*Proceedings of the Edinburgh Mathematical Society* (2001) **44**, 571–583 ©

# DUALITY FOR SOME LARGE SPACES OF ANALYTIC FUNCTIONS

H. JARCHOW[1], V. MONTESINOS[2], K. J. WIRTHS[3] AND J. XIAO[4,5]

[1] *Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH 8057 Zürich, Switzerland* (jarchow@math.unizh.ch)

[2] *Departamento de Matematica Aplicada, ETSI Telecomunicacion, Universidad Politecnica de Valencia, C/vera, s/n., E-46071 Valencia, Spain* (vmontesinos@mat.upv.es)

[3] *Institut für Analysis, TU-Braunschweig, Pockelsstr. 14, D-38106 Braunschweig, Germany* (kjwirths@ws.rz.tu-bs.de)

[4] *Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneue Blvd West, Montreal, Quebec, Canada* (jxiao@discrete.Concordia.CA)

[5] *Department of Mathematics, Peking University, Beijing 100871, People's Republic of China*

(Received 24 September 1999)

*Abstract* We characterize the duals and biduals of the $L^p$-analogues $\mathcal{N}_\alpha^p$ of the standard Nevanlinna classes $\mathcal{N}_\alpha$, $\alpha \geqslant -1$ and $1 \leqslant p < \infty$. We adopt the convention to take $\mathcal{N}_{-1}^p$ to be the classical Smirnov class $\mathcal{N}^+$ for $p = 1$, and the Hardy–Orlicz space $LH^p$ $(= (\mathrm{Log}^+ H)^p)$ for $1 < p < \infty$. Our results generalize and unify earlier characterizations obtained by Eoff for $\alpha = 0$ and $\alpha = -1$, and by Yanigahara for the Smirnov class.

Each $\mathcal{N}_\alpha^p$ is a complete metrizable topological vector space (in fact, even an algebra); it fails to be locally bounded and locally convex but admits a separating dual. Its bidual will be identified with a specific nuclear power series space of finite type; this turns out to be the 'Fréchet envelope' of $\mathcal{N}_\alpha^p$ as well.

The generating sequence of this power series space is of the form $(n^\theta)_{n\in\mathbb{N}}$ for some $0 < \theta < 1$. For example, the $\theta$s in the interval $(\frac{1}{2}, 1)$ correspond in a bijective fashion to the Nevanlinna classes $\mathcal{N}_\alpha$, $\alpha > -1$, whereas the $\theta$s in the interval $(0, \frac{1}{2})$ correspond bijectively to the Hardy–Orlicz spaces $LH^p$, $1 < p < \infty$. By the work of Yanagihara, $\theta = \frac{1}{2}$ corresponds to $\mathcal{N}^+$.

As in the work by Yanagihara, we derive our results from characterizations of coefficient multipliers from $\mathcal{N}_\alpha^p$ into various smaller classical spaces of analytic functions on $\Delta$.

*Keywords:* coefficient multipliers; weighted area Nevanlinna classes; weighted Bergman spaces; nuclear power series spaces

AMS 2000 *Mathematics subject classification:* Primary 46E10; 46A11; 47B38
Secondary 30D55; 46A45; 46E15

## 1. Introduction

We denote by $\Delta$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$ and by $\mathcal{H}(\Delta)$ the space of all analytic functions $\Delta \to \mathbb{C}$. With respect to uniform convergence on compact subsets of $\Delta$ (i.e. local uniform convergence), $\mathcal{H}(\Delta)$ is a Fréchet space.

Let $m$ be the normalized area measure on $\Delta$; so $\mathrm{d}m(x+\mathrm{i}y) = \pi^{-1}\,\mathrm{d}x\,\mathrm{d}y$. We shall work with the probability measures $\mathrm{d}m_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha\,\mathrm{d}m(z)$ on (the Borel sets of) $\Delta$ ($-1<\alpha<\infty$). Given $1 \leqslant p < \infty$, we define $\mathcal{N}_\alpha^p$ to consist of all $f \in \mathcal{H}(\Delta)$ such that $\log^+|f|$ is in $L^p(m_\alpha)$. With respect to the $F$-norm

$$|||f|||_{\alpha,p} := \left(\int_\Delta [\log(1+|f|)]^p\,\mathrm{d}m_\alpha\right)^{1/p},$$

this is a complete metrizable topological vector space (even an algebra). If $p=1$, then we obtain the usual weighted area *Nevanlinna class* $\mathcal{N}_\alpha$ discussed, for example, in [**11**].

The *Hardy–Orlicz space* $LH^p$, $1<p<\infty$, consists of all $f \in \mathcal{H}(\Delta)$ such that

$$\|f\|_{LH^p} := \sup_{0\leqslant r<1}\left[\int_0^{2\pi}(\log(1+|f(r\mathrm{e}^{\mathrm{i}t})|))^p\left(\frac{\mathrm{d}t}{2\pi}\right)\right]^{1/p}$$

is finite. Again we get a complete metrizable topological vector space (also an algebra) with $F$-norm $\|\cdot\|_{LH^p}$ (see Stoll [**19**], where $LH^p$ is denoted $(\mathrm{Log}^+H)^p$).

The case $p=1$ is a bit delicate. The *Nevanlinna class* $\mathcal{N}$ consists of all $f \in \mathcal{H}(\Delta)$ such that

$$\sup_{0\leqslant r<1}\int_0^{2\pi}\log(1+|f(r\mathrm{e}^{\mathrm{i}t})|)\left(\frac{\mathrm{d}t}{2\pi}\right) < \infty.$$

This is still an algebra on which a complete metric is defined by

$$d_{\mathcal{N}}(f,g) := \sup_{0\leqslant r<1}\int_0^{2\pi}\log(1+|(f-g)(r\mathrm{e}^{\mathrm{i}t})|)\left(\frac{\mathrm{d}t}{2\pi}\right).$$

But $d_{\mathcal{N}}$ does not generate a linear topology on $\mathcal{N}$ (see Shapiro and Shields [**16**]). The largest subspace of $\mathcal{N}$ on which $d_{\mathcal{N}}$ defines a linear topology is the *Smirnov class*, denoted by $\mathcal{N}^+$. For more on $\mathcal{N}$ and $\mathcal{N}^+$, see Duren's book [**6**].

