Proceedings of the Royal Society of Edinburgh, **128A**, 1261–1280, *1998*

On the different convex hulls of sets involving singular values

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(MS received 2 June 1997. Revised MS received 5 December 1997)

We give a representation formula for the convex, polyconvex and rank one convex hulls of a set of $n \times n$ matrices with prescribed singular values.

1. Introduction

Let $\xi \in \mathbb{R}^{n \times n}$ and denote by $0 \leqq \lambda_1(\xi) \leqq \lambda_2(\xi) \leqq \ldots \leqq \lambda_n(\xi)$ the singular values of the matrix ξ (i.e. the eigenvalues of $(\xi^t \xi)^{\frac{1}{2}}$; this implies in particular that $|\xi|^2 = \Sigma_{i=1}^n [\lambda_i(\xi)]^2$ and $|\det \xi| = \Pi_{i=1}^n [\lambda_i(\xi)]$. Let $0 < a_1 \leqq a_2 \leqq \ldots \leqq a_n$ and

$$E = \{\xi \in \mathbf{R}^{n \times n} : \lambda_i(\xi) = a_i,\ i = 1, \ldots, n\}. \tag{1.1}$$

The main results of this article (cf. Theorem 3.1) are that

$$coE = \left\{\xi \in \mathbf{R}^{n \times n} : \sum_{i=v}^{n} \lambda_i(\xi) \leqq \sum_{i=v}^{n} a_i,\ v = 1, \ldots, n\right\}, \tag{1.2}$$

$$PcoE = RcoE = \left\{\xi \in \mathbf{R}^{n \times n} : \prod_{i=v}^{n} \lambda_i(\xi) \leqq \prod_{i=v}^{n} a_i,\ v = 1, \ldots, n\right\}, \tag{1.3}$$

where coE denotes the convex hull of E, and $PcoE$ (respectively $RcoE$) the polyconvex (respectively the rank one convex) hull of E. The first notion corresponds to the classical one (cf. [**9**]) while the two others will be defined in Section 2.

It is interesting to note that, if $a_1 = a_2 = \ldots = a_n$, then it turns out that

$$coE = PcoE = RcoE = \{\xi \in \mathbf{R}^{n \times n} : \lambda_n(\xi) \leqq a_n\}$$

as already observed in [**4, 6**]. The case where the a_i are not all equal is more involved and has already been considered in [**5, 7**] when $n = 2$.

An important application of the above representations is for attainment results in problems of the calculus of variations. A direct consequence of the results of [**7**] (in particular Theorems 6.1 and 6.4) leads to the following existence theorem: let $\Omega \subset \mathbf{R}^n$

be an open set, $a_i : \bar{\Omega} \times \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, n$ be continuous functions satisfying

$$0 < c \leqq a_1(x, s) \leqq \ldots \leqq a_n(x, s)$$

for every $(x, s) \in \bar{\Omega} \times \mathbf{R}^n$ and let $\varphi \in C^1(\bar{\Omega}; \mathbf{R}^n)$ satisfy

$$\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} a_i(x, \varphi(x)), \quad x \in \Omega, \quad \nu = 1, \ldots, n,$$

(in particular $\varphi \equiv 0$); then there exists $u \in \mathbf{W}^{1,\infty}(\Omega; \mathbf{R}^n)$ such that

$$\begin{cases} \lambda_i(Du(x)) = a_i(x, u(x)), & \text{a.e. } x \in \Omega, \quad i = 1, \ldots, n \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

2. The different convex hulls

Before proceeding with the proofs of our main results, we introduce the following definition and properties (cf. [7] for more details).

DEFINITION 2.1. Let $E \subset \mathbf{R}^{m \times n}$ and

$$F_E = \{ f : \mathbf{R}^{m \times n} \to \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}, f|_E = 0 \}.$$

Define

$$coE = \{ \xi \in \mathbf{R}^{m \times n} : f(\xi) \leqq 0, \forall f \in F_E, f \text{ convex} \},$$

called the convex hull of E;

$$PcoE = \{ \xi \in \mathbf{R}^{m \times n} : f(\xi) \leqq 0, \forall f \in F_E, f \text{ polyconvex} \},$$

called the polyconvex hull of E;

$$RcoE = \{ \xi \in \mathbf{R}^{m \times n} : f(\xi) \leqq 0, \forall f \in F_E, f \text{ rank one convex} \},$$

called the rank one convex hull of E.

REMARK 2.2. The first one corresponds to the classical definition of convex hull (cf. [9]).

From the above definition, we can easily deduce the following propositions:

PROPOSITION 2.3. *Let $E \subset \mathbf{R}^{m \times n}$; then*

$$E \subset RcoE \subset PcoE \subset coE.$$

PROPOSITION 2.4. *Let $E \subset \mathbf{R}^{m \times n}$ and define by induction*

$$R_0 coE = E,$$

$$R_{i+1} coE = \{ \xi \in \mathbf{R}^{m \times n} : \xi = tA + (1-t)B,\ t \in (0, 1),\ A, B \in R_i coE,\ rank \{A - B\} = 1 \}.$$

Then $RcoE = \bigcup_{i \in \mathbf{N}} R_i\, coE$.

REMARK 2.5. We can observe that the above proposition is a weaker version of the result obtained in the characterisation of convex and polyconvex hulls. For example, using Carathéodory's Theorem, we have (cf. [9]):

$$coE = \left\{ \xi \in \mathbf{R}^{m \times n} : \xi = \sum_{i=1}^{mn+1} t_i \xi_i,\ \xi_i \in E,\ t_i \geqq 0, \text{ with } \sum_{i=1}^{mn+1} t_i = 1 \right\}.$$

PROPOSITION 2.6. *Let* $0 \leqq \lambda_1(\xi) \leqq \lambda_2(\xi) \leqq \ldots \leqq \lambda_n(\xi)$ *be the singular values of the matrix* $\xi \in \mathbf{R}^{n \times n}$. *Then*

(i) $\xi \to \Sigma_{i=v}^{n} \lambda_i(\xi)$ *is a convex function, for every* $v = 1, \ldots, n$;

(ii) $\xi \to \Pi_{i=v}^{n} \lambda_i(\xi)$ *is a polyconvex function, for every* $v = 1, \ldots, n$.

For a proof of the first result, we refer to [**2**, **3**, **8**]; for the last one, see [**2**] and [**1**], when $n = 2$ and $n = 3$ (the general case follows similarly).

