WATER RESOURCES RESEARCH, VOL. 30, NO. 11, PAGES 2925–2929, NOVEMBER 1994

Three-dimensional analysis of infiltration from the disc infiltrometer
1. A capillary-based theory

K. R. J. Smettem,[1,2] J. Y. Parlange,[3] P. J. Ross,[2] and R. Haverkamp[4]

Abstract. The hydraulic properties of an unsaturated homogenous and isotropic soil can be obtained from the unconfined flux out of a disc infiltrometer into the soil over the depth of wetting. The disc infiltrometer is becoming increasingly popular, but methods of analysis have generally relied on the restrictive assumptions of one-dimensional flow at early times or quasi-steady state flow at large times. We provide an approximate analytical expression for three-dimensional unsteady, unconfined flow out of a disc infiltrometer, and this includes the geometric effect of the circular source but ignores gravity. This physically based solution is tested against data obtained from laboratory experiments on repacked material. The results illustrate that the difference between three-dimensional and one-dimensional flow is linear with time.

Introduction

In situ measurement of hydraulic properties of soil surface horizons is a fundamental requirement for physically based modeling of field infiltration and runoff processes. The disc, or tension infiltrometer [*Perroux and White*, 1988], has become a widely used device for obtaining soil-surface hydraulic properties, usually at a small negative pressure head [*Clothier and White*, 1981; *Ankeny et al.*, 1988; *Wilson and Luxmoore*, 1988; *Smettem and Clothier*, 1989; *Smettem and Kirkby*, 1990; *Reynolds and Elrick*, 1991; *Smettem and Ross*, 1992; *Logsdon et al.*, 1993].

The mathematical analysis for unconfined, three-dimensional flow from a disc has rested heavily on *Wooding*'s [1968] steady state approximation, given here in the form used by *Warrick* [1992] as

$$Q = \pi r^2 K_0 + 4 \frac{K_0}{K_0 - K_n} \phi_0 r \tag{1}$$

In (1), Q is the flow volume per unit time (L^3T^{-1}); r is the disc radius (L); K_0 is the hydraulic conductivity (LT^{-1}) at the imposed pressure head; K_n is the initial hydraulic conductivity (LT^{-1}); ϕ_0 is the value at h_0 of the linearized Kirchhoff transform, or matric flux potential [*Gardner*, 1958] defined by

$$\phi = \int_{h_n}^{h} K(h)\, dh = \int_{\theta_n}^{\theta} D(\bar{\theta})\, d\bar{\theta} \tag{2}$$

where h is the soil water pressure head (L); D is the soil water diffusivity (L^2T^{-1}); θ is the volumetric water content (L^3L^{-3}); $\bar{\theta}$ is a variable of integration; and the subscripts n and 0 refer respectively to the initial condition and surface boundary condition under the disc infiltrometer.

From his complicated series solution, *Wooding* [1968] noted that to a good approximation, the flow could be given as the gravity-driven flow ($\pi r^2 K_0$) added to the capillary-induced flow $4K_0\phi_0 r/(K_0 - K_n)$. The latter, with $K_n = 0$, is the steady state solution without gravity [*Carslaw and Jaeger*, 1959, p. 215].

Wooding's [1968] solution is based on the exponential hydraulic conductivity function of *Gardner* [1958]

$$K = K_0 \exp(\alpha h) \tag{3}$$

where K_0 is the hydraulic conductivity at h_0 (LT^{-1}) and α (L^{-1}) is an empirical constant. Providing K_n is negligible, solutions from (1) can be found when Q is measured for two or more r values [e.g., *Smettem and Clothier*, 1989; *Smettem and Ross*, 1992] or, alternatively, for a single r value at multiple tensions [e.g., *Ankeney et al.*, 1991; *Reynolds and Elrick*, 1991].

Another method of estimation is to obtain ϕ_0 from the sorptivity S_0 ($LT^{-1/2}$), the latter being found from the short time flow from the disc. Then from the general expression [*Parlange et al.*, 1980; *Elrick and Robin*, 1981]

$$\phi_0 = \frac{S_0^2}{(\theta_0 - \theta_n)} \frac{1}{2(1 - \varepsilon)} \tag{4}$$

ϕ_0 can be found, and from the steady flow, K_0 can be found subsequently.

Although ε is a function of the "shape" of the diffusivity function, its value is, in practice, quite small. It is of the order of $\varepsilon < 1/15$, so that theoretically, its boundaries are given by $0 < \varepsilon < 1 - 2/\pi$, which are equivalent to the bounds given by *White and Sully* [1987]. This approach has the attraction of requiring only a single disc and a single supply potential to obtain S_0 and K_0 at any supply potential (provided θ_0 and θ_n are measured and the value for ε is assumed).

