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ASYMPTOTICS FOR A DISCRETE-TIME RISK MODEL WITH THE EMPHASIS ON FINANCIAL RISK

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This paper focuses on a discrete-time risk model in which both insurance risk and financial risk are taken into account. We study the asymptotic behavior of the ruin probability and the tail probability of the aggregate risk amount. Precise asymptotic formulas are derived under weak moment conditions of involved risks. The main novelty of our results lies in the quantification of the impact of the financial risk.

1. INTRODUCTION AND PRELIMINARIES

In this paper, for every $i \geq 1$, let X_i be an insurer's net loss (the total amount of claims less premiums) within period i and let Y_i be the stochastic discount factor (the reciprocal of the stochastic return rate) over the same time period. Then the stochastic present values of aggregate net losses of the insurer can be specified as

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n \geq 1, \tag{1.1}$$

with their maxima

$$M_n = \max_{0 \leq k \leq n} S_k, \quad n \geq 1. \tag{1.2}$$

We are concerned with the asymptotic behavior of the tail probabilities $\mathbb{P}(S_n > x)$ and $\mathbb{P}(M_n > x)$ as $x \to \infty$, in which $\mathbb{P}(M_n > x)$ coincides with the insurer's finite-time ruin probability within period n given that the initial wealth is x.

In the literature, $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ are usually called the insurance risk and the financial risk, respectively. Under certain independence and/or identical distribution assumptions imposed on X_i's and Y_i's, the asymptotic tail behavior of S_n and M_n has been extensively studied by many researchers. See, for example, Tang and Tsitsiashvili [30,31], Konstantinides and Mikosch [16], Tang [28], Zhang, Shen, and Weng [33], Chen [4], and Yang and Wang [32] for some recent findings. Since the products of Y_i's appearing in Eq. (1.1) essentially cause technical problems in the derivation of explicit asymptotic formulas, most of the existing works assumed that the financial risk is dominated by the insurance risk, that is, the tails of Y_i's are lighter than the tails of X_i's, usually through imposing strong moment conditions on Y_i's. Then the problem becomes relatively tractable and the final results are mainly determined by the tails of X_i's.

However, as shown by empirical data and the most recent financial crisis, the financial risk may impair the insurer's solvency as seriously as does the insurance risk and, hence, it should not be underestimated as before; see Norberg [17], Frolova, Kabanov, and Pergamenshchikov [10], Kalashnikov and Norberg [15], and Pergamenshchikov and Zeitouny [22]. Therefore, in the current contribution, we focus on the other directions where the financial risk dominates the insurance risk or no dominating relationship exists between the two kinds of risk. We aim at capturing the impact of the financial risk (the products of Y_i's) on the tail behavior of S_n and M_n. Loosening some independence and identical distribution constraints, we derive precise asymptotic formulas under weak moment conditions of Y_i's and X_i's.

Throughout this paper, an underlying assumption is the following:

Assumption A: $\{X_i; i \geq 1\}$ is a sequence of real-valued rv's (random variables) with distribution functions F_i's, $\{Y_i; i \geq 1\}$ is a sequence of positive and independent rv's with distribution functions G_i's, and $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ are mutually independent.

It is worth mentioning that, if we further assume that both $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ are sequences of iid (independent and identically distributed) rv's in Eq. (1.1), then there is a natural connection between this discrete-time risk model and the general bivariate Lévy-driven risk model with the form

$$U_t = \int_0^t e^{Q_s} \mathrm{d}P_s, \quad t \geq 0,$$

where $\{Q_s; s \geq 0\}$ and $\{P_s; s \geq 0\}$ are two independent Lévy processes; see Paulsen [20,21], Hao and Tang [12], and references therein. To see this, arbitrarily embed an increasing sequence of stopping times, say $\{\tau_i; i \geq 1\}$, to the Lévy-driven model. Then, after such a discretization procedure, U_{τ_n} takes the form as S_n in Eq. (1.1). Due to this reason, the results obtained in this paper can provide us with some valuable insights to the general bivariate Lévy-driven case.

We restrict our discussions within the scope that Y_i's are regularly varying. A real-valued rv Z with distribution function H is said to be regularly varying if its survival function $\overline{H} = 1 - H$ is regularly varying at infinity, that is, $\lim_{x\to\infty} \overline{H}(xy)/\overline{H}(x) = y^{-\alpha}$ for every $y > 0$ and some $\alpha \geq 0$. In this case, we write $Z \in \mathcal{R}_{-\alpha}$ or $\overline{H} \in \mathcal{R}_{-\alpha}$. A positive regularly varying function with $\alpha = 0$ is also called slowly varying function. See Bingham, Goldie, and Teugels [1], Resnick [23], or Embrechts, Klüpelberg, and Mikosch [8] for more details on regularly varying functions.

Hereafter, all limit relations hold as $x \to \infty$ unless otherwise specified. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \gtrsim b(x)$ or $b(x) \lesssim a(x)$ if $\liminf_{x\to\infty} a(x)/b(x) \geq 1$ and write $a(x) \sim b(x)$ if both $a(x) \lesssim b(x)$ and $a(x) \gtrsim b(x)$.

Our first result below shows that, in a special case of regular variation, the moment conditions of involved rv's can be dropped thanks to a Rootzén-type lemma stated in Section 3 (Lemma 3.1).

Theorem 1.1. *Under Assumption* **A**, *let X_i's be independent. If, for every $i \geq 1$, $\overline{F}_i(x) \sim \ell_i^*(\ln x) \cdot (\ln x)^{\gamma^*-1} x^{-\alpha}$ and $\overline{G}_i(x) \sim \ell_i(\ln x)(\ln x)^{\gamma_i-1} x^{-\alpha}$ for some positive constants $\alpha, \gamma^*, \gamma_i$ and some slowly varying functions $\ell_i^*(\cdot), \ell_i(\cdot)$ then, for every $n \geq 1$, letting $\bar{\gamma}_n = \gamma^* + \sum_{i=1}^n \gamma_i$,* we have

$$\begin{aligned}\mathbb{P}(S_n > x) \sim \mathbb{P}(M_n > x) \sim \mathbb{P}\left(X_n \prod_{j=1}^n Y_j > x\right) \\ \sim \frac{\alpha^n \Gamma(\gamma^*) \prod_{i=1}^n \Gamma(\gamma_i)}{\Gamma(\bar{\gamma}_n)} \ell_n^*(\ln x) \left(\prod_{i=1}^n \ell_i(\ln x)\right) (\ln x)^{\bar{\gamma}_n-1} x^{-\alpha}. \qquad \textbf{(1.3)}\end{aligned}$$

Remark 1.1. A well-known folklore in risk theory is that the ruin of an insurer, that is, the tail of M_n, will be determined by one of the insurance risk and the financial risk which has a heavier tail. Nevertheless, Theorem 1.1 provides a counterexample violating the folklore. To see this more clearly, let both $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be sequences of iid rv's with common survival functions $\overline{F}(x) \sim \ell^*(\ln x)(\ln x)^{\gamma^*-1} x^{-\alpha}$ and $\overline{G}(x) \sim \ell(\ln x)(\ln x)^{\gamma-1} x^{-\alpha}$, respectively. Then, according to the different selections of $\gamma^*, \ell^*(\cdot)$ and $\gamma, \ell(\cdot)$, Theorem 1.1 covers various asymptotic relationships between $\overline{F}$ and $\overline{G}$. However, we have the unified asymptotic expansion determined by both $\overline{F}$ and $\overline{G}$.