3. The main results

In this section we will proceed with the proof of the main result of this article:

THEOREM 3.1. *Let* $\xi \in \mathbf{R}^{n \times n}$ *and denote by* $0 \leqq \lambda_1(\xi) \leqq \lambda_2(\xi) \leqq \ldots \leqq \lambda_n(\xi)$ *the singular values of the matrix* ξ. *Let* $0 < a_1 \leqq a_2 \leqq \ldots \leqq a_n$,

$$E = \{\xi \in \mathbf{R}^{n \times n} : \lambda_i(\xi) = a_i,\ i = 1, \ldots, n\}.$$

Then:

(i) $coE = \{\xi \in \mathbf{R}^{n \times n} : \Sigma_{i=v}^{n} \lambda_i(\xi) \leqq \Sigma_{i=v}^{n} a_i,\ v = 1, \ldots, n\}$;

(ii) $PcoE = RcoE = \{\xi \in \mathbf{R}^{n \times n} : \Pi_{i=v}^{n} \lambda_i(\xi) \leqq \Pi_{i=v}^{n} a_i,\ v = 1, \ldots, n\}$;

(iii) $intRcoE = \{\xi \in \mathbf{R}^{n \times n} : \Pi_{i=v}^{n} \lambda_i(\xi) < \Pi_{i=v}^{n} a_i,\ v = 1, \ldots, n\}$.

REMARK 3.2. When $n = 2$ and $E = \{\xi \in \mathbf{R}^{2 \times 2} : \lambda_1(\xi) = a_1, \lambda_2(\xi) = a_2\}$, the theorem reads as

$$coE = \{\xi \in \mathbf{R}^{2 \times 2} : \lambda_2(\xi) \leqq a_2,\ \lambda_1(\xi) + \lambda_2(\xi) \leqq a_1 + a_2\}$$

and

$$PcoE = RcoE = \{\xi \in \mathbf{R}^{2 \times 2} : \lambda_2(\xi) \leqq a_2,\ \lambda_1(\xi) \cdot \lambda_2(\xi) \leqq a_1 \cdot a_2\}.$$

Proof of Theorem 3.1(i). Let $K = \{\xi \in \mathbf{R}^{n \times n} : \Sigma_{i=v}^{n} \lambda_i(\xi) \leqq \Sigma_{i=v}^{n} a_i,\ v = 1, \ldots, n\}$.

We show that $coE = K$. We divide the proof into two steps.

Step 1. $coE \subset K$. The inclusion $coE \subset K$ is easy. In fact, $E \subset K$ and from Proposition 2.6, the functions $\xi \to \Sigma_{i=v}^{n} \lambda_i(\xi)$ are convex. Therefore K is convex and hence $coE \subset K$.

Step 2. $K \subset coE$. Let $\xi \in K$; we will prove that ξ can be expressed as a convex combination of elements of E, i.e. $\xi \in coE$.

Since the functions $\xi \to \lambda_i(\xi)$ are invariant by orthogonal transformations, we can assume, without loss of generality, that

$$\xi = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix},$$

with $0 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_n$ and $\Sigma_{i=v}^{n} x_i \leqq \Sigma_{i=v}^{n} a_i$, $v = 1, \ldots, n$.

We proceed by induction. We start with the proof in dimension $n = 2$.

(i) $n = 2$. We subdivide this case into two parts:

(a) $x_1 \leqq a_1$ and, since $\xi \in K$, then $x_2 \leqq a_2$ and $x_1 + x_2 \leqq a_1 + a_2$.

Since $-a_1 \leqq x_1 \leqq a_1$, then $x_1 = ta_1 + (1-t)(-a_1)$ with $t = (x_1 + a_1)/2a_1$. We can write:

$$\xi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = t \begin{pmatrix} a_1 & 0 \\ 0 & x_2 \end{pmatrix} + (1-t) \begin{pmatrix} -a_1 & 0 \\ 0 & x_2 \end{pmatrix}. \tag{3.1}$$

We proceed similarly for x_2, i.e. $x_2 = sa_2 + (1-s)(-a_2)$, where $s = (x_2 + a_2)/2a_2$. Thus we obtain

$$\begin{pmatrix} \pm a_1 & 0 \\ 0 & x_2 \end{pmatrix} = s\begin{pmatrix} \pm a_1 & 0 \\ 0 & +a_2 \end{pmatrix} + (1-s)\begin{pmatrix} \pm a_1 & 0 \\ 0 & -a_2 \end{pmatrix}. \tag{3.2}$$

Combining (3.1) and (3.2), we get that

$$\xi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \sum_{i=1}^{I} t_i \xi_i,$$

with $\lambda_1(\xi_i) = a_1$, $\lambda_2(\xi_i) = a_2$ (i.e. $\xi_i \in E$). Therefore

$$\xi \in coE.$$

(b) $x_1 \geqq a_1$, i.e. since $\xi \in K$, $a_1 \leqq x_1 \leqq x_2 \leqq a_2$ and $x_1 + x_2 \leqq a_1 + a_2$. This implies that

$$a_1 \leqq x_1 \leqq a_1 + a_2 - x_2.$$

In this case we just interpolate x_1 between a_1 and $a_1 + a_2 - x_2$, i.e.

$$x_1 = ta_1 + (1-t)(a_1 + a_2 - x_2),$$

which implies that

$$\xi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = t\begin{pmatrix} a_1 & 0 \\ 0 & x_2 \end{pmatrix} + (1-t)\begin{pmatrix} a_1 + a_2 - x_2 & 0 \\ 0 & x_2 \end{pmatrix}. \tag{3.3}$$

The first matrix is treated in case (a). For the second matrix, we interpolate x_2 between a_1 and a_2, i.e. $x_2 = sa_2 + (1-s)a_1$, to obtain

$$\begin{pmatrix} a_1 + a_2 - x_2 & 0 \\ 0 & x_2 \end{pmatrix} = s\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + (1-s)\begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}. \tag{3.4}$$

Combining (3.3) and (3.4), we have proved that

$$\xi = \sum_{i=1}^{I} t_i \xi_i,$$

with $\lambda_1(\xi_i) = a_1$, $\lambda_2(\xi_i) = a_2$ (i.e. $\xi_i \in E$). Therefore $\xi \in coE$. In conclusion, we have obtained, for $n = 2$, that

$$K \subset coE.$$

(ii) $n > 2$. We suppose that the result has been established up to $n-1$, i.e. every ξ such that $\Sigma_{i=\nu}^{n-1} \lambda_i(\xi) \leqq \Sigma_{i=\nu}^{n-1} a_i$, $\nu = 1, 2, \ldots, n-1$ (i.e. $\xi \in K$) can be expressed as a convex combination of elements of $\{\xi \in \mathbf{R}^{(n-1)\times(n-1)} : \lambda_i(\xi) = a_i,\ i = 1, \ldots, n-1\}$, i.e.

$$\xi = \sum_{\mu=1}^{I} t_\mu \xi_\mu,$$

with ξ_μ such that $\lambda_i(\xi_\mu) = a_i$, $i = 1, 2, \ldots, (n-1)$. We divide the proof into five parts:

Part 1. $0 \leqq x_1 \leqq x_2 \leqq x_1 + x_2 \leqq a_2$. Note that these conditions imply that $x_1 + x_2 \leqq a_1 + a_2$ and $x_2 \leqq a_2$. We can therefore apply the case $n = 2$ to $\{x_1, x_2\}$ and to $\{a_1, a_2\}$. We then use the hypothesis of induction on $\{x_3, \ldots, x_n\}$ and on $\{a_3, \ldots, a_n\}$. Combining these two decompositions, we get the result, i.e. $\xi \in coE$.

