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A BOUNDEDNESS THEOREM IN $\mathrm{ID}_1(W)$

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§0. In this paper we prove a boundedness theorem in the theory $\mathrm{ID}_1(W)$. This answers a question asked by Feferman, for example in [3]. The background is the following.

Let $A[X, x]$ be an X-positive formula arithmetic in X. The theory $\mathrm{ID}_1(P^A)$ is an extension of Peano arithmetic PA by the following axioms:

$$(\mathrm{ID}_A.1) \qquad A[P^A, x] \to P^A(x),$$

$$(\mathrm{ID}_A.2) \qquad \forall x(A[F, x] \to F(x)) \to \forall x(P^A(x) \to F(x)),$$

for arbitrary formulas F; P^A is a constant for the least fixed point of $A[X, x]$. Set-theoretically, P^A can be defined by recursion on the ordinals as follows:

$$P^A_\alpha := \{x: A[\bigcup\{P^A_\xi: \xi < \alpha\}, x]\}, \qquad P^A := \bigcup\{P^A_\xi: \xi < \omega_1^{ck}\},$$

where ω_1^{ck} is the first nonrecursive ordinal.

Now let $a \prec b$ be the arithmetic relation which expresses that the recursive tree coded by a is a proper subtree of the tree coded by b, and define

$$\mathrm{Tree}[X, x] :\Leftrightarrow \forall y \prec x(y \in X).$$

The least fixed point of $\mathrm{Tree}[X, x]$ is the set P^{Tree} of all well-founded recursive trees. We write W or W_α for P^{Tree} or P^{Tree}_α, respectively. Since W is Π^1_1 complete we have $W_\alpha \subsetneq W$ for all $\alpha < \omega_1^{ck}$. If we define for each element $a \in W$ its inductive norm $|a|$ by $|a| := \min\{\xi: a \in W_\xi\}$, then we have $\omega_1^{ck} = \{|a|: a \in W\}$ and the elements of W can be used as codes for the ordinals less than ω_1^{ck}.

Assume that $B[X, x]$ is an X-positive formula arithmetic in X with the only free variables X and x, and assume that Q^B is a relation that satisfies

$$Q^B(a, b) \leftrightarrow B[\{x: (\exists y \prec a) Q^B(y, x)\}, b].$$

If we define

$$I^B(x) :\Leftrightarrow (\exists a \in W) Q^B(a, x),$$

then we obviously have $P^B = I^B$. It was an open question whether a weak theory like

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$\mathrm{ID}_1(W)$ is strong enough to prove the following boundedness theorem:

$$\text{(BT)} \qquad \forall x(B[I^B, x] \to I^B(x)),$$

which corresponds to $(\mathrm{ID}_B.1)$ with P^B replaced by I^B. The translation of the second axiom of $\mathrm{ID}_1(P^B)$

$$\forall x(B[F, x] \to F(x)) \to \forall x(I^B(x) \to F(x))$$

is provable in $\mathrm{ID}_1(W)$ by induction on W.

Our proof of boundedness in $\mathrm{ID}_1(W)$ essentially uses a second order version of $\mathrm{ID}_1(W)$ due to Feferman [3], Spector's boundedness theorem for Σ_1^1 subsets of W and finally a lemma of Kreisel [4] that states that for every Π_1^1 predicate $F(x, y)$

$$\forall x \exists y F(x, y) \to \exists \alpha_{\mathrm{HYP}} \forall x F(x, \alpha(x)).$$

§1. In the following we assume that the reader is familiar with the theory ID_1 of one inductive definition as presented for example in [1], [2], and [3]. Our notation mostly follows [3].

Let L be the usual first order language of arithmetic with constants for all primitive recursive functions and relations. The language L_2 has in addition set variables $X, Y, Z, \ldots$. $\lambda x, y.\langle x, y\rangle$ is the usual pairing function; s, t, s_1, t_1 range over sequence numbers; $\langle\ \rangle$ is the number of the empty sequence; $s = \langle (s)_0, \ldots, (s)_{\mathrm{lh}(s)-1}\rangle$; $s * t$ is concatenation; $s \subseteq t$ $(s \subset t)$ holds if s is an initial (proper) segment of t. Functions are defined by sets X satisfying $\forall x \exists! y(\langle x, y\rangle \in X)$, and we take the function variables α, β, γ to range over such sets; $\bar{\alpha}(n)$ is the sequence number $\langle \alpha(0), \ldots, \alpha(n-1)\rangle$.