Remark 1.2. Tang and Tsitsiashvili [30] gave a similar result for M_n in their Theorem 6.2. Their result does not cover, and is not covered by, our Theorem 1.1, since their conditions of X_i's and ours are mutually exclusive. However, their assumptions imply $\overline{F}(x) = o\left(\overline{G}(x)\right)$, whereas our Theorem 1.1, as stated in Remark 1.1, is valid for various relationships between $\overline{F}$ and $\overline{G}$.

Theorem 1.1 presents an elegant result which is due to the special forms of $\overline{F}_i$'s and $\overline{G}_i$'s. In the subsequent sections, we focus on asymptotic analysis of S_n and M_n for general regularly varying conditions, while the price to pay for it is the lack of elegance and the high technicalities of the proofs. Our main results presented in Theorem 2.1 below show that, as expected, similarly to Theorem 1.1, both S_n and M_n are regularly varying rv's under some general conditions. Furthermore, we derive precise tail asymptotics for both S_n and M_n. One remarkable feature of our Theorem 2.1 is the weakening of the moment assumptions commonly imposed on X_i's and Y_i's in the literature.

The rest of the paper is organized as follows. Section 2 shows our main theorem with several interesting remarks. Section 3 gives the lemmas and proofs related to the results presented in Sections 1 and 2. Finally, Appendix A discusses the constant weighted sums of the products of Y_i's ($X_i \equiv c_i > 0$ for every $i \geq 1$ in Eq. (1.1)), which model the stochastic present values of some risk-free bond with fixed income c_i in period i. We derive an asymptotic formula with the uniformity of the constant weights in this case.

2. MAIN RESULTS AND REMARKS

Hereafter, the summation and the product over an empty set of indices are considered as 0 and 1, respectively. Moreover, to avoid triviality, every individual real-valued rv is assumed to be not only concentrated on $(-\infty, 0]$. For a real number a, we write $a_+ = a \vee 0$.

Under the framework specified in Assumption **A**, we continue to study the tail behavior of S_n and M_n defined in Eqs. (1.1) and (1.2). For the conciseness in writing and presentation, we further define

$$S_0^{(l)} = 0, \quad S_n^{(l)} = \sum_{i=l}^{n+l-1} X_i \prod_{j=l}^{i} Y_j, \quad l, n = 1, 2, \ldots,$$

and

$$M_n^{(l)} = \max_{0 \le k \le n} S_k^{(l)}, \quad l, n = 1, 2, \ldots.$$

Clearly, $S_n^{(l)}$ describes the stochastic present value at time $l-1$ of aggregate net losses occurring from time l to time $n+l-1$. Note in passing that $S_n^{(1)} = S_n$, $M_n^{(1)} = M_n$, and further

$$S_n^{(l)} = Y_l \left(X_l + S_{n-1}^{(l+1)} \right) \text{ and } M_n^{(l)} = Y_l \left(X_l + M_{n-1}^{(l+1)} \right)_+, \quad l, n = 1, 2, \ldots. \tag{2.1}$$

Our main results are given in the following Theorem 2.1, in which assertion (i) is valid for arbitrarily dependent X_i's, assertion (ii) drops the dominating relationship between $\overline{F}_i$'s and $\overline{G}_i$'s, and neither assertion (i) nor (ii) requires $\mathbb{E}(X_i)_+^\beta < \infty$ or $\mathbb{E}Y_i^\beta < \infty$ for every $i \ge 1$ and some $\beta > \alpha$.

Theorem 2.1. *Under Assumption* **A**, *assume that $\overline{G}_i \in \mathcal{R}_{-\alpha}$ for every $i \ge 1$ and some $\alpha \ge 0$, and $\mathbb{E}Y_i^\alpha < \infty$ for every $i \ge 2$.*

(i) *If $X_i Y_i \in \mathcal{R}_{-\alpha}$ and*

$$\mathbb{P}(|X_i| > x) = o\left(\overline{G}_{i+1}(x)\right) \tag{2.2}$$

for every $i \ge 1$ then, for every $n \ge 1$, $S_n \in \mathcal{R}_{-\alpha}$, $M_n \in \mathcal{R}_{-\alpha}$, and further

$$\mathbb{P}(S_n > x) \sim \sum_{i=1}^{n-1} B_{n,i} \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) + \mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right) \tag{2.3}$$

and

$$\mathbb{P}(M_n > x) \sim \sum_{i=1}^{n-1} D_{n,i} \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) + \mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right), \tag{2.4}$$

where

$$B_{n,i} = \mathbb{E}\left(X_i + S_{n-i}^{(i+1)}\right)_+^\alpha - \mathbb{E}\left(S_{n-i}^{(i+1)}\right)_+^\alpha \quad \text{and} \quad D_{n,i} = \mathbb{E}\left(X_i + M_{n-i}^{(i+1)}\right)_+^\alpha - \mathbb{E}\left(M_{n-i}^{(i+1)}\right)^\alpha.$$

(ii) *If X_i's are independent and $\overline{F}_i \in \mathcal{R}_{-\alpha}$ with $\mathbb{E}(X_i)_+^\alpha < \infty$ for every $i \ge 1$ then, for every $n \ge 1$, $S_n \in \mathcal{R}_{-\alpha}$, $M_n \in \mathcal{R}_{-\alpha}$, and further*

$$\mathbb{P}(S_n > x) \sim \sum_{i=1}^{n-1} \left(B_{n,i} - \mathbb{E}(X_i)_+^\alpha\right) \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) + \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \tag{2.5}$$

and

$$\mathbb{P}(M_n > x) \sim \sum_{i=1}^{n-1} \left(D_{n,i} - \mathbb{E}(X_i)_+^\alpha\right) \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) + \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right). \tag{2.6}$$

One important theoretical merit of Theorem 2.1 lies in that, through the transparent expansions (2.3)–(2.6), it gives new criteria for the regular-variation membership of S_n and M_n. A common shortcoming of formulas (2.3)–(2.6) is the involved constants which cannot be accurately calculated in general. However, this is the price we have to pay for highlighting the impact of the financial risk Y_i's and weakening the moment conditions. Moreover, our explicit expressions of $B_{n,i}$ and $D_{n,i}$ enable us to easily conduct numerical estimates.