Recently, *Warrick* [1992] has refined the short time solution by adding the geometric effect of the disc source to the

[1]Department of Soil Science and Plant Nutrition, University of Western Australia, Nedlands, Western Australia, Australia.
[2]Division of Soils, CSIRO, Townsville, Queensland, Australia.
[3]School of Earth Sciences, Flinders University, Adelaide, South Australia, Australia.
[4]Laboratoire d'Etude des Transferts en Hydrologie et Environnement, Grenoble, France.

Paper number 94WR01787.
0043-1397/94/94WR-01787$05.00

one-dimensional solution. His analysis is partially empirical and assumes a constant average diffusivity, which is appropriate for the "linear" soil model [*Philip*, 1973]. The linear soil model defines one bound to the envelope of possible soil wetting patterns. The other bound is given by the sharp Green-Ampt type model, which assumes a Dirac δ function in which $D(\theta) = 0$, except at θ_0, where $D \rightarrow \infty$.

In this, the first in a series of papers, we provide an analytical expression for unconfined three-dimensional flow from the disc infiltrometer into soil characterized by a diffusivity that increases sharply with increasing water content. Initial application of the solution is illustrated using data obtained from disc infiltrometer experiments on repacked soil in the laboratory.

Theory

The general approach adopted is that of *Turner and Parlange* [1974]. We examine first infiltration without gravity. Our point of departure is the nonlinear diffusion equation describing absorption of water by nonswelling homogenous isotropic soils at early times:

$$\partial \theta/\partial t = \nabla \cdot (K \nabla h) \quad (5)$$

Introducing (2) into (5) and neglecting $\partial\theta/\partial t$ and considering steady state conditions as a first approximation gives Laplace's equation

$$\nabla^2 \phi = 0 \quad (6)$$

For one-dimensional flow, (6) has the solution [*Parlange*, 1971a, b]

$$\phi_0 - \phi = 1/2\hat{S}_0 y/t^{1/2} \quad (7)$$

where y is depth and $\hat{S}_0$ is an estimate of the sorptivity S_0, given by

$$\hat{S}_0^2 = 2(\theta_0 - \theta_n)\phi_0 \quad (8)$$

i.e., (4) with $\varepsilon = 0$.

Conformal mapping in two dimensions is then used to solve Laplace's equation at the edge of the source [*Parlange*, 1971b]. The mapping

$$(X + iY)^2 = (x + iy) \quad (9)$$

transforms the half line, $x > 0$, $y = 0$, from the physical space into the complete line $Y = 0$ in the transformed plane. To obtain the two-dimensional solution, we follow *Turner and Parlange* [1974] and take ϕ as the real part of the complex variable Φ, and combined with (7), then

$$\phi_0 - \Phi = -i\lambda \hat{S}_0^{3/2} t^{-1/4}(x + iy)^{1/2} \quad (10)$$

where λ is a dimensionless scaling parameter and $i^2 = -1$. Taking the real part of each side gives the two-dimensional solution

$$\phi_0 - \phi = \lambda \hat{S}_0^{3/2} t^{-1/4}\{[(x^2 + y^2)^{1/2} - x]/2\}^{1/2} \quad (11)$$

In one dimension, the cumulative infiltration is given approximately by $S_0 t^{1/2}$. Thus if at depth y_1 there is a wetting front at which θ changed abruptly from θ_0 to θ_n, then

$$y_1 = \hat{S}_0 t^{1/2}/(\theta_0 - \theta_n) \quad (12)$$

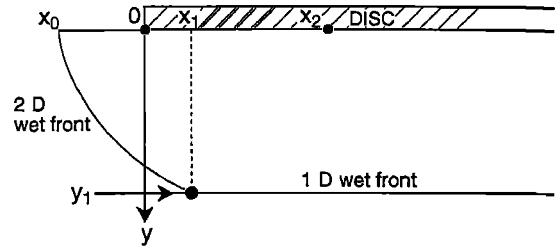

Figure 1. Diagram showing the axes described in the text. The edge of the disc infiltrometer is at 0. Here, x_0 is the lateral extent of the two-dimensional front; x_1, y_1 is the point where the one- and two-dimensional wetting fronts meet; x_2 is the point of equality between the one- and two-dimensional fluxes; and y_1 is the position of the one-dimensional wetting front.

We let (x_1, y_1) be the position where this one-dimensional front meets the two-dimensional front (Figure 1). Note the discontinuity in the slope where the two fronts meet, which is acceptable to the lowest order used here.

