Mon. Not. R. Astron. Soc. **391,** 815–824 (2008) doi:10.1111/j.1365-2966.2008.13916.x

Geometrical and physical properties of circumbinary discs in eccentric stellar binaries

Barbara Pichardo,[1,2]★ Linda S. Sparke[3]★ and Luis A. Aguilar[4]★

[1]*Instituto de Astronomía, Universidad Nacional Autónoma de México, Apdo. postal 70-264, Ciudad Universitaria, 04510 D.F., México*
[2]*Institute for Theoretical Physics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland*
[3]*Department of Astronomy, 475 North Charter Street, University of Wisconsin-Madison, Madison, WI 53706-1582, USA*
[4]*Observatorio Astronómico Nacional, Universidad Nacional Autónom a de México, Apdo. postal 877, 22800 Ensenada, México*

Accepted 2008 September 4. Received 2008 July 22; in original form 2008 February 7

ABSTRACT

In a previous work, we studied stable configurations for circumstellar discs in eccentric binary systems. We searched for 'invariant loops': closed curves (analogous to stable periodic orbits in time-independent potentials) that change shape with the binary orbital phase, as test particles in them move under the influence of the binary potential. This approach allows us to identify stable configurations when pressure forces are unimportant, and dissipation acts only to prevent gas clouds from colliding with one another. We now extend this work to study the main geometrical properties of circumbinary discs. We have studied more than 100 cases with a range in eccentricity $0 \leq e \leq 0.9$ and mass ratio $0.1 \leq q \leq 0.9$. Although gas dynamics may impose further restrictions, our study sets lower stable bounds for the size of the central hole in a simple and computationally cheap way, with a relation that depends on the eccentricity and mass ratio of the central binary. We extend our previous studies and focus on an important component of these systems: circumbinary discs. The radii for stable orbits that can host gas in circumbinary discs are sharply constrained as a function of the binary's eccentricity. The circumbinary disc configurations are almost circular, with eccentricity $e_{\mathrm{d}} < 0.15$, but if the mass ratio is unequal the disc is offset from the centre of mass of the system. We compare our results with other models, and with observations of specific systems like GG Tauri A, UY Aurigae, HD 98800 B, and Fomalhaut, restricting the plausible parameters for the binary.

Key words: binaries: general – circumstellar matter.

1 INTRODUCTION

It is currently believed that fragmentation is the most probable mechanism for star formation, and the main products of fragmentation are multiple stellar systems with preference for wide eccentric binaries with separations ≥10 au (Bonnell & Bastien 1992; Bate 1997; Bate & Bonnell 1997; Bodenheimer, Hubickyj & Lissauer 2000). Even in isolated stars, there is evidence that the majority of Sun-like stars formed in clusters (Carpenter 2000; Lada & Lada 2003), including the Sun (Looney, Tobin & Fields 2006).

In the last decade, the interest in binary systems has increased. This is in part because of the discovery that many T-Tauri and other pre-main sequence binary stars possess circumstellar and circumbinary discs as inferred from observations of excess radiation at infrared to millimetre wavelengths, polarization and both Balmer and forbidden emission lines (Mathieu et al. 2000; Itoh et al. 2002, for a review see Mathieu 1994). On the other hand, recent observations of binary star systems, using the *Spitzer Space Telescope*, show evidence of debris discs in these environments (Trilling et al. 2007) and planets (Fischer et al. 2008). In their studies, they find that 60 per cent of the observed close binary systems (separations smaller than 3 au) have excess in their thermal emission, implying on-going collisions in their planetesimal regions.

Over 150 extrasolar planets have been identified in surveys using the Doppler technique. Of the first 131 extrasolar planetary systems that have been confirmed, at least 40 are in binary or multiple systems (for an up-to-date list see Haghighipour 2006). Approximately 30 of them are on S-type orbits (around one of the components: circumstellar discs) with wide stellar separations (between 250 and 6500 au), including at least three that orbit one member of a triple star (Raghavan et al. 2006). Although most of these binaries are very wide, a few have separations smaller than 20 au (Els et al. 2001; Hatzes et al. 2003), challenging standard ideas of Jovian planet formation. Some interesting ideas try to explain the formation of Jovian planets within close binaries, but they could only explain few cases (Pfahl & Muterspaugh 2006), if

★E-mail:barbara@astroscu.unam.mx (BP); sparke@astro.wisc.edu (LSS); aguilar@astrosen.unam.mx (LAA)

more are discovered soon, these theories would not be sufficient. Although close binaries are not included in precise Doppler radial velocity search programs because of their complex and varying spectra, at least one planet with a minimum of 2.5 Jupiter masses has been detected in a P-type orbit (around both components: circumbinary discs), with a distance from the centre of mass of 23 au. The source is a radio pulsar binary comprised by a neutron star and a white dwarf in a 191 d stellar orbit (Lyne et al. 1988; Sigurdsson & Phinney 1993; Sigurdsson et al. 2003). An example of accretion in P-type orbits about close binaries is given by Quintana & Lissauer (2006) who note the observation of the two small moons orbiting in nearly circular/planar orbits about the binary system Pluto–Charon (Weaver et al. 2006). Specifically regarding to circumbinary disc material, millimetre and mid-infrared excess emission has been detected around several spectroscopic pre-main sequence binary star systems including GW Ori (Mathieu et al. 1995), UZ Tau E (Jensen, Koerner & Mathieu 1996), DQ Tau (Mathieu et al. 1997).

In this work, we have followed the same steps as in Pichardo, Sparke & Aguilar (2005, hereafter Paper I), where we opted for a simpler approach, analogous to using the structure of periodic orbits in a circular binary, to predict the gas flow. The path followed by a gas parcel in a stable disc around a star must not intersect itself or the path of a neighbouring parcel (unlike the case of planets, where the paths may cross). In our work, we follow Rudak & Paczynski (1981) and explore those non-crossing orbits of test particles that could be interpreted as gas particles in the low-pressure regime, or as protoplanets or planets. An important issue in Celestial Mechanics is to determine the regions around a stellar binary system where accretion discs can form. Important theoretical effort carried out to answer this question is reviewed in Paper I, where we studied circumstellar and circumbinary discs in binaries of arbitrary eccentricity and mass ratio. In this work, we extend those studies, which were based on identifying families of stable *invariant loops*, a concept introduced by Maciejewski & Sparke (1997, 2000) in studies of nested galactic bars. We focus this time specifically on the geometry of circumbinary discs. We employ for this approach a test particle method probing the orbital structure of binaries of various eccentricities and mass ratios.

In Section 2, we briefly review the concept of an *invariant loop*, describe the method to solve the motion equations and the strategy used to find invariant loops. The geometry of the circumbinary discs including a fit for the inner radii of the circumbinary disc and a fit for the lopsidedness, are presented in Section 3. In Sections 4 and 5, we apply this study to compare with theoretical work and observations of some well-known systems, respectively. Our conclusions are presented in Section 6.

2 THE METHOD AND NUMERICAL IMPLEMENTATION

A more detailed description of the invariant loops method and its numerical implementation is given in Paper I (also in Maciejewski & Sparke 1997, 2000). We give in this section a brief description.

In the well-studied circular three-body problem, one known integral of motion is conserved: the Jacobi constant, defined in the rotating reference frame of the stars. Stable periodic orbits in this rotating system are defined and represent the 'backbone' of the orbital structure. On the other hand, when the eccentricity is non-zero, there are no known integrals of motion to facilitate any analytical studies. However, the lack of a global integral of motion does not preclude the existence of restrictions that apply to particular orbits. For motion in the plane of the binary, an additional integral of motion would confine an orbit to lie on a one-dimensional curve every time the system comes back to the initial orbital phase. For example, if we look at the system every time the binary is at periastron, a particle following this orbit will land in a different spot but on the same one-dimensional curve, which we call an *invariant loop*. In this manner, *invariant loops* represent the generalization of periodic orbits for periodically time-varying potentials. This means that an invariant loop is not a simple orbit but an ensemble of orbits that lie on a 3-torus in this extended phase-space, but supported by an additional isolating integral of motion that forces the particles to have a one-dimensional intersection with the orbital plane at a fixed binary phase.