Part 2. $0 \leqq x_1 \leqq x_2 \leqq a_2 \leqq x_1 + x_2$. We can write

$$\xi = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 & \lambda & & \\ \lambda & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 & -\lambda & & \\ -\lambda & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$

$$= \frac{1}{2} A_+ + \frac{1}{2} A_-, \tag{3.5}$$

where we have chosen

$$\lambda^2 = (x_2 - a_2)(x_1 - a_2).$$

Note that by hypothesis ($x_1 \leqq x_2 \leqq a_2$) the right-hand side is positive. The choice of λ allows us to find $O_\pm, O'_\pm \in O(n)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} a_2 & & & & \\ & x_1 + x_2 - a_2 & & & \\ & & x_3 & & \\ & & & \ddots & \\ & & & & x_n \end{pmatrix}.$$

We next apply the hypothesis of induction to

$$\{y_1 = x_1 + x_2 - a_2,\ y_2 = x_3, \ldots, y_{n-1} = x_n\}$$

and to

$$\{b_1 = a_1,\ b_2 = a_3, \ldots, b_{n-1} = a_n\}.$$

To do this, we first observe that

$$0 \leqq y_1 = x_1 + x_2 - a_2 \leqq x_1 \leqq x_3 = y_2 \leqq y_3 \leqq \ldots \leqq y_{n-1}$$

and

(1) if $\nu \geqq 2$, then $\Sigma_{i=\nu}^{n-1} y_i = \Sigma_{i=\nu+1}^{n} x_i \leqq \Sigma_{i=\nu+1}^{n} a_i = \Sigma_{i=\nu}^{n-1} b_i$;

(2) if $\nu = 1$, then $\Sigma_{i=1}^{n-1} y_i = -a_2 + \Sigma_{i=1}^{n} x_i \leqq -a_2 + \Sigma_{i=1}^{n} a_i = \Sigma_{i=1}^{n-1} b_i$.

We can therefore deduce (by hypothesis of induction) that

$$\begin{pmatrix} a_2 & & & & \\ & x_1 + x_2 - a_2 & & & \\ & & x_3 & & \\ & & & \ddots & \\ & & & & x_n \end{pmatrix} \in coE.$$

Since coE is invariant up to orthogonal transformations, we obtain that

$$A_\pm = \begin{pmatrix} x_1 & \pm\lambda & & \\ \pm\lambda & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \in coE, \tag{3.6}$$

which leads, combining (3.5) and (3.6), to

$$\xi \in coE,$$

which is the claimed result.

Part 3. $x_{n-1} \geqq a_{n-1}$. We write

$$\xi = \begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & \\ & & & x_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & \lambda \\ & & \lambda & x_n \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & -\lambda \\ & & -\lambda & x_n \end{pmatrix}$$

$$= \frac{1}{2} A_+ + \frac{1}{2} A_-, \tag{3.7}$$

where we have chosen

$$\lambda^2 = (x_n - a_{n-1})(x_{n-1} - a_{n-1}).$$

Note that by hypothesis ($x_n \geqq x_{n-1} \geqq a_{n-1}$) the right-hand side is positive. As above, the choice of λ leads to the existence of $O_\pm, O'_\pm \in O(n)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} x_1 & & & & \\ & \ddots & & & \\ & & x_{n-2} & & \\ & & & x_n + x_{n-1} - a_{n-1} & \\ & & & & a_{n-1} \end{pmatrix}.$$

We next apply the hypothesis of induction to

$$\{y_1 = x_1, \ldots, y_{n-2} = x_{n-2}, y_{n-1} = x_n + x_{n-1} - a_{n-1}\}$$

and to

$$\{b_1 = a_1, \ldots, b_{n-2} = a_{n-2}, b_{n-1} = a_n\}.$$

To do this, we can observe that

$$0 \leqq y_1 \leqq \ldots \leqq y_{n-2} = x_{n-2} \leqq x_n \leqq x_n + x_{n-1} - a_{n-1} = y_{n-1}.$$

By hypothesis and since $\xi \in K$, we have:

(1) if $\nu = n-1$, $y_{n-1} = x_n + x_{n-1} - a_{n-1} \leqq a_n$;

(2) if $1 \leqq \nu \leqq n-2$,

$$\sum_{i=\nu}^{n-1} y_i = x_n + x_{n-1} - a_{n-1} + \sum_{i=\nu}^{n-2} x_i = -a_{n-1} + \sum_{i=\nu}^{n} x_i$$
$$\leqq -a_{n-1} + \sum_{i=\nu}^{n} a_i = a_n + \sum_{i=\nu}^{n-2} a_i = \sum_{i=\nu}^{n-1} b_i.$$

We can therefore deduce by hypothesis of induction and by invariance of coE under orthogonal transformations that

$$A_\pm \in coE,$$

which combined with (3.7) lead to

$$\xi \in coE.$$

Part 4. $a_2 \leqq x_2 \leqq \dots \leqq x_{n-1} \leqq a_{n-1}$. Note that this case occurs only if $n \geqq 4$. We first observe that we can therefore find $k \in \{2, \dots, n-2\}$ such that

$$a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}. \tag{3.8}$$

Hence we can write

$$\xi = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} = \frac{1}{2} A_+ + \frac{1}{2} A_-$$

$$= \frac{1}{2} \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_k & \lambda & & \\ & & \lambda & x_{k+1} & & \\ & & & & \ddots & \\ & & & & & x_n \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_k & -\lambda & & \\ & & -\lambda & x_{k+1} & & \\ & & & & \ddots & \\ & & & & & x_n \end{pmatrix}, \tag{3.9}$$

where we have chosen

$$\lambda^2 = (x_k - b)(x_{k+1} - b) \tag{3.10}$$

with $b = a_k$ (Part 4.1) or $b = a_{k+1}$ (Part 4.2). Note that, from the above assumption (3.8), the right-hand side is positive in both cases.

