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Averaging of flows with capillary hysteresis in stochastic porous media

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Fluids in unsaturated porous media are described by the relationship between pressure (p) and saturation (u). Darcy's law and conservation of mass provides an evolution equation for u, and the capillary pressure provides a relation between p and u of the form $p \in p_c(u, \partial_t u)$. The multi-valued function p_c leads to hysteresis effects. We construct weak and strong solutions to the hysteresis system and homogenize the system for oscillatory stochastic coefficients. The effective equations contain a new dependent variable that encodes the history of the wetting process and provide a better description of the physical system.

1 Introduction

Our aim is an effective description of fluid flow in porous media, where only part of the pore space is occupied by the fluid, say water, while the rest of the pore space is occupied by air at a constant pressure. We are not aiming at a description of the microscopic situation, but rather use the two macroscopic scalar variables of fluid pressure $p = p(x, t)$ and water content $u = u(x, t)$. Here, $u(x, t) \in [0, 1]$ is a measure for the volume fraction of liquid in the pore space, looking in the vicinity of the point x at time t. It is standard to relate velocity and pressure with Darcy's law, which imposes a linear relation between velocity and pressure gradient. Conservation of mass then implies

$$\partial_t u = \nabla \cdot (K \nabla p). \tag{1.1}$$

We allow K to depend on the position x, but for simplicity, we assume that K is independent of u.

We must now consider the microscopic situation in order to understand the capillary relation between u and p. If the volume fraction of water is increased, the liquid must fill smaller and smaller pores; in order to do so, an increasing local capillary pressure must be overcome (we describe the case of a non-wetting fluid). Since the gas phase is under a constant pressure, we find a monotone relation between p and u.

Capillary hysteresis

A more detailed study of the microscopic interfaces in a single pore reveals an additional property: the bottleneck effect. If the water content increases, water–air interfaces must repeatedly pass very small pores. To overcome these 'bottlenecks', a high pressure is needed. In the opposite case of a decreasing water content, the interfaces must repeatedly be pulled out of large pores, which means that a lower pressure is needed. If, instead, the water content is constant, the pressure has the freedom to adjust at any value in between [8]. A rigorous derivation is performed in [17] and [18]. Choosing an affine function as a simple monotone relation, the arguments justify

$$p \in au + b + \gamma \operatorname{sign}(\partial_t u). \tag{1.2}$$

The parameters $a, b, \gamma : \Omega \to \mathbb{R}$ satisfy $a, \gamma > 0$. We use the multi-valued sign function defined as $\operatorname{sign}(\xi) = \pm 1$ for $\pm\xi > 0$ and $\operatorname{sign}(0) = [-1, 1]$. Formally, (1.1)–(1.2) defines an evolution equation for u. The system must be complemented with appropriate initial and boundary conditions. We consider evolutions that are driven by imposed pressures on the boundary. Given $g \in C^1([0, T], H^2(\Omega))$ and $U_0 \in L^2(\Omega)$, we impose

$$u(., t = 0) = U_0 \qquad \text{in } \Omega, \tag{1.3}$$

$$p(., t) = g(., t) \qquad \text{on } \partial\Omega,\ \forall t \in [0, T]. \tag{1.4}$$

On the initial conditions, we have to assume some compatibility. For simplicity, we restrict to initial values that are compatible with a vanishing pressure. We demand

$$g(., t = 0) = 0, \tag{1.5}$$

$$a(x)U_0(x) + b(x) \in [-\gamma(x), \gamma(x)] \quad \forall x \in \Omega. \tag{1.6}$$

Recent studies of the *play-type hysteresis* system (1.1)–(1.2) are due to Beliaev. In [3], he introduces a concept of weak solutions and shows existence and uniqueness results by means of semigroup theory of Barbu [1].

The model was developed further in [4] and [5] in order to include dynamic effects and rate-dependent laws, essentially by replacing the sign function in (1.2) with a strictly monotone function. In this work, we use such a modification as a regularization. We rediscover existence and uniqueness properties of (1.1)–(1.2) and provide a Galerkin approximation.

Homogenization

The next step in the analysis of the hysteresis system regards homogenization. A first homogenization result was derived by Beliaev [2] for a periodic setting. He considered a situation in which the physical parameters have a finite range of values, K_i, a_i, b_i and γ_i, where $i = 1, \ldots, N$. These values are repeated periodically across the medium, with a period $\varepsilon > 0$. Beliaev was able to derive the homogenized system that describes the limit $\varepsilon \to 0$. If the values indexed by i are chosen in a region with volume fraction c_i, the limit system for $p = p(x, t)$ and $u_i = u_i(x, t)$ reads

$$\sum_{i=1}^{N} c_i \partial_t u_i = \nabla \cdot (K^* \nabla p), \qquad p \in a_i u_i + b_i + \gamma_i \operatorname{sign}(\partial_t u_i) \quad \forall i = 1, \ldots, N, \tag{1.7}$$

where K^* is a homogenized diffusion matrix obtained from cell problems.

Our aim in this contribution is to study the stochastic situation. It is interesting to note that in the stochastic situation, the limit system is more accessible in some respects. We study the situation where the parameters a, b, K and γ can take all values in given intervals. In cells of size ε, the four values are chosen randomly and independent of each other, and we consider the limit $\varepsilon \to 0$. We expect two modifications with respect to system (1.7).

- The discrete variable γ_i is replaced by a real variable y with values in an interval.
- The parameters a and b are averaged.

We further note that in the discrete case, the values γ_i either vanish or have a finite distance from 0. In our study, we allow all values of $\gamma \in [0,1]$; this difference leads to smooth scanning curves for the upscaled system.

Our main result is Theorem 4.2. It is shown that the following is the upscaled hysteresis system in the stochastic case. With expected values denoted by $\langle . \rangle$, we introduce the averaged quantities

$$a^* := \langle a^{-1} \rangle^{-1}, \quad b^* := \langle b \rangle,$$

and an effective permeability matrix K^* that is defined by the standard stochastic cell problem. We denote

$$\Gamma(x,.) \in \mathscr{M}([0,1])$$

as the distribution of γ in the point x.

We seek for functions $p(x,t)$, $w(x,y,t)$, such that the saturation

$$u(x,t) = \int_0^1 \frac{w(x,y,t) - b^*}{a^*} \, d\Gamma(x,y) \tag{1.8}$$

satisfies the hysteresis system

$$\partial_t u = \nabla \cdot (K^* \nabla p) \qquad \text{in } \Omega \times (0,T), \tag{1.9}$$

$$p(x) \in w(x,y) + y \,\mathrm{sign}(\partial_t w(x,y)) \qquad \forall x \in \Omega, y \in \mathrm{supp}(\Gamma(x,.)). \tag{1.10}$$

We see that two new variables are introduced. The dependent variable $w(x,y,t)$ can be regarded as an expected pressure at points with the γ value y. The new independent variable y substitutes the parameter γ. The parameter a is homogenized to the harmonic mean a^*. The system is complemented by boundary and initial conditions

$$w(x,.,t=0) = W_0(x,.) \in \mathrm{Lip}_1([0,1]) \quad \forall x \in \Omega, \tag{1.11}$$

$$p(.,t) = g(.,t) \text{ on } \partial\Omega, \qquad \forall t \in [0,T], \tag{1.12}$$

where Lip_1 denotes the space of Lipschitz continuous functions with Lipschitz constant bounded by 1. For compatibility, we demand that the initial condition can be realized with a vanishing pressure,

$$g(.,t=0) = 0, \tag{1.13}$$

$$W_0(x,y) \in [-y,y] \qquad \forall y \in \mathrm{supp}(\Gamma(x,.)), \quad x \in \Omega. \tag{1.14}$$

Equations (1.8)–(1.10) with the general measure Γ include the two equations of interest as special cases. Setting $\Gamma(x,.) = \delta_{\gamma(x)}(.)$ and $W_0(x,.) = a(x)U_0(x) + b(x)$, we recover the original system (1.1)–(1.2). On the other hand, the homogenized system will be of the form (1.8)–(1.10) with the one-dimensional Lebesgue measure $\Gamma(x,.) = dy$. In particular, existence and uniqueness results and a priori estimates for the homogenized system (1.8)–(1.10) imply the same results for the original system (1.1)–(1.2).

In the language of hysteresis theory [20], we may state our main result as follows: The evolution equation (1.1) with a play-type hysteresis relation between u and p is homogenized with a Prandtl–Ishlinskii hysteresis relation.

Outline and further literature

This article is organized as follows. In section 2, we analyze a Galerkin scheme that provides approximate solutions for the general equations (1.8)–(1.12). For the approximate solutions, we prove a priori estimates and the fundamental structure property (2.18). In section 3, we perform the limit procedure. We find weak and strong solutions of (1.8)–(1.12) and show the uniqueness. Section 4 is devoted to the homogenization. In the limit $\varepsilon \to 0$, strong solutions of (1.1)–(1.4) converge almost surely to solutions of the homogenized system (1.8)–(1.12). In this theorem, we exploit the bounds for strong solutions of Section 3 and use the approximate solutions of Section 2 in the construction of test functions.

We restrict here to an affine underlying p–u relation; nonlinear and degenerate problems are studied [10, 16, 19]. A construction of approximate solutions to a one-dimensional unsaturated flow problem can be found in [14]. Homogenization of two-phase flows is performed, for example, in [6, 7], a filtration model with hysteresis is studied in [15]. Regarding homogenization of stochastic flow problems, we mention [9, 11, 13].

Interpretation: Effective scanning curves

In imbibition/drainage experiments, one increases/decreases the water content u in a porous material and measures the pressure p. Up to transitional behaviour, one finds a fixed relation between p and u for both processes. In our model, the two relations are $p = au + b + \gamma$ and $p = au + b - \gamma$. The curves that are obtained when changing from imbibition to drainage (or vice versa) are called scanning curves. In the play-type hysteresis of (1.2) with constant parameters, these scanning curves are vertical lines—in contrast to experimental results.

To understand better the homogenized system, we now calculate a scanning curve after a drainage process, assuming $b^* = 0$, $a^* = 1$ and the homogeneous distribution $\Gamma = dy$. For homogeneous fields $p(x,t) = p(t)$, $w(x,y,t) = w(y,t)$, we find $w(0,t) = p(t)$ by (1.10) and $u(t) = \int_0^1 w(y,t)\,dy$. After drainage with $\partial_t w < 0$, we have $w(y,0) = p(0) + y$, again by (1.10).

Starting from this drainage situation, we study an evolution with $\partial_t p(t) = 1$. For small values of y, the value $w(y,t)$ must increase after a short time, since equation (1.10) does not allow larger differences between $w(y,t)$ and $p(t)$. The qualitative picture is that of Figure 1(a). To be precise, the value

$$s(t) := \sup\{y_0 | \partial_t w(y,t) > 0 \forall y < y_0\} \tag{1.15}$$

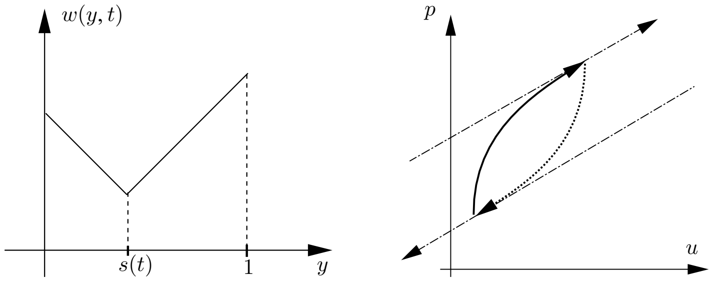

FIGURE 1. (a) The function $w(.,t)$; (b) effective scanning curves.

increases, and the function w has the form

$$w(y,t) = \begin{cases} p(t) - y & y < s(t), \\ p(t) - 2s(t) + y & y \geqslant s(t). \end{cases} \tag{1.16}$$

For $y > s(t)$, we find $0 = \partial_t w(y,t) = \partial_t p(t) - 2\partial_t s(t)$, and thus $\partial_t s(t) = \frac{1}{2}$ for the position of the free boundary. We can therefore calculate for the water content u

$$u(t) = p(t) - 2s(t) + \frac{1}{2} + s(t)^2, \quad \partial_t u(t) = 1 - 2\partial_t s(t) + 2s(t)\partial_t s(t) = s(t) = \frac{t}{2}.$$

This yields the qualitative scanning curves of Figure 1(b) for the upscaled equations. In the original system of play-type hysteresis, the scanning curves are vertical, and, in particular, independent of the history. We see that after homogenization, the function $w(x,.,t)$ contains the relevant information about the history of the process and determines the shape of the scanning curves.

