SOME REMARKS ON QUASI-UNIFORM SPACES

by HANS-PETER A. KÜNZI

(Received 7 July, 1988)

Introduction. A topological space is called a *uqu space* [**10**] if it admits a unique quasi-uniformity. Answering a question [**2**, Problem B, p. 45] of P. Fletcher and W. F. Lindgren in the affirmative we show in [**8**] that a topological space X is a *uqu* space if and only if every interior-preserving open collection of X is finite. (Recall that a collection $\mathscr{C}$ of open sets of a topological space is called *interior-preserving* if the intersection of an arbitrary subcollection of $\mathscr{C}$ is open (see e.g. [**2**, p. 29]).) The main step in the proof of this result in [**8**] shows that a topological space in which each interior-preserving open collection is finite is a transitive space. (A topological space is called *transitive* (see e.g. [**2**, p. 130]) if its fine quasi-uniformity has a base consisting of transitive entourages.) In the first section of this note we prove that each hereditarily compact space is transitive. The result of [**8**] mentioned above is an immediate consequence of this fact, because, obviously, a topological space in which each interior-preserving open collection is finite is hereditarily compact; see e.g. [**2**, Theorem 2.36]. Our method of proof also shows that a space is transitive if its fine quasi-uniformity is quasi-pseudo-metrizable. We use this result to prove that the fine quasi-uniformity of a T_1 space X is quasi-metrizable if and only if X is a quasi-metrizable space containing only finitely many nonisolated points. This result should be compared with Proposition 2.34 of [**2**], which says that the fine quasi-uniformity of a *regular* T_1 space has a countable base if and only if it is a *metrizable* space with only finitely many nonisolated points (see e.g. [**11**] for related results on uniformities). Another by-product of our investigations is the result that each topological space with a countable network is transitive.

Recently there has been some interest in the construction of *uqu* spaces (compare [**4**]). In this connection our observation that the product of finitely many *uqu* spaces is a *uqu* space may be useful. We prove this result in the second section of this note. Topological spaces admitting a unique quasi-proximity are called *uqp spaces* in [**10**]. Each hereditarily compact space is a *uqp* space [**10**, Theorem 2.4]. Answering a question [**2**, Problem B, p. 45] of P. Fletcher and W. F. Lindgren in the negative, we show in [**6**] that a (first-countable) *uqp* space need not be hereditarily compact. In fact, it is proved in [**9**, Proposition 4] that a *uqp* space X is hereditarily compact if and only if each ultrafilter on X has an irreducible convergence set. (Recall that a nonempty subset A of a topological space is called *irreducible* (see e.g. [**9**, p. 238]) if each pair of nonempty A-open subsets has a nonempty intersection.) In particular each *uqp* Hausdorff space is hereditarily compact (and, thus, finite) [**2**, Theorem 2.36]. It seems to be unknown whether the *uqp* T_1 spaces can be characterized in a similar way. The last result contained in this paper shows that, at least, each *uqp* T_1 space with *countable pseudo-character* is hereditarily compact (and, thus, countable). (Recall that a topological space X has *countable pseudo-character* if each point of X is a G_δ-set in X.)

Glasgow Math. J. **31** (1989) 309–320.

Throughout this note we use the terminology of [**2**]. In particular all separation axioms used are explicitly mentioned. By $\mathbb{N}$ we denote the set of positive integers.

1. Transitive spaces. It does not seem easy to construct nontransitive topological spaces. In fact, essentially, the only nontransitive space known is the so-called Kofner plane (see [**2**, p. 147]). For readers not familiar with this space, we will give an "analytic" variant of Kofner's construction at the end of this section. Maybe this variant, although closely related to Kofner's original idea, will turn out to be helpful in future research.

On the other hand not too many classes of topological spaces are known to contain only transitive members (see [**2**, chapter 6]). As we are going to prove in this section, the class of hereditarily compact spaces is of this kind. We begin with some auxiliary results that seem to be of independent interest.

First it seems useful to analyze the proof given in [**8**] that a topological space in which each interior-preserving open collection is finite is a transitive space. To this end we have to formulate two facts explicitly that are contained (implicitly) in [**8**].

To begin we characterize the topological spaces that have the property that the class of the quasi-uniformities inducing the Pervin quasi-proximity contains a quasi-pseudo-metrizable member. This class of rather peculiar topological spaces will turn out to be helpful in the following investigations. (In this connection let us mention that the Pervin quasi-uniformity of a topological space X is quasi-pseudo-metrizable if and only if the topology of X is countable [**3**, Proposition 1].)

LEMMA 1.1. *A topological space X has a σ-interior-preserving topology if and only if there exists a quasi-uniformity with a countable base on X that induces the Pervin quasi-proximity for X.*

Proof. Let X be a topological space with a σ-interior-preserving topology $\mathcal{T}$. Then $\mathcal{T} = \bigcup \{\mathcal{T}_n : n \in \mathbb{N}\}$ where we can assume that $\mathcal{T}_n$ is interior preserving whenever $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and each $x \in X$ set $S_n(x) = \bigcap \{G : x \in G \in \mathcal{T}_n\}$. (We use the convention that $\bigcap \varnothing = X$.) Moreover for each $n \in \mathbb{N}$ set $S_n = \bigcup \{\{x\} \times S_n(x) : x \in X\}$. Suppose that $G \in \mathcal{T}_n$ for some $n \in \mathbb{N}$. Then $S_n \subseteq [G \times G] \cup [(X \setminus G) \times X]$. Thus the Pervin quasi-uniformity for X is coarser than the (compatible) quasi-uniformity $\mathcal{U}$ generated by the subbase $\{S_n : n \in \mathbb{N}\}$ on X. We conclude that $\mathcal{U}$ induces the Pervin quasi-proximity for X, because the Pervin quasi-proximity is the finest compatible quasi-proximity on X [**2**, §2.11]. Hence $\mathcal{U}$ is a quasi-uniformity on X with a countable base that induces the Pervin quasi-proximity for X.