Part 4.1. $\begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1} \\ x_k + x_{k+1} + \sum_{i=\nu+1}^{n} x_i \leqq a_k + \sum_{i=\nu}^{n} a_i, \quad \nu = k+2, \dots, n \end{cases}$

(with the convention $\Sigma_{i=n+1}^{n} x_i = 0$).

Part 4.2. $\begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1} \\ \sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i \leqq \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i, \quad \mu = 1, \dots, k-1. \end{cases}$

Before proceeding with the study of the above cases, we show that Part 4.1 and Part 4.2 cover all possibilities. In fact, if $0 \leqq x_1 \leqq \dots \leqq x_n$ and if $\Sigma_{i=\nu}^{n} x_i \leqq \Sigma_{i=\nu}^{n} a_i$, $\nu = 1, \dots, n$, then at least one of the following sets of inequalities holds:

$$x_k + x_{k+1} + \sum_{i=\nu+1}^{n} x_i \leqq a_k + \sum_{i=\nu}^{n} a_i, \quad \nu = k+2, \dots, n;$$

$$\sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i \leqq \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i, \quad \mu = 1, \dots, k-1.$$

We proceed by contradiction and we assume that there exists $\nu \in \{k+2, \dots, n\}$ and

$\mu \in \{1, \ldots, k-1\}$ such that

$$x_k + x_{k+1} + \sum_{i=v+1}^{n} x_i > a_k + \sum_{i=v}^{n} a_i,$$
$$\sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i > \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i.$$

Summing up these two inequalities and using the assumptions, we get

$$\sum_{i=\mu}^{n} a_i + \sum_{i=v+1}^{n} a_i \geqq \sum_{i=\mu}^{n} x_i + \sum_{i=v+1}^{n} x_i > a_k - a_{k+1} + \sum_{i=\mu+1}^{n} a_i + \sum_{i=v}^{n} a_i$$

i.e.

$$a_\mu + a_{k+1} > a_k + a_v.$$

However, $\mu \in \{1, \ldots, k-1\}$, hence $a_\mu \leqq a_k$ and $v \in \{k+2, \ldots, n\}$, therefore $a_v \geqq a_{k+1}$. We therefore get

$$a_k + a_{k+1} \geqq a_\mu + a_{k+1} > a_k + a_v \geqq a_k + a_{k+1},$$

which is the claimed contradiction. In conclusion, Part 4.1 and Part 4.2 cover all possibilities. We now separately study these two cases:

$$\text{Part 4.1.} \begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1} \\ x_k + x_{k+1} + \sum_{i=v+1}^{n} x_i \leqq a_k + \sum_{i=v}^{n} a_i, \quad v = k+2, \ldots, n \end{cases}$$

(with the convention $\Sigma_{i=n+1}^{n} x_i = 0$). We choose here $b = a_k$ in (3.9) and (3.10). We can, as above, find $O_\pm, O'_\pm \in O(n)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} x_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x_{k-1} & & & & & \\ & & & x_k + x_{k+1} - a_k & & & & \\ & & & & a_k & & & \\ & & & & & x_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & x_n \end{pmatrix}.$$

We apply the hypothesis of induction to

$$\{y_1 = x_1, \ldots, y_{k-1} = x_{k-1}, y_k = x_k + x_{k+1} - a_k, y_{k+1} = x_{k+2}, \ldots, y_{n-1} = x_n\}$$

and to

$$\{b_1 = a_1, \ldots, b_{k-1} = a_{k-1}, b_k = a_{k+1}, \ldots, b_{n-1} = a_n\}.$$

Observe that, since $a_k \leqq x_k$, then $0 \leqq y_1 \leqq \ldots \leqq y_{k-1} = x_{k-1} \leqq x_k + x_{k+1} - a_k = y_k$. On the contrary, *a priori*, we cannot compare y_k to $y_{k+1} \leqq \ldots \leqq y_{n-1}$. We next verify the hypothesis of induction.

(1) Let $v = n-1$. We must show that $y_{n-1} = x_n \leqq b_{n-1} = a_n$ and $y_k \leqq b_{n-1} = a_n$. The first inequality is valid by assumption, while the second is also true since it is equivalent to $x_k + x_{k+1} \leqq a_k + a_n$ which is the assumption of Part 4.1 with $v = n$.

(2) Let $n-2 \geqq v \geqq k+1$. We have again by hypothesis of Part 4.1 and since $\xi \in K$

$$\begin{cases} \sum_{i=v}^{n-1} y_i = \sum_{i=v+1}^{n} x_i \leqq \sum_{i=v+1}^{n} a_i = \sum_{i=v}^{n-1} b_i \\ y_k + \sum_{i=v+1}^{n-1} y_i = x_k + x_{k+1} - a_k + \sum_{i=v+2}^{n} x_i \leqq \sum_{i=v+1}^{n} a_i = \sum_{i=v}^{n-1} b_i. \end{cases}$$

(3) If $k \geqq v \geqq 1$,

$$\sum_{i=v}^{n-1} y_i = \sum_{i=v}^{k-1} y_i + \sum_{i=k}^{n-1} y_i = \sum_{i=v}^{k-1} x_i + \sum_{i=k}^{n} x_i - a_k \leqq \sum_{i=v}^{n} a_i - a_k = \sum_{i=v}^{n-1} b_i.$$

Therefore we can apply the hypothesis of induction and the invariance of coE under orthogonal transformations to get

$$A_{\pm} \in coE. \tag{3.11}$$

Combining (3.9) and (3.11), we indeed get that

$$\xi \in coE.$$

Part 4.2. $$\begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}, \\ \sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i \leqq \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i, \quad \mu = 1, \dots, k-1. \end{cases}$$

We choose here $b = a_{k+1}$ in (3.9) and (3.10). We can, as above, find $O_{\pm}, O'_{\pm} \in O(n)$ such that

$$O_{\pm} A_{\pm} O'_{\pm} = \begin{pmatrix} x_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x_{k-1} & & & & & \\ & & & x_k + x_{k+1} - a_{k+1} & & & & \\ & & & & a_{k+1} & & & \\ & & & & & x_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & x_n \end{pmatrix}.$$

We apply the hypothesis of induction to

$$\{y_1 = x_1, \dots, y_{k-1} = x_{k-1}, y_k = x_k + x_{k+1} - a_{k+1}, y_{k+1} = x_{k+2}, \dots, y_{n-1} = x_n\}$$

and to

$$\{b_1 = a_1, \dots, b_{k-1} = a_{k-1}, b_k = a_k, b_{k+1} = a_{k+2}, \dots, b_{n-1} = a_n\}.$$

