Universal Gröbner Bases for Maximal Minors

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Bernstein, Sturmfels, and Zelevinsky proved in 1993 that the maximal minors of a matrix of variables form a universal Gröbner basis. We present a very short proof of this result, along with broad generalization to matrices with multihomogeneous structures. Our main tool is a rigidity statement for radical Borel fixed ideals in multigraded polynomial rings.

1 Introduction

A set G of polynomials in a polynomial ring S over a field is said to be a universal Gröbner basis if it is a Gröbner basis with respect to every term order on S. Twenty years ago Bernstein, Sturmfels, and Zelevinsky proved in [3, 14] that the set of the maximal minors of an $m \times n$ matrix of variables X is a universal Gröbner basis. Indeed, in [14], the assertion is proved for certain values of m, n and the general problem is reduced to a combinatorial statement that it is then proved in [3]. Kalinin gave in [10] a different proof of this result. Boocher proved in [4] that any initial ideal of the ideal $I_m(X)$ of maximal minors of X has a linear resolution (or, equivalently in this case, defines a Cohen–Macaulay ring).

The goal of this paper is two-fold. First, we give a quick proof of the results mentioned above. Our proof is based on a specialization argument, see Section 2, and,

unlike the proofs given in [3, 10], does not involve combinatorial techniques. Secondly, we show that similar statements hold in a more general setting, for matrices of linear forms satisfying certain homogeneity conditions. More precisely, in Section 4, we show that the set of maximal minors of an $m \times n$ matrix $L = (L_{ij})$ of linear forms is a universal Gröbner basis, provided that L is column-graded. By this, we mean that the entries L_{ij} belong to a polynomial ring with a standard $\mathbb{Z}^n$-graded structure, and that $\deg L_{ij} = e_j \in \mathbb{Z}^n$. Under the same assumption, we show that every initial ideal of $I_m(L)$ has a linear resolution. Furthermore, the projective dimension of $I_m(L)$ and of its initial ideals is $n - m$, unless $I_m(L) = 0$ or a column of L is identically 0 (note that, under these assumptions, the codimension of $I_m(L)$ can be smaller than $n - m + 1$).

If instead L is row-graded, that is, $\deg L_{ij} = e_i \in \mathbb{Z}^m$, then we prove in Section 5 that $I_m(L)$ has a universal Gröbner basis of elements of degree m and that every initial ideal of $I_m(L)$ has a linear resolution, provided that $I_m(L)$ has the expected codimension. Note that in the row-graded case, the maximal minors do not form a universal Gröbner basis in general (since every maximal minor might have the same initial term).

The proofs of the statements in Sections 4 and 5 are based on a rigidity property of radical Borel fixed ideals in a multigraded setting. This property has been observed already in special cases, for example, by Cartwright and Sturmfels [5, Proof of Theorem 2.1] in their studies of the multigraded Hilbert scheme associated to the Segre product of two projective spaces and by Aholt *et al.* [1, Lemmas 2.4 and 2.5] in the study of varieties associated with multilinear constructions arising in computer vision. In a polynomial ring with a standard $\mathbb{Z}^m$-grading, one can take generic initial ideals with respect to the product of general linear groups preserving the grading. Such generic initial ideals are Borel fixed. The main theorem of Section 3 asserts that if two $\mathbb{Z}^m$-graded Borel fixed ideals I, J have the same Hilbert series and I is radical, then $I = J$. This is the rigidity property that we referred to, and which has very strong consequences. For instance, if I is Cohen–Macaulay, radical, and Borel fixed, then all the multihomogeneous ideals with the same multigraded Hilbert series are Cohen–Macaulay and radical as well.

Extensive computations performed with CoCoA [6] led to the discovery of the results and examples presented in this paper.

2 A Simple Proof of the Universal Gröbner Bases Theorem

Let K be a field, $S = K[x_{ij} : 1 \le i \le m, \ 1 \le j \le n]$. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates, and let $I_m(X)$ be the ideal generated by the maximal minors of X. The goal of

this section is giving a quick proof of the following result of Bernstein, Sturmfels, and Zelevinsky [3, 14], and Boocher [4].

Theorem 2.1. The set of maximal minors of X is a universal Gröbner basis of $I_m(X)$, that is, a Gröbner basis of $I_m(X)$ with respect to all the term orders. Furthermore, every initial ideal of $I_m(X)$ has the same Betti numbers as $I_m(X)$.   □

Let R be a standard graded K-algebra. We denote by

$$\mathrm{HS}(M, y) = \sum_{i \in \mathbb{Z}} (\dim_K M_i) y^i \in \mathbb{Q}[|y|]\left[y^{-1}\right]$$

the Hilbert series of a finitely generated graded R-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$. We need the following "Hilfssatz".

Lemma 2.2. Let R be a standard graded K-algebra, let M, T be finitely generated graded R-modules, and $x_1, \ldots, x_s \in R$ be homogeneous elements of positive degree. Set $J = (x_1, \ldots, x_s)$. Suppose that:

(1) $\mathrm{HS}(T, y) \geq \mathrm{HS}(M, y)$ coefficientwise,
(2) $\mathrm{HS}(T/JT, y) = \mathrm{HS}(M/JM, y)$,
(3) $x_1, \ldots, x_s$ is an M-regular sequence.

