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Gradient-like parabolic semiflows on BUC($\mathbb{R}^N$)

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We prove that a class of weighted semilinear reaction diffusion equations on $\mathbf{R}^N$ generates gradient-like semiflows on the Banach space of bounded uniformly continuous functions on $\mathbf{R}^N$. If $N = 1$ we show convergence to a single equilibrium. The key for getting the result is to show the exponential decay of the stationary solutions, which is obtained by means of a decay estimate of the kernel of the underlying semigroup.

1. Introduction

We consider a class of semilinear reaction diffusion equations on the whole of $\mathbf{R}^N$. A typical exponent is given by

$$\begin{cases} \partial_t u - \Delta u = m(x)u - u^3 & \text{in } \mathbf{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases} \tag{1.1}$$

The positive and negative parts m^+ and m^- of the bounded weight function m are assumed to have the following properties:

- m^+ vanishes at infinity.
- There exist constants $r, \eta > 0$ such that for all $x \in \mathbf{R}^N$ the estimate $\int_{B(x,r)} m^-(x)\, dx \geqq \eta$ holds.

The non-negative function m^- is called a strongly absorbing potential. We are interested in continuous initial data $u_0(x)$ that do not have any specified decay at infinity but are required to be bounded. More precisely, we choose the phase space for (1.1) as BUC($\mathbf{R}^N$), the Banach space of bounded uniformly continuous real-valued functions on $\mathbf{R}^N$. It is shown in [**13**] that (1.1) generates a global continuous semiflow (φ, BUC($\mathbf{R}^N$)) of class $\mathscr{AK}$ on BUC($\mathbf{R}^N$) that consists of classical solutions of (1.1) if m is Hölder-continuous.

We remark that in general the canonical Liapunov function associated with (1.1)

$$\mathscr{L}(u(x)) := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u(x)|^2\, dx - \int_{\mathbf{R}^N} \left(\frac{1}{2} m(x)u(x)^2 - \frac{1}{4} u(x)^4\right) dx \tag{1.2}$$

is of little use for orbits starting in $\mathrm{BUC}(\mathbf{R}^N)$. For example, if the initial condition $u_0(x)$ is a nonzero constant and if m^+ has compact support, then

$$\mathscr{L}(\varphi(t, u_0)) = \infty$$

for $t \geqq 0$. To overcome this difficulty, we will make use of the results in [**13**], where it is shown that the semiflow $(\varphi, \mathrm{BUC}(\mathbf{R}^N))$ possesses a compact global B-attractor $\mathscr{M}$ of finite Hausdorff dimension. Furthermore, $\mathscr{M}$ is contained in an order interval $V := [\underline{w}, \bar{w}]$ of the ordered Banach space $\mathrm{BUC}(\mathbf{R}^N)$. Here $\underline{w}$ is the smallest and $\bar{w}$ is the greatest stationary solution of (1.1). The principal part of this paper is devoted to proving that the stationary solutions of (1.1) have exponential decay at infinity. The underlying idea is to write (1.1) as

$$\partial_t u - \Delta u + m^-(x)u = m^+(x)u - u^3,$$

and to study the kernel $k(t, x, y)$ of the semigroup associated with the generator $\Delta - m^-$. In Section 2, we will prove the estimate

$$0 \leqq k(t, x, y) \leqq ct^{-N/2} \exp(-\eta t) \exp\left(-\frac{|x-y|^2}{ct}\right)$$

for some constants $c, \eta > 0$. This estimate will then imply the exponential decay at infinity of the stationary solutions of (1.1). As a consequence, the elements of the order interval V and, in particular, of the B-attractor $\mathscr{M}$ have exponential decay. This will imply that the Liapunov function is uniformly bounded below on the B-attractor $\mathscr{M}$. As a consequence of the LaSalle invariance principle (e.g. [**12**, Theorem 2.3]) the ω-limit set $\omega(u_0)$ of each element $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$ is contained in the set of stationary points of the semiflow, i.e. the semiflow induced by (1.1) is gradient-like. In one dimension it turns out that $\omega(u_0)$ consists of a single equilibrium.