The equations of motion for the binary system are solved in terms of the eccentric anomaly ψ (Goldstein, Poole & Safko 2002, section 3.7). We use units where the gravitational constant G, the binary semimajor axis a, and its total mass $m_1 + m_2$ are set to unity so that, the binary period is 2π, and its frequency $\omega = 1$. The separation between the stars at time t, measured from periastron where the azimuthal angle $\theta = 0$, is given by the radius r,

$$r = a(1 - e\cos\psi), \tag{1}$$

$$\omega t = (\psi - e\sin\psi), \tag{2}$$

$$\cos\theta = a(\cos\psi - e)/r. \tag{3}$$

The binary eccentricity, defined as $e = \sqrt{1 - b^2/a^2}$ where a and b are the semimajor and semiminor axes and the mass ratio $q = m_2/(m_1 + m_2)$ are the only free parameters. We use an Adams integrator (from the Numerical Algorithms Group (NAG) FORTRAN library) to follow the motion of a test particle moving in the orbital plane of the two stars. Kepler's equation (2) is solved with a tolerance of 10^{-9}.

The equations of motion of the test particle are solved in an inertial reference frame using Cartesian coordinates, with their origin at the centre of mass of the binary. All test particle trajectories are launched when the binary is at periastron, with the two components lying on the x-axis. The computation is halted if the particle runs away, moving further than 10 times the semimajor axis from the centre of mass, or if it comes within a distance of either star that results in a high number of force computations, in general due to close approaches to the stars.

To find stable *invariant loops*, for which the phase space coordinates of our test particle, traces a one-dimensional curve on successive passes through periastron, we launch particles from a chosen position along the x-axis joining the two stars at periastron, and examine the iterates in some two-dimensional subspace, such as the x–y plane. We plot the positions of the test particle at each complete binary period, and adjust the starting velocity v_y until the iterates converge on a one-dimensional curve. In practice, we look at the scatter along the radial direction for those iterates that lie within a sector that spans a small angle (5°) about the x-axis when viewed from the centre of mass of the system. We adjust the launch velocity v_y until the radial scatter of periastron positions of the test particle in the 5° sector drops within a threshold value. A value of $10^{-4}a$ is used for circumbinary loops and $10^{-6}a$ for circumstellar loops. These values are consistent with the numerical errors in the orbit integration. For the majority of orbits, 10 points within the sector suffice and no more than five attempts are necessary to identify a given loop. While we are in a region of stable invariant loops, the required launch velocity v_y is a continuous function of the starting point x.

The numerical strategy we employ to solve the problem will allow us to find only stable invariant loops. Particles launched close to an unstable loop would diverge and the code would not be able to find this kind of loops. However, it is the stable orbits we are interested in. We do not calculate all the possible loops, but restrict our attention to those that are symmetric about the line joining the two stars when they are at periastron. When the binary orbit is circular, these are exactly the closed periodic orbits of a circumbinary disc. Although our figures show a set of discrete curves, invariant loops form continuous families in the same way as periodic orbits. We show only a few of the possible invariant loops, for clarity.

3 GEOMETRICAL CHARACTERISTICS OF CIRCUMBINARY DISCS

Discs in multiple stellar systems have attracted attention because of their high abundance, and also due to the interesting effects of the interaction on the disc morphology, better studied every day with the improvement of observations. Of particular interest are the circumbinary discs that, because of their low density, are very difficult to observe. However, their importance arises from the possibility that these envelope discs might be feeding putative circumstellar discs or harbour protoplanets, planets or any kind of debris.

As in section 3 of Paper I, the inner radius of the circumbinary disc is set by the criterion that the stable loops exist and do not intersect each other. If the binary eccentricity is not small ($e > 0.1$) then the loops become unstable before they begin to intersect each other; we find no more one-dimensional curves or the test particles fall towards the stars or go out of the system.

To characterize the geometry of circumbinary discs, we use the coefficients of the Fourier expansion of the innermost loop:

$$A_k = \frac{1}{N}\sum_{i=1}^{N} s(\phi_i)\cos(k\phi_i),$$
$$B_k = \frac{1}{N}\sum_{i=1}^{N} s(\phi_i)\sin(k\phi_i), \qquad (4)$$

where (s_i, ϕ_i) are the polar coordinates of N evenly spaced (in ϕ) points along the innermost stable loop, measured from the binary centre of mass. The modulus $\sqrt{A_k^2 + B_k^2}$ is used to determine the mean distance to the barycentre ($k = 0$) and the lopsidedness ($k = 1$).

More than one hundred simulations were included to calculate the fits we present in the next two subsections that provide the main geometrical characteristics of the discs. These simulations were performed in the eccentricity interval $e = [0.0, 0.9]$ and mass ratio $q = [0.1, 0.5]$ (equivalent to sample the whole range $q = [0.1, 0.9]$ because of the symmetry in the definition of q). It is worth to mention that we have not included in the fits values for $q < 0.1$ since the behaviour of the radius at those extreme values of q, changes abruptly and requires many more calculations. We will produce fits for extreme q values (as is the case of planets) in a further paper. The loops technique, however, allows us to reach these extreme cases and we present in this paper an example of it (Section 5).

At any time, the invariant loops form closed curves, which deform as the binary follows its orbit, and return to their original shape when the binary returns to the same phase. But we found in Paper I that in practice even the circumstellar loops do not deform strongly, and the circumbinary loops even less. Thus, we measure the properties of the cirumbinary discs, and show their shapes in the figures, at the time of periastron passage.

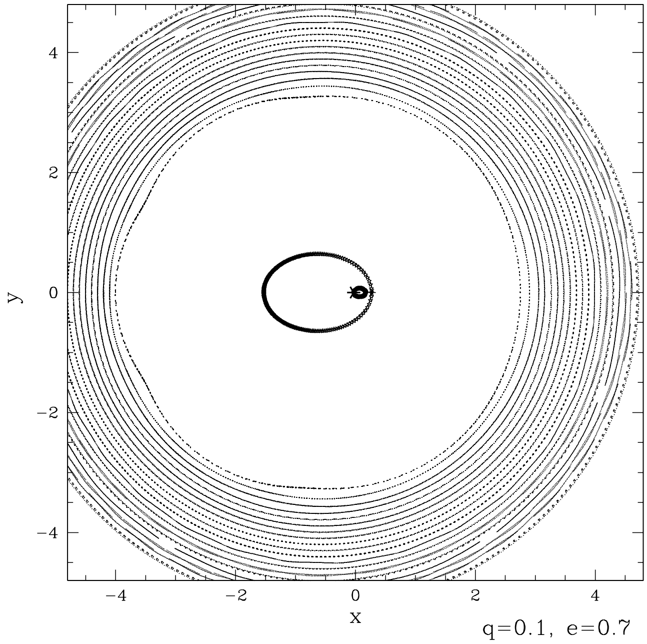

Figure 1. Circumbinary disc computed with invariant loops for a binary with $q = 0.1$ and $e = 0.7$, viewed at the moment of periastron marked with the two stars at $(x, y) \approx (0, 0)$. The orbits of the stars are shown with the darker curves. The disc is not centred about the centre of mass of the system.

3.1 Lopsidedness of circumbinary discs

An interesting effect produced by a high binary eccentricity combined with a large mass contrast, is the displacement of the geometric centre of the circumbinary disc (inner edge) with respect to the barycentre (See for example Fig. 1). This effect is a physical characteristic that could explain some observed asymmetries in discs (Duchêne et al. 2004; Boden et al. 2005; Kalas, Graham & Clampin 2005). The displacement of the disc centre may affect calculations of the disc's inclination (e.g. Itoh et al. 2002) since it is usual to link asymmetries to inclination effects rather than to intrinsic asymmetries in the geometrical centre of the discs.