Observe that, since $x_{k+1} \leqq a_{k+1}$, we have $y_k = x_k + x_{k+1} - a_{k+1} \leqq x_k \leqq x_{k+2} = y_{k+1} \leqq \dots \leqq y_{n-1}$. On the contrary, *a priori*, we cannot compare y_k to $0 \leqq y_1 \leqq \dots \leqq y_{k-1}$. We next verify the hypothesis of induction. Since $\xi \in K$ and by assumption of Part 4.2, we get:

(1) if $v \geqq k+1$, $\Sigma_{i=v}^{n-1} y_i = \Sigma_{i=v+1}^{n} x_i \leqq \Sigma_{i=v+1}^{n} a_i = \Sigma_{i=v}^{n-1} b_i$;

(2) if $\nu = k$,

$$\begin{cases} \sum_{i=k}^{n-1} y_i = -a_{k+1} + \sum_{i=k}^{n} x_i \leqq -a_{k+1} + \sum_{i=k}^{n} a_i = \sum_{i=k}^{n-1} b_i, \\ y_{k-1} + \sum_{i=k+1}^{n-1} y_i = x_{k-1} + \sum_{i=k+2}^{n} x_i \leqq a_k + \sum_{i=k+2}^{n} a_i = \sum_{i=k}^{n-1} b_i; \end{cases}$$

(3) if $k-1 \geqq \nu \geqq 1$,

$$\begin{cases} \sum_{i=\nu}^{n-1} y_i = -a_{k+1} + \sum_{i=\nu}^{n} x_i \leqq -a_{k+1} + \sum_{i=\nu}^{n} a_i = \sum_{i=\nu}^{n-1} b_i, \\ \sum_{i=\nu-1}^{k-1} y_i + \sum_{i=k+1}^{n-1} y_i = \sum_{i=\nu-1}^{k-1} x_i + \sum_{i=k+2}^{n} x_i \leqq \sum_{i=\nu}^{k} a_i + \sum_{i=k+2}^{n} a_i \\ \qquad = \sum_{i=\nu}^{n-1} b_i. \end{cases}$$

We can therefore apply the hypothesis of induction to obtain

$$\begin{pmatrix} x_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x_{k-1} & & & & & \\ & & & x_k + x_{k+1} - a_{k+1} & & & & \\ & & & & a_{k+1} & & & \\ & & & & & x_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & x_n \end{pmatrix} \in coE.$$

The invariance under orthogonal transformations leads immediately to

$$A_{\pm} \in coE. \tag{3.12}$$

Combining (3.9) and (3.12), we have indeed obtained

$$\xi \in coE.$$

This achieves the proof of Step 2, i.e. $K \subset coE$, and thus part (i) of the theorem. □

Proof of Theorem 3.1(ii). Let $X = \{\xi \in \mathbf{R}^{n\times n} : \Pi_{i=\nu}^{n} \lambda_i(\xi) \leqq \Pi_{i=\nu}^{n} a_i,\ \nu = 1, \ldots, n\}$. We prove that $X = RcoE$. We divide the proof into two steps.

Step 1. $RcoE \subset X$. Observe that $E \subset X$ and, from Proposition 2.6, the functions $\xi \to \Pi_{i=\nu}^{n} \lambda_i(\xi)$, $\nu = 1, \ldots, n$ are polyconvex (and hence rank one convex). Therefore we deduce that X is polyconvex and hence

$$RcoE \subset PcoE \subset X.$$

Step 2. $X \subset RcoE$. Let $\xi \in X$; we will prove that $\xi \in RcoE$. Since the functions $\xi \to \lambda_i(\xi)$ are invariant by orthogonal transformations, we can assume, without loss of generality, that

$$\xi = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix},$$

with $0 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_n$ and $\Pi_{i=\nu}^n x_i \leqq \Pi_{i=\nu}^n a_i$, $\nu = 1, \ldots, n$.

We show the result by induction. We start with the proof in dimension $n = 2$. Note that the proof of this case is simpler than the one in [**6**].

(i) $n = 2$. We write

$$\xi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 & \lambda \\ 0 & x_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 & -\lambda \\ 0 & x_2 \end{pmatrix} = \frac{1}{2} A_+ + \frac{1}{2} A_- \tag{3.13}$$

(observe that *rank* $\{A_+ - A_-\} \leqq 1\}$ and we choose

$$\lambda^2 = \frac{(a_2^2 - x_2^2)(a_2^2 - x_1^2)}{a_2^2}.$$

Note that the right-hand side is positive by assumption $(0 \leqq x_1 \leqq x_2 \leqq a_2)$. This leads to

$$\lambda_1(A_\pm) = \frac{x_1 x_2}{a_2}, \quad \lambda_2(A_\pm) = a_2.$$

Therefore $\exists\, O_\pm, O'_\pm \in O(2)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} \dfrac{x_1 x_2}{a_2} & 0 \\ 0 & a_2 \end{pmatrix}.$$

However, we have

$$\begin{pmatrix} \dfrac{x_1 x_2}{a_2} & 0 \\ 0 & a_2 \end{pmatrix} = \left(\frac{1}{2} + \frac{x_1 x_2}{2a_1 a_2}\right)\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \left(\frac{1}{2} - \frac{x_1 x_2}{2a_1 a_2}\right)\begin{pmatrix} -a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

and hence

$$\begin{pmatrix} \dfrac{x_1 x_2}{a_2} & 0 \\ 0 & a_2 \end{pmatrix} \in R_1 coE \subset RcoE.$$

Since $RcoE$ is invariant up to orthogonal transformations, we deduce that

$$A_\pm = \begin{pmatrix} x_1 & \pm\lambda \\ 0 & x_2 \end{pmatrix} \in RcoE. \tag{3.14}$$

Finally, combining (3.13) and (3.14), we obtain that

$$\xi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \in RcoE,$$

which is the claimed result.

(ii) $n > 2$. We divide this case into four parts.

Part 1. $x_2 \leqq a_2$. We write

$$\xi = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 & \lambda & & \\ 0 & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 & -\lambda & & \\ 0 & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$

$$= \frac{1}{2}A_+ + \frac{1}{2}A_- \tag{3.15}$$

(observe that *rank* $\{A_+ - A_-\} \leqq 1$) and we define λ by:

$$\lambda^2 = \frac{(a_2^2 - x_2^2)(a_2^2 - x_1^2)}{a_2^2}.$$

Note that the right-hand side is positive by assumption ($0 \leqq x_1 \leqq x_2 \leqq a_2$). The choice of λ (as in the case $n = 2$) leads to the existence of $O_\pm, O'_\pm \in O(n)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} a_2 & & & & \\ & \dfrac{x_1 x_2}{a_2} & & & \\ & & x_3 & & \\ & & & \ddots & \\ & & & & x_n \end{pmatrix}.$$

We apply the hypothesis of induction to

$$\left\{ y_1 = \frac{x_1 x_2}{a_2},\ y_2 = x_3, \ldots, y_{n-1} = x_n \right\}$$

and to

$$\{b_1 = a_1,\ b_2 = a_3, \ldots, b_{n-1} = a_n\}.$$

Note that, since $x_2 \leqq a_2$, then $0 \leqq y_1 \leqq \ldots \leqq y_{n-1}$.

