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On universal unfoldings of certain real functions on a Banach space

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The aim of this article is to prove a result which has been thought true for some time. Roughly speaking, if you take a universal unfolding of a germ in finitely many variables, and add to it a non-degenerate quadratic form on an infinite-dimensional space, you still have a universal unfolding.

NOTATION

For a pair of Banach spaces X and Y, the space of germs at 0 of C^∞ mappings from X to Y will be denoted by $\mathscr{E}(X, Y)$. $\mathscr{E}(X, \mathbb{R})$ will be abbreviated to $\mathscr{E}(X)$. Germs will generally be confused with mappings defined on a neighbourhood of 0. This is for brevity and to avoid awkward expressions. If X is a Banach space and X^* its normed dual, the number $x^*(x)$, where $x^* \in X^*$ and $x \in X$, will be denoted by $\langle x^*, x\rangle$.

LEMMA 1. *Let X, A be Banach spaces and let $f \in \mathscr{E}(X \times A)$ be such that $f(0, a) = 0$ for all $a \in A$. Then $f(x, a) = \langle h(x, a), x\rangle$ where*

$$h \in \mathscr{E}(X \times A, X^*).$$

Proof.

$$f(x,a) = \int_0^1 \frac{d}{dt} f(tx, a)\, dt$$

$$= \int_0^1 \langle f_x(tx, a), x\rangle\, dt$$

$$= \left\langle \int_0^1 f_x(tx, a)\, dt, x \right\rangle.$$

LEMMA 2. *Let X and A be Banach spaces with X reflexive, let $d \in \mathscr{E}(X \times A, X^*)$ be such that $d(0,0) = 0$ and $D_x d(0,0)$ is an invertible linear mapping from X onto X^*. Then for every $f \in \mathscr{E}(X \times A)$, there exist $h \in \mathscr{E}(X \times A, X)$ and $r \in \mathscr{E}(A)$ such that*

$$f(x, a) = \langle d(x, a), h(x, a)\rangle + r(a).$$

Proof. The mapping $(x, a) \to (d(x, a), a)$ is by the inverse function theorem a diffeomorphism of a neighbourhood of $(0, 0)$ in $X \times A$ to a neighbourhood of $(0, 0)$ in $X^* \times A$. Let its inverse be the mapping $(x^*, a) \to (\gamma(x^*, a), a)$ where $\gamma \in \mathscr{E}(X^* \times A, X)$. By Lemma 1

$$f(\gamma(x^*, a), a) = f(\gamma(0, a), a) + \langle x^*, k(x^*, a)\rangle,$$

where $k \in \mathscr{E}(X^* \times A, X)$ (here we use the identification of X and X^{**}). Now set

$$r(a) = f(\gamma(0, a), a)$$

and
$$h(x, a) = k(d(x, a), a).$$

This concludes the proof.

To introduce the main theorem we announce some puzzling terminology. Let X and A be Banach spaces. Then we define $\mathscr{U}(X; A)$ to be $\mathscr{E}(X \times A)$. Why we should want to do this is given by the next definition. Two members f and g of $\mathscr{U}(X; A)$ shall be called equivalent if

$$f(x, a) = g(\phi(x, a), \psi(a)) + r(a),$$

where $\phi \in \mathscr{E}(X \times A, X)$, $\psi \in \mathscr{E}(A, A)$, $r \in \mathscr{E}(A)$, $\phi(\cdot, a)$ is for each a, a diffeomorphism on a neighbourhood of $0 \in X$, ψ is a diffeomorphism on a neighbourhood of $0 \in$ A, and finally $\phi(0, 0) = 0$, $\psi(0) = 0$. Equivalence is an equivalence relation. This definition is slightly different from the usual definition of isomorphism of unfoldings, (see (1)), since f and g need not be unfoldings of the same germ.

THEOREM 1. *Let $f \in \mathscr{U}(X; A)$, where X is a reflexive Banach space. Assume that $D_x f(0, 0) = 0$ and $D_x^2 f(0, 0) = T$ is a Fredholm operator (since T is symmetric it suffices to assume that it has closed range and finite-dimensional null-space). Let $u_1^*, \ldots, u_n^*$ be elements of X^* whose projections into X^*/TX form a basis of the latter space. Then there is a germ $g \in \mathscr{E}(\mathbb{R}^n \times A)$ such that f is equivalent to a member of $\mathscr{U}(X; A)$ given by*

$$(x, a) \to \tfrac{1}{2}\langle Tx, x\rangle + g(\langle u_1^*, x\rangle, \ldots, \langle u_n^*, x\rangle, a).$$

Suppose further that $\{u_1, \ldots, u_n\}$ is a basis of $N(T)$ dual to $\{u_1^*, \ldots, u_n^*\}$. If $D_x f(x, 0) \in \operatorname{sp}\{u_1^*, \ldots, u_n^*\}$ whenever $x \in N(T)$ then

$$g(\lambda_1, \ldots, \lambda_n, 0) = f(\lambda_1 u_1 + \ldots + \lambda_n u_n, 0).$$

Proof. Let $Z \subset X$ be the annihilator of $\{u_1^*, \ldots, u_n^*\}$. Every x in X can be written uniquely in the form $z + \Sigma_{j=1}^n \lambda_j u_j$ where $z \in Z$ and $\lambda_j = \langle u_j^*, x\rangle$. Furthermore T is an invertible linear mapping of Z onto TX, and the latter space is in a natural way the dual of Z. Let

$$F(z, \underline{\lambda}, a) = f(z + \Sigma \lambda_j u_j, a)$$

where $z \in Z$, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $a \in A$.