The one-dimensional front is given by setting $\phi_0 - \phi = 1/2\hat{S}_0 y_1/t^{1/2} = 1/2\hat{S}_0^2/(\theta_0 - \theta_n)$ using (7) and (12). Substituting in (11) and simplifying, we get

$$4(\theta_0 - \theta_n)^4 y_1^2 \lambda^4 t^{-1}\hat{S}_0^{-2} = 1 + 4(\theta_0 - \theta_n)^2\lambda^2 x t^{-1/2}\hat{S}_0^{-1} \quad (13)$$

In particular, the value of $-x_0$, where $y = 0$, on the parabola given by (11) when $\phi_0 - \phi = 1/2\hat{S}_0^2/(\theta_0 - \theta_n)$ is given by

$$-x_0 t^{-1/2} = \hat{S}_0/[4\lambda^2(\theta_0 - \theta_n)^2] \quad (14)$$

Replacing y_1 in (13) by its value in (12), we obtain

$$4(\theta_0 - \theta_n)^2\lambda^4 = 1 + 4(\theta_0 - \theta_n)^2\lambda^2 x_1 t^{-1/2}\hat{S}_0^{-1} \quad (15)$$

We now have three equations, (12), (14), and (15), and four unknowns (λ, x_0, x_1, y_1). The missing equation is obtained from consideration of mass conservation.

Let x_2 be the point where the two-dimensional flux $q = -\partial\phi/\partial y$ is equal to the one-dimensional flux, $\frac{1}{2}\hat{S}_0 t^{-1/2}$ (Figure 1), or since from (11),

$$q = \tfrac{1}{2}\lambda \hat{S}_0^{3/2} t^{-1/4}/x_2^{1/2} \quad (16)$$

$$x_2 = \lambda^2 \hat{S}_0 t^{1/2} \quad (17)$$

The total amount of water in excess of the one-dimensional flow $\Delta A(L^3L^{-1})$ is

$$\Delta A = \int_0^t d\bar{t}\left\{\int_0^{x_2(\bar{t})} [q(\bar{t}, x) - \tfrac{1}{2}\hat{S}_0\bar{t}^{-1/2}]\, dx\right\} \quad (18)$$

where $\bar{t}$ identifies the variable of integration. After integration, (18) is

$$\Delta A = \tfrac{1}{2}\lambda^2\hat{S}_0^2 t \quad (19)$$

This must be equal to the extra amount of water behind the wetting front, relative to one-dimensional flow or, since the front is parabolic and vertical at x_0,

$$\Delta A = \tfrac{2}{3}(\theta_0 - \theta_n)y_1(x_1 - x_0) - (\theta_0 - \theta_n)x_1 y_1 \quad (20)$$

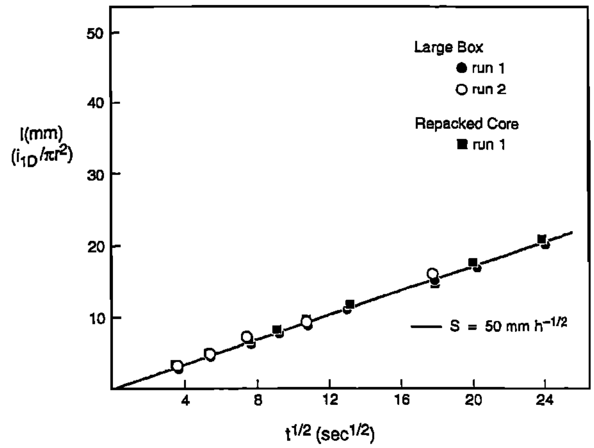

Figure 2. One-dimensional infiltration at short times into Redlands soil (h_0 = −30 mm; $\Delta\theta$ = 0.28).

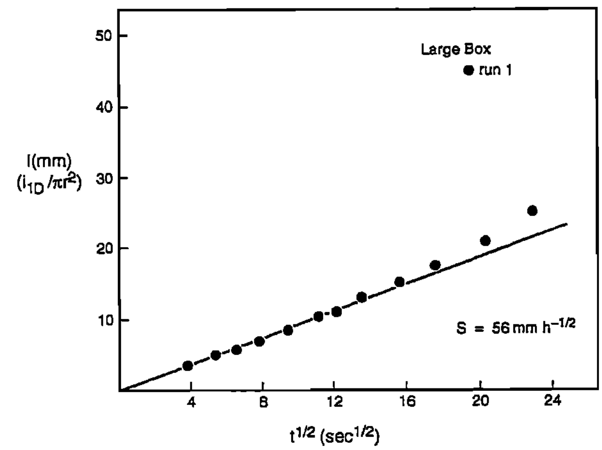

Figure 3. One-dimensional infiltration at short times into Redlands soil (h_0 = −20 mm; $\Delta\theta$ = 0.29).