In order to prove the converse we assume that $(X, \mathcal{T})$ is a topological space, the Pervin quasi-proximity of which is induced by a quasi-uniformity $\mathcal{U}$ on X with a countable base $\{U_n : n \in \mathbb{N}\}$. Since the Pervin quasi-uniformity $\mathcal{P}(\mathcal{T})$ for X is totally bounded, we have that $\mathcal{P}(\mathcal{T}) \subseteq \mathcal{U}$ by Theorem 1.33 of [**2**]. Therefore for each $G \in \mathcal{T}$ there is an $n \in \mathbb{N}$ such that $U_n \subseteq [G \times G] \cup [(X \setminus G) \times X]$. Thus, for each $G \in \mathcal{T}$ there is an $n \in \mathbb{N}$ such that $U_n(G) = G$. We conclude that $\mathcal{T} = \bigcup \{\mathcal{T}_n : n \in \mathbb{N}\}$ where $\mathcal{T}_n = \{G \in \mathcal{T} : U_n(G) = G\}$ is interior preserving for each $n \in \mathbb{N}$ (compare [**8**, p. 41]).

Another rather technical result implicitly used in [**8**] is the following. (Recall that a topological space X is called *quasi-sober* (see e.g. [**9**, p. 238]) if the only irreducible closed subsets of X are the closures of singletons.)

LEMMA 1.2. *Let X be a topological space with a σ-interior-preserving base $\mathscr{B}$ such that no strictly decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of open subsets of X has an open intersection. Then X is quasi-sober.*

Proof. By our assumption we have that $\mathscr{B} = \bigcup\{\mathscr{B}_n : n \in \mathbb{N}\}$ where we can assume that $\mathscr{B}_n$ is interior preserving and that $\mathscr{B}_n \subseteq \mathscr{B}_{n+1}$ for each $n \in \mathbb{N}$. Let F be an irreducible closed subset of X and for each $n \in \mathbb{N}$, set $\mathscr{S}_n = \{G \in \mathscr{B}_n : G \cap F \neq \varnothing\}$. Let $n \in \mathbb{N}$. Since X has no strictly decreasing sequence of open sets with an open intersection and since $\mathscr{B}_n$ is interior preserving, there exists a finite subcollection $\mathscr{P}_n$ of $\mathscr{S}_n$ such that $\bigcap \mathscr{P}_n = \bigcap \mathscr{S}_n$. Therefore $\bigcap \mathscr{S}_n \cap F \neq \varnothing$, because F is irreducible. Consider $\{(X\backslash F) \cup (\bigcap \mathscr{S}_n) : n \in \mathbb{N}\}$, an open family. Since X has no strictly decreasing sequence of open sets with an open intersection, we conclude that there is an $x \in \bigcap\{\bigcap \mathscr{S}_n : n \in \mathbb{N}\} \cap F$. Thus $F = \overline{\{x\}}$. We have shown that X is quasi-sober.

REMARK 1.3. We can now easily understand the basic ideas of the proof given in [**8**] that the fine quasi-uniformity of a topological space in which each interior-preserving open collection is finite has a transitive base.

To this end let $(X, \mathscr{T})$ be a topological space in which each interior-preserving open collection is finite and let $\mathscr{S}$ be an arbitrary quasi-pseudo-metrizable topology on X that is coarser than $\mathscr{T}$. In order to prove that $(X, \mathscr{T})$ is transitive, it clearly suffices to show that $(X, \mathscr{S})$ is transitive. For in this case each entourage of the fine quasi-uniformity of the space $(X, \mathscr{T})$ belongs to a transitive quasi-uniformity of a topology on X that is coarser than $\mathscr{T}$ and, hence, contains a transitive $\mathscr{T}$-entourage.

It remains to show that $(X, \mathscr{S})$ is transitive. In [**8**] this is done by showing that $(X, \mathscr{S})$ is a *uqu* space. (Of course, then $(X, \mathscr{S})$ is transitive, because the transitive Pervin quasi-uniformity is the unique quasi-uniformity that $(X, \mathscr{S})$ admits.) The details are as follows. First observe that since $\mathscr{S}$ is coarser than $\mathscr{T}$, each interior-preserving open collection of $(X, \mathscr{S})$ is finite, too. Then note that, because for the hereditarily compact (hence *uqp*) space $(X, \mathscr{S})$ the Pervin quasi-uniformity $\mathscr{P}(\mathscr{S})$ is the coarsest compatible quasi-uniformity, the topology $\mathscr{S}$ is σ-interior-preserving according to Lemma 1.1. Hence $(X, \mathscr{S})$ is quasi-sober by Lemma 1.2. Therefore $(X, \mathscr{S})$ is a *uqu* space, because a hereditarily compact and quasi-sober space is a *uqu* space [**9**, Proposition 3(a)].

Let us observe that in the last step of the proof outlined above, instead of citing Lemma 1.2 and Proposition 3(a) of [**9**], we could also use Corollary 1.6 below in order to show that $(X, \mathscr{S})$ is transitive. (Obviously $(X, \mathscr{S})$ has a countable base, since it is a topological space with a σ-interior-preserving topology in which each interior-preserving open collection is finite.)

Now we are ready to show that a variant of the argument given above proves that the fine quasi-uniformity of every hereditarily compact space has a transitive base.

Recall that a binary relation V on a topological space X is called a (n open) *neighbornet* of X (see e.g. [**2**, p. 4]), if $V(x) = \{y \in X : (x, y) \in V\}$ is a (n open) neighborhood at x for each $x \in X$. We begin with a useful technical lemma.

LEMMA 1.4. *Let X be a topological space and let V be a neighbornet of X. Assume that X has a sequence $(\mathscr{B}_n)_{n\in\mathbb{N}}$ of interior-preserving open collections and a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ of closure-preserving collections so that for each $x \in X$ there exist $z \in X$, $n(x) \in \mathbb{N}$, $m(x) \in \mathbb{N}$, $G_x \in \mathscr{B}_{n(x)}$ and $H_x \in \mathscr{H}_{m(x)}$ such that $x \in G_x \subseteq V(z)$ and $x \in H_x \subseteq V^{-1}(z)$. Then V^3 contains a transitive neighbornet of X.*

Proof. As usual we use the convention that $\bigcap \varnothing = X$. For each $n \in \mathbb{N}$ and each $x \in X$ set $T_n(x) = \bigcap \{B : x \in B \in \mathscr{B}_n\}$, $H_n(x) = X \setminus \bigcup \{\bar{H} : H \in \mathscr{H}_n$ and $x \notin \bar{H}\}$ and $S_n(x) = \bigcap \{T_k(x) \cap H_k(x) : k = 1, \ldots, n\}$. Note that $H_n^{-1}(x) = \bigcap \{\bar{H} : H \in \mathscr{H}_n$ and $x \in \bar{H}\}$ whenever $x \in X$ and $n \in \mathbb{N}$ and that $S_n = \bigcup \{\{x\} \times S_n(x) : x \in X\}$ is a transitive neighbornet of X whenever $n \in \mathbb{N}$. For each $x \in X$ set $h(x) = \max\{n(x), m(x)\}$. Let $P = \bigcup \{S_{h(x)}^{-1}(x) \times S_{h(x)}(x) : x \in X\}$. Since $(S_n)_{n\in\mathbb{N}}$ is a decreasing sequence of transitive neighbornets of X, P is a transitive neighbornet of X. (A similar idea is used in the proof of Theorem 5 of **[5]**.) Let $x \in X$. Then $S_{h(x)}(x) \subseteq T_{n(x)}(x) \subseteq G_x \subseteq V(z)$ and $S_{h(x)}^{-1}(x) \subseteq H_{m(x)}^{-1}(x) \subseteq \overline{H_x} \subseteq V^{-2}(z)$ by our assumption. Thus $P \subseteq V^3$.