We write $\mathbf{X}$ and $\mathbf{x}$ for finite strings $X_1, \ldots, X_n$ and $x_1, \ldots, x_n$ of set and number variables. The notation $A[\mathbf{X}, \mathbf{x}]$ is used to indicate that all free variables of A come from the lists $\mathbf{X}$ and $\mathbf{x}$; $A(\mathbf{X}, \mathbf{x})$ may contain other free variables besides $\mathbf{X}$ and $\mathbf{x}$. $A(F)$ denotes the formula that results from $A(X)$ if we replace each occurrence of $(y \in X)$ by $F(y)$.

$\{e\}$ is the eth recursive function; $\mathrm{Tot}(e)$ expresses that $\{e\}$ is total. If $\mathrm{Tot}(e)$, then e codes a recursive tree $T_e := \{s : \forall t \subseteq s(\{e\}(t) = 0)\}$. Given e and s one effectively associates $e \restriction s$ which codes the subtree of T_e below s, $T_{e \restriction s} = \{t : s * t \in T_e\}$. For simplicity we write $e \restriction x$ instead of $e \restriction \langle x\rangle$. By $a \prec e$ we express that T_a is a proper subtree of T_e.

The set W of all well-founded recursive trees is the least set X such that $\forall x(\mathrm{Tree}[X, x] \to x \in X)$ for the formula

$$\mathrm{Tree}[X, x] :\Leftrightarrow (\mathrm{Tot}(x) \mathbin{\&} \{x\}(\langle\ \rangle) \neq 0) \vee (\mathrm{Tot}(x) \mathbin{\&} \forall y(x \restriction y \in X)).$$

Now suppose that $A[X, x]$ is an arbitrary (but fixed) X-positive formula arithmetic in X. The language $L(W, Q^A)$ is the language L extended by the unary predicate constants W and Q^A, and we write $x \in W$ and $x \in Q^A$ for $W(x)$ and $Q^A(x)$. The theory $\mathrm{ID}(W, A)$ is given by the following axioms where F is an arbitrary formula of $L(W, Q^A)$.

1. Axioms of primitive recursive arithmetic PRA;
2. $F(0) \mathbin{\&} \forall x(F(x) \to F(x')) \to \forall x F(x)$;
3. $\forall x(\mathrm{Tree}[W, x] \to x \in W)$;

4. $\forall x(\text{Tree}[F,x] \to F(x)) \to (\forall x \in W)F(x)$;

5. $\forall a \forall x[\text{Tot}(a) \to (\langle a,x\rangle \in Q^A \leftrightarrow A[\exists y \prec a(\langle y,\cdot\rangle \in Q^A),x])]$.

Axioms 3 and 4 formalize that W is the set of all well-founded recursive trees. Axiom 5 expresses that the sets $Q_a^A = \{x: \langle a,x\rangle \in Q^A\}$ with $a \in W$ are the stages of the inductive definition given by $A[X,x]$. Now we define

$$I^A(x) :\Leftrightarrow \exists a \in W(\langle a,x\rangle \in Q^A).$$

By an obvious induction on W we can show in $\text{ID}(W,A)$ that

$$\forall x(A[F,x] \to F(x)) \to \forall x(I^A(x) \to F(x))$$

for arbitrary formulas F of $L(W,Q^A)$. In the following we will see that $\text{ID}(W,A)$ also proves the boundedness principle

(BT) $$\forall x(A[I^A,x] \to I^A(x)).$$

§2. For the proof of (BT) it is more convenient to work in the second order version $\text{ID}^{(2)}(W,A)$ of $\text{ID}(W,A)$ introduced by Feferman in [3]. $\text{ID}^{(2)}(W,A)$ is formulated in the second order language $L_2(W,Q^A)$ and has the following axioms:

I. Axioms of PRA;

II. $\forall X[0 \in X \;\&\; \forall x(x \in X \to x' \in X) \to \forall x(x \in X)]$;

III. $\forall x(\text{Tree}[W,x] \to x \in W)$;

IV. $\forall X[\forall x(\text{Tree}[X,x] \to x \in X) \to W \subset X]$;

V. $\forall a \forall x[\text{Tot}(a) \to (\langle a,x\rangle \in Q^A \leftrightarrow A[\exists y \prec a(\langle y,\cdot\rangle \in Q^A),x])]$;

VI. $\exists X \forall x(x \in X \leftrightarrow G(x))$ for each formula G of $L_2(W,Q^A)$ without bound set variables.

By VI each $L(W,Q^A)$ formula defines a set and, consequently, the axioms 2 and 4 of $\text{ID}(W,A)$ are derivable from the axioms II and IV of $\text{ID}^{(2)}(W,A)$. Hence $\text{ID}(W,A)$ is contained in $\text{ID}^{(2)}(W,A)$. In [3] it is proved that $\text{ID}^{(2)}(W,A)$ actually is the second order version of $\text{ID}(W,A)$. The following theorem is obtained by an obvious modification of Feferman's proof that $\text{ID}^{(2)}(W)$ is a conservative extension of $\text{ID}(W)$.