Then $\mathrm{HS}(M, y) = \mathrm{HS}(T, y)$ and $x_1, \ldots, x_s$ is a T-regular sequence.   □

Proof.   For $i = 1, \ldots, s$, set $J_i = (x_1, \ldots, x_i)$, $T_i = T/J_i T$, $d_i = \deg(x_i)$, and $g_i(y) = \prod_{j=1}^{i} (1 - y^{d_j}) \in \mathbb{Q}[y]$. Furthermore, set $T_0 = T$ and for $i = 0, \ldots, s - 1$ denote by K_{i+1} the submodule $\{m \in T_i : x_{i+1} m = 0\}$ of T_i shifted by $-d_{i+1}$.

The four terms exact complex induced the multiplication by x_{i+1} on T_i yields

$$\mathrm{HS}(T_{i+1}, y) = (1 - y^{d_{i+1}})\mathrm{HS}(T_i, y) + \mathrm{HS}(K_{i+1}, y),$$

and hence

$$\mathrm{HS}(T/JT, y) = g_s(y)\mathrm{HS}(T, y) + \sum_{j=1}^{s} g_{s-j}(y)\mathrm{HS}(K_j, y).$$

Since $\mathrm{HS}(T/JT, y) = \mathrm{HS}(M/JM, y) = g_s(y)\mathrm{HS}(M, y)$ by assumption, we have

$$g_s(y)(\mathrm{HS}(T, y) - \mathrm{HS}(M, y)) + \sum_{j=1}^{s} g_{s-j}(y)\mathrm{HS}(K_j, y) = 0.$$

Since $\mathrm{HS}(T, y) - \mathrm{HS}(M, y)$ and $\mathrm{HS}(K_j, y)$ are powers series with nonnegative coefficients and $g_i(y)$ are polynomials with positive least degree term coefficient, we obtain that $\mathrm{HS}(T, y) = \mathrm{HS}(M, y)$ and $K_j = 0$ for $j = 1, \ldots, s$. ∎

Proof of Theorem 2.1.   We may assume without loss of generality that K is infinite. Let $A = (a_{ij})$ be an $m \times n$ matrix with entries in K^*, such that all its m-minors are nonzero. It exists because K is infinite. Consider the K-algebra map

$$\Phi : S = K[x_{ij} : 1 \le i \le m, \ 1 \le j \le n] \to K[y_1, \ldots, y_n]$$

induced by $\Phi(x_{ij}) = a_{ij} y_j$ for every i, j. By construction, the kernel of Φ is generated by $n(m-1)$ linear forms. Let $Y = \Phi(X) = (a_{ij} y_j)$. Denote by $[c_1, \ldots, c_m]_W$ the minor with column indices $c_1, \ldots, c_m$ of an $m \times n$ matrix W. By construction

$$\Phi([c_1, \ldots, c_m]_X) = [c_1, \ldots, c_m]_A y_{c_1} \cdots y_{c_m}.$$

Hence, by our assumption on A, we have that

$$\Phi(I_m(X)) = I_m(Y) = (y_{c_1} \cdots y_{c_m} : 1 \le c_1 < \cdots < c_m \le n),$$

that is, $I_m(Y)$ is generated by all the square-free monomials in $y_1, \ldots, y_n$ of total degree m. In particular, it has codimension $n - m + 1$. It follows that $I_m(Y)$ is resolved by the Eagon–Northcott complex, hence $\mathrm{Ker}\,\Phi$ is generated by a regular sequence on $S/I_m(X)$. Now let $\prec$ be any term order on S and let D be the ideal generated by the leading terms of the maximal minors of X with respect to $\prec$. We have $D \subseteq \mathrm{in}_\prec(I_m(X))$ and

$$\Phi(\mathrm{in}_\prec([c_1, \ldots, c_m]_X)) = \Phi(x_{\sigma_1 c_1} \cdots x_{\sigma_m c_m}) = a_{\sigma_1 c_1} \cdots a_{\sigma_m c_m} y_{c_1} \cdots y_{c_m},$$

for some $\sigma \in S_m$. Hence,

$$\Phi(D) = I_m(Y).$$

We apply Lemma 2.2 to the following data:

$$M = S/I_m(X), \quad T = S/D, \ \text{and} \ J = \mathrm{Ker}\,\Phi,$$

to conclude that $D = \mathrm{in}_\prec(I_m(X))$. The Betti numbers of $I_m(X)$ equal those of D since, in this case, $T/JT = M/JM$ and J is generated by a sequence of linear forms which is regular on both T and N. ∎

Can one generalize Theorem 2.1 to ideals of maximal minors of matrices of linear forms? In Sections 4 and 5, we will give positive answers to the question by assuming that the matrix is multigraded, either by rows or by columns. In general, however, one cannot expect too much, as the following remark shows.

Remark 2.3.   One can consider various properties related to the existence of Gröbner bases and various families of matrices of linear forms. For instance, we can look at the following properties for the ideal $I_m(L)$ of m-minors of an $m \times n$ matrix L of linear forms in a polynomial ring S:

(a) $I_m(L)$ has a Gröbner basis of elements of degree m with respect to some term order and possibly after a change of coordinates.

(b) $I_m(L)$ has a Gröbner basis of elements of degree m with respect to some term order and in the given coordinates.