Next we state several corollaries to Lemma 1.4 that seem interesting enough to be included here, although we will not make any use of them in this paper.

COROLLARY 1.5. *Let V be an open neighbornet of a topological space X. If there exists a countable cover $\{A_n : n \in \mathbb{N}\}$ of X such that $\bigcup \{A_n \times A_n : n \in \mathbb{N}\} \subseteq V$, then V^3 contains a transitive neighbornet of X.*

Proof. For each $n \in \mathbb{N}$ such that $A_n \neq \varnothing$ choose $h_n \in A_n$ and set $\mathscr{B}_n = \{V(h_n)\}$ and $\mathscr{H}_n = \{V^{-1}(h_n)\}$. Let $x \in X$. Then there exists $k(x) \in \mathbb{N}$ such that $x \in A_{k(x)}$. Thus $x \in V(h_{k(x)})$ and $x \in V^{-1}(h_{k(x)})$. Set $n(x) = m(x) = k(x)$, $z = h_{k(x)}$, $G_x = V(h_{k(x)})$ and $H_x = V^{-1}(h_{k(x)})$. We conclude that all conditions of Lemma 1.4 are satisfied. Hence V^3 contains a transitive neighbornet of X.

Recall that a collection $\mathscr{B}$ of subsets of a topological space X is called a *network* for X if for each point x of X and each neighborhood G of x there is a $C \in \mathscr{B}$ such that $x \in C \subseteq G$.

COROLLARY 1.6. *Let X be a topological space with a countable network and let V be a neighbornet of X. Then V^3 contains a transitive neighbornet of X. In particular, each space with a countable network is transitive.*

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a countable network for X. Set $W = \bigcup \{\{x\} \times (\text{int } V(x)) : x \in X\}$. For each $n \in \mathbb{N}$ set $A_n = \{x \in X : x \in B_n \subseteq W(x)\}$. Then $\{A_n : n \in \mathbb{N}\}$ is a cover of X such that $\bigcup \{A_n \times A_n : n \in \mathbb{N}\} \subseteq W$. The assertion follows from Corollary 1.5.

COROLLARY 1.7. *A preorthocompact space with a countable network is orthocompact.*

Proof. The assertion is an immediate consequence of Corollary 1.6 (see **[2**, p. 100 and p. 104] for the definition of the notion of (pre)orthocompactness).

COROLLARY 1.8(a). *Each topological space with a σ-interior-preserving base and a σ-locally finite network is transitive.*

(b) *Each topological space with a σ-locally finite base is transitive.*

Proof. (a) Let X be a topological space with a σ-interior-preserving base and with a σ-locally finite network $\mathcal{H}$. Thus $\mathcal{H} = \bigcup \{\mathcal{H}_n : n \in \mathbb{N}\}$ where we can assume that $\mathcal{H}_n$ is locally finite for each $n \in \mathbb{N}$. Let V be a neighbornet of X. For each $H \in \mathcal{H}$ set $H' = \{x \in H : x \in H \subseteq V(x)\}$ and for each $n \in \mathbb{N}$, set $\mathcal{H}'_n = \{H' : H \in \mathcal{H}_n\}$. Then each $\mathcal{H}'_n$ is closure preserving, because each $\mathcal{H}_n$ is locally finite. Let $x \in X$. There are an $m(x) \in \mathbb{N}$ and a $P \in \mathcal{H}_{m(x)}$ such that $x \in P \subseteq V(x)$, because $\mathcal{H}$ is a network for X. Then $P' \times P' \subseteq V$ and $x \in P' \subseteq V^{-1}(x)$. Set $z = x$. Hence the sequence $(\mathcal{H}'_n)_{n\in\mathbb{N}}$ satisfies the second part of the condition of Lemma 1.4. Since X has a σ-interior preserving base, we conclude by Lemma 1.4 that V^3 contains a transitive neighbornet of X. Hence X is transitive.

(b) The assertion follows immediately from part (a).

REMARK 1.9. It does not seem to be known whether a topological space with a σ-interior-preserving base is transitive (compare [**2**, Problem P, p. 155]).

Finally we will now use Lemma 1.4 to prove that hereditarily compact spaces are transitive.

PROPOSITION 1.10. *Each topological space with a σ-interior-preserving topology is transitive.*

Proof. Let X be a topological space with a σ-interior-preserving topology. Obviously X has a σ-interior-preserving base. Furthermore, of course, $\{\overline{\{x\}} : x \in X\}$ is a σ-closure-preserving collection in X. Let $x \in X$ and let V be a neighbornet of X. Note that $V^{-1}(x) \supseteq \overline{\{x\}}$. Set $z = x$. Then we see that the conditions of Lemma 1.4 can be satisfied. We conclude that X is transitive.

PROPOSITION 1.11. *Each hereditarily compact space is transitive.*

Proof. (The idea of the proof was outlined in Remark 1.3. Because of Proposition 1.10 a further simplification is now possible.) Let $(X, \mathcal{T})$ be a hereditarily compact space and let V be an entourage belonging to the fine quasi-uniformity of $(X, \mathcal{T})$. Then there is a sequence $(V_n)_{n\in\mathbb{N}}$ of neighbornets of $(X, \mathcal{T})$ such that $V_{n+1}^2 \subseteq V_n$ for each $n \in \mathbb{N}$ and such that $V_1 \subseteq V$. Let $\mathcal{S}$ be the quasi-pseudo-metrizable topology induced by the quasi-uniformity generated by $\{V_n : n \in \mathbb{N}\}$ on X. Since $\mathcal{S}$ is coarser than $\mathcal{T}$, the space $(X, \mathcal{S})$ is hereditarily compact. Hence the Pervin quasi-uniformity $\mathcal{P}(\mathcal{S})$ is the coarsest compatible quasi-uniformity for $(X, \mathcal{S})$. Since $(X, \mathcal{S})$ admits a quasi-uniformity with a countable base, the space $(X, \mathcal{S})$ has a σ-interior preserving topology by Lemma 1.1 and, thus, is transitive by Proposition 1.10. Hence V contains a transitive $\mathcal{S}$-neighbornet of X. Since $\mathcal{S}$ is coarser than $\mathcal{T}$, we conclude that $(X, \mathcal{T})$ is transitive.