The following remarks and Corollary 2.1 contain some interesting special cases of Theorem 2.1, from which one can realize to some extents the flexibility and generalization of our Theorem 2.1.

Remark 2.1. If $\alpha = 0$ then assertion (i) gives

$$\mathbb{P}\left(S_n > x\right) \sim \mathbb{P}\left(M_n > x\right) \sim \mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right),$$

and assertion (ii) reduces to

$$\mathbb{P}\left(S_n > x\right) \sim \mathbb{P}\left(M_n > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) - \sum_{i=1}^{n-1} \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right).$$

Remark 2.2. Clearly, if $\mathbb{E}\left|X_i\right|^{\beta} < \infty$ for every $i \geq 1$ and some $\beta > \alpha$ then the two special conditions of assertion (i) hold in view of Lemma 3.2(a) below. In this case, the last term of Eqs. (2.3) and (2.4) can be expanded as follows by Breiman's lemma; see Breiman [2],

$$\mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right) \sim \mathbb{E}\left(X_n\right)_+^{\alpha} \cdot \mathbb{P}\left(\prod_{j=1}^{n} Y_j > x\right).$$

Plugging this relation into Eqs. (2.3) and (2.4) and noting that $\mathbb{E}\left(X_n\right)_+^{\alpha} = B_{n,n} = D_{n,n}$ yield

$$\mathbb{P}\left(S_n > x\right) \sim \sum_{i=1}^{n} B_{n,i} \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) \text{ and } \mathbb{P}\left(M_n > x\right) \sim \sum_{i=1}^{n} D_{n,i} \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right).$$

Remark 2.3. By the proofs of Theorem 2.1(i) and Lemma 3.3 below, if X_i's are independent then Eq. (2.2) in assertion (i) can be weakened to $\overline{F}_i(x) = o\left(\overline{G}_{i+1}(x)\right)$.

In what follows, for a sequence $\{Z_i; i \geq 1\}$ of iid rv's, we always denote by Z its generic rv.

Remark 2.4. By Lemma 3.2(a) and Remark 2.3, if both $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ are sequences of iid rv's then only $\overline{F}(x) = o\left(\overline{G}(x)\right)$ suffices for assertion (i). Moreover, in this case, we have

$$B_{n,i} = B_{n-i} = \mathbb{E}\left(X_1 + S_{n-i}^{(2)}\right)_+^{\alpha} - \mathbb{E}\left(S_{n-i}^{(2)}\right)_+^{\alpha} = \mathbb{E}\left(S_{n-i+1}\right)_+^{\alpha}\left(\mathbb{E}Y^{\alpha}\right)^{-1} - \mathbb{E}\left(S_{n-i}\right)_+^{\alpha},$$

and

$$D_{n,i} = D_{n-i} = \mathbb{E}\left(X_1 + M_{n-i}^{(2)}\right)_+^{\alpha} - \mathbb{E}\left(M_{n-i}^{(2)}\right)^{\alpha} = \mathbb{E}M_{n-i+1}^{\alpha}\left(\mathbb{E}Y^{\alpha}\right)^{-1} - \mathbb{E}M_{n-i}^{\alpha}.$$

Remark 2.5. The conditions of assertion (ii) do not exclude the simultaneous occurrence of $\overline{F}_i(x) = o\left(\overline{G}_{i+1}(x)\right)$ for every $i \geq 1$. In such an intersectional case, Lemma 3.2(b) and

Remark 2.3 imply that assertion (i) also holds and, hence, Eqs. (2.5) and (2.6) should be equivalent to Eqs. (2.3) and (2.4), respectively. The fact can be easily shown through Lemma 3.5 below. Actually, for every $1 \le i \le n-1$, by $\overline{F}_i(x) = o\left(\overline{G}_{i+1}(x)\right)$ and Lemma 3.5, we have

$$\mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) - \mathbb{E}\left(X_i\right)_+^{\alpha} \cdot \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) = o(1)\mathbb{P}\left(\prod_{j=1}^{i+1} Y_j > x\right).$$

On the other hand, it follows from Fatou's lemma that, for every $1 \le i \le n-1$,

$$\mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right) \gtrsim \mathbb{E}\left(X_n \prod_{j=i+2}^{n} Y_j\right)_+^{\alpha} \cdot \mathbb{P}\left(\prod_{j=1}^{i+1} Y_j > x\right).$$

Hence,

$$\sum_{i=1}^{n-1}\left(\mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) - \mathbb{E}\left(X_i\right)_+^{\alpha} \cdot \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right)\right) = o(1)\mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right),$$

which implies that Eqs. (2.5) and (2.6) are equivalent to Eqs. (2.3) and (2.4), respectively.

The following corollary concerns another special case of Theorem 2.1, in which the more explicit asymptotics can be derived. The assertion for M_n was partially given by Theorem 6.1 of Tang and Tsitsiashvili [30]. Recall that a real-valued rv Z with survival function $\overline{H}$ is said to belong to the class $\mathcal{S}(\alpha)$ for some $\alpha \ge 0$ if

$$\lim_{x\to\infty} \frac{\overline{H}(x-y)}{\overline{H}(x)} = e^{\alpha y}, \quad y \in (-\infty, \infty), \tag{2.7}$$

and

$$\lim_{x\to\infty} \frac{\overline{H_+^{2*}}(x)}{\overline{H}(x)} = 2\mathbb{E}e^{\alpha Z} < \infty,$$

where $H_+(x) = H(x)\mathbf{1}_{\{x\ge 0\}}$ and H_+^{2*} stands for the 2-fold convolution of H_+. In the literature, relation Eq. (2.7) itself defines a larger class denoted by $\mathcal{L}(\alpha)$. See, for example, Cline [6] and Pakes [18,19] for more details on the classes $\mathcal{S}(\alpha)$ and $\mathcal{L}(\alpha)$. Note that, for a positive rv Z, $\ln Z \in \mathcal{S}(\alpha)$ implies $Z \in \mathcal{R}_{-\alpha}$ and $\mathbb{E}Z^{\alpha} < \infty$.