We conclude that the experimental observations can be described well with the effective equations (1.8)–(1.12); the history variable w provides a rich variety of possible scanning curves. In this work, we rigorously derive the effective equations in a homogenization process, starting from the elementary hysteresis model (1.1)–(1.4).

2 Approximate solutions

The aim of this section is to find uniform estimates for the approximate solutions of the homogenized system with a Galerkin scheme. These estimates, in turn, provide us with estimates for the solutions of the limit system. Moreover, the approximate solutions are well-suited for the construction of test functions in the homogenization procedure. We emphasize that all the results on existence of solutions and estimates carry over to the original problem with the special choice of the distribution function $\Gamma_x = \delta_{\gamma(x)}$.

For notational convenience, we choose a rectangle $\Omega \subset \mathbb{R}^n$ as macroscopic domain and fix a time interval $[0,T]$. On the physical parameters, we assume $K^* \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ to be uniformly positive definite, $a^*, b^* \in L^\infty(\Omega, \mathbb{R})$ with $a^* \geqslant \alpha > 0$ bounded from below. We furthermore assume that for a triangulation $\mathscr{T}_0$ of the domain, the functions a^*, b^* and

K^* are constant on each triangle $A \in \mathscr{T}_0$ and that the probability distributions

$$\Gamma(x,.) \in \mathscr{M}([0,1])$$

are independent of x in each triangle $A \in \mathscr{T}_0$. Our aim is to study (1.8)–(1.10) to find a discrete approximation of the equations, as well as to find strong solutions. Our main result is the existence of approximate solutions that satisfy the structure condition (2.18). These are the approximate solutions that are used in the construction of test functions in the homogenization procedure.

Spatial discretization

We consider a sequence of triangulations of the domain Ω with vertices $\Omega_h := \{x_1, \ldots, x_K\}$, where $h > 0$ is the maximal distance between neighbours. We assume that each triangulation $\mathscr{T}_h$ is a refinement of the coarse triangulation $\mathscr{T}_0$. In this way, we achieve that the coefficients are x independent on each triangle $A \in \mathscr{T}_h$. In addition, we discretize the interval $I := [0,1]$ with equidistant nodes $I_\eta := \{y_1, \ldots, y_L\}$, $0 = y_0 < y_1 = \eta < \ldots < y_L = 1$, with $\eta > 0$ the distance between neighbours. The weights for the discretization are

$$\Gamma_\eta(x,y) := \Gamma_x((y-\eta, y] \cap I) \qquad \forall y \in I_\eta, \tag{2.1}$$

with the closed interval for $y = y_1 = \eta$.

Regularization

We replace the inequalities of (1.10) by a dynamic condition. For $\delta > 0$, we use the following approximation of the inverse sign-function. For $y \in I$ and $\delta > 0$, let $\psi_\delta^y : \mathbb{R} \to \mathbb{R}$ be the function

$$\psi_\delta^y(r) := \begin{cases} \delta r & \text{for } \; r \in [-y, y], \\ y\delta + \frac{1}{\delta}(r-y) & \text{for } \; r > y, \\ -y\delta + \frac{1}{\delta}(r+y) & \text{for } \; r < -y. \end{cases} \tag{2.2}$$

Given the triangulation $\mathscr{T}_h$ of Ω, we can associate to every triangle $A \in \mathscr{T}_h$ a corner $x \in \Omega_h$. This provides us with an interpolation operator Q, which maps a discrete function $u : \Omega_h \to \mathbb{R}$ to piecewise linear interpolations $\bar{u}$. Furthermore, we have the L^2-orthogonal projection P, which maps functions $v \in L^2(\Omega)$ to piecewise constant functions $\bar{v} \in L^2(\Omega)$. To every piecewise constant function $\bar{v}$, we can associate a discrete map $\hat{v} : \Omega_h \to \mathbb{R}$ such that $Q\hat{v} = \bar{v}$. In such a situation, we do not distinguish between $\hat{v}$ and $\bar{v}$. On the initial values W_0, we assume that they are x independent on triangles $A \in \mathscr{T}_0$ as are a^*, b^*, K^* and Γ.

Definition 2.1 (Galerkin scheme) We consider the following system of ordinary differential equations for $p_\delta = p_\delta^{h,\eta} : \Omega_h \times [0,T] \to \mathbb{R}$ and $w_\delta = w_\delta^{h,\eta} : \Omega_h \times I_\eta \times [0,T] \to \mathbb{R}$.

$$\partial_t w_\delta(x,y,t) = -\psi_\delta^y\left(w_\delta(x,y,t) - p_\delta(x,t)\right) \qquad \forall x \in \Omega_h, y \in I_\eta, \tag{2.3}$$

$$w_\delta(.,y,t=0) = W_0^\eta(.,y) := \frac{1}{\eta}\int_{y-\eta}^{y} W_0(.,\zeta)\,d\zeta \quad \forall y \in I_\eta. \tag{2.4}$$

It remains to describe how the pressure p_δ is reconstructed from w_δ. We identify w_δ with its piecewise constant interpolation, and solve the following elliptic problem for $\tilde{p}_\delta(.,t) : \Omega \to \mathbb{R}$ and $p_\delta := P\tilde{p}_\delta$,

$$\nabla(K^*\nabla\tilde{p}_\delta)(x) = -\frac{1}{a^*(x)} \sum_{y\in I_\eta} \Gamma_\eta(x,y)\, \psi_\delta^y\,(w_\delta(x,y) - p_\delta(x))\,, \tag{2.5}$$

$$\tilde{p}_\delta(.,t) = g(.,t) \quad \text{on } \partial\Omega, \forall t \in [0,T]. \tag{2.6}$$

We will see that these solutions can be used to find solutions of (1.8)–(1.10). But first we have to study the solvability of the equations and a priori estimates.

Lemma 2.2 (*Existence for the ODE*) *The solution map $w_\delta \mapsto p_\delta$ defined by equations* (2.5), (2.6) *is well-defined and is Lipschitz continuous. In particular, Definition* 2.1 *describes a system of ordinary equations. There is a unique local solution (p_δ, w_δ) for all positive δ, h and η.*

Proof We show the argument for $g = 0$; the general case is analogous. We define the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\begin{aligned}\langle Au, v\rangle &:= \langle K^*\nabla u, \nabla v\rangle_{L^2(\Omega)} \\ &\quad - \left\langle \frac{1}{a^*(x)} \sum_{y\in I_\eta} \Gamma_\eta(x,y)\, \psi_\delta^y\,(w_\delta(x,y) - Pu(x))\,, v \right\rangle_{L^2(\Omega)}.\end{aligned}$$

We claim that A is monotone, coercive and continuous on finite dimensional subspaces. Once this is shown, the theory of monotone operators (e.g. [12], Chapter III, Cor. 1.8) yields the existence of a solution to the equation $Au = 0$. For the monotonicity, we calculate

$$\begin{aligned}\langle Au - Av, u - v\rangle &= \langle K^*\nabla(u-v), \nabla(u-v)\rangle_{L^2(\Omega)} \\ &\quad - \left\langle \sum_{y\in I_\eta} \frac{\Gamma_\eta(.,y)}{a^*(x)} \left[\psi_\delta^y\,(w_\delta(.,y) - Pu) - \psi_\delta^y\,(w_\delta(.,y) - Pv)\right], u - v \right\rangle_{L^2(\Omega)} \\ &= \langle K^*\nabla(u-v), \nabla(u-v)\rangle_{L^2(\Omega)} - \sum_{y\in I_\eta} \sum_{T\in\mathscr{T}_h} \frac{1}{a^*(T)} |T| \Gamma_\eta(T,y) \cdot \\ &\quad \left\langle \left[\psi_\delta^y(w_\delta(.,y) - Pu(.)) - \psi_\delta^y(w_\delta(.,y) - Pv(.))\right], Pu(.) - Pv(.)\right\rangle_{L^2(T)} \\ &\geqslant \langle K^*\nabla(u-v), \nabla(u-v)\rangle_{L^2(\Omega)}.\end{aligned}$$

In the last step, we exploited that all ψ_δ^y are monotonically increasing. The right-hand side of the equation is non-negative and we conclude the monotonicity of A. The Poincaré inequality yields the coerciveness of A. The continuity on finite dimensional subspaces follows from the continuity of ψ_δ^y and P.

For a sequence $w \to w_0 \in L^\infty(\Omega_h \times I_\eta, \mathbb{R})$, we consider the corresponding operators A_w and A_{w_0} and find solutions u_w and u_{w_0} of $A_w u_w = 0$ and $A_{w_0} u_{w_0} = 0$. By uniform coerciveness of A_w, the solution u_w is bounded. From the Poincaré inequality, we calculate

$$\begin{aligned} c\|u_w - u_{w_0}\|^2 &\leqslant \langle A_w u_w - A_w u_{w_0}, u_w - u_{w_0}\rangle \\ &= \langle A_{w_0} u_{w_0} - A_w u_{w_0}, u_w - u_{w_0}\rangle \\ &\leqslant C\frac{1}{\delta}\|w - w_0\|\,\|u_w - u_{w_0}\|. \end{aligned}$$

Dividing by $\|u_w - u_{w_0}\|$, we conclude the local Lipschitz continuity of the map $w \mapsto u$. □

Lemma 2.3 (*Estimates and global solutions*) *Every solution w_δ, p_δ to the scheme of Definition* 2.1 *satisfies for every $t \in [0, T]$ the estimate*

$$\begin{aligned} &\int_\Omega \sum_{y\in I_\eta} \Gamma_\eta(x,y)\,|\partial_t w_\delta(x,y,t)|^2\,dx + \int_0^t \int_\Omega |\nabla \partial_t \tilde{p}_\delta(x,t')|^2\,dx\,dt' \\ &\quad \leqslant C_1(g) + C_2(\delta, h, \eta). \end{aligned} \tag{2.7}$$

The constants depend on the bounds for a^* and K^*. We can choose C_2 with

$$\lim_{\delta\to 0} C_2(\delta, h, \eta) = 0 \qquad \forall h, \eta > 0. \tag{2.8}$$

The function w_δ is Lipschitz in y with constant 1, *for all $x \in \Omega_h$ and all $t \in [0, T]$,*

$$w_\delta(x, ., t) \in \mathrm{Lip}_1(I_\eta). \tag{2.9}$$

A consequence of the lemma is that we can extend the local solutions to the ODE to the whole interval $[0, T]$.