Let us remark that the only known example of a nontransitive compact Hausdorff space depends on the set-theoretic axiom $b = c$ [**7**].

It seems interesting to note that with some additional work one can strengthen Proposition 1.10 considerably.

Each topological space X such that $\{\overline{\{x\}} : x \in X\}$ is a σ-closure-preserving collection is transitive.

Proof. By our assumption there is an increasing sequence $(H_n)_{n\in\mathbb{N}}$ of subsets of X such that $\bigcup \{H_n : n \in \mathbb{N}\} = X$ and such that for each $n \in \mathbb{N}$, $\mathcal{T}_n = \{X \setminus \overline{\{x\}} : x \in H_n\} \cup \{X\}$ is interior preserving. For each $x \in X$ choose an $n(x) \in \mathbb{N}$ such that $x \in H_{n(x)}$. Set $T_n(x) = \bigcap \{G : x \in G \in \mathcal{T}_n\}$ whenever $n \in \mathbb{N}$ and $x \in X$; set $T_n = \bigcup \{\{x\} \times T_n(x) : x \in X\}$ whenever $n \in \mathbb{N}$. Consider an arbitrary entourage V of the fine quasi-uniformity of X. Let $(V_n)_{n \in \mathbb{N}}$ be a sequence of neighbornets of X such that $V_1^2 \subseteq V$ and such that $V_{n+1}^2 \subseteq V_n$ for each $n \in \mathbb{N}$. Set $S_n = V_{n+1} \cap T_n$ whenever $n \in \mathbb{N}$. Note that for each $x \in X$ and each $k \in \mathbb{N}$ we have that $S_{n(x)}^{-k}(x) = S_{n(x)}^{-1}(x)$, because $S_{n(x)}^{-k}(x) \subseteq T_{n(x)}^{-k}(x) \subseteq T_{n(x)}^{-1}(x) = \overline{\{x\}} \subseteq S_{n(x)}^{-1}(x)$ by the definition of $T_{n(x)}$. Set $H = \bigcup \{S_{n(x)}^{-1}(x) \times S_{n(x)}(x) : x \in X\}$ and $T = \bigcup \{H^n : n \in \mathbb{N}\}$. Observe that T is a transitive neighbornet of X. We wish to show that $T \subseteq V$. Assume the contrary. Then there is a minimal $s \in \mathbb{N}$ such that $H^s \nsubseteq V$. Clearly $s \neq 1$. Let $(x_0, x_s) \in H^s \setminus V$. There are points x_i $(i = 1, \ldots, s-1)$ and a_i $(i = 0, \ldots, s-1)$ of X such that $(x_i, x_{i+1}) \in S_{n(a_i)}^{-1}(a_i) \times S_{n(a_i)}(a_i)$ whenever $i \in \{0, \ldots, s-1\}$. Note that if $x, y \in X$, $(a, b) \in S_{n(x)}^{-1}(x) \times S_{n(x)}(x)$, $(b, c) \in S_{n(y)}^{-1}(y) \times S_{n(y)}(y)$ and $n(x) \geq n(y)$, then $(a, c) \in S_{n(y)}^{-1}(y) \times S_{n(y)}(y)$, since $S_{n(y)}^{-3}(y) = S_{n(y)}^{-1}(y)$. Thus if there is a $j \in \{0, \ldots, s-2\}$ such that $n(a_j) \geq n(a_{j+1})$, then $(x_0, x_s) \in H^{s-1} \setminus V$—a contradiction to the minimality of s. Therefore we conclude that $n(a_i) < n(a_{i+1})$ for each $i \in \{0, \ldots, s-2\}$. However since

$$x_0 \in S_{n(a_0)}^{-2} S_{n(a_1)}^{-2} \ldots S_{n(a_{s-1})}^{-2}(x_s) \subseteq V_{n(a_0)}^{-1} V_{n(a_1)}^{-1} \ldots V_{n(a_{s-1})}^{-1}(x_s) \subseteq V_{n(a_0)}^{-2}(x_s) \subseteq V_1^{-2}(x_s) \subseteq V^{-1}(x_s),$$

we have reached another contradiction. We deduce that $T \subseteq V$ and that X is transitive.

As an application of our results we want to characterize the topological spaces that have the property that their fine quasi-uniformity is quasi-metrizable.

PROPOSITION 1.12. *The fine quasi-uniformity of a T_1 space X has a countable base if and only if X is a quasi-metrizable space with only finitely many nonisolated points.*

In order to prove this proposition we need some auxiliary results.

PROPOSITION 1.13. *If the fine quasi-uniformity of a topological space X has a countable base, then X is transitive.*

Proof. Since the fine quasi-uniformity of a topological space induces its Pervin quasi-proximity, we conclude by Lemma 1.1 that X has a σ-interior-preserving topology. Hence X is transitive by Proposition 1.10.

REMARK 1.14. The fine quasi-uniformity of each first-countable space $(X, \mathcal{T})$ in which all but finitely many points have a smallest neighborhood has a countable base.

Proof. Assume that $x_1, \ldots, x_k$ are the finitely many points of X without a smallest neighborhood. For each $i \in \{1, \ldots, k\}$ let $\{g(n, x_i) : n \in \mathbb{N}\}$ be an open decreasing neighborhood base at x_i. For each $n \in \mathbb{N}$ set $\mathcal{T}_n = \{G \in \mathcal{T} : x_i \in G$ and $i \in \{1, \ldots, k\}$ imply that $g(n, x_i) \subseteq G\} \cup \{g(n, x_i) : i = 1, \ldots, k\} \cup \{X\}$. Clearly $\mathcal{T} = \bigcup \{\mathcal{T}_n : n \in \mathbb{N}\}$ and $\mathcal{T}_n$ is an interior-preserving open cover of X whenever $n \in N$. In particular X has a σ-interior-preserving topology. For each $x \in X$ and $n \in \mathbb{N}$ set $S_n(x) = \bigcap \{G : x \in G \in \mathcal{T}_n\}$. Let C be an arbitrary transitive neighbornet of X. There is an $n \in \mathbb{N}$ such that for each $i \in \{1, \ldots, k\}$ we have that $g(n, x_i) \subseteq C(x_i)$. Let $y \in X$. If $x_i \in C(y)$ for some $i \in \{1, \ldots, k\}$, then $g(n, x_i) \subseteq C(x_i) \subseteq C(y)$. Thus $C(y) \in \mathcal{T}_n$. Hence we have shown that

$S_n(y) \subseteq C(y)$ whenever $y \in X$. We conclude that the fine transitive quasi-uniformity for X has a countable base. Since X has a σ-interior-preserving topology, X is transitive by Proposition 1.10. Hence the fine quasi-uniformity for X has a countable base.