LEMMA 3. *There exists a C^∞ mapping h with range in Z of the variables z, $\underline{\lambda}$, a, and an additional real variable t. The domain of h is defined by relations of the form*

$$\max(\|z\|, |\underline{\lambda}|, \|a\|) < \epsilon; \quad -\epsilon < t < 1 + \epsilon.$$

The following equation holds:

$$F(z, \underline{\lambda}, a) - \tfrac{1}{2}\langle Tz, z\rangle = \langle tD_z F(z, \underline{\lambda}, a) + (1-t) Tz, h(z, \underline{\lambda}, a, t)\rangle + \psi(\underline{\lambda}, a, t). \qquad (1)$$

In this equation ψ is defined on the same domain as h but does not depend on z.

Proof of lemma. By Lemma 2 we can define 'h' and 'ψ' for t in a neighbourhood of any point $t_0 \in [0, 1]$. Just take Z to be the space X of Lemma 2 and amalgamate the other variables into the A of Lemma 2. Pick a finite number of these neighbourhoods, say $V_1, \ldots, V_k$, and a partition of unity of $[0, 1]$ relative to them, say, $\phi_1, \ldots, \phi_k$.

Let the 'h' and 'ψ' on V_j be denoted by h_j and ψ_j, these mappings being extended by setting them equal to zero for $t \notin V_j$. Finally we define $h = \Sigma\phi_j h_j$ and $\psi = \Sigma\phi_j \psi_j$. These satisfy the required conditions.

Completion of proof of theorem. Let $w(z, \underline{\lambda}, a, t)$ be the solution of the differential equation

$$\dot{\zeta} = -h(\zeta, \underline{\lambda}, a, t) \quad (\zeta \in Z) \tag{2}$$

such that $w(z, \underline{\lambda}, a, 0) = z$. Define

$$g(\underline{\lambda}, a) = \int_0^1 \psi(\underline{\lambda}, a, t)\, dt.$$

Set

$$M(z, \underline{\lambda}, a, t) = tF(z, \underline{\lambda}, a) + (1-t)\tfrac{1}{2}\langle Tz, z\rangle.$$

Equation (1) now implies

$$(d/dt)\, M(w(z, \underline{\lambda}, a, t), \underline{\lambda}, a, t) = \psi(\underline{\lambda}, a, t).$$

Integrating between 0 and 1

$$F(w(z, \underline{\lambda}, a, 1), \underline{\lambda}, a) = \tfrac{1}{2}\langle Tz, z\rangle + g(\underline{\lambda}, a).$$

This implies the required equivalence, since the mapping $z \to w(z, \underline{\lambda}, a, 1)$ is a diffeomorphism.

A note on the existence of solutions of (2): by considering Lemmas 1 and 2 it may be seen that $h(0,0,0,t) \equiv 0$. Hence the solution of (2) with initial value 0 exists for all t when $\underline{\lambda} = 0$ and $a = 0$. It is just $\zeta = 0$. The set of quadruples $(z, \underline{\lambda}, a, t)$ such that the solution of (2) with initial values z exists on the interval $[0, t]$ is open. Hence this set contains an open neighbourhood of the set $\{(0,0,0,t) : 0 \leqslant t \leqslant 1\}$, which is what we require.

To obtain the last part, note that the stated assumption is equivalent to $D_z F(0, \underline{\lambda}, 0) = 0$. Then by (1), $\psi(\underline{\lambda}, 0, t) = F(0, \underline{\lambda}, 0) = f(\Sigma\lambda_j u_j, 0)$, whence the result.

We shall now state two corollaries which use the idea of a universal unfolding. Let $\eta \in \mathscr{E}(X)$ and let $f \in \mathscr{U}(X; A)$ be such that $f(x, 0) = \eta(x)$. f is called an unfolding of η. f is called a universal unfolding of η if given $g \in \mathscr{U}(X; B)$ such that $g(x, 0) = \eta(x)$, we have

$$g(x, b) = f(\phi(x, b), \psi(b)) + r(b), \tag{3}$$

where $\phi \in \mathscr{E}(X \times B, X)$, $\psi \in \mathscr{E}(B, A)$, $r \in \mathscr{E}(B)$, $\phi(\cdot, b)$ is, for each b, a diffeomorphism of a neighbourhood of $0 \in X$ such that $\phi(\cdot, 0)$ is the identity, and $\psi(0) = 0$. Since ψ need not be invertible the relation expressed by (3) is quite different from equivalence. If for some $f \in \mathscr{U}(X; A)$ and $g \in \mathscr{U}(X; B)$ the relation (3) holds, except that $\phi(\cdot, 0)$ need not be the identity, we shall say that g is induced from f.

