of the generators A_{kl}, B_{kl}, C_{kl} and D_{kl} we have been using. What we have to do now is to translate the D_k and E_k, which we dealt with in the previous section, into expressions involving our generators. To do this, first let us recall the picture we were using:



(I have replaced i and j by k and l, respectively, to avoid confusion with the fact that i and j now have specific meanings, whereas the indices in the picture above are generic.) Now, the horizontal component in each individual triangle is precisely $E \times \{P\}$, and we are now using i to denote the component on which the reduction of P lies. This means that, if the labelling of each axis on the above picture is consistent with our labelling of the components D_k of the special fibre of $\mathcal{E}$, then the D_k on the horizontal component becomes $B_{ki} + C_{k+1,i} + D_{ki}$, if we recall the divisors of the co-ordinate functions on the octagons. So the total contribution from the horizontal under the boundary map is

$$\sum_k a_k^H (B_{ki} + C_{k+1,i} + D_{ki}).$$

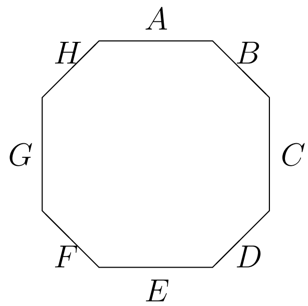
The superscript H here refers to the fact that this comes from the horizontal component – I will use V and D for the same purposes on the other two components. It is important to keep these separate, as the sets of a_k need not be the same on each, given that we want to be able to choose independent constants on each of the three curves, if necessary, to multiply the function by.

We can read off the contribution of the vertical just as easily; it is:

$$\sum_k a_k^V (A_{jk} + C_{j,k-1} + D_{jk})$$

in this notation. As for the diagonal, recall that its closure inside the integral model we are using is the same as the blowup of $\mathcal{E}$ at each of the n singular points of its fibre, which we considered in the previous section. The D_k correspond to the diagonals of the $\mathbb{P}^1 \times \mathbb{P}^1$'s which became the octagons after the blowup, while the E_k are the intersections of the strict transform of the diagonal with the exceptional divisors – we shall need to compute both of these.

Inside each octagon, we compute the divisor of the function $y - x$ in the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$. This has a simple zero along the diagonal, and simple poles along A and C – we use the same labelling of the edges of the octagon as we did before, clockwise from A at the top (ie. $\mathbb{P}^1 \times \{\infty\}$). I will again repeat the earlier diagram:



The order of $y - x$ along the edges which come from the exceptional divisors of blowups can be found in the same way as we did when we computed the divisors of x and y, but now using the functions $x - y$, $1/x - y$, $x - 1/y$ and $1/x - 1/y$ for F, D, H and B respectively. We find a simple zero along F, the lower-left corner, and simple poles along the other three (as would be expected naively from treating them as points with coordinates zero and infinity), and hence that, in the Chow group, the diagonal of the octagon is equivalent to $A + B + C + D + H - F$.

We also need to compute the E_k. It turns out – see Proposition 5.1 later, and its proof in the appendix – that this corresponds to the side of the "square" (which is the exceptional divisor) labelled as a D_{kl} rather than a C_{kl}. (Again, this corresponds to our intuition, as it is the sides D_{kl} which are parallel to the diagonal in our picture.)

Putting these together, and referring to the picture, we see then that the contribution from the diagonal, from a single triangle, is

$$\sum_{k=1}^{n} \left(-a_k^D(A_{k+1,k} + B_{kk} + C_{k+1,k} - C_{k,k-1} + D_{kk} + D_{k+1,k-1}) - b_k^D D_{k+1,k}\right)$$

(The minus sign is because we have the function f^{-1} on the diagonal, whereas all the computations already done were for f. Note that replacing f by f^{-1} replaces i and Δ with $i+\Delta$ and $n-\Delta$ respectively – the reader can convince himself that the formulae given earlier for the a_k are unchanged if we replace each a_k by $-a_k$ as well as altering i and Δ as just mentioned. This is most easily seen from the difference formulae in Proposition 3.11 – the two ranges for k switch round, and each difference becomes the other multiplied by -1, so that, for f^{-1}, the differences are exactly as they would be if we had replaced each a_k by $-a_k$.)

Therefore, by putting all three expressions together, we can write down what the image of one of our triangle configurations is in the Chow group of the fibre, in terms of the $a_k^\bullet$. (And the b_k^D too, but we already know that $b_k^D = a_k^D + a_{k+1}^D$, which we shall make use of.) In fact, we shall do so for an arbitrary linear combination of triangles, of the form $\sum_h e_h \alpha_{f_h}$. So each h comes with its own d_h ($\in \mathbb{Z}$), i_h and Δ_h ($\in \{0, \ldots, n-1\}$). Further, the three sets of a_k can be different on each, so I shall write $a_{k,h}^\bullet$. I shall use a_{kl} to denote the coefficient of A_{kl} which appears as the image under the boundary map of this linear combination, and analagously b_{kl}, c_{kl} and d_{kl} for the other coefficients. Then, we find the following results (here each δ denotes the Kronecker delta):

$$\begin{aligned}
a_{kl} &= \sum_h e_h \left(\delta_{i_h+\Delta_h,k} a_{l,h}^V - \delta_{k,l+1} a_{l,h}^D\right) \\
b_{kl} &= \sum_h e_h \left(\delta_{i_h,l} a_{k,h}^H - \delta_{kl} a_{k,h}^D\right) \\
c_{kl} &= \sum_h e_h \left(\delta_{i_h,l} a_{k-1,h}^H + \delta_{i_h+\Delta_h,k} a_{l+1,h}^V - \delta_{k,l+1}(a_{l,h}^D - a_{k,h}^D)\right) \\
d_{kl} &= \sum_h e_h \left(\delta_{i_h,l} a_{k,h}^H + \delta_{i_h+\Delta_h,k} a_{l,h}^V - \delta_{kl} a_{k,h}^D - \delta_{k-1,l+1} a_{k-1,h}^D - \delta_{k,l+1}(a_{l,h}^D + a_{k,h}^D)\right).
\end{aligned}$$

3.5.5 Conditions for integrality

Finally, in this chapter, we shall note some conditions on the coefficients in a general linear combination of the generators A_{kl}, B_{kl}, C_{kl} and D_{kl} which are necessary for the resulting element to be zero in the Chow group – that is, in the span of the relations already given. We know that the relations span a subspace of dimension $2n^2-2$, so that any necessary and sufficient set of conditions on the coefficients for the element to be zero must have $4n^2 - (2n^2 - 2) = 2n^2 + 2$ independent conditions. It is not clear, in general, what these will be. However, it is easy to pick out $4n$ conditions which are certainly necessary (although not usually sufficient), and it turns out that these will be enough to allow us to prove that any integral linear combination of triangles has vanishing regulator.

These conditions are as follows. It is clear that all of them are necessary, as they are true of each of the $2n^2$ relations, and hence for anything in their span.

Proposition 3.14 Let $\sum_{k,l}(a_{kl}A_{kl} + b_{kl}B_{kl} + c_{kl}C_{kl} + d_{kl}D_{kl})$ be an element of the codimension 1 Chow group of the fibre of our model for $E \times E$ (at a prime of split multiplicative reduction). If it is zero, then we must have, for each $k \in \mathbb{Z}/n\mathbb{Z}$:

$$\begin{aligned} \sum_m a_{mk} &= 0 \\ \sum_m b_{km} &= 0 \\ \sum_m c_{m,m+k} &= 0 \\ \sum_m d_{m,k-m} &= 0. \end{aligned}$$

□

We now simply apply these conditions to the coefficients of a linear combination of triangle configurations, which we have just written down. We find that:

$$\begin{aligned} 0 &= \sum_m a_{mk} \\ &= \sum_h e_h \left(\sum_m \delta_{i_h+\Delta_h,m} a^V_{k,h} - \sum_m \delta_{m,k+1} a^D_{k,h} \right) \\ &= \sum_h e_h \left(a^V_{k,h} - a^D_{k,h} \right); \\ 0 &= \sum_m b_{km} \\ &= \sum_h e_h \left(\sum_m \delta_{i_h,m} a^H_{k,h} - \sum_m \delta_{km} a^D_{k,h} \right) \\ &= \sum_h e_h \left(a^H_{k,h} - a^D_{k,h} \right); \end{aligned}$$

Similar computations for the other two yield:

$$\begin{aligned} 0 &= \sum_m c_{m,m+k} \\ &= \sum_h e_h \left(a^H_{i_h-k-1,h} + a^V_{i_h+\Delta_h+k+1,h}\right); \end{aligned}$$

$$\begin{aligned} 0 &= \sum_m d_{m,k-m} \\ &= \sum_h e_h \left(a^H_{k-i_h,h} + a^V_{k-i_h-\Delta_h,h} - 2\sum_{m|2m=k} a^D_{m,h} - \sum_{m|2m=k-1} a^D_{m,k} - \sum_{m|2m=k+1} a^D_{m,k} \right). \end{aligned}$$

What we will do next is, for each of these four sets of relations (each set contains n relations, one for each k), is to sum them up over all k – so really the only conditions for triviality in the Chow group which we are using are that the sum of all the a_{kl} (and b_{kl} etc.) must be zero. If we do this, we find the following conditions:

$$\begin{aligned} \sum_h e_h \left(\sum_k a^V_{k,h} - \sum_k a^D_{k,h}\right) &= 0 \\ \sum_h e_h \left(\sum_k a^H_{k,h} - \sum_k a^D_{k,h}\right) &= 0 \\ \sum_h e_h \left(\sum_k a^H_{k,h} + \sum_k a^V_{k,h}\right) &= 0 \\ \sum_h e_h \left(\sum_k a^H_{k,h} + \sum_k a^V_{k,h} - 4\sum_k a^D_{k,h}\right) &= 0. \end{aligned}$$

It is now clear from these that we must have that $\sum_h e_h \left(\sum_k a^\bullet_{k,h}\right) = 0$, for $\bullet = H$, V or D. Hence, using the formula (3.3), we discover the following crucial result, which will be essential in the following chapter:

Theorem 3.15 If $\sum_h e_h \alpha_{f_h}$ is integral, then we must have

$$\sum_h e_h a^\bullet_{i_h,h} = \sum_h e_h \frac{\Delta_h(n-\Delta_h)d_h}{2n}$$

for $\bullet = H$, V or D.

Chapter 4

The Vanishing of the Regulator

4.1 The regulator image of a triangle, and the norm of a function

Having examined in some detail under what conditions a linear combination of triangle configurations is integral (or might be integral), we are now going to have a look at what the image of such a linear combination will be under the regulator map – and in particular, at certain conditions under which this regulator image will be zero.

Recall from Chapter 2 that the target group of the Beilinson regulator from $K_1^{(2)}(X)$ is $H^3_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(2))$ (or the "plus space" which is a subspace of this), which we saw was isomorphic to the dual space of $H^{2d-2}(X, \mathbb{R}(1)) \cap F^{d-1}H^{2d-2}(X, \mathbb{C})$ (where d is the dimension of X). In our case, X is a surface, so that the regulator image of a triangle, like that of any other element of $K_1^{(2)}(E \times E)$, is a linear map on $H^2((E \times E)_{\mathbb{C}}, \mathbb{R}(1)) \cap F^1H^2((E \times E)_{\mathbb{C}}, \mathbb{C})$, which as we saw in Chapter 2 is a space of dimension $[K : \mathbb{Q}](2g^2 + 2)$, or $4[K : \mathbb{Q}]$ in the case we are dealing with, where E is an elliptic curve.

In order to simplify our discussion to start with, we shall assume that the base field K is $\mathbb{Q}$, so that the target group of the regulator map is the dual space to a 4-dimensional space of differential forms. We recall that a basis for this space consists of the four forms $\pi_1^*(\omega \wedge \overline{\omega})$, $\pi_1^*(\omega) \wedge \pi_2^*(\overline{\omega}) - \pi_1^*(\overline{\omega}) \wedge \pi_2^*(\omega)$, $i\pi_1^*(\omega) \wedge \pi_2^*(\overline{\omega}) +$

$i\pi_1^*(\overline{\omega}) \wedge \pi_2^*(\omega)$ and $\pi_2^*(\omega \wedge \overline{\omega})$, where π_1 and π_2 are the projections $E \times E \to E$ and ω is any representative for a non-zero element of $H^{1,0}(E_{\mathbb{C}})$. These forms on an elliptic curve all have a particular special property – ω is an *invariant* differential form, meaning that, if R is any point on E and τ_R the translation map $S \mapsto S + R$, then $\omega \circ \tau_R = \omega$ (see chapter 3 of [21]). This property, as we shall see, has some striking consequences for the regulator image of our triangles.

So, let us look at what the regulator of a single triangle is going to do to each of these four differential forms. For ease of notation, we shall actually multiply the regulator by $2\pi i$, which has no effect because our only concern here is whether or not the regulator is zero. We shall write I_f for the quantity $\int_{E_{\mathbb{C}}} \log|f|\omega \wedge \overline{\omega}$. Then $\pi_1^*(\omega \wedge \overline{\omega})$ vanishes on $\{Q\} \times E$, and becomes $\omega \wedge \overline{\omega}$ on both $E \times \{P\}$ and Δ – so the result is $I_f + I_{f^{-1}}$. Stated this way, it looks like this is zero – but recall that we can replace each of our copies of f by a suitable constant multiple. So all we know is that this gives a constant times $I = \int_{E_{\mathbb{C}}} \omega \wedge \overline{\omega}$. And $\pi_2^*(\omega \wedge \overline{\omega})$ gives the same result, as the roles of the "horizontal" and "vertical" curves are merely switched round. Let us note here that I must be non-zero, as the integration pairing between homology and (de Rham) cohomology is non-degenerate, and the differential form $\omega \wedge \overline{\omega}$ is a basis for the second cohomology group.

The second form, $\pi_1^*(\omega) \wedge \pi_2^*(\overline{\omega}) - \pi_1^*(\overline{\omega}) \wedge \pi_2^*(\omega)$, vanishes on both $E \times \{P\}$ and $\{Q\} \times E$, and becomes $2\omega \wedge \overline{\omega}$ on Δ (because both $\pi_1^*(\omega)$ and $\pi_2^*(\omega)$ restrict to ω on the diagonal). So the regulator image of the triangle sends this second form to $2I_f$. The third form, $i\pi_1^*(\omega) \wedge \pi_2^*(\overline{\omega}) + i\pi_1^*(\overline{\omega}) \wedge \pi_2^*(\omega)$, vanishes on all three of the curves concerned, so gives zero – this is not unexpected, as we know our regulator is really mapping into a three-dimensional space rather than a four-dimensional one, and this third form is precisely the one of the four which is *not* in the "plus" subspace.

These computations make it clear that it is the quantity I_f which controls everything that happens here, so we shall spend some time seeing if we can compute it. By definition, it is $\int_{E_{\mathbb{C}}} \log|f|\omega \wedge \overline{\omega}$. But, by the invariance of ω, we can pick any point R on $E(K)$ and also have that $I_f = \int_{E_{\mathbb{C}}} \log|f \circ \tau_R|\omega \wedge \overline{\omega}$. In particular, recalling that $(f) = d(P) - d(Q)$, let us apply this with $R = P - Q$. This doesn't tell us much in itself, but if we also apply it with $R = i(P - Q)$ and add these together from $i = 0$ up to $i = d-1$, we have that $I_f = \frac{1}{d}\int_{E_{\mathbb{C}}} \log\left|\prod_{i=0}^{d-1}\left(f\circ\tau_{i(P-Q)}\right)\right| \omega\wedge\overline{\omega}$. The significance of this formula is shown by the following simple result:

Lemma 4.1 The function $\prod_{i=0}^{d-1}\left(f\circ\tau_{i(P-Q)}\right)$ is constant.

Proof This is a simple matter of writing out the divisor of the function, and observing that it is zero. The divisor of $f\circ\tau_{i(P-Q)}$ is $d(iQ-(i-1)P) - d((i+1)Q - i(P))$, so the negative contribution from $f\circ\tau_{i(P-Q)}$ is always cancelled out by the positive one from $f\circ\tau_{(i+1)(P-Q)}$. As $P-Q$ is a d-torsion point, this also happens with the negative contribution from $i = d-1$ and the positive one from $i = 0$. □

Definition 4.2 Let $f \in K(E)$ have divisor $d(P) - d(Q)$. Then we define the *norm* of f to be $N(f) := \prod_{i=0}^{d-1}\left(f\circ\tau_{i(P-Q)}\right)$.

Note that, if P and Q are defined over K, then $N(f) \in K$. We can now state the result just reached in a more enlightening way:

Proposition 4.3 We have that $I_f = \frac{\log|N(f)|}{d} I$, where $I = \int_{E_{\mathbb{C}}} \omega\wedge\overline{\omega}$. In particular, $I_f = 0$ if and only if $N(f)$ is a root of unity.

Let us now recall the definition of decomposability from Chapter 1 (Definition 2.1). Note here that it is possible for elements to be decomposable even if they don't "look it", as two element of $K_1^{(2)}(X)$ are equal whenever their difference is in the image of the tame symbol map from K_2 of the function field of X.

So our proposition above tells us that:

Corollary 4.4 A triangle configuration in $K_1^{(2)}(E\times E)$ has the same regulator image as a decomposable element. (And therefore, if the regulator is injective as is believed, triangle configurations are in fact decomposable!)

Proof Let our triangle configuration be $(E\times\{P\}, f^H) + (\{Q\}\times E, f^V) + (\Delta, (f^D)^{-1})$, where f^H, f^V and f^D are all obtained from multiplying the original function f by constants, chosen arbitrarily in each case. Then, from the above discussion, it can be seen that the decomposable element consisting of the constants $N(f^H)$, $N(f^V)$ and $1/N(f^D)$ on the horizontal, vertical and diagonal curves, respectively, has exactly the same regulator image. □

As an illustration of this result, let us consider the case of a 2-torsion point – ie. a function f with divisor $2(P) - 2(Q)$. After a change of co-ordinates, such a curve will always have an affine equation of the form $y^2 = x(x^2 + ax + b)$, with a and b in K, and we take $f(x, y) = x$; then $P = (0, 0)$ and $Q = O$ (the point at infinity). Then translation by $P - Q = (0, 0)$ sends (x, y) to $(\frac{b}{x}, -\frac{by}{x^2})$, so $N(f) = b$. Hence $I_f = 0$ if and only if $|b| = 1$. And I_f can in this case be written as an integral over the complex numbers: $I_f = 2\int_{\mathbb{C}} \frac{\log|z|}{|z||z^2+az+b|}\mathrm{d}x \wedge \mathrm{d}y$. When $|b| = 1$, one find that the substitution $z \mapsto \frac{b}{z}$ gives $I_f = -I_f$, so that $I_f = 0$. (The example I gave at the end of Chapter 2 to show that triangles do not always vanish was of this type; we have now confirmed the statement I made then, that that integral is zero if and only if $|s| = 1$.)