For the purposes of this paper, it is convenient to set

$$LH^1 := \mathcal{N}^+ \quad \text{and} \quad \mathcal{N}_{-1}^p := LH^p \quad \text{for } 1 \leqslant p < \infty.$$

It is clear that $\mathcal{N}_\alpha^p \subset \mathcal{N}_\beta^q$ for all $\beta \geqslant \alpha \geqslant -1$ and $p \geqslant q$. If $(\alpha,p) \neq (\beta,q)$, then these inclusions are clearly proper.

Let $\beta > -1$ and $0<q<\infty$. The corresponding *weighted Bergman space* is defined to be $\mathcal{A}_\beta^q := L^q(m_\beta) \cap \mathcal{H}(\Delta)$. This is a closed subspace of $L^q(m_\beta)$. We denote the canonical norm ($q$-norm if $q<1$) on $\mathcal{A}_\beta^q$ by $\|\cdot\|_{\beta,q}$. Again, it will be convenient to include the classical Hardy spaces $H^q$ by labelling them $\mathcal{A}_{-1}^q$. Clearly, $\mathcal{A}_\beta^q$ embeds continuously into $\mathcal{N}_\beta^p$, for any $0<q<\infty$ and $1 \leqslant p < \infty$.

Point evaluations are easily seen to be continuous on any $\mathcal{N}_\alpha^p$, so that the dual $(\mathcal{N}_\alpha^p)^*$ separates points. This dual has been characterized by Yanagihara [**20**] in the case in which $\alpha = -1$, $p=1$. Using somewhat different methods, Eoff [**7**] extended this to characterize the duals $(LH^p)^*$ for $p>1$ and $(\mathcal{N}_0^q)^*$ for $1 \leqslant q < \infty$. In this paper, we settle the case of arbitrary $\mathcal{N}_\alpha^p$ by returning to Yanagihara's approach. However, we need to use stronger tools. The $(\mathcal{N}^p_\alpha)^*$ coincide for all $(\alpha, p)$ on any straight line $p = c(\alpha + 2)$, $c > 0$ a constant, and they are also duals of specific nuclear power series spaces of finite type, $F_\theta$ say. Here, $\theta$ varies over the interval $(0, 1)$. $F_\theta$ is the 'Fréchet envelope' of $\mathcal{N}^p_\alpha$; its topology is obtained by convexifying the neighbourhoods of zero in $\mathcal{N}^p_\alpha$ and then passing to the completion. If we take into account only the spaces $\mathcal{N}^1_\alpha$ and $LH_p$, then the full interval is covered in this way in a one-to-one fashion: $\theta = \frac{1}{2}$ corresponds to the Smirnov class [**20**], $\theta = (\alpha + 2)/(\alpha + 3)$ corresponds to the scale of spaces $\mathcal{N}^1_\alpha$ $(\alpha > -1)$, and $\theta = 1/(p + 1)$ corresponds to the classes $LH^p$ $(1 < p < \infty)$.

## 2. Main results

As in [**20**], the key result is a characterization of the coefficient multipliers from $\mathcal{N}^p_\alpha$ into various small spaces of analytic functions.

Suppose that $X$ and $Y$ are complete and metrizable topological vector spaces which consist of analytic functions on $\Delta$ and whose topology is finer than that of local uniform convergence. Let $[\![X, Y]\!]$ be the collection of all sequences $(\lambda_n)_0^\infty$ in $\mathbb{C}$ such that $\sum_{n=0}^\infty \lambda_n a_n z^n$ belongs to $Y$ whenever $\sum_{n=0}^\infty a_n z^n$ is a member of $X$. This is a linear space whose elements are called the (*coefficient*) *multipliers* from $X$ into $Y$. By the Closed Graph Theorem, each $(\lambda_n)_n \in [\![X, Y]\!]$ gives rise to a continuous linear operator

$$\Lambda : X \to Y : \sum_{n=0}^\infty a_n z^n \mapsto \sum_{n=0}^\infty \lambda_n a_n z^n.$$

The case of interest is when $X$ is a space $\mathcal{N}^p_\alpha$ and $Y$ is a 'small' space of analytic functions, such as $\mathcal{A}^q_\beta$. We shall also consider multipliers from $X$ into $Y$ if one of the spaces is as above, and the other is a complete metrizable sequence space. Such multipliers are defined in an analogous fashion, and they also form a linear space $[\![X, Y]\!]$.

The first main result of this paper is the following theorem.

**Theorem 2.1.** *Let $\alpha \geqslant -1$ and $1 \leqslant p < \infty$. For every sequence $(\lambda_n)$ in $\mathbb{C}$, the following are equivalent:*

(i) $(\lambda_n) \in [\![\mathcal{N}^p_\alpha, \mathcal{A}^q_\beta]\!]$ *for some $\beta \geqslant -1$ and some $0 < q < \infty$;*

(ii) $(\lambda_n) \in [\![\mathcal{N}^p_\alpha, \mathcal{A}^q_\beta]\!]$ *for all $\beta \geqslant -1$ and all $0 < q < \infty$;*

(iii) $(\lambda_n) \in [\![\mathcal{N}^p_\alpha, W]\!]$*; and*

(iv) $\lambda_n = O(\exp[-cn^{(\alpha+2)/(\alpha+2+p)}])$ *for some $c > 0$.*

Here $W$ is the *Wiener algebra* consisting of all $f \in \mathcal{H}(\Delta)$ whose Taylor coefficients $\hat{f}(n)$ form an $\ell^1$-sequence. It is a Banach algebra with respect to the norm $\|f\| = \sum_n |\hat{f}(n)|$. Other Banach algebras, such as the disk algebra and even appropriate $F$-algebras, can be used instead.

In the case in which $p = 1$, we are dealing with the function $\alpha \mapsto (\alpha+2)/(\alpha+3)$, which maps the interval $[-1, \infty)$ onto $[\frac{1}{2}, 1)$. If $\alpha = -1$, we are dealing with $p \mapsto 1/(p+1)$, and $[1, \infty)$ is mapped onto $(0, \frac{1}{2}]$. As mentioned before, Eoff [**7**] has settled this case earlier, as well as the case $\alpha = 0$. It does not seem, however, that her approach can easily be modified to deal with the general case.