(c) Property (b) holds and the associated initial ideal has a linear resolution.

(d) $I_m(L)$ has a universal Gröbner basis of elements of degree m.

We consider the following families of matrices of linear forms:

(1) No further assumption on L is made.

(2) $I_m(L)$ has codimension $n - m + 1$.

(3) The entries of L are linearly independent over the base field (i.e., L arises from a matrix of variables by a change of coordinates).

What we know (and do not know) is summarized in the following table:

	(a)	(b)	(c)	(d)
(1)	No	No	No	No
(2)	Yes	No	No	No
(3)	Yes	?	?	No

There are ideals of 2-minors of 2×4 matrices of linear forms that define non-Koszul rings (see [7, Remark 3.6]). Hence, those ideals cannot have a single Gröbner bases of quadrics (not even after a change of coordinates). This explains the four "no" in the first row of the table.

Every initial ideal of the ideal of 2-minors of

$$\begin{pmatrix} x_1 + x_2 & x_3 & x_3 \\ 0 & x_1 & x_2 \end{pmatrix}$$

has a generator in degree 3 if the characteristic of the base field is $\neq 2$. The codimension of $I_2(L)$ is 2. This example explains the three "no" in the second row of the table. The "yes" in the second row follows because the generic initial ideal with respect to the reverse lexicographic order is generated in degree m under assumption (2).

Finally, the matrix

$$\begin{pmatrix} x_1 & x_4 & x_3 \\ x_5 & x_1 + x_6 & x_2 \end{pmatrix}$$

belongs to the family (3) and the initial ideal with respect to any term order satisfying $x_1 > x_2 > \cdots > x_6$ has a generator in degree 3. This explains the "no" in the third row. The "yes" is there because (3) is contained in (2).

It remains open whether the ideal of maximal minors of a matrix in the family (3) has at least a Gröbner basis of elements of degree m in the given coordinates, and whether the associated initial ideal has a linear resolution. □

3 Radical and Borel Fixed Ideals

The goal of the section is to prove Theorem 3.5, a rigidity result for multigraded Hilbert series associated to radical multigraded Borel fixed ideals. Special cases of it appeared already in [1, 5]. We will introduce the G-multidegree, a generalization of the notion of multidegree of Miller and Sturmfels [11, Chapter 8], that allows us to deal with minimal components of various codimensions in the case of Borel fixed ideals.

Given $m \in \mathbb{N}$ and $(n_1, \ldots, n_m) \in \mathbb{N}^m$, let S be the polynomial ring in the set of variables x_{ij} with $1 \leq i \leq m$ and $1 \leq j \leq n_i$ over an infinite field K, with grading induced by $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. Let M be a finitely generated, $\mathbb{Z}^m$-graded S-module. The multigraded Hilbert series of M is

$$\mathrm{HS}(M, y) = \mathrm{HS}(M, y_1, \ldots, y_m) = \sum_{a \in \mathbb{Z}^m} (\dim M_a) y^a \in \mathbb{Q}\, [|y_1, \ldots, y_m|] \left[y_1^{-1}, \ldots, y_m^{-1} \right].$$

The group $G = \mathrm{GL}_{n_1}(K) \times \cdots \times \mathrm{GL}_{n_m}(K)$ acts on S as the group of $\mathbb{Z}^m$-graded K-algebra automorphisms. Let $B = B_{n_1}(K) \times \cdots \times B_{n_m}(K)$ be the Borel subgroup of G consisting of the upper triangular matrices with arbitrary nonzero diagonal entries. An ideal I is said to be Borel fixed (or $\mathbb{Z}^m$-graded Borel fixed to avoid confusion with the standard graded setting) if $g(I) = I$ for every $g \in B$. Borel fixed ideals are monomial ideals that can be characterized in a combinatorial way by means of exchange properties as it is explained in [8, Theorem 15.23]. Indeed in [8, Theorem 15.23], details are given in

the standard graded setting but, as observed in [2, Section 1], the same characterization holds also in the multigraded setting.

Given a term order $\prec$ such that $x_{ik} \prec x_{ij}$ for every $j > k$, one can associate a (multi-graded) generic initial ideal $\mathrm{gin}_{\prec}(I)$ to any $\mathbb{Z}^m$-graded ideal of I of S. As in the standard graded setting, it turns out that $\mathrm{gin}_{\prec}(I)$ is a $\mathbb{Z}^m$-graded Borel fixed ideal.

The prime $\mathbb{Z}^m$-graded Borel fixed ideals are easy to describe. Set

$$U = \{(b_1, \ldots, b_m) \in \mathbb{N}^m : b_i \leq n_i \text{ for every } i = 1, \ldots, m\}.$$

The following assertion follows immediately from the definition.