Proof We insert (2.3) into (2.5). Omitting the dependence on t, we can write

$$\nabla(K^*\nabla \tilde{p}_\delta(x)) = \sum_{y\in I_\eta} \frac{\Gamma_\eta(x,y)}{a^*(x)}\partial_t w_\delta(x,y) \qquad \forall x \in \Omega,$$

where the right-hand side of the equation is the piecewise constant in x. We differentiate with respect to t and find

$$\nabla(K^*\nabla \partial_t \tilde{p}_\delta(x)) = \sum_{y\in I_\eta} \frac{\Gamma_\eta(x,y)}{a^*(x)}\partial_t^2 w_\delta(x,y).$$

Multiplication with $\partial_t(\tilde{p}_\delta - g)$ and an integration over Ω yields

$$\begin{aligned} &-\int_\Omega K^*\nabla \partial_t \tilde{p}_\delta \cdot \nabla \partial_t \tilde{p}_\delta + \int_\Omega K^*\nabla \partial_t g \cdot \nabla \partial_t \tilde{p}_\delta \\ &\quad = \sum_{y\in I_\eta} \int_\Omega \frac{\Gamma_\eta(x,y)}{a^*(x)}\partial_t^2 w_\delta(x,y)\partial_t p_\delta(x)\,dx. \end{aligned} \tag{2.10}$$

The function ψ_δ^y is invertible and we denote the inverse by Φ_δ^y. Note that Φ_δ^y is a regularized and scaled sign-function. Relation (2.3) can be written as

$$-\Phi_\delta^y(\partial_t w_\delta(x,y)) = w_\delta(x,y) - p_\delta(x).$$

We can differentiate with respect to t and find

$$\partial_t p_\delta(x) = \partial_t w_\delta(x,y) + D\Phi_\delta^y(\partial_t w_\delta(x,y)) \cdot \partial_t^2 w_\delta(x,y).$$

We can now insert this expression into (2.10),

$$\begin{aligned}
&-\int_\Omega K^*\nabla\partial_t\tilde{p}_\delta \cdot \nabla\partial_t\tilde{p}_\delta + \int_\Omega K^*\nabla\partial_t g \cdot \nabla\partial_t\tilde{p}_\delta \\
&\quad= \sum_{y\in I_\eta}\int_\Omega \frac{\Gamma_\eta(x,y)}{a^*(x)}\partial_t^2 w_\delta(.,y)\left[\partial_t w_\delta(.,y) + D\Phi_\delta^y(\partial_t w_\delta(.,y))\cdot\partial_t^2 w_\delta(.,y)\right] \\
&\quad= \sum_{y\in I_\eta}\int_\Omega \frac{\Gamma_\eta(x,y)}{a^*(x)}\partial_t\frac{1}{2}|\partial_t w_\delta(.,y)|^2 + D\Phi_\delta^y(\partial_t w_\delta(.,y))\cdot|\partial_t^2 w_\delta(.,y)|^2 \\
&\quad\geqslant \sum_{y\in I_\eta}\int_\Omega \frac{\Gamma_\eta(x,y)}{a^*(x)}\partial_t\frac{1}{2}|\partial_t w_\delta(.,y)|^2,
\end{aligned}$$

where in the last step we used that $D\Phi_\delta^y$ is positive. An integration over $(0,t)$ yields the a priori estimate (2.7) with

$$C_2(\delta,h,\eta) := C\sum_{y\in I_\eta}\int_\Omega \frac{\Gamma_\eta(x,y)}{a^*(x)}|\partial_t w_\delta(.,y)|^2\bigg|_{t=0}.$$

δ-dependence of C_2. To show (2.8), it remains to verify for the initial values $\partial_t w_\delta(.,y,t=0)\to 0$ for $\delta\to 0$, for all $y\in I_\eta$ with $\Gamma_\eta(.,y)>0$. Since the spatial variables are discrete, $\tilde{p}_\delta(t=0)$ is contained in a finite dimensional subspace of $H^2(\Omega)$. It therefore suffices to show $\tilde{p}_\delta(t=0)\to 0$. At this point, we exploit the compatibility condition (1.14) on the initial values. We must study the monotone operator $A^\delta_{W_0^\eta}$ and the solution $\tilde{p}_\delta$ of $A^\delta_{W_0^\eta}\tilde{p}_\delta = 0$. We use this for the trivial pressure distribution by compatibility, $A^\delta_{W_0^\eta}0\to 0$, where we exploit the direction of the discretization I_η of I. The uniform coerciveness of $A^\delta_{W_0^\eta}$ yields

$$c\|\tilde{p}_\delta - 0\|^2 \leqslant C\left\langle A^\delta_{W_0^\eta}0 - A^\delta_{W_0^\eta}\tilde{p}_\delta, 0-\tilde{p}_\delta\right\rangle \leqslant C\left\|A^\delta_{W_0^\eta}0\right\|\,\|\tilde{p}_\delta\|.$$

Dividing by $\|\tilde{p}_\delta\|$, we verify the claim.

Lipschitz property. The initial values satisfy the Lipschitz estimate. We claim that the Lipschitz constant can never exceed the value 1. To this end, let t is a time instance, x a point in Ω_h, and $0\leqslant y_1<y_2\leqslant 1$ such that

$$w_\delta(x,y_2,t) - w_\delta(x,y_1,t) = y_2 - y_1. \tag{2.11}$$

Our claim is proven once we find that the time derivative on the left-hand side is negative. We restrict here to the case $w_\delta(x, y_2, t) > w_\delta(x, y_1, t)$, the other sign is treated in the same way.

First case If $w_\delta(x, y_1, t) \leqslant p_\delta(x) + y_1$, then $w_\delta(x, y_2, t) \leqslant p_\delta(x) + y_2$. We find

$$\partial_t[w_\delta(x, y_2, t) - w_\delta(x, y_1, t)] = -\delta[w_\delta(x, y_2, t) - w_\delta(x, y_1, t)] < 0.$$

Second case If $w_\delta(x, y_1, t) > p_\delta(x) + y_1$, then also $w_\delta(x, y_2, t) > p_\delta(x) + y_2$. We find

$$\begin{aligned}
&\partial_t\,[w_\delta(x, y_2, t) - w_\delta(x, y_1, t)] \\
&= -\delta y_2 - \frac{1}{\delta}(w_\delta(x, y_2, t) - p_\delta(x) - y_2) + \delta y_1 + \frac{1}{\delta}(w_\delta(x, y_1, t) - p_\delta(x) - y_1) \\
&= -\delta(y_2 - y_1) < 0.
\end{aligned}$$

This shows the Lipschitz estimate for all $\delta > 0$. □

We can now study the limit $\delta \to 0$ in order to find spatially discrete approximate solutions.

Theorem 2.4 (*Approximate solutions*) *For x and y discrete, there exists a solution* $(u^{h,\eta}, p^{h,\eta}, w^{h,\eta})$ *of the following discretization of* (1.8)–(1.10).

$$u^{h,\eta} = \sum_{y \in I_\eta} \Gamma_\eta(., y)\, \frac{w^{h,\eta}(., y) - b^*}{a^*} \tag{2.12}$$

$$\nabla(K^* \nabla \tilde{p}^{h,\eta}) = \partial_t u^{h,\eta} \tag{2.13}$$

$$p^{h,\eta} \in w^{h,\eta}(., y) + y\,\mathrm{sign}(\partial_t w^{h,\eta}(., y)) \quad \forall y \in I_\eta \text{ with } \Gamma_\eta(., y) > 0, \tag{2.14}$$

for almost all $t \in (0, T)$, *together with the initial values* $w^{h,\eta} = W_0^\eta$ *and the boundary values* $\tilde{p}^{h,\eta} = g$ *on* $\partial\Omega$.

The solutions satisfy uniform a priori bounds in the norms of

$$\partial_t w^{h,\eta} \in L^\infty L^2(\Omega \times I, dx \otimes d\Gamma_\eta(x, y)), \tag{2.15}$$

$$\partial_t \tilde{p}^{h,\eta} \in L^2 H^1(\Omega), \tag{2.16}$$

$$\tilde{p}^{h,\eta} \in L^\infty H^2(\Omega). \tag{2.17}$$

For some $z^{h,\eta} \in L^\infty(\Omega \times (0, T))$ *the solution satisfies the structure condition*

$$\partial_t w^{h,\eta}(x, y, t) = \begin{cases} \partial_t p^{h,\eta}(x, t) & \text{for } y \leqslant z^{h,\eta}(x, t), \\ 0 & \text{else,} \end{cases} \tag{2.18}$$

for almost every t and all y with $\Gamma_\eta(x, y) > 0$.

Proof We use the approximations $w_\delta^{h,\eta}$ and $p_\delta^{h,\eta}$ of Definition 2.1. For a subsequence, we find a weak-* limit in $W^{1,\infty}((0, T), L^\infty)$ and a weak limit in the space $H^1((0, T), L^\infty)$ (we use that x and y are discrete),

$$\left(w_\delta^{h,\eta}, p_\delta^{h,\eta}\right) \rightharpoonup \left(w^{h,\eta}, p^{h,\eta}\right) \quad \text{for } \delta \to 0.$$

A priori estimates (2.15) and (2.16) are guaranteed by Lemma 2.3. The estimate (2.17) is a consequence of equation (2.13) and the bound of (2.15). All bounds depend only on $C_1(g)$, and are therefore independent of h and η.

To derive the equations, we once more insert (2.3) into (2.5),

$$\nabla\left(K^*\nabla\tilde{p}_\delta^{h,\eta}\right) = \frac{1}{a^*}\sum_{y\in I_\eta}\Gamma_\eta(.,y)\,\partial_t w_\delta^{h,\eta}(.,y).$$

We can take weak limits for $\delta \to 0$ and find equation (2.13).

Relation (2.14) We study (2.3),

$$\partial_t w_\delta^{h,\eta}(x,y,t) = -\psi_\delta^y\left(w_\delta^{h,\eta}(x,y) - p_\delta^{h,\eta}(x)\right).$$

The left-hand side of the above equation is bounded in $L^\infty((0,T),L^\infty)$, with a bound that is independent of δ, since x and y are discrete. By the estimates for their time derivatives, $w_\delta^{h,\eta} \to w^{h,\eta}$ and $p_\delta^{h,\eta} \to p^{h,\eta}$ are weak convergences in $H^1((0,T))$, and can therefore be assumed to be pointwise convergences. We use $|\psi_\delta^y(\xi)| \geqslant \delta^{-1}(\xi - y)_+$ to find for fixed x, y, t, $\Gamma_\eta(x,y) > 0$,

$$\begin{aligned} 0 \leftarrow \delta\left|\psi_\delta^y\left(w_\delta^{h,\eta}(x,y,t) - p_\delta^{h,\eta}(x,t)\right)\right| \\ \geqslant \left(w_\delta^{h,\eta}(x,y,t) - p_\delta^{h,\eta}(x,t) - y\right)_+ \to \left(w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) - y\right)_+. \end{aligned}$$

The same calculation for $-y$ yields for all t and all the (discrete) values of x and y in the relation

$$w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) \in [-y, y]. \tag{2.19}$$

Let now (x,y,t) be a point as above, with $w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) > -y$. Then, for all small δ, by the pointwise convergence, also

$$w_\delta^{h,\eta}(x,y,t) - p_\delta^{h,\eta}(x,t) > -y,$$

whence the positive part $(\partial_t w_\delta^{h,\eta}(x,y,t))_+ = (-\psi_\delta^y)_+ \leqslant \delta y$. We find for all x, y, t

$$\left(\partial_t w_\delta^{h,\eta}(x,y,t)\right)_+ 1_{\{w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) > -y\}} \to 0 \quad \text{for } \delta \to 0.$$

Since $\partial_t w_\delta^{h,\eta}$ are bounded, independent of δ, we can apply the Lebesgue convergence theorem to conclude

$$\left(\partial_t w_\delta^{h,\eta}(x,y,t)\right)_+ 1_{\{w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) > -y\}} \to 0 \quad \text{in } L^2((0,T)),$$

for $\delta \to 0$. But by the definition of the limit function $w^{h,\eta}$ and the L^2-weak lower semicontinuity of the positive part, we find in the limit for the left hand side

$$(\partial_t w^{h,\eta}(x,y,t))_+ 1_{\{w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) > -y\}} \leqslant 0 \tag{2.20}$$

in the sense of L^2-functions. We have verified the implication

$$\partial_t w^{h,\eta}(x,y,t) > 0 \Rightarrow w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) = -y \tag{2.21}$$

for almost every t and all x, y with $\Gamma_\eta(x,y) > 0$. The conclusion for the other sign is calculated in the same way by replacing the positive part with the negative part. Relation (2.14) is shown.