LEMMA 1.15. *Let X be a topological space, the fine quasi-uniformity of which has a countable base. Let Y be the subspace of X consisting of the points of X without a smallest neighborhood in X. Then each strictly decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of Y-open subsets of Y has a nonempty intersection.*

Proof. Assume the contrary. Let $(G_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of Y-open subsets of Y that has an empty intersection. Since X is transitive by Proposition 1.13, we can assume that the fine quasi-uniformity for X has a countable decreasing base $\{T_n : n \in \mathbb{N}\}$ consisting of transitive neighbornets. Inductively we define a sequence $(x_n)_{n\in\mathbb{N}}$ of points of Y and a strictly increasing sequence of positive integers $(k(n))_{n\in\mathbb{N}}$ such that $T_{k(n)+1}(x_n) \subset T_{k(n)}(x_n)$ and $(T_{k(n)}(x_n) \cap Y) \subseteq G_n$ whenever $n \in \mathbb{N}$. Note that $\mathscr{C} = \{X\} \cup \{T_{k(n)+1}(x_n) : n \in \mathbb{N}\}$ is an interior-preserving open cover of X, as $\bigcap\{G_n : n \in \mathbb{N}\} = \varnothing$ and each point of $X \setminus Y$ has a smallest neighborhood in X. Since $\{T_n : n \in \mathbb{N}\}$ is a base of the fine quasi-uniformity for X, there is an $n \in \mathbb{N}$ such that $T_n(x) \subseteq \bigcap\{D : x \in D \in \mathscr{C}\}$ whenever $x \in X$. Choose an $s \in \mathbb{N}$ such that $k(s) \geq n$. Then $T_{k(s)+1}(x_s) \subset T_{k(s)}(x_s) \subseteq T_n(x_s) \subseteq T_{k(s)+1}(x_s)$, because $T_{k(s)+1}(x_s) \in \mathscr{C}$—a contradiction. This completes the proof of the lemma.

Proof of Proposition 1.12. Let X be a T_1 space the fine quasi-uniformity of which has a countable (decreasing) base $\{U_n : n \in \mathbb{N}\}$. Then X is quasi-metrizable (see e.g. [**2**, p. 4]). Let x be a nonisolated point of X. Consider the sequence $([\operatorname{int} U_n(x)] \setminus \{x\})_{n\in\mathbb{N}}$ of open sets in X. Since X is a T_1 space, this sequence has an empty intersection. We conclude by Lemma 1.15 that x is an isolated point in the subspace Y of the nonisolated points of X. Furthermore, by the same lemma, each collection of pairwise disjoint nonempty Y-open subsets of Y must be finite. We deduce that X is a quasi-metrizable T_1 space containing only finitely many nonisolated points. The converse follows from Remark 1.14.

We conclude this section with the promised construction of a nontransitive space.

EXAMPLE. Let $X = \mathscr{C}[-1, 1]$ be the set of the continuous real-valued functions defined on the compact interval $[-1, 1]$ of real numbers equipped with the well-known norm defined by

$$\|f\| = \max\{|f(x)| : x \in [-1, 1]\}, \quad \text{for each} \quad f \in X.$$

For each $n \in \mathbb{N}$ and each $f \in X$ set $H_n(f) = \{g \in X : \|f - g\| < 2^{-n}\}$. $\mathscr{S}$ will denote the topology induced on X by the norm $\| \ \|$. As usual, by id we will denote the element of X defined by $\operatorname{id}(x) = x$ for each $x \in [-1, 1]$. For each $n \in \mathbb{N}$ set

$$V_n = \left\{(f, g) \in X \times X : g \in \left[H_n\left(f + \frac{\operatorname{id}}{2^n}\right) \cup \{f\}\right]\right\}.$$

First we want to check that $\{V_n : n \in \mathbb{N}\}$ is a countable base for a quasi-uniformity $\mathscr{U}$ on X.

Clearly by definition V_n is reflexive for each $n \in \mathbb{N}$. It is also easy to show that the sequence $(V_n)_{n\in\mathbb{N}}$ is decreasing. (To this end the inequality

$$\left\|f+\frac{\mathrm{id}}{2^n}-g\right\| \le \left\|f+\frac{\mathrm{id}}{2^{n+1}}-g\right\| + \left\|\frac{\mathrm{id}}{2^{n+1}}\right\|$$

can be used.) Let $n\in\mathbb{N}$ and $(f,g)\in V_{n+1}$, $(g,h)\in V_{n+1}$, where $f,g,h\in X$ such that $f\neq g$ and $g\neq h$. Then

$$\left\|f-h+\frac{\mathrm{id}}{2^n}\right\| = \left\|\left(f-g+\frac{\mathrm{id}}{2^{n+1}}\right)+\left(g-h+\frac{\mathrm{id}}{2^{n+1}}\right)\right\| < 2^{-(n+1)}+2^{-(n+1)} = 2^{-n}.$$

Hence $(f,h)\in V_n$. Thus $V_{n+1}^2\subseteq V_n$ for each $n\in\mathbb{N}$.

Next we want to show that $(X, \mathcal{T}(\mathcal{U}))$ is not transitive. It will suffice to prove that V_3 does not contain a transitive $\mathcal{T}(\mathcal{U})$-neighbornet.