Theorem 2.1 shows that our multipliers on $\mathcal{N}_\alpha^p$ are largely independent of the range space. (iv) reveals a certain independence of the domain space as well: $\mathcal{N}_\alpha^p$ can be replaced by $\mathcal{N}_\beta^q$ as long as $(\alpha+2)/p = (\beta+2)/q$. We shall see below that certain power series spaces can be used instead of $\mathcal{N}_\alpha^p$. This complements a number of similar results presented in the survey article [**2**] by Campbell and Leach.

The second result characterizes the duals $(\mathcal{N}_\alpha^p)^*$ of our spaces. These spaces are rather small, as is to be expected. Let $F_\theta$ be the collection of all sequences $(a_n)_n$ in $\mathbb{C}$ such that, for each $k \in \mathbb{N}$, the sequence $(|a_n|^2 \exp(-n^{\theta/(\theta+1)}/k))_n$ belongs to $\ell^1$. In a natural fashion, $F_\theta$ is a Fréchet space (see below). We are interested in the case in which $\theta = (\alpha+2)/p$.

**Theorem 2.2.** *Let $\alpha, \beta \geqslant -1$ and $1 \leqslant p, q < \infty$ be given.*

(a) *$\mathcal{N}_\alpha^p$ embeds densely and continuously into $F_{(\alpha+2)/p}$.*

(b) *The duals of $\mathcal{N}_\alpha^p$ and of $F_{(\alpha+2)/p}$ coincide and can be identified with the space of all $g \in \mathcal{H}(\Delta)$ whose Taylor coefficients satisfy $b_n = O(\exp[-cn^{(\alpha+2)/(\alpha+2+p)}])$. The action of $g$ on $f(z) = \sum_{n=0}^\infty a_n z^n$ in $\mathcal{N}_\alpha^p$ is given by*

$$\langle g, f\rangle = \sum_n a_n b_n = \lim_{r\to 1-} \frac{1}{2\pi}\int_0^{2\pi} g(r\mathrm{e}^{\mathrm{i}t}) f(r\mathrm{e}^{-\mathrm{i}t})\,\mathrm{d}t.$$

(c) *$F_{(\alpha+2)/p}$ induces on $\mathcal{N}_\alpha^p$ the Mackey topology $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$.*

(d) *$F_{(\alpha+2)/p}$ is the Fréchet envelope of $\mathcal{N}_\alpha^p$.*

(e) *$\mathcal{N}_\alpha^p$ is neither locally bounded nor locally convex; in particular, $\mathcal{N}_\alpha^p \neq F_{(\alpha+2)/p}$.*

For a special case of (a), see [**19**, Corollaries 5.6 and 4.4].

Our third result is related to earlier investigations in [**11**] on composition operators with domain a space $\mathcal{N}_\alpha^1$. Given an analytic map $\varphi : \Delta \to \Delta$, we write $C_\varphi$ for the associated composition operator $f \mapsto f \circ \varphi$, acting on appropriate spaces of functions.

**Theorem 2.3.** *With $\alpha$ and $p$ as in Theorem 2.2 and $\varphi : \Delta \to \Delta$ an analytic function, the following are equivalent statements:*

(i) *$C_\varphi$ exists as a bounded operator $\mathcal{N}_\alpha^p \to \mathcal{A}_\beta^q$ for some $\beta \geqslant -1$ and $0 < q < \infty$;*

(ii) *$C_\varphi$ exists as a nuclear operator $\mathcal{N}_\alpha^p \to \mathcal{A}_\beta^q$ for every $\beta \geqslant -1$ and $0 < q < \infty$;*

(iii) *for some $c > 0$, $\sum_{n=0}^\infty \exp[cn^{(\alpha+2)/(\alpha+2+p)}]\|\varphi^n\|_{\beta,1} < \infty$;*

(iv) *the sequence $(\|\varphi^n\|_{\beta,1})_n$ belongs to $[\![\mathcal{N}_\alpha^p, \mathcal{A}_\beta^q]\!]$ for some $0 < q < \infty$; and*

(v) *the sequence $(\|\varphi^n\|_{\beta,1})_n$ belongs to $[\![\mathcal{N}_\alpha^p, W]\!]$.*

Of course, $\varphi^n$ is the usual multiplicative $n$th power of $\varphi$.

Nuclear and order bounded operators on $F$-spaces will be defined later in this paper. Since, for appropriate range spaces, nuclearity implies order boundedness, Theorem 2.3 also provides access to some of the results in [**11**]. Whereas composition operators with domain $\mathcal{N}^+$, and even $\mathcal{N}$, have already been investigated by Roberts and Stoll [**14**] and in particular by Jaoua [**9**] and by Choa *et al*. [**3**], the general case does not seem to have received attention yet.

## 3. Nuclear power series spaces of finite type

Before we get to the proofs, we recall some basic facts on nuclear power series spaces of finite type. We refer to [**10**] for any unexplained terminology and notation on topological vector spaces.

Let $(\pi_n)_{n\in\mathbb{N}_0}$ be an unbounded increasing sequence of positive numbers. For each $k \in \mathbb{N}$ and $(a_n) \in \mathbb{C}^{\mathbb{N}_0}$, put $q_k((a_n)) := [\sum_{n=0}^{\infty} |a_n|^2 \exp(-\pi_n/k)]^{1/2}$. It is clear that each $F^{(k)} := \{(a_n) \in \mathbb{C}^{\mathbb{N}_0} : q_k((a_n)) < \infty\}$ is a Hilbert space with norm $q_k$, that $F^{(k+1)} \subset F^{(k)}$ (densely) with $q_k \leqslant q_{k+1}$; in fact, the canonical embedding $F^{(k+1)} \hookrightarrow F^{(k)}$ is even a nuclear (i.e. trace class) operator. Consequently, $F := \bigcap_{k\in\mathbb{N}} F^{(k)}$ is a nuclear Fréchet space with respect to the corresponding projective limit topology: a so-called *nuclear power series space of finite type* (see [**10**, ch. 21]). In the canonical fashion, the dual of $F^{(k)}$ can be identified with $(F^{(k)})^* = \{(a_n) \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} |a_n|^2 \exp(\pi_n/k) < \infty\}$, and the dual of $F$ can be written $F^* = \bigcup_k (F^{(k)})^*$. By nuclearity, the natural locally convex inductive limit topology on $F^*$ is the strong topology $\beta(F^*, F)$ [**10**, § 13.4]. Moreover, the strong dual of $[F^*, \beta(F^*, F)]$ is the original space $F$.