THEOREM 1 (FEFERMAN). *$\text{ID}^{(2)}(W,A)$ is a conservative extension of $\text{ID}(W,A)$.*

It will be shown that $\text{ID}^{(2)}(W,A)$ proves (BT). For simplicity we now write $\text{ID}^{(2)}$ instead of $\text{ID}^{(2)}(W,A)$.

Let $\Pi_0^1(W,Q^A)$ or Π_0^1 be the class of all arithmetic formulas of $L_2(W,Q^A)$ or L_2, respectively. A formula is called strict Π_1^1 (strict Σ_1^1) if it is in Π_0^1 or has the form $\forall X F(X)$ ($\exists X F(X)$) where $F(X)$ is in Π_0^1 and contains no set variables besides X. The class of strict Π_1^1 formulas (strict Σ_1^1 formulas) is denoted by Π^s (Σ^s). Observe that strict Π_1^1 and strict Σ_1^1 formulas of the form $\forall X F(X)$ and $\exists X F(X)$ may contain free number variables but no free set variables. Finally, a formula is $\Pi_0^1(W,Q^A)$ in Π^s (Π_0^1 in Π^s) if it is of the form $G(F_1,\ldots,F_n)$ where $F_1(x),\ldots,F_n(x)$ are strict Π_1^1 formulas and $G(X_1,\ldots,X_n)$ is in $\Pi_0^1(W,Q^A)$ (in Π_0^1). Now we list some properties of $\text{ID}^{(2)}$:

LEMMA 1. *Suppose that $\forall X F[X,\mathbf{x}]$ is strict Π_1^1. Then we can find a primitive recursive π for which $\text{ID}^{(2)}$ proves*

$$\forall X F[X,\mathbf{x}] \leftrightarrow \pi(\mathbf{x}) \in W.$$

LEMMA 2. $\mathrm{ID}^{(2)}$ *proves* (Π^s-CA), *i.e.*

$$\mathrm{ID}^{(2)} \vdash \exists X \forall x (x \in X \leftrightarrow F(x))$$

for every strict Π^1_1 *formula* $F(x)$.

LEMMA 3. $\mathrm{ID}^{(2)}$ *proves* (Σ^s-AC), *i.e.*

$$\mathrm{ID}^{(2)} \vdash \forall x \exists X \exists Y F(x, X, Y) \to \exists Z \forall x \exists Y F(x, (Z)_x, Y)$$

for every Π^1_0 *formula* $F(x, X, Y)$ *with no set variables besides* X *and* Y.

LEMMA 4. *If* $F(x)$ *is* $\Pi^1_0(W, Q^A)$ *in* Π^s, *then the following is a theorem of* $\mathrm{ID}^{(2)}$:

$$\exists X \forall x (x \in X \leftrightarrow F(x)).$$

LEMMA 5. *If* $F_1[\mathbf{x}, y], \ldots, F_n[\mathbf{x}, y]$ *are* Π^s *formulas* (Σ^s *formulas*) *and* $G[X_1, \ldots, X_n, \mathbf{x}]$ *is a* Π^1_0 *formula positive in* $X_1, \ldots X_n$, *then there exists a* Π^s *formula* (Σ^s *formula*) $C[\mathbf{x}]$ *such that*

$$\mathrm{ID}^{(2)} \vdash G[F_1[\mathbf{x}, \cdot], \ldots, F_n[\mathbf{x}, \cdot]] \leftrightarrow C[\mathbf{x}].$$

LEMMA 6. *If* $F(x)$ *is* $\Pi^1_0(W, Q^A)$ *in* Π^s, *then the following are provable in* $\mathrm{ID}^{(2)}$:

(a) $F(0) \;\&\; \forall x(F(x) \to F(x')) \to \forall x F(x)$;

(b) $\forall x(\mathrm{Tree}[F, x] \to F(x)) \to (\forall x \in W) F(x)$.

Lemma 1 and Lemma 2 are proved in [3]; Lemma 3 is a standard consequence of Lemma 2 (see for example [6]). Lemma 4 follows from Lemma 1 and axiom VI of $\mathrm{ID}^{(2)}$; Lemma 5 is proved by induction on the complexity of G using Lemma 3. Lemma 6 follows from Lemma 4 and the axioms II and IV of $\mathrm{ID}^{(2)}$.