Lemma 3.1.   For every vector $b \in U$, the ideal

$$P_b = (x_{ij} : i = 1, \ldots, m \text{ and } 1 \leq j \leq b_i)$$

is prime and $\mathbb{Z}^m$-graded Borel fixed, and every prime $\mathbb{Z}^m$-graded Borel fixed ideal is of this form.   □

Lemma 3.2.   The associated prime ideals of a $\mathbb{Z}^m$-graded Borel fixed ideal I are $\mathbb{Z}^m$-graded Borel fixed.   □

Proof.   Let P be an associated prime to S/I. Clearly, P is monomial (i.e., generated by variables) because I is monomial. We have to prove that if $x_{ij} \in P$, then also $x_{ik} \in P$ for all $k < j$. We may write $P = I : f$ for some monomial f. Let α be the exponent of x_{ij} in f. Consider $g \in B$ such that $g(x_{ij}) = x_{ij} + x_{ik}$ and fixes all the other variables. Then $g(x_{ij} f) \in I$ because $x_{ij} f \in I$. The monomial $x_{ik}^{\alpha+1} f / x_{ij}^{\alpha}$ appears with nonzero coefficient in $g(x_{ij} f)$. Hence, $x_{ik}^{\alpha+1} f / x_{ij}^{\alpha} \in I$ and $x_{ik}^{\alpha+1} f \in I$. In other words, $x_{ik}^{\alpha+1} \in I : f = P$ and hence $x_{ik} \in P$.   ■

Lemma 3.3.   Let I be a radical $\mathbb{Z}^m$-graded Borel fixed ideal. Then every minimal generator of I has multidegree bounded above by $(1, 1, \ldots, 1) \in \mathbb{Z}^m$.   □

Proof.   Consider a generator f of I of degree $(a_1, \ldots, a_m) \in \mathbb{N}^m$. Assume that one of the a_i's is positive, say $a_1 \geq 1$. We will show that $a_1 = 1$. We may write $f = ug$ with u a monomial of degree $a_1 e_1$ and g a monomial of degree $(0, a_2, \ldots, a_m)$. Set $j = \min\{k : x_{1k} | u\}$. By construction, $x_{1j} g$ divides f. Since I is $\mathbb{Z}^m$-graded Borel fixed, we have $x_{1j}^{a_1} g \in I$. Since I is radical, we have $x_{1j} g \in I$ and $x_{1j} g$ is a proper divisor of f, unless $a_1 = 1$.   ■

Lemma 3.4. Let I be a radical $\mathbb{Z}^m$-graded Borel fixed ideal and let $\{P_{b_1}, \ldots, P_{b_c}\}$, with $b_1, \ldots, b_c \in U$, be the minimal primes of I. Then I is the Alexander dual of the polarization of

$$J = \left(\prod_{b_{ij} > 0} x_j^{b_{ij}} : i = 1, \ldots, c \right) \subset K[x_1, \ldots, x_m].$$

In particular, if all the generators of I have the same multidegree, then I has a linear resolution. $\qquad\square$

Proof. The first assertion follows immediately from the definition of polarization and Alexander duality, see [11, Chapter 5]. For the second, one observes that if all the generators of I have degree, say, $\sum_{i \in A} e_i \in \mathbb{Z}^m$ with $A \subset \{1, \ldots, m\}$, then I is the Alexander dual of the polarization of an ideal $J \subset K[x_1, \ldots, x_m]$ involving only variables x_i with $i \in A$ and whose radical is $(x_i : i \in A)$. Hence, J defines a Cohen–Macaulay ring, and so does its polarization. Finally, one applies [9, Theorem 8.1.9]. $\qquad\blacksquare$

The goal of this section is to prove the following theorem.

Theorem 3.5. Let $I, J \subset S$ be $\mathbb{Z}^m$-graded Borel fixed ideals such that $\mathrm{HS}(I, y) = \mathrm{HS}(J, y)$. If I is radical, then $I = J$. $\qquad\square$

Remark 3.6. In the case $m = 1$, the assertion of Theorem 3.5 is a simple consequence of the fact that (ordinary) Borel fixed radical ideals are indeed prime ideals of the form $(x_1, \ldots, x_j)$. Similarly, the case $n_i = 1$ for every $i = 1, \ldots, m$ is also obvious because in that case $\mathbb{Z}^m$-graded Borel ideals are simply monomial ideals in m variables, and they are determined by their $\mathbb{Z}^m$-graded Hilbert series. $\qquad\square$

The most important consequence of Theorem 3.5 is the following rigidity result.

Corollary 3.7. Let I be a radical $\mathbb{Z}^m$-graded Borel fixed ideal. For every multigraded ideal J with $\mathrm{HS}(J, y) = \mathrm{HS}(I, y)$, one has:

(a) $\mathrm{gin}_\prec(J) = I$ for every term order $\prec$.
(b) J is radical.
(c) J has a linear resolution whenever I has a linear resolution.
(d) S/J is Cohen–Macaulay whenever S/I is Cohen–Macaulay.
(e) $\beta_{i,a}(S/J) \leq \beta_{i,a}(S/I)$ for every $i \in \mathbb{N}$ and $a \in \mathbb{Z}^m$ and $\beta_{i,a}(S/J) = 0$ if $a \not\leq (i, i, \ldots, i) \in \mathbb{Z}^m$. $\qquad\square$

Proof. The ideal $\text{gin}_\prec(J)$ is a $\mathbb{Z}^m$-graded Borel fixed ideal and $\text{HS}(J, y) = \text{HS}(\text{gin}_\prec(J), y)$. Since, by assumption, $\text{HS}(J, y) = \text{HS}(I, y)$, we may conclude, by virtue of Theorem 3.5 that $\text{gin}_\prec(J) = I$. This proves (a). Statements (b)–(d) are standard applications of well-known principles. Finally, (e) follows from Lemma 3.3 and from the bounds derived from the Taylor complex, see [11, Chapter 6]. ∎

In order to prove Theorem 3.5, we need the following definitions.