Multipliers from $F$ into weighted Bergman spaces can be characterized as follows.

**Proposition 3.1.** *Let $(\lambda_n) \in \mathbb{C}^{\mathbb{N}_0}$ be given. The following statements are equivalent.*

(i) *For some $-1 \leqslant \beta < \infty$ and $0 < q < \infty$, $(\lambda_n) \in [\![F, \mathcal{A}^q_\beta]\!]$.*

(ii) *For all $-1 \leqslant \beta < \infty$ and $0 < q < \infty$, $(\lambda_n) \in [\![F, \mathcal{A}^q_\beta]\!]$.*

(iii) $(\lambda_n) \in [\![F, W]\!]$.

(iv) *There exist $k \in \mathbb{N}$ and $C > 0$ such that $|\lambda_n| \leqslant C \exp(-\pi_n/k)$ for every $n \in \mathbb{N}_0$.*

It follows from (iv) that $F^*$ is just the space $[\![F, W]\!]$ of multipliers from $F$ to $W$.

For the spaces $F$ related to the $\mathcal{N}^p_\alpha$, Proposition 3.1 will also be a consequence of results to be proved below. Nevertheless, we believe that the following direct proof is interesting enough to be included here.

**Proof.** (iv) ⇒ (iii) follows from the definition of $F$, whereas (iii) ⇒ (ii) and (ii) ⇒ (i) are trivial. So we are left with proving that (i) implies (iv).

Fix $\beta \geqslant -1$. We settle the case $q = 2$ first. Set $\kappa_n := \sqrt{(n!)^{-1}(\beta+1)\dots(\beta+n+1)}$, so that $(\kappa_n z^n)_n$ is an orthonormal basis in $\mathcal{A}^2_\beta$. Use Stirling's formula to find a constant $C = C(\beta)$ such that $\kappa_n^2 \leqslant C(n+1)^{\beta+2}$ for all $n$.

By continuity, the adjoint $\Lambda^*$ of the multiplier $\Lambda$ generated by $(\lambda_n)$ maps $\mathcal{A}^2_\beta$ into some $(F^{(k)})^*$. The induced operator is also a multiplier, and is given by $(\lambda_n)$. Accordingly, there is a $C > 0$ such that, for all $(a_n) \in \mathbb{C}^{(\mathbb{N}_0)}$,

$$\sum_{n=0}^{\infty} |\lambda_n a_n|^2 \exp\left(\frac{\pi_n}{k}\right) \leqslant C \sum_{n=0}^{\infty} \kappa_n |a_n|^2.$$

This is equivalent to $\sum_n [(|\lambda_n|^2/\kappa_n)\exp(\pi_n/k)]^2 < \infty$. So $|\lambda_n| \leqslant c\kappa_n^{1/2}\exp(-\pi_n/(2k))$ holds for all $n$ with some constant $c$, and our claim follows from what was observed above.

To settle the general case we only need to look at $0 < q \leqslant 1$. It was shown by Shapiro [**15**] that $\mathcal{A}^q_\beta \hookrightarrow \mathcal{A}^1_\sigma$, where $\sigma = ((\beta+2)/q) - 2$, and, from a result in [**4**], we know that $\mathcal{A}^1_\sigma \hookrightarrow \mathcal{A}^2_\tau$, where $\tau = 2(\sigma+1)$. Here we use '$\hookrightarrow$' to denote continuous set theoretic inclusion. □

The analogy with Theorem 2.1 is by no means accidental. The spaces $F$ of relevance for our topic are obtained by choosing $\pi_n = n^{\theta/(\theta+1)}$ for $0 < \theta < \infty$. Stoll [**19**] labelled these spaces $F_\theta$ and derived several of their properties. Note that $(a_n) \mapsto \sum a_n z^n$ defines a continuous embedding of $F_\theta$ into $\mathcal{H}(\Delta)$ and that $\mathcal{H}(\Delta)$ itself can be identified with the nuclear power series space of finite type obtained by taking $\pi_n = n$ for each $n$. Hence, allowing $\theta = \infty$, we can write $F_\infty = \mathcal{H}(\Delta)$.

## 4. Proof of Theorem 2.1

We require three lemmas. They are essentially known but of different degrees of difficulty.

**Lemma 4.1.** *Let $\beta \geqslant -1$ and $0 < q < \infty$ be given. Then there is a constant $C = C(\beta, q)$ such that if $g = \sum_{n=0}^\infty a_n z^n$ belongs to $\mathcal{A}^q_\beta$, then*

$$|a_n| \leqslant C\|g\|_{\beta,q} n^{(\beta+2)/q} \quad \text{for all } n.$$

This is easily derived from Cauchy's Theorem (see, for example, Lemma 2.5 in Smith [**18**]).

**Lemma 4.2.** *Let $\alpha \geqslant -1$, $1 \leqslant p < \infty$. If $f(z) = \sum_{n=0}^\infty a_n z^n$ belongs to $\mathcal{N}^p_\alpha$, then*

$$a_n = O(\exp[o(n^{(\alpha+2)/(\alpha+2+p)})]).$$

Special cases of this are treated in [**17**] and [**19**]; the proof of the general case requires only minor changes.

The last lemma is crucial and is technically the most difficult one. Up to an inessential modification (we require an extra parameter), an even more precise version was obtained by Beller. We refer to [**1**] for a proof.

**Lemma 4.3.** *Let $a > 0$ and $r_0 > 0$ be given. For $0 < r \leqslant r_0$, define*

$$f(z) = \exp[r/((1-z)^a)].$$

*Then there exists a constant $K$, depending only on $a$ and $r_0$, such that the Taylor coefficients $a_n$ of $f$ satisfy*

$$a_n \geqslant K \exp[r^{1/(a+1)} a^{-a/(a+1)} n^{a/(a+1)}] \quad (n \in \mathbb{N}_0).$$

(The $a_n$ are in fact positive: power series expansion!)

**Proof of Theorem 2.1.** For notational convenience, we confine ourselves to the case in which $\alpha > -1$. The (known) case $\alpha = -1$ can be settled in a similar fashion; the use of the measure $\mathrm{d}t/2\pi$ just requires some natural changes.