Assume the contrary. Let T denote a transitive $\mathcal{T}(\mathcal{U})$-neighbornet contained in V_3. For each $m,n\in\mathbb{N}$ set $A_{m,n}=\{f\in X: H_m(f)\cap V_n(f)\subseteq T(f)$ and $H_k(f)\cap V_{n-1}(f)\nsubseteq T(f)$ whenever $k\in\mathbb{N}\}$. We want to show that $\bigcup\{A_{m,n}: m,n\in\mathbb{N}\}=X$. Let $f\in X$. Then there exists an $n\in\mathbb{N}$ such that $V_n(f)\subseteq T(f)$, because T is a $\mathcal{T}(\mathcal{U})$-neighbornet of X. Hence there is a minimal $n\in\mathbb{N}$ such that $H_m(f)\cap V_n(f)\subseteq T(f)$ for some $m\in\mathbb{N}$. Note that $n\neq 1$, because $H_m(f)\cap V_1(f)\subseteq T(f)$ for some $m\in\mathbb{N}$ is impossible. Assume the contrary. Then we have that $H_m(f)\cap V_1(f)\subseteq V_3(f)$. Consider now the function g defined by

$$g(x)=\begin{cases} f(x)+\dfrac{x+1}{4}-\dfrac{1}{2^{m+4}}, & \text{if } -1\le x<-1+\dfrac{1}{2^{m+1}},\\[2ex] f(x)+\dfrac{1}{2^{m+4}}, & \text{if } -1+\dfrac{1}{2^{m+1}}\le x\le 1.\end{cases}$$

A straightforward calculation shows that $g\in X$, $g\in H_m(f)$ and $g\in V_1(f)$. However for $x=-1+1/2^{m+1}$ we have that $|f(x)+x/8-g(x)|=\frac{1}{8}$. Hence $g\notin V_3(f)$—a contradiction. Thus $n\neq 1$. We conclude that $\bigcup\{A_{m,n}: m,n\in\mathbb{N}\}=X$. By Baire's Category Theorem there are a nonempty $\mathcal{S}$-open set G and $m,n\in\mathbb{N}$ such that $G\subseteq \mathrm{cl}_{\mathcal{S}} A_{m,n}$.

Let $f\in G\cap A_{m,n}$. There is a $k\in\mathbb{N}$ such that $k\ge m$ and such that $H_k(f)\subseteq G$. Set $B=A_{m,n}\cap H_k(f)\cap V_n(f)$. We want to argue that $H_k(f)\cap V_n(f)\subseteq \mathrm{cl}_{\mathcal{S}} B$: Note that $f\in B$. Let $g\in H_k(f)\cap V_n(f)$ and $g\neq f$. Then $g\in H_k(f)\subseteq G\subseteq \mathrm{cl}_{\mathcal{S}} A_{m,n}$ and, thus, for an arbitrary $\mathcal{S}$-neighborhood V of g we have that $[V\cap H_k(f)\cap V_n(f)]\cap A_{m,n}\neq\varnothing$, since $H_k(f)\cap V_n(f)$ is an $\mathcal{S}$-neighborhood of g. Hence $g\in \mathrm{cl}_{\mathcal{S}} B$.

We observe next that it will suffice to show that

$$H_k(f)\cap V_{n-1}(f)\subseteq \bigcup\{H_k(s)\cap V_n(s): s\in B\}. \qquad (*)$$

Since in this case we get that $H_k(f)\cap V_{n-1}(f)\subseteq\bigcup\{H_k(s)\cap V_n(s): s\in B\}\subseteq T(B)\subseteq T[H_k(f)\cap V_n(f)]\subseteq TT(f)\subseteq T(f)$, because $B\subseteq A_{m,n}$, $k\ge m$, $f\in A_{m,n}$ and T is transitive. However this contradicts our assumption that $f\in A_{m,n}$. We will conclude that V_3 cannot contain a transitive $\mathcal{T}(\mathcal{U})$-neighbornet of X as soon as we know that inequality $(*)$ is correct.

Let $h\in H_k(f)\cap V_{n-1}(f)$. If $h=f$, then $h\in B$. Thus $h\in\bigcup\{H_k(s)\cap V_n(s): s\in B\}$ in this case. Hence we assume that $h\neq f$. Then for $p=(h+f)/2$ one easily checks that

$h \in H_k(p) \cap V_n(p)$ and $p \in H_k(f) \cap V_n(f)$. For instance $p \in V_n(f)$, because

$$\left\|f + \frac{\mathrm{id}}{2^n} - p\right\| = \left\|\frac{f}{2} + \frac{\mathrm{id}}{2^n} - \frac{h}{2}\right\| = \frac{1}{2}\left\|f + \frac{\mathrm{id}}{2^{n-1}} - h\right\| < 2^{-n}.$$

In particular, since, as we have shown above, $H_k(f) \cap V_n(f) \subseteq \mathrm{cl}_{\mathcal{S}} B$, we get that $p \in \mathrm{cl}_{\mathcal{S}} B$. Clearly there is $r \in \mathbb{N}$ such that $\|h - p\| + 2^{-r} < 2^{-k}$ and $\|p + (\mathrm{id}/2^n) - h\| + 2^{-r} < 2^{-n}$. Let $t \in H_r(p) \cap B$. Note that

$$\|h - t\| \leq \|h - p\| + \|p - t\| < 2^{-k}$$

and

$$\left\|t + \frac{\mathrm{id}}{2^n} - h\right\| \leq \|t - p\| + \left\|p + \frac{\mathrm{id}}{2^n} - h\right\| < 2^{-n}.$$

Hence $h \in H_k(t) \cap V_n(t)$. We have shown that inequality $(*)$ is satisfied.

2. *uqu* and *uqp* spaces. It is known that a topological space is a *uqu* space if and only if it is a hereditarily compact space in which there is no strictly decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of open sets with an open intersection (see [**8**, Remark on p. 41] and [**1**]). Let us begin this section with another characterization of *uqu* spaces in the class of hereditarily compact spaces. Clearly it is motivated by Proposition 2.2 of [**4**]. Recall that a topological space X is a *Baire* space if the intersection of countably many open dense subsets of X is dense in X.

Proposition 2.1. *A hereditarily compact space is a uqu space if and only if each of its closed subspaces is a Baire space.*

Proof. Using the characterization of *uqu* spaces given above it is immediately clear that each closed subspace of a *uqu* space is a *uqu* space. Hence each closed subspace of a *uqu* space is a Baire space by Lemma 2.1 of [**4**].