Note that, apart from this $d = 2$ case, computing $N(f)$ directly from the definition, for a specific curve and function f, is quite involved. Much the easiest way is to use the fact that it is a constant function, and simply evaluate it at any point – so pick a point R on E, such that none of the points $R + i(P - Q)$ are equal to P or Q, and then $N(f) = \prod_{i=0}^{d-1} f(R + i(P - Q))$. The restriction on R is to ensure that none of the terms in this product are zero or infinity – it may be that there are no R in $E(K)$ which satisfy this, but we can always find a suitable R over some extension field, and then calculate $N(f)$ in exactly the same way using that.

With this in mind, we can list some of the properties of $N(f)$:

Proposition 4.5

1. For any $c \in K$, $N(cf) = c^d N(f)$.
2. $N(f^{-1}) = N(f)^{-1}$.
3. For any positive integer a, $N(f^a) = N(f)^{a^2}$.
4. $N(f) = N(f \circ \tau_R)$, for any $R \in E(K)$.

Proof

1. This is clear from the definition.
2. Again, clear.
3. We have that $N(f^a) = \sum_{i=0}^{ad-1}(f \circ \tau_{i(P-Q)})^a$, because the divisor of f^a is $ad(P) - ad(Q)$. If the sum were just up to $d - 1$, then this would be equal to $N(f)^a$.

But since $P - Q$ is a d-torsion point, the terms of this product just run over those from $i = 0$ to $i = d - 1$, a times, which gives a further power of a.

4. Note that $(f \circ \tau_R) = d(P - R) - d(Q - R)$, so that $N(f \circ \tau_R) = \prod_{i=0}^{d-1}(f \circ \tau_R)(S + i(P - Q)) = \prod_{i=0}^{d-1} f(R + S + i(P - Q))$ for some (in fact, almost all) $S \in E(\overline{K})$. But this is exactly the computation that produces $N(f)$, as it is the (constant) function $N(f)$ evaluated at $R + S$.

$\square$

Returning to what we saw above, the regulator image of a single triangle will be zero only if we have that $N(f)$, for the copy of f on the diagonal, is a root of unity, and that the two other copies of f are both some root of unity times the diagonal copy. However, for our purposes it will be enough to show that these quantities (which we have just said must be roots of unity) are in $\mathcal{O}_K^*$. This is because, as discussed in Chapter 2, we already know about "uninteresting" (that is, decomposable) non-zero integral elements of $K_1^{(2)}(E \times E)$, given by elements of $\mathcal{O}_K^*$ along any horizontal, vertical or diagonal curve in $E \times E$. The regulator image of the unit u along a horizontal curve sends the first three of the four forms listed above to 0, and the fourth to $(\log |u|)I$. On the vertical, it sends the first to $(\log |u|)I$ and the last three to zero, while on the diagonal, it sends them to $((\log |u|)I, 2(\log |u|)I, 0, (\log |u|)I)$. As u varies over all of $\mathcal{O}_K^*$, these three images together span, over $\mathbb{Q}$, the space of all vectors of the form $(aI, bI, 0, cI)$, where a, b and c are rational multiples of the logarithms of units. But clearly, if the quantities above are units, then the regulator image of our triangle will be of this form. So if this holds, then the triangle has the same regulator image as one of our "uninteresting" integral elements. Another way of saying the same thing is that, as we can multiply each of the copies of f involved in the triangle by any unit, without affecting integrality, we can choose these units in such a way as to make its regulator image zero.

All of the above of course only applies for a single triangle, over the base-field $\mathbb{Q}$. We shall deal with general linear combinations in a moment – but what happens over larger number fields? Then there are $4[K : \mathbb{Q}]$ different differential forms, which consist of the four given above restricted to each fo the $[K : \mathbb{Q}]$ connected components of $(E \times E)_{\mathbb{C}}$. This means that everywhere where I have written I above, we really get $I^\sigma := \int_{E^\sigma} (\omega \wedge \overline{\omega})^\sigma$, where E^σ is the component corresponding to the embedding σ, and ω^σ the image of ω on this embedding. In particular, the factors of $N(f)$ and so on stay in place, so it is once again enough for these quantities to always be units.

4.2 Proof of the main result

We now have all the tools at our disposal to prove the following result:

Theorem 4.6 Let E/K be an elliptic curve defined over the number field K, and let $\alpha = \sum_h e_h \alpha_{f_h}$ be a linear combination of triangle configurations in $K_1^{(2)}(E \times E)$. Then if $\alpha \in K_1^{(2)}(E \times E)_{\mathbb{Z}}$, then the regulator of α is equal to the regulator of an element of $K_1^{(2)}(E \times E)_{\mathbb{Z}}$ of the form $\sum_k (C_k, u_k)$, where the C_k are curves on $E \times E$ and the u_k are in $\mathcal{O}_K^*$.

We have already seen that any triangle configuration – and hence any linear combination of them – necessarily has the same regulator image as a decomposable element. This theorem provides the further statement that, if the linear combination of triangles is also integral, then it has the same regulator image as a decomposable integral element. This does not follow from the previous statement, as we have no guarantee that, if two elements have the same regulator image and one is integral, the other must be too. The Beilinson regulator map is conjectured to be injective when restricted to integral elements, but it is not thought to be injective on the whole of $K_1^{(2)}(X)$, and thus we have no compelling reason to believe that it isn't possible for an integral and a non-integral element to share a regulator image.

So, we really do have something to prove. We will begin the proof with the following important formula, which is related to (3.3) from the last chapter. I make use of the notation given in Definition 3.10.

Proposition 4.7 Let the $a_k^\bullet$ ($\bullet = H$, V or D) be as in the previous chapter. Then, for any $k \in \mathbb{Z}/n\mathbb{Z}$,

$$\sum_{j=1}^{d} a_{k+j\Delta} = d\left(a_i - \frac{\Delta(n-\Delta)d}{2n}\right).$$

Proof Given what we already know about the $a_k^\bullet$, it is clearly sufficient to prove that the sum is zero when $a_i = \frac{\Delta(n-\Delta)d}{2n}$. Note next that when j exceeds $\frac{n}{\gcd(n,\Delta)}$ then $k + j\Delta$ starts repeating (as the indices are interpreted modulo n), so that the sum is $\frac{d\gcd(n,\Delta)}{n}\sum_{j=1}^{\frac{n}{\gcd(n,\Delta)}} a_{k+j\Delta}$. (This is an integer multiple, as we have $d(P-Q) = O$ on $E(K)$, so that $d\overline{P} = d\overline{Q}$ in the fibre of $\mathcal{E}$; passing to the group of components, we see that $n|d\Delta$, and hence $\frac{n}{\gcd(n,\Delta)}|d\frac{\Delta}{\gcd(n,\Delta)}$; thus $\frac{n}{\gcd(n,\Delta)}|d$.) Thus it is enough to prove that $\sum_{j=1}^{\frac{n}{\gcd(n,\Delta)}} a_{k+j\Delta} = 0$. Further, it is clear that we can rewrite this sum as $\sum_{j=1}^{\frac{n}{\gcd(n,\Delta)}} a_{k+j\gcd(n,\Delta)}$. Note that if $\gcd(n,\Delta) = 1$, then the result we want is the same as (3.3).

Firstly, we shall suppose that $k = i$. Then we can use the formulae (3.1) and (3.2) derived in the previous chapter to compute the sum; let us write G for $\gcd(n,\Delta)$ for ease of notation:

$$\begin{aligned}
& a_i + \sum_{j=1}^{\frac{\Delta}{G}} a_{i+jG} + \sum_{j=\frac{\Delta}{G}+1}^{\frac{n}{G}-1} a_{i+jG} \\
= \; & \frac{n}{G}a_i - \sum_{j=1}^{\frac{\Delta}{G}} \frac{jG(n-\Delta)d}{n} \\
& - \sum_{j=\frac{\Delta}{G}+1}^{\frac{n}{G}-1} \frac{\Delta(n-jG)d}{n} \\
= \; & \frac{n}{G}a_i - \frac{G(n-\Delta)d}{n}\sum_{j=1}^{\frac{\Delta}{G}} j \\
& - \left(\frac{n}{G} - \frac{\Delta}{G} - 1\right)\Delta d + \frac{\Delta G d}{n}\sum_{j=\frac{\Delta}{G}+1}^{\frac{n}{G}-1} j \\
= \; & \frac{\Delta(n-\Delta)d}{2G} - \frac{G(n-\Delta)d}{2n}\left(\frac{\Delta}{G}\left(\frac{\Delta}{\gcd(n,\Delta)} + 1\right)\right) \\
& - \left(\frac{n}{G} - \frac{\Delta}{G} - 1\right)\Delta d + \frac{\Delta G d}{2n} \times \\
& \left[\left(\frac{n}{G} - 1\right)\frac{n}{G} - \frac{\Delta}{G}\left(\frac{\Delta}{G} + 1\right)\right].
\end{aligned}$$

It is now a routine, if tedious, matter to put this all over a common denominator and find that all the terms cancel to give 0.

For other values of i, we shall use the difference formulae from Proposition 3.11. The difference between the sum we are interested in for $k = l$ and $k = l + 1$ is $\sum_{j=1}^{\frac{n}{G}}(a_{l+jG} - a_{l+jG+1})$, and it will suffice to show that this is 0, for any l. Again, we shall take $l = i$ first, when Proposition 3.11 provides a straightforward proof (the sum is $\frac{\Delta}{G}\frac{(n-\Delta)d}{n} - \frac{(n-\Delta)}{G}\frac{\Delta d}{n}$). And if we look at differences again – in other words at

$$\sum_{j=1}^{\frac{n}{G}}\left((a_{l+jG} - a_{l+jG+1}) - (a_{l+jG+1} - a_{l+jG+2})\right),$$

it is easy to see that the sum will always be zero, as the terms are all zero apart from those when $l = i + \Delta - 1$ and when $l = i + n - 1$, which are d and $-d$ respectively, and each sum must contain either none or both of these two. □

We will need just one more result before proving the main theorem, which I am going to state separately because it will be needed several times in the main proof.

Lemma 4.8 Let $\mathcal{E}$ be the Néron model for E over the localisation of $\mathcal{O}_K$ at the prime $\mathfrak{p}$ at which E has either good or (split) multiplicative reduction. Let $f \in K(E)$ be a function with divisor $d(P) - d(Q)$. Then there exists a point T on E, defined over some finite extension field of K, such that, for any integer k, the order of $f(T + k(P - Q)$ at $\mathfrak{p}$ is equal to the order of the function f (considered now as a function on $\mathcal{E}$) along the component of the fibre along which the reduction of $T + k(P - Q)$ lies.

More generally, given any finite set of such functions, there exists a point T which satisfies the above condition for all of the functions at once.

Proof Letting π be a uniformiser for the localisation of $\mathcal{O}_K$ at $\mathfrak{p}$ as before, we know that π has order 1 along each component of the fibre – this is clear in the case of good reduction (where there is only one component), and in the case of multiplicative reduction it follows from our computation of (π) in the previous chapter. So, if we denote by D_l the component of the fibre to which $T + k(P - Q)$ reduces (so l will depend on the choice of T, which hasn't been made yet; it also depends on

k), then $\pi^{-\mathrm{ord}(D_l)(f)} f$ has order 0 along D_l, and thus can be considered as a genuine function on D_l. There will be n of these functions in all (in the case of multiplicative reduction; there is only one in the good reduction case) – let $R_1, \ldots, R_s$ be the finite set of all the zeros and poles of all n of them. Further, let $S_1, \ldots, S_t$ be the set of all d-torsion points of the fibre (the fibre is an elliptic curve over the residue field k, if we have good reduction, or has a group structure isomoprhic to $\frac{\mathbb{Z}}{n\mathbb{Z}} \times k^*$ if we have multiplicative reduction). Note that not only are both of these sets finite, but that their size is bounded above universally, no matter how big a finite extension of K we take (by the size of these sets over the algebraic closure of k, which are both still finite). On the other hand, the number of points on the fibre itself grows without bound as the size of the field increases. This means that, if we take a large enough extension field L/K, there exists some point $\overline{T}$ on the fibre of $E(L)$ which is not any of the finitely many points $R_m + S_n$. We can then lift this to $E(L)$ (although we may have to take a further finite extension to do this). Call this point T. I claim that it satisfies the property needed for the lemma.

For, suppose that $\mathrm{ord}_{\mathfrak{p}}(f(T + k(P - Q))) \neq \mathrm{ord}_{D_l}(f)$. Then we would have that $\mathrm{ord}_{\mathfrak{p}}(\pi^{-\mathrm{ord}_{D_l}(f)} f(T+k(P-Q))) \neq 0$, since π is a uniformiser for $\mathfrak{p}$. This would mean that $\pi^{-\mathrm{ord}_{D_l}(f)} f(T + k(P - Q))$ is either divisible by π or has denominator divisible by π, and hence that the function $\pi^{-\mathrm{ord}_{D_l}(f)} f$, when restricted to D_l, has a zero or a pole at $\overline{T} + k(\overline{P - Q})$ (the bar denotes reduction mod $\mathfrak{p}$) – so it would mean that $\overline{T} + k(\overline{P - Q}) = R_s$ for some s. This in turn would mean that $\overline{T} - R_s = k(\overline{Q - P})$, which is a d-torsion point of the fibre – contrary to the way we found $\overline{T}$. This completes the proof.

The generalisation is proved in exactly the same way – we still have infinitely many points from which to choose $\overline{T}$, and only have to avoid finitely many. □

Now we can put together several of these past few results. We will be interested in quantities such as $\mathrm{ord}_{\mathfrak{p}}(N(f))$, where $\mathfrak{p}$ is a prime of either good or multiplicative reduction. By definition of $N(f)$ (and the way we noted in the first section that we can compute it), this will be $\sum_{k=0}^{d-1} \mathrm{ord}_{\mathfrak{p}} \left(f(T + k(P - Q)\right)$, for any suitable T. And the lemma I have just stated and proved tells us that, for an appropriate choice of T, this will be equal to $\sum_{k=0}^{d-1} \mathrm{ord}_{D_l}(f)$, where D_l is the component of the fibre on which the reduction $T+k(P-Q)$ lies. But, in the multiplicative case, the order of f along D_k is precisely the quantity we have been calling a_k. So, if D_k is the component on which the reduction of T lies, then we see that $\mathrm{ord}_{\mathfrak{p}}(N(f)) = \sum_{j=0}^{d-1} a_{k+j\Delta}$. And Proposition 4.7 tells us what this is (as $j=0$ is the same term as $j=d$, since $n|d\Delta$). Thus we have the following:

Proposition 4.9

$$\mathrm{ord}_{\mathfrak{p}}(N(f)) = d\left(a_i - \frac{\Delta(n-\Delta)d}{2n}\right)$$

Note that this statement still holds in the good reduction case, if we interpret the notation correctly – a_i is the order of f along the single component of the fibre, or in other words the order of f at π, and we must take $\Delta = 0$.

Let us now return at last to the regulator image of a general linear combination $\sum_h e_h \alpha_{f_h}$ of triangles. We already observed the effect of the regulator on each of the three relevant types of differential form (there are actually four, but one of them always vanishes on the regulator of any triangle) in the case of a single triangle. Using this, we see that the first of our forms will be sent to $\sum_h e_h \left(I_{f_h^H} - I_{f_h^D}\right)$, where the new superscripts, as with the a_k, are there to keep track of whether this is the f on the horizontal or on the diagonal. (In this notation, α_{f_h} is $(E \times \{P\}, f^H) + (\{Q\} \times E, f^V) + (\Delta, (f^D)^{-1})$.) By our computations in section one, this is equal to $\sum_h \frac{e_h}{d_h}\left(\log|N(f_h^H)| - \log|N(f_h^D)|\right)$ times the integral I on the image of E under one of the embeddings of the base field K into $\mathbb{C}$. We can also write the sum as $\frac{1}{m}\log\left|\prod_h \frac{N(f_h^H)^{\frac{me_h}{d_h}}}{N(f_h^D)^{\frac{me_h}{d_h}}}\right|$, where m is any common multiple of all the d_h (m is not important, it is chosen simply to ensure that all the powers stay integral). By the comments I made at the end of the first section of this chapter, it is enough to prove that the product is in $\mathcal{O}_K^*$, or in other words that it has order 0 at each prime $\mathfrak{p}$ of K.

To do this, notice that we can once again replace K by any finite extension of K in doing this, and this changes nothing in the product we are interested in, as everything is defined over K. So we may assume that E has either good reduction or split multiplicative reduction at all primes of K. Then we can use the proposition above, to see that the order of the product at $\mathfrak{p}$ is $\sum_h me_h\left(a^H_{i_h,h} - a^D_{i_h,h}\right) = m\sum_h e_h\left(a^H_{i_h,h} - a^D_{i_h,h}\right)$.

And this is zero for an integral element, by Theorem 3.15

Clearly, the same happens for the other type of form, involving π_2^* instead of π_1^* – all that changes in the above is that the "H" superscripts become "V"s, but here Theorem 3.15 still applies.

Finally, we have to consider the image of $\pi_1^*(\omega) \wedge \pi_2^*(\overline{\omega}) - \pi_1^*(\overline{\omega}) \wedge \pi_2^*(\omega)$ under the regulator of our (integral) linear sum of triangle configurations. We have already seen that this is $\sum_h e_h I_{f_h^D}$, which is equal to $\sum_h \frac{e_h}{d_h} \log |N(f_h^D)| I$, or $\frac{1}{m} \log \left| \prod_h N(f_h^D)^{\frac{me_h}{d_h}} \right|$ for m any common multiple of all the d_h. Again, it is enough to show that the product has order zero at each prime of good or split multiplicative reduction; and, using the same proposition, this order is $m \sum_h e_h \left(a^D_{i_h,h} - \frac{\Delta_h(n-\Delta_h)d_h}{2n} \right)$ – and this is zero, again by Theorem 3.15.