2. Kernel estimates

Suppose that we are given a self-adjoint strongly uniformly elliptic operator of second order on $\mathbf{R}^N$ of the form

$$\mathscr{A} = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right), \tag{2.1}$$

with $a_{ij} = a_{ji} \in L_\infty(\mathbf{R}^N)$. By 'strongly uniformly elliptic', we mean as usual that there exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqq \alpha_0|\xi|^2, \tag{2.2}$$

for all $x, \xi \in \mathbf{R}^N$. We consider perturbations of the above operator with a non-negative potential satisfying a condition introduced in [**1**].

Definition 2.1. We say that a potential $V \in L_\infty(\mathbf{R}^N)$ is *strongly absorbing* if $V \geqq 0$ and there exist constants $r, \eta > 0$ such that

$$\int_{B(x,r)} V(\xi)\, d\xi \geqq \eta \tag{2.3}$$

for all $x \in \mathbf{R}^N$. Here, $B(x, r) = \{y \in \mathbf{R}^N : |x - y| < r\}$.

Using form methods, one can show that an appropriate restriction of $-\mathscr{A}$ to $L_p(\mathbf{R}^N)$ generates a strongly continuous analytic semigroup on $L_p(\mathbf{R}^N)$ for all $p \in [1, \infty)$. A similar statement is true for the operator $-(\mathscr{A} + V)$. One of the main results in [**1**] is that if $V \geqq 0$, then the semigroup $T_V(t)$ associated with $-(\mathscr{A} + V)$ is exponentially stable on $L_p(\mathbf{R}^N)$ for some (or equivalently all) $p \in [1, \infty)$ if and only if V is strongly absorbing. By 'exponentially stable', we mean that there exist $\gamma_0, c_0 > 0$ such that

$$\| T_V(t) \|_{\mathscr{L}(L_p, L_p)} \leqq c_0 \exp(-\gamma_0 t), \tag{2.4}$$

for all $t \geqq 0$. Alternative proofs of the result can be found in [**2, 5**]. The key in the proof is that the kernel $k_0(t; x, y)$ of $T_0(t)$ satisfies upper and lower Gaussian estimates; that is, for some constant $c > 0$,

$$c^{-1} t^{-N/2} \exp\left(-\frac{c|x-y|^2}{t}\right) \leqq k_0(t; x, y) \leqq c t^{-N/2} \exp\left(-\frac{|x-y|^2}{ct}\right). \tag{2.5}$$

By a simple comparison argument, it is clear that for any non-negative potential the upper estimate for the corresponding semigroup kernel remains valid. For strongly absorbing potentials, we shall prove the following better upper estimate.

THEOREM 2.2. *Suppose that the above assumptions on $\mathscr{A}$ are satisfied and that V is a strongly absorbing potential. Denote the kernel of the semigroup $T_V(t)$ by $k_V(t; x, y)$. Then there exist constants $c, \eta > 0$ such that*

$$0 \leqq k_V(t; x, y) \leqq c t^{-N/2} \exp(-\eta t) \exp\left(-\frac{|x-y|^2}{ct}\right) \tag{2.6}$$

for all $t > 0$ and $x, y \in \mathbf{R}^N$.