We have to show that $\Pi_{i=v}^{n-1} y_i \leqq \Pi_{i=v}^{n-1} b_i$, $v = 1, \ldots, n-1$.

(1) By assumption, if $v \geqq 2$, we have $\Pi_{i=v}^{n-1} y_i = \Pi_{i=v+1}^{n} x_i \leqq \Pi_{i=v+1}^{n} a_i = \Pi_{i=v}^{n-1} b_i$.

(2) If $v = 1$, we have

$$\prod_{i=1}^{n-1} y_i = \frac{x_1 x_2}{a_2} \prod_{i=3}^{n} x_i = \frac{1}{a_2} \prod_{i=1}^{n} x_i \leqq \frac{1}{a_2} \prod_{i=1}^{n} a_i = a_1 \prod_{i=3}^{n} a_i = \prod_{i=1}^{n-1} b_i.$$

Therefore we can deduce that (by hypothesis of induction)

$$\begin{pmatrix} a_2 & & & & \\ & \dfrac{x_1 x_2}{a_2} & & & \\ & & x_3 & & \\ & & & \ddots & \\ & & & & x_n \end{pmatrix} \in RcoE.$$

Since $RcoE$ is invariant up to orthogonal transformations, we obtain

$$A_{\pm} = \begin{pmatrix} x_1 & \pm\lambda & & \\ 0 & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \in RcoE \tag{3.16}$$

and therefore, combining (3.15) and (3.16), we get

$$\xi \in RcoE,$$

which is the claimed result.

Part 2. $x_{n-1} \geqq a_{n-1}$. We write (as in Part 1, but interchanging the role of (x_n, x_{n-1}) and (x_1, x_2))

$$\xi = \begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & \\ & & & x_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & \lambda \\ & & 0 & x_n \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & -\lambda \\ & & 0 & x_n \end{pmatrix}$$

$$= \frac{1}{2} A_+ + \frac{1}{2} A_- \tag{3.17}$$

(observe that $rank\,\{A_+ - A_-\} \leqq 1$) and we choose λ to be:

$$\lambda^2 = \frac{(x_n^2 - a_{n-1}^2)(x_{n-1}^2 - a_{n-1}^2)}{a_{n-1}^2}.$$

Note that the right-hand side is positive by assumption ($a_{n-1} \leqq x_{n-1} \leqq x_n$). As above, the choice of λ leads to the existence of $O_{\pm}, O'_{\pm} \in O(n)$ such that

$$O_{\pm} A_{\pm} O'_{\pm} = \begin{pmatrix} x_1 & & & & & \\ & x_2 & & & & \\ & & \ddots & & & \\ & & & x_{n-2} & & \\ & & & & \dfrac{x_{n-1} x_n}{a_{n-1}} & \\ & & & & & a_{n-1} \end{pmatrix}.$$

We apply the hypothesis of induction to

$$\left\{ y_1 = x_1, \ldots, y_{n-2} = x_{n-2}, \; y_{n-1} = \frac{x_{n-1} x_n}{a_{n-1}} \right\}$$

and to

$$\{b_1 = a_1, \ldots, b_{n-2} = a_{n-2}, b_{n-1} = a_n\}.$$

Note that, since $a_{n-1} \leqq x_{n-1}$, then $0 \leqq y_1 \leqq \ldots \leqq y_{n-1}$.

We verify the hypothesis of induction.

(1) First, observe that $y_{n-1} \leqq b_{n-1}$ because $x_{n-1}x_n \leqq a_{n-1}a_n$.

(2) If $1 \leqq v \leqq n-1$, then

$$\prod_{i=v}^{n-1} y_i = \prod_{i=v}^{n-2} y_i \cdot y_{n-1} = \prod_{i=v}^{n-2} x_i \frac{x_{n-1}x_n}{a_{n-1}} = \frac{1}{a_{n-1}} \prod_{i=v}^{n} x_i$$
$$\leqq \frac{1}{a_{n-1}} \prod_{i=v}^{n} a_i = \prod_{i=v}^{n-2} a_i \cdot a_n = \prod_{i=v}^{n-1} b_i.$$

Therefore (by hypothesis of induction)

$$\begin{pmatrix} x_1 & & & & & \\ & x_2 & & & & \\ & & \ddots & & & \\ & & & x_{n-2} & & \\ & & & & \dfrac{x_{n-1}x_n}{a_{n-1}} & \\ & & & & & a_{n-1} \end{pmatrix} \in RcoE$$

and, since $RcoE$ is invariant up orthogonal transformations, we obtain that $A_\pm \in RcoE$, which combined with (3.17) leads to the claimed result,

$$\xi \in RcoE.$$

Part 3. $a_2 \leqq x_2 \leqq \ldots \leqq x_{n-1} \leqq a_{n-1}$. Note that this case occurs only if $n \geqq 4$. We first observe that we can therefore find $k \in \{2, \ldots, n-2\}$ such that

$$a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}. \tag{3.18}$$

Hence we can write

$$\xi = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} = \frac{1}{2} A_+ + \frac{1}{2} A_-$$

$$= \frac{1}{2} \begin{pmatrix} x_1 & & & & \\ & \ddots & & & \\ & & x_k & \lambda & & \\ & & 0 & x_{k+1} & & \\ & & & & \ddots & \\ & & & & & x_n \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 & & & & \\ & \ddots & & & \\ & & x_k & -\lambda & & \\ & & 0 & x_{k+1} & & \\ & & & & \ddots & \\ & & & & & x_n \end{pmatrix} \tag{3.19}$$

(observe that $rank\,\{A_+ - A_-\} \leqq 1$) where λ is given by

$$\lambda^2 = \frac{(b^2 - x_k^2)(b^2 - x_{k+1}^2)}{b^2}, \tag{3.20}$$

where $b = a_k$ (Part 3.1) or $b = a_{k+1}$ (Part 3.2). Note that, from the above assumptions (3.18), the right-hand side is positive in both cases.

Part 3.1. $\begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}, \\ x_k x_{k+1} \prod_{i=v+1}^{n} x_i \leqq a_k \prod_{i=v}^{n} a_i, \quad v = k+2, \ldots, n \end{cases}$

(with the convention $\Pi_{i=n+1}^{n} x_i = 1$).