This completes the proof of Theorem 4.6.

Chapter 5

Some generalisations, and more on integrality

The main result, at the end of the previous chapter, demonstrates that the triangle configurations cannot generate any integral elements of $K_1^{(2)}(E \times E)$ which can be distinguished, via the regulator map, from those arising from units in the ring of integers of the base field. But there are still several questions which have been left unanswered. For I mentioned right at the start that these triangles are merely one special case of a more generalised triangle configuration, involving the curves Δ_a in place of the diagonal. Does the vanishing result still hold for all linear combinations of such triangles, or are there some integral linear combinations of such triangles whose regulators do not vanish (even when we multiply the functions involved by any units of our choice), thus providing non-zero integral elements of the relevant K-group which do not arise from units?

We shall see that this turns out to be a much more difficult question, and one which I have so far been unable to answer – although I strongly suspect the same vanishing result to hold. (I will explain my reason for this belief in a Section 5.4.) The main problem arises from one other question which I have so far left unanswered, even in the simpler $a = 1$ case we have been dealing with so far – what are the conditions on such a linear combination of triangles which determine its integrality? So far, I have only listed some necessary conditions, which proved to be sufficient to demonstrate the required result on the regulator – in general these will not be sufficient for integrality. In fact, it is quite surprising that I was able to derive so strong a result from so few conditions. For the target group of the boundary map – the Chow group of the fibre – has, in the case of split multiplicative reduction, $4n^2$ generators and $2n^2 - 2$ independent relations. This means that any (linearly independent) set of necessary and sufficient conditions for integrality on the coefficients of these generators must have size $2n^2 + 2$. But a glance at the arguments used so far shows that we have used much fewer than this – I have so far only mentioned $4n$ of these conditions, namely the ones given in Proposition 3.14. But in fact we have not even used that many, as we then proceeded to sum up each set of n relations, leaving us with just four! And even then, it can easily be seen that these are linearly dependent when applied to our triangle configurations, so we have so far been relying on only three relations out of a theoretically existing set of $2n^2 + 2$ – and there is of course no upper bound on what n could be.

This explains the difficulty arises with attempting to generalise our result to linear combinations of arbitrary triangle configurations – the three or four conditions that sufficed before do not suffice in general. So, it seems that a necessary step for such a generalisation would be to determine a complete set of $2n^2 + 2$ independent necessary conditions on the coefficients of the generators of the Chow group for an element to be zero, and then apply this to the image of a linear combination of triangles. But finding such a complete set does not seem to be as easy as one might think. So far I have only been able to discover another $2n^2$ relations to add to the $4n$ given earlier – and these are not all independent. These may or may not be sufficient, as well as necessary, for integrality. But it turns out that they certainly are enough – in fact, the original $4n$ are already enough, although the four used before are not – to prove an integrality result for certain *individual* triangles. The main bulk of this chapter is devoted to a proof of this result, which applies to triangles for which $a = 1$ or $a = -1$. So far, all we have said about the (potential) integrality of such triangles is that it will hold whenever $\Delta = 0$; here, we will see that the converse does not hold, for there do exist at least some cases of integral individual triangles for which $\Delta \neq 0$.

For $|a| > 1$, the question becomes a lot harder to treat in a general manner –

I will explain why in section 5.4, illustrating it by looking at what happens in the case when $a = \pm 2$.

5.1 The image of a triangle in the Chow group

Recall first the definition of the generalised triangle elements. We begin as before with a function $f \in K(E)$ with divisor $(f) = d(P) - d(Q)$. We also take an arbitrary integer a, and consider the function f_a with divisor $d(aP) - d(aQ)$ (where the multiplication by a takes place in the group structure of E) – this exists, and is unique up to multiplication by constants. Then we defined the generalised triangle configuration $\alpha_{a,f}$ to be

$$\alpha_{a,f} := (E \times \{aP\}, f) + (\{Q\} \times E, f_a) + (\Delta_a, f^{-1}).$$

We now want to consider what the image of such an element will be under the boundary map in the localisation sequence, in order to perform the same computations as we have already done in the previous three chapters for the case $a = 1$. We shall discuss what happens for general a, and see the difficulties which arise, before restricting to the special case where a is either 1 or -1.

For the "horizontal" and "vertical" components of this element, it is easy to generalise the previous results from the $a = 1$ case. On the horizontal, the only difference is that P has been replaced by aP. If we make sure that we organised our numbering of the components so that the identity component is labelled as D_0, then the natural map from the fibre of the Néron model to its group of components agrees with that taking a point on D_k to $k \in \mathbb{Z}/n\mathbb{Z}$, so in particular, if the reduction of P lies on D_i then the reduction of aP lies on D_{ai} (the indices are to be interpreted as being classes in $\mathbb{Z}/n\mathbb{Z}$). Hence the image of the horizontal component will be:

$$\sum_k a_k^H (B_{k,ai} + C_{k+1,ai} + D_{k,ai}).$$

As for the vertical, the curve has remained the same, but the function has changed. However, f_a has the defining property that its divisor is $d(aP) - d(aQ)$, so that the elements i and j, and therefore Δ too, will here be a times what they were before. This will change the relationships among the various a_k^D (as they were crucially dependent on these quantities, and in particular on Δ), but it will not make any difference to the image of the "vertical" component in terms of these (as the j in the following expression comes from the curve on which the function lies, rather than the function itself). Thus, this contribution will be (formally) exactly as it was before:

$$\sum_k a_k^V (A_{jk} + C_{j,k-1} + D_{jk}).$$

The "diagonal" contribution, however (which now comes from Δ_a rather than the diagonal itself), causes considerably more problems. Firstly, the fibre of the Néron model is isomorphic to $k^* \times \frac{\mathbb{Z}}{n\mathbb{Z}}$ as a group – it is certainly an extension of $\mathbb{Z}/n\mathbb{Z}$ by k^* (see [22]), but all such extensions are trivial due to lemma 1 in section 23 of [16]. From this it follows, since the reduction map is a group homomorphism, that the image in the fibre of the closure of Δ_a must lie only in those components (l, m) where $m = al$, and that on those components (each naturally isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) it coincides with the curve $y = x^a$, where y and x are the two co-ordinate functions. But here a problem arises, as may be seen for example by considering a case such as $a = 2$ and $n = 2$. For here the image lies only in the $(0,0)^{th}$ and $(1,0)^{th}$ components, and is $y = x^2$ in each – if we try to draw the intuitive picture that arises here, we see that it is disconnected; there being no path in this image from (∞, ∞) on the $(0,0)^{th}$ component, which is the same as $(0, \infty)$ on the $(1,0)^{th}$ component, to $(0,0)$ on the $(1,0)^{th}$ component. But this would appear to be a contradiction, for it is clear that the image of Δ_a in the fibre of our regular model for $E \times E$ must be connected.

The resolution of this apparent contraduction comes from realising that our argument for this being the image of Δ_a was purely group-theoretical, and thus comes from considering only the Néron model – but this is not the whole of the regular model which we have been using, as it comes without the singular points on the fibre, in other words the points of intersection of any two adjacent $\mathbb{P}^1$'s. Thus, in our intutitive picture of an n by n grid of $\mathbb{P}^1 \times \mathbb{P}^1$'s, we have so far been ignoring any of the horizontal and vertical "edges" of each of these. And it is clear that, if we are allowed to add some of these back in, we can always produce something connected (for example, by adding the edge $\{0\} \times \mathbb{P}^1$ of the $(1,0)^{th}$ component, in the example above - we'd also have to add the same on the $(0,0)^{th}$ component).

This, of course, isn't a proof that this particular configuration in the special fibre of $\mathcal{E} \times \mathcal{E}$ is the precise image of Δ_2 – in fact, it turns out that these "vertical" edges each have multiplicity two. But it does show that we need to be very careful when $|a|$ is greater than 1, for then there will always be some values for n (and these will occur on some elliptic curves) for which the union of all the required $y = x^a$ curves will not produce a connected picture ("connectedness" occurs if and only if a is either both positive and congruent to 1 mod n, or negative and and congruent to -1 mod n). Indeed, even in the two cases just mentioned, it turns out that we do still need some extra edges in our image – I will demonstrate this in section 5.4 when we look in a bit of detail at the cases $a = \pm 2$. We shall also see there how hard it is to treat all these cases in any kind of generality, which is why, for the main part of this chapter, until then, I am going to restrict to a being ± 1.

It of course still needs to be proved, in the light of what I have just said, that the image of $\Delta_{\pm 1}$ in the fibre does consist of just the curves I am claiming, and no more. Indeed, we have already used this fact in the previous chapters! This follows from the fact that, for any two curves in $E \times E$, their images in the special fibre of the integral model must have the same intersection number as the original curves do inside $E \times E$. Clearly, both the diagonal Δ_1 and the antidiagonal Δ_{-1} have intersection number 1 with any curves of the form $E \times \{P\}$ or $\{P\} \times E$, so these must be preserved on the special fibre. And as the curves we already know about in the image of $\Delta_{\pm 1}$ in the special fibre have the desired intersection numbers with any horizontal or vertical fibre, whereas any new component in either of those directions would add extra intersections, we can conclude that there can be nothing else in this image. (The same argument applied to Δ_a with $|a| > 1$ shows that there must always be such additional curves, even if they are not needed to make the image connected. I shall return to this in section 5.4.)

So, we will now determine the image in the Chow group of the fibre, of the function f^{-1} along either Δ or Δ_{-1}. (We have already done this for Δ in Chapter 3, of course, but I shall go over it again here.) In either case, the closure of $\Delta_{\pm 1}$ in

$\mathcal{E} \times \mathcal{E}$ (that is, before the blowup) will be isomorphic to $\mathcal{E}$ itself, and goes through n of the n^2 singular points of the fibre – those on the "diagonal" for $a > 0$, or the "antidiagonal" for $a < 0$. After blowing up at these points, we will find, just as we did before, that the "vertical" part of the divisor of f (that is, the part which lies in the Chow group of the fibre) takes the form $a_1D_1 + \ldots + a_nD_n + b_1E_1 + \ldots + b_nE_n$, and that the same relations will hold among the coefficients a_k and b_k. So the only thing that may be new is how the D_i and E_i can be expressed in terms of our $4n^2$ generators for the Chow group.

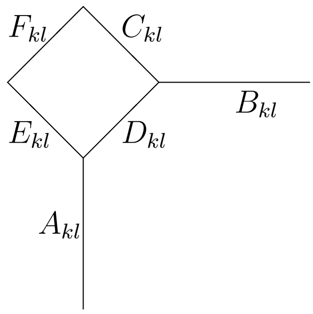
This is relatively easy to do for the D_i – originally we computed the divisor of the function $y - x$ on each "octagon" (x and y being the two co-ordinate functions on the $\mathbb{P}^1 \times \mathbb{P}^1$ of which the octagon is the result of blowing up each of the four corners), and set it to zero, to find that Δ was equivalent in the Chow group to $A + B + C + D + H - F$, where we have adopted the temporary labelling of the sides of the octagon as A to H going clockwise from A at the top. For Δ_{-1}, we do the same with the function $xy - 1$. We can compute that this function has a simple zero along the curve we want, simple poles along A and C, and a double pole along B. Hence the curve $y = x^{-1}$ is equivalent in the Chow group to $A + 2B + C$.

That leaves us with the E_i, which we recall are the strict transforms of the images of these curves in the fibre after having blown up the model at each of the n^2 singular points of its fibre. I will leave a proof until the appendix, but the result is the following

Proposition 5.1 The closure of Δ_a in $\mathcal{X}_{\mathcal{E}}$, when intersected with one of the exceptional divisors, gives a curve which is equivalent, in the Chow group, to something of the form D_{kl}, for $a = 1$, and C_{kl} if $a = -1$.

(The same result in fact holds for $\Delta_{\pm 2}$, with results $2D_{kl}$ and $2C_{kl}$, at least when $n = 1$, and I believe that an analagous result holds for all a, although I have been unable to prove it. This will be discussed further in the appendix.)

Hence, if we revisit our usual diagram from Chapter 3:



we find that the contribution from the $\Delta_{\pm 1}$, in a connected triangle, is:

$$\sum_{k=1}^{n} \left(-a_k^D(A_{k+1,k} + B_{kk} + C_{k+1,k} - C_{k,k-1} + D_{kk} + D_{k+1,k-1}) - b_k^D D_{k+1,k}\right)$$

for Δ_1 (the same result as we used in Chapter 3), and

$$\sum_{k=1}^{n} \left(-a_k^D(A_{k+1,-k} + B_{k,-k} + 2C_{k+1,-k} - b_k^D C_{k,-k}\right)$$

for Δ_{-1}. (Note that the k and l in the diagram satisfy $k = l$ for Δ_1, and $k = -l$ for Δ_{-1}, as we assume that the diagonal or antidiagonal in question goes through the "octagon" in the lower-right portion of the diagram. Other conventions would yield the same sum, but notated slightly differently.)

5.2 An integrality result

We are now going to use a refinement of the methods of Chpater 3, in order to prove the following theorem about the integrality of a triangle configuration $\alpha_{f,\pm 1}$ in $K_1^{(2)}(E \times E)$ at a prime of split multiplicative reduction. (Δ here means the same thing as it did in Chapters 3 and 4.)

Theorem 5.2 Let $\alpha_{f,a}$ be a triangle, with $a = \pm 1$.

- If $a = -1$, then, for potential integrality, we need either $\Delta = 0$ or $n = 2$ and $\Delta = 1$. Further, all triangles satisfying the latter condition are potentially integral.

- If $a = 1$, then there is no restriction on Δ – for $n = 2$ or 3, at least, there exist integral triangles for $\Delta \neq 0$.

The proof of this theorem, except for the statements about the existence of integral triangles with non-zero Δ, is the subject of the rest of this section. In the next section I shall prove the existence statements, and give some examples.

Before beginning the proof, we note the following important corollary:

Corollary 5.3 Any integral triangle involving a Δ_1 or Δ_{-1} has the same regulator image as an integral decomposable element.

Proof We shall see in the proof of the theorem below that we must have $a_i^\bullet = \frac{\Delta(n-\Delta)d}{2n}$ for $\bullet = H$, V or D. Thus, Proposition 4.9 tells us that all three of the functions we are using in $\alpha_{f,\pm1}$ have norms with order 0 at the prime we are interested in. The rest of the argument is the same as in Chapter 4. (Note that, when we take apply the regulator of $\alpha_{f,\pm1}$ to the four differential forms involved, some will now involve f_{-1} rather than f, but as we have just observed this also has a norm whose order is 0 at the relevant prime, so there is no change to the result.) □

We start the proof of Theorem 5.2, as before, by using the formulae of the previous section to tell us what the image of our triangle in $K_1^{(2)}(E \times E)$ in the Chow group of the fibre of the regular model. As in Chapter 3, I shall do this by giving the coefficients of each of the $4n^2$ generators, which can be read off from the expressions in the previous section. I will split he whole proof up into the two cases Δ_1 and Δ_{-1} – the two cases are very similar, but the precise formulae differ from each other, so it is easier to deal with them separately. So, for now, we assume we are dealing with Δ_1 (here we are merely redoing what was already done in Chapter 3, but in the next section we shall use these formulae to prove the new result in Proposition 5.1. The coefficients here are then:

$$\begin{aligned}
a_{kl} &= \delta_{i+\Delta,k} a_l^V - \delta_{k,l+1} a_l^D \\
b_{kl} &= \delta_{il} a_k^H - \delta_{kl} a_k^D \\
c_{kl} &= \delta_{il} a_{k-1}^H + \delta_{i+\Delta,k} a_{l+1}^V - \delta_{k,l+1}(a_l^D - a_k^D) \\
d_{kl} &= \delta_{il} a_k^H + \delta_{i+\Delta,k} a_l^V - \delta_{kl} a_k^D - \delta_{k-1,l+1} a_{k-1}^D - c_a \delta_{k,l+1}(a_k^D + a_l^D).
\end{aligned}$$

We will now use the same $4n$ necessary conditions for integrality as we did in Chapter 3, and see what restrictions these place upon integral "connected" triangles. We find, for each k

$$\begin{aligned} 0 &= \sum_m a_{mk} \\ &= \sum_m \delta_{i+\Delta,m} a_k^V - \sum_{m,k+1} a_k^D \\ &= a_k^V - a_k^D, \end{aligned}$$

so $a_k^V = a_k^D$;

$$\begin{aligned} 0 &= \sum_m b_{km} \\ &= \sum_m \delta_{im} a_k^H - \sum_m \delta_{km} a_k^D \\ &= a_k^H - a_k^D, \end{aligned}$$

so $a_k^H = a_k^D$;

$$\begin{aligned} 0 &= \sum_m c_{m,m+k} \\ &= \sum_m \delta_{i,m+k} a_{m-1}^H + \sum_m \delta_{i+\Delta,m} a_{m+k+1}^V - \sum_m \delta_{m,m+k+1}(a_{m+k}^D - a_m^D) \\ &= a_{i-k-1}^H + a_{i+\Delta+k+1}^V. \end{aligned}$$

Using the previous two sets of equations, this last can be rewritten as

$$0 = a_{i-k-1}^D + a_{i+\Delta+k+1}^D. \tag{5.1}$$

I shall omit for now the equations obtained from the condition $\sum_m d_{m,k-m} = 0$, as these are more complicated, and are not necessary for the first part of the proof of Theorem 5.2.

Notice that if we sum all n of these relations together, we obtain that $\sum_m a_m^D = 0$, which forces $a_i^D = \frac{\Delta(n-\Delta)d}{2n}$, due to (3.3). This proves the $a = 1$ case of:

Proposition 5.4 For any potentially integral triangle $\alpha_{f,a}$ with $a = \pm 1$, we have $a_i^D = \frac{\Delta(n-\Delta)d}{2n}$, or equivalently, $\sum_m a_m^D = 0$.

(Of course, we have already proved the $a = 1$ case in Chapter 3; the $a = -1$ case will follow from the equivalent formulae in that case.)

In combination with the relations given between the a_k^H, a_k^V and a_k^D, this determines all of the $a_k^{\bullet}$. However, as already pointed out, this is a very long way from determining that all triangle configurations which meet these conditions must be integral.

For the proof of the second part of Theorem 5.2 in the next section, we shall also need to show that, when $a = 1$ and $a_i^D = \frac{\Delta(n-\Delta)d}{2n}$, (5.1), which is equivalent to the assertion that $a_k = -a_{2i+\Delta-k}$ for each k, does necessarily hold, for all k – this I will do next.