In theoretical work of dynamics in planetary systems, there have been several studies using linear analysis of perturbations, which consider low mass ratios or low eccentricities (e.g. Wyatt et al. 1999; Kuchner & Holman 2003; Deller & Maddison 2005). In these studies, it is shown how the presence of a second body (like a planet) in a system with an eccentric orbit would impose a forced eccentricity on the orbits of the constituent dust particles, thus shifting the geometric centre of the disc away from the mass centre of the binary. In this manner, the dust in slightly eccentric orbits would glow more brightly when it approaches the pericentre. This could explain the asymmetry in the double-lobed feature in many systems (Holland et al. 1998; Koerner et al. 1998; Schneider et al. 1999; Telesco et al. 2000; Kalas et al. 2005; Freistetter, Krivov & Lohne 2007). In the same direction, Dermott et al. (1999) find that if there is at least one massive perturber in the HR 4796 system that is on an eccentric orbit, then the system's secular perturbations could cause the geometric centre of the disc to be offset from the star.

Our technique of invariant loops allows the displacement of the disc to be calculated for arbitrary mass ratio and eccentricity. For mass ratios $q \geq 0.1$, we have obtained a fit for the displacement of the circumbinary geometric centre with respect to the centre of

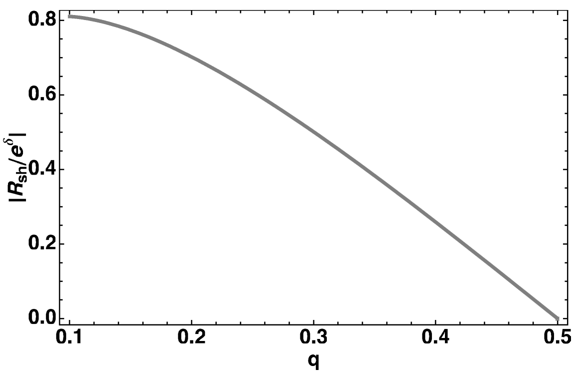

Figure 2. Lopsidedness measured by the quantity $R_{\rm sh}(e, q)/e^{\delta}$, as a function of eccentricity e and mass ratio q, from equation (5).

Table 1. Difference between the shift of the centre of the circumbinary disc computed using invariant loops and the fit (equation 5), in units of the semimajor axis, a for some chosen pairs (e, q).

e	q: 0.1	0.2	0.3	0.4	0.5
0.00	0.000	0.010	0.006	0.010	0.001
0.20	−0.005	−0.025	0.030	0.009	−0.004
0.40	0.012	0.011	−0.001	0.009	−0.001
0.60	−0.012	−0.052	0.012	0.024	0.000
0.80	0.001	0.009	0.081	0.015	0.000

mass of the binary (see Fig. 2),

$$R_{\rm sh}(e, q) = -C_1 a e^{\delta}\,(0.5 - q)[q(1 - q)]^{\eta}, \qquad (5)$$

where a is the semimajor axis $C_1 = 3.7, \delta = 0.8$ and $\eta = 0.25$, give our best fit to the calculated off-centre distance. The displacement in our calculations is always directed to the left of the centre of mass due to the position we chose to place the primary (left-hand side) and secondary (right-hand side) with respect to the centre of mass of the binary at their pericentre (see Fig. 1).

We find the best fit to the proposed equation by searching the minimum residuals (some of them are given in Table 1) produced by the square root of the sum of the squared differences between the proposed function, equation 5, and the computed displacements. The standard deviation comparing the calculated data and the fit is $0.029a$. The residual values in Table 1 are in units of the semimajor axis, a.

The inner circumbinary rim is not exactly elliptical; close to resonances, the loops can even become slightly triangular. However, unless the eccentricity is very close to zero, the shape of the inner rim is close to circular and its maximum and minimum diameters lie almost perpendicular. The eccentricity reaches a maximum value $e_{\rm d} \approx 0.15$, almost independent of the binary eccentricity.

3.2 Inner radii (the 'gap')

In Paper I, we derived a simple relation for the size of circumstellar discs as a function of binary mass ratio and eccentricity. In this paper we extend the study to circumbinary discs and derive a relation now for the inner radius ('gap'). Together, these radii define the region where no loops exist. We found that the change in radius of the circumbinary disc with the binary phase is very small; so here we calculate the inner radius of the circumbinary discs at one phase, when the stars are at their periastron.

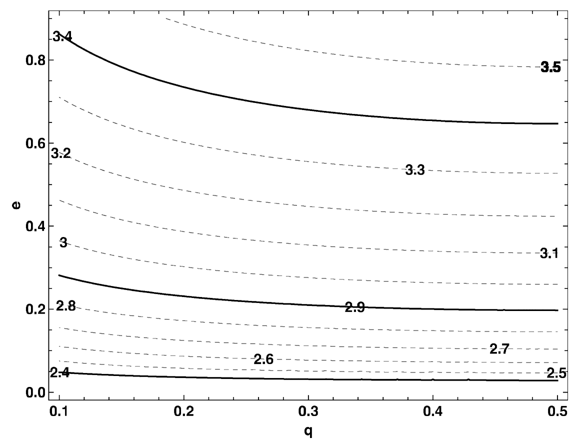

Figure 3. Contour plot of the computed average inner radius $R_{\rm CB}/a$ for the circumbinary discs, from equation (6).

The best fit to our calculated inner radii is given by

$$R_{\rm CB}(e, q) \approx C_2 a(1 + \alpha e^{\beta})\,[q(1 - q)]^{\gamma}, \qquad (6)$$

where a is the semimajor axis, $C_2 = 1.93, \alpha = 1.01, \beta = 0.32$ and $\gamma = 0.043$. The standard deviation obtained is $0.09a$.

In Fig. 3, we show a contour plot of the approximation to the average radii for the circumbinary discs from equation 6. In Table 2, we show the difference between the relation 6, and the computed radii.

The ratio of equations (5) and (6) gives an estimate of the size of the displacement of the circumbinary disc geometric centre, as a fraction of the inner circumbinary (average) radius. In Table 3, we present the ratio between the calculated shifts and the corespondent average circumbinary radius.

4 COMPARISON WITH THEORETICAL WORK

In the problem of accretion and planet formation in binary systems, it is of paramount importance to determine the regions where a

Table 2. Difference between the computed inner radii of the circumbinary discs and the fit (from relation 6), in units of the semimajor axis, a, for some chosen pairs (e, q).

e	q: 0.1	0.2	0.3	0.4	0.5
0.00	−0.051	−0.065	−0.005	−0.034	0.067
0.20	0.000	−0.060	−0.016	0.107	−0.204
0.40	0.009	0.250	0.203	0.216	−0.297
0.60	0.170	−0.022	0.106	0.105	−0.202
0.80	−0.040	−0.120	−0.003	−0.062	−0.392

Table 3. Displacements calculated from invariant loops for some cases used for the fit in equation 5, given as a percentage of the corresponding circumbinary average radius.

e	q: 0.1	0.2	0.3	0.4	0.5
0.00	0.000	0.557	0.320	0.538	0.051
0.20	8.291	7.859	3.785	2.038	0.149
0.40	12.340	9.682	7.177	3.363	0.035
0.60	15.217	15.509	9.235	4.219	0.000
0.80	20.179	17.178	9.577	5.776	0.000

circumbinary disc can exist. Extensive literature exists, but mostly devoted to the circular orbit binary case. Several studies have been done about the dependence on mass ratio and orbital eccentricity for the existence and characteristics of circumbinary discs. In particular, Quintana & Lissauer (2006) have simulated the late stages of terrestrial planet formation within circumbinary discs in close binary systems with a wide range of orbital eccentricities. Quintana & Lissauer (2007) simulated the final stages of terrestrial planet formation in S- and P-type orbits within main-sequence binary star systems.