Part 3.2. $\begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}, \\ \prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leqq \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i, \quad \mu = 1, \ldots, k-1. \end{cases}$

Before proceeding with the study of the above cases, we show that Part 3.1 and Part 3.2 cover all possibilities. In fact, if $0 \leqq x_1 \leqq \ldots \leqq x_n$ and if $\Pi_{i=v}^{n} x_i \leqq \Pi_{i=v}^{n} a_i$, $v = 1, \ldots, n$, then at least one of the following sets of inequalities holds:

$$x_k x_{k+1} \prod_{i=v+1}^{n} x_i \leqq a_k \prod_{i=v}^{n} a_i, \quad v = k+2, \ldots, n;$$

$$\prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leqq \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i, \quad \mu = 1, \ldots, k-1.$$

We proceed by contradiction and we assume that there exist $v \in \{k+2, \ldots, n\}$ and $\mu \in \{1, \ldots, k-1\}$ such that

$$x_k x_{k+1} \prod_{i=v+1}^{n} x_i > a_k \prod_{i=v}^{n} a_i,$$

$$\prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i > \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i.$$

Multiplying together the two inequalities and using the assumptions, we deduce that

$$\prod_{i=\mu}^{n} a_i \cdot \prod_{i=v+1}^{n} a_i \geqq \prod_{i=\mu}^{n} x_i \cdot \prod_{i=v+1}^{n} x_i > a_k \prod_{i=v}^{n} a_i \cdot \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i,$$

i.e.

$$a_\mu \prod_{i=k+1}^{n} a_i \cdot \prod_{i=v+1}^{n} a_i > a_k \prod_{i=v}^{n} a_i \cdot \prod_{i=k+2}^{n} a_i,$$

therefore

$$a_\mu a_{k+1} > a_k a_v.$$

However, $\mu \in \{1, \ldots, k-1\}$, hence $a_\mu \leqq a_k$ and $v \in \{k+2, \ldots, n\}$, therefore $a_v \geqq a_{k+1}$. We therefore get

$$a_k a_{k+1} \geqq a_\mu a_{k+1} > a_k a_v \geqq a_k a_{k+1},$$

which is the claimed contradiction. In conclusion, Part 3.1 and Part 3.2 cover all possibilities. We now study these two cases separately.

$$\text{Part 3.1}\quad \begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}, \\ x_k x_{k+1} \prod_{i=\nu+1}^{n} x_i \leqq a_k \prod_{i=\nu}^{n} a_i \quad \nu = k+2, \dots, n \end{cases}$$

(with the convention $\Pi_{i=n+1}^{n} x_i = 1$). We choose here $b = a_k$ in (3.19) and (3.20); therefore we can find $O_\pm, O'_\pm \in O(n)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} x_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x_{k-1} & & & & & \\ & & & \dfrac{x_k x_{k+1}}{a_k} & & & & \\ & & & & a_k & & & \\ & & & & & x_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & x_n \end{pmatrix},$$

where we recall that

$$A_\pm = \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_k & \pm\lambda & & \\ & & 0 & x_{k+1} & & \\ & & & & \ddots & \\ & & & & & x_n \end{pmatrix}.$$

We apply the hypothesis of induction to

$$\left\{ y_1 = x_1, \dots, y_{k-1} = x_{k-1},\ y_k = \frac{x_k x_{k+1}}{a_k},\ y_{k+1} = x_{k+2}, \dots, y_{n-1} = x_n \right\}$$

and to

$$\{ b_1 = a_1, \dots, b_{k-1} = a_{k-1},\ b_k = a_{k+1}, \dots, b_{n-1} = a_n \}.$$

Observe that, since $a_k \leqq x_k$, then $0 \leqq y_1 \leqq \dots \leqq y_{k-1} \leqq y_k$. On the contrary, *a priori*, we cannot compare y_k to $y_{k+1} \leqq \dots \leqq y_{n-1}$. We next verify the hypothesis of induction.

(1) We must show that $x_n = y_{n-1} \leqq b_{n-1} = a_n$ and $y_k \leqq b_{n-1} = a_n$.
The first inequality is verified by assumption and the second is also verified by the assumption of Part 3.1 with $\nu = n$. The assumption $\xi \in X$ and that of Part 3.1 again ensure that

(2) if $n-1 \geqq \nu \geqq k+1$,

$$\begin{cases} \prod_{i=\nu}^{n-1} y_i = \prod_{i=\nu+1}^{n} x_i \leqq \prod_{i=\nu+1}^{n} a_i = \prod_{i=\nu}^{n-1} b_i, \\ y_k \prod_{i=\nu+1}^{n-1} y_i = \dfrac{x_k\, x_{k+1}}{a_k} \cdot \prod_{i=\nu+2}^{n} x_i \leqq \prod_{i=\nu+1}^{n} a_i = \prod_{i=\nu}^{n-1} b_i. \end{cases}$$

(3) If $k \geqq \nu \geqq 1$,

$$\prod_{i=\nu}^{n-1} y_i = \prod_{i=\nu}^{k} y_i \cdot \prod_{i=k+1}^{n-1} y_i = \frac{1}{a_k} \prod_{i=\nu}^{n} x_i \leqq \frac{1}{a_k} \prod_{i=\nu}^{n} a_i$$
$$= \prod_{i=\nu}^{k-1} a_i \cdot \prod_{i=k+1}^{n} a_i = \prod_{i=\nu}^{n-1} b_i.$$

Therefore we can apply the assumption of induction and deduce that

$$\begin{bmatrix} x_1 &&&&&&& \\ & \ddots &&&&&& \\ && x_{k-1} &&&&& \\ &&& \dfrac{x_k x_{k+1}}{a_k} &&&& \\ &&&& a_k &&& \\ &&&&& x_{k+2} && \\ &&&&&& \ddots & \\ &&&&&&& x_n \end{bmatrix} \in RcoE.$$

As above, we get that

$$A_\pm \in RcoE \tag{3.21}$$

and, finally, combining (3.19) and (3.21), we obtain the claimed result:

$$\xi \in RcoE.$$

Part 3.2 $\begin{cases} a_k \leqq x_k \leqq x_{k+1} \leqq a_{k+1}, \\ \prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leqq \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i \quad \mu = 1, \ldots, k-1. \end{cases}$

We choose here $b = a_{k+1}$ in (3.19) and (3.20); therefore we can find $O_\pm, O'_\pm \in O(n)$ such that

$$O_\pm A_\pm O'_\pm = \begin{bmatrix} x_1 &&&&&&& \\ & \ddots &&&&&& \\ && x_{k-1} &&&&& \\ &&& \dfrac{x_k x_{k+1}}{a_{k+1}} &&&& \\ &&&& a_{k+1} &&& \\ &&&&& x_{k+2} && \\ &&&&&& \ddots & \\ &&&&&&& x_n \end{bmatrix}.$$