Proof. To prove (2.6), we make use of (2.4) and the results in [**6**, Section 3.2]. First of all, note that

$$\| T_V(t) \|_{\mathscr{L}(L_p, L_q)} \leqq c t^{-N/2(1/q - 1/p)} \tag{2.7}$$

whenever $1 \leqq p \leqq q \leqq \infty$. This follows, for instance, from the upper bound in (2.5), which, as we remarked earlier, is also valid for $k_V(t; x, y)$. By using the semigroup property as well as (2.4) and (2.7), we obtain

$$\begin{aligned} \| T_V(t) \|_{\mathscr{L}(L_1, L_\infty)} &\leqq \| T_V(t/3) \|_{\mathscr{L}(L_2, L_\infty)} \| T_V(t/3) \|_{\mathscr{L}(L_2, L_2)} \| T_V(t/3) \|_{\mathscr{L}(L_1, L_2)} \\ &\leqq c_0 (t/3)^{-N/2} \exp\left(-\frac{\eta_0}{3} t\right) = c t^{-N/2} \exp(-\eta t) \end{aligned}$$

for some $c, \eta > 0$ and all $t > 0$. It is well known that this implies

$$|k_V(t; x, y)| \leqq c t^{-N/2} e^{-\eta t} \tag{2.8}$$

for all $x, y \in \mathbf{R}^N$ and $t \geqq 0$. Combining (2.8) and the upper estimate in (2.5) which holds also for k_V, we get

$$\begin{aligned} 0 \leqq k_V(t; x, y) &= k_V(t; x, y)^{1/2} k_V(t; x, y)^{1/2} \\ &\leqq c t^{-N/4} \exp\left(-\frac{\eta}{2} t\right) t^{-N/4} \exp\left(-\frac{|x-y|^2}{2ct}\right) \end{aligned}$$

$$= ct^{-N/2} \exp\left(-\frac{\eta}{2}t\right) \exp\left(-\frac{|x-y|^2}{2ct}\right)$$

for all $x, y \in \mathbf{R}^N$ and $t \geqq 0$. This proves the theorem. □

COROLLARY 2.3. *If the potential V is strongly absorbing, then the weak Green's function $g_V(x, y)$ associated with $-(\mathscr{A} + V)$ satisfies an estimate of the form*

$$0 \leqq g_V(x, y) \leqq c\eta^{(N/2)-1} \left(\frac{|x-y|}{\lambda}\right)^{1-(N/2)} K_{(N/2)-1}(\lambda |x-y|) \tag{2.9}$$

for some $c > 0$ and $x \neq y$, where $K_{(N/2)-1}$ is the modified Bessel function, and $\lambda := 2\sqrt{\eta/c}$.

Proof. First, note that $g_V(x, y)$ is given by the Laplace transform of the kernel $k_V(t; x, y)$. More precisely,

$$g_V(x, y) = \int_0^\infty k_V(t; x, y)\, dt \tag{2.10}$$

for all $x, y \in \mathbf{R}^N$ with $x \neq y$. Hence, by Theorem 2.2, we have

$$0 \leqq g_V(x, y) \leqq c \int_0^\infty t^{-N/2} \exp\left(-\frac{|x-y|^2}{ct}\right) dt. \tag{2.11}$$

Recall that the modified Bessel functions can be represented by

$$K_\nu(\rho) = \frac{1}{2}\left(\frac{\rho}{2}\right)^\nu \int_0^\infty s^{-\nu-1} \exp\left(-\frac{\rho^2}{4s} - s\right) ds \tag{2.12}$$

for all $\nu \in \mathbf{R}$ and $\rho \in \mathbf{R}\backslash\{0\}$ (e.g. [**8**, p. 82, formula (23)]). Hence, by the substitution $s = \eta t$, and setting $\lambda := 2\sqrt{\eta/c}$, we get from (2.11) that (2.9) holds for $x \neq y$. □

REMARK 2.4. Using asymptotic expansions of the modified Bessel functions (see [**8**, Section 7.2]) we get that

$$0 \leqq g_V(x, y) \leqq \begin{cases} c \exp(-|x-y|/c) & \text{if } N = 1, \\ c \log(|x-y|/c) \exp(-|x-y|/c) & \text{if } N = 2, \\ c|x-y|^{-N+2} \exp(-|x-y|/c) & \text{if } N \geqq 3, \end{cases} \tag{2.13}$$

for some constant $c > 0$. In particular, we see that the Green's function of $-\mathscr{A} + V$ decreases exponentially in space if V is strongly absorbing.