In planetary studies, a lot of work has been done (e.g. Holman & Wiegert 1999, hereafter HW99; Wyatt et al. 1999). In the last reference, in particular, the authors address this issue by investigating the long-term stability of planetary orbits numerically using the eccentric restricted three-body problem. They launch circular prograde orbits in the vicinity of the stars and in the circumbinary region, looking for orbits that remain close to the stars for more than 10 000 binary periods. They provide a fit for the outer radii of the circumstellar discs, and for the inner part of the circumbinary disc, that depends only on the parameters of the binary (a, e, q). Their study has the advantage that it is able to examine a full range of eccentricity and mass ratios. The disadvantages are the lower precision of this technique, due to the fact that a disc can live much more than the fiducial 10 000 binary periods used by the authors to qualify an orbit as stable, and the fact that it is expensive computationally.

In our work, we identify stable non-intersecting loops where gas may accumulate, a disc may develop and a planet may form. This condition plays the same role as searching for stable non-intersecting periodic orbits in the potential of a circular binary; in either case, the results could be modified by pressure or viscous forces. In this sense, our search is more stringent than the study of HW99, who find orbits where a planet may survive for long times around a binary, but these orbits are permitted to intersect themselves or neighbouring orbits. Still, a comparison of our results with those of HW99 is relevant to gauge to what extent our different criteria result in similar constraints.

We have calculated the difference between the fits made by HW99 for the circumstellar and circumbinary discs. In the case of circumstellar discs, we find a good agreement with their results. For the circumbinary discs, there are some differences. In the Fig. 4, we show the results of HW99 (filled triangles) and the results obtained from invariant loops (continuous lines) of the calculated average inner radii of circumbinary discs, including the minimum and maximum distance from the centre of mass (open circles), versus the binary eccentricity, for two mass ratios, $q = 0.1$ (left-hand side), $q = 0.3$ (right-hand side). We see in the figure that the fit by HW99 gives in general larger radii for the gap, especially at higher eccentricities of the central binary. For smaller eccentricities, the radii provided for the fit of HW99 are almost the same or even smaller than the ones provided by the invariant loops. It is likely that as the binary becomes more eccentric, the phase space in which orbits can be trapped so that they must remain close to the stable circumbinary loops shrinks in volume. That would make it less probable that the initially circular orbits of HW99 would lie in that trapped region.

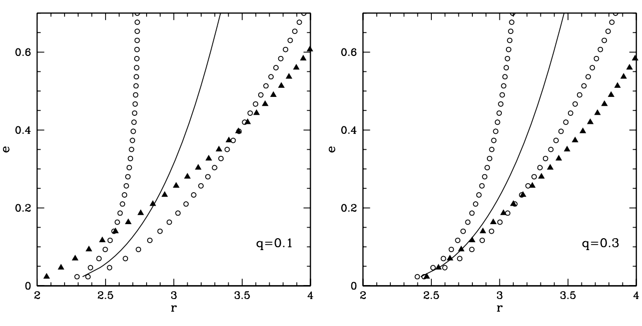

Figure 4. Comparison between the radii calculated by the fit of HW99 (filled triangles) and the fit with invariant loops that gives the mean radius (continuous lines) as eccentricity increases. We have also included the maximum and minimum radius from the centre of mass due to the shift of the disc (empty circles). Left-hand panel: mass ratio $q = 0.1$, right-hand panel: mass ratio $q = 0.3$.

5 APPLICATION TO OBSERVATIONS

Lim & Takakuwa (2006) have already applied our results in Paper I to constrain the eccentricity of the binary in L1551 IRS5 to $e < 0.3$, based on the sizes of the circumstellar and circumbinary discs.

As a further application of our study, we have chosen four systems. The first represents the prototype of circumbinary discs: GG Tauri A. The second is UY Aurigae, the third is HD 98800 B and the last is Fomalhaut.

5.1 The circumbinary disc of GG Tauri A system

GG Tau is a well-known young multiple system. The system has two binary stars: GG Tau Aa/Ab and GG Tau Ba/Bb. GG Tau A is an interesting binary since it possesses a circumbinary disc resolved in the millimetre wavelengths (Kawabe et al. 1993; Guilloteau, Dutrey & Simon 1999). It has been observed at high resolution in the optical (Krist, Stapelfeldt & Watson 2002) and in near-infrared (Roddier et al. 1996; Silber et al. 2000; McCabe, Duchêne & Ghez 2002).

The structure of the circumbinary disc seems to be characterized by an annulus with an inner radius between 180 and 190 au (Guilloteau et al. 1999; Duchêne et al. 2004) and an outer radius extending up to 800 au. The total mass of the circumbinary material (H_2 + dust) is $\approx 0.12\,M_\odot$ (Guilloteau et al. 1999). Itoh et al. (2002) derive an inclination of approximately 37°, assuming the orbit of the binary is coplanar to the circumbinary disc.

The orbital characteristics of the central binary are still controversial. While Roddier et al. (1996), propose an eccentric orbit in which the stars are located near periastron, at the same observation time, Krist et al. (2002) find they are close to apoastron in a highly eccentric orbit. McCabe et al. (2002) deduce an elliptical orbit $e = 0.3 \pm 0.2$ with a semimajor axis of $a = 35^{+22}_{-8}$ au. The mass of each component of GG Tau A *a* and *b*, binary is obtained by White et al. (1999): 0.78 ± 0.1 and $0.68 \pm 0.03\,M_\odot$, respectively (mass ratio $q \approx 0.47$).

Based on equation (6), we have calculated a band of possible solutions for different semimajor axis and eccentricities that could give an inner radius of 180 ± 18 au that we present in Fig. 5. In the same figure, we locate the prediction by McCabe et al. (2002), whose error bar in the semimajor axis locates it inside our error zone (those with $e = 0.3$ and semimajor axis around 55 au). In the same manner, some of the values derived by the predictions based in observations of Itoh et al. (2002), rest inside or close to this region, specifically those with semimajor axis $a = 50$ au, and eccentricities $e = 0.4$ and 0.5.

In Fig. 6, we present two possible configurations for GG Tau constructed with invariant loops. The first plot is using the values of eccentricity and semimajor axis reported by McCabe et al. (2002) (triangle on Fig. 5). Notice that this configuration results in a significantly reduced inner gap which is not compatible with the result reported by Duchêne et al. (2004) and Guilloteau et al. (1999). In

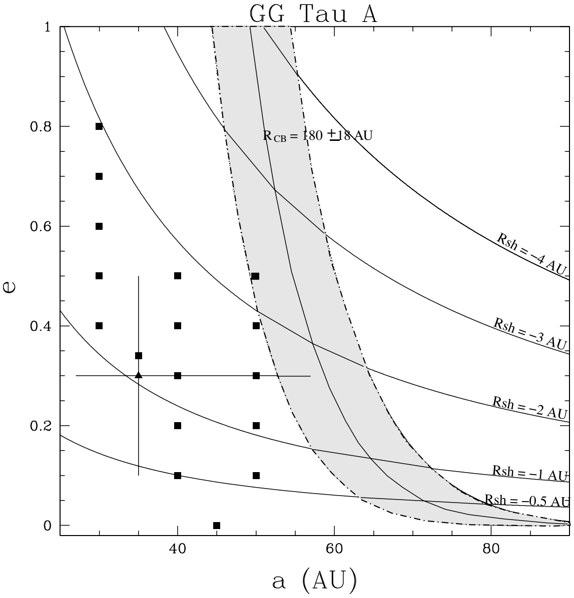

Figure 5. Possible configurations for GG Tau A with our model using the observed inner radius of the circumbinary disc and an error from observations of ±18 au (shadowed region), assuming the disc is only sculpted by the binary. We have also included some observational and model predictions for the configuration. The filled squares are the predictions in the work of Itoh et al. (2002). The filled triangle is the prediction of McCabe et al. (2002) including the error bars. Finally, we show five possible curves (continuous lines labelled with different $R_{\rm sh}$) with the calculated shifts (equation 5), for the given $q = 0.47$, and the corresponding pair (a, e).

the second panel, we present a configuration that is compatible with these two references ($R_{\rm CB} \approx 180$ au) and still within the error box of McCabe around the preferred values of ($e = 0.5, a = 53$ au). In both cases, we have assumed a mass ratio of $q = 0.47$. Although this parameter has little importance in determining the circumbinary disc inner radius, it is important in fixing its lopsidedness.