We have to prove the hypothesis of induction for

$$\left\{ y_1 = x_1, \ldots, y_{k-1} = x_{k-1}, \; y_k = \frac{x_k x_{k+1}}{a_{k+1}}, \; y_{k+1} = x_{k+2}, \ldots, y_{n-1} = x_n \right\}$$

and for

$$\{b_1 = a_1, \ldots, b_k = a_k,\ b_{k+1} = a_{k+2}, \ldots, b_{n-1} = a_n\}.$$

Observe that, since $x_{k+1} \leqq a_{k+1}$, then $y_k \leqq y_{k+1} \leqq \ldots \leqq y_{n-1}$. On the contrary, *a priori*, we cannot compare y_k to $y_1 \leqq \ldots \leqq y_{k-1}$. We verify the hypothesis of induction. From the assumption $\xi \in X$ and from that of Part 3.2 we can write:

(1) if $v \geqq k+1$, then $\Pi_{i=v}^{n-1} y_i = \Pi_{i=v+1}^{n} x_i \leqq \Pi_{i=v+1}^{n} a_i = \Pi_{i=v}^{n-1} b_i$;

(2) if $v = k$, then

$$\begin{cases} \displaystyle\prod_{i=k}^{n-1} y_i = \frac{1}{a_{k+1}} \prod_{i=k}^{n} x_i \leqq \frac{1}{a_{k+1}} \prod_{i=k}^{n} a_i = a_k \prod_{i=k+2}^{n} a_i = b_k \prod_{i=k+1}^{n-1} b_i = \prod_{i=k}^{n-1} b_i, \\ \displaystyle y_{k-1} \prod_{i=k+1}^{n-1} y_i = x_{k-1} \prod_{i=k+2}^{n} x_i \leqq a_k \prod_{i=k+2}^{n} a_i = \prod_{i=k}^{n-1} b_i; \end{cases}$$

(3) if $k-1 \geqq v \geqq 1$, then

$$\begin{cases} \displaystyle\prod_{i=v}^{n-1} y_i = \prod_{i=v}^{k-1} x_i \frac{x_k x_{k+1}}{a_{k+1}} \cdot \prod_{i=k+2}^{n} x_i = \frac{1}{a_{k+1}} \cdot \prod_{i=v}^{n} x_i \\ \displaystyle\qquad \leqq \frac{1}{a_{k+1}} \cdot \prod_{i=v}^{n} a_i = \prod_{i=v}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i = \prod_{i=v}^{n-1} b_i, \\ \displaystyle\prod_{i=v-1}^{k-1} y_i \cdot \prod_{i=k+1}^{n-1} y_i = \prod_{i=v-1}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \\ \displaystyle\qquad \leqq \prod_{i=v}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i = \prod_{i=v}^{k} b_i \cdot \prod_{i=k+1}^{n-1} b_i = \prod_{i=v}^{n-1} b_i. \end{cases}$$

We can apply the hypothesis of induction and deduce that

$$\begin{pmatrix} x_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x_{k-1} & & & & & \\ & & & \dfrac{x_k x_{k+1}}{a_{k+1}} & & & & \\ & & & & a_{k+1} & & & \\ & & & & & x_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & x_n \end{pmatrix} \in RcoE.$$

Since $RcoE$ is invariant up the orthogonal transformations, we can obtain that

$$A_{\pm} \in RcoE. \tag{3.22}$$

Finally, combining (3.19) and (3.22), we can write $\xi \in RcoE$. In conclusion, we have obtained the claimed result: $X \subset RcoE$. □

Proof of Theorem 3.1(iii). Let $Y = \{\xi \in \mathbf{R}^{n\times n} : \Pi_{i=\nu}^{n} \lambda_i(\xi) < \Pi_{i=\nu}^{n} a_i,\ \nu = 1, \ldots, n\}$. We show that *int RcoE* $= Y$. We divide the proof into two steps.

Step 1. $Y \subset$ *int RcoE*, since by continuity Y is open and, by (ii), $Y \subset RcoE$.

Step 2. *int RcoE* $\subset Y$. So let $\xi \in$ *int RcoE*; we can therefore find ε sufficiently small so that $B_\varepsilon(\xi) \subset RcoE$ (where $B_\varepsilon(\xi)$ denotes the ball centred at ξ and of radius ε). Let R, R' be orthogonal matrices so that

$$\xi = R \begin{pmatrix} \lambda_1(\xi) & & & \\ & \lambda_2(\xi) & & \\ & & \ddots & \\ & & & \lambda_n(\xi) \end{pmatrix} R'.$$

Define

$$\eta = R \begin{pmatrix} \lambda_1(\xi) & & & \\ & \lambda_2(\xi) & & \\ & & \ddots & \\ & & & \lambda_n(\xi) + \dfrac{\varepsilon}{2} \end{pmatrix} R'.$$

Since $|\eta - \xi| = (\varepsilon/2) < \varepsilon$, then $\eta \in RcoE$. We then get

$$\lambda_n(\xi) < \lambda_n(\eta) \leqq a_n.$$

Assume that $\lambda_\nu(\xi) \neq 0$ for every ν; we then get for $\nu = 1, \ldots, n$ and with the convention $\Pi_{i=n+1}^{n} \lambda_i(\xi) = 1$,

$$\prod_{i=\nu}^{n} \lambda_i(\xi) = \prod_{i=\nu}^{n-1} \lambda_i(\xi) \cdot \lambda_n(\xi) < \prod_{i=\nu}^{n-1} \lambda_i(\eta) \cdot \lambda_n(\eta) \leqq \prod_{i=\nu}^{n-1} a_i \cdot a_n$$

which implies that $\xi \in Y$.

Finally, if $\exists\, \bar\nu \in \{1, \ldots, n\}$ such that $\lambda_{\bar\nu}(\xi) = 0$, and $\lambda_{\bar\nu+1}(\xi) > 0$, then the same argument as above is valid for $\nu = \bar\nu + 1, \ldots, n$ and is trivial if $\nu = 1, \ldots, \bar\nu$. We therefore also get that $\xi \in Y$. □

REMARK 3.3. We should draw the attention to the following facts.

(1) We have privileged proofs that are as similar as possible for *coE* and *RcoE*, replacing Σ by Π. We did not succeed in doing this for the case $n = 2$.

(2) The above choice forced us, in the convex case, to consider nondiagonal (but symmetric) decompositions of the matrix ξ. If one insists in keeping decompositions with only diagonal matrices, then this is possible and is indeed achieved here for $n = 2$.

Acknowledgments

We would like to thank P. Marcellini for important discussions on this paper. Part of this research was financially supported by Fonds National Suisse (21-50472.97).

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(*Issued* 18 *December* 1998)