The structure property (2.18). We next verify the equality

$$(\partial_t w^{h,\eta}(x,y,t) - \partial_t p^{h,\eta}(x,t)) 1_{\{|w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t)| = y\}} = 0 \tag{2.22}$$

for all $x \in \Omega_h$, $y \in I_\eta$, and almost every t. For fixed x and y, the set $\{t \in [0,T] : |w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t)| = y\}$ is a countable union of closed intervals by the continuity of $p^{h,\eta}$ and $w^{h,\eta}$, and the two functions differ by one constant on these intervals. In particular, the weak derivatives coincide almost everywhere on the intervals.

For every t and x, the sets $\{y \in I_\eta : w^{h,\eta}(x,y,t) - p^{h,\eta}(x,t) = \pm y\}$ are of the form $\{y \in I_\eta : y \leqslant z^{h,\eta}\}$ for some $z^{h,\eta}$ by the Lip_1 estimate for $w^{h,\eta}$. This defines $z^{h,\eta}$. Property (2.18) is a consequence of (2.22) and (2.20) (together with the equality with opposite signs). □

3 Weak and strong solutions

In this section, we show that the approximate solutions of the last section can be used to find continuous solutions of the upscaled system. We proceed in two steps and show that

(i) limits of approximate solutions for $(h,\eta) \to 0$ are weak solutions,

(ii) under regularity assumptions on Γ, weak solutions are strong solutions.

In particular, we find strong solutions to the original system, with uniform bounds that allow the homogenization. For the original system, we essentially recover a result of Beliaev that was obtained with the help of semigroup theory.

To prepare for the limit procedure $(h,\eta) \to 0$, we show a compactness result. For a function $u : \Omega \to \mathbb{R}$ that is piecewise constant on the h grid, we denote by $|\nabla^h u|$ the upper bound for the discrete difference quotient: in every node we take the supremum over the norms of the finite difference quotients along outgoing edges. This function on the nodes is identified with its piecewise constant interpolation $|\nabla^h u| : \Omega \to \mathbb{R}$.

We recall that, by our assumptions, data a^*, b^*, K^*, Γ and W_0 are constant on triangles $A \in \mathcal{T}_0$ covering Ω.

Lemma 3.5 (*Compactness*) *The approximate solutions $p^{h,\eta}, w^{h,\eta}$ of Theorem* 2.4 *satisfy the following pointwise estimate for discrete spatial derivatives. For all triangles $A \in \mathcal{T}_0$, $x \in \Omega_h$ an inner point of A, and all $y \in I_\eta$ with $\Gamma_\eta(A,y) > 0$ there holds*

$$|\nabla^h w^{h,\eta}(x,y,t)| \leqslant \int_0^t |\partial_t \nabla^h p^{h,\eta}(x,t')| \, dt'. \tag{3.1}$$

We define functions $F_j^{h,\eta} : \Omega \times [0,T] \to \mathbb{R}$, $j = 1,2,3$, *by*

$$F_0^{h,\eta}(x,t) := \sum_y \Gamma_\eta(x,y)\, w^{h,\eta}(x,y,t),$$
$$F_1^{h,\eta}(x,t) := \sum_y \Gamma_\eta(x,y)\, y\, w^{h,\eta}(x,y,t),$$
$$F_2^{h,\eta}(x,t) := \sum_y \Gamma_\eta(x,y)\, |w^{h,\eta}(x,y,t)|^2.$$

Then $F_j^{h,\eta}$ *are compact in* $L^1(\Omega \times (0,T))$, *and* $F_j^{h,\eta}(.,t)$ *are compact in* $L^1(\Omega)$ *for all* t, $j = 1,2,3$.

Proof We omit the superscript (h,η) and write shortly (w,p) for $(w^{h,\eta}, p^{h,\eta})$. We fix $A \in \mathcal{T}_0$ and want to show for all x_1, x_2 in A, all $y \in I_\eta$ with $\Gamma_\eta(y) > 0$, all $t \in [0,T]$, for $\delta = 0$, the inequality

$$|w(x_1,y,t) - w(x_2,y,t)| \leqslant \int_0^t |\partial_t p(x_1,t') - \partial_t p(x_2,t')|\, dt' + \delta(1+t). \tag{3.2}$$

Estimate (3.1) follows if we show (3.2) for all $\delta > 0$. Note that W_0 was assumed to be piecewise constant on A such that the estimate holds initially. We claim that the estimate can never cease to hold. For a contradiction argument, let $t < T$ be the last time instance such that the estimate holds up to time t. Interchanging x_1 with x_2 if necessary, we can assume $w(x_1,y,t) > w(x_2,y,t)$. We have to consider two cases.

Case 1 ($\partial_t w(x_1,y,t) > 0$) In this case, we have $\operatorname{sign}(\partial_t w(x_1,y,t)) = 1$, and therefore $w(x_1,y,t) = p(x_1,t) - y$. We can calculate

$$\begin{aligned} w(x_1,y,t) - w(x_2,y,t) &\leqslant p(x_1,t) - y - p(x_2,t) + y = p(x_1,t) - p(x_2,t) \\ &\leqslant \int_0^t |\partial_t p(x_1,t') - \partial_t p(x_2,t')|\, dt'. \end{aligned}$$

Thus, inequality (3.2) holds strictly and case 1 cannot occur.

Case 2 ($\partial_t w(x_1,y,t) \leqslant 0$) In this case, we have either (a) $\partial_t w(x_2,y,t) \geqslant 0$ or (b) $\partial_t w(x_2,y,t) < 0$. In case (a), we find

$$\partial_t [w(x_1,y,t) - w(x_2,y,t)] \leqslant 0.$$

But the time derivative on the right-hand side in (3.2) is positive and the inequality does not cease to hold.

In case (b), we have $\operatorname{sign}(\partial_t w(x_2,y,t)) = -1$, and therefore $w(x_2,y,t) = p(x_2,t) + y$. We then find

$$\begin{aligned} w(x_1,y,t) - w(x_2,y,t) &\leqslant p(x_1,t) + y - p(x_2,t) - y = p(x_1,t) - p(x_2,t) \\ &\leqslant \int_0^t |\partial_t p(x_1,t') - \partial_t p(x_2,t')|\, dt'. \end{aligned}$$

The inequality holds again strictly and case 2 cannot occur either.

Compactness. For the compactness, it suffices to consider a single triangle $A \subset \Omega$ out of the finite number of triangles $A \in \mathscr{T}_0$. The right-hand side of (3.1) is bounded in $L^2(A \times (0,T))$, hence the inequality can be regarded as a replacement for spatial regularity of $w^{h,\eta}(.,y,.)$. To be precise, we claim that $F_j^{h,\eta}$ has temporal and discrete spatial derivatives bounded in $L^1(A \times (0,T))$. Indeed, for F_0,

$$\int_0^T \int_A \left|\nabla^h F_0^{h,\eta}(x,t)\right| dx\,dt = \int_0^T \int_A \sum_y \Gamma_\eta(x,y) |\nabla^h w^{h,\eta}(x,y,t)|\,dx\,dt$$

$$\leqslant \int_0^T \int_A \sum_y \Gamma_\eta(x,y) \left\{ \int_0^t |\partial_t \nabla^h p^{h,\eta}(x,t')|\,dt' \right\} dx\,dt$$

$$\leqslant \int_0^T \int_A \left\{ \int_0^t |\partial_t \nabla^h p^{h,\eta}(x,t')|\,dt' \right\} dx\,dt \leqslant C,$$

where, in the last step, we used (2.16). For temporal derivatives, we calculate

$$\int_0^T \int_A \left|\partial_t F_0^{h,\eta}(x,t)\right| dx\,dt = \int_0^T \int_A \sum_y \Gamma_\eta(x,y)\ |\partial_t w^{h,\eta}(x,y,t)|\,dx\,dt \leqslant C,$$

using (2.15). The other integrals F_j are treated similarly and we find the $L^1(\Omega \times (0,T))$ compactness. For fixed $t \in [0,T]$, the $L^1(\Omega)$ compactness follows along the same lines from (3.1). □

Our next result is on the existence of a weak solution. The solution concept is analogous to that in [2], but we use a stronger formulation in the third term.

Theorem 3.2 (*Weak solutions*) *There exists a pair* (p,w)

$$w \in L^\infty(0,T; L^2(\Omega, \mathrm{Lip}_1(I))), \tag{3.3}$$

$$p \in H^1(0,T; H^1(\Omega, dx)), \tag{3.4}$$

which is a weak solution of equations (1.8)–(1.10) *in the following sense. The relation* $w(x,y,t) - p(x,t) \in [-y,y]$ *holds for* $\mathscr{L}^{n+1}$*—almost every* (x,t) *and all* $y \in \mathrm{supp}(\Gamma(x,.))$. *Moreover, with u defined by* (1.8), *for all* $q \in H^1(\Omega)$ *and all* $0 \leqslant t_1 < t_2 \leqslant T$, *we have*

$$\begin{aligned}0 \geqslant{}& \int_\Omega \left\{ \int_I \frac{1}{2a^*} |w(x,y,t)|^2 d\Gamma(y) - u(x,t)\cdot q(x) \right\} dx \Bigg|_{t=t_1}^{t_2} \\ &+ \int_\Omega \frac{1}{a^*} \int_I y\ |w(x,y,t_2) - w(x,y,t_1)|\ d\Gamma(y)\,dx \\ &+ \int_{t_1}^{t_2} \int_\Omega K^* \nabla p(x,t) \nabla(p(x,t) - q(x))\,dx\,dt \\ &- \int_{t_1}^{t_2} \int_{\partial\Omega} n \cdot (K^* \nabla p(t))(g(t) - q)\,d\mathscr{H}^{n-1}\,dt. \end{aligned} \tag{3.5}$$

The solution (w,p) *is bounded in the above norms by a constant that depends only on* Ω, g, *and the bounds for the parameters.*

Proof We assume $a^* = 1$ and $b^* = 0$ for brevity of the calculations. It suffices to restrict to smooth functions q. We consider the approximate solutions $(p^{h,\eta}, w^{h,\eta})$ of Theorem 2.4. We can choose a sequence $(h, \eta) \to 0$ and limit functions such that the following convergences hold: $p^{h,\eta} \to p$ weakly and weakly-$*$ in the norms of (2.16) and (2.17), and, for all $j = 1, 2, 3$, $F_j^{h,\eta} \to F_j$ strongly in $L^1(\Omega \times (0,T))$, weakly in $H^1((0,T), L^1(\Omega))$ and $F_j^{h,\eta}(.,t) \to F_j(.,t)$ in $L^1(\Omega)$ for rational $t \in (0,T)$.