We start by showing that (i) implies (iv). For this, we look at the functions

$$f_{(s)}(z) = \exp\left[\frac{c(1-s)^{(\alpha+2)/p}}{(1-sz)^{(2\alpha+4)/p}}\right] - 1.$$

By Lemma 4.2.2 in [**21**], there is a constant $K_0 = K_0(\alpha)$ such that

$$\begin{aligned} |||f_{(s)}|||_{\alpha,p}^p &= \int_\Delta [\log(1+|f_{(s)}|)]^p \,\mathrm{d}m_\alpha \leqslant c^p \int_\Delta \frac{(1-s)^{\alpha+2}}{|1-sz|^{2\alpha+4}} \,\mathrm{d}m_\alpha(z) \\ &= c^p(\alpha+1)(1-s)^{\alpha+2} \int_\Delta \frac{(1-|z|^2)^\alpha}{|1-sz|^{2\alpha+4}} \,\mathrm{d}m(z) \leqslant c^p K_0^p. \end{aligned}$$

Let $\Lambda : \mathcal{N}_\alpha^p \to \mathcal{A}_\beta^q$ be the multiplier induced by $(\lambda_n)$ and let $C > 0$ be such that $\|\Lambda(g)\|_{\beta,q} \leqslant 1$ for all $g \in \mathcal{N}_\alpha^p$ with $|||g|||_{\alpha,p} \leqslant C$. Hence, if we choose $c \leqslant C/K_0$ in our definition of $f_{(s)}$, then $\|\Lambda(f_{(s)})\|_{\beta,q} \leqslant 1$ for all $0 < s < 1$. Put

$$R := c(1-s)^{(\alpha+2)/p}$$

and consider the function

$$f(z) = \exp\left[\frac{R}{(1-z)^{(2\alpha+4)/p}}\right] - 1.$$

Note that $f_{(s)}(z) = f(sz)$. Writing

$$f_{(s)}(z) = \sum_{n=0}^\infty a_{n,s} z^n \quad \text{and} \quad f(z) = \sum_{n=0}^\infty c_n z^n$$

we see that $a_{n,s} = s^n c_n$ for all $n \in \mathbb{N}_0$ and $0 < s < 1$. Recall from Lemma 4.3 that these are positive numbers and that there is a constant $C_1$ such that

$$a_{n,s} \geqslant C_1 s^n \exp[L(1-s)^{(\alpha+2)/(2\alpha+4+p)} n^{(2\alpha+4)/(2\alpha+4+p)}];$$

here

$$L = c^{p/(2\alpha+4+p)} \left(\frac{p}{2\alpha+4}\right)^{(2\alpha+4)/(2\alpha+4+p)}.$$

Next we choose $0 < b$, $d < 1$ such that

$$d < b < \tfrac{1}{2} L d^{(\alpha+2)/(2\alpha+4+p)},$$

and let $N_0 \in \mathbb{N}$ be so big that $b \leqslant (\frac{3}{4})N_0^{p/(\alpha+2+p)}$. For $N \geqslant N_0$ and for all $s$ satisfying $dN^{-p/(\alpha+2+p)} \leqslant 1-s \leqslant bN^{-p/(\alpha+2+p)}$, we get (since $1-t > \mathrm{e}^{-2t}$ for $0 < t \leqslant \frac{3}{4}$)

$$\begin{aligned} a_{N,s} &\geqslant C_1(1-bN^{-p/(\alpha+2+p)})^N \exp[L(dN^{-p/(\alpha+2+p)})^{(\alpha+2)/(2\alpha+4+p)}N^{(2\alpha+4)/(2\alpha+4+p)}] \\ &\geqslant C_1 \exp[-2bN^{1-(p/(\alpha+2+p))} + Ld^{(\alpha+2)/(2\alpha+4+p)}N^{(\alpha+2)/(\alpha+2+p)}] \\ &= C_1 \exp[C_2 N^{(\alpha+2)/(\alpha+2+p)}], \end{aligned}$$

where $C_2 := Ld^{(\alpha+2)/(2\alpha+4+p)} - 2b$ $(> 0)$.

By Lemma 4.1 there is a constant $C_3$ such that $|\lambda_n a_{n,s}| \leqslant C_3 n^{(\beta+2)/q}$ for all $n$ and $s$. It follows that

$$|\lambda_N| \leqslant \frac{C_3}{C_1} N^{(\beta+2)/q} \exp[-C_2 N^{(\alpha+2)/(\alpha+2+p)}].$$

for $N \geqslant N_0$. From here our claim is immediate.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. To prove that (iv) implies (iii) consider any function $g(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{N}_\alpha^p$. By Lemma 4.2, there are a constant $C'$ and a null sequence of positive numbers $b_n$ such that $|a_n| \leqslant C' \exp[b_n n^{(\alpha+2)/(\alpha+2+p)}]$ for all $n$. It follows that

$$\sum_{n=0}^{\infty} |\lambda_n a_n| \leqslant CC' \sum_{n=0}^{\infty} \exp[(b_n - c)n^{(\alpha+2)/(\alpha+2+p)}] < \infty.$$

□

**Remark 4.4.** There is no problem in extending Theorem 2.1 to analytic $X$-valued functions when $X$ is a Banach space, i.e. to functions $f : \Delta \to X$ for which there are $x_n \in X$ such that $f(z) = \sum_{n=0}^{\infty} z^n x_n$ on $\Delta$. It is also possible to obtain a version of Theorem 2.1 when $X$ is a quasi-Banach space. Let $A(X)$ consist of all continuous functions $f : \bar{\Delta} \to X$ that are analytic on $\Delta$ (in the above sense). In a natural fashion, $A(X)$ is a quasi-Banach space. An argument of Eoff [**8**], which is based on a theorem of Kalton [**12**], can be used to show that a sequence $(x_n)$ is a multiplier from $\mathcal{N}_\alpha^p$ into $A(X)$ (definition as before) if and only if there are constants $C, c > 0$ such that $\|x_n\| \leqslant C \exp[-cn^{(\alpha+2)/(\alpha+2+p)}]$ for all $n$. The problem here is to overcome the fact that Cauchy's formula, on which Lemma 4.1 relies, is not available for analytic functions with values in a quasi-Banach space. Kalton's theorem just provides an appropriate substitute for Lemma 4.1 for functions in $A(X)$.