REMARK 2.5. To prove (2.9), we only used an estimate of the form (2.6). Therefore, if we have a potential $V = V_1 - V_2$ with V_1 strongly absorbing and $V_2 \in L_\infty(\mathbf{R}^N)$ and $\|V_2\|_\infty$ small enough, we still get an estimate of the form (2.6) and (2.9).

3. Nonlinear elliptic equations

Consider the nonlinear elliptic equation

$$\mathscr{A}u = f(x, u), \quad x \in \mathbf{R}^N. \tag{3.1}$$

For simplicity, we will only discuss properties of non-negative solutions of (3.1). But under further restrictions on the nonlinearity f the arguments remain valid for arbitrary solutions (compare [**13**, Example 3.3b]). Following [**13**, Section 3] we impose the following restrictions on the nonlinearity $f(x, \xi)$:

$f(\cdot, \xi) \in \mathrm{BUC}^{\mu}(\mathbf{R}^N)$, for some $\mu \in (1, 2)$, uniformly for ξ in bounded subsets of $\mathbf{R}$; (3.2)

the derivatives $\partial_{\xi} f$ exists and is uniformly Lipschitz-continuous on sets of the form $\mathbf{R}^N \times B$, where B is a bounded subset of $\mathbf{R}$: (3.3)

there exists a constant $M \geqq 0$ such that $f(x, c) \leqq 0$ but $f(\cdot, c) \not\equiv 0$ for $c \geqq M$ and $x \in \mathbf{R}^N$; (3.4)

f is of the form $f(x, \xi) = h(x, \xi)\xi$. Moreover, for each $u \in X^+$ the positive part $h^+(\cdot, u(\cdot))$ of $h(\cdot, u(\cdot))$ is in $C_0(\mathbf{R}^N)$, whereas the negative part $h^-(\cdot, u(\cdot))$ is strongly absorbing. (3.5)

Under the assumptions considered in the Introduction, the nonlinearity in (1.1) satisfies all the above properties. We shall prove the following theorem:

THEOREM 3.1. *There exists an exponentially decaying function* $\psi \in C_0(\mathbf{R}^N)$ *such that any solution* $u \in L_\infty(\mathbf{R}^N)$ *satisfies* $|u(x)| \leqq \psi(x)$ *for all* $x \in \mathbf{R}^N$.

Proof. Suppose that $u \in L_\infty(\mathbf{R}^N)$ is a non-negative solution of (3.1) and that $\varepsilon > 0$ is given. Setting

$$m_1(x) := h^-(x, u(x)) - \varepsilon, \quad m_2(x) := h^+(x, u(x)) - \varepsilon,$$

we may rewrite (3.1) as

$$\mathscr{A} u + m_1 u = m_2 u. \tag{3.6}$$

Furthermore, note that by (3.5) the potential $m_1 + \varepsilon$ is strongly absorbing. Also, note that

$$m_2(x) \leqq (h(x, u(x)) - \varepsilon)^+.$$

Therefore, by the results of the previous section, we see that

$$|u(x)| \leqq \int_{\mathbf{R}^N} g_{m_1}(x, y)(h(x, u(x)) - \varepsilon)^+(y)\, dy \|u\|_\infty,$$

for all $x \in \mathbf{R}^N$. If we choose $\varepsilon > 0$ small enough and take into account Corollary 2.3 and Remark 2.5, we see that $g_{m_1}(x, y)$ decays exponentially as $|x - y|$ goes to infinity. Due to (3.5) the function $(h^+(x, u(x)) - \varepsilon)^+$ has compact support and thus

$$\tilde{\psi}(x) := \int_{\mathbf{R}^N} g_{m_1}(x, y)(h(x, u(x)) - \varepsilon)^+(y)\, dy \tag{3.7}$$

decays expoentially as $|x|$ goes to infinity. Next, observe that due to (3.4) all non-negative solutions of (3.1) are bounded by the same constant M. Hence, $\psi := M\tilde{\psi}$ has the required properties. This completes the proof of the theorem. □