Equations (19) and (20), together with (12), (14), and (15) yield

$$4\lambda^4(\theta_0 - \theta_n)^2 = 6/5 \tag{21}$$

Hence it follows that

$$\Delta A = \tfrac{1}{2}(0.3)^{1/2}\hat{S}_0^2 t/(\theta_0 - \theta_n) \tag{22}$$

which applies for a unit length.

Then, for a disc of radius r, the cumulative efflux of water from the disc $i_{3D}(L^3)$ is given by

$$i_{3D} = i_{1D} + \pi r(0.3)^{1/2}\hat{S}_0^2 t/(\theta_0 - \theta_n) \tag{23}$$

where the subscripts 3D and 1D refer to the three-dimensional unconfined flow and the one-dimensional flow, respectively. Note that (23) is linear in t because it is independent of gravity.

We also have the following useful wetting front relationships obtained by substituting λ from (21) and y_1 from (12) into (13), (14), and (17):

$$-x_0/y_1 = \tfrac{1}{2}(5/6)^{1/2}(=0.456)$$

$$x_1/y_1 = \tfrac{1}{2}(30)^{1/2}(=0.0913) \tag{24}$$

$$x_2/y_1 = \tfrac{1}{2}(6/5)^{1/2}(=0.548)$$

Note that the maximum value of x_2 may be estimated at a time such that the one-dimensional flux is equal to the hydraulic conductivity difference $(K_0 - K_n)$ at the imposed pressure head, K_1, or $\tfrac{1}{2}\hat{S}_0 t^{-1/2} \approx K_1$, giving

$$x_2 \approx \hat{S}_0^2/4K_1(\theta_0 - \theta_n) \tag{25}$$

Although formulated without gravity, (23) may be useful for all times if it is surmized that the effects of gravity may be incorporated into the i_{1D} term. This will be explored in a subsequent paper.

Experimental Methods

Disc infiltrometer experiments were performed in the laboratory using square perspex boxes with sides of 0.5 m filled with repacked soil. The test material was Redlands sandy loam (Oxic Paleustalf) passed through a 2-mm sieve and carefully packed to a dry bulk density of 1.3 Mg m^{-3}. Prior to sieving, the soil particle size analysis was 10% clay (<2 μm), 7% silt (2–20 μm), 39% fine sand (20 μm to 0.2 mm), and 44% coarse sand (0.2–2 mm).

Because the flow field from a disc is radially symmetric, measurements of flow away from the corner of a box were made primarily using a quarter-disc infiltrometer (an infiltrometer consisting of a 90° segment of a disc) of a radius of 150 mm. A circular disc infiltrometer of 37.5 mm radius was also used to investigate the effect of disc radius. This disc was always placed in the center of a box to allow fully unconfined three-dimensional flow.

One-dimensional infiltration tests were performed on repacked cores of 75-mm diameter and 50-mm length and also in the boxes by inserting a thin impermeable barrier into the

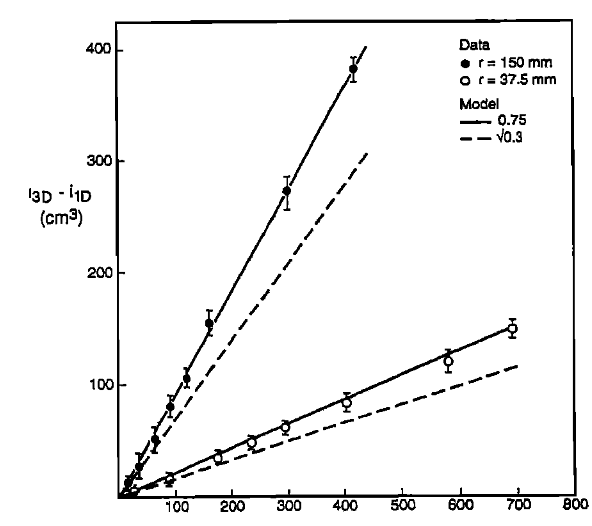

Figure 4. The difference between three- and one-dimensional flow over time into the Redlands soil for two different disc radii at h_0 = −30 mm. The model lines are from (23) with two different values of γ. Bars are ±1 standard deviation.