Firstly, if $k = i$, we need that $a_i = -a_{i+\Delta}$ – but we already know that $a_i = \frac{\Delta(n-\Delta)d}{2n}$, from which it follows (due to the formulae (3.1) and (3.2)) that $a_{i+\Delta} = -\frac{\Delta(n-\Delta)d}{2n}$, so this holds. We shall establish that it holds for all other k by induction. That is, we assume that $a_k = -a_{2i+\Delta-k}$, and wish to show that $a_{k+1} = -a_{2i+\Delta-k-1}$ – this will be enough to prove the desired result. By the inductive assumption, this is equivalent to showing that $a_k - a_{k+1} = a_{2i+\Delta-k-1} - a_{2i+\Delta-k}$, which we can establish using the formulae in Proposition 3.11. Using those formulae, we put k into the range $i, \ldots, i+n-1$, and first suppose that $i \leq k \leq i+\Delta-1$, so that the first of these two differences is equal to $\frac{(n-\Delta)d}{n}$. Our assumptions on k tell us that $i \leq 2i+\Delta-k-1 \leq i+\Delta-1$, so that the second difference also is $\frac{(n-\Delta)d}{n}$. This argument clearly works in both directions, and thus completes the inductive proof.

Next, we will deal with the Δ_{-1} case. We proceed as before – assuming Conjecture 5.1, the formulae for the co-efficients, which we can read off from the expressions in the previous section, are:

$$\begin{aligned}
a_{kl} &= \delta_{i+\Delta,k} a_l^V - \delta_{k-1,-l} a_{k-1}^D \\
b_{kl} &= \delta_{-i,l} a_k^H - \delta_{k,-l} a_k^D \\
c_{kl} &= \delta_{-i,l} a_{k-1}^H + \delta_{i+\Delta,k} a_{l+1}^V - 2\delta_{k-1,-l} a_{k-1}^D + \delta_{k,-l}(a_k^D + a_{k+1}^D) \\
d_{kl} &= \delta_{-i,l} a_k^H + \delta_{i+\Delta,k} a_l^V.
\end{aligned}$$

Thus we obtain the conditions:

$$\begin{aligned} 0 &= \sum_m a_{mk} \\ &= \sum_m \delta_{i+\Delta,m} a_k^V - \sum_m \delta_{m-1,-k} a_{m-1}^D \\ &= a_k^V - a_{-k}^D, \end{aligned}$$

so $a_k^V = a_{-k}^D$;

$$\begin{aligned} 0 &= \sum_m b_{km} \\ &= \sum_m \delta_{-i,m} a_k^H - \sum_m \delta_{k,-m} a_k^D \\ &= a_k^H - a_k^D, \end{aligned}$$

so again $a_k^H = a_k^D$;

$$\begin{aligned} 0 &= \sum_m d_{m,k-m} \\ &= \sum_m \delta_{-i,k-m} a_m^H + \sum_m \delta_{i+\Delta,m} a_{k-m}^V \\ &= a_{k+i}^H + a_{k-i-\Delta}^V. \end{aligned}$$

Again, this can be rewritten using the previous two sets of equations, as

$$0 = a_{i+k}^D + a_{i+\Delta-k}^D.$$

(Once more, one of the four sets of relations has been left out, to be returned to in the following section.)

Note in particular the first of these conditions: $a_k^V = a_{-k}^D$, for all k. Recall, from the formulae (3.1) and (3.2), that the greatest of the $a_k^\bullet$ is always at $k = i$, while the smallest is at $k = i + \Delta$ – this applies whether $\bullet$ is H, V or D. The only way for this to be consistent with $a_k^V = a_{-k}^D$ (other than to have all a_k zero, and thus $\Delta = 0$) is to have $i = -i$ and $i + \Delta = -i - \Delta$. Recall that these are equalities in $\mathbb{Z}/n\mathbb{Z}$, so they have solutions not only when $i = i + \Delta = 0$ (which gives us the trivial case, when $\Delta = 0$), but also, when n is even, when either i or $i+\Delta$ is $\frac{n}{2}$. (Note that there are no other solutions when n is odd.) Thus it is possible to have $\Delta = \frac{n}{2}$, although then we shall also require i to be 0 or $\frac{n}{2}$.

Note that, when $a = -1$, $\Delta = \frac{n}{2}$, $i = 0$ or $\frac{n}{2}$, and $a_i^D = \frac{\Delta(n-\Delta)d}{2n}$, then once more all the conditions dealt with so far are automatically met – the necessary induction argument is identical to that for the case $a = 1$.

Note that we have now proved all of Theorem 5.2 apart from the assertions that integral elements do exist with $\Delta \neq 0$, and that integral triangles with $a = -1$ and $\Delta = \frac{n}{2}$ (n even) must have $n = 2$. That is what we shall do in the next section.

5.3 Further conditions for integrality; completion of the proof

We are now going to return to the issue which I mentioned in the introduction to this chapter – that so far we only have necessary conditions for integrality, not sufficient ones. The problem here is simply one of linear algebra – if we have a general element

$$\sum_{k,l}(a_{kl}A_{kl} + b_{kl}B_{kl} + c_{kl}C_{kl} + d_{kl}D_{kl})$$

of the Chow group, what is a set of linear conditions on the coefficients a_{kl}, b_{kl}, c_{kl}, d_{kl} which will be both necessary and sufficient for the above expression to lie in the span of the known relations:

$$A_{k,l+1} + C_{kl} + D_{k,l+1} - A_{k+1,l+1} - C_{k+1,l+1} - D_{k+1,l}$$

$$B_{kl} + C_{kl} + D_{k+1,l} - B_{k,l+1} - C_{k+1,l+1} - D_{k,l+1},$$

as k and l run over all of $\mathbb{Z}/n\mathbb{Z}$?

As there are $4n^2$ generators and $2n^2 - 2$ independent relations (as we saw in Chapter 3), any linearly independent set of necessary and sufficient conditions will have size $2n^2 + 2$. So far, we have only used $4n$ – that $\sum_m a_{mk} = \sum_m b_{km} = \sum_m c_{m,m+k} = \sum_m d_{m,k-m} = 0$, for each k. Note that these $4n$ conditions are at least linearly independent (each coefficient appears in exactly one of them), but they are clearly not enough, as $4n$ is strictly less than $2n^2 + 2$ whenever n is more than 1 ($2n^2+2-4n = 2(n-1)^2$). This means that, using these conditions alone, we cannot say for sure whether or not any given linear combination of the generators is in fact zero, and thus cannot know for sure that any given triangle is actually integral (except in trivial cases like when all of the coefficients are zero, which happens when $\Delta = 0$ and we ensure all the a_k are zero). For $n = 1$, these conditions are enough – reflecting the fact that the Chow group in this case is simply free on the four generators A, B, C and D – but then we always have $\Delta = 0$ anyway, so we need to look for larger values of n if we want examples of non-trivial (meaning $\Delta > 0$) integral triangles.

So we need to find some further necessary conditions – ideally all $2(n-1)^2$ of the "missing" ones. n^2 cmore are given in the following proposition:

Proposition 5.5 If $\sum_{k,l}(a_{kl}A_{kl} + b_{kl}B_{kl} + c_{kl}C_{kl} + d_{kl}D_{kl})$ is zero in the codimension 1 Chow group, then we must have, for each pair (k, l) of indices,

$$a_{kl} + a_{k,l+1} + b_{k-1,l} + b_{kl} = c_{kl} + d_{kl}.$$

Proof We need only check that this applies to each of the $2n^2$ relations given above (and labelled as X_{kl} and Y_{kl} in Chapter 3.) In the relation X_{mn}, for example, we see that:

$$\begin{aligned}
a_{kl} &= \delta_{l,n+1}(\delta_{km} - \delta_{k,m+1}) \\
b_{kl} &= 0 \\
c_{kl} &= \delta_{km}\delta_{ln} - \delta_{k,m+1}\delta_{l,n+1} \\
d_{kl} &= \delta_{km}\delta_{l,n+1} - \delta_{k,m+1}\delta ln,
\end{aligned}$$

from which it can be seen that the two sides of the proposed condition are indeed equal. The same procedure can be carried out for Y_{mn}, where again we find that the condition holds. □

We now have $n^2 + 4n$ conditions, in total, and would like to investigate these for independence. Firstly, it is clear that, if we sum together all n^2 of the conditions in the preceding proposition, we get something which is depend on the $4n$ conditions which we already knew about (as they tell us that the sums over all k and l of the a_{kl}, b_{kl}, c_{kl} and d_{kl} are all zero). So, we have at most $n^2 + 4n - 1$ conditions. These are still not enough in general, of course – for $2n^2 + 2$ can easily be seen to be greater than $n^2 + 4n - 1$ for all positive integers n greater than 3. However, it turns out that we do have enough for the cases $n = 2$ and $n = 3$:

Proposition 5.6 The conditions listed in Propositions 3.14 and 5.5 are sufficient for $\sum_{k,l} (a_{kl}A_{kl} + b_{kl}B_{kl} + c_{kl}C_{kl} + d_{kl}D_{kl})$ to be zero, when $n = 2$ or $n = 3$.

(They are also sufficient when $n = 1$, but this case is trivial and of no interest to us.)

Proof First, suppose $n = 2$. Then the four "new" relations are:

$$\begin{aligned}
a_{00} + a_{01} + b_{00} + b_{10} &= c_{00} + d_{00} \\
a_{00} + a_{01} + b_{01} + b_{11} &= c_{01} + d_{01} \\
a_{10} + a_{11} + b_{00} + b_{10} &= c_{10} + d_{10} \\
a_{10} + a_{11} + b_{01} + b_{11} &= c_{11} + d_{11}.
\end{aligned}$$

If we take the sum of the first and fourth of these, or the second and third, we can see that we obtain $\sum_{k,l} a_{kl} + \sum_{k,l} b_{kl} = c_{00} + c_{11} + d_{00} + d_{11}$ in one case, while in the other the right-hand side is $c_{01} + c_{10} + d_{01} + d_{10}$. The conditions in Proposition 3.14 tell us that both right-hand sides, and the sum on the left, are zero. Thus we can throw away the third and fourth of the list above, and lose no information. This gives us 10 conditions in total – the 8 previous ones, and 2 new ones. In fact, these 10 are now independent – to see this, note that a_{10}, for example, occurs in only one of them ($a_{00} + a_{10} = 0$), so that one must not be involved in any linear dependence relation. Further, a_{11}, c_{10}, c_{11}, d_{10} and d_{11} are each only involved in one condition, thereby eliminating those from any possible linear dependence. That leaves us with just four – the first two in the four listed above, and the two involving just the b_{kl}. But now c_{00} and c_{01}, for example, also occur in only one, eliminating the two "new" relations, and meaning that any linear dependence has to be among just the two conditions $b_{00} + b_{01} = 0$ and $b_{10} + b_{11} = 0$. Clearly, there are no such dependencies.

Thus, we find that we have 10 independent conditions for $n = 2$ – and the total number we are looking for is $2n^2 + 2$, which equals 10 when $n = 2$. This proves sufficiency for this case.

The argument when $n = 3$ is very similar, but takes up more space, so it will be most economical if I leave it to the reader. Here, we have $12 + 9 = 21$ conditions to start with, and there turns out to be just one dependence, which is the one I mentioned straight after introducing the new conditions. $\square$

Thus, in the cases $n = 2$ and $n = 3$ we are now in a position to decide precisely what conditions are needed on a triangle configuration in order for it to be integral – and thus to complete the proof of Theorem 5.2.

We will first see what these "new" relations say in general about the integrality of a triangle configuration, though – again splitting up into the two cases $a = 1$ and $a = -1$. First, if $a = 1$, then, in our condition $a_{kl} + a_{k,l+1} + b_{kl} + b_{k-1,l} = c_{kl} + d_{kl}$, we find that, for the image of a triangle configuration, the left-hand side is (upon collecting like terms):

$$\delta_{il}(a^H_{k-1} + a^H_k) + \delta_{i+\Delta,k}(a^V_l + a^V_{l+1}) - (\delta_{k,l+1} + \delta_{k,l+2})a^D_{k-1} - (\delta_{kl} + \delta_{k,l+1})a^D_l.$$

The right-hand side gives exactly the same expression, after cancelling like terms, and therefore our "new" relations give us no new information when $a = 1$.

So, let us recap the situation when $a = 1$. We wish to show that, for $n = 2$ or 3, there exist integral triangles with $\Delta > 0$. I will again only go through the simpler $n = 2$ case in detail, as $n = 3$ is very similar. So we need to check the case $n = 2$ and $\Delta = 1$. We know already that we must have $a^D_i = \frac{\Delta(n-\Delta)d}{2n} = \frac{d}{4}$, and hence $a^D_{i+1} = -\frac{d}{4}$,, and further that the same formulae hold for the a^H_k and a^V_k as well. These are enough to satisfy six of the ten required conditions for $n = 2$, and nine of the twenty required for $n = 3$, namely the ones we dealt with in the previous section. Further, we have just seen that the n^2 "new" conditions give no further information – thus there are only two conditions left to check, which are those which I omitted in the previous section, coming from $\sum_m d_{m,k-m} = 0$.

Recall that we had, in general, for $a = 1$,

$$d_{kl} = \delta_{il}a^H_k + \delta_{i+\Delta,k}a^V_l - \delta_{kl}a^D_k - \delta_{k-1,l+1}a^D_{k-1} - \delta_{k,l+1}(a^D_k + a^D_l).$$

Using the above information, we can easily compute each of these four coefficients:

$$\begin{aligned} d_{ii} &= a_i^H - a_i^D - a_{i+1}^D \\ &= \frac{d}{4} \\ d_{i,i+1} &= -(a_i^D + a_{i+1}^D) \\ &= 0 \\ d_{i+1,i} &= a_{i+1}^H + a_i^V - (a_{i+1}^D + a_i^D) \\ &= 0 \\ d_{i+1,i+1} &= a_{i+1}^V - a_{i+1}^D - a_i^D \\ &= -\frac{d}{4}. \end{aligned}$$

So we have that $d_{ii} + d_{i+1,i+1} = d_{i,i+1} + d_{i+1,i} = 0$, as required. As we have now gone through what we know to be all of the necessary conditions when $n = 2$ (and $n = 3$, although I have not written the necessary computations down), this completes the proof of the theorem in the $a = 1$ case.

We now need to go through everything again when $a = -1$, starting with the "new" conditions. Here, we find that the left-hand side is:

$$\delta_{-i,l}(a_{k-1}^H + a_k^H) + \delta_{i+\Delta,k}(a_l^V + a_{l+1}^V) - (\delta_{k-1,-l} + \delta_{k,-l})a_{k-1}^D - (\delta_{k,-l} + \delta_{k,1-l})a_{-l}^D,$$

while the right-hand side is:

$$\begin{aligned} \delta_{-i,l}(a_{k-1}^H + a_k^H) + \delta_{i+\Delta,k}(a_l^V + a_{l+1}^V) - (\delta_{k-1,l} + \delta_{k,-l})a_{k-1}^D - \delta_{k-1,-l}a_{k-1}^D \\ +\delta_{k,-l}(a_{k-1}^D + a_k^D + a_{k+1}^D) \quad . \end{aligned}$$

Equating the two sides thus gives the relation:

$$0 = \delta_{k,-l}(a_{k-1}^D + 2a_k^D + a_{k+1}^D).$$

Recall that this holds for *all* k and l. Since, for each k, we can always choose $-k$ for l, we see that for all k we must have that $a_{k-1}^D + 2a_k^D + a_{k+1}^D = 0$.

We assume that $\Delta \neq 0$ as usual, which allows us to deduce that $a_{i+1}^D = a_i^D - \frac{\Delta d}{n}$, while $a_{i-1}^D = a_i^D - \frac{(n-\Delta)d}{n}$ (formulae (3.1) and (3.2)). Substituting these into the above condition (with $k = i$), we find that we must have $a_i^D = \frac{d}{4}$. But we already know that we also need $a_i^D = \frac{\Delta(n-\Delta)d}{2n}$, so n and Δ must satisfy the condition $n = 2\Delta(n - \Delta)$. This can also be written as $(2\Delta - 1)n = 2\Delta^2$, and since n is an integer, we must have that $2\Delta - 1$ divides $2\Delta^2$. But then $2\Delta - 1 | 2\Delta^2 - \Delta(2\Delta - 1) = \Delta$, which is clearly only possible if $\Delta = 1$ – and this, in turn, forces $n = 2$. This confirms another of the statements of Theorem 5.2.

It remains only to check that, when $a = -1$, $n = 2$, $\Delta = 1$, and $a_i^H = a_i^V = a_i^D = \frac{d}{4}$, all ten of our necessary and sufficient conditions for integrality are met – we have so far dealt with seven of them. Two of the remaining ones are that $d_{ii} + d_{i+1,i+1} = d_{i,i+1} + d_{i+1,i} = 0$, which can be checked just as we did for $a = 1$. The general formulae, for $a = -1$, are:

$$d_{kl} = \delta_{-i,l} a_k^H + \delta_{i+\Delta,k} a_l^V .$$

Plugging in the known conditions, we find that $d_{ii} = a_i^H = \frac{d}{4}$, $d_{i,i+1} = 0$, $d_{i+1,i} = a_{i+1}^H + a_i^V = 0$ and $d_{i+1,i+1} = a_{i+1}^V = -\frac{d}{4}$. So these two conditions are indeed met.

This finally completes the proof of Theorem 5.2.

5.3.1 Some examples

Strictly speaking, actually, we are not quite done with the proof of Theorem 5.2. The above results guarantee that we can find non-trivial integral triangles on certain elliptic curves of split multiplicative reduction of type I_2 and I_3, providing we can find a function on such a curve with divisor $d(P) - d(Q)$, for which P and Q reduce to different components of the fibre of the Néron model. It does still remain to show that there do exist elliptic curves of the required type, which do admit such functions! One would expect there to be many, and this does indeed turn out to be the case, but we still need to exhibit one. In this section, I will give a couple of the many examples.

One can find several suitable examples of such curves, defined over $\mathbb{Q}$, with low conductor, and conveniently small coefficients, from the tables on John Cremona's website [6]. One example is the curve listed as 17a2:

$$y^2 + xy + y = x^3 - x^2 - 6x - 4.$$

This curve has bad reduction only at the prime 17, where the reduction is multiplicative, since the conductor is 17 and not any higher power of 17. Further, since the discriminant of the curve can be computed to be $289 = 17^2$, the precise reduction type is I_2.