4. Convergence to the set of equilibria

In this section we study the semilinear parabolic evolution problem in the Banach space $X := \mathrm{BUC}(\mathbf{R}^N)$. For simplicity, we restrict our attention to non-negative initial conditions u_0 in the positive cone X^+ of X:

$$\begin{cases} \partial_t u - \Delta u = f(x, u) & \text{in } \mathbf{R}^N \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbf{R}^N. \end{cases} \tag{4.1}$$

As in the previous section, we require that f has the properties (3.2)–(3.5). Under these assumptions, the existence of a global semiflow (φ, X^+) that consists of classical solutions of (4.1) as well as the existence of the compact global B-attractor $\mathscr{M} \subset X^+$ is shown in [**13**, Theorem 3.1]. The proof of [**13**, Theorem 3.1] also shows that the attractor $\mathscr{M}$ is contained in an order interval V that is invariant under the flow. V is of the form

$$V := [0, \bar{w}] \subset C_0(\mathbf{R}^N), \tag{4.2}$$

where $\bar{w}$ is a steady state of (4.1). Making use of the results of the previous sections, we will verify that the functional

$$\mathscr{L}(u) := \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \int_{\mathbf{R}^N} F(x, u) \, dx \tag{4.3}$$

is a Liapunov function, where for $(x, \xi) \in \mathbf{R}^N \times \mathbf{R}$ we have set

$$F(x, \xi) := \int_0^\xi f(x, \eta) \, d\eta.$$

Will will see that the functional $\mathscr{L}$ is well defined on the B-attractor $\mathscr{M}$. This will be sufficient to prove that the ω-limit set of each point in X^+ is contained in the set of equilibria of the semiflow (φ, X^+). To prove that $\mathscr{L}$ is a Liapunov function, we show some regularity properties of solutions of (4.1).

LEMMA 4.1. *Let $u_0 \in L_2(\mathbf{R}^N) \cap \mathrm{BUC}(\mathbf{R}^N)$. Then, for some $\eta > 0$, the solution u of* (4.1) *lies in*

$$C^1([\varepsilon, T], W_2^1(\mathbf{R}^N)) \cap \mathrm{BUC}^{3+\eta,(3+\eta)/2}(\mathbf{R}^N \times [\varepsilon, T])$$

for all $\varepsilon > 0$ and $T > \varepsilon$.

Proof. Fix $u_0 \in L_2(\mathbf{R}^N) \cap \mathrm{BUC}(\mathbf{R}^N)$, $\varepsilon > 0$ and $T > \varepsilon$. We first show Hölder regularity. As a consequence of the classical regularity theorems for parabolic equations with Hölder-continuous coefficients and the abstract theory of parabolic evolution equations, we conclude that, for some $\eta > 0$, the solution u of (4.1) lies in $\mathrm{BUC}^{2+\eta,1+\eta/2}(\mathbf{R}^N \times [\varepsilon, T])$ (see [**4**, Theorem 25.2]). Hence, by assumption (3.2) this implies that

$$m(x, t) := h(\cdot, u(t, \cdot)) \in \mathrm{BUC}^{1+\eta}(\mathbf{R}^N \times [\varepsilon, T]), \tag{4.4}$$

where we possibly need to decrease η. Applying [**11**, Theorem IV.5.1], we get the claimed Hölder regularity. For the Sobolev regularity, we study the equation

$$\begin{cases} \partial_t u - \Delta u - m(x, t) u = 0 & \text{in } \mathbf{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \mathbf{R}^N, \end{cases} \tag{4.5}$$

whose solution lies in $C((0, T], W_2^1(\mathbf{R}^N))$. To see this, define the multiplication operator $M(t)u(x) := m(x, t)u(x)$. Then,

$$M(\cdot) \in C^\eta([0, T], \mathscr{L}(L_2) \cap \mathscr{L}(W_2^1)). \tag{4.6}$$