In order to prove the converse let X be a hereditarily compact space, each closed subspace of which is a Baire space. Assume that there exists a strictly decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of open sets in X with an open intersection. Since X is hereditarily compact, the closed set $X \setminus \bigcap\{G_n : n \in \mathbb{N}\}$ is the union of finitely many irreducible closed sets F_j $(j = 1, \ldots, k$ and $k \in \mathbb{N})$ [**12**, p. 903]. Hence there exists an $i \in \{1, \ldots, k\}$ such that $G_n \cap F_i \neq \emptyset$ whenever $n \in \mathbb{N}$. Since F_i is irreducible, $G_n \cap F_i$ is dense in F_i for each $n \in \mathbb{N}$. Since by our assumption F_i is a Baire space, we deduce that $\bigcap\{G_n : n \in \mathbb{N}\} \cap F_i \neq \emptyset$—a contradiction. Hence $\bigcap\{G_n : n \in \mathbb{N}\}$ is not open. We conclude that X is a *uqu* space.

Next we want to prove the result about finite products of *uqu* spaces mentioned in the introduction.

Proposition 2.2. *The product of finitely many uqu spaces is a uqu space.*

Proof. Let X and Y be *uqu* spaces. It suffices to show that $X \times Y$ is a *uqu* space. Since X and Y are hereditarily compact and the property of hereditary compactness is finitely multiplicative [**12**, Theorem 9], it suffices to show that $\bigcap\{G_n : n \in \mathbb{N}\}$ is not open whenever $(G_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of open sets in $X \times Y$.

Assume the contrary. Let $(G_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of open sets in

$X \times Y$ with an open intersection. Since $X \times Y$ is hereditarily compact and the set of open rectangles in $X \times Y$ is a base for $X \times Y$, each open set in $X \times Y$ is the union of finitely many open rectangles. Hence by our assumption $\bigcap \{G_n : n \in \mathbb{N}\} = \bigcup \{B_j : j = 1, \ldots, m\}$ where $m \in \mathbb{N}$, $B_j = \mathrm{pr}_X B_j \times \mathrm{pr}_Y B_j$ and B_j is open in $X \times Y$ $(j = 1, \ldots, m)$. Then

$$(X \times Y) \setminus \bigcap \{G_n : n \in \mathbb{N}\} = \bigcap \{[(X \setminus \mathrm{pr}_X B_j) \times Y] \cup [X \times (Y \setminus \mathrm{pr}_Y B_j)] : j = 1, \ldots, m\}$$
$$= \bigcup \{F_p : p = 1, \ldots, 2^m\}$$

where we can assume that each F_p $(p = 1, \ldots, 2^m)$ is a closed rectangle in $X \times Y$. Since the sequence $(G_n)_{n \in \mathbb{N}}$ is strictly decreasing, there is an $i \in \{1, \ldots, 2^m\}$ such that $G_n \cap F_i \neq \emptyset$ for each $n \in \mathbb{N}$. By induction we wish to define a sequence of open rectangles $(A_k)_{k \in \mathbb{N}}$ in $X \times Y$ such that $A_k \subseteq G_k$ whenever $k \in \mathbb{N}$ and such that $G_n \cap \bigcap \{A_l : l = 1, \ldots, k\} \cap F_i \neq \emptyset$ whenever $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

Assume now that $k \in \mathbb{N}$ and that A_l is defined for each $l \in \mathbb{N}$ such that $l < k$. Since G_k is the union of finitely many open rectangles, say $G_k = \bigcup \{K_i : i = 1, \ldots, f\}$ where $f \in \mathbb{N}$ and each K_i is a nonempty open rectangle in $X \times Y$, and since by our hypothesis we have that

$$\emptyset \neq G_n \cap \bigcap \{A_l : l = 1, \ldots, k-1\} \cap F_i = G_n \cap \bigcup \{K_i : i = 1, \ldots, f\}$$
$$\cap \bigcap \{A_l : l = 1, \ldots, k-1\} \cap F_i$$

whenever $n \geq k$ (for $k = 1$ the expression $\bigcap \{A_l : l = 1, \ldots, k-1\}$ means $X \times Y$), there is an $s \in \{1, \ldots, f\}$ such that $G_n \cap K_s \cap \bigcap \{A_l : l = 1, \ldots, k-1\} \cap F_i \neq \emptyset$ whenever $n \in \mathbb{N}$. Set $A_k = K_s$. This completes the construction of the sequence $(A_k)_{k \in \mathbb{N}}$.

For each $n \in \mathbb{N}$ set $H_n = \bigcap \{\mathrm{pr}_X A_k \cup (X \setminus \mathrm{pr}_X F_i) : k = 1, \ldots, n\}$ and $P_n = \bigcap \{\mathrm{pr}_Y A_k \cup (Y \setminus \mathrm{pr}_Y F_i) : k = 1, \ldots, n\}$. Then $(H_n)_{n \in \mathbb{N}}$ is a decreasing sequence of open sets in X and $(P_n)_{n \in \mathbb{N}}$ is a decreasing sequence of open sets in Y. Furthermore by the construction of the sequence $(A_k)_{k \in \mathbb{N}}$ we have that $H_n \cap \mathrm{pr}_X F_i \neq \emptyset$ and $P_n \cap \mathrm{pr}_Y F_i \neq \emptyset$ whenever $n \in \mathbb{N}$. Since X and Y are *uqu* spaces and since $\mathrm{pr}_X F_i$ is closed in X and $\mathrm{pr}_Y F_i$ is closed in Y, we conclude by the characterization of *uqu* spaces given in the beginning of this section that there are an $x \in \bigcap \{H_n \cap \mathrm{pr}_X F_i : n \in \mathbb{N}\}$ and a $y \in \bigcap \{P_n \cap \mathrm{pr}_Y F_i : n \in \mathbb{N}\}$. Hence $(x, y) \in \bigcap \{A_k : k \in \mathbb{N}\} \cap F_i$. On the other hand $(\bigcap \{A_k : k \in \mathbb{N}\}) \cap F_i \subseteq (\bigcap \{G_n : n \in \mathbb{N}\}) \cap F_i = \emptyset$—a contradiction. We have shown that $\bigcap \{G_n : n \in \mathbb{N}\}$ cannot be open. Hence $X \times Y$ is a *uqu* space.

COROLLARY 2.3. *If X and Y are uqu spaces, then the fine quasi-uniformity for $X \times Y$ is the product quasi-uniformity of the fine quasi-uniformities for X and Y.*

Proof. Since $X \times Y$ is a *uqu* space, the (compatible) product quasi-uniformity on $X \times Y$ of the fine quasi-uniformities for X and Y is the unique compatible quasi-uniformity for $X \times Y$.

We finish this paper with a result on *uqp* spaces. We recall that by a result of **[6]** a topological space X is a *uqp* space if and only if its topology is the unique base of open sets that is closed under finite unions and finite intersections. (In this result it is assumed that $\bigcap \emptyset = X$). Although there are simple examples of first-countable *uqp* spaces that are not hereditarily compact [**6**, p. 561], we are going to show now that such examples cannot be T_1 spaces.