Rewriting the above Weierstrass equation as $(y+\frac{1}{2}x+\frac{1}{2})^2 = x^3 - \frac{3}{4}x^2 - \frac{11}{2}x - \frac{15}{4} = (x-3)(x+1)(x+\frac{5}{4})$, we see that there are three 2-torsion points, at $(3,-2)$, $(-1,0)$ and $(-\frac{5}{4},\frac{1}{8})$. (In fact, these three points, together with the identity, form the whole group of torsion points defined over $\mathbb{Q}$, which is therefore isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.) So we can find three functions of the desired type, namely $x-3$ (with divisor $2(3,-2)-2(O)$), $x+1$ (with divisor $2(-1,0)-2(O)$) and $x+\frac{5}{4}$ (divisor $2(-\frac{5}{4},\frac{1}{8})-2(O)$).

If we reduce the curve modulo 17, we see from the above form of the equation that we obtain the equation $(y-8x-8)^2 = (x-3)^2(x+1)$, from which it follows that the singular point of the reduction is at $(3,-2)$. (This also allows us to check that the reduction is split – applying a translation to the curve mod 17 so that the singularity is at the origin, it is $(y-2)^2+(x+3)(y-2)+(y-2) = (x+3)^3-(x+3)^2-6(x+3)-4$, or $y^2+xy = x^3+2x^2$, so the tangents at the singular point are given by the quadratic $0 = y^2+xy-2x^2$, which splits over the prime field as $(y-x)(y+2x)$.) Since the reduction type is I_2, The Néron model is obtained by a single blowup of $\mathrm{Spec}\frac{\mathbb{Z}[x,y]}{(y^2+xy+y-x^3+x^2+6x+4)}$ (or, more precisely, its closure inside the projective plane over $\mathbb{Z}$) at this singular point – one of the two resulting components is the exceptional divisor, and the other consists of all the points on the fibre before the blowup other than the singular point. This tells us that, in order to tell which of the two components of the fibre a rational point on the curve reduces to, we need only take the reduction mod 17 of its co-ordinates, and see whether or not the result is $(3,-2)$. If it is not, the reduction lies on the identity component (as the point at infinity on E, acting as the identity for the group structure, reduces to this component); otherwise, it lies on the other component (the exceptional divisor of the blowup).

Knowing this, we can see that any triangle built on the functions $x-3$ or $x+\frac{5}{4}$ will have $\Delta = 1$, as the poles of these functions (namely O, in both cases) reduce to the identity component, while the zeros reduce to the other one. So, for example, the triangle:

$$\alpha_{x-3} = (E \times \{(3,-2)\}, x-3) + (\{O\} \times E, x-3) + (\Delta, (x-3)^{-1})$$

is potentially integral.

We shall also figure out how to get an *actually* integral element from this triangle. Recall that we will have actual integrality only when $a_i = \frac{d}{4}$, which here is $\frac{1}{2}$. So we need to choose a multiple of the function $x-3$ which has order $\frac{1}{2}$ along the exceptional divisor, and $-\frac{1}{2}$ along the identity component. Of course, these fractional orders are not possible, which is why the definition of potential integrality was extended to allow for some multiple of the element being "adjustable" in this matter. So we will replace $x-3$ by $(x-3)^2$ (thereby taking $\alpha_{(x-3)^2} = 2\alpha_{x-3}$), and need this to have orders 1 and -1 along the exceptional divisor and the identity component, respectively.

It is clear that, before the blowup which produced the Néron model, the function $(x-3)^2$ had order 0 along the fibre, as it neither vanishes nor blows up everywhere on the fibre. Hence, after the blowup, this function still has order 0 along the identity component. It must therefore have order 2 along the exceptional divisor, as we know that the $a_{i+1} = a_i - \frac{\Delta(n-\Delta)d}{2n}$, where here $n = 2$, $\Delta = 1$ and $d = 4$ (since our function is now $(x-3)^2$), so $a_i = a_{i+1} + 2$ – and a_i and a_{i+1} are the orders of $(x-3)^2$ along respectively the exceptional divisor (the component containing the reduction of the zero, $(3,-2)$) and the identity component. Thus, we can conclude that the triangle configuration

$$(E \times \{(3,-2)\}, \frac{1}{17}(x-3)^2) + (\{O\} \times E, \frac{1}{17}(x-3)^2) + (\Delta, 17(x-3)^{-2})$$

is integral (at the prime 17), despite having $\Delta > 0$. ($\alpha_{\frac{1}{17}(x+\frac{5}{4})^2}$ is also integral, by exactly the same argument.)

A slightly more complex example comes on the curve listed as 14a6, which is the one given by the Weierstrass equation

$$y^2 + xy + y = x^3 - 11x + 12.$$

This has conductor 14, and has reduction of type I_2 at the prime 7 (the discriminant is $98 = 2 \times 7^2$, so it has type I_1 reduction at 2). The singularity this time, one can check, occurs at $(0,3)$ on the fibre, and the reduction is split multiplicative.

The torsion subgroup this time is cyclic of order 6, a generator being $(0,3)$. Notice that this 6-torsion point reduces to the non-identity component of the fibre of the Néron model, while O of course reduces to the identity component – thus, the triangle built from a function with divisor $6(0,3)-6(O)$ (which necessarily exists, since $(0,3)$ is a 6-torsion point) will once again have $\Delta = 1$.

It is easy to find such a function – one simply keeps on computing multiples of $(0,3)$ on E, and note down the various straight lines that occur in the process; these give functions whose divisors give the desired $6(0,3)-6(O)$ in some linear combination. Here, for example, we find the following divisors:

$$\begin{aligned}(2x+y-3) &= 2(0,3)+(2,-1)-3(O)\\ (x-2) &= (2,-1)+(2,-2)-2(O)\\ (3x-y-4) &= 3(2,-2)-3(O),\end{aligned}$$

and thus the function

$$f(x,y) = \frac{(2x+y-3)^3(3x-y-4)}{(x-2)^3}$$

has the desired divisor.

Thus, with f as just defined, α_f is potentially integral. This time, for integrality we need $a_i = \frac{3}{2}$, and $a_{i+1} = -\frac{3}{2}$, which means that $\alpha_{\frac{1}{7^3}f^2}$ will be integral.

Finally, we can use this same function to find an integral triangle of the form $\alpha_{-1,f'}$, where f' is some function closely related to f above. Recall the definition of $\alpha_{a,f}$:

$$\alpha_{a,f} := (E \times \{aP\}, f) + (\{Q\} \times E, f_a) + (\Delta_a, f^{-1}),$$

where, if $(f) = d(P) - d(Q)$, f_a is a function with divisor $d(aP) - d(aQ)$. So, for this particular function f, we are looking for a function f_{-1} whose divisor is $6(-(0,3)) - 6(-O)$, or $6(0,-4) - 6(O)$. We can find this function in exactly the same way we found f; it turns out that one choice is:

$$f_{-1}(x,y) = \frac{(x-y-4)^3}{(3x-y-4)}.$$

Then we know, by the main theorem (Theorem 5.2) that $\alpha_{a,f}$ is potentially integral – and the proof tells us that in order to do this, we need to use a multiple of f with order $-\frac{3}{2}$ along the identity component, and the same with f_{-1}. As these orders are again zero for the functions I just wrote down, it follows that

$$(E \times \{(0,-4)\}, \frac{1}{7^3}f^2) + (\{O\} \times E, \frac{1}{7^3}(f_{-1})^2) + (\Delta_{-1}, 7^3(f^{-1})^{-1})$$

is also integral.

Our theorem also tells us that any triangle with $a = 1$, on a curve with reduction type I_3 at a given prime, is potentially integral at that prime. There are once again many examples of such curves – one is the curve 14a1 with equation $y^2 + xy + y = x^3 + 4x - 6$, whose torsion subgroup is also cyclic of order 6, generated by $(9, 23)$.

Needless to say, more examples could easily be found if needed. (Although I do not know whether there are finitely or infinitely many.)

5.4 $|a| > 1$

In the original version of this thesis, this final chapter included an examination of individual triangles for any value of a (subject to the "connectedness" condition I mentioned in section 5.1), and Theorem 5.2 included a statement about such triangles (I thought I had proved that no such triangles could be integral, except when $\Delta = 0$). This result might well still be correct – I have no real intuition about whether or not it ought to be – but the proof I had given was quite incorrect, as was pointed out to me by my thesis examiner Dr. Matt Kerr. I shall in this section give an indication of where the difficulties lie in attempting to generalise the above results to this situation.

Many of the computations in this chapter would in fact be easy to generalise to arbitrary values of a, but the problem comes before any of this, in working out the image of the configuration – and, of course, the Δ_a part in particular – in the Chow group. We already know that, in those components of the fibre which Δ_a passes through (which are all copies of $\mathbb{P}^1 \times \mathbb{P}^1$), we obtain the curve $y = x^a$, in appropriate co-ordinates, and it is relatively easy to work out what happens to these, in terms of the A_{kl}, B_{kl}, C_{kl} and D_{kl}, when we blow up at each of the four corners, to produce one of our "octagons". Furthermore, one can also see what the image of Δ_a in the exceptional divisors of each of the necessary blowups must be (at least for $a = \pm 2$); see the comment immediately following Proposition 5.1.

The difficulty comes because it turns out that, when $|a| > 1$, there are *always* extra components – "horizontal" and "vertical" ones (in fact, always "vertical" as we shall see) – before the blowup has taken place. This is clear (and was already remarked upon) if our triangle was not "connected"; what is less clear (but still true) is that the same holds even in the connected cases (such as when $n = 1$). I will now demonstrate this, using the image of Δ_2 in the fibre for a curve of reduction type I_1 (which is therefore connected) – the following arguments can be made more general, but I will only do it for one specific class of curves.

Consider, then, a curve defined over $\mathbb{Q}$ by the Weierstrass equation $y^2 = x^3 + x^2 + p$, where p is an arbitrary odd prime. This curve can be easily seen to have reduction of type I_1 at the prime p, and therefore its regular model $\mathcal{E}$ is simply Spec $\frac{\mathbb{Z}[x,y]}{(y^2-x^3-x^2-p)}$. One can also show (for example by using the duplication formula, given in the appendix) that, in $E \times E$, with affine coordinates x, y, u and v, the equations for Δ_2 include the relation $u(4x^3 + 4x^2 + p) = x^4 - 4x^2 - 8px - 4p$ (and one other involving v, to distinguish Δ_2 from $\Delta_{\pm 2}$, as inverses on a Weierstrass elliptic curve share the same x co-ordinate). This must therefore be the one of the equations for the Zariski closure of Δ_2 inside $\mathcal{E} \times \mathcal{E}$. On the special fibre, we reduce the coefficients modulo p, to obtain, after factorisation, $4x^2(x+1)u = x^2(x+2)(x-2)$. Since the "vertical" curve in our picture of the special fibre has equation $x = y = 0$, we see that this particular equation holds everywhere on that curve. Of course, $x = 0$ forces $y = 0$ too, and with multiplicity two. Therefore, this particular equation, which is in fact the equation for the union of (the Zariski closres of) Δ_2 and Δ_{-2}, as well as picking out the two curves we already know about in the "interior" of our $\mathbb{P}^1 \times \mathbb{P}^1$, also picks out no fewer than four copies of the "vertical" curve $x = y = 0$ ($x = 0$ would give two, but our equation, when restricted to the special fibre, is divisible by x^2). As one would expect, when one puts in the further equation for specifically Δ_2 (or specifically Δ_{-2}) one is left with exactly two of these. Therefore, the "vertical" curve here occurs with multiplicity two in the special fibre of this integral model.

This fits in with the arguments I gave earlier, in section 5.1, about intersections – that the intersection number of the total image of Δ_a in the special fibre with any horizontal or vertical fibre has to equal that of Δ_a with the corresponding curve in the generic fibre. In the generic fibre we clearly have intersection number precisely one with any vertical fibre – the only place where $\{P\} \times E$ intersects Δ_a is at (P, aP). But for a horizontal fibre, which has the form $(E \times \{P\})$, we have intersections with Δ_a at all points (Q, P) for which $aQ = P$. For a given P, it is well-known that, at least over the algebraic closure of K, there are a^2 such points. It is rare, in general, for all of these to be defined over K, but if they are defined over an extension field of degree m, then it can be shown that, while there is only one "true" point of intersection (corresponding to a maximal ideal in the relevant polynomial ring defined over K), the intersection will have degree m, so that the total intersection number remains a^2.

Meanwhile, the curves "$y = x^a$" inside the $\mathbb{P}^1 \times \mathbb{P}^1$ components of the special fibre (I put it in quotes to emphasize that this y and x are different from the y and x in the Weierstrass equation of the elliptic curve) each contribute 1 towards the intersection number with vertical fibres, and $|a|$ towards that with horizontal fibres (as the equation $y = x^a$ has $|a|$ roots, up to multiplicity, for any given non-zero y). Note that the above computations for $\Delta_{\pm 2}$ on a curve of reduction type I_1 are compatible with these numbers – the total vertical intersection is 1, while the horizontal intersection is 4 (two from the "$y = x^{\pm 2}$" curve, and one from each of the two copies of the vertical fibre), as required.

What can be seen from this is that there is no easy way to work out, in general, what the image of Δ_a is going to be in the special fibre of $\mathcal{E} \times \mathcal{E}$, at a prime of multiplicative reduction. At best, the above discussion allows it to be computed for specific values of a and n – but these do not fall easily into general patterns. Suppose, for the moment, that we focus on the specific value $a = 2$, and see what this image will be for various values of n – in particular, which vertical fibres must be included, and with what multiplicities (there can be no horizontal fibres ever occuring, or the intersection number with a vertical fibre would then be greater than 1). For $n = 1$, we have already computed this, finding that the single vertical curve occurs with multiplicity 2. For $n = 2$, the configuration we already have (two copies of a "$y = x^2$" curve) already has the desired intersection number of 4 with any horizontal fibre $E \times \{P\}$ for which P lies in the identity component, but zero if P lies in the other component. So one needs at least one of the two vertical fibres which such a curve would pass through, and they need to have multiplicities totalling four. It is easy to see that, for our image to be connected, both fibres must occur with non-zero multiplicities, and it seems clear that, for reasons of symmetry, both must have multiplicity exactly 2. Simiilar arguments occur for $a = 2$ and any value of n, although the result which I would expect depends on whether n is even or odd. For n odd, I would expect to see n of the n^2 different vertical fibres occuring, all with multiplicity two – these will occur on the "right-hand-side" (that is, $\infty \times \mathbb{P}^1$) in the components $(k, 2k+1)$ for each value of k in $\mathbb{Z}/n\mathbb{Z}$ (thereby connecting the two copies of "$y = x^2$" in the $(k, 2k)$ and $(k+1, 2k+2)$ components). Each horizontal fibre here will intersect exactly one of these vertical fibres, thereby contributing 2 to the intersection number, as well as twice with exactly one of the other curves. For n even, the picture I expect will again have (just as for $n = 2$) n such components, each of multiplicy 2, this time on the "right-hand-sides" of all components of the form $(k, 2k+1)$ and $(k, 2k+1\frac{n}{2})$, where k is now restricted to $0 \leq k < \frac{n}{2}$. If one draws a diagram of this, one will see that this is both connected, and has the correct intersection numbers. Further, this is the only way to fulfil both of these conditions.

Very similar results would be expected for Δ_{-2}, for exactly the same reasons. (I hesitate to describe any of these as actual "results", as I have not done any explicit computations for any examples with $n > 2$ – things soon get extremely messy if one tries, as one needs to perform more and more blowups to produce $\mathcal{E}$ as n increases). Clearly, one could try to redo the computations of earlier in this chapter for the cases $a = \pm 2$, using this information – this is an obvious direction for further work. I had hoped to do something in this direction myself, but the fact that, as can be sean from the previous paragraph, things are rather different for different values of n (at the very least, one would need to divide into separate cases according to whether n is even or odd) means that one must be quite careful before making general assertions.

As for those values of a with $|a| > 3$, these seem to create further problems not encountered with $\Delta_{\pm 2}$. Say we try to determine, using the same methods as above, the image of Δ_3 in the special fibre, for a curve of reduction type I_2. We now need an intersection number of 9 with each horizontal fibre, of which are "$y = x^3$" curves will only provide 3. So, we need a total of six extra intersections from the two vertical fibres which each horizontal fibre will cross (as each crosses exactly one of the other curves). And *any* such configuration will be connected, provided each of the two connecting pairs of vertical fibres have the same multiplicity – but these multiplicities could be 3 and 3, 6 and 0, 0 and 6, or even 1 and 5, and both of our conditions (connectedness, and having the necessary intersections) will be met.

Of course, one could also try to use a symmetry argument here to argue that the multiplicities must all be 3, and this may well be correct (although I am not so convinced that the situation here is so necessarily symmetrical). But, if one looks at the case where $n = 4$, still with $n = 3$, then one can still come up with two possible configurations which meet all the conditions I have set out, for which there seems to be no compelling reason to choose between then. Here, in any given vertical fibre, the ends of the two "$y = x^3$" curves which are contained within it are exactly two units apart, meaning that there are two different and complementary choices for which two of the four component fibres one chooses to connect them with (although in either case, both would have to have multiplicity 3). So one these that there are several further problems to be taken into account before one can try to extend the arguments of this chapter to general triangle configurations.

5.5 Some further unanswered questions

I have now completed the presentation of all of the results of this thesis. In this final section, I wish to mention, and briefly comment on, a few questions related to this work, other than the one discussed in the preious section which I have not been able to answer, whether through lack of ideas or lack of time (and often both). No doubt several of these questions have already occurred to the reader!

5.5.1 General conditions for integrality

Firstly, it would be nice to be able to give, for a single triangle configuration, a set of necessary and sufficient conditions for its integrality. The main theorem of this chapter, Theorem 5.2, comes close to doing this, at least for the limited case when $\Delta = \pm 1$, but still leaves open the question of whether or not triangles with $a = 1$ and $\Delta \neq 0$, or $a = -1$, $n = 2$, $\Delta = 1$, are actually integral for $n > 3$. As already discussed, this would need more conditions among the coefficients of a general element of the Chow group which are necessary for it to be zero. In theory we want $2n^2 + 2$ such conditions, but so far we have only found $n^2 + 4n - 1$, at most – the -1 is because of the obvious linear dependence, that the sum of the "new" n^2 conditions clearly lies in the linear span of the other $4n$.