Corollary 2.1. *Under Assumption* **A**, *let both* $\{X_i; i \ge 1\}$ *and* $\{Y_i; i \ge 1\}$ *be sequences of iid rv's. If* $\ln Y \in \mathcal{S}(\alpha)$ *for some* $\alpha \ge 0$ *and* $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = \theta \in [0,\infty)$ *then, for every* $n \ge 1$,

$$\mathbb{P}\left(S_n > x\right) \sim K_n \overline{G}(x) \text{ and } \mathbb{P}\left(M_n > x\right) \sim L_n \overline{G}(x), \tag{2.8}$$

where

$$K_n = \sum_{i=1}^{n}\left(\mathbb{E}\left(S_{n-i+1}\right)_+^{\alpha}\left(\mathbb{E}Y^{\alpha}\right)^{i-2} + \theta\left(\mathbb{E}Y^{\alpha}\right)^{i}\right)$$

and

$$L_n = \sum_{i=1}^{n}\left(\mathbb{E}M_{n-i+1}^{\alpha}\left(\mathbb{E}Y^{\alpha}\right)^{i-2} + \theta\left(\mathbb{E}Y^{\alpha}\right)^{i}\right).$$

Particularly, if $\alpha = 0$ *then, for every* $n \ge 1$,

$$\mathbb{P}\left(S_n > x\right) \sim \mathbb{P}\left(M_n > x\right) \sim \left(\theta + 1\right) n\overline{G}(x).$$

3. LEMMAS AND PROOFS

The following result is due to Corollary 2.1 of Hashorva and Li [13], which is motivated by Lemma 7.1 of Rootzén [25]; see also Rootzén [26]. Note that for iid Z_i's such that $\mathbb{P}(Z > x) \sim cx^{-\alpha}$ the assertion was shown in Lemma 4.1(4) of Jessen and Mikosch [14].

Lemma 3.1. *Let $Z_1, \ldots, Z_n$ be n positive and independent rv's. If, for every $1 \le i \le n$, $\mathbb{P}(Z_i > x) \sim \ell_i(\ln x)(\ln x)^{\gamma_i - 1}x^{-\alpha}$ for some positive constants α, γ_i and some slowly varying function $\ell_i(\cdot)$ then we have*

$$\mathbb{P}\left(\prod_{i=1}^{n} Z_i > x\right) \sim \frac{\alpha^{n-1}\prod_{i=1}^{n}\Gamma(\gamma_i)}{\Gamma\left(\sum_{i=1}^{n}\gamma_i\right)}\left(\prod_{i=1}^{n}\ell_i(\ln x)\right)(\ln x)^{\sum_{i=1}^{n}\gamma_i - 1}x^{-\alpha}.$$

Proof of Theorem 1.1: The last relation in Eq. (1.3) follows immediately from Lemma 3.1. It remains to verify that both the tails of S_n and M_n are asymptotically equivalent to the right-hand side of Eq. (1.3). We only prove the assertion for S_n, since the counterpart of M_n can be obtained similarly.

By Lemma 3.1, it is clear that the assertion holds for $S_1 = X_1 Y_1$. Now we assume by induction that the assertion holds for $n - 1 \ge 1$ and prove it for n. Recalling Eq. (2.1), it holds that

$$\mathbb{P}\left(S_n > x\right) = \mathbb{P}\left(Y_1\left(X_1 + S_{n-1}^{(2)}\right) > x\right). \tag{3.1}$$

From the induction assumption, we know that $S_{n-1}^{(2)} \in \mathcal{R}_{-\alpha}$ and $\overline{F}_1(x) = o(1)\mathbb{P}\left(S_{n-1}^{(2)} > x\right)$. Noting also that $\overline{F}_1 \in \mathcal{R}_{-\alpha}$ and X_1 is independent of $S_{n-1}^{(2)}$, we have (see, e.g., Feller [9], pp. 278)

$$\begin{aligned}\mathbb{P}\left(X_1 + S_{n-1}^{(2)} > x\right) &\sim \mathbb{P}\left(S_{n-1}^{(2)} > x\right)\\ &\sim \frac{\alpha^{n-1}\Gamma(\gamma^*)\prod_{i=2}^{n}\Gamma(\gamma_i)}{\Gamma\left(\gamma^* + \sum_{i=2}^{n}\gamma_i\right)}\ell_n^*(\ln x)\left(\prod_{i=2}^{n}\ell_i(\ln x)\right)(\ln x)^{\gamma^* + \sum_{i=2}^{n}\gamma_i - 1}x^{-\alpha}.\end{aligned}$$

Then, applying Lemma 3.1 to Y_1 and $X_1 + S_{n-1}^{(2)}$ in (3.1) completes the proof. ∎

The next lemma is a restatement of the Corollary of Theorem 3 in Embrechts and Goldie [7].

Lemma 3.2. *Let Y be a positive rv with survival function $\overline{G} \in \mathcal{R}_{-\alpha}$ for some $\alpha \ge 0$ and let Z be a real-valued rv with survival function $\overline{H}$. Assume that Y and Z are independent. Then $YZ \in \mathcal{R}_{-\alpha}$ if either (a) $\overline{H}(x) = o(\overline{G}(x))$ or (b) $\overline{H} \in \mathcal{R}_{-\alpha}$.*

The first assertion of Lemma 3.3 below is borrowed from Lemma 3.3 of Hao and Tang [12]; see also Lemma 4.4.2 of Samorodnitsky and Taqqu [27], and the second assertion is a special case of Proposition 2 of Rogozin and Sgibnev [24].

Lemma 3.3. *Let Y and Z be two real-valued rv's with survival functions $\overline{G}$ and $\overline{H}$, respectively. If $\overline{G} \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$ and*

$$\mathbb{P}\left(|Z| > x\right) = o\left(\overline{G}(x)\right) \tag{3.2}$$

then

$$\mathbb{P}\left(Y + Z > x\right) \sim \overline{G}(x).$$

Particularly, if Y and Z are independent then Eq. (3.2) can be weakened as $\overline{H}(x) = o\left(\overline{G}(x)\right)$.

Lemma 3.4 below is crucial for the proof of our main theorem.

Lemma 3.4. *Let Y be a positive rv with survival function $\overline{G} \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$ and let $Z_1, \ldots, Z_n$ be n real-valued rv's satisfying $\mathbb{E}\left(Z_i\right)_+^\alpha < \infty$ for every $1 \leq i \leq n$ and*

$$\mathbb{P}\left(\sum_{i=1}^n Z_i > x\right) \sim \sum_{i=1}^n c_i \mathbb{P}\left(Z_i > x\right) \tag{3.3}$$

for n non-negative constants $c_1, \ldots, c_n$ such that $\max_{1 \leq i \leq n} c_i > 0$. Assume further that Y and $\{Z_1, \ldots, Z_n\}$ are independent. Then

$$\mathbb{P}\left(Y\sum_{i=1}^n Z_i > x\right) \sim \left(\mathbb{E}\left(\sum_{i=1}^n Z_i\right)_+^\alpha - \sum_{i=1}^n c_i \mathbb{E}\left(Z_i\right)_+^\alpha\right)\mathbb{P}\left(Y > x\right) + \sum_{i=1}^n c_i \mathbb{P}\left(YZ_i > x\right). \tag{3.4}$$

One merit of Lemma 3.4 is that we do not require $\mathbb{E}\left(Z_i\right)_+^\beta < \infty$ for every $1 \leq i \leq n$ and some $\beta > \alpha$. In return, the tails of products $\mathbb{P}\left(YZ_i > x\right)$ for $1 \leq i \leq n$ cannot be expanded further. Otherwise, relation Eq. (3.4) will reduce to Breiman's formula. If Z_i 's are independent then relation Eq. (3.3) with $c_1 = \cdots = c_n = 1$ is usually called the max-sum equivalence property; see, for example, Cai and Tang [3] for some heavy-tailed distribution classes satisfying such a property. Moreover, even under some special dependence structures, including the pairwise negative dependence and (quasi-) asymptotic independence, relation Eq. (3.3) still holds with $c_1 = \cdots = c_n = 1$ for Z_i's belonging to certain heavy-tailed distribution classes; see Chen and Yuen [5], Geluk and Tang [11], and Tang [29], among others.