Definition 3.8. Let M be a finitely generated $\mathbb{Z}^m$-graded S-module. The Hilbert series $\text{HS}(M, y)$ is rational, that is, it can be written as

$$\text{HS}(M, y) = \frac{\mathcal{K}(M, y)}{\prod_{i=1}^m (1 - y_i)^{n_i}},$$

where $\mathcal{K}(M, y) \in \mathbb{Z}[y_1, \ldots, y_m][y_1^{-1}, \ldots, y_m^{-1}]$ is a uniquely determined Laurent polynomial that is called the $\mathcal{K}$-polynomial of M. □

Definition 3.9. For every finitely generated $\mathbb{Z}^m$-graded S-module M, we set

$$\mathcal{C}(M, y) = \mathcal{K}(M, 1 - y_1, \ldots, 1 - y_m) \in \mathbb{Z}[|y_1, \ldots, y_m|].$$

We define the G-multidegree of M as

$$\mathcal{G}(M, y) = \sum c_a y^a \in \mathbb{Z}[y_1, \ldots, y_m],$$

where the sum runs over the $a \in \mathbb{Z}^m$ which are minimal in the support of $\mathcal{C}(M, y)$ and c_a is the coefficient of y^a in $\mathcal{C}(M, y)$. □

Example 3.10. Let $m = 2$, $n_1 = 2$, and $n_2 = 2$. Let $M = S/I$ where

$$I = (x_{11}^2, x_{11}x_{12}, x_{12}x_{21}, x_{21}x_{22}).$$

Then
$$\text{HS}(S/I, y_1, y_2) = \frac{1 - y_2^2 - y_1 y_2 - 2y_1^2 + y_1 y_2^2 + y_1^2 y_2 + y_1^2 y_2^2 + y_1^3 - y_1^3 y_2^2}{(1 - y_1)^2 (1 - y_2)^2}.$$

Hence,
$$\mathcal{K}(S/I, y_1, y_2) = 1 - y_2^2 - y_1 y_2 - 2y_1^2 + y_1 y_2^2 + y_1^2 y_2 + y_1^2 y_2^2 + y_1^3 - y_1^3 y_2^2$$

and
$$\mathcal{C}(S/I, y_1, y_2) = y_1^3 y_2^2 - 2y_1^3 y_2 - 2y_1^2 y_2^2 + 3y_1^2 y_2 + y_1 y_2.$$

Therefore,

$$\mathcal{G}(M, y) = y_1 y_2.$$ □

The following result follows immediately from the definition above.

Proposition 3.11. (1) Let P be a prime ideal generated by variables and let $a(P)$ be the vector whose ith coordinate is $\#(P \cap \{x_{i1}, \dots, x_{in_i}\})$. Then

$$\mathcal{G}(S/P, y) = y^{a(P)}.$$

(2) One has $a(P_b) = b$ for every $b \in U$ and for $b_1, b_2 \in U$ one has $P_{b_1} \subseteq P_{b_2}$ if and only if $y^{b_1} \mid y^{b_2}$. □

The key observation is the following proposition.

Proposition 3.12. Let I be a $\mathbb{Z}^m$-graded Borel fixed ideal. One has

$$\mathcal{G}(S/I, y) = \sum_{i=1}^{c} \mathrm{length}((S/I)_{P_{b_i}}) y^{b_i},$$

where $\mathrm{Min}(I) = \{P_{b_1}, \dots, P_{b_c}\}$ with $b_1, \dots, b_c \in U$. □

Proof. In order to compute the $\mathcal{K}$-polynomial of $M = S/I$, consider a filtration of $\mathbb{Z}^m$-graded modules

$$0 = M_0 \subset M_1 \subset \dots \subset M_h = M,$$

such that $M_i/M_{i-1} \simeq S/P_i(-v_i)$. Here P_i is a $\mathbb{Z}^m$-graded monomial prime ideal and $v_i = (v_{i1}, \dots, v_{im}) \in \mathbb{Z}^m$. Existence of such a filtration follows from basic commutative algebra facts, see [8, Proposition 3.7]. Furthermore,

$$\mathrm{Min}(I) \subseteq \mathrm{Ass}(S/I) \subseteq \{P_1, \dots, P_h\},$$

and the set of minimal elements in $\{P_1, \dots, P_h\}$ is exactly $\mathrm{Min}(I)$. Hence, we have

$$\mathcal{K}(S/I, y) = \sum_{i=1}^{h} \mathcal{K}(S/P_i(-v_i), y) = \sum_{i=1}^{h} y^{v_i} \mathcal{K}(S/P_i, y).$$

It follows that

$$\mathcal{C}(S/I, y) = \sum_{i=1}^{h} \prod_{j=1}^{m} (1 - y_j)^{v_{ij}} \mathcal{C}(S/P_i, y).$$