In fact, we can say precisely how many independent conditions we have among this set:

Proposition 5.7 The $n^2 + 4n$ conditions given in Propositions 3.14 and 5.5 have precisely one linear dependence when n is odd, and precisely two when n is even.

Proof Let us write out all of the conditions and label them

$$\begin{aligned}
\sum_m a_{mk} &= 0 \qquad (I_{a,k}) \\
\sum_m b_{km} &= 0 \qquad (I_{b,k}) \\
\sum_m c_{m,k+m} &= 0 \qquad (I_{c,k}) \\
\sum_m d_{m,k-m} &= 0 \qquad (I_{d,k}) \\
a_{kl} + a_{k,l+1} + b_{k-1,l} + b_{kl} &= c_{kl} + d_{kl} \qquad (II_{kl})
\end{aligned}$$

For specific (but arbitrary) k and l, c_{kl} occurs in only two of these conditions, namely $I_{c,l-k}$ and II_{kl}. This means that, in any linear dependence relation among the conditions, the coefficient of II_{kl} depends only on the difference $l - k$. By the same reasoning applied to d_{kl}, it also depends only on the sum $l + k$.

If we assume that n is odd, then given any two pairs (k_1, l_1) and (k_1, l_2), one can always find a third pair (k_3, l_3) for which $k_3 + l_3 = k_1 + l_1$ and $l_3 - k_3 = l_2 - k_2$; for there is a unique inverse of 2 in $\mathbb{Z}/n\mathbb{Z}$, so we can take $k_3 = \frac{1}{2}(k_1 + l_1 - k_2 - l_2)$ and $l_3 = \frac{1}{2}(k_1 + l_1 + k_2 + l_2)$. Then, by the observations made in the previous paragraph, the coefficient of II_{k_1,l_1} in any linear dependence relation equal that of II_{k_3,l_3}, and therefore also that of II_{k_2,l_2}. As this applies to any two such pairs, all of the II_{kl} must have the same coefficient. It then follows that all of the $I_{a,k}$ do too, as do the $I_{b,k}$, $I_{c,k}$ and $I_{d,k}$, and that this common coefficient is twice that of all the II_{kl}, for the $I_{a,k}$ and $I_{b,k}$, and minus that of the II_{kl} for the other two. This means that the assumed linear dependence must be a multiple of the one which I already mentioned.

When n is even, it makes sense to talk about whether a given $k \in \mathbb{Z}/n\mathbb{Z}$ is even or odd, as all of its integer representatives will have the same parity. Then, if we sum the relations II_{kl} over those pairs (k, l) of the same parity, or over those of opposite parity, we will obtain $\sum_{k,l}(a_{kl} + b_{kl}) = \sum_{k,l|k\sim l}(c_{kl} + d_{kl})$, where the sum on the left is over all pairs (k, l), and the relation $\sim$ on the right means that k and l have either the same (in one of the two cases) or opposite (in the other) parity. Both sides of this expression are in the span of the $4n$ conditions $I_{x,k}$ ($x = a$, b, c, d) – this is clear on the left-hand side, and on the right follows because the pairs $(m, k \pm m)$ have the same parity when k is even and opposite parity when k is odd. This gives us two linear dependencies when n is even, which are clearly independent of each other.

One can now use the same argument as we did when n was odd to prove that these are the only two dependencies – it comes down to showing that the coefficient of II_{kl} in such a dependency depends only on the relative parities of k and l. When the two pairs (k_1, l_1) and (k_2, l_2) have either both the same parity or both opposite parities, then the expressions $k_1 + l_1 \pm (k_2 + l_2)$ are both even, and thus we can halve them, and proceed as we did before. The “half” is not unique this time, of course, but for two of the four possible choices of both, $k_3 = \frac{1}{2}(k_1 + l_1 - k_2 - l_2)$ and $l_3 = \frac{1}{2}(k_1 + l_1 + k_2 + l_2)$ will have the desired sum and difference. $\square$

Note that this is a generalisation of properties used in the proof of Proposition 5.6. For $n > 3$ though, we have only $n^2 + 4n - 1$ independent conditions, at most (one fewer than this if n is even), when we need $2n^2 + 2$, which is strictly greater when $n > 3$ (the difference is $(n - 1)(n - 3)$). So what other conditions can we obtain?

Proposition 5.8 In addition to the conditions already given in Propositions 3.14 and 5.5, we also must have

$$\begin{aligned} a_{kl} + a_{k+1,l+2} + b_{k+1,l+1} + b_{k-1,l} + c_{k,l+1} + c_{k+1,l} &= a_{k,l+1} + a_{k+1,l+1} + b_{kl} + b_{k,l+1} \\ &\quad + c_{kl} + c_{k+1,l+1} \end{aligned}$$

for each k and l.

The proof is as for the others; one simply computes that the conditions hold for each of the relations X_{kl} and Y_{kl}. As it is both simple and a little tedious, and will not be used for any important subsequent work, I will leave it to the reader to check.

Including this condition may get us closer to the full set of $2n^2+2$ necessary and sufficient ones – although how close I am not yet sure. Once again, if we sum up all n^2 conditions in this new set then we obtain something in the span of the already known ones. But it is not at all clear to me how many conditions this adds to an independent set – clearly none for $n = 2$ or 3. I have been unable to determine the answer to this in general, and thus do not know whether or not there may be more conditions to be found. (I would suspect that there are, actually – the above set of relations involves none of the d_{kl} at all, and it feels likely that there would be a similar set involving the d_{kl} but not the c_{kl}.)

I am grateful to my supervisor, Herbert Gangl, for coming up with both this last set of relations, and the set from Proposition 5.5 – he found these by analysing cases for small values of n in PARI, and writing down the conditions needed; it was then easy for us to guess what the general expressions were, and then for me to prove that they are valid in general. The set in Proposition 5.5 came from analysing $n = 2$ and $n = 3$, and the set given in the last proposition from $n = 4$ – Dr Gangl tells me that there seems to be yet another new type needed or $n = 5$, but I have not pursued this.

Of course, we would like to apply any new set of conditions, such as the one just given, to the image of a triangle configuration. When one does this, it turns out that no further conditions are imposed upon the integrality of a triangle, and thus one gets no new information, other than extending the results already obtained on integral triangles with $\Delta \neq 0$ to any cases for which all the conditions used so far are sufficient. This probably includes the case $n = 4$ (it does if the PARI computations are correct), and perhaps some others, and but I have not had time to investigate this matter.

5.5.2 Linear combinations of arbitrary triangles

The main result of Chapter 4 – and of the whole thesis – was that any linear combination of triangles (with $a = 1$), if integral, has vanishing regulator (after possibly multiplying the functions by units). In this chapter, we have proven a similar result for any *individual* triangle with $a = -1$. Clearly, it would be nice to be able to generalise these both at once, and be able to say that the same result is true for an arbitrary linear combination of triangles, whatever the values of a are.

Of course, this is a more general problem than that discussed in the last section, so it is no surprise that I have not made any real progress on it. Indeed, even when I believed that I was able to say things about the image in the Chow group of arbitrary connected triangles (before I discovered that I had neglected an important part of this image), I was unable to prove the result which I desired, as many difficulties were caused in the algebraic computations were caused by the fact that one was looking at sums (over h, to return to the notation of Chapters 3 and 4) over which the values of a, as well as those of i and Δ, could vary.

Nevertheless, I do expect that the desired result is in fact true (and that it should be possible to show it) – here is why. Firstly, if the base field is $\mathbb{Q}$, then Beilinson's conjecture, as we saw in Chapter 2, predicts that there should be no non-trivial integral elements of $K_1^{(2)}(E \times E)$. This means, that, if the conjecture is true – as seems likely, if very uncertain – any linear combination of triangle configurations must, if integral, vanish, and therefore have zero regulator. But all the methods I have been using apply over any number field whatsoever, as I have essentially reduced it to a problem of linear algebra. So, if there is a method to prove that these linear combinations of triangles, for curves defined over $\mathbb{Q}$, vanish (or their regulators do) whenever they are integral, it will also be applicable over any number field at all. As I certainly expect it to be demonstrable over $\mathbb{Q}$, using these linear-algebraic methods, it will therefore be demonstrable over any number field.

5.5.3 Injectivity of the regulator map

I have shown that all (connected) triangle configurations, and certain linear combinations of them, have vanishing regulator whenever they are integral (subject to Conjecture 5.1). If, as seems likely (and as Beilinson predicts), the regulator map is injective on integral elements, then it follows that there are no non-trivial integral triangle configurations. However, as we do not know for sure that the regulator is injective, it would be nice to be able to show directly that these elements are trivial in $K_1^{(2)}(E \times E)$ – that is, to construct an element of K_2 of the function field of $E \times E$ whose tame symbol is the given triangle configuration. This is something else which I have attempted, but in this case I got absolutely nowhere with it. It seems that in order to do this, one needs a function on $E \times E$ with a zero or pole along the diagonal. There are obviously infinitely many such functions – $\pi_1^*(g) - \pi_2^*(g)$ for any function g on E, for example – but all have further zeros and poles along other curves. I presume it is possible to construct the desired elements of K_2 of the function field, but one would need to somehow account for all the various zeros and poles that arise, and cancel them out along all curves other than the three involved in the triangle. I am not aware of how to do this.

5.5.4 Curves of higher genus

The original aim of my research project was to investigate triangle configurations in $K_1^{(2)}(C \times C)$, where C is any hyperelliptic curve (or perhaps even any curve) – there was no restriction on E having genus 1. But I gradually concentrated on this case, and for two principal reasons. One is that much more is known about elliptic curves than about curves of genus 2 or higher, and in particular there is a much smaller list of all the possible reduction types. Further, the group structure on an elliptic curve is a powerful tool to be able to use, which is not available – at least no immediately – for higher genus curves.

If we wish to try to obtain similar results for curves of higher genus, the first problem is in determining a regular model for $C \times C$ over the ring of integers – and this in turn needs one to know about the minimal regular model for C itself. For elliptic curves, the theory of Néron models is well-known, and the number of different types not too large – although even here, recall that we could not cope directly with many of the reduction types, and instead had to deal with them via field extensions. For curves of higher genus, there are enormously more reduction types, greatly increasing the amount of work which would need to be done here.

Nevertheless, for those reduction types whose fibres consist of copies of $\mathbb{P}^1$ intersecting in ordinary double points, one can do the same procedures as I have done in the multiplicative reduction case for elliptic curves. One would hope to obtain a good description of the fibre of the regular model of $C \times C$, and be able to figure out generators and relations for its codimension 1 Chow group.

The one part of the argument which really relies on E being an elliptic curve was the "translation" argument which allowed us derive an explicit formula for the regulator of the triangle configurations, and to show that this is the same as the regulator image of a decomposable element. In fact, it is not clear whether or not any equivalent to $N(f)$ exists, as that too was defined using translations in the group structure of E. One other place where the group structure was used was in constructing the triangles $\alpha_{a,f}$ for $a > 1$ – on a higher genus curve, Δ_a does not exist, and nor do the functions f_a for $a \neq 1$.

Of course, while the curve C has no group structure when its genus is greater than 1, C will always have a Jacobian, which does have such a structure. It might be possible to use the Jacobian to make some of these arguments work in the higher genus cases – although this is not an issue I have thought about in any detail. But it would not surprise me if arguments using the Jacobian were able to produce similar results to the ones in this thesis.

Appendix A

The blowup of $\mathcal{E} \times \mathcal{E}$, in the case of split multiplicative reduction

This appendix will be taken up with proofs of the statements made in chapters 3 and 5 concerning the regular model for $E \times E$, where E is an elliptic curve over a number field which has split multiplicative reduction at the prime we are interested in.

We shall be performing very explicit computations on a general elliptic curves with split multiplicative reduction. The general Weierstrass equation has the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6;$$

we shall assume that this gives an elliptic curve whose fibre, mod π, has a singularity at $(0,0)$ (if it is anywhere else we can simply translate) which is split multiplicative. The condition that the origin is a point of the reduction mod π means that $\pi | a_6$, while the assumption that the origin is singular further means that $\pi | a_3$ and $\pi | a_4$. The reduction being multiplicative means precisely that π does *not* divide $b_2 = a_1^2 + 4a_2$, and our assumption that it is *split* multiplicative means that $\sqrt{b_2}$ exists in the residue field k. For the rest of this appendix, we shall assume that all of these conditions hold.

If $\pi^2 \nmid a_6$, then $\mathbb{P}^2_R$ modulo the above Weierstrass equation is already regular (as the equation is in the maximal ideal (x, y, π) but not in $(x, y, \pi)^2$), and thus we can take this as our regular model $\mathcal{E}$ – this is reduction type I_1, whose special fibre is a copy of $\mathbb{P}^1_k$ with two points identified. If $\pi^2 | a_6$, on the other hand, this model will not be regular at the origin of the fibre, so we must blow it up here, and then repeatedly at any new singularities which arise. This process always ends after finitely many steps, and produces a model whose fibre consists of n copies of $\mathbb{P}^1_k$ joined in a "circle" (reduction type I_n), where n is the order at π of the discriminant of the original Weierstrass equation. (See [22], Chapter 4, and especially the description of Tate's Algorithm 9.4.)

The repeated blowings-up, when $\pi^2 | a_6$, are an inductive process, which means that we need only prove our results in the cases of reduction types I_1 and I_2, as it is clear that the local structure of $\mathcal{E}$ about each of the singularities of the fibre will be identical. But I will do I_1 and I_2 separately, as type I_1 is somewhat different locally; when $n > 1$, we have, in the fibre, two separate copies of $\mathbb{P}^1$ intersecting at a double point in the fibre of the Néron model of a curve with reduction type I_n, while if $n = 1$ we have just one copy of $\mathbb{P}^1$ which intersects itself. So we will perform detailed explicit calculations for reduction tpes I_1 and I_2, for which we need a concrete description of the model $\mathcal{E}$. For I_1 it is just $\mathbb{P}^2_R$ modulo the above Weierstrass equation (with $\pi^2 \nmid a_6$), but for I_2 that is singular at the origin of the special fibre, so we must first perform a blowup there.

So, let us assume that $\pi^2 | a_6$ in our Weierstrass equation, and blow up our model ($\mathbb{P}^2_R$ modulo a projectivised form of the equation) at the singular point corresponding to the maximal ideal (x, y, π). We will in fact take an affine part, Spec $R[x, y]$ modulo the equation, which is legitimate since blowing up is a local process. This means that we take Proj of the graded ring $G := \sum_{i \geq 0} \mathcal{M}^i$, where $\mathcal{M}$ is the maximal ideal (x, y, π), $\mathcal{M}^0$ is the whole ring $R[x, y]$, and the grading is given by the index i. (See [12].) If we take the polynomial ring $R[x, y, X, Y, P]$, considered as a graded ring in which x, y and all elements of R have degree 0 while X, Y and P each have degree 1, then there is a natural and surjective map of graded rings from G to the one we are interested in, sending X, Y and P to x, y and π respectively (each in degree 1). Thus our blowup will be Proj of G modulo some relations, and these relations will be given precisely by the elements in the kernel of the map I just described.

So, what is in this kernel? Firstly, it will contain $y^2+a_1xy+a_3y-x^3-a_2x^2-a_4x-a_6$, since the Weierstrass equation tells us that this is zero in G, in degree 0. But because every term in this equation is in the maximal ideal, it will also give rise to a degree 1 element of the kernel, namely $yY+a_1xY+a_3Y-x^2X-a_2xX-a_4X-\frac{a_6}{\pi}P$ – recall that $\pi|a_6$, and that P is sent to π in degree 1. And because we have further assumed that a_6 is divisible by π^2, and that a_3 and a_4 are divisible by π, there is also a degree 2 elemen in the kernel, induced by the Weierstrass equation: $Y^2+a_1XY+\frac{a_3}{\pi}YP-xX^2-a_2X^2-\frac{a_4}{\pi}XP-\frac{a_6}{\pi^2}P^2$. Clearly there is nothing of degree higher than 2 induced by the Weierstrass equation, as the term y^2 is not in the cube of the maximal ideal (x, y, π).

There are three more elements of the kernel, each in degree 1, coming from the fact that elements such as xy in the degree 1 part of G have two pre-emages – xY and yX. So, $xY-yX$ is in the kernel, and so, by the same reasoning, are $xP-\pi X$ and $yP-\pi Y$. It is easy to see that we have now found generators for the whole kernel – and therefore that our blowup will consist of Proj of the graded ring $R[x, y, Z, Y, P]$ modulo the ideal generated by the six elements I have just listed.

This new model is itself not regular in general. But it can be shown that it is in the case when the reduction type is I_2 (which is equivalent to the discriminant of the Weierstrass equation having order exactly 2 at π), and hence that our regular model $\mathcal{E}$ is the result of the blowup we just performed (with the addition of points at infinity on both the generic and the special fibres, as we first took an affine chart before we blew up). Notice that the special fibre (which we obtain by putting $\pi=0$, which makes a_3, a_4 and $\frac{a_6}{\pi}$ vanish too) is subject to the equations $xP=yP=0$, so that either $x=y=0$ or $P=0$. The first of these options gives the exceptional divisor, which can be seen to be isomorphic to the projective line over k. The second likewise gives a quadratic in $\mathbb{P}^2_k$ – when the reduction type is I_2, this quadratic is irreducible (and thus gives a curve isomorphic to $\mathbb{P}^1$), and further the two points at which the two components meet, which are singular on the special fibre, are regular on the arithmetic surface $\mathcal{E}$. These facts are well-known, and straightforward but tedious to check, so I will not go through them in detail here. I am now going to use these models to prove the various assertions I have made about $\mathcal{E} \times \mathcal{E}$ and its blowup, in the cases I_1 and I_2.