Proof of Lemma 3.4: For every $0 < \varepsilon < 1$, by relation Eq. (3.3), there is some $M > 0$ such that the relations

$$(1-\varepsilon)\sum_{i=1}^n c_i \mathbb{P}\left(Z_i > x\right) \leq \mathbb{P}\left(\sum_{i=1}^n Z_i > x\right) \leq (1+\varepsilon)\sum_{i=1}^n c_i \mathbb{P}\left(Z_i > x\right) \tag{3.5}$$

hold for all $x \geq M$. By this large M, we rewrite the left-hand side of (3.4) as

$$\begin{aligned}\mathbb{P}\left(Y\sum_{i=1}^n Z_i > x\right) &= \mathbb{P}\left(Y\sum_{i=1}^n Z_i > x, Y > \frac{x}{M}\right) + \mathbb{P}\left(Y\sum_{i=1}^n Z_i > x, Y \leq \frac{x}{M}\right)\\ &= I_1(M,x) + I_2(M,x).\end{aligned}$$

Applying Remark A.1(a) below to $I_1(M,x)$, we have, for M large enough,

$$1 - \varepsilon \leq \lim_{x\to\infty} \frac{I_1(M,x)}{\mathbb{E}\left(\sum_{i=1}^n Z_i\right)_+^\alpha \cdot \mathbb{P}\left(Y > x\right)} \leq 1 + \varepsilon. \tag{3.6}$$

Consider $I_2(M,x) = \int_0^{x/M} \mathbb{P}\left(\sum_{i=1}^n Z_i > x/y\right) G(\mathrm{d}y)$. It follows from Eq. (3.5) that

$$(1-\varepsilon)J(M,x) \le I_2(M,x) \le (1+\varepsilon)J(M,x), \tag{3.7}$$

where

$$J(M,x) = \sum_{i=1}^n c_i \mathbb{P}\left(YZ_i > x\right) - \sum_{i=1}^n c_i \mathbb{P}\left(YZ_i > x, Y > \frac{x}{M}\right).$$

Using Remark A.1(a) again to each summand of the second summation, we obtain that, for M large enough,

$$1-\varepsilon \le \lim_{x\to\infty} \frac{J(M,x)}{\sum_{i=1}^n c_i \mathbb{P}\left(YZ_i > x\right) - \sum_{i=1}^n c_i \mathbb{E}\left(Z_i\right)_+^\alpha \cdot \mathbb{P}\left(Y > x\right)} \le 1+\varepsilon. \tag{3.8}$$

Combining Eqs. (3.6)–(3.8) and noting the arbitrariness of ε complete the proof. ■

Lemma 3.5. *Let Y be a positive rv with survival function $\overline{G} \in \mathcal{R}_{-\alpha}$ for some $\alpha \ge 0$ and let Z_1, Z_2 be two real-valued rv's with survival functions H_1, H_2 satisfying $\overline{H}_1(x) = o\left(\overline{H}_2(x)\right)$ and $\mathbb{E}\left(Z_2\right)_+^\alpha < \infty$. Assume that Y and $\{Z_1, Z_2\}$ are independent. Then*

$$\mathbb{P}\left(YZ_1 > x\right) - \mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x) = o(1)\mathbb{P}\left(YZ_2 > x\right).$$

Proof: For every $0 < \varepsilon < 1$, since $\overline{H}_1(x) = o\left(\overline{H}_2(x)\right)$, there is some M such that for all $x \ge M$ the relation $\overline{H}_1(x) \le \varepsilon \overline{H}_2(x)$ holds. Write

$$\mathbb{P}\left(YZ_1 > x\right) = \mathbb{P}\left(YZ_1 > x, Y > \frac{x}{M}\right) + \mathbb{P}\left(YZ_1 > x, Y \le \frac{x}{M}\right) = I_1(M,x) + I_2(M,x).$$

By Remark A.1(a), choosing M large enough, it holds that

$$\lim_{x\to\infty} \frac{I_1(M,x) - \mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x)}{\mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x)} \le \varepsilon. \tag{3.9}$$

For $I_2(M,x)$, by conditioning on Y and noting that $\overline{H}_1(x) \le \varepsilon \overline{H}_2(x)$ for $x \ge M$, we have

$$I_2(M,x) \le \varepsilon \mathbb{P}\left(YZ_2 > x, Y \le \frac{x}{M}\right) \le \varepsilon \mathbb{P}\left(YZ_2 > x\right). \tag{3.10}$$

Moreover, Fatou's lemma gives

$$\mathbb{P}\left(YZ_2 > x\right) \gtrsim \mathbb{E}\left(Z_2\right)_+^\alpha \cdot \overline{G}(x). \tag{3.11}$$

Therefore,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathbb{P}\left(YZ_1 > x\right) - \mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x)}{\mathbb{P}\left(YZ_2 > x\right)} \\
&\quad = \limsup_{x\to\infty} \left(\frac{I_1(M,x) - \mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x)}{\mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x)} \cdot \frac{\mathbb{E}\left(Z_1\right)_+^\alpha \cdot \overline{G}(x)}{\mathbb{P}\left(YZ_2 > x\right)} + \frac{I_2(M,x)}{\mathbb{P}\left(YZ_2 > x\right)} \right) \\
&\quad \le \varepsilon \left(\frac{\mathbb{E}\left(Z_1\right)_+^\alpha}{\mathbb{E}\left(Z_2\right)_+^\alpha} + 1 \right),
\end{aligned}$$

where in the last step we used Eqs. (3.9)–(3.11) in turn. Noting the arbitrariness of ε completes the proof. ■

Proof of Theorem 2.1(i): We only derive relation (2.3) which implies that $S_n \in \mathcal{R}_{-\alpha}$ by Lemma 3.2(b), then the assertions regarding M_n follow from the similar procedures with obvious modifications.