## 5. Proof of Theorem 2.2

We begin by showing that $\mathcal{N}_\alpha^p$ is in fact a linear subspace of $F_{(\alpha+2)/p}$. Let $f \in \mathcal{N}_\alpha^p$ be given, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. By Lemma 4.2, there are a null sequence of scalars $s_n > 0$ and a constant $C$ such that $|a_n| \leqslant C \exp(s_n n^{(\alpha+2)/(\alpha+2+p)})$. Our claim follows, since, for each $k \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} |a_n| \exp\left[-\frac{n^{(\alpha+2)/(\alpha+2+p)}}{k}\right] \leqslant C \sum_{n=0}^{\infty} \exp\left[\left(s_n - \frac{1}{k}\right) n^{(\alpha+2)/(\alpha+2+p)}\right] < \infty.$$

It is clear that the resulting embedding $\mathcal{N}_\alpha^p \hookrightarrow F_{(\alpha+2)/p}$ has dense range (polynomials) and is continuous (closed graph theorem). Accordingly, $F^*_{(\alpha+2)/p}$ can be identified with a linear subspace of $(\mathcal{N}_\alpha^p)^*$.

To show that this is all of $(\mathcal{N}_\alpha^p)^*$, take any $g \in (\mathcal{N}_\alpha^p)^*$. Clearly, the monomials $z^n$ belong to $\mathcal{N}_\alpha^p$. Given $s > 0$ we can choose $\lambda > 0$ in such a way that $\log(1 + |\lambda z^n|) \leqslant s$. It follows that $(z^n)$ is a bounded sequence in $\mathcal{N}_\alpha^p$, and so the $\lambda_n := g(z^n)$ form a bounded sequence of scalars.

Take any $f(z) = \sum_{n=0}^\infty a_n z^n$ from $\mathcal{N}_\alpha^p$. Clearly, $\lim_{|w|\to 1} |||f - f_w|||_{\alpha,p} = 0$, where $f_w(z) = f(wz)$ for $w, z \in \Delta$. It follows that

$$g(f_w) = \lim_{m\to\infty} \sum_{n=0}^m \lambda_n a_n w^n = \sum_{n=0}^\infty \lambda_n a_n w^n,$$

and so

$$g(f) = \lim_{|w|\to 1} g(f_w) = \lim_{|w|\to 1} \sum_{n=0}^\infty \lambda_n a_n w^n.$$

Since $g \in (\mathcal{N}_\alpha^p)^*$, the right-hand side defines an element of $H^\infty$. Moreover, $(\lambda_n)$ multiplies $\mathcal{N}_\alpha^p$ into $H^\infty$ (which embeds into any $\mathcal{A}_\beta^q$). By Theorem 2.1, there is a constant $c > 0$ such that $\sup_n |\lambda_n| \exp[cn^{(\alpha+2)/(\alpha+2+p)}] < \infty$, which signifies that $(\lambda_n)$ belongs to $(F_{(\alpha+2)/p}{}^{(k)})^*$ for sufficiently large integers $k$ (see § 3 for notation). Therefore, the corresponding analytic function $\sum_{n=0}^\infty \lambda_n z^n$, which is nothing but $g$, belongs to $F^*_{(\alpha+2)/p}$, and for $f(z) = \sum_{n=0}^\infty a_n z^n$ in $\mathcal{N}_\alpha^p$ we obtain what was asserted:

$$\sum_n a_n \lambda_n = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}t}) g(r\mathrm{e}^{-\mathrm{i}t})\,\mathrm{d}t.$$

This proves (a) and (b). Statements (c) and (d) are easy consequences. Let $X$ be any metrizable topological vector space, let $\tau$ be its topology and $|||\cdot|||$ a defining $F$-norm. The sets $U(s) := \{x \in X : |||x||| \leqslant s\}$, $s > 0$, form a 0-basis for $\tau$. Their convex hulls, $\operatorname{conv} U(s)$, form a 0-basis for the finest locally convex topology on $X$ that is coarser than $\tau$; we label it $\tau_0$. If $Y$ is any locally convex space, then every continuous linear map $[X, \tau] \to Y$ is also continuous as a map $[X, \tau_0] \to Y$. In particular, $[X, \tau]$ and $[X, \tau_0]$ have the same dual, $X^*$.

$\tau_0$ has a countable 0-basis since $\tau$ has. It is metrizable whenever $X^*$ separates the points in $X$. In that case, it coincides with the Mackey topology $\mu(X, X^*)$ determined by the dual pairing $\langle X, X^*\rangle$. The completion of $[X, \mu(X, X^*)]$ is a Fréchet space, the so-called *Fréchet envelope* of $X$. Typically, this envelope is strictly bigger than $X$: if $\tau$ is complete but not locally convex, then $\tau_0$ cannot be complete.

Again let $X$ be $\mathcal{N}_\alpha^p$, let $\beta(X^*, X)$ be the usual strong topology on $X^*$ and let $X^{**}$ be the dual of $[X^*, \beta(X^*, X)]$. We endow $X^{**}$ with the strong topology $\beta(X^{**}, X^*)$. It is a simple consequence of our considerations and standard duality theory that $X^{**}$ can naturally be identified with $F_{(\alpha+2)/p}$.

To prove (e), observe that if $\mathcal{N}^p_\alpha$ is locally bounded, then $\mu(\mathcal{N}^p_\alpha, (\mathcal{N}^p_\alpha)^*)$ is locally bounded too. But this is impossible, since we are dealing with an infinite-dimensional nuclear space.

$\mathcal{N}^p_\alpha$ is properly contained in $F_{(\alpha+2)/p}$: it is in fact easy to construct sequences in $F_{(\alpha+2)/p}$ that do not satisfy the conclusion of Lemma 4.2. Thus $\mathcal{N}^p_\alpha$ cannot be locally convex. □

**Remark 5.1.** By the extension of Theorem 2.1 mentioned in Remark 4.4, even every continuous linear map from $\mathcal{N}^p_\alpha$ into a quasi-Banach space $X$ admits an extension to a continuous operator $F_{(\alpha+2)/p} \to X$. It follows that $\mathcal{N}^p_\alpha$ even fails to be locally pseudoconvex: no matter how we choose a sequence $(s_n)$ in $(0, 1]$, $\mathcal{N}^p_\alpha$ does not admit a 0-basis $(U_n)_n$ consisting of $s_n$-convex sets. In other words, $\mathcal{N}^p_\alpha$ cannot be represented as a projective limit of quasi-Banach spaces.