First, though, we will change co-ordinates in the I_2 case so that the singularity of the special fibre lies at the origin. (Here of course there are two, and we want to move one of them to the origin.) By the observations made above, these occur when $x = y = P = 0$, which means that the homogeneous co-ordinates X and Y are subject to the relation $Y^2 + a_1XY = a_2X^2$. Here, we recall that the reduction was assumed to be *split* multiplicative, and therefore that $a_1^2 + 4a_2$ has a square root in k. This is equivalent to saying that the above quadratic factorises over k (and the two factors are distinct, since the reduction is multiplicative and not additive). Let ω be an element of R whose reduction mod π is one of these roots in k. Then one of the two singular points of the fibre is at $(x, y) = (0, 0)$ and $[X, Y, P] = [1, \omega, 0]$ in homogeneous co-ordinates. So, to find a local picture with a singularity at the origin of the fibre, we can go to the affine chart $X = 1$, and translate Y to $Y + \omega$. This gives, after eliminating redundant conditions, Spec of the ring $R[x, Y, P]$ modulo the ideal generated by the two equations $xP = \pi$ and $(Y+\omega)^2 + a_1(Y+\omega) + \frac{a_3}{\pi}(Y+\omega)P = x + a_2 + \frac{a_4}{\pi}P + \frac{a_6}{\pi^2}P^2$. The second of these can be used to write x in terms of Y and P, so leaving us with a surface in the affine plane over R given by a single equation, of the form $Pf(Y, P) = \pi$, where $f(Y, P)$ is:

$$Y^2 + \frac{a_3}{\pi}YP - \frac{a_6}{\pi^2}P^2 + (a_1 + 2\omega)Y + (\frac{a_3}{\pi}\omega - \frac{a_4}{\pi})P + (\omega^2 + a_1\omega - a_2).$$

The constant term here, by choice of ω, is divisible by π. Notice, though, that if we replace ω by $\omega + c\pi$, as we are free to do for any c in R, then $\omega^2 + a_1\omega - a_2$ is replaced by $(\omega^2 + a_1\omega - a_2) + c(2\omega + a_1 + c\pi)\pi$. Notice that $2\omega + a_1 + c\pi$ is not divisible by π – for if it is, we would have $\omega = -\frac{a_1}{2}$ in k (unless k has characteristic 2, in which case we would need that $\pi | a_1$, which means $\pi | a_1^2 + 4a_2$ in this case too), and hence that $0 = \omega^2 + a_1\omega - a_2 = -\frac{1}{4}(a_1^2 + 4a_2)$, contrary to the reduction being multiplicative. Thus, if $\omega^2 + a_1\omega - a_2$ is divisible by π^2 for a particular choice of ω, we can always find another choice for which this is not the case (for example by replacing ω by $\omega + \pi$). Therefore, changing the names of the variables to make things look more visually intuitive, we find the following:

Proposition A.1 If E has reduction of type I_2, then, locally near a singularity of the fibre, $\mathcal{E}$ is isomorphic to Spec $\frac{R[x,y]}{(xf(x,y)-\pi)}$, where $f(x,y)$ is a quadratic, irreducible mod π (over $\overline{k}$, the algebraic closure of k), whose constant term has order 1 at π.

A.1 Singularities of $\mathcal{E} \times \mathcal{E}$

Recall that the statement here was that $\mathcal{E} \times \mathcal{E}$ has non-regular points at each of the n^2 points (P,Q) where P and Q are both singularities on the special fibre of $\mathcal{E}$, and at no other places. I shall, as always in this appendix, prove this in two separate cases, for reduction types I_n where $n=1$ and then where $n=2$ (which also accounts for $n>2$, as these cases are all the same locally).

A.1.1 $n = 1$

Recall here that $\mathcal{E}$ is just Spec $\frac{R[x,y]}{(y^2+a_1xy+a_3y-x^3-a_2x^2-a_4x-a_6)}$, where π divides each of a_3, a_4 and a_6, but π^2 does not divide a_6. (Strictly speaking, $\mathcal{E}$ is a projectivisation of this, but this will not concern us as we are only talking about local properties.) Thus $\mathcal{E} \times \mathcal{E}$ is

$$\text{Spec } \frac{R[x,y,u,v]}{(w(x,y),w(u,v))},$$

where $w(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$. We want to show that the only place where this fails to be regular is at the origin of the special fibre.

First, we look for singular points of the fibre itself. These occur when both of the pairs of equations $a_1y - 3x^2 - 2a_2x = 2y + a_1x = 0$ and $a_1v - 3u^2 - 2a_2u = 2v + a_1u = 0$ hold simultaneously (in the fibre, that is, modulo π). The only pairs of solution for (x,y) can be easily seen to be $(0,0)$ and $(-\frac{a_1^2+4a_2}{6}, \frac{a_1(a_1^2+4a_2)}{12})$, providing the characteristic of k is neither 2 nor 3 – if k has one of these characteristics, $(0,0)$ is the only solution. But the other point is easily seen not to actually satisfy the required equation, $y^2 + a_1xy = x^3 + a_2x^2$, since $a_1^2 + 4a_2$ does not vanish in the fibre. So, the only non-regular points on $\mathcal{E} \times \mathcal{E}$ can occur at points (P,Q) on the fibre, $\mathcal{E}_\mathfrak{p} \times \mathcal{E}_\mathfrak{p}$, for which either P or Q is the singular point of $\mathcal{E}_\mathfrak{p}$.

We will now check all such points for regularity on the threefold $\mathcal{E} \times \mathcal{E}$. We can do this by considering our two equations as being in the five variables x, y,

u, v and π, and considering the matrix whose rows are the linear parts of each equation in each of these five variables – the point (x_0, y_0, u_0, v_0) will fail to be regular precisely when, after changing variables to translate (x_0, y_0, u_0, v_0) to the origin, this matrix has less than the maximum rank (here 2). (See section 2.2 of [1].) We have just seen that we need only check those points where either $x_0 = y_0 = 0$ or $u_0 = v_0 = 0$ (on the fibre), and there is clearly no loss of generality in assuming that $x_0 = y_0 = 0$. Then the first of our two equations is not translated at all, so stays as $y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$. Since a_3 and a_4 are assumed to vanish on the fibre, this has no linear part in either x or y (and clearly none in u or v either), and linear part $\frac{a_6}{\pi}$ in π – this does not vanish since we assumed that $\pi^2 \nmid a_6$.

In particular, we see that, in order for our 5x2 matrix to have rank less than 2, and therefore for (x_0, y_0, u_0, v_0) to be non-regular, the second equation, which after translation becomes $(v + v_0)^2 + a_1(u + u_0)(v + v_0) + a_3(v + v_0) - (u + u_0)^3 - a_2(u + u_0)^2 - a_4(u + u_0) - a_6$, must have zero linear parts in both u and v. These are easily seen to be (the reductions mod π of) $a_1v_0 - 3u_0^2 - 2a_2u_0$ and $2v_0 + a_1u_0$ respectively. And we just saw that the only simultaneous solution to these two equations which also satisfies the defining Weierstrass equation is $(u_0, v_0) = (0, 0)$. Therefore, all points apart from the origin of the fibre are regular, as I claimed.

Finally, the origin itself does fail to be regular, as both equations then have no non-vanishing linear parts in x, y, u and v, and linear part $\frac{a_6}{\pi}$ in π, and thus the matrix has rank just 1.

A.1.2 $n = 2$

By Proposition A.1, $\mathcal{E} \times \mathcal{E}$ has the form Spec $\frac{R[x,y,u,v]}{(xf(x,y)-\pi, uf(u,v)-\pi)}$, where $f(x, y)$ is a quadratic which, modulo π, is irreducible and goes through the origin. Checking first for singular points of the fibre, we see for the same reasons as before that one (or both) of the two pairs (x, y) and (u, v) must be roots of $xf(x, y)$ and both of its partial derivatives, which are $f(x, y) + xf_x(x, y)$ and $xf_y(x, y) = 0$ (f_x amd f_y here denote the partial derivatives of f with respect to x and y respectively). So, if $x \neq 0$, we must have $f(x, y) = 0$, $f_y(x, y) = 0$ and $f_x(x, y) = 0$, meaning that (x, y) is a singular point of the quadratic $f(x, y) = 0$ – and this is impossible since

f is irreducible, and an irreducible quadratic is always non-singular. Thus we must have $x = 0$, and also $f(x, y) = 0$ – this will have two roots in y (distinct, since the reduction is multiplicative), one of which is $y = 0$, as the constant term of f is divisible by π. These correspond, of course, to the two singular points of the fibre, where the two components of the Néron 2-gon – given in this model by $x = 0$ and $f(x, y) = 0$ – intersect. So, once again, the only singular points on the fibre of $\mathcal{E} \times \mathcal{E}$ are when one of the two co-ordinates is the singular point on the fibre of $\mathcal{E}$.

And once more, such a point (x_0, y_0, u_0, v_0) is non-regular on $\mathcal{E} \times \mathcal{E}$ only when all the four co-ordinates are zero (on the fibre) – the argument is exactly the same as in the $n = 1$ case.

A.2 Proof of Proposition 3.7

Recall that the statement here was that, when we blow $\mathcal{E} \times \mathcal{E}$ up at one of the singular points, the exceptional divisor is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Again, it is enough to prove this in the cases $n = 1$ and $n = 2$.

A.2.1 $n = 1$

Recall that, here Spec $\frac{R[x,y,u,v]}{(w(x,y),w(u,v))}$ is an affine chart for $\mathcal{E} \times \mathcal{E}$, with $w(x, y) = y^2 + a_1xy + a_3y - x^2 - a_2x^2 - a_4x - a_6$, and the singularity at the origin of the fibre, which corresponds to the maximal ideal (x, y, u, v, π). So the blowup will be Proj of $R[x, y, u, v, X, Y, U, V, P]$ – a ring graded by the upper-case variables having degree 1 and lower-case ones degree 0 – modulo the relations induced by the two Weierstrass equations, and by each pair of generators of the maximal ideal. The exceptional divisor is then found by setting all of the lower-case variables, and π, to zero.

The relations induced by the Weierstrass equations in degree 1 are that $(y + a_1x + a_3)Y = (x^2 + a_2x + a_4)X + \frac{a_6}{\pi}P$ and $(v + a_1u + a_3)V = (u^2 + a_2u + a_4)U + \frac{a_6}{\pi}P$. So, in the exceptional divisor, since π vanishes and hence so do a_3 and a_4, as well as x, y, u and v, either of these equations tells us that $\frac{a_6}{\pi}P = 0$, and hence that $P = 0$, since our assumption here is that $\pi^2 \nmid a_6$, meaning that $\frac{a_6}{\pi}$ does not vanish in k. Meanwhile, while each Weierstrass equation on its own does not induce any relation in degree 2 (precisely because $\pi^2 \nmid a_6$, so a_6 is not in $(x, y, \pi)^2$), between them they do – because each one allows us to express a_6 in terms of things which *are* in $(x, y, \pi)^2$, and these two expressions must therefore coincide. Thus, we deduce that, in the blowup of $\mathcal{E} \times \mathcal{E}$, we have the following relation:

$$Y^2 + a_1 XY + \frac{a_3}{\pi} YP - xX^2 - a_2 X^2 - \frac{a_4}{\pi} XP = V^2 + a_1 UV + \frac{a_3}{\pi} VP - uU^2 - \frac{a_4}{\pi} UP.$$

On the exceptional divisor, this gives us the equation $Y^2 + a_1 XY - a_2 X^2 = V^2 + a_1 UV - a_2 V^2$, inside Proj $R[X, Y, U, V]$, or $\mathbb{P}^3_k$. We already know that both sides factorise, to give $(Y - \omega_1 X)(Y - \omega_2 X) = (V - \omega_1 U)(V - \omega_2 U)$. And this is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, via the Segre embedding which takes $([X_1, Y_1], [X_2, Y_2])$ in $\mathbb{P}^1 \times \mathbb{P}^1$ to $[X, Y, U, V] = [X_1 X_2 - Y_1 Y_2, \omega_2 X_1 X_2 - \omega_1 Y_1 Y_2, X_1 Y_2 - Y_1 X_2, \omega_2 X_1 Y_2 - \omega_1 Y_1 X_2]$.

A.2.2 $n = 2$

Here, we again recall the affine description of $\mathcal{E} \times \mathcal{E}$, which is Spec $\frac{R[x,y,u,v]}{(xf(x,y)-\pi, uf(u,v)-\pi)}$ – $f(x, y)$ is a quadratic which goes through the origin mod π. We form the blowup, at the origin, just as we did for $n = 1$; here the Weierstrass equations induce $Xf(x, y) = Uf(u, v) = P$ in degree 1 – so, in the exceptional divisor, we see that $P = 0$. Let us now write out $f(x, y)$ in more detail – we know it is quadratic, so has the form $ax^2 + bxy + cy^2 + dx + ey + f$, where f has order 1 at π. So, in degree 2 we obtain $X(axX + bxY + cyY + dX + eY + \frac{f}{\pi}P) = U(auU + buV + cvV + dU + eV + \frac{f}{\pi}P)$, which simplifies to $X(dX + eY) = U(dU + eV)$ in the exceptional divisor – this lying inside $\mathbb{P}^3$, with homogeneous co-ordinates X, Y, U and V. This is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via a Segre embedding, exactly as in the $n = 1$ case. Note that this needs d and e to not both be divisible by π, but here we saw that the "e" is $a_1 + 2\omega$, which as noted earlier cannot be divisible by π.

A.3 The order of π along the exceptional divisor of a blowup of $\mathcal{E}$

In this section I shall prove a statement I made at the end of Section 3.5.3 – that, if we blow up the model $\mathcal{E}$ of the curve E at a singular point of its fibre, then π has order 2 along the exceptional divisor of this blowup. Note that here we are working with just $\mathcal{E}$, and not with $\mathcal{E} \times \mathcal{E}$. Again, it will be enough to consider just the two cases $n = 1$ and $n = 2$.

A.3.1 $n = 1$

I once again remind you that $\mathcal{E}$ is given by the Weierstrass equation $y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6$ in Spec $R[x, y]$ (or, strictly speaking, the projective closure of this in $\mathbb{P}^2_R$), where π divides each of a_3, a_4 and a_6 but $\pi^2 \nmid a_6$. To blow this up at the origin of the special fibre means to take Proj of $R[x, y, X, Y, P]$ (with grading given as before by the upper-case variables), modulo the relations $xY = yX$, $xP = \pi X$, $yP = \pi Y$, $y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6$ and $yY+a_1xY+a_3Y = x^2X+a_2xX+a_4X+\frac{a_6}{\pi}P$. (There is no equation in degree higher than 1, because a_6 has order exactly 1.) The exceptional divisor is determined by the equations $x = y = \pi = 0$. To make things more transparent, we shall take an affine part of the blowup, say that given by $X = 1$. Then we have $y = xY$ and $\pi = xP$, so that the ideal corresponding to the exceptional divisor becomes principal, generated by x. Our question then becomes, what is the highest power of x to divide π? Since $\pi = xP$, this must be one more than the highest power of x to divide P. And the last of the relations in the blowup which I just listed becomes $\frac{a_6}{\pi}P = x[Y^2+a_1Y+\frac{a_3}{\pi}YP-x-a_2-\frac{a_6}{\pi}P]$ when we make the necessary substitutions to put everything in terms of the independent variables x, Y and P. This shows us that x divides P, since $\frac{a_6}{\pi}$ is not divisible by π, and hence not by x either. But it is equally clear from this expression that x^2 cannot divide P, and therefore that P has order exactly 1 along the exceptional divisor. Therefore $\pi = xP$ has order exactly 2, as claimed.

A.3.2 $n = 2$

$\mathcal{E}$ is now (a projectivisation of) Spec $\frac{R[x,y]}{xf(x,y)-\pi}$, where $f(x,y)$ has the form $ax^2 + bxy + cy^2 + dx + ey + f$, and f is divisible by π but not by π^2. When we blow $\mathcal{E}$ up at the origin of the fibre, as we did in the $n = 1$ case, and take the affine chart $X = 1$, this is the same as replacing y by xy' and π by $x\pi'$, where y' and π' are new variables, corresponding to what I called Y and P before, and then cancelling the highest power of x which occurs in the equation. Here, the equation becomes $xf(x, xy') = x\pi'$, so we obtain $f(x, xy') = \pi'$. Since $\pi = x\pi'$, and we want to show that π has order 2 at x, we must show that π' has order 1. But $\pi' = f(x, xy') = ax^2 + bx^2y' + cx^2y'^2 + dx + exy' + \frac{f}{\pi}x\pi' = x(ax + bxy' + cxy'^2 + d + ey' + \frac{f}{\pi}\pi')$, which has order exactly one at x, since $\pi^2 \nmid f$.

A.4 Proof of (a slight generalisation of) Proposition 5.1

The last task which is left for this appendix is to prove this proposition, concerning the image in the exceptional divisor of the blowup of $\mathcal{E} \times \mathcal{E}$ of the curves $\Delta_{\pm 1}$ in $E \times E$. Once more we deal with just the two cases $n = 1$ and $n = 2$, as the local structure of the blowup at a singular point of $\mathcal{E} \times \mathcal{E}$ when $n > 2$ is the same as that when $n = 2$. I will prove not only Proposition 5.1 itself by these methods, but also, for $n = 1$, its generalisation to $\Delta_{\pm 2}$, which I mentioned immediately after giving the proposition. I will then finish with a few closing remarks about a possible generalisation of this result to other values of a and n.

A.4.1 $n = 1$

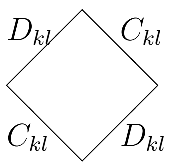
We saw already in section A.2.1 a concrete description of the blowup. We are now going to take the curve Δ_a in $E \times E$, the generic fibre of our blown-up model, and ask what the closure of this is in the blowup, and in particular how this intersects with the exceptional divisor of the blowup. We will do the case $a = 1$ – the diagonal – first of all.

Here, $\mathcal{E}\times\mathcal{E}$ before the blowup was Spec $\frac{R[x,y,u,v]}{(w(x,y),w(u,v))}$, where $w(x,y)$ is the Weierstrass equation for E. The diagonal is given in the generic fibre by the equations $u=x$ and $v=y$, and therefore these equations also determine the closure of diagonal inside $\mathcal{E}\times\mathcal{E}$.

Now, let us impose these equations in the blowup. We have that $xY=yX=vX=xV$, and $yV=vY=yY$, and therefore that $V=Y$ unless both x and y are zero. But $x=y=0$ occurs only in the exceptional divisor, so when we take the strict transform of this subscheme in the blowup – that is, remove the exceptional divisor, and then take the closure – we see that it will satisfy the equation $V=Y$. By similar arguments, $U=X$ will also be satisfied – these are the equations for the closure of the diagonal in the blowup.