The next step is to define the limit object $w : \Omega \times I \times (0,T) \to \mathbb{R}$. For a fixed triangle $A \in \mathscr{T}_0$ and $t \in (0,T)$, we consider $y \in \operatorname{supp}(\Gamma(A,.))$. By (3.1) and the Lip_1 continuity in y, we have the compactness of the sequence $w^{h,\eta}(.,y,t)$ in the space $L^2(A)$. We can therefore assume on our sequence $(h,\eta) \to 0$ additionally that $w^{h,\eta}(.,y,t) \to w(.,y,t)$ in $L^2(A)$ for all y in a dense subset of $\operatorname{supp}(\Gamma(A,.))$ and all $t \in (0,T) \cap \mathbb{Q}$. This defines a limit function $w(x,y,t)$ for almost all $x \in \Omega$ for all t in a dense subset, and, by the uniform Lipschitz estimate in y for all $y \in \operatorname{supp}(\Gamma(A,.))$. We claim that for all such y, the function $t \mapsto w(.,y,t) \in L^2(A)$ is uniformly continuous. Indeed, the approximations satisfy with a Dirac family $\Phi_\varepsilon(\zeta) := \Phi_0(y + \zeta/\varepsilon)$, for $\varepsilon \to 0$,

$$
\begin{aligned}
&\|w^{h,\eta}(.,y,t_2) - w^{h,\eta}(.,y,t_1)\|_{L^2(A)} \\
&\quad \leqslant O(\varepsilon) + \left\| \frac{1}{\Gamma(\Phi_\varepsilon)} \int_I [w^{h,\eta}(.,\zeta,t_2) - w^{h,\eta}(.,\zeta,t_1)]\, \Phi_\varepsilon(\zeta)\, d\Gamma(\zeta) \right\|_{L^2(A)} \\
&\quad \leqslant O(\varepsilon) + \frac{1}{\Gamma(\Phi_\varepsilon)} \left\| \int_{t_1}^{t_2} \sum_{\zeta \in I_\eta} \partial_t w^{h,\eta}(.,\zeta,t)\, \Phi_\varepsilon(\zeta)\, \Gamma_\eta(\zeta)\, dt \right\|_{L^2(A)} \\
&\quad \leqslant O(\varepsilon) + \frac{1}{\Gamma(\Phi_\varepsilon)} C\, |t_2 - t_1|
\end{aligned}
$$

by (2.15). In particular, w extends uniquely to all of $[0,T]$ to a function w as in (3.3). We claim that for w the strong L^1 limits of $F_j^{h,\eta}$ coincide almost everywhere with the expressions

$$
\begin{aligned}
F_0(x,t) &= \int_I w(x,y,t)\; d\Gamma(x,y), \qquad F_1(x,t) = \int_I y\, w(x,y,t)\; d\Gamma(x,y), \\
F_2(x,t) &= \int_I |w(x,y,t)|^2\; d\Gamma(x,y).
\end{aligned}
$$

For rational $t \in (0,T)$, this follows by the strong convergence of $w^{h,\eta}(.,y,t) \to w(.,y,t)$ for y in a dense set of $\operatorname{supp}(\Gamma)$ and the uniform Lipschitz continuity in y. The equality for general t follows by the continuity of both sides in t.

After these preparations, we can now derive inequality (3.5). We multiply (2.13) with $\tilde{p}^{h,\eta} - q$ and integrate over Ω to find at an arbitrary time instance $t \in (0,T)$

$$
\begin{aligned}
&\int_{\partial\Omega} n \cdot (K^* \nabla \tilde{p}^{h,\eta})(g - q) - \int_\Omega K^* \nabla \tilde{p}^{h,\eta} \nabla(\tilde{p}^{h,\eta} - q) + \int_\Omega \partial_t u^{h,\eta}(x) \cdot q\, dx \\
&\quad = \int_\Omega \partial_t u^{h,\eta} \cdot \tilde{p}^{h,\eta} = \int_\Omega \sum_y \Gamma_\eta(.,y) \partial_t w^{h,\eta}(.,y) \cdot p^{h,\eta}
\end{aligned}
$$

$$\overset{(2.14)}{\in} \int_\Omega \sum_y \Gamma_\eta(.,y)\partial_t w^{h,\eta}(.,y)\cdot[w^{h,\eta}(.,y)+y\,\mathrm{sign}(\partial_t w^{h,\eta}(.,y))]$$
$$=\partial_t \int_\Omega \sum_y \Gamma_\eta(.,y)\frac{1}{2}|w^{h,\eta}(.,y)|^2+\int_\Omega\sum_y \Gamma_\eta(.,y)y\,|\partial_t w^{h,\eta}(t,y)|.$$

We integrate over (t_1,t_2) and find

$$\int_{t_1}^{t_2}\int_{\partial\Omega} n\cdot(K^*\nabla\tilde p^{h,\eta}(t))(g(t)-q)\,dt-\int_{t_1}^{t_2}\int_\Omega K^*\nabla\tilde p^{h,\eta}(t)\nabla(\tilde p^{h,\eta}(t)-q)\,dt+\int_\Omega u^{h,\eta}(.,t)\cdot q\Big|_{t=t_1}^{t_2}$$
$$=\int_\Omega\sum_y\Gamma_\eta(.,y)\frac{1}{2}|w^{h,\eta}(.,y)|^2\Big|_{t=t_1}^{t_2}+\int_{t_1}^{t_2}\int_\Omega\sum_y\Gamma_\eta(.,y)y|\partial_t w^{h,\eta}(.,y)|$$
$$\geqslant\int_\Omega\sum_y\Gamma_\eta(.,y)\frac{1}{2}|w^{h,\eta}(.,y)|^2\Big|_{t=t_1}^{t_2}+\int_\Omega\sum_y\Gamma_\eta(.,y)y\,|w^{h,\eta}(.,y,t_2)-w^{h,\eta}(.,y,t_2)|. \tag{3.6}$$

By the strong L^1-convergence of the $F_j^{h,\eta}$, we can take the limit $(h,\eta)\to 0$ and find (3.5) for t_1,t_2 in a dense subset of $(0,T)$. As all terms in (3.5) are continuous in t_1 and t_2, the inequality holds for all $t_1,t_2\in[0,T]$.

The equality $(w^{h,\eta}(x,y,t)-p^{h,\eta}(x,t)-y)_+=0$ carries over to the limit (also for reversed sign). Therefore, $w(x,y,t)-p(x,t)\in[-y,y]$ is valid almost everywhere. The Lipschitz continuity of w in y implies the inclusion for all $y\in I$. □

We are particularly interested in two special cases of the equations. The first is the original problem that we recover by setting $\Gamma(x,.)=\delta_{\gamma^\varepsilon(x)}(.)$. The second is the homogenized problem in which the measure $\Gamma(x,.)=\mathscr{L}^1\lfloor I$ appears. In both cases, the above constructed weak solutions are indeed strong solutions. As a corollary to the above proof, we find the following.

Corollary 3.3 (*Strong solutions*) *Let Γ be one of the following.*

(i) *$d\Gamma(x,y)=\varphi(x,y)\,dy$, with a positive function $\varphi:\Omega\times[0,1]\to\mathbb{R}_+$, piecewise constant in x and continuous in y.*

(ii) *$\Gamma(x,.)=\delta_{\gamma(x)}(.)$ with $\gamma\in L^\infty(\Omega,[0,1])$ piecewise constant.*

Then the weak solution (p,w) found in Theorem 3.2 is a strong solution, that it,

$$\partial_t w\in L^\infty((0,T),L^2(\Omega\times I,dx\otimes d\Gamma)), \tag{3.7}$$

in particular $\partial_t u\in L^\infty((0,T),L^2(\Omega))$, and relations (1.8)–(1.10) hold almost everywhere.

Proof We consider once more the approximate solutions $(u^{h,\eta},p^{h,\eta},w^{h,\eta})$ of (2.12)–(2.14) and identify them with their piecewise constant interpolations. By estimate (2.15), we find $u\in W^{1,\infty}(0,T;L^2(\Omega))$ such that $\partial_t u^{h,\eta}\overset{*}{\rightharpoonup}\partial_t u$ in $L^\infty(0,T;L^2(\Omega))$. Furthermore, the compactness of $F_0^{h,\eta}$ implies the strong convergence $u^{h,\eta}\to u$ in $L^1(\Omega\times(0,T))$.

In case (i), we find, starting again from estimate (2.15), the convergence $w^{h,\eta}\overset{*}{\rightharpoonup} w$ in $W^{1,\infty}(0,T;L^2(\Omega\times I))$, and, in particular, the regularity (3.7). In case (ii), by the characterization of $F_0 = L^1 - \lim_{h,\eta} F_0^{h,\eta}$, we find that w essentially coincides with u, $w(x,\gamma(x),t) = u(x,t)$. This implies the regularity (3.7) in case (ii).

We now verify the equations. By the characterization of F_0, relation (1.8) is a consequence of (2.12) and relation (1.9) is the limit of (2.13). It remains to check (1.10). We recall that $w(x,y,t) - p(x,t) \in [-y,y]$ was already verified in Theorem 3.2. The main point is therefore to show for Lebesgue almost every point $(x,t) \in \Omega \times (0,T)$ and for every $y \in I$, that

$$|(w-p)(x,y,t)| < y \quad \Rightarrow \quad \partial_t w(x,y,t) = 0. \tag{3.8}$$

An improved characterizing inequality. The principal idea is to improve the calculation of (3.6). We do not have to take the norm out of the integral in the term

$$\int_{t_1}^{t_2}\int_\Omega F^{h,\eta} \quad \text{with} \quad F^{h,\eta}(x,t) := \sum_y \Gamma_\eta(x,y)\; y\,|\partial_t w^{h,\eta}(x,y,t)|.$$

Case (*i*) By continuity of $\varphi(x,.)$, we may rewrite $F^{h,\eta}$ up to a uniformly small error as

$$F^{h,\eta}(x,t) = \int_I y\,|\partial_t w^{h,\eta}(x,y,t)|\varphi(y)\,dy + o(1)$$

for $\eta \to 0$. We use the lower semicontinuity of convex functionals to find

$$\liminf_{(h,\eta)\to 0} \int_{t_1}^{t_2}\int_I y\,|\partial_t w^{h,\eta}(.,y,t)|\,\varphi(y)\,dy\,dt \geqslant \int_{t_1}^{t_2}\int_I y\,|\partial_t w(.,y,t)|\,\varphi(y)\,dy\,dt.$$

Thus, (3.6) yields the following stronger version of the characterizing inequality.

$$\begin{aligned} 0 \geqslant{}& \int_\Omega \left\{ \int_I \frac{1}{2a^*}|w(x,y)|^2\,d\Gamma(x,y) - u(x)\cdot q \right\} dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \frac{1}{a^*}\int_I y|\partial_t w(x,y,t)|\,d\Gamma(x,y)\,dx\,dt \\ &+ \int_{t_1}^{t_2}\int_\Omega K^*\nabla p(t)\nabla(p(t)-q)\,dt - \int_{t_1}^{t_2}\int_{\partial\Omega} n\cdot(K^*\nabla p(t))(g(t)-q)\,dt. \end{aligned} \tag{3.9}$$

Case (ii) We write

$$F^{h,\eta}(x,t) = \gamma(x)\,|\partial_t w^{h,\eta}(x,\gamma(x),t)| + o(1)$$

for $\eta \to 0$. The lower semicontinuity of convex functionals yields

$$\liminf_{(h,\eta)\to 0} \int_{t_1}^{t_2} F^{h,\eta} \geqslant \textstyle\int_{t_1}^{t_2}\int_I \gamma(.)\,|\partial_t w(.,\gamma(.),t)|\,dt,$$

and therefore again inequality (3.9).

Verification of (*3.8*) We give all arguments for case (i) and $\Gamma(x,.) = \mathscr{L}^1$, the other cases are similar. We assume again $a^* = 1$ and $b^* = 0$ for notational convenience. We can write for the first two integrals of (3.9)

$$\int_\Omega\int_I |w(.,y,t_2)|^2 - |w(.,y,t_1)|^2\,dy \;=\; \int_{t_1}^{t_2}\int_\Omega\int_I 2w(.,y,s)\partial_t w(.,y,s)\,dy\,ds,$$

$$\int_\Omega \{u(t_2) - u(t_1)\}\,q = \int_{t_1}^{t_2}\int_\Omega \partial_t u(s)\,q\,ds.$$

We now choose a countable family of test functions $q \in H^1(\Omega)$. To be specific, we choose the family $\{p(t) : t \in (0,T) \cap Q\}$. Almost every $t \in (0,T)$ is a Lebesgue point for the (countable family of) L^1 functions $\int_\Omega \int_I w\partial_t w$, $\int_\Omega \int_I y\,|\partial_t w|$, $\int_\Omega \partial_t u\, q$ and $\int_\Omega K^*\nabla p\cdot\nabla(p-q)$.