Since locally bounded spaces are locally $s$-convex for some $0 < s \leqslant 1$, this also improves upon (e) in Theorem 2.2. That the spaces $\mathcal{N}^p_\alpha$ cannot be locally bounded can also be derived from general results (see Nawrocki [**13**, p. 170]).

Let $\alpha, \beta \geqslant -1$ and $1 \leqslant p$, $q < \infty$ are such that $(\alpha, p) \neq (\beta, q)$. Then $\mathcal{A}^p_\alpha \neq \mathcal{A}^q_\beta$, and so $\mathcal{N}^p_\alpha \neq \mathcal{N}^q_\beta$ (take exponentials). On the other hand, we have the following corollary.

**Corollary 5.2.** *The spaces $\mathcal{N}^p_\alpha$ and $\mathcal{N}^q_\beta$ have the same Fréchet envelopes if and only if*

$$\frac{\alpha+2}{p} = \frac{\beta+2}{q}.$$

This has an interesting consequence. Let $X$ be a Banach space. Every continuous linear map $\mathcal{N}^p_\alpha \to X$ extends uniquely to an operator $F_{(\alpha+2)/p} \to X$, so that we may identify the corresponding spaces of continuous linear maps, $L(\mathcal{N}^p_\alpha, X)$ and $L(F_{(\alpha+2)/p}, X)$.

**Corollary 5.3.** *If*

$$\frac{\alpha+2}{p} = \frac{\beta+2}{q},$$

*then $L(\mathcal{N}^p_\alpha, X)$ and $L(\mathcal{N}^q_\beta, X)$ can be identified in a canonical fashion.*

By the preceding remarks, this remains true even if $X$ is only a quasi-Banach space.

Several questions remain open. For example, it is easy to check that every locally convex quotient space of $\mathcal{N}^p_\alpha$ is nuclear. In particular, no infinite-dimensional Banach space can be isomorphic to a quotient of $\mathcal{N}^p_\alpha$. But we do not know if any infinite-dimensional Banach space can be isomorphic to a subspace of $\mathcal{N}^p_\alpha$.

## 6. Proof of Theorem 2.3

Let $X$ be an $F$-space with a separating dual and with $F$-norm $|||\cdot|||$, and let $Y$ be a quasi-Banach space with quasinorm $\|\cdot\|_Y$. Extending the standard Banach space definition (compare, for example, with [**5**]), we say that a linear map $u : X \to Y$ is *nuclear* if it admits a representation

$$ux = \sum_{n=1}^{\infty} \langle x^*_n, x\rangle y_n \quad \text{for all } x \in X$$

(convergent series in $Y$), where $(x_n^*)$ and $(y_n)$ are sequences in $X^*$ and $Y$, respectively, such that, for some $s > 0$, all $|x_n^*|_s := \sup_{x \in U(s)} |\langle x_n^*, x\rangle|$ exist and satisfy

$$\sum_{n=1}^{\infty} |x_n^*|_s \|y_n\|_Y < \infty.$$

Since $|x_n^*|_s = \sup\{|\langle x_n^*, x\rangle| : x \in \operatorname{conv} U(s)\}$ for each $s > 0$, such an operator is also nuclear as an operator $[X, \tau_0] \to Y$, and it even admits a nuclear extension mapping the completion of $[X, \tau_0]$ into $Y$. Here $\tau_0$ is as before.

Now let $X$ be a space $\mathcal{N}_\alpha^p$, $\alpha \geqslant -1$, $p \geqslant 1$, and let $u$ be a continuous linear map from $\mathcal{N}_\alpha^p$ to a Banach space $Y$. Its continuous extension, $\tilde{u} : F_{(\alpha+2)/p} \to Y$, is a nuclear operator in the usual sense, since the domain is a nuclear locally convex space.

Let $\varphi : \Delta \to \Delta$ be analytic and let $\beta \geqslant -1$, $0 < q < \infty$. If $f \mapsto f \circ \varphi$ defines a bounded linear map ('composition operator') $C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{A}_\beta^q$ for some $0 < q < \infty$, then it does so for every $q$, and in such a case $C_\varphi$ has the additional property of being 'order bounded': every $\mathcal{N}_\alpha^p$-ball $\{f \in \mathcal{N}_\alpha^p : |||f|||_{\alpha,p} \leqslant s\}$ $(s > 0)$ is mapped into an order interval in the lattice $L^q(m_\beta)$. (For more information on order boundedness of Banach space operators, see [**5**].) We can always assume that $C_\varphi$ maps into a Banach space. This was proved in [**11**] for $p = 1$. The proof could be carried over to the general case, but the following provides another argument.

**Proof of Theorem 2.3.** By Theorem 2.1, (iii), (iv) and (v) are equivalent. We are going to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Suppose that $C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{A}_\beta^q$ exists as a bounded operator for some $0 < q < \infty$. We want to show that $C_\varphi(\mathcal{N}_\alpha^p) \subset \mathcal{A}_\beta^{q'}$, for any $q < q' < \infty$. To this end, let $N \in \mathbb{N}$ be such that $q' \leqslant Nq$. By assumption, there is a $c > 0$ such that $\|C_\varphi f\|_{\beta,q} \leqslant 1$ whenever $f \in \mathcal{N}_\alpha^p$ satisfies $|||f|||_{\alpha,p} \leqslant c$. But since $\log(1 + |f|^N) \leqslant N \log(1 + |f|)$, we have $|||f^N|||_{\alpha,p} \leqslant c$ whenever $|||f|||_{\alpha,p} \leqslant c/N$. The assertion is now immediate from

$$\|C_\varphi f\|_{\beta,q'} \leqslant \|C_\varphi f\|_{\beta,Nq} = \|(C_\varphi f)^N\|_{\beta,q}^{1/N} = \|C_\varphi(f^N)\|_{\beta,q}^{1/N} \leqslant 1.$$

Since we may take $q' \geqslant 1$, $C_\varphi$ maps $F_{(\alpha+2)/p}$ into $\mathcal{A}_\beta^{q'}$, and so

$$C_\varphi : \mathcal{N}_\alpha^p \hookrightarrow F_{(\alpha+2)/p} \to \mathcal{A}_\beta^{q'} \hookrightarrow \mathcal{A}_\beta^q$$

is nuclear.