Corollary 1. *Let Z be a reflexive Banach space, $T: Z \to Z^*$ a symmetric linear homeomorphism. Let $\eta \in \mathscr{E}(R^n)$ such that $\eta'(0) = 0$ have a universal unfolding $f \in \mathscr{U}(\mathbb{R}^n; A)$. Then the germ $(z, \underline{\lambda}) \to \tfrac{1}{2}\langle Tz, z\rangle + \eta(\underline{\lambda})$ which is a member of $\mathscr{E}(Z \times \mathbb{R}^n)$ has a universal unfolding $(z, \underline{\lambda}, a) \to \tfrac{1}{2}\langle Tz, z\rangle + f(\underline{\lambda}, a)$.*

Corollary 2. *Let $\eta \in \mathscr{E}(X)$, where X is reflexive, be such that $\eta'(0) = 0$ and $\eta''(0) = T$ is a Fredholm operator. Let $\{u_1, \ldots, u_n\}$ be a basis for $N(T)$ and suppose there exists a* *topological supplement* Z *of* $N(T)$ *with the property that for any* $x \in N(T)$ *and* $z \in Z$, $\langle \eta'(x), z \rangle = 0$. *Let* $\gamma(\underline{\lambda}) = \eta(\lambda_1 u_1 + \ldots + \lambda_n u_n)$ *and let* $g(\underline{\lambda}, a)$ *be a universal unfolding of* γ. *Then any unfolding of* η *is induced from the unfolding germ*

$$(x, a) \to \tfrac{1}{2}\langle Tz, z \rangle + g(\underline{\lambda}, a)$$

where $x = z + \Sigma \lambda_j u_j$, $z \in Z$.

Example. This is taken from (2). *Let* $X = \{x \in H^2[-1, 1] : x(-1) = x(1) = 0\}$. $H^2[-1, 1]$ *is the space of* L^2*-functions on* $[-1, 1]$, *whose first- and second-order distribution derivatives are* L^2*-functions. Such functions are continuous, and it is a Hilbert space. Define* $\eta \in \mathscr{E}(X)$ *by*

$$\eta(x) = \frac{1}{2}\int_{-1}^{1}\left(|x''(s)|^2 - \frac{\pi^2}{4}|x'(s)|^2\right) ds + \frac{k}{8}\left(\int_{-1}^{1}|x'(s)|^2 ds\right)^2,$$

where k *is a constant. The physical meaning of* η *according to* (2) *is the following. Consider an elastic beam of small cross-section fixed between the points* -1 *and* $+1$, *and subjected to an increasing compressive stress exactly in line with it. Then at a certain value of the stress the beam buckles. If, with this value of the stress, the shape of the beam happened to be described by the function* $x(s)$, *then* $\eta(x)$ *would be its elastic energy. In reality the configuration of the beam is supposed to be a function* x *such that* $\eta'(x) = 0$.

We have

$$\langle \eta'(x), u \rangle = \int_{-1}^{1}\left(x''(s)u''(s) - \frac{\pi^2}{4}x'(s)u'(s)\right) ds + \frac{k}{2}\left(\int_{-1}^{1}|x'(s)|^2 ds\right)\left(\int_{-1}^{1}x'(s)u'(s) ds\right)$$

and

$$\langle \eta''(0)v, u \rangle = \int_{-1}^{1}\left(v''(s)u''(s) - \frac{\pi^2}{4}v'(s)u'(s)\right) ds.$$

That $T = \eta''(0)$ *is a Fredholm operator and that its kernel is spanned by* $\cos\frac{1}{2}\pi s$ *is shown in* (2). *Now*

$$\langle \eta'(\lambda \cos\tfrac{1}{2}\pi s), u \rangle = \text{const}\,\lambda^3 \int_{-1}^{1} u(s)\cos\tfrac{1}{2}\pi s\, ds.$$

Hence if we define

$$Z = \left\{z \in X : \int_{-1}^{1} z(s)\cos\tfrac{1}{2}\pi s\, ds = 0\right\}$$

then Z *has the properties required in Corollary* 2. *Finally*

$$\eta(\lambda\cos\tfrac{1}{2}\pi s) = \text{const}\,\lambda^4.$$

The germ λ^4 *has a universal unfolding* $(\lambda, a_1, a_2) \to \lambda^4 + a_1\lambda^2 + a_2\lambda$, (*see* (3)). *Hence any unfolding of* $\eta(x)$ *is induced from*

$$x \to \lambda^4 + a_1\lambda^2 + a_2\lambda + Q(z),$$

where

$$\lambda = \int_{-1}^{1} x(s)\cos\tfrac{1}{2}\pi s\, ds,$$

$z = x - \lambda\cos\frac{1}{2}\pi s$, *and* Q *is a non-degenerate quadratic form on* Z.

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