Thus, taking into account that Δ generates an analytic semigroup on L_2, we find that the above claim is true. (This follows e.g. from [**4**, Theorem 5.10 and Corollary 5.4], noting that $D(\Delta) = W_2^2(\mathbf{R}^N)$ and $[L_2, W_2^2]_{1/2} = W_2^1$.) This implies that $u(\varepsilon) \in W_2^1(\mathbf{R}^N)$. As Δ also generates a strongly continuous analytic semigroup on W_2^1, it follows from (4.6) by the same argument that $u \in C^1([\varepsilon, T], W_2^1(\mathbf{R}^N))$. This concludes the proof of the lemma. □

LEMMA 4.2. *For $u_0 \in V$, the function $\mathscr{L}(\varphi(t, u_0))$ is well defined for $t > 0$. Moreover, it is bounded from below on V, and*

$$\frac{d}{dt}\mathscr{L}(\varphi(t, u_0)) < 0,$$

unless u_0 is an equilibrium of (φ, X^+).

Proof. Let u_0 be an arbitrary element in V. We first show that $\mathscr{L}(\varphi(t, u_0))$ is finite for $t > 0$. By Theorem 3.1, the function $\bar{w}$ in (4.2) is exponentially decreasing and, since V is invariant,

$$0 \leqq u(t, x) := \varphi(t, u_0)(x) \leqq Ce^{-\alpha|x|}, \quad x \in \mathbf{R}^N, \tag{4.7}$$

for some constants $C, \alpha > 0$ independent of t and u_0. In particular, this implies that $V \subset L_2(\mathbf{R}^N)$. It follows now from Lemma 4.1 that the gradient term in (4.3) is finite for all $t > 0$. We next show the finiteness of the other term. Using (4.7), we conclude that

$$|F(x, u(x, t))| \leqq \int_0^{u(x,t)} |f(x, \xi)|\, d\xi \leqq \int_0^{Ce^{-\alpha|x|}} |f(x, \xi)|\, d\xi.$$

The assumption (3.5) implies that $|f(x, \xi)| \leqq c_1 < \infty$ for $(x, \xi) \in \mathbf{R}^N \times [0, C]$. We thus find that

$$|F(x, u(x, t))| \leqq c_1 e^{-\alpha|x|}. \tag{4.8}$$

This shows that

$$\left| \int_{\mathbf{R}^N} F(x, u(x, t))\, dx \right| \leqq c_0 < \infty,$$

with $c_0 > 0$ independent of $u_0 \in V$, whence (4.3) is defined and bounded from below on V. This proves the first assertion. Next we show differentiability of $\mathscr{L}(u(t))$ with respect to t. By Lemma 4.1 and the fact that the inner product on $W_2^1(\mathbf{R}^N)$ is differentiable, we get that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^N} |\nabla u(t)|^2\, dx = \int_{\mathbf{R}^N} (\nabla u \mid \nabla u_t)\, dx. \tag{4.9}$$

Regarding the other term, we already know the uniform bound (4.8). On the other hand,

$$\frac{d}{dt} F(x, u(x, t)) = f(x, u(x, t))u_t(x, t) = m(x, t)u(x, t)u_t(x, t)$$

is bounded by $\|m(t)u_t(t)\|_\infty |u(x, t)|$, and by (4.7) and Lemma 4.1 the derivative of $F(x, u(x, t))$ is bounded uniformly with respect to t in compact subsets of $(0, \infty)$ by an integrable function. Hence, we can interchange the order of integration and differentiation to get

$$\frac{d}{dt} \int_{\mathbf{R}^N} F(x, u(x, t))\, dx = \int_{\mathbf{R}^N} f(x, u(x, t))u_t\, dx.$$