DEFINITION. (a) $\tilde{W}(a) :\Leftrightarrow \forall X[\forall x(\mathrm{Tree}[X, x] \to x \in X) \to a \in X]$;

(b) $H^A(X) :\Leftrightarrow \forall a \forall x[\mathrm{Tot}(a) \to (\langle a, x\rangle \in X \leftrightarrow A[\exists y \prec a(\langle y, \cdot\rangle \in X), x])]$;

(c) $\tilde{Q}^A(a, x) :\Leftrightarrow \forall X(H^A(X) \to \langle a, x\rangle \in X)$.

LEMMA 7. $\mathrm{ID}^{(2)}$ *proves the following*:

(a) $\forall x(x \in W \leftrightarrow \tilde{W}(x))$;

(b) $H^A(X) \;\&\; a \in W \to \forall x(\langle a, x\rangle \in X \leftrightarrow \langle a, x\rangle \in Q^A)$;

(c) $(\forall a \in W)\forall x(\langle a, x\rangle \in Q^A \leftrightarrow \tilde{Q}^A(a, x))$.

PROOF. Since $\tilde{W}(x)$ is a Π^s formula, (a) follows from Lemma 2 and axiom III of $\mathrm{ID}^{(2)}$. (b) is proved by induction on W. The direction from left to right of (c) follows from (b); for the converse direction observe that Q^A is a set in $\mathrm{ID}^{(2)}$ which satisfies $H^A(X)$ by axiom V.

§3. In order to state and prove Spector's boundedness theorem in $\mathrm{ID}^{(2)}$ we need some familiar notions from recursion theory as they can be found for example in Shoenfield [5].

If $\{a\}$ and $\{b\}$ are total recursive functions, then a number-theoretic function α is a monotone function from the tree T_a to the tree T_b if α: $T_a \to T_b$ and $s \subset t \to \alpha(s) \subset \alpha(t)$ for all s, t in T_a. The predicate "α is a monotone function from T_a to T_b" is arithmetic in α.

DEFINITION. (a) $T_{\leqq}(a, b) :\Leftrightarrow \mathrm{Tot}(a) \;\&\; \mathrm{Tot}(b) \;\&\; [\neg \tilde{W}(b) \vee \exists \alpha(\alpha$ is monotone from T_a to $T_b)]$;

(b) $T_{<}(a, b) :\Leftrightarrow \mathrm{Tot}(a) \;\&\; \mathrm{Tot}(b) \;\&\; [\neg \tilde{W}(b) \vee \exists \alpha \exists x(\langle x\rangle \in T_b \;\&\; \alpha$ is monotone from T_a to $T_{b \restriction x})]$.

By Lemma 5 there exist Σ^s formulas provably equivalent to $T_{\leqq}(a,b)$ and $T_{<}(a,b)$; we denote them by $|a| \leqq |b|$ and $|a| < |b|$.

LEMMA 8. *The following are theorems of* $\mathrm{ID}^{(2)}$:

(a) $b \in W$ & $|a| \leqq |b| \rightarrow a \in W$;

(b) $a, b \in W \rightarrow |a| \leqq |b| \vee |b| < |a|$;

(c) $b \in W$ & $|a| \leqq |b| \rightarrow \forall x(\langle a, x\rangle \in Q^A \rightarrow \langle b, x\rangle \in Q^A)$.

PROOF. All three assertions follow by induction on W. The proofs of (a) and (b) can essentially be found in [5]. For the proof of (c) we use axiom V of $\mathrm{ID}^{(2)}$ and the fact that $A[X, x]$ is positive in X.

THEOREM 2 (SPECTOR). *If* $F(x)$ *is a* Σ^s *formula, then* $\mathrm{ID}^{(2)}$ *proves*

$$\forall x(F(x) \rightarrow x \in W) \rightarrow (\exists e \in W)\forall x(F(x) \rightarrow |x| < |e|).$$

PROOF. Again we follow [5]. Choose a Π^s predicate $G(x)$ which is provably not equivalent to a Σ^s formula and assume that $\forall x(F(x) \rightarrow x \in W)$ and $(\forall e \in W)\exists x(F(x)$ & $|x| \nless |e|)$. By Lemma 8(b), the last formula is equivalent to

$$(1) \qquad (\forall e \in W)\exists x(F(x) \;\&\; |e| \leqq |x|).$$

On the other hand, Lemma 1 gives a primitive recursive function π such that

$$(2) \qquad G(a) \leftrightarrow \pi(a) \in W.$$

Since $F \subset W$, (1) and (2) imply

$$G(a) \leftrightarrow \exists x(F(x) \;\&\; |\pi(a)| \leqq |x|).$$

The right side is equivalent to a Σ^s formula by Lemma 5; a contradiction.