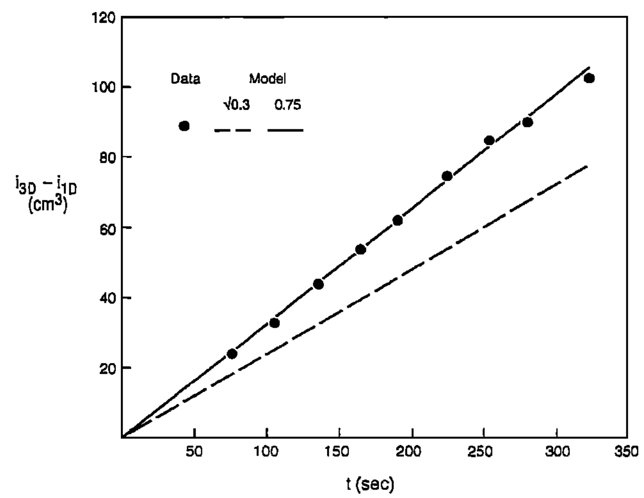

Figure 5. The difference between three- and one-dimensional flow into the Redlands soil at $h_0 = -20$ mm and $r = 45$ mm. The model lines are from (23) with two different values of γ.

soil to a depth of 0.1 m around the perimeter of the quarter disc. Samples for water content were also obtained from the surface of the repacked cores at the cessation of an infiltration experiment. Initial gravimetric water contents were measured on subsamples prior to packing and converted to volumetric water contents (θ_n) using the measured bulk density.

For the purpose of the tests, the disc infiltrometer was maintained at initial supply potentials h_0 of −30 mm and −20 mm H_2O, with θ_n constant at 0.04. The reservoir tower of the disc infiltrometer was filled, and the assembly was placed onto an absorbent tissue to bring the permeameter base to h_0 prior to placement on the soil surface. No contact material was required for these experiments.

Results and Discussion

Cumulative one-dimensional infiltration versus square root of time at supply potentials of −30 mm and −20 mm H_2O are shown in Figures 2 and 3; the sorptivity $\hat{S}_0$ was estimated from the linear early time slopes as 50 and 56 mm $h^{-1/2}$, respectively. Infiltration into the repacked cores and into the larger boxes gave the same results, indicating that the packing of the boxes was adequate for these experiments. Note that at −20 mm, departure from linearity due to the effect of gravity is evident after a short time ($t = 256$ s).

The difference between the three-dimensional unconfined flow and the one-dimensional confined flow for pressure heads of −30 mm and −20 mm are shown in Figures 4 and 5. Results from the quarter disc of 150-mm radius have been multiplied by 4 to allow direct comparison with the 37.5-mm-radius disc. As predicted from (23), the slope of $i_{3D} - i_{1D}$ is linear with time. Empirical observation suggests that a better fit of the model to data for all experiments is obtained by substituting 0.75 for $(0.3)^{1/2}$ in (23). Because of the sharp wetting front approximation used in developing the analytic solution, it is not surprizing that $(0.3)^{1/2}$ underpredicts the experimental data. Using the experimentally determined sorptivity to estimate $\hat{S}_0^2$ would also contribute to the difference (see (4) and (8)). This is explained further in paper 2 of the series [*Haverkamp et al.*, this issue].

Conclusions

An analytic expression for unconfined gravity-free, three-dimensional flow from the disc infiltrometer into soils that are homogenous and isotropic over the depth of wetting has been developed. We have shown that $i_{3D} - i_{1D}$ is proportional to time, but that our solution incorporating $(0.3)^{1/2}$ underpredicts the experimental data due to the assumptions made in developing the analytic expression and due to using measured sorptivity to estimate $\hat{S}_0^2$.

With this analysis, the one- and three-dimensional components of the flow are separated. Infiltration parameters can be obtained over time periods that are experimentally convenient without any recourse to assumptions concerning attainment of the quasi-steady state flow.

Acknowledgments. This study was funded by the CSIRO Coastal Zone Program. Belinda Wilson is thanked for technical assistance.

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R. Haverkamp, Laboratoire d'Etude des Transferts en Hydrologie et Environnement, (LTHE/IMG, UJF, CNRS URA 1512), B.P. 53X, 38041, Grenoble Cedex, France.

J. Y. Parlange, School of Earth Sciences, Flinders University, Adelaide, South Australia 5001, Australia.

P. J. Ross and K. R. J. Smettem, Division of Soils, CSIRO, Private Bag, Aitkenvalle, Townsville, Queensland 4814, Australia.

(Received August 2, 1993; revised June 29, 1994;
accepted July 6, 1994.)