Together with (4.9), we arrive at

$$\frac{d}{dt} \mathscr{L}(u(t)) = \int_{\mathbf{R}^N} (\nabla u \,|\, \nabla u_t)\, dx - \int_{\mathbf{R}^N} f(x, u)u_t\, dx. \tag{4.10}$$

Using that $\operatorname{div}[u_t \nabla u] = u_t \Delta u + (\nabla u \,|\, \nabla u_t)$ and applying the Divergence Theorem in the form

$$f \in W_1^1(\mathbf{R}^N, \mathbf{R}^N) \Rightarrow \int_{\mathbf{R}^N} \operatorname{div} f(x)\, dx = 0,$$

we obtain

$$\int_{\mathbf{R}^N} (\nabla u \,|\, \nabla u_t)\, dx = -\int_{\mathbf{R}^N} u_t \Delta u\, dx.$$

Finally, using that u is a solution of (4.1), we conclude from (4.10) that

$$\frac{d}{dt} \mathscr{L}(\varphi(t, u_0)) = -\int_{\mathbf{R}^N} u_t^2\, dx.$$

The last term is clearly negative if u is not stationary, which finishes the proof. □

As a consequence of [**12**, Theorem 2.3] and the previous lemma, we obtain the following result:

THEOREM 4.3. *The ω-limit set $\omega(u_0)$ of each point $u_0 \in X^+$ is contained in the set of stationary points of the semiflow (φ, X^+).*

Proof. The global B-attractor $\mathscr{M}$ is contained in the order interval V. Hence the function $\mathscr{L}$ is well defined on $\mathscr{M}$. The result is now obtained similarly as in [**12**, Theorems 2.3 and 3.2]. Instead of using the continuity of the Liapunov function, we make use of the fact that $\mathscr{L}$ is bounded below uniformly on V. Hence, by the previous lemma and the invariance of V,

$$\lim_{t \to \infty} \mathscr{L}(\varphi(t, u_0))$$

exists for each u_0 in V. This is the only nontrivial modification needed in [**12**, Theorem 2.3] to match the present situation. We also note that the semiflow (φ, X^+) is of class $\mathscr{AK}$ and that as a consequence the ω-limit set $\omega(u_0)$ is not empty and compact. □

REMARK 4.4. A class of problems with polynomial nonlinearities satisfying the conditions (3.2)–(3.5) is given in [**13**, Example 3.3.a].

We note that the theorem above does not imply that $\omega(u_0)$ consists of a single stationary point of the semiflow. However, we will see in the next section that this is true in the scalar case $N=1$. In [**3**, **14**], it is shown that the corresponding result is true for compact intervals.

5. Convergence to a single equilibrium

In this section we shall employ a result due to Hale and Raugel [**10**] to show that in one dimension any trajectory converges to a single equilibrium. Suppose that u_0 is any equilibrium of (4.1). We need to show that zero is an eigenvalue of

$$\Delta\varphi + f_u(x, u_0(x))\varphi = \lambda\varphi \quad \text{in } \mathbf{R}^N \tag{5.1}$$

of algebraic multiplicity at most one, and that there exist no purely imaginary eigenvectors.

LEMMA 5.1. *For all* $\lambda \in \mathbf{C}$ *with* $\operatorname{Re}\lambda \geqq 0$, *the operator*

$$\Delta + f_u(\cdot, u_0(\cdot)) - \lambda \tag{5.2}$$

is Fredholm of index zero in $\mathrm{BUC}(\mathbf{R}^N)$, $C_0(\mathbf{R}^N)$ *and* $L_2(\mathbf{R}^N)$. *Moreover, all eigenvalues* λ *with* $\operatorname{Re}\lambda \geqq 0$ *of* (5.2) *are real. They are algebraically simple in one dimension.*