We can now consider $t_1 = t-\tau$, $t_2 = t+\tau$ and the limit $0 < \tau \to 0$. We divide the weak equation (3.9) by $t_2 - t_1$. In the limit $\tau \to 0$, we find in all Lebesgue points t

$$0 \geqslant \int_\Omega \int_I w\,\partial_t w - \partial_t w\, q + \int_\Omega \int_I y\,|\partial_t w| + \int_\Omega K^*\nabla p\cdot\nabla(p-q) - \int_{\partial\Omega} n\cdot(K^*\nabla p)(g-q).$$

By continuity of p in t, we can choose the test function $q \in H^1(\Omega)$ arbitrarily close to $p(t)$. We conclude

$$0 \geqslant \int_\Omega \int_I (w-p)\partial_t w + y|\partial_t w|.$$

By $|w-p| \leqslant y$, the integrand is nonnegative. We conclude that the integrand vanishes almost everywhere. This yields (3.8) almost everywhere and $\mathrm{sign}(\partial_t w) = \mathrm{sign}(p-w)$. □

We have seen for strong solutions that either $\partial_t w$ vanishes or $w-p$ is constant. Formally, this is equivalent to the structure property (2.18). But we need the strong formulation of (2.18) for the homogenization limit. This is the main reason why we work with the space discrete solutions as test functions.

We conclude the analysis of the original problem (related to $\Gamma = \delta_{\gamma(x)}$) and of the limit problem (related to $\Gamma = \varphi\, dy$) with a uniqueness result.

Remark 3.4 (Uniqueness) Let Γ be as in (i) or (ii) of Corollary 3.3. Then there exists only one strong solution (p,w) of (1.8)–(1.12).

Proof Let (p_1,w_1) and (p_2,w_2) be two strong solutions of (1.8)–(1.12) as characterized in Corollary 3.3. We consider here case (i) with $\varphi \equiv 1$ and $a^* = 1$, $b^* = 0$ and $K^* = 1$ for notational convenience. The equations imply

$$\Delta(p_1-p_2) = \partial_t(u_1-u_2) = \int_I \partial_t(w_1-w_2)\,dy.$$

We multiply with (p_1-p_2) and integrate over Ω to find

$$\begin{aligned} -\int_\Omega |\nabla(p_1-p_2)|^2 &= \int_\Omega \int_I (p_1-p_2)\partial_t(w_1-w_2) \\ &\in \int_\Omega \int_I (w_1 + y\ \mathrm{sign}(\partial_t w_1) - w_2 - y\ \mathrm{sign}(\partial_t w_2))\partial_t(w_1-w_2) \\ &= \int_\Omega \int_I \frac{1}{2}\partial_t|w_1-w_2|^2 + y\ (\mathrm{sign}(\partial_t w_1) - \mathrm{sign}(\partial_t w_2))\partial_t(w_1-w_2). \end{aligned}$$

This yields

$$\begin{aligned} &\int_0^T \int_\Omega |\nabla(p_1-p_2)|^2 + \int_\Omega \int_I \frac{1}{2}|(w_1-w_2)(T)|^2 \\ &\quad \in -\int_\Omega \int_I y\ (\mathrm{sign}(\partial_t w_1) - \mathrm{sign}(\partial_t w_2))\partial_t(w_1-w_2) \leqslant 0, \end{aligned}$$

which provides $p_1 = p_2$ and $w_1 = w_2$. □

Our uniqueness result is for strong solutions. We emphasize that, by Corollary 3.3, this also implies a uniqueness result for weak solutions as soon as we incorporate the initial values in the solution concept. Regarding Corollary 3.3 and Remark 3.4, we note that we restricted to the two cases (i) and (ii) in order to keep the proofs accessible. With the help of some additional tools of measure theory, the case of a general measure Γ can also be treated.

4 Homogenization

In this section, we consider flow in unsaturated porous media described by the hysteresis system (1.1)–(1.2). The material parameters a, b, γ and K are assumed to vary across the medium and are chosen randomly. Our aim is to derive upscaled equations that describe the averaged behaviour almost surely. We assume for simplicity that the material parameters are piecewise constant in the medium, and that the different values are chosen independently according to a stochastic law.

We consider again a rectangle $\Omega \subset \mathbb{R}^n$. For every $\varepsilon > 0$, we subdivide Ω into cells

$$Q_k^\varepsilon := \varepsilon[k + (0,1)^N] \cap \Omega, \quad k \in \mathbb{Z}^N.$$

For given bounds $0 < a_l < a_u$, $b_l < b_u$ and $K_l < K_u$, in each cell $Q_k^\varepsilon \subset \Omega$, we choose randomly $a_k \in J_a := [a_l, a_u]$, $b_k \in J_b := [b_l, b_u]$, $K_k \in J_K := [K_l, K_u]$ and $\gamma_k \in I := [0,1]$, all independently and, for simplicity, uniformly distributed. We define

$$\gamma^\varepsilon \in L^\infty(\Omega, \mathbb{R}), \qquad \text{by} \qquad \gamma(x) = \gamma_k \ \forall x \in Q_k^\varepsilon,$$

and similarly for a^ε, b^ε and K^ε. We consider (1.1)–(1.2) in the stochastic geometry, that is,

$$\partial_t u^\varepsilon = \nabla \cdot (K^\varepsilon \nabla p^\varepsilon), \tag{4.1}$$

$$p^\varepsilon \in a^\varepsilon u^\varepsilon + b^\varepsilon + \gamma^\varepsilon \operatorname{sign}(\partial_t u^\varepsilon), \tag{4.2}$$

with the initial and boundary values of (1.3) and (1.4). Corollary 3.3 (ii) provides the existence of a solution to this problem, with bounds independent of ε. The characterization of weak solutions in Theorem 3.2 implies $|p^\varepsilon - a^\varepsilon u^\varepsilon - b^\varepsilon| \leqslant \gamma^\varepsilon$ almost everywhere and, by evaluating $\frac{1}{2a^\varepsilon}|a^\varepsilon u^\varepsilon + b^\varepsilon|^2$,

$$\begin{aligned}
&\int_\Omega \left(\frac{a^\varepsilon}{2}|u^\varepsilon|^2 + b^\varepsilon u^\varepsilon - u^\varepsilon q \right)\Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega K^\varepsilon (\nabla p^\varepsilon - \nabla q) \cdot (\nabla p^\varepsilon - \nabla q) \\
&\qquad + \int_\Omega \gamma^\varepsilon |u^\varepsilon(.,t_2) - u^\varepsilon(.,t_1)| + \int_{t_1}^{t_2} \int_\Omega K^\varepsilon \nabla q \cdot (\nabla p^\varepsilon - \nabla q) \\
&\leqslant \int_{t_1}^{t_2} \int_{\partial\Omega} n \cdot (K^\varepsilon \nabla p^\varepsilon(t))(g(t) - q)\, dt,
\end{aligned} \tag{4.3}$$

for all $q \in H^1(\Omega)$ and all $[t_1, t_2] \subset [0, T]$.

The above described model of a stochastic medium can be realized as in [11]. The independent distributions of the coefficients (a, b, K, γ) can be realized with a probability space $(\Sigma, \mathscr{A}, P)$ such that

$$\begin{aligned}
\Sigma = \{ &\omega \in L^\infty(\mathbb{R}^n, [a_l, a_u] \times [b_l, b_u] \times [K_l, K_u] \times [0,1]) : \\
&\omega \text{ constant in all cells } x + k + (0,1)^N,\ k \in \mathbb{Z}^n \text{ for some } x \in [0,1]^n \big\}.
\end{aligned}$$

We use the shift operator $T(x) : \omega(.) \mapsto \omega(. + x)$. The coefficients of the equations are determined for an element $\omega \in \Sigma$ as $a^\varepsilon(x) := \omega_1(x/\varepsilon) = [T(x/\varepsilon)\omega]_1(0)$, and similarly for $b^\varepsilon, K^\varepsilon$ and γ^ε.

To homogenize the diffusion operator, we use the following cell solutions on unbounded domains. With $K(\omega) := \omega_3(0)$, our aim is to study for $\omega \in \Sigma$ a solution Q_j^ω, $j = 1, \dots, n$, of the cell problem

$$\nabla \cdot \left[K(T(x)\omega) \cdot \left(e_j + \nabla Q_j^\omega(x) \right) \right] = 0. \tag{4.4}$$

Following the approach of [11], we use the spaces $L^2_{pot}(\Sigma)$ and $L^2_{sol}(\Sigma)$ of vector fields $v \in L^2(\Sigma)^n$, such that for almost all $\omega \in \Sigma$, the realizations $v(T(x)\omega)$ are potential and solenoidal, respectively. Instead of searching for $\nabla_x Q$ for fixed ω, we then search for $v_j = v_j(\omega)$, such that almost all realizations are potential. We can write the family of problems (4.4) as

$$v_j \in L^2_{pot}(\Sigma) \cap \{f | \mathbb{E} f = 0\}, \quad K \cdot (e_j + v_j) \in L^2_{sol}(\Sigma), \tag{4.5}$$

and this can be solved with the Lax–Milgram theorem. The homogenized diffusion matrix K^* is defined by

$$\mathbb{E}(K \cdot (e_j + v_j)) = K^* \cdot e_j. \tag{4.6}$$

As a preparation for the homogenization, we collect some consequences of the ergodicity of the system.

Lemma 4.1 *For every $\alpha \geqslant 1$ and almost all $\omega \in \Sigma$ we have*

$$b^\varepsilon \rightharpoonup b^* \qquad \text{in } L^\alpha(\Omega), \tag{4.7}$$

$$\frac{1}{a^\varepsilon} 1_{\{\gamma^\varepsilon \leqslant z\}} \rightharpoonup \frac{1}{a^*} z \qquad \text{in } L^\alpha(\Omega), \tag{4.8}$$

$$K^\varepsilon \cdot \left(e_j + \nabla Q_j^\omega \right) \rightharpoonup K^* \cdot e_j \quad \text{in } L^2(\Omega). \tag{4.9}$$

Furthermore, for almost every $\omega \in \Sigma$, there exists a continuous potential Q_j^ω with

$$\varepsilon \| Q_j^\omega(./\varepsilon) \|_{L^\infty(\Omega)} \to 0, \tag{4.10}$$

and for all $\varepsilon_n < \varepsilon_0$ along a sequence $\varepsilon_n \to 0$ we have

$$|\{x \in \Omega | \gamma^{\varepsilon_n}(x) < y\}| < 2|\Omega| y. \tag{4.11}$$

Proof The probability measure $\mathscr{P}$ is ergodic with respect to the translations T. Therefore, by the Birkhoff ergodic theorem (cp. e.g. [11], Theorem 7.2) the oscillating function b^ε converges weakly to its expected value $b^* = \langle b^\varepsilon \rangle$, hence (4.7). The same argument shows (4.9). To show (4.8), we first notice that for a fixed $z \in I$, for almost all $\omega \in \Sigma$, the limit follows from the fact that a^ε and γ^ε are independently distributed. Since $\mathbb{Q}$ is countable, we conclude that for almost all ω, convergence (4.8) is valid for all $z \in I \cap \mathbb{Q}$. Using the fact that the left-hand side is monotone in z and the right-hand side is continuous in z, we conclude the result for all $z \in I$.

For almost every ω, the realization v_j is indeed a gradient. We can choose $Q_j^\omega(./\varepsilon)$ with vanishing average on Ω such that $\nabla(\varepsilon Q_j^\omega(./\varepsilon)) = v_j(./\varepsilon)$. The Birkhoff theorem yields

$v_j \rightharpoonup \mathbb{E}v_j = 0$ in L^2 by definition (4.5). This implies the strong L^2 convergence of $\varepsilon Q_j^\omega(./\varepsilon)$. The functions $\varepsilon Q_j^\omega(./\varepsilon)$ are solutions of uniform elliptic equation, and we can estimate the L^∞ norm on a compact set by the L^2 norm on a larger set. This argument provides (4.10). The argument is taken from [13], Lemma 2, and we refer to this article for more details.