Then the support of the polynomial $\prod_{j=1}^{m}(1-y_j)^{v_{ij}}\mathcal{C}(S/P_i, y)$ contains exactly one minimal element, namely $y^{a(P_i)}$, which appears in the polynomial with coefficient 1. It follows that $\mathcal{G}(S/I, y)$ is obtained as the sum of the terms which are minimal in the support of the polynomial

$$\sum_{i=1}^{h} y^{a(P_i)}. \tag{3.1}$$

Now the elements that are minimal in the support of (3.1) are exactly the y^{b_i} corresponding to the minimal primes P_{b_i}. This follows from Proposition 3.11, since if $P \subseteq P'$, then $y^{a(P)}|y^{a(P')}$. Finally, by standard localization arguments, we have that each minimal prime P_{b_i} appears in the multiset $\{P_1, \dots, P_h\}$ as many times as length$((S/I)_{P_{b_i}})$. $\blacksquare$

Remark 3.13. The notion of geometric degree is discussed in the paper [13] as a variant of the ordinary degree that takes into consideration the presence of minimal primes of different codimension. As proved in Proposition 3.12, the G-multidegree is a variant of Miller and Sturmfels' multidegree which encodes minimal associated primes of different codimension, for ideals which are $\mathbb{Z}^m$-graded Borel fixed (but unfortunately not in general, as Example 3.10 shows). $\square$

We are finally ready to prove Theorem 3.5.

Proof of Theorem 3.5. Since I and J have the same Hilbert series, we have that $\mathcal{C}(S/I, y) = \mathcal{C}(S/J, y)$ and hence

$$\mathcal{G}(S/I, y) = \mathcal{G}(S/J, y).$$

It follows by Proposition 3.12 that $\mathrm{Min}(I) = \mathrm{Min}(J)$. Since I is radical, we deduce that $J \subseteq I$ and the Hilbert series forces the equality $I = J$. $\blacksquare$

4   Column-Graded Ideals of Maximal Minors

Consider $S = K\left[x_{ij} : 1 \leq i \leq m, \ 1 \leq j \leq n\right]$ graded by $\deg(x_{ij}) = e_j \in \mathbb{Z}^n$. The group of $\mathbb{Z}^n$-graded K-algebra automorphism is $G = \mathrm{GL}_m(K)^n$ acting by linear substitution on the columns. The generic initial ideals computed below refer to this multigraded structure.

Let $L = (L_{ij})$ be an $m \times n$ matrix of linear forms which is column-graded, that is, whose entries L_{ij} satisfy $\deg(L_{ij}) = e_j$. In other words,

$$L_{ij} = \sum_{k=1}^{m} \lambda_{ijk} x_{kj},$$

where $\lambda_{ijk} \in K$. As a first direct application of Corollary 3.7, we have the following theorem.

Theorem 4.1. Let $L = (L_{ij})$ be an $m \times n$ matrix which is column-graded and assume that the codimension of $I_m(L)$ is $n - m + 1$. Then $I_m(L)$ is radical and the maximal minors of L form a universal Gröbner basis of it. Furthermore, every initial ideal of $I_m(L)$ is radical, has a linear resolution, and its Betti numbers equal those of $I_m(L)$. □

Proof. We may assume without loss of generality that K is infinite. Let $I = (x_{1j_1} x_{1j_2} \cdots x_{1j_m} : 1 \le j_1 < j_2 < \cdots < j_m \le n)$. Then I is generated by the maximal minors of a column-graded matrix whose (i, j)th entry is $a_{ij} x_{1j}$ with sufficiently general scalars a_{ij}. Since the codimension of I is $n - m + 1$, by the Eagon–Northcott complex it follows that I and $I_m(L)$ have the same multigraded Hilbert series and the same Betti numbers. Since I is radical and $\mathbb{Z}^n$-graded Borel fixed, we may apply Corollary 3.7 with $J = I_m(L)$ or J equal any initial ideal of $I_m(L)$. It follows that $I_m(L)$ and all its initial ideals are radical and they have a linear resolution (and hence the same Betti numbers). Finally, the maximal minors of L form a universal Gröbner basis since distinct maximal minors have distinct multidegree. ■

We want now to generalize Theorem 4.1 and get rid of the assumption on the codimension of $I_m(L)$.

Theorem 4.2. Let $L = (L_{ij})$ be an $m \times n$ matrix which is column-graded. Then:

(a) The maximal minors of L form a universal Gröbner basis of $I_m(L)$.

(b) $I_m(L)$ is radical and it has a linear resolution.

(c) Any initial ideal J of $I_m(L)$ is radical and has a linear resolution. In particular, $\beta_{i,j}(I_m(L)) = \beta_{i,j}(J)$ for all i, j.

(d) Assume that $I_m(L) \ne 0$ and that no column of L is identically 0. Then the projective dimension of $I_m(L)$ (and hence of all its initial ideals) is $n - m$. □

Proof.   Again we may assume that K is infinite. Fix a term order $\prec$. It is not restrictive to assume that $x_{1j} \succ x_{ij}$ for all $i \neq 1$ and j; set for simplicity $x_j = x_{1j}$. Let

$$I = (x_{j_1} \cdots x_{j_m} \mid [j_1, \ldots, j_m]_L \neq 0).$$

We claim that $I = \mathrm{gin}_{\prec}(I_m(L))$. First we note that $I \subseteq \mathrm{gin}_{\prec}(I_m(L))$. This is because if $[j_1, \ldots, j_m]_L \neq 0$, then $I_m(L)$ contains a nonzero element of degree $e_{j_1} + \cdots + e_{j_m}$ and its initial term in generic coordinates is $x_{j_1} \cdots x_{j_m}$.