PROOF. Throughout this paper, X denotes one of the spaces $L_2(\mathbf{R}^N)$, $C_0(\mathbf{R}^N)$ or $\mathrm{BUC}(\mathbf{R}^N)$. By assumption (3.5), we have

$$m(x) := f_u(x, u_0) = h(x, u_0) + h_u(x, u_0)u_0.$$

As u_0 is exponentially decaying, and again using (3.5), it follows that m^- is strongly absorbing and $m^+ \in C_0(\mathbf{R}^N)$. Due to Corollary 2.3 and Remark 2.5, we can write $m = -V + g$ where g having compact support, and $[\operatorname{Re} z \geqq -\omega_0] \subset \rho(\Delta - V)$ for some $\omega_0 > 0$ for all spaces under consideration. By interior regularity (e.g. [**9**, Theorem 8.22]), it follows that for all $u \in L_\infty(\mathbf{R}^N)$ the function $v := (\Delta - V)^{-1}u$ is Hölder-continuous with a Hölder-constant independent on $\|u\|_\infty \leqq M$. Also, $\|\nabla v\|_2$ is bounded uniformly in $\|u\|_2 \leqq \infty$. Hence, embedding theorems for bounded domains imply that

$$v \to g(\Delta - V)^{-1}v$$

is compact on X. Hence, due to [**7**, Theorem IX.2.1], the first assertion of the lemma follows.

Note that the operator under consideration is selfadjoint in $L_2(\mathbf{R}^N)$, and therefore has real spectrum. From the above we conclude in particular that any point in the spectrum with positive real part is an eigenvalue. If we show that each eigenvalue in BUC is in L_2, it follows that this part of the spectrum is real. Observe that if $\lambda \in \mathbf{C}$, then the heat kernel associated with $-\Delta + V + \lambda$ is $k_V(t; x, y)e^{\lambda t}$ and thus satisfies the estimate

$$|k_{(V+\lambda)}(t; x, y)| \leqq |k_{(V+\operatorname{Re}\lambda)}(t; x, y)|.$$

If now $\operatorname{Re}\lambda \geqq 0$, we conclude as in the proof of Theorem 3.1 that any eigenvector to λ decays exponentially and thus belongs to $L_2(\mathbf{R}^N)$. This proves the second part of the lemma.

Suppose now that the dimension is one. We first show that all eigenvalues with $\operatorname{Re}\lambda \geqq 0$ must be geometrically simple. Suppose to the contrary that (5.1) has two decaying solutions φ_1, φ_2. Then the Wronskian for the associated first-order system of ordinary differential equations is

$$W(x) = \varphi_1(x)\varphi_2'(x) - \varphi_2(x)\varphi_1'(x) = \text{constant},$$

for all $x \in \mathbf{R}$. Note that by Lemma 4.1 the derivatives φ_1' and φ_2' are bounded. If φ_1, φ_2 are both decaying also $W(x)$ has to decay, which is impossible. Hence λ is geometrically simple. As algebraic and geometric multiplicity coincide for selfadjoint operators, any eigenvalue $\lambda \geqq 0$ is algebraically simple in L_2. The same is true in BUC and C_0. Indeed, suppose an eigenvalue $\lambda \geqq 0$ has higher algebraic multiplicity. Then there exists $u \in \mathrm{BUC}$ such that $(\Delta - m - \lambda)^2 u = 0$ and $v = (\Delta - m - \lambda)u \neq 0$. Hence v is an eigenvalue of $\Delta + m$, and, according to what we proved in the previous paragraph, $v \in L_2$. This means that λ has a higher multiplicity in L_2 which is a contradiction. □

THEOREM 5.2. *Suppose that* $N = 1$. *Then, any solution of* (4.1) *with non-negative* $u_0 \in \mathrm{BUC}(\mathbf{R})$ *converges to an equilibrim in* $\mathrm{BUC}(\mathbf{R})$.

Proof. From Theorem 4.3 we know that all solutions converge to the set of equilibria. Using Lemma 5.1 it is easy to see that all assumptions of [**10**, Theorem 2.4] satisfied. Hence, the assertion of the theorem follows. □

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