We proceed by the mathematical induction. Trivially, relation (2.3) holds for $n = 1$ with a by-product

$$\mathbb{P}\left(S_1 > x\right) \gtrsim \mathbb{E}\left(X_1\right)_+^\alpha \cdot \mathbb{P}\left(Y_1 > x\right).$$

Assume by induction that relation (2.3) holds for $n-1 \geq 1$ with

$$\mathbb{P}\left(S_{n-1} > x\right) \gtrsim \mathbb{E}\left(X_1 + S_{n-2}^{(2)}\right)_+^\alpha \cdot \mathbb{P}\left(Y_1 > x\right).$$

Now we consider S_n and recall that relation (3.1) holds. Applying the induction assumption to $\{Y_2, \ldots, Y_n\}$ and $\{X_2, \ldots, X_n\}$ leads to

$$\mathbb{P}\left(S_{n-1}^{(2)} > x\right) \gtrsim \mathbb{E}\left(X_2 + S_{n-2}^{(3)}\right)_+^\alpha \cdot \mathbb{P}\left(Y_2 > x\right). \tag{3.12}$$

Combining Eq. (3.12) with (2.2) gives

$$\mathbb{P}\left(|X_1| > x\right) = o\left(1\right)\mathbb{P}\left(S_{n-1}^{(2)} > x\right),$$

which, together with Lemma 3.3, gives

$$\mathbb{P}\left(X_1 + S_{n-1}^{(2)} > x\right) \sim \mathbb{P}\left(S_{n-1}^{(2)} > x\right).$$

Applying Lemma 3.4 to Eq. (3.1) with Y, Z_1, Z_2 replaced by Y_1, X_1, $S_{n-1}^{(2)}$, respectively, and $c_1 = 0$, $c_2 = 1$, we have

$$\begin{aligned}\mathbb{P}\left(S_n > x\right) &\sim \left(\mathbb{E}\left(X_1 + S_{n-1}^{(2)}\right)_+^\alpha - \mathbb{E}\left(S_{n-1}^{(2)}\right)_+^\alpha\right)\mathbb{P}\left(Y_1 > x\right) + \mathbb{P}\left(Y_1 S_{n-1}^{(2)} > x\right) \\ &= B_{n,1}\mathbb{P}\left(Y_1 > x\right) + \mathbb{P}\left(\widehat{S}_{n-1}^{(2)} > x\right),\end{aligned} \tag{3.13}$$

where $\widehat{S}_{n-1}^{(2)}$ stands for $S_{n-1}^{(2)}$ with Y_2 replaced by $Y_1 Y_2$. Clearly, $\{Y_1Y_2, Y_3, \ldots, Y_n\}$ and $\{X_2, \ldots, X_n\}$ also satisfy all the conditions of assertion (i). Thus, using the induction assumption to $\widehat{S}_{n-1}^{(2)}$ yields

$$\mathbb{P}\left(\widehat{S}_{n-1}^{(2)} > x\right) \sim \sum_{i=2}^{n-1} B_{n,i}\mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) + \mathbb{P}\left(X_n \prod_{j=1}^{n} Y_j > x\right). \tag{3.14}$$

A combination of Eqs. (3.13) and (3.14) gives relation Eq. (2.3). ■

Proof of Theorem 2.1(ii): Similarly as before, we only derive relation Eq. (2.5) by the mathematical induction. Trivially, relation (2.5) holds for $n = 1$. Assume by induction that relation (2.5) holds for $n-1 \geq 1$, which implies that $S_{n-1}^{(2)} \in \mathcal{R}_{-\alpha}$. Since $F_1 \in \mathcal{R}_{-\alpha}$ and X_1

is independent of $S_{n-1}^{(2)}$, it holds that

$$\mathbb{P}\left(X_1 + S_{n-1}^{(2)} > x\right) \sim \mathbb{P}\left(X_1 > x\right) + \mathbb{P}\left(S_{n-1}^{(2)} > x\right).$$

Now, applying Lemma 3.4 to Eq. (3.1) with Y, Z_1, Z_2 replaced by Y_1, X_1, $S_{n-1}^{(2)}$, respectively, and $c_1 = c_2 = 1$, we have

$$\begin{aligned}\mathbb{P}\left(S_n > x\right) &\sim \left(\mathbb{E}\left(X_1 + S_{n-1}^{(2)}\right)_+^\alpha - \mathbb{E}\left(X_1\right)_+^\alpha - \mathbb{E}\left(S_{n-1}^{(2)}\right)_+^\alpha\right)\mathbb{P}\left(Y_1 > x\right)\\ &\quad + \mathbb{P}\left(X_1 Y_1 > x\right) + \mathbb{P}\left(Y_1 S_{n-1}^{(2)} > x\right)\\ &= \left(B_{n,1} - \mathbb{E}\left(X_1\right)_+^\alpha\right)\mathbb{P}\left(Y_1 > x\right) + \mathbb{P}\left(X_1 Y_1 > x\right) + \mathbb{P}\left(\widehat{S}_{n-1}^{(2)} > x\right). \qquad \textbf{(3.15)}\end{aligned}$$

Since $\{Y_1 Y_2, Y_3, \ldots, Y_n\}$ and $\{X_2, \ldots, X_n\}$ also satisfy all the conditions of assertion (ii), using the induction assumption on $\widehat{S}_{n-1}^{(2)}$ yields

$$\mathbb{P}\left(\widehat{S}_{n-1}^{(2)} > x\right) \sim \sum_{i=2}^{n-1}\left(B_{n,i} - \mathbb{E}\left(X_i\right)_+^\alpha\right)\mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) + \sum_{i=2}^{n}\mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right). \qquad \textbf{(3.16)}$$

A combination of Eqs. (3.15) and (3.16) gives relation (2.5). ∎

Proof of Corollary 2.1: Since $\ln Y \in \mathcal{S}(\alpha)$ and $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = \theta$, we can derive by Proposition 2 of Rogozin and Sgibnev [24] that, for every $i \geq 1$,

$$\mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \sim \left(i\mathbb{E}X_+^\alpha + \theta\mathbb{E}Y^\alpha\right)\left(\mathbb{E}Y^\alpha\right)^{i-1}\overline{G}(x), \qquad \textbf{(3.17)}$$

and, particularly,

$$\mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right) \sim i\left(\mathbb{E}Y^\alpha\right)^{i-1}\overline{G}(x). \qquad \textbf{(3.18)}$$

If $\theta = 0$, that is, $\overline{F}(x) = o\left(\overline{G}(x)\right)$, then Remark 2.4 indicates that Theorem 2.1(i) holds. Plugging Eqs. (3.17) and (3.18) into (2.3) and (2.4), and then rearranging the constants with keeping in mind the two relations specified in Remark 2.4, we obtain the relations in Eq. (2.8) with $\theta = 0$. On the other hand, if $\theta > 0$ then Theorem 2.1(ii) is valid. Plugging Eqs. (3.17) and (3.18) into (2.5) and (2.6), and then rearranging the constants, we complete the proof. ∎

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APPENDIX A

In this section, we derive some asymptotic results for the constant weighted sums of partial products of Y_i's with the uniformity of the constant weights; see Theorem A.1 below. We first prepare two important lemmas.