Next note that I is the Stanley–Reisner ideal of the Alexander dual of the matroid dual M_L^* of the matroid M_L associated to L. As such, I has a linear resolution by a result of Eagon and Reiner [9, Theorem 8.1.9], since M_L^* is Cohen–Macaulay, see [12, Chapter III, Section 3]. By Buchberger's Algorithm, in order to prove that $I = \mathrm{gin}_{\prec}(I_m(L))$, it suffices to show that any S-pair associated to a linear syzygy among the generators of I reduces to 0. Any such linear syzygy involves at most $m + 1$ column indices in total. After renaming the column indices, we may assume that the syzygy in question involves the column indices $\{1, 2, \ldots, m + 1\}$. Set

$$d = e_1 + e_2 + \cdots + e_{m+1}.$$

To prove that the S-polynomial reduces to 0, we may as well prove that $\dim I_m(L)_d \leq \dim I_d$. Let

$$W = \left\{ u : 1 \leq u \leq m + 1 \text{ and } [\{1, \ldots, m + 1\} \setminus \{u\}]_L \neq 0 \right\}.$$

Renaming if needed, we may assume that

$$W = \{1, 2, \ldots, s\}.$$

By definition, I_d is generated by the set of monomials

$$\left\{ \frac{x_1 x_2 \cdots x_{m+1}}{x_j} x_{ij} : j = 1, \ldots, s \text{ and } i = 1, \ldots, m \right\},$$

whose cardinality is easily seen to be $sm - s + 1$. Hence, it remains to prove that

$$\dim I_m(L)_d \leq sm - s + 1.$$

Denote by Ω the first syzygy module of $\left\{ [\{1, \ldots, m + 1\} \setminus \{u\}]_L : u = 1, \ldots, s \right\}$. Since

$$\dim I_m(L)_d = sm - \dim \Omega_d,$$

it suffices to show that

$$\dim \Omega_d \geq s - 1.$$

Let L_1 be the submatrix of L consisting of the first s columns of L. Since the rows of L_1 are elements of Ω_d, it is enough to show that L_1 has at least $s - 1$ linearly independent rows over K. By contradiction, if this is not the case, by applying invertible K-linear operations to the rows of L we may assume that the last $m - s + 2$ rows of L_1 are identically zero. In particular, the minor $[2, \ldots, m + 1]_L = 0$, contradicting our assumptions.

Since I is $\mathbb{Z}^n$-graded Borel fixed and radical with $\mathrm{HS}(I, y) = \mathrm{HS}(I_m(L), y)$, we may apply Corollary 3.7 and deduce (b) and (c). Then (a) follows, as in the proof of Theorem 4.1, from the fact that each nonzero maximal minor of L has a distinct multidegree. Finally, for (d) one observes that, under the assumption that no column of L is 0 and $I_m(L) \neq 0$, the ideal I is nonzero and each of the variables $x_1, \ldots, x_n$ is involved in some generator. Then M_L^* has dimension $n - m$ and has no cone-points. This implies that the Stanley–Reisner ring of M_L^* has regularity $n - m$, as it is 2-Cohen–Macaulay (see [12, p. 94] for details). By [9, Proposition 8.1.10], the projective dimension of I (that is, the Alexander dual of M_L^*) is $n - m$.   ∎

5 Row-Graded Ideals of Maximal Minors

In this section, we treat ideals of maximal minors of row-graded matrices. Consider $S = K[x_{ij} : i = 1, \ldots, m$ and $j = 1, \ldots, n]$ graded by $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. The group of $\mathbb{Z}^m$-graded K-algebra automorphism is $G = \mathrm{GL}_n(K)^m$ acting by linear substitution on the rows. The generic initial ideals computed below refer to this multigraded structure.

Let $L = (L_{ij})$ be an $m \times n$ matrix of linear forms with $m \leq n$. We assume that L is row-graded, that is, the entries L_{ij} satisfy $\deg(L_{ij}) = e_i$. In other words,

$$L_{ij} = \sum_{k=1}^{m} \lambda_{ijk} x_{ik},$$

where $\lambda_{ijk} \in K$. Observe that in the row-graded case, we cannot expect that the maximal minors of X form a Gröbner basis simply because every maximal minor might have the same leading term. Nevertheless, we can prove the following theorem.

Theorem 5.1. Let $L = (L_{ij})$ be an $m \times n$ matrix which is row-graded and assume that the codimension of $I_m(L)$ is $n - m + 1$. Then $I_m(L)$ is radical and every initial ideal is generated by elements of total degree m (equivalently, there is a universal Gröbner basis

of elements of degree m). Furthermore, every initial ideal of $I_m(L)$ is radical, has a linear resolution, and its Betti numbers equal those of $I_m(L)$. □

Set

$$I = (x_{1j_1} \cdots x_{mj_m} : j_1 + \cdots + j_m \leq n).$$

Theorem 5.1 follows immediately from Corollary 3.7 and from the following proposition, by observing that I is radical and $\mathbb{Z}^m$-graded Borel fixed. Note that Corollary 3.7 also implies that $I = \text{gin}_{\prec}(I_m(L))$ for every term order $\prec$.