The circumbinary disc seems to be shifted away from the centre of mass of the binary (Duchêne et al. 2004) by ∼0.16 arcsec or 22 au for a distance of 140 pc. We predict that the shift of the disc geometrical centre with respect to the barycentre should be small since the stars have almost the same mass. As shown in Fig. 5, we find that the shift produced by the binary system on the circumbinary disc, would be less than 4 au. We calculate that a mass ratio of $q < 0.3$ would be necessary to produce both the observed shift and the observed inner radius of the circumbinary disc. This is unlikely because the effective temperature and luminosity of the two stars are very similar (White et al. 1999).

Because the observational parameters determined by McCabe et al. (2002) and Itoh et al. (2002) imply that the circumbinary disc should extend much closer to the binary and the lopsidedness should be much smaller than is predicted by considering orbits around the binary GG Tau A, alternative theories have arisen. An interesting idea is the possibility that the smaller GG Tau B binary (10 arcsec away and with masses 0.12 ± 0.02 and $0.044 \pm 0.006\,{\rm M}_{\odot}$ for GG Tau B a and b, respectively) has something to do with the sculpting of the circumbinary disc (Beust & Dutrey 2006). Unfortunately, these authors find that, although there are possible arrangements to explain the outer edge of the disc, the binary B is not massive enough, and not close enough to explain the wide inner radius in this way.

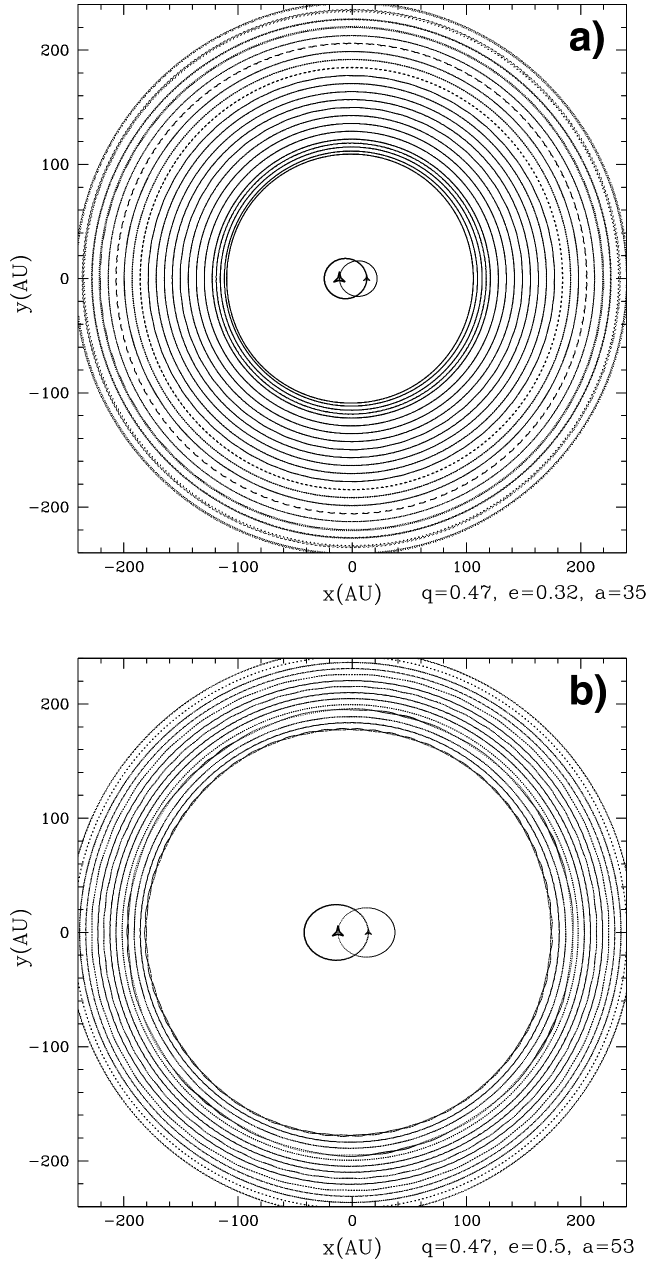

Figure 6. (a) Circumbinary disc computed with invariant loops for the central value of eccentricity ($e = 0.32$) and semimajor axis ($a = 35$ au) derived by McCabe et al. (2002) for a mass ratio $q = 0.47$. (b) Same as (a) but for a plausible invariant loops solution for eccentricity (0.5) and semimajor axis ($a = 53$ au) that reproduces the inner edge observed, of 180 au, assuming that the only factor sculpting the inner edge of the circumbinary disc is the main binary, GG Tau A.

5.2 The binary system UY Aur

UY Aur is a binary system of classical T Tauri stars (Duchêne et al. 1999). It is located in the Taurus–Aurigae star-forming region at an approximate distance of 140 pc (Elias 1978). The projected separation on the sky between UY Aur primary and secondary (A and B) is 120 au (Close et al. 1998). The spectral types of UY Aur A and B are estimated to be M0 and M2.5, respectively (Hartigan & Kenyon 2003). Assuming a circular orbit, the binary period is $\sim$1640 $\pm$ 90 yr and the total mass of the binary is $\sim$1.73 $\pm$ 0.29 $M_{\odot}$ (Hioki et al. 2007).

A circumbinary disc around the UY Aur binary was detected by near-infrared, polarimetric and millimetre interferometric ^{13}CO emission observations (Duvert et al. 1998). Its inner radius is about 520 au (Hioki et al. 2007). These authors deproject the circumbinary disc assuming that its inner edge is circular and the binary is coplanar with it, to get a binary separation of 167 au and an inclination of 42° $\pm$ 3°. The disc seems to be not uniform showing clumpy structure, circumstellar material inside the inner cavity and an arm-like structure, probably created by accretion from the outer region of the disc or stellar encounters (Hioki et al. 2007).

We calculated stable invariant loops in a system with the orbital parameters proposed for the binary by Hioki et al. (2007), a binary separation of 167 au and $e = 0$. We have set $q = 0.5$, but Fig. 3 shows that the edge of the CB disc does not change significantly for any $q > 0.1$. This model would give an inner radius for the circumbinary disc of no more than 340 au, far less than the observed 520 au for this system.

We have proceeded by constructing a family of solutions as in the case of GG Tau A (Section 5.1). This is, we provide a family of solutions which reproduce the approximate inner radius of the circumbinary disc. We construct two different possible configurations, with different mass ratios, q = [0.25, 0.4] (Figs 7 and 8), which bracket $q = 0.36$ as given by Hartigan & Kenyon (2003). Although the inner radius of the circumbinary disc is almost independent of the mass ratio (for $q \geq 0.1$), the shift of the circumbinary disc increases as the mass ratio becomes more unequal. The present separation is 167 au, which must lie between the periastron and apastron separations. This restricts e, a to points above the curves given by $a(1 - e)$ and $a(1 + e)$ in Figs 7 and 8. Thus, the binary cannot be circular, but must have $e > 0.1$ and for a mass ratio of

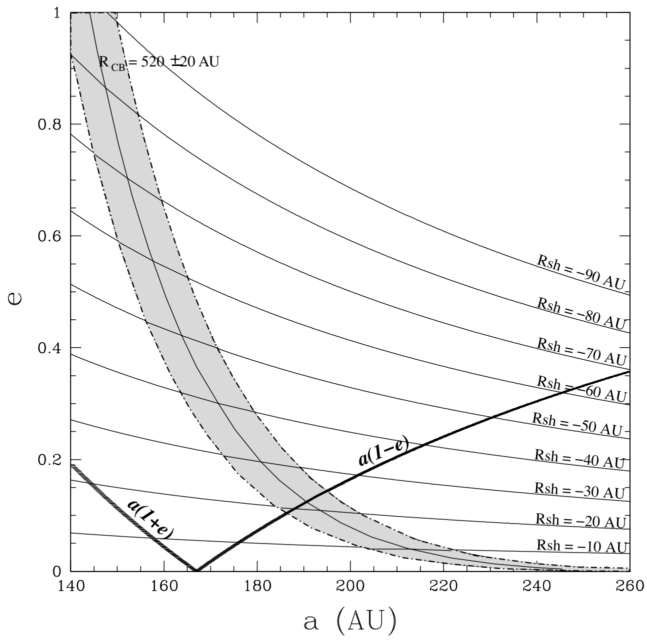

Figure 7. Possible configurations for UY Aurigae with our fit for the circumbinary radius for a mass ratio $q = 0.25$, using the observed circumbinary disc inner radius of 520 au and an error of $\pm$ 20 au (shadowed region), assuming the disc is only sculpted by the binary. We show nine possible curves (continuous lines labelled with different $R_{\rm sh}$) with the calculated shifts (equation 5), for the given $q = 0.25$, and the corresponding pair (a, e). The darker curves are the lines $a(1 + e) = 167$ au and $a(1 - e) = 167$ au.