(ii) $\Rightarrow$ (iii). Recall from § 3 that $F_{(\alpha+2)/p}$ is the projective limit of the Hilbert spaces $F_{(\alpha+2)/p}^{(k)}$. Suppose that $C_\varphi$ exists as a map $\mathcal{N}_\alpha^p \to \mathcal{A}_\beta^1$. Then there is a $k \in \mathbb{N}$ such that $C_\varphi$ extends to a bounded operator $u : F_{(\alpha+2)/p}^{(k)} \to \mathcal{A}_\beta^1$. For $a = (a_n) \in F_{(\alpha+2)/p}^{(k)}$ $(\subset {F_{(\alpha+2)/p}}^{(2k)})$, with

$$C^2 = \sum_{n=0}^{\infty} \exp\biggl[-\frac{n^{(\alpha+2)/(\alpha+2+p)}}{2k}\biggr],$$

we have

$$\sum_{n=0}^{\infty} q_k(a_n e_n) = \sum_{n=0}^{\infty} |a_n| \exp\left[-\frac{n^{(\alpha+2)/(\alpha+2+p)}}{2k}\right] \leqslant C q_{2k}(a) < \infty;$$

here $e_n$ is the $n$th standard unit vector in $F^{(k)}_{(\alpha+2)/p}$. But $ue_n = \varphi^n$ for each $n$, so that, by continuity of $u$, the sequence $(\|a_n\varphi^n\|_{\beta,1})_n$ belongs to $\ell^1$. This holds for every $(a_n)$ for which

$$\sum_{n=0}^{\infty} |a_n|^2 \exp\left[-\frac{n^{(\alpha+2)/(\alpha+2+p)}}{k}\right] < \infty.$$

Hence $\sup_n \exp[n^{(\alpha+2)/p}p/(2k)]\|\varphi^n\|_{\beta,1} < \infty$, and taking $c = (2k+1)^{-1}$ we get what we wanted.

(iii) $\Rightarrow$ (i). By our hypothesis,

$$c_k^2 := \sum_{n=0}^{\infty} \exp\left[\frac{n^{(\alpha+2)/p}p}{k}\right]\|\varphi^n\|^2_{\beta,1}$$

is finite for some $k$. It follows that $\|C_\varphi f\|_{\beta,1} \leqslant c_k q_k(f)$ for all $f \in F_{(\alpha+2)/p}$: $C_\varphi$ maps $F_{(\alpha+2)/p}$, and *a fortiori* $\mathcal{N}^p_\alpha$, continuously into $\mathcal{A}^1_\beta$. □

As mentioned above, the $\mathcal{N}^1_\alpha$-part of Theorem 2.3 can also be derived from Theorem 1.4 in [**11**]. But an operator between Hilbert function spaces is order bounded if and only if it is Hilbert–Schmidt, so that, conversely, Theorem 1.4 in [**11**] appears as a consequence of Theorem 2.3 as well.

**Acknowledgements.** H.J. was supported by the Swiss National Funds. V.M. was supported in part by project pb96-0758 and by the Universidad Politecnica de Valencia. J.X. was supported by the AvH Foundation of Germany and by the NNSF and SEC of the People's Republic of China. Research for this paper started while V.M. and J.X. were visiting the University of Zürich in October 1998. They gratefully acknowledge support and excellent working conditions.

## References

1. E. Beller, Zeros of $A^p$ functions and related classes of analytic functions, *Israel J. Math.* **22** (1975), 68–80.
2. D. M. Campbell and R. J. Leach, A survey of $H^p$ multipliers as related to classical function theorey, *Complex Variables* **3** (1984), 85–111.
3. J. S. Choa, H. O. Kim and J. H. Shapiro, Compact composition operators on the Smirnov class, *Proc. Am. Math. Soc.*, in press.
4. T. Domenig, H. Jarchow and R. Riedl, The domain space of an analytic composition operator, *J. Aust. Math. Soc.* **66** (1999), 56–65.
5. J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators* (Cambridge University Press, 1995).
6. P. Duren, *Theory of $H^p$ spaces* (Academic Press, New York, 1970).

7. C. M. Eoff, Fréchet envelopes of certain algebras of analytic functions, *Mich. Math. J.* **35** (1988), 413–426.
8. C. M. Eoff, $p$-envelopes on non-locally convex $F$-spaces, *Ann. Polon. Math.* **57** (1992), 121–134.
9. N. Jaoua, Order bounded composition operators on the Hardy spaces and the Nevanlinna class, *Studia Math.* **134** (1999), 35–55.
10. H. Jarchow, *Locally convex spaces* (Teubner, Stuttgart, 1981).
11. H. Jarchow and J. Xiao, Composition operators between Nevanlinna classes and Bergman spaces with weights, *J. Operator Theory*, in press.
12. N. J. Kalton, Analytic functions in non-locally convex spaces and applications, *Studia Math.* **83** (1986), 275–303.
13. M. Nawrocki, The Fréchet envelopes of vector-valued Smirnov classes, *Studia Math.* **94** (1989), 163–177.
14. J. W. Roberts and M. Stoll, Composition operators on $F^+$, *Studia Math.* **57** (1976), 217–228.
15. J. H. Shapiro, Mackey topologies, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces, *Duke Math. J.* **43** (1976), 187–202.
16. J. H. Shapiro and A. L. Shields, Unusual topological properties of the Nevanlinna class, *Am. J. Math.* **97** (1976), 915–936.
17. S. V. Shvedenko, Rate of increase and Taylor coefficients of functions of the Nevanlinna area class, *Izvestiya VUZ Matematika* **30** (1986), 40–43.
18. W. Smith, Composition operators between Bergman and Hardy spaces, *Trans. Am. Math. Soc.* **348** (1996), 2331–2348.
19. M. Stoll, Mean growth and Taylor coefficients of some topological algebras of holomorphic functions, *Ann. Polon. Math.* **35** (1977), 141–158.
20. N. Yanagihara, Multipliers and linear functionals for the class $N^+$, *Trans. Am. Math. Soc.* **180** (1973), 449–461.
21. K. Zhu, *Operator theory in function spaces* (Marcel Dekker, New York, 1990).