Proposition 5.2. Under the assumptions of Theorem 5.1, the $\mathbb{Z}^m$-graded Hilbert series of $I_m(L)$ equals that of I. □

Proof. The Hilbert series of $I_m(L)$ equals that of $I_m(X)$ with $X = (x_{ij})$, because both ideals are resolved by the multigraded version of the Eagon–Northcott complex. Hence, we may assume without loss of generality that $L = X$. We will show that $S/I_m(X)$ and S/I have the same $\mathcal{K}$-polynomial.

Let $\mathcal{K}_{m,n}(y)$ be the $\mathcal{K}$-polynomial of $S/I_m(X)$. By looking at the diagonal initial ideal of $I_m(X)$, one obtains the recursion:

$$\mathcal{K}_{m,n}(y) = (1 - y_m)\mathcal{K}_{m,n-1}(y_1, \ldots, y_m) + y_m \mathcal{K}_{m-1,n-1}(y_1, \ldots, y_{m-1}).$$

Solving the recursion or, alternatively, by looking directly at the multigraded version of the Eagon–Northcott complex, one obtains

$$\mathcal{K}_{m,n}(y) = 1 - \left(\prod_{i=1}^{m} y_i\right) \sum_{k=0}^{n-m} (-1)^k \binom{n}{m+k} h_k(y_1, \ldots, y_m), \tag{5.1}$$

where $h_k(y_1, \ldots, y_m)$ is the complete symmetric polynomial of degree k, that is, the sum of all the monomials of degree k in the variables $y_1, \ldots, y_m$.

We now compute the $\mathcal{K}$-polynomial of S/I. For $b \in [n]^m$, set $x_b = x_{1b_1} x_{2b_2} \cdots x_{mb_m}$ so that

$$I = (x_b : b \in \mathbb{N}_{>0}^m \text{ and } |b| \leq n).$$

Extend the natural partial order, that is, $x_b \leq x_c$ if $b \leq c$ coefficientwise, to a total order $<$ (no matter how). For every $b \in [n]^m$, we have

$$(x_c : x_c < x_b) : x_b = (x_{ij} : i = 1, \ldots, m \text{ and } 1 \leq j < b_i). \tag{5.2}$$

Filtering I according to $<$ and using (5.2), one obtains

$$\mathcal{K}(S/I, y) = 1 - y_1 \cdots y_m \sum_b \prod_{i=1}^m (1 - y_i)^{b_i - 1}, \tag{5.3}$$

where the sum $\sum_b$ is over all the $b \in \mathbb{N}_{>0}^m$ and $|b| \leq n$. Setting $c = b - (1, \ldots, 1)$ and replacing b with c in (5.3), we obtain

$$\mathcal{K}(S/I, y) = 1 - y_1 \cdots y_m \sum_c \prod_{i=1}^m (1 - y_i)^{c_i}, \tag{5.4}$$

where the sum $\sum_c$ is over all the $c \in \mathbb{N}^m$ and $|c| \leq n - m$. We may rewrite the last expression as

$$\mathcal{K}(S/I, y) = 1 - y_1 \cdots y_m \sum_{k=0}^{n-m} h_k(1 - y_1, \ldots, 1 - y_m). \tag{5.5}$$

Taking into consideration (5.1) and (5.5), it remains to prove that

$$\sum_{k=0}^{n-m} h_k(1 - y_1, \ldots, 1 - y_m) = \sum_{k=0}^{n-m} (-1)^k \binom{n}{m+k} h_k(y_1, \ldots, y_m), \tag{5.6}$$

or equivalently, by replacing y_i with $-y_i$ in (5.6), it is left to show that

$$\sum_{k=0}^{n-m} h_k(1 + y_1, \ldots, 1 + y_m) = \sum_{k=0}^{n-m} \binom{n}{m+k} h_k(y_1, \ldots, y_m). \tag{5.7}$$

Setting $t = n - m$, (5.7) is equivalent to the assertion that the equality:

$$\sum_{k=0}^{t} h_k(1 + y_1, \ldots, 1 + y_m) = \sum_{k=0}^{t} \binom{m+t}{m+k} h_k(y_1, \ldots, y_m) \tag{5.8}$$

holds for every m and t. Formula (5.8) can be derived from the more precise:

$$h_t(1 + y_1, \ldots, 1 + y_m) = \sum_{k=0}^{t} \binom{m+t-1}{m+k-1} h_k(y_1, \ldots, y_m). \tag{5.9}$$

Equation (5.9) can be proved by (long and tedious) induction on m. The following simple argument using generating functions was suggested by Christian Krattenthaler.

First note that

$$\sum_{t \geq 0} h_t(y_1, \ldots, y_m) z^t = \prod_{i=1}^{m} \frac{1}{1 - y_i z}. \tag{5.10}$$

Replacing in (5.10) y_i with $y_i + 1$ and observing that

$$\prod_{i=1}^{m} \frac{1}{1 - (y_i + 1)z} = \frac{1}{(1-z)^m} \prod_{i=1}^{m} \frac{1}{1 - y_i \frac{z}{(1-z)}},$$

we have

$$\sum_{t \geq 0} h_t(1 + y_1, \ldots, 1 + y_m) z^t = \sum_{t \geq 0} h_t(y_1, \ldots, y_m) \frac{z^t}{(1-z)^{t+m}}. \tag{5.11}$$

Expanding the right-hand side of (5.11), one obtains (5.9).　　　　■

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