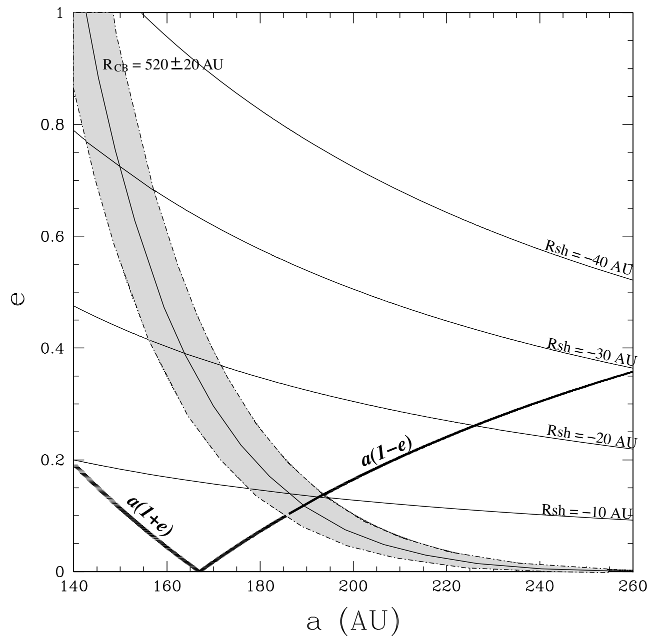

Figure 8. Same as Fig. 7 but for $q = 0.4$.

$q = 0.36$ (Hartigan & Kenyon 2003), the centre of the disc should be offset from the mass centre of the stars in the direction towards the centre of the secondary star's orbit by $R_{\rm sh} \gtrsim 0.05a$. Since the apparent separation has remained constant since 1944 (Hioki et al. 2007), the system is unlikely to be very close to periastron, implying a larger eccentricity.

5.3 The binary system HD 98800 B

HD 98800 (HIP 55505, TWA 4A) is a hierarchical quadruple star system in the TW Hya association. The separation between A and B components is approximately 0.8 arcsec on a north-south line (Prato et al. 2001). Torres et al. (1995) find that both visual components are themselves spectroscopic binaries. Tokovinin (1999) derived orbital parameters $a = 62$ au, $e = 0.5$ for the orbit of HD 98800 B around HD 98800 A.

The system HD 98800 B has excess flux in the mid-infrared, which was interpreted by Soderblom et al. (1998) and by Prato et al. (2001) as a circumbinary dust disc. Their estimates of the inner radius of the disc were 1.5 to 2 au. New observations of the stellar binary were reported by Boden et al. (2005). They estimated visual and physical orbits of the HD 98800 B subsystem with interferometric observations combined with astrometric measurements by the *Hubble Space Telescope* Fine Guidance Sensors. The orbital and physical parameters obtained in that work are given in Table 4.

Using the parameters inferred by Boden et al. (2005), Akeson et al. (2007) used the results in our Paper I to argue that the average inner radius of the circumbinary disc should be at

Table 4. Approximate orbital parameters of HD 98800 B from Boden et al. (2005).

q	0.45
e	0.78
a	0.98 au
Period (d)	314

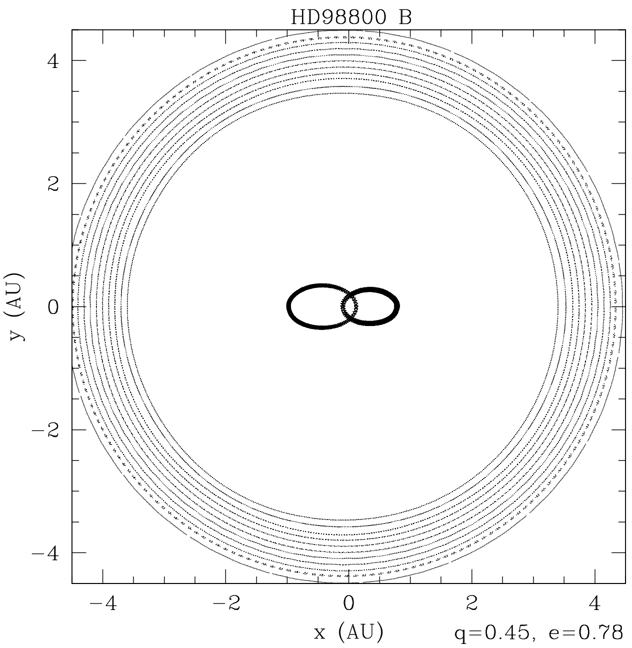

Figure 9. Stable invariant loops making up the circumbinary disc of HD98800 B, using the parameters of Boden et al. (2005), given in Table 4. The orbits of the stars are shown with the darker curves.

$R_{CB} \approx 3.4$ au. Thus, the central hole in the cirumbinary disc should be significantly larger than suggested by Soderblom et al. (1998) and Prato et al. (2001). Treating both binaries (A and B) as point masses, each with the combined mass of its two stars, $q \approx 0.5$ and from Table 1 in Paper I we predict that the maximum outer radius of the circumbinary disc of B is about a 10th of the semimajor axis of the orbit of A and B or 6.2 au. Akeson et al. compare these results with a dynamical simulation of the system, in which the two stars of binary B are followed separately while binary A is modelled as a single star, and the circumbinary disc of B is represented by test particles. After the equivalent of 1 Myr, the inner edge of the disc was at about 3 au and the outer edge at 10 au, in approximate agreement with predictions from our invariant loops.

We have constructed the corresponding circumbinary disc for HD98800 B (Fig. 9). Using equations 5 and 6, we find that the disc centre is shifted with respect to the centre of mass of the system by $R_{sh} \approx -0.1$ au. The shift is small because the two stars have nearly equal mass.

5.4 The Fomalhaut dust belt

Fomalhaut (HD 216956 or α Pisces Austrinus) is a bright nearby A3 V star of 2 $M_\odot$, an age of 200 ± 100 Myr (Barrado y Navascues 1998) at a distance of 7.7 pc. It shows a dust ('debris') ring around it between 133 and 158 au from the central star (Aumann 1985; Holland et al. 1998, 2003; Dent et al. 2000) with an approximate mass between 50 and 100 Earth mass (Kalas et al. 2005). This structure represents one of the best-observed extrasolar analogue to our Kuiper belt. Fomalhaut's ring has an inclination of 24° away from edge on. The disc presents an asymmetry in the brightness with the southern side nearer the star than the opposite side (Stapelfeldt et al. 2004; Marsh et al. 2005). The deprojected asymmetry (off-centred) of ≈15 au, the sharp inner cut at 133 au, and the slight eccentricity of the disc $e_d \approx 0.1$ has been studied recently by (Kalas et al. 2005) and Quillen (2006) who have proposed that all the characteristics of this system can be explained by the presence of a planet just interior to the ring inner edge.

Several theories to form the sharp inner edge of the disc in Fomalhaut have been proposed by different studies, the most accepted one until this moment is the presence of inner planets. The solution for Fomalhaut is degenerate in the sense that several combinations of the main parameters (mass ratios between the central star and the planet, eccentricities and semimajor axes) could reproduce the observed values for the inner radius of the dust disc R_{CB}, and the displacement R_{sh} of its centre from the star's position. We explore here three possible sets of parameters given by other authors.