We also saw that the exceptional divisor is $(Y-\omega_1X)(Y-\omega_2X)=(V-\omega_1U)(V-\omega_2U)$, which is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ via $([X_1,Y_1],[X_2,Y_2])$ in $\mathbb{P}^1\times\mathbb{P}^1\mapsto[X,Y,U,V]=[X_1X_2-Y_1Y_2,\omega_2X_1X_2-\omega_1Y_1Y_2,X_1Y_2-Y_1X_2,\omega_2X_1Y_2-\omega_1Y_1X_2]$. So, when $X=U$ and $Y=V$, we find that we have $X_2(X_1+Y_1)=Y_2(X_1+Y_1)$ and $X_2(\omega_2X_1+\omega_1Y_1)=Y_2(\omega_2X_1+\omega_1Y_1)$. Since $\omega_1\neq\omega_2$ and we can't have $X_1=Y_1=0$ (as they are homogeneous co-ordinates for the projective line), the only solution to these equations is $X_2=Y_2$. This tells us that the image of the diagonal in the exceptional divisor, $\mathbb{P}^1\times\mathbb{P}^1$, is $\mathbb{P}^1\times\{1\}$.

Note that this does correspond to the direction which I have labelled, in chapters 3 and 5, as D_{kl}, rather than the other one which is labeled as C_{kl} – for in the picture,



the image of the diagonal must come in along the diagonal of the "octagon", from both directions, and there is clearly no way this could be connected if it intersected the exceptional divisor (above) along the "C_{kl}" direction.

When dealing with Δ_{-1}, the equations in the generic fibre – and hence for the closure of Δ_{-1} in the blowup – are $u=x$ and $v=-y-a_1x-a_3$, as the inverse of (x,y) on the elliptic curve $y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$ is $(x,-y-a_1x-a_3)$.

We now have, in the blowup, that $xV = vX = -yX - a_1xX - a_3X$. Since $xY = yX$ and $xP = \pi X$, while $\pi | a_3$, this can also be written as $xV = -x(Y + a_1X + \frac{a_3}{\pi}P)$. Similarly, from $yV = vY$ we obtain $yV = -y(Y + a_1X + \frac{a_3}{\pi}P)$. As before, these between them tell us that the strict transform of Δ_{-1} intersects the exceptional divisior in the curve $V = -Y - a_1X - \frac{a_3}{\pi}P$, or $V = -Y - a_1X$, since P vanishes on the exceptional divisior.

Putting these conditions into the Segre embedding, we find that $(X_1 + Y_1)(X_2 - Y_2) = 0$ and $\omega_2X_1Y_2 - \omega_1Y_1X_2 = (\omega_1 + a_1)Y_1Y_2 - (\omega_2 + a_1)X_1X_2$. Since ω_1 and ω_2 are the two roots of $\omega^2 + a_1\omega - a_2$, we have that $\omega_1 + \omega_2 = -a_1$. Hence the latter condition can be rewritten as $(\omega_1X_2 - \omega_2Y_2)(X_1 + Y_1) = 0$. As in the $a = 1$ case, these two conditions have only one simultaneous solution, $Y_1 = -X_1$, since $\omega_1 \neq \omega_2$ and it is impossible to have $X_2 = Y_2 = 0$. Thus Δ_{-1} has image $\{-1\} \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ – the opposite "direction" to that of Δ_1. So this must give us a C_{kl}, as claimed.

Next, as promised, let us also consider the cases $a = \pm 2$. That is, we are considering the curve Δ_2 in $E \times E$, defined as the set of all points (P, Q) where $Q = 2P$ in the group structure on E. Writing (x, y) and (u, v), as before, for the affine co-ordinates of P and Q respectively, the duplication formula for elliptic curves (see [21]) tells us that Δ_2 – and hence its closure in $\mathcal{E} \times \mathcal{E}$ – satisfies the equation $u = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6}$, where the b_k are certain standard algebraic expressions in the a_k (also defined in [21]). Note that this does not completely define Δ_2, as it only gives us the x-coordinate. Since, on a curve given by a Weierstrass equation, the two points with a given x-coordinate are inverses under the group law, the above equation is in fact shared by Δ_2 and Δ_{-2}. There is a further equation, expressing v in terms of x and y, which distinguishes which of the two cases we are in (this equation is different in the two different cases) – but we will not worry about this for the time being.

In the blowup, we have $xU = uX$. As in section A.3, I shall take the affine chart $X = 1$ to make the computations easier. So we have $xU = u = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6}$, or, clearing denominators, $x(4x^3 + b_2x^2 + 2b_4x + b_6)U = x^4 - b_4x^2 - 2b_6x - b_8$. When we restrict this to the exceptional divisor, we get $0 = 0$, as $b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$ is divisible by π. But, because we are interested in the strict transform, or the closure of the part of this subscheme *outside* the exceptional divisor, we may divide through by x as many times as we need to get a non-trivial result (on this affine part, x is a uniformiser for the exceptional divisor). If we divide through by x^2, and use the facts that b_6 and b_8 are both divisible by π ($b_6 = a_3^2 + 4a_6$), and that $\pi = xP$, we find that $(4x^2 + b_2x + 2b_4 + \frac{b_6}{\pi}P)U = (x^2 - b_4 - 2\frac{b_6}{\pi}P - \frac{b_8}{\pi}\frac{P}{x})$.

The degree 1 analogue of the Weierstrass equation says that $(y + a_1x + a_3)Y = (x^2 + a_2x + a_4)X + \frac{a_6}{\pi}P$. Note first that, if we restrict to the exceptional divisor, this gives $\frac{a_6}{\pi}P = 0$, hence $P = 0$ since $\pi^2 \nmid a_6$. And then $\frac{P}{x} = \frac{(Y+a_1+\frac{a_3}{\pi}P)Y-(x+a_2+\frac{a_4}{\pi}P)}{\left(\frac{a_6}{\pi}\right)}$, which therefore restricts to $\frac{Y^2+a_1Y-a_2}{\left(\frac{a_6}{\pi}\right)} = \frac{(Y-\omega_1)(Y-\omega_2)}{\left(\frac{a_6}{\pi}\right)}$ on the exceptional divisor. If we combine this with the equation at the end of the previous paragraph, and recall that $b_4 = a_1a_3 + 2a_4$, so that $\pi | b_4$, and that $\pi^2 \nmid b_8$ because we are talking about reduction type I_1, we see that the closure of the union of Δ_2 and Δ_{-2} in the blowup of $\mathcal{E} \times \mathcal{E}$ intersects the exceptional divisor in the subscheme $(Y - \omega_1 X)(Y - \omega_2 X) = 0$. That is, we have two "components" here, one with equation $Y = \omega_1 X$ and one with equation $Y = \omega_2 X$.

Since the equation of the exceptional divisor is $(Y - \omega_1 X)(Y - \omega_2 X) = (V - \omega_1 U)(V - \omega_2 U)$, this condition alone would give us four curves in all. But, as already noted, what we have just done applies equally to Δ_2 and Δ_{-2}, and it turns out that each gives exactly two of the four curves here.

To see this, we shall need the aforementioned formulae for v in terms of the other three variables. We first recall how to double a point on an elliptic curve – one takes the tangent line to the curve at the given point, and find its third point of intersection with the curve. This point will be -2 times the given one; to find two times it, we apply the formula already mentioned for inversion on an elliptic curve, $(x, y) \mapsto (x, -y - a_1x - a_3)$.

The slope of the tangent at (x, y) is $\lambda(x, y) := \frac{3x^2+2a_2a-a_1y+a_4}{a_1x+2y+a_3}$, and the equation of Δ_{-2} is $v = \lambda(x, y)(u - x) + y$, while that of Δ_2 is $v = \lambda(x, y)(x - u) - y - a_1u - a_3$, by the remark in the previous paragraph. We will first evaluate $\lambda(x, y)$ on the exceptional divisor, again by using the affine chart $X = 1$. As already noted, this has the advantage that the exceptional divisor is defined by the single equation $x = 0$, so to find the strict transform of any subscheme of $\mathcal{E} \times \mathcal{E}$ in the blowup, we can just keep dividing its defining equations by x until we get a relation other than $0 = 0$. In particular, notice that the expression given for $\lambda(x, y)$ seems to give $\frac{0}{0}$ on the exceptional divisor, but if we divide numerator and denominator by x then we get $\lambda(x, y) = \frac{3x+2a_2-a_1Y+\frac{a_4}{\pi}P}{a_1+2Y+\frac{a_3}{\pi}P}$, which becomes $\frac{2a_2-a_1Y}{a_2+2Y}$ on the exceptional divisor. Since we've just seen that, on the closure of Δ_2 or Δ_{-2}, we must have $Y = \omega_i$ (on this particular affine chart) for $i = 1$ or 2, we see further than λ becomes $\frac{2a_2-a_1\omega_i}{a_1+2\omega_i}$. Using the defining equation for ω_i once more, the numerator is $2(\omega_i^2 + a_1\omega_i) - a_1\omega_i = 2\omega_i^2 + a_1\omega_1$, which is ω_i times the denominator. Since $a_1 + 2\omega_i$ can never be zero (in characteristic 2, a_1 must be non-zero, and otherwise $\frac{-a_1}{2}$ is not a root of $Y^2 + a_1Y - a_2$, both because $a_1^2 + 4a_2$ is non-zero in k), we find that λ is equal to ω_i on the images of both Δ_2 and Δ_{-2} in the exceptional divisor.

Δ_{-2} has equation $v = \lambda(u - x) + y$, which becomes $0 = 0$ when restricted to the exceptional divisor (inevitably so, since the closure of Δ_{-2} in $\mathcal{E} \times \mathcal{E}$ is known to pass through the singularity at which we are blowing up); so we pass to the $X = 1$ affine part and divide both sides by x, to give $V = \lambda(U - 1) + Y$. Since we know that we must have $Y = \omega_i$ for $i = 1$ or 2, and have just seen that that forces $\lambda = \omega_i$ too, we must then have $V = \omega_i U$. Thus, on Δ_{-2} we do indeed get two of the four possible curves when Y is one of the ω_i times X, namely $Y = \omega_1 X$, $V = \omega_1 U$, and $Y = \omega_2 X$, $V = \omega_2 U$. On Δ_2, on the other hand, we have $v = \lambda(x - u) - y - a_1u - a_3$, or $V = \lambda(1 - U) - Y - a_1U$ on dividing by x (a_3 becomes $\frac{a_3}{\pi}P$, which vanishes on the exceptional divisor). So if $Y = \omega_i$, meaning $\lambda = \omega_i$ too, we find that $V = -\omega_i U - a_1 U = \omega_j U$, where $\{i, j\} = \{1, 2\}$. Thus Δ_2 gives the two curves $Y = \omega_1 X$, $V = \omega_2 U$, and $Y = \omega_2 X$, $V = \omega_1 U$.

Finally, we shall use the Segre embedding to check that these curves go in the right "directions" in $\mathbb{P}^1 \times \mathbb{P}^1$. Since ω_1 and ω_2 are distinct, we find that $Y = \omega_1 X$ corresponds to $X_1X_2 = 0$, $Y = \omega_2 X$ to $Y_1Y_2 = 0$, $V = \omega_1 U$ to $X_1Y_2 = 0$ and $V = \omega_2 U$ to $Y_1X_2 = 0$. So, the two curves which form the image of Δ_2 correspond to $X_2 = 0$ and $Y_2 = 0$, respectively; that is, $\mathbb{P}^1 \times \{0\}$ and $\mathbb{P}^1 \times \{\infty\}$. Similarly, the two for Δ_{-2} give the two fibres in the "vertical" direction. So, in each case, we get precisely two curves in the same direction. And, looking back over the computations above for Δ_1 and Δ_{-1}, we see that the directions for Δ_2 and Δ_{-2} are the same as those for Δ_1 and Δ_{-1} respectively – so we get $2D_{kl}$ for Δ_2, and $2C_{kl}$ for Δ_{-2}, as claimed.

A.4.2 $n = 2$

Recall that here, we have modelled $\mathcal{E}$ as Spec $\frac{R[x,y]}{(xf(x,y)-\pi)}$, where $f(x,y) = ax^2+bxy+xy^2+dx+ey+f$, with e not divisible by π, and f divisible by π but not by π^2. Further, the exceptional divisor of the blowup of $\mathcal{E}\times\mathcal{E}$ at its singular point at the origin had equation $X(dX+eY) = U(dU+eV)$.

In this case, we need to be a bit careful with our notation, as the x and y coordinates on $\mathcal{E}$ do not correspond to the x and y coordinates on the curve E with which we started, with Weierstrass equation $y^2+a_1xy+a_3Y = x^3+a_2x^2+a_4x+a_6$. Reading through the discussion leading up to Proposition A.1, it can be seen that the x-coordinate to which the Weierstrass equation refers is equal to $f(x,y)$ on $\mathcal{E}$, while the y-coordinate becomes $(y+\omega)f(x,y)$ – recall that ω is one of the two distinct roots in k of the quadratic $\omega^2+a_1\omega-a_2$.

Let us start with the image of $\Delta_1 = \Delta$ in the exceptional divisor. Here, we have $u = x$ and $v = y$ as coordinates on the curve E, so, by the previous paragraph, these become $f(x,y) = f(u,v)$ and $(y+\omega)f(x,y) = (v+\omega)f(u,v)$ in our blown-up model for $E\times E$. Substituting the first of these conditions into the second gives $y = v$, from which we can conclude that $Y = V$ on the strict transform. And, from $f(x,y) = f(u,v)$ itself, if we use the affine chart $Y = 1$ (not $X = 1$ this time, as we may find important curves have their image in the exceptional divisor contained in $X = 0$), and divide by y which is now the uniformiser for the exceptional divisor, we find $axX+bx+cy+dX+e+\frac{f}{\pi}P = auU+buV+cvV+dU+eV+\frac{f}{\pi}P$. Since the equation $xf(x,y) = \pi$ gives $x(axX+bxY+cyY+dX+eY+\frac{f}{\pi}P) = P$ in degree 1, we see that $P = 0$ in the exceptional divisor, and hence that our condition $f(x,y) = f(u,v)$ gives $dX+eY = dU+eV$ here, if we pass back to homogeneous coordinates. So the image of Δ_1 has equations $Y = V$ and $dX+eY = dU+eV$. Our exceptional divisor is, as we have seen, isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, the two coordinate functions corresponding to $\frac{X}{U} = \frac{dU+eV}{dX+eY}$ and $\frac{X}{dU+eV} = \frac{U}{dX+eY}$ respectively – so when $dX+eY = dU+eV$, then we have $\{1\}\times\mathbb{P}^1$ (unless $dX+eY$ and $dU+eV$ are both zero, which doesn't happen here; see the next paragraph, however), a curve in just one of the two "directions". As in the $n = 1$ case, it is easy to see that this must be the direction that I labelled D_{kl}.

And for Δ_{-1}, as the x- and u-coordinates (with respect to the Weierstrass equations) are still equal, we again obtain that $dX + eY = dU + eV$. This time, though, the y and v coordinates (again, with respect to the Weierstrass equations) are not equal, but sum to $-a_1u - a_3$. Thus we have, in our new coordinates on $\mathcal{E} \times \mathcal{E}$ and its blowup, $f(x, y)(y+\omega)+f(u, v)(v+\omega) = -a_1 f(u, v)-a_3$. This, of course, gives no information on the exceptional divisor, so we again restrict to the $Y = 1$ chart, divide by y, and only then set $y = 0$. This gives $\omega(dX + eY + dU + eV) = -a_1(dX + eY)$, or, using that $dX + eY = dU + eV$, $0 = (2\omega + a_1)(dX + eY)$. Since, as we have noted several times already in this appendix, $2\omega + a_1$ is not zero in the residue field, we find that Δ_{-1} has image $dX + eY = dU + eV = 0$. And this gives $\mathbb{P}^1 \times \{\infty\}$ in the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ used above – thus a fibre in the opposite direction to the image of Δ_1, or a C_{kl}, as desired.

A.4.3 Possible generalisations of the Proposition

The obvious case to deal with next would be $\Delta_{\pm 2}$ for reduction type I_2, which would simultaneously take care of all types I_n for $n > 2$ as well, as usual. However, I have spent quite a while trying to show that the desired result holds here, without success, but also without demonstrating that the desired result was incorrect; the computations required are much harder than in any of the cases dealt with above. So I have had to leave this case open for now; it is after all only one more case among all those Δ_a with $|a| > 2$!

When $|a| > 2$, of course, things become more complicated still, even for $n = 1$. Previously, we have used specific formulae for the curves $\Delta_{\pm 1}$ and $\Delta_{\pm 2}$ – other Δ_a would need different formulae. Δ_3, for example, would need a formula for the triple of a point on an elliptic curve, similar to the well-known duplication formula which I used for $\Delta_{\pm 2}$. And a new such formula would be needed for each Δ_a – these can all in principle be computed, but this would not only rapidly become very tedious, it would only allow those individual cases to be done, as there is no general formula in a simple form for the coordinates of a times (x, y) in terms of a, x and y.

Nevertheless, there might be a way to prove a full generalisation of Proposition 5.1 in general (or perhaps to disprove it!), by an inductive argument – if $a > 0$, and one knows the formulae for Δ_a, then one can work it out for Δ_{a+1} using the usual addition formula for elliptic curves. Once again, this is something which I found tricky to do in detail, and found myself unable to see any sensible results in time, which is one reason why I have chosen only to focus on the cases of $\Delta_{\pm 1}$ (the other reason, of course, is that the results of Chapter 5 only deal with $a = \pm 1$ as it stands anyway, even if such a generalisation could be proven).

I will, however, finish by saying a few words about why I believe that this generalisation of Propositio 5.1 – that one obtains aD_{kl} when $a > 0$, and $-aC_{kl}$ when $a < 0$. My reasoning is the same as that which I have already used to deduce that the one fibre which is the image of Δ_1 is a D_{kl} rather than a C_{kl} – that it seems impossible, from looking naively at the pictures anyway, that there could be any C_{kl}'s involved when a Δ_a, for $a > 0$, crosses one of the exceptional divisors, as it is plainly "in the wrong direction". The same applies in reverse when $a < 0$, of course. However, I am well aware that this falls well short of the rigorous argument which would be required, which is why I do not offer it as a proof. Nonetheless, I cannot help but feel that there ought to be some quite simple argument which proves the conjecture while preserving the naive flavour of this particular argument. If there is one, though, I have been unable to see it.

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