Lemma A.1. *Let Y be a positive rv with survival function $\overline{G} \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$ and let $\mathcal{Z} = \{Z\}$ be a set of positive rv's satisfying $\inf \mathcal{Z} > 0$ and $\mathbb{E}(\sup \mathcal{Z})^{\alpha} < \infty$, where $\inf / \sup \mathcal{Z} = \inf / \sup_{Z \in \mathcal{Z}} Z$. Assume that Y and $\mathcal{Z}$ are independent. Then it holds uniformly for $Z \in \mathcal{Z}$ that*

$$\lim_{M \to \infty} \lim_{x \to \infty} \frac{\mathbb{P}(YZ > x, Y > x/M)}{\mathbb{E}Z^{\alpha} \cdot \overline{G}(x)} = 1. \tag{A1}$$

Proof: For every $M > 1 > \delta > 0$ and $x > 0$, we have

$$\begin{aligned}\mathbb{P}\left(YZ > x, Y > \frac{x}{M}\right) &= \mathbb{P}\left(Y > \frac{x}{M}, Z > M\right) + \mathbb{P}(YZ > x, 0 < Z \leq \delta) + \mathbb{P}(YZ > x, \delta < Z \leq M) \\ &= I_1(M,x) + I_2(M,x) + I_3(M,x).\end{aligned}$$

Since Y and $\mathcal{Z}$ are independent, it holds that

$$\begin{aligned}&\lim_{M \to \infty} \lim_{x \to \infty} \sup_{Z \in \mathcal{Z}} \frac{I_1(M,x) + I_2(M,x)}{\mathbb{E}Z^{\alpha} \cdot \overline{G}(x)} \\ &\leq \lim_{M \to \infty} \lim_{x \to \infty} \sup_{Z \in \mathcal{Z}} \frac{\mathbb{P}(Z > M)\overline{G}(x/M) + \mathbb{P}(Z \leq \delta)\overline{G}(x/\delta)}{\mathbb{E}Z^{\alpha} \cdot \overline{G}(x)} \\ &\leq \lim_{M \to \infty} \lim_{x \to \infty} \frac{\mathbb{P}(\sup \mathcal{Z} > M)\overline{G}(x/M) + \mathbb{P}(\inf \mathcal{Z} \leq \delta)\overline{G}(x/\delta)}{\mathbb{E}(\inf \mathcal{Z})^{\alpha} \cdot \overline{G}(x)} \\ &= \lim_{M \to \infty} \frac{\mathbb{P}(\sup \mathcal{Z} > M) M^{\alpha} + \mathbb{P}(\inf \mathcal{Z} \leq \delta)\delta^{\alpha}}{\mathbb{E}(\inf \mathcal{Z})^{\alpha}} \\ &\leq \frac{\mathbb{P}(\inf \mathcal{Z} \leq \delta)}{\mathbb{E}(\inf \mathcal{Z})^{\alpha}},\end{aligned} \tag{A2}$$

where in the third and the fourth steps we used $G \in \mathcal{R}_{-\alpha}$ and $\mathbb{E}(\sup \mathcal{Z})^{\alpha} < \infty$, respectively. For $I_3(M,x)$, we have

$$\begin{aligned}&\lim_{M \to \infty} \lim_{x \to \infty} \sup_{Z \in \mathcal{Z}} \left| \frac{I_3(M,x)}{\mathbb{E}Z^{\alpha} \cdot \overline{G}(x)} - 1 \right| \\ &\leq \lim_{M \to \infty} \lim_{x \to \infty} \sup_{Z \in \mathcal{Z}} \frac{\left| \int_{\delta}^{M} \left(\overline{G}(x/y)/\overline{G}(x) - y^{\alpha}\right) \mathbb{P}(Z \in dy) \right| + \mathbb{E}Z^{\alpha} \mathbf{1}_{\{Z > M\} \cup \{Z \leq \delta\}}}{\mathbb{E}Z^{\alpha}} \\ &\leq \lim_{M \to \infty} \lim_{x \to \infty} \frac{\sup_{\delta < y \leq M} \left| \overline{G}(x/y)/\overline{G}(x) - y^{\alpha} \right| + \mathbb{E}(\sup \mathcal{Z})^{\alpha} \mathbf{1}_{\{\sup \mathcal{Z} > M\}} + \mathbb{P}(\inf \mathcal{Z} \leq \delta)\delta^{\alpha}}{\mathbb{E}(\inf \mathcal{Z})^{\alpha}} \\ &\leq \frac{\mathbb{P}(\inf \mathcal{Z} \leq \delta)}{\mathbb{E}(\inf \mathcal{Z})^{\alpha}},\end{aligned} \tag{A3}$$

where in the last step we used Theorem 1.5.2 of Bingham, Goldie, and Teugels [1] to neglect the first term of the numerator as $x \to \infty$. Combining Eq. (A2) with (A3) and noting the arbitrariness of δ complete the proof. ■

Remark A.1. Going along the same lines of the above proof with corresponding modifications, we can obtain two variants of Lemma A.1: Let Y be that in Lemma A.1 and let $\mathcal{Z}$ be a set of real-valued rv's independent of Y, then (a) relation (A1) with $\mathbb{E}Z^\alpha$ replaced by $\mathbb{E}Z_+^\alpha$, denoted by (4.1′), holds for every fixed Z with $\mathbb{E}Z_+^\alpha < \infty$; (b) relation (4.1′) holds uniformly for $Z \in \mathcal{Z}$ if $\alpha > 0$ and $0 < \mathbb{E}\,(\inf \mathcal{Z})_+^\alpha \le \mathbb{E}\,(\sup \mathcal{Z})_+^\alpha < \infty$.

Using Lemma A.1 and the same idea as in the proof of Lemma 3.4, we have the following:

Lemma A.2. *In addition to the other conditions of Lemma A.1, if $\mathbb{P}\,(Z > x-1) \sim \mathbb{P}(Z > x)$ holds uniformly for $Z \in \mathcal{Z}$ then it holds uniformly for $Z \in \mathcal{Z}$ that*

$$\mathbb{P}\,(Y\,(1+Z) > x) \sim \left[\mathbb{E}\,(1+Z)^\alpha - \mathbb{E}Z^\alpha\right]\mathbb{P}\,(Y > x) + \mathbb{P}\,(YZ > x)\,.$$

Theorem A.1. *Let $\{Y_i; i \ge 1\}$ be a sequence of positive and independent rv's with survival functions $\overline{G}_i \in \mathcal{R}_{-\alpha}$ for every $i \ge 1$ and some $\alpha \ge 0$. Assume that $\mathbb{E}Y_i^\alpha < \infty$ for every $i \ge 2$. Then, for every $n \ge 1$ and $0 < a \le b < \infty$, it holds uniformly for $(c_1, \ldots, c_n) \in [a,b]^n$ that*

$$\mathbb{P}\left(\sum_{i=1}^n c_i \prod_{j=1}^i Y_j > x\right) \sim \sum_{i=1}^n A_{n,i}\mathbb{P}\left(\prod_{j=1}^i Y_j > x\right), \tag{A4}$$

where

$$A_{n,i} = \mathbb{E}\left(\sum_{k=i}^n c_k \prod_{j=i+1}^k Y_j\right)^\alpha - \mathbb{E}\left(\sum_{k=i+1}^n c_k \prod_{j=i+1}^k Y_j\right)^\alpha.$$