In Table 5, we show the selected parameters for which we have calculated the invariant loops that would make up a circumbinary disc, and their corresponding references.

Under the assumption that a secondary body (a giant planet) is the only source of asymmetry of the ring in Fomalhaut we have calculated sets of invariant loops corresponding to both models in Table 5. In the upper panel of Fig. 10, the results are given with invariant loops for the approximation of Deller & Maddison (2005). From Fig. 10 and Table 6, we show that the inferred value of eccentricity from Deller & Maddison (2005) is considerably larger than it should be to obtain the observed inner radius of Fomalhaut's disc, and the shift, if the mass of the planet is $2M_J$. In the lower panel of Fig. 10 and second row of Table 6, we show a solution for a planet mass of $2\,M_J$ for which R_{CB} and R_{sh} are closer to the observed values for Fomalhaut. We have also computed the eccentricity e_d of the inner rim of the circumbinary disc presented in the last column of Table 6.

Quillen (2006) proposed solutions with a less-massive planet, using the formulae of Wyatt et al. (1999) to calculate the expected eccentricity. For two extreme values of the range given in that work, this is, the mass of Saturn and the mass of Neptune, given in Table 5, we have calculated the invariant loops to find the average and the shift radii. The values given by Quillen (2006) are close to the results that we derive from the invariant loops, as can be seen in Table 6. This shows that in the limit of small eccentricity, calculations based on invariant loops agree with those from earlier methods which are valid only in those regimes.

6 CONCLUSIONS

We have extended the studies started in Paper I to a more detailed analysis of circumbinary discs in eccentric binary systems from the

Table 5. Orbital parameters proposed for Fomalhaut. The first column is the mass, the second is the planetary mass (in terms of the mass of Jupiter, Saturn or Neptune), eccentricity e and semimajor axis a are in the third and fourth columns, the employed technique and reference is given in the last two columns.

M_{Fomal} ($M_\odot$)	M_{Pl}	e	a (au)	Technique	Reference
2.3	$2\,M_J$	0.4	59	N-body	Deller & Maddison (2005)
2	$1\,M_S$ to $1\,M_N$	0.1	119	Secular perturbations	Quillen (2006)

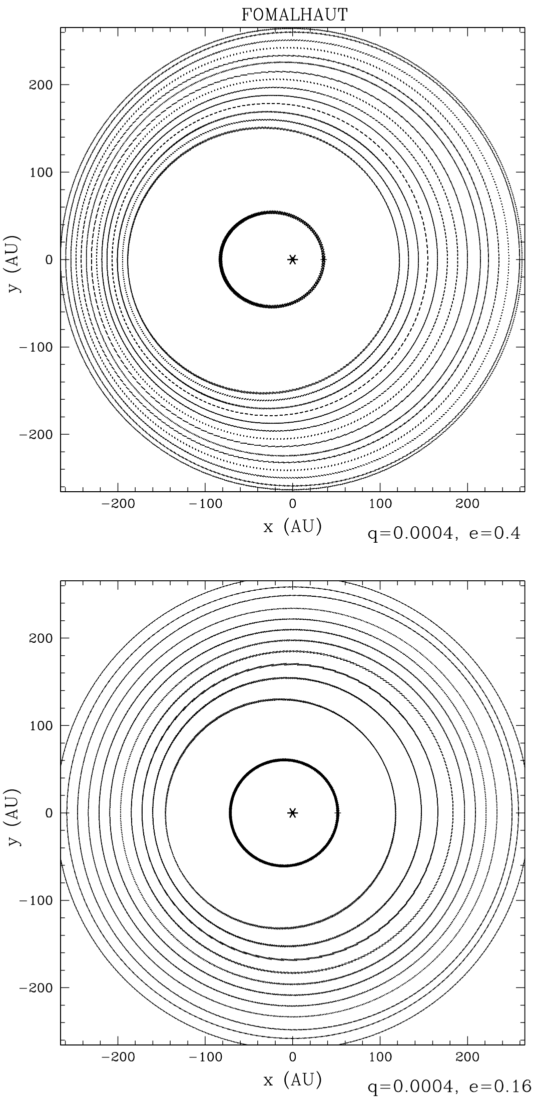

Figure 10. Circumbinary disc of Fomalhaut. Upper panel: invariant loops disc using the parameters ($e = 0.4, a = 59$ au) in Table 5 by Deller & Maddison (2005). Lower panel: our solution for the same planetary mass but with ($e = 0.16, a = 61.5$ au). The trajectory of the planet is shown with the darker curves.

geometrical and physical point of view. The discs are defined by a family of stable *invariant loops*, the analogues to stable periodic orbits around a circular binary, which do not cross each other or themselves. Thus, they define paths that can be followed by clouds of gas, which dissipate energy when they run into each other. Just as with a binary in circular orbit, the circumbinary disc is truncated at its inner edge when there are no longer any stable non-crossing orbits for the gas to follow. We have used this property of the invariant loops to define the inner edge of the circumbinary disc.

We showed already in Paper I that the inner radius of a circumbinary disc depends strongly on eccentricity, opening wider gaps for higher eccentricities. The size of the inner hole depends only slightly on the mass ratio. The geometric centre of the circumbinary disc is off-centre with respect to the centre of mass of the binary system. The disc is closer to being symmetrical around the whole orbit of the secondary star.

Here, we have explored the range of parameters to quantify both the off-centring of the circumbinary disc with respect to the centre of mass of the system, and the average inner radius of circumbinary discs, as a function of mass ratio and eccentricity. We compare our results with the work of HW99 who searched for initially circular orbits that survive more than 10 000 periods of the binary. When the eccentricity is small this procedure gives similar results to ours, but at larger eccentricity HW99 find fewer stable orbits close to the binary, and hence larger inner gaps. This could be related to the fact that the larger the eccentricity, the smaller the available phase space of orbits that are trapped so that they must remain close to a stable invariant loop.

If the properties of a circumbinary disc are observable, it is possible to constrain the binary system properties by comparing the predictions based on invariant loops with what is observed. Likewise, if the orbital parameters of a binary are known, the geometry of the circumbinary and circumstellar regions permitted for stable orbits are readily obtained.

For the well-known circumbinary disc of the binary system GG Tau A, we use the inner radius of the circumbinary disc to restrict the possibilities for the binary parameters. Since two stars have nearly equal masses, we would expect the ring to be nearly symmetrical about the centre of mass of the stars. The observed offset is substantial and much larger than can be explained by orbital dynamics; effects such as the finite ring thickness may be important (e.g. Duchêne et al. 2004).

Table 6. Disc characteristics from invariant loops for Fomalhaut assuming the proposed masses of Deller & Maddison (2005) and Quillen (2006) for the planet, and our best approximation to the solution taking the proposed masses by these authors. The first column shows the assumed mass for Fomalhaut, the second is the proposed mass for the planet around Fomalhaut (J = Jupiter, S = Saturn and N = Neptune), the third and fourth are the eccentricity and the semimajor axis of the planetary orbit, the fifth and sixth are the shift radii calculated with invariant loops, the seventh column is the value of the circumbinary inner rim eccentricity ($e_{\rm d}$) and the last column indicates the author of the paper we have taken the parameters from or our best approximation to the parameters that approach the most to the observed values of Fomalhaut.

$M_{\rm Fomal}$ ($M_\odot$)	$M_{\rm Pl}$	e	a (au)	$R_{\rm sh}$ (au)	$R_{\rm CB}$ (au)	$e_{\rm d}$	Work
2.3	$2M_{\rm J}$	0.4	59	36	155		Deller & Maddison (2005)
2.3	$2M_{\rm J}$	0.16	61.5	16	133	0.15	Our best approximation
2	$M_{\rm N}$	0.1	119	12.5	142.5		Quillen (2006)
2	$M_{\rm N}$	0.13	90.5	14.5	136	0.13	Our best approximation
2	$M_{\rm S}$	0.1	119	12.5	148		Quillen (2006)
2	$M_{\rm S}$	0.13	106.4	14	135	0.13	Our best approximation

In the case of UY Aurigae, we show that the observed radius of the circumbinary disc requires that the binary orbit be noncircular, with eccentricity $e > 0.1$. The centre of the disc should be offset from the mass centre of the stars in the direction towards the centre of the secondary star's orbit by $R_{\rm sh} \gtrsim 0.05a$.