For all $y \in I \cap \mathbb{Q}$ and almost all $\omega \in \Sigma$, the characteristic function $1_{\{x\in\Omega|\gamma^{\varepsilon_n}(x)\leqslant y\}}$ converges weakly to its expected value y. Therefore, its average converges to $y|\Omega|$. We find (4.11) first for all rational y, but this implies the estimate for all $y \in [0,1]$. □

The next theorem is the main result of this article. We find the averaged equations for the hysteresis problem in unsaturated porous media. The principal idea is to construct test functions on the basis of the limit problem. To be precise, we use the solution $(u^{h,\eta}, p^{h,\eta}, w^{h,\eta})$ of the discretized limit problem to construct the test function w^ε in step 2 of the proof. The use of the discretized equation is essential since we want to exploit the structure property (2.18) in step 3.

Theorem 4.2 (*Homogenization*) *Let a sequence of stochastic geometries be given as above, let the pressure boundary values g satisfy* (1.5), *and let, for compatibility, the initial values for the saturation* $U_0^{(\varepsilon)}$ *result from a drainage process at the point of vanishing pressure, i.e.*

$$a^\varepsilon(x)U_0^{(\varepsilon)}(x) + b^\varepsilon(x) = \gamma^\varepsilon(x). \tag{4.12}$$

We study a strong solution $(p^\varepsilon, u^\varepsilon)$ *of the original* ε *equations* (1.1)–(1.4), *and a strong solution* (u, p, w) *of the limit system* (1.8)–(1.12) *with initial values* $W_0(x, y) := y$, *both as constructed in Corollary* 3.3.

Then, for any sequence $\varepsilon \to 0$, *almost surely we find*

$$p^\varepsilon \rightharpoonup p \text{ in } H^1((0,T), H^1(\Omega)), \tag{4.13}$$

$$u^\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty((0,T), L^2(\Omega)). \tag{4.14}$$

Let us note that the drainage assumption (4.12) can be replaced by an imbibition assumption without changes in the result. Much more general initial values U_0 can be considered; necessary is that W_0 can be defined consistently satisfying (1.14).

Proof We note that the compatibilities (1.6) and (1.14) are satisfied; thus Theorem 2.4 and Corollary 3.3 are applicable.

Let $\varepsilon = \varepsilon_n \to 0$ be a fixed sequence. Corollary 3.3 provides solutions with uniform estimates for $p^\varepsilon \in H^1((0,T), H^1(\Omega))$ and $u^\varepsilon \in W^{1,\infty}((0,T), L^2(\Omega))$. We can assume for a subsequence corresponding weak and weak-∗ convergences $p^\varepsilon \rightharpoonup p^0$ and $u^\varepsilon \to u^0$ and we have to show $u^0 = u$ and $p^0 = p$. We fix $\omega \in \Sigma$ such that the convergences of Lemma 4.1 hold. We use the function $\tilde{p}^{h,\eta}$ from Theorem 2.4 to construct an oscillating test function for the homogenization procedure. The bounds (2.16)–(2.17) provide uniform estimates for $\tilde{p}^{h,\eta} \in L^\infty((0,T), H^2(\Omega)) \cap H^1((0,T), H^1(\Omega))$ and $u^{h,\eta} \in W^{1,\infty}((0,T), L^2(\Omega))$.

Step 1 (*Appropriate choice of a test function in the weak equation*). For arbitrary $s \in (0,T)$, we set

$$q(x) := \tilde{p}^{h,\eta}(x,s) + \varepsilon \sum_j Q_j\left(\frac{x}{\varepsilon}\right) \partial_{x_j}\tilde{p}^{h,\eta}(x,s).$$

In the subsequent calculations, we decompose one integral as

$$\int_{t_1}^{t_2}\int_\Omega K^\varepsilon\nabla q\cdot(\nabla p^\varepsilon-\nabla q)=\int_{t_1}^{t_2}\int_\Omega K^\varepsilon\sum_j(e_j+\nabla Q_j)\,\partial_{x_j}\tilde p^{h,\eta}(s)\,(\nabla p^\varepsilon-\nabla q)$$
$$+\int_{t_1}^{t_2}\int_\Omega K^\varepsilon\varepsilon\sum_j Q_j\,\nabla\partial_{x_j}\tilde p^{h,\eta}\,(\nabla p^\varepsilon-\nabla q),$$

and exploit that the last integral is small. We insert q in the weak equation (4.3) to find

$$\int_\Omega\left(\frac{a^\varepsilon}{2}|u^\varepsilon|^2+b^\varepsilon u^\varepsilon-u^\varepsilon\tilde p^{h,\eta}(s)\right)\Bigg|_{t_1}^{t_2}+c_0\int_{t_1}^{t_2}\|p^\varepsilon-q\|_{H^1}^2+\int_\Omega\gamma^\varepsilon|u^\varepsilon(.,t_2)-u^\varepsilon(.,t_1)|$$
$$\leqslant-\int_{t_1}^{t_2}\int_\Omega\sum_j K^\varepsilon\,(e_j+\nabla Q_j)\,\partial_{x_j}\tilde p^{h,\eta}(s)\,(\nabla p^\varepsilon-\nabla\tilde p^{h,\eta}(s))$$
$$+\int_{t_1}^{t_2}\sum_{j,k}\int_\Omega[K^\varepsilon\,(e_j+\nabla Q_j)\partial_{x_j}\tilde p^{h,\eta}(s)]\nabla(\varepsilon Q_k(./\varepsilon)\,\partial_{x_k}\tilde p^{h,\eta}(s))+q_1(t_1,t_2,\varepsilon),\qquad(4.15)$$

with

$$q_1(t_1,t_2,\varepsilon):=C\varepsilon\|Q(./\varepsilon)\|_{L^\infty(\Omega)}\|u^\varepsilon(t_2)-u^\varepsilon(t_1)\|_{L^2}+C\varepsilon\|Q(./\varepsilon)\|_{L^\infty(\Omega)}\,(t_2-t_1)+o(t_2-t_1).$$

To treat the second integral on the right-hand side, we have to make use of the theorem of compensated compactness. The divergence of the squared bracket converges weakly in $L^2(\Omega)$, and therefore strongly in $H^{-1}(\Omega)$, since the divergence of $K^\varepsilon(e_j+\nabla Q_j)$ vanishes. The gradient of the other bracket is obviously curl free. We can apply the theorem on compensated compactness (compare, e.g. [11]). On a dense set of time instances s, the Ω integral converges to zero. By the estimates for $\tilde p^{h,\eta}$, the Ω integral is continuous in s, with modulus of continuity independent of ε. We therefore have convergence of the Ω integral to zero, uniformly in s.

In the first integral, we replace $K^\varepsilon\,(e_j+\nabla Q_j)$ by K^*, leading to the error term

$$\int_{t_1}^{t_2}\left|\int_\Omega\sum_j[K^*_{.j}-K^\varepsilon\,(e_j+\nabla Q_j)]\partial_{x_j}\tilde p^{h,\eta}(s)\cdot\nabla(p^\varepsilon-\tilde p^{h,\eta}(s))\right|$$
$$=:q_2'(t_1,t_2,\varepsilon)=o_\varepsilon(1)\,(t_2-t_1).\qquad(4.16)$$

For this last estimate, we use the same argument as above based on the theorem on compensated compactness, and exploit estimate (3.4) for $p^{h,\eta}$ and for p^ε.

On the right-hand side of (4.15), we have now after an integration by parts and (2.13)

$$-\int_{t_1}^{t_2}\int_\Omega\sum_j K^\varepsilon\,(e_j+\nabla Q_j)\,\partial_{x_j}\tilde p^{h,\eta}(s)\,(\nabla p^\varepsilon-\nabla\tilde p^{h,\eta}(s))$$
$$\leqslant-\int_{t_1}^{t_2}\int_\Omega K^*\nabla\tilde p^{h,\eta}(s)\,(\nabla p^\varepsilon-\nabla\tilde p^{h,\eta}(s))+q_2'(t_1,t_2,\varepsilon)$$
$$=\int_{t_1}^{t_2}\int_\Omega\partial_t u^{h,\eta}\cdot(p^\varepsilon-\tilde p^{h,\eta}(s))+q_2'(t_1,t_2,\varepsilon)+o(t_2-t_1),$$

where the last error term is introduced by the boundary integral. We have thus transformed (4.15) into

$$\int_\Omega \left(\frac{a^\varepsilon}{2} |u^\varepsilon|^2 + b^\varepsilon u^\varepsilon - u^\varepsilon \tilde{p}^{h,\eta}(s) \right) \Big|_{t_1}^{t_2} + c_0 \int_{t_1}^{t_2} \|p^\varepsilon - q\|_{H^1}^2$$
$$+ \int_\Omega \gamma^\varepsilon |u^\varepsilon(.,t_2) - u^\varepsilon(.,t_1)| - \int_{t_1}^{t_2} \int_\Omega \partial_t u^{h,\eta} \cdot (p^\varepsilon - \tilde{p}^{h,\eta}(s))$$
$$\leqslant q_1(t_1,t_2,\varepsilon) + q_2(t_1,t_2,\varepsilon), \tag{4.17}$$

where $q_2(t_1,t_2,\varepsilon) = o_\varepsilon(1)\,(t_2 - t_1)$ contains both error terms that were treated by the method of compensated compactness.

We next replace in (4.17) the function $\tilde{p}^{h,\eta}$ by its piecewise averages $p^{h,\eta}$. This introduces an error

$$q_3(t_1,t_2,\varepsilon) := C o_h(1) \left(\|u^\varepsilon(.,t_2) - u^\varepsilon(.,t_1)\|_{L^2} + \int_{t_1}^{t_2} \|\partial_t u^{h,\eta}\|_{L^2} \right), \tag{4.18}$$

with $o_h(1) \to 0$ for $h \to 0$ independent of ε.

Step 2 (An energy decay result) We next calculate for an appropriate energy function a decay result on the basis of the $p^{h,\eta}$ version of (4.17). To shorten the calculations, we write $p(s)$ for $p^{h,\eta}(s)$ and perform the computations in the case $b^\varepsilon \equiv 0$.