PROPOSITION 2.4. *A uqp T_1 space with countable pseudo-character is hereditarily compact (and, thus, countable).*

Proof. Let X be a *uqp* T_1 space with countable pseudo-character. First let us note that X is compact, because if x is any point in an arbitrary *uqp* space Z, then $Z\setminus\overline{\{x\}}$ is compact [**6**, p. 561]. Consider the set $\mathscr{H}=\{G\subseteq X:G$ is open in X and not compact$\}$ ordered by set-theoretic inclusion. We want to show that $\mathscr{H}$ is empty. Of course this will mean that X is hereditarily compact [**12**, Theorem 1]. Assume that $\mathscr{H}\neq\emptyset$. Since for any nonempty chain $\mathscr{K}\subseteq\mathscr{H}$ we have that $\bigcup\mathscr{K}\in\mathscr{H}$, we conclude by Zorn's lemma that there exists a maximal element $Y\in\mathscr{H}$. Let $\mathscr{C}$ be a collection of open subsets of Y covering Y such that no finite subcollection of $\mathscr{C}$ covers Y. We observe that since X is compact, there is an $x\in(X\setminus Y)$. Since X is a space with countable pseudo-character, there is a sequence $(G_n)_{n\in\mathbb{N}}$ of open sets of X such that $\bigcap\{G_n:n\in\mathbb{N}\}=\{x\}$. By the maximality of Y the set $Y\cup G_n$ is compact whenever $n\in\mathbb{N}$. Hence for each $n\in\mathbb{N}$ there exists a finite subcollection $\mathscr{C}_n$ of $\mathscr{C}$ covering $Y\setminus G_n$. Since $\bigcap\{G_n:n\in\mathbb{N}\}=\{x\}$ and $x\notin Y$, the collection $\mathscr{D}=\bigcup\{\mathscr{C}_n:n\in\mathbb{N}\}$ is a countable subcollection of $\mathscr{C}$ covering Y. Since no finite subcollection of $\mathscr{C}$ covers Y, it is clear by induction we can define an open cover $\{D_n:n\in\mathbb{N}\}$ of Y and a sequence $(x_n)_{n\in\mathbb{N}}$ of points of Y such that $D_n\in\mathscr{D}$ and $x_n\in D_n\setminus(\bigcup\{D_k:k<n,\ k\in\mathbb{N}\})$ whenever $n\in\mathbb{N}$. Let $H_1=Y\setminus\{x_{2n}:n\in\mathbb{N}\}$ and $H_2=Y\setminus\{x_{2n-1}:n\in\mathbb{N}\}$. Note that H_1 and H_2 are open in X, because X is a T_1 space and $\{D_n:n\in\mathbb{N}\}$ is a cover of Y. Moreover, of course, $H_1\cup H_2=Y$. Let $\mathscr{U}$ be an ultrafilter on X that contains the collection $\{Y\setminus C:C\in\mathscr{C}\}$. Since $Y\in\mathscr{U}$, there is a $t\in\{1,2\}$ such that $H_t\in\mathscr{U}$. Since X is a *uqp* space, there is a (nonempty) finite collection $\mathscr{M}$ of open sets in X such that $\bigcap\mathscr{M}\subseteq H_t$ and such that for each $M\in\mathscr{M}$ the set M contains a limit point of $\mathscr{U}$ [**6**, Proposition]. Since Y does not contain any limit points of $\mathscr{U}$, we conclude that $Y\subset Y\cup M$ whenever $M\in\mathscr{M}$. Therefore by the maximality of Y we see that for each $M\in\mathscr{M}$ the set $Y\cup M$ is compact. Hence for each $M\in\mathscr{M}$ there is a finite subcollection of $\{D_n:n\in\mathbb{N}\}$ covering $Y\setminus M$. Since $\bigcap\mathscr{M}\subseteq H_t\subseteq Y$ and $\mathscr{M}$ is finite, it follows that there exists a finite subcollection $\mathscr{P}$ of $\{D_n:n\in\mathbb{N}\}$ such that $(Y\setminus H_t)\subseteq\bigcup\mathscr{P}$. Hence $x_n\in\bigcup\mathscr{P}$ for infinitely many $n\in\mathbb{N}$—a contradiction to the construction of the sequence $(x_n)_{n\in\mathbb{N}}$. We conclude that $\mathscr{H}$ is empty. Thus X is hereditarily compact. The last assertion of the proposition follows from the fact that a hereditarily compact space with countable pseudo-character is countable [**12**, Proof of Theorem 11].

REFERENCES

1. L. M. Brown, On topological spaces with a unique compatible quasi-uniformity, *Glasgow Math. J.* **18** (1977), 11–12.

2. P. Fletcher and W. F. Lindgren, *Quasi-uniform spaces,* Lectures Notes Pure Appl. Math. 77, (New York, 1982).

3. V. Gregori and J. Ferrer, Quasi-metrization and completion for Pervin's quasi-uniformity, *Stochastica* **6** (1982), 151–155.

4. V. Gregori and J. Ferrer, Some classes of topological spaces with unique quasi-uniformity, *Canad. Math. Bull.* **29** (1986), 446–449.

5. H. P. A. Künzi, On strongly quasi-metrizable spaces, *Arch. Math.* **41** (1983), 57–63.

6. H. P. A. Künzi, Topological spaces with a unique compatible quasi-proximity, *Arch. Math.* **43** (1984), 559–561.

7. H. P. A. Künzi, Transitivity is neither hereditary nor finitely productive, *Top. Appl.* **19** (1985), 165–168.

8. H. P. A. Künzi, Topological spaces with a unique compatible quasi-uniformity, *Canad. Math. Bull.* **29** (1986), 40–43.

9. H. P. A. Künzi and G. C. L. Brümmer, Sobrification and bicompletion of totally bounded quasi-uniform spaces, *Math. Proc. Camb. Phil. Soc.* **101** (1987), 237–247.

10. W. F. Lindgren, Topological spaces with a unique compatible quasi-uniformity, *Canad. Math. Bull.* **14** (1971), 369–372.

11. J. Rainwater, Spaces whose finest uniformity is metric, *Pacific J. Math.* **9** (1959), 567–570.

12. A. H. Stone, Hereditarily compact spaces, *Amer. J. Math.* **82** (1960), 900–916.

Department of Mathematics
University of Berne
Sidlerstrasse 5
3012 Berne
Switzerland