In [4] Kreisel proves that for every Π^1_1 predicate $F(x, y)$ with $\forall x\exists y F(x, y)$ there exists an hyperarithmetical function α such that $\forall x F(x, \alpha(x))$. His theorem and its proof can easily be adapted to our present context, and we obtain the following Theorem 3.

THEOREM 3. *For every* Π^s *formula* $F(x, y)$ *there exists a* Σ^s *formula* $G(x, y)$ *with the same free variables as* $F(x, y)$ *such that* $\mathrm{ID}^{(2)}$ *proves*

$$\forall x\exists y F(x, y) \rightarrow \forall x\exists! y G(x, y) \;\&\; \forall x\forall y(G(x, y) \rightarrow F(x, y)).$$

§4. In this last section we finally prove the boundedness theorem for $\mathrm{ID}^{(2)}$ and $\mathrm{ID}(W, A)$.

LEMMA 9. *Suppose that* $F(x, y)$ *is a* Π^s *formula. Then*

$$\mathrm{ID}^{(2)} \vdash \forall x(\exists y \in W)F(x, y) \rightarrow (\exists e \in W)\forall x\exists y(|y| < |e| \;\&\; F(x, y)).$$

PROOF. We work in $\mathrm{ID}^{(2)}$ and assume $\forall x(\exists y \in W)F(x, y)$, which is equivalent to

$$\forall x\exists y(\tilde{W}(y) \;\&\; F(x, y)).$$

By Lemma 5 and Theorem 3 there is a Σ^s formula $G(x, y)$ such that

$$\forall x\exists! y G(x, y) \;\&\; \forall x\forall y(G(x, y) \rightarrow \tilde{W}(y) \;\&\; F(x, y)).$$

Now define $C(y) :\Leftrightarrow \exists x G(x, y)$ and apply Lemma 5 and Theorem 2. Hence we have an $e \in W$ such that $\forall y(C(y) \rightarrow |y| < |e|)$, and therefore $\forall x\exists y(|y| < |e|$ & $F(x, y))$.

LEMMA 10. *Let* $B(X)$ *be an* X*-positive* Π^1_0 *formula. Then* $\mathrm{ID}^{(2)}$ *proves*

$$B(\exists a \in W[\langle a, \cdot\rangle \in Q^A]) \rightarrow (\exists e \in W)B(\langle e, \cdot\rangle \in Q^A).$$

PROOF (by induction on the complexity of $B(X)$). The only critical case is that when $B(X)$ has the form $\forall x C(X, x)$. Now let us work in $\mathrm{ID}^{(2)}$ and assume that

$$\forall x C(\exists a \in W[\langle a, \cdot\rangle \in Q^A], x).$$

By the induction hypothesis we obtain $\forall x(\exists b \in W)C(\langle b, \cdot\rangle \in Q^A, x)$ and therefore by Lemma 7(c)

$$\forall x(\exists b \in W)C(\tilde{Q}^A(b, \cdot), x).$$

Since $B(X)$ is X-positive, $C(\tilde{Q}^A(b, \cdot), x)$ is equivalent to a Π^s formula, and we obtain by Lemma 9

$$(\exists e \in W)\forall x \exists b[|b| < |e| \;\&\; C(\tilde{Q}^A(b, \cdot), x)],$$

i.e.

$$(\exists e \in W)\forall x \exists b[|b| < |e| \;\&\; C(\langle b, \cdot\rangle \in Q^A, x)].$$

Again the X-positivity of $C(X, x)$ and Lemma 8(c) yield

$$(\exists e \in W)\forall x C(\langle e, \cdot\rangle \in Q^A, x).$$

THEOREM 4 (BOUNDEDNESS IN $\mathrm{ID}^{(2)}$).

$$\mathrm{ID}^{(2)} \vdash \forall x(A[I^A, x] \to I^A(x)).$$

This theorem immediately follows from the X-positivity of $A[X, x]$, Lemma 10 and axiom V of $\mathrm{ID}^{(2)}$. The boundedness theorem for $\mathrm{ID}(W, A)$ is a corollary of Theorem 4 by Feferman's result (Theorem 1).

THEOREM 5 (BOUNDEDNESS IN $\mathrm{ID}(W, A)$).

$$\mathrm{ID}(W, A) \vdash \forall x(A[I^A, x] \to I^A(x)).$$

Questions. 1. Can a boundedness theorem be proved in ID_1^i where we have intuitionistic logic instead of classical logic?

2. Is it possible to obtain boundedness theorems for the theories ID_α and ID_α^i with $\alpha > 0$?

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