We can evaluate $w^{h,\eta}$ only in points $y \in I_\eta$. To an arbitrary point $y \in I$, we therefore define $y_\eta(y) := \eta[y/\eta + 1] \in I_\eta$, which is the node in I_η corresponding to y. We can now introduce $w^\varepsilon(x,t) := w^{h,\eta}(x, y_\eta(\gamma^\varepsilon(x)), t)$ to find

$$\int_\Omega \frac{1}{2a^\varepsilon} |a^\varepsilon u^\varepsilon + b^\varepsilon - w^\varepsilon|^2 \Big|_{t_1}^{t_2} + c_0 \int_{t_1}^{t_2} \|p^\varepsilon - q\|_{H^1}^2$$
$$= \int_\Omega \frac{a^\varepsilon}{2} |u^\varepsilon|^2 - u^\varepsilon w^\varepsilon + \frac{1}{2a^\varepsilon} |w^\varepsilon|^2 \Big|_{t_1}^{t_2} + c_0 \int_{t_1}^{t_2} \|p^\varepsilon - q\|_{H^1}^2$$
$$\overset{(4.17)}{\leqslant} \int_\Omega u^\varepsilon p(s) \Big|_{t_1}^{t_2} - \int_\Omega \gamma^\varepsilon |u^\varepsilon(.,t_2) - u^\varepsilon(.,t_1)| + \int_{t_1}^{t_2} \int_\Omega \partial_t u^{h,\eta}\, (p^\varepsilon - p(s))$$
$$- \int_\Omega u^\varepsilon w^\varepsilon \Big|_{t_1}^{t_2} + \int_\Omega \frac{1}{2a^\varepsilon} |w^\varepsilon|^2 \Big|_{t_1}^{t_2} + \sum_{j=1}^{3} q_j(t_1,t_2,\varepsilon)$$
$$= \int_\Omega u^\varepsilon [p(s) - w^\varepsilon(s)] \Big|_{t_1}^{t_2} - \int_\Omega \gamma^\varepsilon |u^\varepsilon(.,t_2) - u^\varepsilon(.,t_1)|$$
$$+ \int_{t_1}^{t_2} \int_\Omega \left[\partial_t u^{h,\eta} - \frac{1}{a^\varepsilon} \partial_t w^\varepsilon \right] p^\varepsilon - \int_{t_1}^{t_2} \int_\Omega \left[\partial_t u^{h,\eta} - \frac{1}{a^\varepsilon} \partial_t w^\varepsilon \right] p(s)$$
$$- \int_\Omega u^\varepsilon [w^\varepsilon - w^\varepsilon(s)] \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \frac{1}{a^\varepsilon} [w^\varepsilon - p(s)]\; \partial_t w^\varepsilon$$
$$+ \int_{t_1}^{t_2} \int_\Omega \frac{1}{a^\varepsilon} \partial_t w^\varepsilon\, p^\varepsilon + \sum_{j=1}^{3} q_j(t_1,t_2,\varepsilon).$$

We start by studying the first two integrals together. Exploiting (2.14), we find

$$\begin{aligned}
&\int_\Omega u^\varepsilon [p(s) - w^\varepsilon(s)]\Big|_{t_1}^{t_2} - \int_\Omega \gamma^\varepsilon |u^\varepsilon(., t_2) - u^\varepsilon(., t_1)| \\
&\in \int_\Omega u^\varepsilon\, y_\eta(\gamma^\varepsilon)\mathrm{sign}(\partial_t w^\varepsilon(s))\Big|_{t_1}^{t_2} - \int_\Omega \gamma^\varepsilon |u^\varepsilon(., t_2) - u^\varepsilon(., t_1)| \\
&\leqslant \eta \int_\Omega |u^\varepsilon(., t_2) - u^\varepsilon(., t_1)|.
\end{aligned}$$

The last two integrals of the above calculation can be written as

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_\Omega \frac{1}{a^\varepsilon} [w^\varepsilon - p(s)]\ \partial_t w^\varepsilon + \int_{t_1}^{t_2} \int_\Omega \frac{1}{a^\varepsilon} \partial_t w^\varepsilon\ p^\varepsilon \\
&= \int_{t_1}^{t_2} \int_\Omega u^\varepsilon \partial_t [w^\varepsilon - w^\varepsilon(s)] + \int_{t_1}^{t_2} \int_\Omega (p^\varepsilon - a^\varepsilon u^\varepsilon) \frac{\partial_t w^\varepsilon}{a^\varepsilon} + \int_{t_1}^{t_2} \int_\Omega [w^\varepsilon - p(s)]\ \frac{\partial_t w^\varepsilon}{a^\varepsilon} \\
&\leqslant \int_{t_1}^{t_2} \int_\Omega u^\varepsilon \partial_t [w^\varepsilon - w^\varepsilon(s)] + q_4(t_1, t_2, \varepsilon).
\end{aligned}$$

Here we estimated the last two integrals by the error term q_4. We use that $(p^\varepsilon - a^\varepsilon u^\varepsilon) \in \gamma^\varepsilon \mathrm{sign}(\partial_t u^\varepsilon)$ by (4.2) and $w^\varepsilon - p^{h,\eta} \in -y_\eta(\gamma^\varepsilon)\mathrm{sign}(\partial_t w^\varepsilon)$ by (2.14). This makes the error negative up to $p^{h,\eta}(t) \mp p(s) = p^{h,\eta}(s)$. We can set

$$q_4(t_1, t_2, \varepsilon) := C \int_{t_1}^{t_2} \int_{t_1}^{s} \int_\Omega \|\partial_t p^{h,\eta}(\xi)\|_{L^2(\Omega)}\, d\xi\, ds.$$

The last error term already shows that we must deal with the whole time interval $(0, T)$ in one estimate. We consider discretizations $\mathscr{F}$ of $(0, T)$ given by families $0 = t_0 < \ldots < t_N = T$, and apply the above estimate with $t_i, t_{i+1} \in \mathscr{F}$ and $s = t_i$. We fix $\Delta t > 0$ and use only discretizations $\mathscr{F}$ such that $|t_{i+1} - t_i| \leqslant \Delta t$ for all i. In the above inequality, we take the positive part and sum over i. Taking the supremum over all $\mathscr{F}$ as above, we find essentially a BV norm on the left-hand side—the factor 2 stems from the fact that we sum only the positive increments. We exploit here that the integral vanishes initially. With $t_i(t)$ denoting the point $s = t_i \leqslant t$ closest to t, we can write

$$\begin{aligned}
&\frac{1}{2} \left\| \int_\Omega \frac{1}{2a^\varepsilon} |a^\varepsilon u^\varepsilon + b^\varepsilon - w^\varepsilon|^2 \right\|_{BV([0,T],\mathbb{R})} + c_0 \|p^\varepsilon - q\|_{L^2 H^1}^2 \\
&\leqslant C\eta + \sup_{\mathscr{F}} \int_0^T \left| \int_\Omega \left[\partial_t u^{h,\eta} - \frac{1}{a^\varepsilon} \partial_t w^\varepsilon \right]\ (p^\varepsilon - p(t_i(.))) \right| \\
&\quad + \sup_{\mathscr{F}} \sum_i \left| - \int_\Omega u^\varepsilon [w^\varepsilon - w^\varepsilon(t_i)] \Big|_{t_i}^{t_{i+1}} + \int_{t_i}^{t_{i+1}} \int_\Omega u^\varepsilon \partial_t [w^\varepsilon - w^\varepsilon(t_i)] \right| \\
&\quad + \sup_{\mathscr{F}} \sum_i \sum_{j=1}^4 q_j(t_i, t_{i+1}, \varepsilon). \qquad (4.19)
\end{aligned}$$

It remains to analyze this inequality (4.19).

Step 3 (Conclusion) We consider one after another the limits $\Delta t \to 0$, then $\varepsilon \to 0$, then $h \to 0$, then $\eta \to 0$.

The second supremum on the right-hand side of (4.19) vanishes for $\Delta t \to 0$, as can be seen with one integration by parts and using the uniform estimates for derivatives of u^ε and of $w^{h,\eta}$.

Concerning the first supremum on the right-hand side of (4.19), it suffices to show that for every sequence φ^ε bounded in L^2H^1, we have

$$F^\varepsilon := \int_0^T \int_\Omega \left(\partial_t u^{h,\eta} - \frac{1}{a^\varepsilon} \partial_t w^\varepsilon \right) \cdot \varphi^\varepsilon \to 0. \tag{4.20}$$

We calculate for the first factor with (2.12) and the structure property (2.18)

$$\begin{aligned}
\partial_t u^{h,\eta} - \frac{1}{a^\varepsilon}\partial_t w^\varepsilon &= \frac{1}{a^*} \sum_{y \in I_\eta, y \leqslant z^{h,\eta}} \Gamma_\eta(y)\, \partial_t w^{h,\eta}(., y) - \frac{1}{a^\varepsilon} \partial_t w^{h,\eta}(., y_\eta(\gamma^\varepsilon(x))) \\
&= \partial_t p^{h,\eta} \left[\frac{1}{a^*} z^{h,\eta} - \frac{1}{a^\varepsilon} 1_{\{\gamma^\varepsilon \leqslant z^{h,\eta}\}} \right].
\end{aligned}$$

The ergodicity result (4.8) implies, since $z^{h,\eta}$ takes only finitely many values, that

$$Z^\varepsilon := \frac{1}{a^*} z^{h,\eta} - \frac{1}{a^\varepsilon} 1_{\{\gamma^\varepsilon \leqslant z^{h,\eta}\}} \rightharpoonup 0,$$

for $\varepsilon \to 0$, weakly in every $L^\alpha(\Omega)$ and uniformly in $t \in [0, T]$. For every $q > 1$, there is $\alpha < \infty$ such that the embedding $W^{1,q}(\Omega) \subset (L^\alpha(\Omega))' = L^{\alpha^*}(\Omega)$ is compact. Choosing a subsequence, we may therefore assume $Z^\varepsilon \to 0$ in $C^0((0,T), W^{1,q}(\Omega)')$.

On the other hand, for $q > 1$ depending on the dimension n, the product of two bounded $H^1(\Omega)$ functions is an $W^{1,q}(\Omega)$ function with corresponding bound. Therefore,

$$\partial_t p^{h,\eta}\, \varphi^\varepsilon \in L^1((0,T), W^{1,q}(\Omega))$$

is a bounded sequence. Integrals of their product with Z^ε vanish in the limit. This verifies (4.20).

In the limit $\varepsilon \to 0$, we find from (4.19)

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \left\{ \left\| \int_\Omega \frac{1}{2a^\varepsilon} |a^\varepsilon u^\varepsilon + b^\varepsilon - w^\varepsilon|^2 \right\|_{BV(0,T)} + c_0 \|p^\varepsilon - p^{h,\eta}\|^2_{L^2H^1} \right\} \\
&\qquad \leqslant o_h(1) + o_\eta(1),
\end{aligned} \tag{4.21}$$

that is, the right-hand side of equation (4.21) is arbitrary small for h and η. In particular, since $p^\varepsilon \rightharpoonup p^0$ for $\varepsilon \to 0$ and $p^{h,\eta} \rightharpoonup p$ for $(h, \eta) \to 0$,

$$\|p^0 - p\|^2_{L^2H^1} = 0.$$

This shows the claim for (4.13).

For the convergence of u^ε, we once more study (4.19). Almost surely, functions $w^\varepsilon = w^{h,\eta}(., y_\eta(\gamma^\varepsilon), .)$ converge weakly to the expected value for γ ranging in $(0, 1)$ and, by independency,

$$\frac{1}{a^\varepsilon}(w^\varepsilon - b^\varepsilon) \rightharpoonup \frac{1}{a^*}\left(\sum_{y\in I_\eta}\Gamma_\eta(.,y)\,w^{h,\eta}(y) - b^*\right) \tag{4.22}$$
$$= \frac{1}{a^*}(a^* u^{h,\eta} + b^* - b^*) = u^{h,\eta}$$

in $L^2(\Omega\times(0,T))$. Let now u^0 be a weak limit of u^ε in the same space. Then (4.19) yields

$$\|u^0 - u^{h,\eta}\|^2_{L^2(\Omega\times(0,T))} \leqslant \liminf_{\varepsilon\to 0}\|u^\varepsilon - \frac{1}{a^\varepsilon}(w^\varepsilon - b^\varepsilon)\|^2_{L^2(\Omega\times(0,T))}$$
$$\leqslant C\liminf_{\varepsilon\to 0}\|a^\varepsilon u^\varepsilon - w^\varepsilon + b^\varepsilon\|^2_{BV([0,T],L^2(\Omega))}$$
$$\leqslant o_h(1) + o_\eta(1).$$

This implies $u^0 = u$, and thus (4.14). □

5 Conclusion

Starting from simple play-type hysteresis equations for unsaturated porous media, we derived an effective hysteresis model. The model contains a new variable w that can be regarded as an expected pressure. It encodes the wetting history of the process.

The mathematical derivation was based on Galerkin approximations. The approximations were used first to construct weak solutions, then to construct test functions. The crucial point is that the approximate solutions satisfy the structure property (2.18), that we could not verify for strong solutions due to missing regularity properties. Our analysis is restricted to independent stochastic coefficients because of the argument in (4.22).

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