We have modelled the system HD 98800 B with the parameters given by Boden et al. (2005): the shift of the disc with respect to the centre of mass of the system should be about 0.1 au.

Although the fits we provide here are valid for values of $q \geq 0.1$, our technique is also applicable to extreme cases of $q \leq 0.001$. For the disc around Fomalhaut, we have used invariant loops to propose plausible solutions for the orbital parameters of a planet that explains the morphology of the debris disc of this system.

We reach similar results to Quillen (2006), but for the larger planetary mass proposed by Deller & Maddison (2005) we find that the planet's orbit must be less eccentric.

ACKNOWLEDGMENTS

B. P. acknowledges Antonio Peimbert for useful discussions and acknowledges projects CONACYT through grant 50720 and UNAM through grant PAPIIT IN119708.

REFERENCES

Akeson R. L., Rice W. K. M., Boden A. F., Sargent A. I., Carpenter J. M., Bryden G., 2007, ApJ, 670, 1240
Aumann H. H., 1985, PASP, 97, 885
Barrado y Navascues D., 1998, A&A, 339, 831
Bate M. R., 1997, MNRAS, 285, 16
Bate M. R., Bonnell I. A., 1997, MNRAS, 285, 33
Beust H., Dutrey A., 2006, A&A, 446, 137
Boden A. F. et al., 2005, ApJ, 635, 442
Bodenheimer P., Hubickyj O., Lissauer J. J., 2000, Icarus, 143, 2
Bonnell I., Bastien P., 1992, ApJ, 401, 654
Carpenter J. M., 2000, AJ, 120, 3139
Close L. M. et al., 1998, ApJ, 499, 883
Deller A. T., Maddison S. T., 2005, ApJ, 625, 398
Dent W. R. F., Walker H. J., Holland W. S., Greaves J. S., 2000, MNRAS, 314, 702
Dermott S. F., Grogan K., Holmes E., Kortenkamp S., 1999, in Greenberg J. M., Li A., eds, Formation and Evolution of Solids in Space. Kluwer, Dordrecht, p. 565
Duchêne G., Monin J.-L., Bouvier J., Menard F., 1999, A&A, 351, 954
Duchêne G., McCabe C., Ghez A. M., Macintosh B. A., 2004, ApJ, 606, 969
Duvert G., Dutrey A., Guilloteau S., Menard F., Schuster K., Prato L., Simon M., 1998, A&A, 332, 867
Elias J. H., 1978, ApJ, 224, 857
Els S. G., Sterzik M. F., Marchis F., Pantin E., Endl M., Kürster M., 2001, A&A, 370, L1
Fischer D. A. et al., 2008, ApJ, 675, 790
Freistetter F., Krivov A. V., Lohne T., 2007, A&A, 466, 389
Goldstein H., Poole C., Safko J., 2002, Classical Mechanics, 3rd edn. Addison Wesley
Guilloteau S., Dutrey A., Simon M., 1999, A&A, 348, 570
Haghighipour N., 2006, ApJ, 644, 543
Hartigan P., Kenyon S. J., 2003, ApJ, 583, 334
Hatzes A. P., Cochran W. D., Endl M., McArthur B., Paulson D. B., Walker G. A. H., Campbell B., Yang S., 2003, ApJ, 599, 1383
Hioki T. et al., 2007, AJ, 134, 880
Holland W. S. et al., 1998, Natur, 392, 788
Holman M. J., Wiegert P. A., 1999, AJ, 117, 621 (HW99)
Itoh Y. et al., 2002, PASJ, 54, 963
Jensen E. L. N., Koerner D. W., Mathieu R. D., 1996, AJ, 111, 2431
Kalas P., Graham J. R., Clampin M., 2005, Nat, 435, 1067
Kawabe R., Ishiguro M., Omodaka T., Kitamura Y., Miyama S. M., 1993, ApJ, 404, 63
Koerner D. W., Ressler M. E., Werner M. W., Backman D. E., 1998, ApJ, 503, 83
Krist J. E., Stapelfeldt K. R., Watson A. M., 2002, ApJ, 570, 785
Kuchner M. J., Holman M. J., 2003, ApJ, 588, 1110
Lada C. J., Lada E. A., 2003, ARA&A, 41, 57
Lim J., Takakuwa S., 2006, ApJ 653, 425
Looney L. W., Tobin J. J., Fields B. D., 2006, ApJ, 652, 1755
Lyne A. G., Biggs J. D., Brinklow A., McKenna J., Ashworth M., 1988, Nat, 332, 45
McCabe C., Duchêne G., Ghez A. M., 2002, ApJ, 575, 974
Maciejewski W., Sparke L. S., 1997, ApJ, 484, L117
Maciejewski W., Sparke L. S., 2000, MNRAS, 313, 745
Marsh K. A., Velusamy T., Dowell C. D., Grogan K., Beichman C. A., 2005, ApJ, 620, L47
Mathieu R. D., 1994, ARA&A, 32, 465
Mathieu R. D., Adams F. C., Fuller G. A., Jensen E. L. N., Koerner D. W., Sargent A. I., 1995, AJ, 109, 2655
Mathieu R. D., Stassun K., Basri G., Jensen E. L. N., Johns-Krull C. M., Valenti J. A., Hartmann L. W., 1997, AJ, 113, 1841
Mathieu R. D., Ghez A. M., Jensen E. L. N., Simon M., 2000, in Mannings V., Boss A. P., Russell S. S., eds, Protostar and Planets IV. University of Arizona Press, Tucson, p. 731
Pfahl E., Muterspaugh M., 2006, ApJ, 652, 1694
Pichardo B., Sparke L. S., Aguilar L. A., 2005, MNRAS, 359, 521 (Paper I)
Prato L. et al., 2001, ApJ, 549, 590
Quillen A. C., 2006, MNRAS, 372, 14
Quintana E. V., Lissauer J. J., 2006, Icar, 185, 1
Quintana E. V., Lissauer J. J., 2007, preprint (arXiv:0705.3444)
Raghavan D., Henry T. J., Mason B. D., Subasavage J. P., Jao W.-C., Beaulieu T. D., Hambly N. C., 2006, ApJ, 646, 523
Roddier C., Roddier F., Northcott M. J., Graves J. E., Jim K., 1996, ApJ, 463, 326
Rudak B., Paczynski B., 1981, Acta Astron., 31, 13
Schneider G. et al., 1999, ApJ, 513, 127
Sigurdsson S., Phinney E. S., 1993, 1993, ApJ, 415, 631
Sigurdsson S., Richer H. B., Hansen B. M., Stairs I. H., Thorsett S. E., 2003, Sci, 301, 193
Silber J., Gledhill T., Duchene G., Menard F., 2000, ApJ, 536, 89
Soderblom D. R. et al., 1998, ApJ, 498, 385
Stapelfeldt K. R. et al., 2004, ApJS, 154, 458
Telesco C. M. et al., 2000, ApJ, 530, 329
Tokovinin A. A., 1999, Astron. Lett., 25, 669
Torres G., Stefanik R. P., Latham D. W., Mazeh T., 1995, ApJ, 452, 870
Trilling D. E. et al., 2007, ApJ, 658, 1289
Weaver H. A. et al., 2006, Nat, 439, 943
White R. J., Ghez A. M., Reid I. N., Schultz G., 1999, ApJ, 520, 811
Wyatt M. C., Dermott S. F., Telesco C. M., Fisher R. S., Grogan K., Holmes E. K., Piña R. K., 1999, ApJ, 527, 918

This paper has been typeset from a TEX/LATEX file prepared by the author.