Particularly, if $\alpha = 1$ then it holds uniformly for $(c_1, \ldots, c_n) \in [a,b]^n$ that

$$\mathbb{P}\left(\sum_{i=1}^n c_i \prod_{j=1}^i Y_j > x\right) \sim \sum_{i=1}^n c_i\mathbb{P}\left(\prod_{j=1}^i Y_j > x\right),$$

and if $\alpha = 0$ then it holds uniformly for $(c_1, \ldots, c_n) \in [a,b]^n$ that

$$\mathbb{P}\left(\sum_{i=1}^n c_i \prod_{j=1}^i Y_j > x\right) \sim \mathbb{P}\left(\prod_{j=1}^n Y_j > x\right).$$

Proof: We prove relation (A4) by mathematical induction. For $n = 1$, by Theorem 1.5.2 of Bingham, Goldie, and Teugels [1], it holds uniformly for $c_1 \in [a,b]$ that

$$\mathbb{P}\,(c_1 Y_1 > x) \sim c_1^\alpha \mathbb{P}\,(Y_1 > x) = A_{1,1}\mathbb{P}\,(Y_1 > x)\,.$$

Hence, the assertion holds for $n = 1$. Now we assume by induction that the assertion holds for $n - 1 \ge 1$ and prove it for n. Define a set of positive rv's as

$$\mathcal{Z} = \left\{\sum_{i=2}^n \frac{c_i}{c_1} \prod_{j=2}^i Y_j : (c_1, \ldots, c_n) \in [a,b]^n\right\}.$$

It follows from Lemma 3.2(b) that $\prod_{j=2}^{i} Y_j \in \mathcal{R}_{-\alpha} \subset \mathcal{L}(0)$ for every $2 \le i \le n$. Observing that $(c_2/c_1, \ldots, c_n/c_1) \in [a/b, b/a]^{n-1}$, we obtain by the induction assumption that, uniformly for $(c_1, \ldots, c_n) \in [a,b]^n$,

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=2}^{n} \frac{c_i}{c_1} \prod_{j=2}^{i} Y_j > x - 1\right) &\sim \sum_{i=2}^{n} c_1^{-\alpha} A_{n,i} \mathbb{P}\left(\prod_{j=2}^{i} Y_j > x - 1\right) \\
&\sim \sum_{i=2}^{n} c_1^{-\alpha} A_{n,i} \mathbb{P}\left(\prod_{j=2}^{i} Y_j > x\right) \\
&\sim \mathbb{P}\left(\sum_{i=2}^{n} \frac{c_i}{c_1} \prod_{j=2}^{i} Y_j > x\right).
\end{aligned}$$

Moreover, it is obvious that

$$\inf \mathcal{Z} = \sum_{i=2}^{n} \frac{a}{b} \prod_{j=2}^{i} Y_j > 0 \text{ and } \mathbb{E}\left(\sup \mathcal{Z}\right)^{\alpha} = \mathbb{E}\left(\sum_{i=2}^{n} \frac{b}{a} \prod_{j=2}^{i} Y_j\right)^{\alpha} < \infty.$$

Hence, $\mathcal{Z}$ satisfies the conditions of Lemma A.2, which implies that, uniformly for $(c_1, \ldots, c_n) \in [a,b]^n$,

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} c_i \prod_{j=1}^{i} Y_j > x\right) &= \mathbb{P}\left(Y_1\left(1 + \sum_{i=2}^{n} \frac{c_i}{c_1} \prod_{j=2}^{i} Y_j\right) > \frac{x}{c_1}\right) \\
&\sim c_1^{-\alpha} A_{n,1} \mathbb{P}\left(Y_1 > \frac{x}{c_1}\right) + \mathbb{P}\left(Y_1 \sum_{i=2}^{n} \frac{c_i}{c_1} \prod_{j=2}^{i} Y_j > \frac{x}{c_1}\right) \\
&\sim A_{n,1} \mathbb{P}\left(Y_1 > x\right) + \mathbb{P}\left(\sum_{i=2}^{n} c_i Y_1 \prod_{j=2}^{i} Y_j > x\right). \qquad \textbf{(A5)}
\end{aligned}$$

For the second term of Eq. (A5), regarding $Y_1 Y_2$ as a whole and applying the induction assumption to $Y_1Y_2, Y_3, \ldots, Y_n$, we have, uniformly for $(c_2, \ldots, c_n) \in [a,b]^{n-1}$,

$$\mathbb{P}\left(\sum_{i=2}^{n} c_i Y_1 \prod_{j=2}^{i} Y_j > x\right) \sim \sum_{i=2}^{n} A_{n,i} \mathbb{P}\left(\prod_{j=1}^{i} Y_j > x\right). \qquad \textbf{(A6)}$$

A combination of Eqs. (A5) and (A6) completes the proof. ■

Similarly as in Corollary 2.1, assuming further that $\{Y_i; i \ge 1\}$ is a sequence of iid rv's and $\ln Y \in \mathcal{S}(\alpha)$ for some $\alpha \ge 0$ leads to a series of explicit results. We conclude them in the following Corollary A.1.

Corollary A.1. *Let $\{Y_i; i \ge 1\}$ be a sequence of positive and iid rv's with common survival function $\overline{G}$. If $\ln Y \in \mathcal{S}(\alpha)$ for some $\alpha \ge 0$ then, for every $n \ge 1$ and $0 < a \le b < \infty$, it holds uniformly for*

$(c_1, \ldots, c_n) \in [a, b]^n$ *that*

$$\mathbb{P}\left(\sum_{i=1}^{n} c_i \prod_{j=1}^{i} Y_j > x\right) \sim \sum_{i=1}^{n} \mathbb{E}\left(\sum_{k=i}^{n} c_k \prod_{j=1}^{k-i+1} Y_j\right)^{\alpha} \left(\mathbb{E}Y^{\alpha}\right)^{i-2} \cdot \overline{G}(x).$$

Particularly, if $\alpha = 1$ *then it holds uniformly for* $(c_1, \ldots, c_n) \in [a, b]^n$ *that*

$$\mathbb{P}\left(\sum_{i=1}^{n} c_i \prod_{j=1}^{i} Y_j > x\right) \sim \sum_{i=1}^{n} i c_i \left(\mathbb{E}Y\right)^{i-1} \cdot \overline{G}(x),$$

and if $\alpha = 0$ *then it holds uniformly for* $(c_1, \ldots, c_n) \in [a, b]^n$ *that*

$$\mathbb{P}\left(\sum_{i=1}^{n} c_i \prod_{j=1}^{i} Y_j > x\right) \sim n\overline{G}(x).$$